Golden Ratio Based Partitions of the Integers

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Introduction

- Let $\mathbb{Z}^+ = \{1, 2, 3, 4, 5, \ldots\}$.
- A partition of $\mathbb{Z}^+$ is a way of breaking $\mathbb{Z}^+$ into non-overlapping groupings.
- The even and odd integers are a two set partition of $\mathbb{Z}^+$.
- $1, 2, 3, 4, 5, \ldots \cup 2, 4, 6, 8, \ldots$.

Arithmetic Progressions

- A simple way of creating partitions is to take distinct arithmetic progressions in the integers.

Definition

Let $x, r \in \mathbb{Z}^+$. An arithmetic progression is a sequence of the form $f(k) = ak + x$, where $0 \leq r < s$.

- The even integers are given by $E(k) = 2k$ while the odd integers are $O(k) = 2k + 1$.
- More complex sets of progressions give more complex partitions. For instance, we can create a 2 and 3 set case by dividing $\mathbb{Z}^+$ into the groups below.

<table>
<thead>
<tr>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$B_1$</th>
<th>$B_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3m+1$</td>
<td>$3m+2$</td>
<td>$3m+3$</td>
<td>$3m+4$</td>
<td>$3m+5$</td>
</tr>
<tr>
<td>$3m+6$</td>
<td>$3m+7$</td>
<td>$3m+8$</td>
<td>$3m+9$</td>
<td>$3m+10$</td>
</tr>
<tr>
<td>$3m+11$</td>
<td>$3m+12$</td>
<td>$3m+13$</td>
<td>$3m+14$</td>
<td>$3m+15$</td>
</tr>
</tbody>
</table>

The first integers of the sets of the 3 part case are $C_1 = 1, 3, 5, 7, 9, 10, 12, 14, 15, 17, 19, 21, \ldots$.
- $C_2 = 2, 6, 9, 13, 16, 20, 23, 27, 30, 34, 37, 41, 44, \ldots$.
- $C_3 = 4, 11, 18, 25, 32, 39, 46, 53, 60, 67, 74, 81, \ldots$.

While the first in the sets of the 2 part case are $B_1 = 1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, \ldots$.
- $B_2 = 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, \ldots$.

If we classify the rows of the 3 part partition by whether the elements fall into the first or second column of the 2 part partition, we only get 5 out of 8 possibilities. Moreover, this classification has period 12 together with reflection symmetry.

Arithmetic progressions are easy to study because they are periodic. Their partitioning structure is simple. We study partitions composed of semi-periodic sequences.

Beatty Sequences

- The floor function of a number, denoted by $[x]$, is the integer part of $x$.
- $[\alpha] = \lfloor \alpha \rfloor = \frac{1}{\alpha^2} \cdot \lfloor \frac{1}{\alpha^2} \rfloor = 1$ is called the Golden Ratio. We have $[\alpha] = \lfloor \frac{1}{\alpha^2} \rfloor$.

Theorem (Beatty’s theorem)

Let $\alpha, \beta$ be two positive irrational numbers. Let $A$ and $B$ be two sequences such that $g(x) = [\alpha x]$ and $h(x) = [\beta x]$. Then $g(x)$ and $h(x)$ partition $\mathbb{Z}^+$ if and only if $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

- The special property of $\alpha$ is that $\frac{1}{\alpha} = \phi - 1 = 1.61803\ldots$.

- So $\alpha = \lfloor \alpha \rfloor$ and $\beta = \lfloor \alpha \beta \rfloor$ partition $\mathbb{Z}^+$.

Almost Beatty Partition

Beatty’s Theorem does not hold for three (or more) sequences. That is, if $\alpha, \beta, \gamma$ are arbitrary positive numbers, then $[\alpha x], [\beta x], [\gamma x]$ do not partition the positive integers.

Our work concerns constructions we have created which extend the $A, B$ partition. The following construction is in 3 parts.

- $d(k) = \frac{1}{2}([\alpha k]+1)$.
- $d(k) = \frac{1}{2}([\beta k]+1)$.
- $d(k) = \frac{1}{2}([\gamma k]+1)$.

2-Column $\phi$ Partition

- Define the sets $A, B$ as $A = \{(\lfloor \alpha x \rfloor)\}_{x \in \mathbb{Z}}$ and $B = \{(\lfloor \beta x \rfloor)\}_{x \in \mathbb{Z}}$.
- The sequences $a(k)$ and $b(k)$ give a partition of $\mathbb{Z}^+$ with similar structure to the $A, B$ partition.

Grid of 2-Column $\phi$ Partition

<table>
<thead>
<tr>
<th>$k$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
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<th>$8$</th>
<th>$9$</th>
<th>$10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a(k)$</td>
<td>$1$</td>
<td>$2$</td>
<td>$3$</td>
<td>$4$</td>
<td>$5$</td>
<td>$6$</td>
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<td>$8$</td>
<td>$9$</td>
<td>$10$</td>
</tr>
<tr>
<td>$b(k)$</td>
<td>$1$</td>
<td>$2$</td>
<td>$3$</td>
<td>$4$</td>
<td>$5$</td>
<td>$6$</td>
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</table>

Results: Properties of the 3-Column $\phi$ Partition

- Let $\alpha(x)$ denote the fractional part of $x$.
- Let $a(x) = [\alpha x]$ and $b(x) = [\gamma x]$. Then $a(x) + \alpha([a(x)]) = 1$.
- Let $d(k) = \gamma(k) + \beta$ and $e(k) = [\delta x] + 2k$ as above. Then $\{e(k)\} + \alpha([e(k)]) = \{1, 2\}$.
- $\alpha(\phi) = \frac{1}{2}([\phi]+1)$ with $\phi$ on takes one of 8 values:

$\{1, 2, 3, 4, 5, 6, 7, 8\}$.

3-Column $\phi$ Partition

- Define the sets $S, C, D$ as $S = \{\lfloor \phi x \rfloor\}_{x \in \mathbb{Z}}$, $C = \{\lfloor \phi x \rfloor\}_{x \in \mathbb{Z}}$, and $D = \{\lfloor \phi x \rfloor\}_{x \in \mathbb{Z}}$.
- The sequences $d(k), c(k)$, and $s(k)$ give a partition of $\mathbb{Z}^+$ with similar structure to the $A, B$ partition.

Grid of 3-Column $\phi$ Partition

<table>
<thead>
<tr>
<th>$k$</th>
<th>$1$</th>
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<th>$3$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$s(k)$</td>
<td>$1$</td>
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<td>$10$</td>
</tr>
<tr>
<td>$c(k)$</td>
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<td>$2$</td>
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<td>$4$</td>
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<td>$6$</td>
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<td>$9$</td>
<td>$10$</td>
</tr>
<tr>
<td>$d(k)$</td>
<td>$1$</td>
<td>$2$</td>
<td>$3$</td>
<td>$4$</td>
<td>$5$</td>
<td>$6$</td>
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<td>$10$</td>
</tr>
</tbody>
</table>

Column Densities

How do the 2 and 3 columns overlap with one another?

- For a given integer $a(k)$ or $b(k)$, can we figure out whether $\alpha$ lies in $D, C$, or $S$?
- If we mark the rows of $C, D, S$ with $A$ and $B$ dependent on whether the integers in that row lie in $A$ or $B$, only 5 of 8 possibilities occur. What are the frequencies and why?

We can instead mark the $A, B$ integer pairs by how they appear in $D, C$, and $S$. Numerical data suggests the following density values:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$2$</td>
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References