GENERALIZATIONS OF NO K-EQUAL SPACES

BY

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ABSTRACT

We consider generalizations of no k-equal spaces as well as their relations to other concepts. For any topological space X, the n^{th} no k-equal space of X is the space of n points from X such that no k are the same. First, we consider a generalization where each of the points is assigned one of m colors; the interactions between various points are governed by a subset of $\mathbb{N}^m$. We call these spaces polychromatic configuration spaces. We find the homology groups and cohomology rings for two classes of polychromatic configuration spaces of $\mathbb{R}^d$.

Next, we consider the relation between no k-equal spaces of $\mathbb{R}$ and k-trees of simplicial complexes. It was noticed that the first non-trivial homology group of the n^{th} no k-equal space of $\mathbb{R}$ has rank equal to the number of facets in a k-dimensional spanning tree of the n-dimensional hypercube. We give a proof of this that is not reliant on knowledge of these numbers. Furthemore, we prove the analogous fact for a generalization of no k-equal spaces: comb no k-equal spaces.

The k-equal arrangements are a generalization of the braid arrangements. In another direction, Manin and Schectman defined discriminantal arrangements as a generalization of braid arrangements. In the final chapter, we combine these two to define codimension-c discriminantal arrangements. These arise geometrically as no $(d+c)$-intersecting translates of hyperplanes. We give results on the first two non-trivial homology groups of no $(d+c)$-intersecting translates of hyperplanes in $\mathbb{R}^d$.

To my parents, for their constant encouragement.

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CONTENTS

Chapter 1

INTRODUCTION

For any topological space X, the n^{th} *no k-equal space of* X is the subspace of X^n where no k coordinates are equal. No k-equal spaces were initially introduced in work in complexity theory by Björner, Lovász, and Yao [5]. The problem they were considering was given a polyhedron $P \in \mathbb{R}^n$, how hard is it to determine if a given point x is in P? They used linear decision trees to model this complexity. As an example of a polyhedron, they considered the space where at least k coordinates were equal, the k-equal arrangement. Further work by Björner and Lovász gave bounds for the depth of these decision trees in terms of the sum of the Betti numbers of $\mathbb{R}^n - P$ [4]. Thus, to find bounds on the depth of linear decision trees for determining membership in the k-equal arrangement, it is sufficient to find the Betti numbers of their complements, the no k-equal spaces of $\mathbb{R}$.

Björner and Welker found these Betti numbers [6]. Their work used the techniques of Goresky-MacPherson [14] and Ziegler-Živaljević [27] which give methods to find the topology of complements of arrangements using the combinatorics of posets. These methods do not shed light on cohomology rings. Further work by Yuzvinski [26], Baryshnikov [2], and Dobrinskaya and Turchin [9] determined the cohomology rings of the no k-equal spaces of $\mathbb{R}^2$, $\mathbb{R}$, and, finally, $\mathbb{R}^d$ for all $d \geq 1$, respectively. In addition, Baryshnikov and Dobrinskaya and Turchin give explicit geometric representatives for homology.

In Chapter 2, we consider a polychromatic generalization of no k-equal spaces. As with no k-equal spaces, polychromatic configuration spaces of a topological space X arise from points on X. Instead of removing all subspaces where k points are equal, we assign each point one of m colors and remove subspaces prescribed by a set $I \subset \mathbb{N}^m$. When $m = 1$, these are no k-equal spaces. These spaces have been seldom studied thus far. In a talk at IMA, Baryshnikov discussed work that determined the generating function for the Euler characteristic of polychromatic configuration spaces of X whenever X is a compact definable set in some o-minimal structure. We compute the homology groups and cohomology rings of two classes of polychromatic configuration spaces of $\mathbb{R}^d$, decreasing polychromatic configuration spaces and bicolored configuration spaces ($m = 2$).

In Chapter 3, we consider a relation between the n^{th} no k-equal space of $\mathbb{R}$ and spanning trees of the k-skeleton of the n-dimensional hypercube. For any n-dimensional cell complex X and any $k \leq n$, one defines a *k-dimensional spanning tree of X* to be a subset of its k-skeleton satisfying properties analogous to those for spanning trees of graphs. The notion of higher dimensional spanning trees began in the work of Bolker [7] and Kalai [16]. More recently, there has been activity in developing the theory. For an overview of the topic, see work by Duval-Klivans-Martin [10, 12] and Lyons [19].

We show the first non-zero Betti number of the n^{th} no k-equal space of $\mathbb{R}$ is equal to the size of spanning trees of the k-skeleton of n-dimensional hypercubes. This result can be seen numerically, independent of any connection between the two. We provide a geometric relationship between the two objects via a construction we call the *simplicial resolution*. Hence, we achieve the equality without needing any prior knowledge of the values involved. Additionally, we show a second situation where this construction may be used by generalizing to an arrangement that has not yet been studied: the *comb no k-equal arrangement*.

The k-equal arrangements are a generalization of braid arrangements (which are recovered when $k = 2$). In another direction, discriminantal arrangements, originally introduced by Manin and Schectman, also generalize the braid arrangements of a field [20]. These arise geometrically as complements to no $(d + 1)$-intersecting translates of n hyperplanes in $\mathbb{R}^d$. It has been shown that the topology of the complements to discriminantal arrangements is not independent of the choice of hyperplanes [13]. However, Athanasiadis showed that this topology is dependent only on n and d for "very generic" hyperplanes [1] (proving a conjecture of Bayer and Brandt [3]). Bayer and Brandt showed a connection between discriminantal arrangements and the fiber zonotopes of hyperplane arrangements [3]. Work of Koizumi-Numata-Takemura [17], Libgober-Settepanella [18], and Numata-Takemura [22] do computations related to the intersection lattices, strata, and characteristic polynomials, respectively, of discriminantal arrangements.

In Chapter 4, we combine these two generalizations to define codimension-c discriminantal arrangements. These arise geometrically as complements to no $(d+c)$-intersecting translates of hyperplanes in $\mathbb{R}^d$. We give results on the first two non-trivial homology groups of no $(d+c)$-intersecting translates of hyperplanes in $\mathbb{R}^d$.

Chapter 2

POLYCHROMATIC CONFIGURATION SPACES (PCS)

Throughout this chapter, we include 0 as a natural number.

2.1 Preliminaries

Definition. *Let* $m \in \mathbb{N}^{>0}, I \subset \mathbb{N}^m$. *We say* I *is an* ideal *if* $(n_1, \ldots, n_m) \in I$ *and* $n'_i \leq n_i$ *for all* i, *implies* $(n'_1, \ldots, n'_m) \in I$

For each ideal, I, let $\mathcal{B}_{I,d}(n_1, \ldots, n_m)$ denote the space of labeled discs in $\mathbb{R}^d$, n_i of color i, satisfying the following property: for all $(n_1, \ldots, n_m) \notin I$, any intersection containing n_i discs of color i for each i is empty.

Let $\mathcal{B}_d$ denote the little d-discs operad. There exists a left action of $\mathcal{B}_d$ on $\mathcal{B}_{I,d}$:

$$\mathcal{B}_d(r) \times \mathcal{B}_{I,d}(\vec{n}_1) \times \ldots \times \mathcal{B}_{I,d}(\vec{n}_r) \to \mathcal{B}_{I,d}(\vec{n}_1 + \ldots + \vec{n}_r)$$

where the i^{th} disc in $\mathcal{B}_d(r)$ is replaced by the configuration of discs from $\mathcal{B}_{I,d}(\vec{n}_i)$. One can also define a right action; however, it will not be necessary for the discussion here.

The Künneth Theorem for homology gives a map:

$$H_*\mathcal{B}_d(r) \times H_*\mathcal{B}_{I,d}(\vec{n}_1) \times \ldots \times H_*\mathcal{B}_{I,d}(\vec{n}_r) \to H_*(\mathcal{B}_d(r) \times \mathcal{B}_{I,d}(\vec{n}_1) \times \ldots \times \mathcal{B}_{I,d}(\vec{n}_r))$$

Combining this with the induced map on homology from the above action gives an action on homology groups

$$H_*\mathcal{B}_d(r) \times H_*\mathcal{B}_{I,d}(\vec{n}_1) \times \ldots \times H_*\mathcal{B}_{I,d}(\vec{n}_r) \to H_*\mathcal{B}_{I,d}(\vec{n}_1 + \ldots + \vec{n}_r)$$

If $f : X \to \mathcal{B}_d(r)$ is a simplicial map representing $[\alpha] \in H_*\mathcal{B}_d(r)$ and $f_i : X_i \to B_{I,d}(\vec{n}_i)$ are simplicial maps representing $[\alpha_i] \in H_*\mathcal{B}_{I,d}(\vec{n}_i)$, then $g : X \times X_1 \times \cdots \times X_r \to \mathcal{B}_{I,d}(\vec{n}_1 + \ldots + \vec{n}_r)$ defined by

$$g(x, x_1, \ldots, x_r) = f(x) \cdot (f_1(x_1), \ldots, f_r(x_r))$$

is a simplicial map representing $[\alpha] \cdot ([\alpha_1], \ldots, [\alpha_r])$ where $\cdot$ denotes the aforementioned actions. The space $\mathcal{B}_{I,d}(n_1, \ldots n_m)$ is homotopy equivalent to a similar space replacing discs with points.

Definition. *Let $I \subset \mathbb{N}^m$ be an ideal. Let $\vec{n} = (n_1, \ldots, n_m)$. The $\vec{n}$* polychromatic configuration space *of $\mathbb{R}^d$ corresponding to I is the space of labeled points, n_i of color i for all i, such that for all $(\ell_1, \ldots, \ell_m) \notin I$, any intersection containing ℓ_i points of color i for all i is empty. We denote this space by $\mathcal{M}_{I,d}(\vec{n})$.*

This space is the complement in $\mathbb{R}^{(n_1+\ldots+n_m)d}$ to a linear subspace arrangement. We will denote the i^{th} point of color j by x_i^j.

Lemma 2.1. *For all ideals $I \subset N^m$ and all $\vec{n} \in \mathbb{N}^m$, $\mathcal{M}_{I,d}(\vec{n})$ is homotopy equivalent to $\mathcal{B}_{I,d}(\vec{n})$.*

A homotopy equivalence is given by taking the centers of the discs in the arrangement from $\mathcal{B}_{I,d}(\vec{n})$. Because $\mathcal{M}_{I,d}(\vec{n})$ is homotopy equivalent to $\mathcal{B}_{I,d}(\vec{n})$, the action of $H_*\mathcal{B}_d$ on $H_*\mathcal{B}_{I,d}$ gives an action of $H_*\mathcal{M}_d$ on $H_*\mathcal{M}_{I,d}$ where $\mathcal{M}_d(n)$ is the n^{th} configuration space of $\mathbb{R}^d$.

As mentioned in the introduction, we will at times restrict to particular classes of polychromatic configuration spaces, one of which is decreasing polychromatic configuration spaces.

Definition. *Let $I \subset \mathbb{N}^m$ be an ideal. We will call I* decreasing *if for all $i \leq m$, if $(n_1, \ldots, n_i, 0, \ldots, 0) \notin I$, $n_i > 0$, and $(n_1, \ldots, n_{i-1}, n_i - 1, 0, \ldots, 0) \in I$, then we have $(n_1, \ldots, n_j - 1, \ldots, n_i, 0, \ldots, 0) \in I$ for all $j < i$ such that $n_j > 0$.*

If X is the polychromatic configuration space of a decreasing ideal, we call it decreasing.

The term decreasing comes from the functions $f_I^j : \mathbb{N}^j \to \mathbb{N} \cup \{\infty\}$ defined by

$$f_I(n_1, \ldots, n_j) = \sup\{i \mid (n_1, \ldots, n_j, i, 0, \ldots, 0) \in I\}$$

The condition of I being decreasing is equivalent to these functions being strictly decreasing in each coordinate.

Examples. *1. When $m = 1$, any ideal is decreasing. Thus, the no k-equal spaces are decreasing polychromatic configuration spaces.*

2. Consider points of m colors such that each color c has an associated weight $w_c > 0$. Suppose $w_{j-1} \geq w_j$ for all $j \leq m$. Let $M \in \mathbb{R}$. Let $Y \subset \mathbb{R}^{(n_1+\ldots+n_m)d}$ be the space consisting of colored points in $\mathbb{R}^d$ satisfying the following property:

$$\text{for all } x \in \mathbb{R}^d, \sum_{(i,j)\in N_x} w_j < M \text{ where } N_x = \{(i,j) \mid x_i^j = x\}$$

Then Y is a decreasing polychromatic configuration space. Y is a weighted analogue of no k-equal spaces which are obtained when all weights are one and $M = k$.

Definition. *Let $I \subset \mathbb{N}^m$ be any ideal. Call an m-tuple $\vec{n} = (n_1, \ldots, n_i, 0, \ldots 0) \notin I$* critical *if $n_i > 0$ and for all $j \leq i$ with $n_j > 0$, $(n_1, \ldots, n_j - 1, \ldots, n_i, 0, \ldots 0) \in I$. Denote the set of critical m-tuples by $\mathcal{C}_I$. For each critical m-tuple, let its* weight *be $\sum n_i$, denoted by $w_{\vec{n}}$.*

Definition. *Let $\vec{e}_i$ denote the m-tuple with a one in the i^{th} coordinate and zeros everywhere else. Let $E = \{\vec{e}_i, \ldots, \vec{e}_m\}$.*

Our main theorem regarding decreasing polychromatic configuration spaces is the follow:

Theorem 2.2. *Let $m > 0$ and I be a decreasing ideal. The left module $H_*\mathcal{M}_{I,d}(\cdot)$ is generated by $H_0\mathcal{M}_{I,d}(\vec{n})$ for $\vec{n} \in E$ and $H_{(w_{\vec{n}}-1)d-1}\mathcal{M}_{I,d}(\vec{n})$ for $\vec{n} \in \mathcal{C}_I$.*

In the case where $m = 1$, this is exactly the theorem of Dobrinskaya and Turchin [9]. Moreover, it is in the same vein in that for each $\vec{n} \in \mathcal{C}_I$, $\mathcal{M}_{I,d}(\vec{n})$ is homotopy equivalent to a sphere. In more general ideals, the above theorem does not hold. To show this fact, we will discuss the case where $m = 2$.

Definition. *For each ideal $I \subset \mathbb{N}^2$, let*

$$\mathcal{D}_I = \{(n,m) \in I \mid (n+1,m), (n,m+1) \notin I \cup \mathcal{C}_I\}$$

For each $(n_1, n_2) \in \mathcal{D}_I$, let its weight *be $n_1 + n_2$, denoted by $w_{(n_1,n_2)}$*

Definition. *Let* $I \subset \mathbb{N}^2$ *be an ideal. We call* I rectangular *if there exists* $m_1, m_2 \in \mathbb{N} \cup \{\infty\}$ *such that* $I = \{(n_1, n_2) \mid 0 \leq n_1 \leq m_1, 0 \leq n_2 \leq m_2\}$.

Bicolored configuration spaces arising from rectangular ideals are products of two no k-equal spaces. Thus, their homology and cohomology can be computed using results on no k-equal spaces.

Theorem 2.3. *Let* $I \subset \mathbb{N}^2$ *be an ideal that is not rectangular. The left module* $H_*\mathcal{M}_{I,d}(\cdot,\cdot)$ *is generated by* $H_0\mathcal{M}_{I,d}(1,0)$, $H_0\mathcal{M}_{I,d}(0,1)$, $H_{(w_{(\ell_1,\ell_2)}-1)d-1}\mathcal{M}_{I,d}(\ell_1,\ell_2)$ *for* $(\ell_1,\ell_2) \in \mathcal{C}_I$, *and* $H_{(w_{(\ell_1,\ell_2)}+1)d-2}\mathcal{M}_{I,d}(\ell_1+1,\ell_2+1)$ *for* $(\ell_1,\ell_2) \in \mathcal{D}_I$.

Just as the general $m = 2$ case is fundamentally different from the $m = 1$ case, we will conclude this chapter by highlighting some differences between the $m = 2$ and $m = 3$ cases.

2.2 Homology of Configuration Spaces of $\mathbb{R}^d$

Our proofs regarding the homology of polychromatic configuration spaces uses knowledge of the homology of configuration spaces of $\mathbb{R}^d$. Thus, we will now give a brief overview of the homology of configuration spaces of $\mathbb{R}^d$. Let $\mathcal{M}_d$ denote the n^{th} configuration space of $\mathbb{R}^d$. For a more extensive look at $H_*\mathcal{M}_d$, I direct the reader to an expository paper written by Sinha [24].

Definition (May [21])**.** *Let* $\mathcal{S}$ *be a symmetric monoidal category with multiplication* $\otimes$ *and unit* κ. *An* operad, $\mathcal{C}$, *over* $\mathcal{S}$ *consists of objects indexed by natural numbers:* $\mathcal{C}(j)$, *a unit map* $\eta : \kappa \to \mathcal{C}(1)$, *a right action by the symmetric group* S_j *on* $\mathcal{C}(j)$ *for all* j, *and product maps:*

$$\mathcal{C}(k) \otimes \mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k) \to \mathcal{C}(j_1 + \ldots + j_k)$$

These maps are required to satisfy associative, unital, and equivarience conditions.

Intuitively, one thinks of $\mathcal{C}(n)$ as being the set of n-ary operations for some algebra. The product maps encode how to compose these operations. In order to define two operads that are of interest to us, we first must introduce algebras over operads.

Definition. *Let $\mathcal{C}$ be an operad. An algebra over $\mathcal{C}$ is an object, A, together with maps*

$$\mathcal{C}(j) \otimes A^j \to A$$

that satisfy associative, unital, and equivarience conditions.

Intuitively, A is an algebra whose operations are encoded by $\mathcal{C}$.

Definition. *The associative operad,* Assoc*, is the operad whose algebras over it are monoids. The degree d Poisson operad,* $\mathrm{Pois_d}$*, is the operad whose algebras over it are graded unital Poisson algebras with bracket degree d.*

Theorem 2.4 (Cohen [8])**.** *For $d = 1$, $H_*\mathcal{M}_d$ is* Assoc*. For $d > 1$, $H_*\mathcal{M}_d$ is* $\mathrm{Pois_{d-1}}$*.*

In the case $d = 1$, $\mathcal{M}_d(n)$ is homeomorphic to a disjoint union of $n!$ cells of dimension $n-1$. Thus, its only non-zero homology is in dimension zero. The contractible connected components of $\mathcal{M}_d(n)$ are indexed by elements of S_n. For $\sigma \in S_n$, a corresponding generator is any point in $\mathbb{R}^d$ such that for all $i, j \in \{1, \ldots, n\}$, if $\sigma(i) < \sigma(j)$, then $x_i < x_j$. Similarly, elements of Assoc are indexed by elements of S_n thought of as describing in which order n elements from the algebra are multiplied. We write the element indexed by $\sigma \in S_n$ as $x_{\sigma(1)} \cdot \ldots \cdot x_{\sigma(n)}$.

Recall that the degree d Poisson operad is generated by three elements: a nullary operation, 1, and two binary operations, $[x_1, x_2]$ and $x_1 \cdot x_2$. For $d > 1$, $\mathcal{M}_d(d)$ is homotopy equivalent to S^{d-1}. Thus, we have non-zero homology in dimensions zero and $d-1$. These correspond to $x_1 \cdot x_2$ and $[x_1, x_2]$, respectively. The preferred generator of $\mathcal{M}_0(0)$ corresponds to 1. More concretely, a cycle representing $[x_1, x_2]$ is the $S^{d-1} \subset \mathcal{M}_d$ where x_1, x_2 are on the unit $(d-1)$-sphere and $x_1 = -x_2$. Recall, the elements of $\mathrm{Pois_{d-1}}$ satisfy Leibniz, Jacobi, and anti-symmetry relations. The Leibniz rule allows any element to be written such that all Lie multiplication occurs first. The Jacobi and anti-symmetry relations will be used to find a basis for $H_*\mathcal{M}_{I,d}$.

2.3 Decreasing PCS

Throughout this section, I will be a decreasing ideal. Furthermore, we will assume that for all $\vec{n} = (n_1, \ldots, n_m)$ with $\sum n_i \leq 2$, we have $\vec{n} \in I$. This assumption is added only to avoid unnecessary complications. The proofs only require minor adjustments to go through if this assumption is not satisfied.

2.3.1 Homology of Decreasing PCS

We will be concerned with homology with $\mathbb{Z}_2$ coefficients, ignoring the orientations of homology representatives. However, a generalization to $\mathbb{Z}$ coefficients is straightforward if one is careful with signs.

As is evident in the statement of Theorem 2.2, there is one class of non-trivial building blocks for $H_*\mathcal{M}_{I,d}$, elements from $H_{(w_{\vec{n}_c}-1)d-1}\mathcal{M}_{I,d}(\vec{n})$ for $\vec{n} \in \mathcal{C}_I$.

Let $\vec{n} = (n_1, \ldots, n_m)$ be critical. Then $\mathcal{M}_{I,d}(\vec{n})$ is homotopy equivalent to $S^{(w_{\vec{n}}-1)d-1}$. This homotopy equivalence is given by retracting $\mathcal{M}_{I,d}(\vec{n})$ onto the sphere given by the equations:

$$\sum_{j=1}^{m}\sum_{i=1}^{n_j} x_i^j = 0 \qquad\qquad \sum_{j=1}^{m}\sum_{i=1}^{n_j} |x_i^j|^2 = 1$$

Thus, elements of $H_{(w_{\vec{n}_c}-1)d-1}\mathcal{M}_{I,d}(\vec{n})$ can be realized by spheres.

Definition. *Denote the sphere described above by* $\{x_1^1, \ldots, x_{n_1}^1, \ldots, x_1^m, \ldots, x_{n_m}^m\}$.

To see that $\{x_1^1, \ldots, x_{n_1}^1, \ldots, x_1^m, \ldots, x_{n_m}^m\}$ is in fact non-trivial, consider the chain in $\mathcal{M}_{I,d}(\vec{n})$ given by the following equations:

$$\begin{aligned} x_1^1 &= x_i^j \text{ for all } j < m, i \leq n_j \\ x_1^1 &= x_i^m \text{ for all } i < n_m \\ (x_1^1)_1 &< (x_{n_m}^m)_1 \\ (x_1)_\ell &= (x_{n_m}^m)_\ell \text{ for all } \ell > 1 \end{aligned}$$

where $(z)_\ell$ denotes the ℓ^{th} coordinate of z. The boundary of this chain is in the complement to

$\mathcal{M}_{I,d}(\vec{n})$ in $\mathbb{R}^{(n_1+\ldots+n_m)d}$. Thus, it represents an element in $H^*(\mathcal{M}_{I,d}(\vec{n}), \mathbb{Z}_2)$. The intersection pairing between this element and $\{x_1^1, \ldots, x_{n_1}^1, \ldots, x_1^m, \ldots, x_{n_m}^m\}$ is non-zero.

Definition. *Define* local classes *to be classes of one of the following forms:*

- $\{x_1^1, \ldots, x_{n_1}^1, \ldots, x_1^m, \ldots, x_{n_m}^m\} \in H_{(w_{\vec{n}_c}-1)d-1}\mathcal{M}_{I,d}(\vec{n})$ *for* $\vec{n} \in \mathcal{C}_I$
- $x_1^j \in H_0\mathcal{M}_{I,d}(\vec{e}_j)$ *for* $\vec{e}_j \in E$

The action of $H_*\mathcal{M}_d$ on $H_*\mathcal{M}_{I,d}$ is very similar to the action of $H_*\mathcal{M}_d$ on itself. That is, if B_1 and B_2 are two elements of $H_*\mathcal{M}_{I,d}$, a representative for $[B_1, B_2]$ is given by considering a representative for $[x_1, x_2]$ and replacing x_i with sufficiently scaled representatives of B_i. We will show that all homology classes of $H_*\mathcal{M}_{I,d}$ can be built up using the left action of $\mathcal{M}_d$ on local classes.

Our proof will follow very similarly to that of Dobrinskaya and Turchin [9]. As with their proof, our proof will use a more general space. Consider the ideal $I' \subset \mathbb{N}^{m+1}$ consisting of the following $(m+1)$-tuples:

- $(n_1, \ldots, n_m, 0)$ for $(n_1, \ldots, n_m) \in I$ (We will denote such tuples by $(\vec{n}, 0)$)
- $(0, \ldots, 0, 1)$

To emphasize the importance of points of color $m+1$, we will denote them by z_i rather than x_i^{m+1}.

Definition. *Define* augmented local classes *to be classes of one of the following forms:*

- $\{x_1^1, \ldots, x_{n_1}^1, \ldots, x_1^m, \ldots, x_{n_m}^m\} \in H_{(w_{\vec{n}_c}-1)d-1}\mathcal{M}_{I',d}(\vec{n}, 0)$ *for* $\vec{n} \in \mathcal{C}_I$
- $x_1^j \in H_0\mathcal{M}_{I',d}(\vec{e}_j)$ *for* $j \leq m$
- $z_1 \in H_0\mathcal{M}_{I',d}(\vec{e}_{m+1})$

We will prove the following:

Theorem 2.5. *For all* $m \geq 1$ *and all decreasing ideals* $I \subset \mathbb{N}^m$*, the module* $H_*\mathcal{M}_{I',d}(\cdot, \ldots, \cdot)$ *is generated by augmented local classes.*

As a corollary of this theorem, we get Theorem 2.2.

Definition. *Call a class* organized *if it can be written as a sum of products of augmented local classes.*

Before proving Theorem 2.5, we define some additional notation.

Definition. *For any $N \in H_*\mathcal{M}_{I',d}(\vec{n})$, let $N|_{a=A}$ be the class in $H_*\mathcal{M}_{I,d}(\vec{n}')$ given by substituting A for a where a is some z coordinate and A is some element in $H_*\mathcal{M}_{I,d}$.*

Example. $[x_1^1, z_1]|_{z_1=\{x_1^2,x_2^2,x_3^2\}}$ *is the class* $[x_1^1, \{x_1^2, x_2^2, x_3^2\}]$.

We now prove Theorem 2.5.

Proof. The proof will be by induction on m. The case $m = 1$ was done by Baryshnikov for $d = 1$ [2] and Dobrinskaya and Turchin for $d > 1$ [9].

Suppose $m > 1$ and that the claim holds for all $m' < m$. Let I be a decreasing ideal in $\mathbb{N}^m$. We will show that for all $(n_1, \ldots, n_{m+1})$, organized classes span $H_*\mathcal{M}_{I',d}(n_1, \ldots, n_{m+1})$. This will be done by induction on n_m. First suppose $n_m = 0$. Then $H_*\mathcal{M}_{I',d}(n_1, \ldots, n_{m+1})$ is homeomorphic to $\mathcal{M}_{J',d}(n_1, \ldots, n_{m-1}, n_{m+1})$ for the decreasing ideal $J \subset \mathbb{N}^{m-1}$ given by $(\ell_1, \ldots, \ell_{m-1}) \in J$ if and only if $(\ell_1, \ldots, \ell_{m-1}, 0) \in I$. All classes that are organized in $\mathcal{M}_{J',d}(n_1, \ldots, n_{m-1}, n_{m+1})$ are also organized in $H_*\mathcal{M}_{I',d}(n_1, \ldots, n_{m+1})$. Inductively, the claim holds when $n_m = 0$.

Now suppose $n_m > 0$ and that the claim holds whenever $n'_m < n_m$. Let γ be a closed s-chain in $\mathcal{M}_{I',d}(n_1, n_2, n_3, \ldots, n_{m+1})$. Consider the homotopy of γ affecting only the $x^m_{n_m}$ coordinate, $\gamma_t = \gamma + v \cdot t$ where v is a vector that is non-zero only in the $x^m_{n_m}$ coordinate. For large enough t, say $t = M$, the $x^m_{n_m}$ coordinate is always far away from all other points. Call the $(s+1)$-chain given by this homotopy Γ. Γ may not be a chain in $\mathcal{M}_{I',d}(n_1, n_2, \ldots, n_{m+1})$. It may intersect forbidden subspaces of the forms:

$$x^m_{n_m} = x^j_i \text{ for all } j \leq m, i \in J_j \text{ where } |J_j| = \ell_j \text{ for some } (\ell_1, \ldots, \ell_{m-1}, \ell_m + 1) \in \mathcal{C}_I$$

$$x^m_{n_m} = z_j$$

In the first case, remove a sufficiently small tubular neighborhood. The intersection of Γ with the boundary of this neighborhood is $N|_{z_{(n_{m+1}+1)}=\{x^1_{i_{1,1}},\ldots,x^1_{i_{1,\ell_1}},\ldots,x^m_{i_{m,1}},\ldots,x^m_{i_{m,\ell_m}},x^m_{n_m}\}}$ where $i_{j,1},\ldots,i_{j,\ell_j}$ is an enumeration of J_j and $N \in H_*\mathcal{M}_{I',d}(n_1-\ell_1,\ldots,n_m-(\ell_m+1),n_{m+1}+1)$.

In the second case, again remove a sufficiently small tubular neighborhood. The intersection of Γ with the boundary of this tubular neighborhood produces a class $N|_{z_j=[x^m_{n_m},z_j]}$ where $N \in H_*\mathcal{M}_{I',d}(n_1,\ldots,n_{m-1},n_m-1,n_{m+1})$.

For $t = M$, we have a class $N \cdot x^m_{n_m}$ where $N \in H_*\mathcal{M}_{I',d}(n_1,\ldots,n_{m-1},n_m-1,n_{m+1})$.

In each of these cases, the resultant classes are organized. Thus, Γ with its intersection with these tubular neighborhoods removed gives a relation which allows $[\gamma]$ to be written as a sum of organized classes. Thus, for all $(n_1,\ldots,n_{m+1})$, organized classes span $H_*\mathcal{M}_{I',d}(n_1,\ldots,n_{m+1})$. □

Theorem 2.5 produces a generating set for $H_*\mathcal{M}_{I,d}(\vec{n})$; we would like a basis. For this, relations between various elements in the generating set are needed.

Lemma 2.6. *Let $\vec{\ell} = (\ell_1,\ldots,\ell_k,0,\ldots,0) \in \mathbb{N}^m$, $\ell_k > 0$ be such that $(\ell_1,\ldots,\ell_k-1,0,\ldots,0)$ is critical. Let $d > 1$. Let $J = \{i \mid \vec{\ell}-\vec{e}_i \in \mathcal{C}_I\}$. Then the elements of $H_*(\mathcal{M}_{I,d}(n_1,n_2),\mathbb{Z}_2)$ satisfy the following relation:*

$$\sum_{j\in J}\sum_{i=1}^{\ell_j}[\{x^1_1,\ldots,x^1_{\ell_1},x^j_1,\ldots,\hat{x}^j_i,\ldots,x^j_{\ell_j},x^k_1,\ldots,x^k_{\ell_k}\},x^j_i] = 0$$

Proof. Consider the sphere, S, given by the following equations:

$$\sum_{j=1}^{k}\sum_{i=1}^{\ell_k} x^j_i = 0 \qquad\qquad \sum_{j=1}^{k}\sum_{i=1}^{\ell_k} |x^j_i|^2 = 1$$

Remove from S tubular neighborhoods of points on S that are not in $\mathcal{M}_{I,d}$. This gives the above relation. □

Using these relations, along with the Jacobi and anti-symmetry relations from $H_*\mathcal{M}_d$, we can find a smaller generating set for $H_*\mathcal{M}_{I,d}(\vec{n})$.

Theorem 2.7. *For all $d > 1, \vec{n} \in \mathbb{N}^m$, let S be the set of elements of $H_*\mathcal{M}_{I,d}(\vec{n})$ that can be written as a product where each factor is an x_i^j or of the form:*

$$[\ldots[[B_1, B_2], B_3]\ldots B_\ell],\ \ell \geq 1 \tag{2.1}$$

where each B_s is of the following form:

$$[\ldots[[\ldots[\{x^1_{i_{1,1}}, \ldots, x^1_{i_{1,\ell_1}}, \ldots, x^k_{i_{k,1}}, \ldots, x^k_{i_{k,\ell_k}}\}, x_{r_{1,1}}]\ldots x_{r_{1,s_1}}], \ldots x_{r_{k,1}}]\ldots x_{r_{k,s_k}}]$$

where $(\ell_1, \ldots, \ell_k, 0, \ldots, 0) \in \mathcal{C}_I, \ell_k > 0, i_{j,1} < \ldots < i_{j,\ell_j}$, and $r_{j,1} < \ldots < r_{j,s_j}$. Furthermore, if $s_k > 0$, then $i_{k,\ell_k} > r_{k,s_k}$. Additionally, we require that the smallest x^1 index in $B_1, \ldots, B_\ell$ be in B_1. Then S is a generating set for $H_*\mathcal{M}_{I,d}(\vec{n})$.

Proof. Throughout this proof, *items* will refer to either a set of curly brackets or a singleton coordinate not in any curly brackets. Recall, as mentioned in the previous section, we may assume that all multiplication occurs outside of Lie brackets.

If an element of $\alpha \in H_*\mathcal{M}_{I,d}(\vec{n})$ has no Lie brackets, then it is already in the desired form. Thus, we may assume it has Lie brackets. Consider one Lie bracket factor, F. The proof will follow by induction on the number of items in F. A small case analysis gives that if F contains at most 3 items, then it can be expressed in the desired form. Thus, suppose it contains n items for some $n > 3$. We may write $F = [F_1, F_2]$. There are a three cases.

Case 1: F_1 and F_2 each have at least 2 items: Inductively, F_1 and F_2 can be expressed in the desired form. Thus, $F = [[\ldots[[B_1, B_2], B_3]\ldots B_\ell], [\ldots[[B'_1, B'_2], B'_3]\ldots B'_{\ell'}]]$. Without loss of generality, we can assume the smallest x^1 index is in B_1. Using the Jacobi and anti-symmetry relations, we may write F as

$$[[F_1, B'_{\ell'}], [\ldots[[B'_1, B'_2], B'_3]\ldots B'_{\ell'-1}]] + [[F_1, [\ldots[[B'_1, B'_2], B'_3], \ldots B'_{\ell'-1}]], B'_{\ell'}]$$

In the first summand, we reduced the number of B_i blocks on the right side of the outer most Lie bracket. The second summand can be expressed as $[F', B'_{\ell'}]$ where F' has fewer items than F. Thus, inductively F' can be written in the desired form. Thus, continuing this procedure, we may write F in the desired form.

Case 2: F_2 is a curly bracket: Inductively, F_1 can be expressed in the desired form. Thus, F is written in the form $[[\dots[[B_1, B_2], B_3]\dots B_\ell], B_{\ell+1}]$ where $B_{\ell+1} = F_2$. If the smallest x^1 index is not in F_2, then we are done. If the smallest x^1 index is in $B_{\ell+1}$, then F may be expressed as:

$$[[\dots[[B_1, B_2], B_3]\dots B_{\ell-1}], [B_\ell, B_{\ell+1}]] + [[[\dots[[B_1, B_2], B_3]\dots B_{\ell-1}], B_{\ell+1}], B_\ell]$$

The first summand can be treated as case 1. The second summand can either be treated as case 1 or as case 2 where the smallest x^1 index is not in F_2.

Case 3: F_2 is a single x_i^j: Inductively, we may write F_1 in the desired form. There are now two subcases: either F_1 contains a single B block or it contains multiple. In the latter case, we may write F as:

$$[[F_1', x_i^j], B_\ell] + [[B_\ell, x_i^j], F_1']$$

where $F_1' = [\dots[[B_1, B_2], B_3]\dots B_{\ell-1}]$. Both of these summands can be treated by previous cases.

Thus, we may suppose F_1 contains only a single B block. That is, F is of the form:

$$[[\dots[F', x^1_{i_{1,1}}]\dots x^k_{i_{k,s_k}}], x_i^j]$$

where F' is some curly bracket expression. If this is not in the desired form, F may be expressed as:

$$[[[\dots[F', x^1_{i_{1,1}}]\dots, x^k_{i_{k,s_k-1}}], x_i^j], x^k_{i_{k,s_k}}] + [[\dots[F', x^1_{i_{1,1}}]\dots, x^k_{i_{k,s_k-1}}], [x^k_{i_{k,s_k}}, x_i^j]]$$

The second summand is zero. If we order all x coordinates such that all x^i come before x^{i+1}, all in their natural linear order, then the first summand has lesser last coordinate than the previous expression. Thus, repeating this process eventually terminates.

Thus, F may be written in the desired form. Doing this for each factor of α completes the proof. $\square$

In the case $d = 1$, there is a similar relation to that from Lemma 2.6. The only difference is $[B_1, B_2]$ is replaced with $B_1 \cdot B_2 + B_2 \cdot B_1$. Using this relation, we get the $d = 1$ analogue to Theorem 2.7.

Theorem 2.8. *For $d = 1$ and any $\vec{n} \in \mathbb{N}^m$, let S be the set of elements of $H_*\mathcal{M}_{I,d}(\vec{n})$ that can be written in the form:*

$$A_{I_0} \cdot B_{J_1} \cdot A_{I_1} \cdot \ldots \cdot B_{J_\ell} \cdot A_{I_\ell}$$

where $I_0, J_1, \ldots, J_\ell, I_\ell$ is a partition of $\{x_i^j \mid 1 \leq j \leq m, 1 \leq i \leq n_j\}$.

A_{I_s} is of the form:

$$x^1_{i_{1,1}} \cdot \ldots \cdot x^1_{i_{1,\ell_1}} \cdot \ldots \cdot x^m_{i_{m,1}} \cdot \ldots \cdot x^m_{i_{m,\ell_m}}$$

where $I_s = \{x_i^j \mid 1 \leq j \leq m, i \in M_j \subset [n_j]\}$ and $i_{j,1}, \ldots, i_{j,\ell_j}$ is an enumeration of M_j.

B_{J_s} is of the form:

$$\{x^1_{i_{1,1}}, \ldots, x^1_{i_{1,\ell_1}}, \ldots, x^k_{i_{k,1}}, \ldots, x^k_{i_{k,\ell_k}}\}$$

where J_s is the set of elements $\{x^1_{i_{1,1}}, \ldots, x^1_{i_{1,\ell_1}}, \ldots, x^k_{i_{k,\ell_k}}\}$ for some $(\ell_1, \ldots, \ell_k, 0, \ldots, 0) \in \mathcal{C}_I$ with $\ell_k > 0$.

Furthermore, if k is the maximum color that appears in J_s, we require that I_s has no color ℓ coordinates for all $\ell > k$ and that the greatest index of a color k coordinate in J_s is greater than any index of any color k coordinate in I_s.

The proof of this follows by induction. First, use relations to ensure B_{J_ℓ} and A_{I_ℓ} satisfy the desired restrictions. Next, use relations to ensure $B_{J_{\ell-1}}$ and $A_{I_{\ell-1}}$ satisfy the desired restrictions. Doing this does not undo the previous step. Continuing inductively we get each B_{J_i} and A_{I_i} satisfy the restrictions.

In the next section, we show that the generating sets given in Theorems 2.7 and 2.8 are actually bases.

2.3.2 Cohomology of Decreasing PCS

As in the no k-equal spaces studied by Dobrinskaya and Turchin [9], the cohomology ring of $\mathcal{M}_{I,d}(\vec{n})$ can be described by a set of forests. We will be computing cohomology with integer coefficients.

Let $N_j = \{x_i^j \mid 1 \leq i \leq n_j\}$.

Definition. *An* admissible forest *is a forest satisfying the following: it has two types of vertices: rectangles and circles. Each circle contains exactly one element of $\bigcup_{j=1}^m N_j$. Each circle is connected to at most one rectangle and nothing else. Each rectangle is connected to at least one circle. For each rectangle, there exists $(\ell_1, \ldots, \ell_k, 0, \ldots, 0) \in \mathcal{C}_I, (\ell_k > 0)$, such that the rectangle contains ℓ_j elements from N_j for all $j < k$ and $\ell_k - 1$ elements from N_k. All circles attached to this rectangle are from $\bigcup_{j=1}^k N_j$.*

An orientation *of an admissible forest is:*

- *an orientation of each edge*
- *an ordering of elements within each rectangle*
- *an ordering of the set of rectangles and edges*

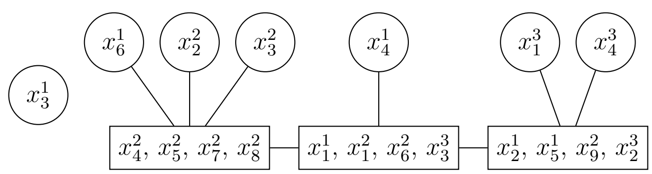


Figure 2.1: An example of an unoriented admissible forest for an ideal $I \subset \mathbb{N}^3$ such that $\{(0, 5, 0), (1, 2, 2), (2, 1, 2)\} \subset \mathcal{C}_I$.

To each admissible forest, we associate a chain in $\mathbb{R}^{(n_1 + \ldots + n_m)d}$ whose boundary lies in the complement of $\mathcal{M}_{I,d}$. Thus, the forest will represent a cocycle in $H^*\mathcal{M}_{I,d}(\vec{n})$. The chain associated to a forest is the set of all points satisfying the following:

- for each rectangle A and each $x, x' \in A$, $x = x'$
- if there is an edge from A to B and $x \in A, x' \in B$, then $(x)_1 \leq (x')_1$ and $(x)_\ell = (x')_\ell$ for all $\ell > 1$ where $(x)_\ell$ is the ℓ^{th} coordinate of x

The rest of the orientation data is used to coorient the chain. We coorient the chain by giving an explicit basis for the normal bundle. Suppose there exists an edge from vertex A

to vertex B. Suppose x is the first element in vertex A and x' is the first element in vertex B. Then this edge contributes:

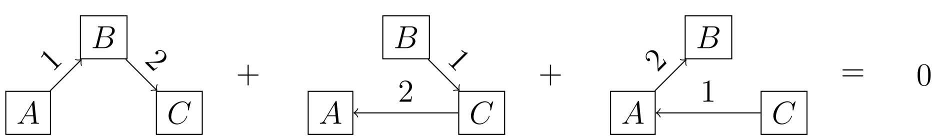
$$\partial(x')_2 - \partial(x)_2, \ldots, \partial(x')_d - \partial(x)_d$$

Suppose there exists a rectangle vertex with ordered elements $(x_1, \ldots, x_\ell)$. This rectangle contributes:

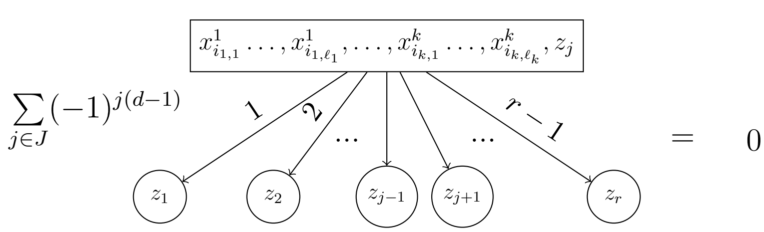
$$\partial(x_2)_1 - \partial(x_1)_1, \ldots, (x_2)_d - \partial(x_1)_d, \ldots, \partial(x_\ell)_d - \partial(x_1)_d$$

Lemma 2.9. *For $d > 1$, the cohomology classes given by admissible forests have the following relations:*

1. *Orientation Relations:*
   - (a) *Changing the order of the orientation set produces the Koszul sign of the permutation*
   - (b) *A permutation $\sigma \in S_n$ of elements inside a rectangle produces a sign $(-1)^{|\sigma|d}$.*
   - (c) *Changing the orientation of an edge produces the sign $(-1)^d$.*
2. *3-term relation:*



3. *Relation exchanging values in rectangles: Let $\{z_i\} \subset \bigcup N_j$. Let $c(j)$ be the color of z_j. Let c' be the maximum color of any z_i. Suppose $c' \geq k$ and that there exists $s \geq c'$ such that $(\ell_1, \ldots, \ell_k, 0 \ldots, 0) + \vec{e}_{c'} + \vec{e}_s \in \mathcal{C}_I$, then:*

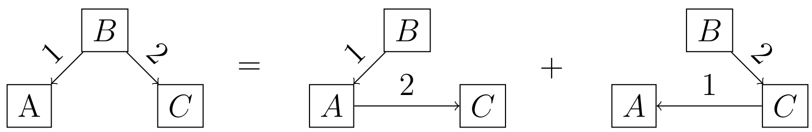


where $J = \{j \mid (\ell_1, \ldots, \ell_k, 0, \ldots, 0) + \vec{e}_{c(j)} \in I\}$.

In relations 2 and 3, the rectangles may be attached to other rectangles.

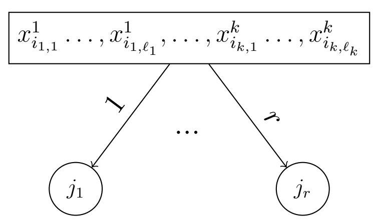
Proof. Relations 1(a) and 1(b) come from changing the coorientation. For 1(c), the inequality changes from $(i)_1 \leq (j)_1$ to $(i)_1 \geq (j)_1$. To see these are homologous (up to a sign), consider the chain given by the the inequality $(i)_2 < (j)_2$. Its boundary is a sum of the two chains in question.

Relation 2 is equivalent to



The chain corresponding to the left hand side is the union of the chains corresponding to the trees on the right hand size.

Relation 3 comes from looking at the boundary of the chain corresponding to:



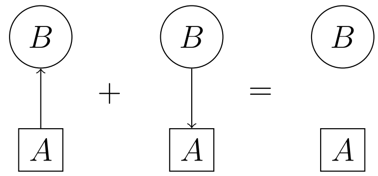
The boundary of the subspace corresponding to the above tree has a component for each circle where the coordinate in the circle is equal to all coordinates in the rectangle. For elements not in J, these subspaces are not in $\mathcal{M}_{I,d}$ and, thus, contribute zero. For $j \in J$, we need to show that the resultant tree is admissible.

Thus, suppose $c' \geq k$ and there exists $s \geq c'$ with $(\ell_1, \ldots, \ell_k, 0 \ldots, 0) + \vec{e}_{c'} + \vec{e}_s \in \mathcal{C}_I$. I claim that for all $c \leq c'$, if $(\ell_1, \ldots, \ell_k, 0, \ldots, 0) + \vec{e}_c \in I$, then $(\ell_1, \ldots, \ell_k, 0, \ldots, 0) + \vec{e}_c + \vec{e}_{c'} \in \mathcal{C}_I$.

Let $\vec{\ell} = (\ell_1, \ldots, \ell_k, 0, \ldots, 0)$. Because $\vec{\ell} + \vec{e}_{c'} + \vec{e}_s \in \mathcal{C}_I$, it is not in I. Thus, $\vec{\ell} + \vec{e}_{c'} + \vec{e}_s + \vec{e}_c \notin I$. If $\vec{\ell} + \vec{e}_{c'} + \vec{e}_c$ were in I, then because I is decreasing, we would have $\vec{\ell} + \vec{e}_{c'} + \vec{e}_s + \vec{e}_c \in \mathcal{C}_I$ However, this can't be since $\vec{\ell} + \vec{e}_{c'} + \vec{e}_s \notin I$. Thus, $\vec{\ell} + \vec{e}_{c'} + \vec{e}_c \notin I$.

In summary, $\vec{\ell} + \vec{e}_c \in I$ and $\vec{\ell} + \vec{e}_c + \vec{e}_{c'} \notin I$. Because I is decreasing, $\vec{\ell} + \vec{e}_c + \vec{e}_{c'} \in \mathcal{C}_I$. Thus, the term for each $j \in J$ is an admissible forest. □

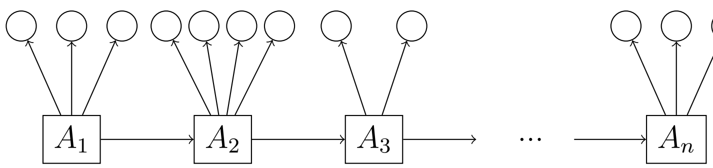
The only relation that does not work when $d = 1$ is relation 1(c). There is a substitute for 1(c) in the case that $d = 1$:



Here, the circle may be replaced with a rectangle.

Using these relations, we produce bases for cohomology.

Definition. *Define a* linear I-tree *to be an admissible tree of the following form:*



such that

- *The elements in A_i appear in their natural order*
- *The circles attached to each rectangle are ordered similarly*
- *The minimal N_1 element in the tree is in B_1*
- *For each i, suppose c is the maximum color present in B_i. Then the maximum element from N_c in B_i is not in A_i.*

where B_i is the set of elements in A_i or in circles attached to A_i.

Using the relations from Lemma 2.9, any admissible forest can be written as a forest whose components are linear I-trees and singleton circles. We will show that this is a basis for $H^*\mathcal{M}_{I,d}(\vec{n})$. For $d > 1$, this basis will be dual to the generating set for homology from Theorem 2.7.

Definition. *Let $\mathcal{H}$ be the set of generators given in Theorem 2.7. Let $\mathcal{C}$ be the set of cohomology classes represented by products of linear I-trees and singleton circles. Define $f : \mathcal{H} \to \mathcal{C}$ as follows:*

Let $A \in \mathcal{H}$. Let $f(A)$ be the forest satisfying the following:

- *For each x_i^j factor in A, $f(A)$ has a singleton circle containing x_i^j*

- *Each other factor in A has a corresponding linear I-tree as follows: Suppose the factor is given by $[\dots[[B_1, B_2], B_3]\dots B_\ell]$, then each B_i has a corresponding rectangle vertex, A_i. These rectangle vertices form a path from A_1 to A_ℓ. Recall, each B_i, is of the following form:*

$$B_i = [\dots[[\dots[\{x^1_{i_{1,1}}, \dots, x^1_{i_{1,\ell_1}}, \dots, x^k_{i_{k,1}}, \dots, x^k_{i_{k,\ell_k}}\}, x_{r_{1,1}}]\dots x_{r_{1,s_1}}], \dots x_{r_{k,1}}]\dots x_{r_{k,s_k}}]$$

  where $(\ell_1, \dots, \ell_k, 0, \dots, 0) \in \mathcal{C}_I, i_{j,1} < \dots < i_{j,\ell_j}, r_{j,1} < \dots < r_{j,s_j}$. The rectangle corresponding the B_i satisfies the following:

  - *it contains $x^1_{i_{1,1}}, \dots, x^1_{i_{1,\ell_1}}, \dots, x^k_{i_{k,1}}, \dots, x^k_{i_{k,\ell_k-1}}$*
  - *all other elements from B_i are circles attached to it.*

It is straight forward to check f is a bijection. Order $\mathcal{H}$ such that if F_1 has more rectangles than F_2, then F_1 comes after F_2. Order $\mathcal{C}$ according to the ordering of corresponding elements in $\mathcal{H}$.

Theorem 2.10. *With the above ordering, the intersection pairing matrix is diagonal such that each diagonal element is* 1 *or* -1*.*

Proof. We will first show that the intersection pairing matrix contains ± 1 along the diagonal, then that every entry off the diagonal is 0.

For $x^1_1 \cdot \dots \cdot x^1_{n_1} \cdot \dots \cdot x^m_1 \cdot \dots \cdot x^m_{n_m} \in H_0\mathcal{M}_{I,d}(\vec{n})$, the claim is obvious.

Consider a product of singletons with a single $\{x^1_{i_{1,1}}, \dots, x^1_{i_{1,\ell_1}}, \dots, x^k_{i_{k,1}}, \dots, x^k_{i_{k,\ell_k}}\}$ where $(\ell_1, \dots, \ell_k, 0, \dots, 0) \in \mathcal{C}_I$. Solving the system of equations from the sphere $\{x^1_{i_{1,1}}, \dots, x^k_{i_{k,\ell_k}}\}$ and the chain corresponding to $f(\{x^1_{i_{1,1}}, \dots, x^k_{i_{k,\ell_k}}\})$ gives one solution.

Similarly, the system of equations arising from Lie brackets produces a single solution. Combining these will produce a single solution for any product of singletons and elements of the form $[\dots[[B_1, B_2], B_3], \dots B_s]$. Below is a diagram showing geometrically what the point of intersection is in the case $[[[\{x^1_1, x^1_4, x^2_1\}, x^1_2], x^1_3], x^2_2] \cdot x^2_3 \in H_*(\mathcal{M}_{I,2}(4,3))$.

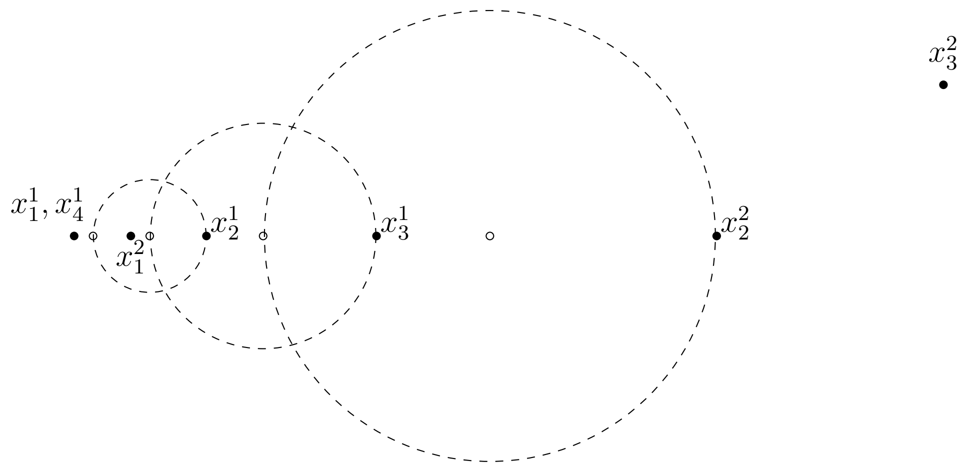


Figure 2.2: Intersection of $[[[\{x_1^1, x_4^1, x_1^2\}, x_2^1], x_3^1], x_2^2] \cdot x_3^2$ and $f([[[\{x_1^1, x_4^1, x_1^2\}, x_2^1], x_3^1], x_2^2] \cdot x_3^2)$ when $d = 2$.

Arbitrary products of elements as in Theorem 2.7 follows from noticing that the different factors correspond to different trees. Indices in different trees do not give any restrictions between the corresponding coordinates. Thus, there still exists a single point of intersection.

Next, we need to show that all entries off the diagonal are zero. Let $[\alpha] \in \mathcal{H}$, $F \in \mathcal{C}$. Suppose there exist two coordinates, z_1, z_2, in the same tree in F but in different factors of $[\alpha]$. Being in the same tree implies $(z_1)_2 = (z_2)_2$ on the chain corresponding to F. It is possible to find a representative of $[\alpha]$ so that $(z_1)_2$ and $(z_2)_2$ are never equal on α. Because there exists a representative of $[\alpha]$ that does not intersect the chain corresponding to F, we have reduced to the situation where each tree in F has indices from exactly one $[\ldots[[B_1, B_2], B_3], \ldots B_s]$ factor.

Consider a factor of α: $[\ldots[[B_1, B_2], B_3], \ldots B_s]$. Let T be the subforest of F that is the trees that contains coordinates from $[\ldots[[B_1, B_2], B_3], \ldots B_s]$. First notice that if T has a rectangle with coordinates that are not a subset of a single curly bracket of some B_i, then there exists a representative of $[\alpha]$ such that the corresponding chain for T does not intersect α.

If T does not have exactly s rectangles, then the preceding fact plus degree considerations tells us that chains corresponding to T and $[\alpha]$ do not have complementary dimensions and, thus, have intersection pairing zero. Thus, T must contain exactly s rectangles. Degree considerations further imply T must be a tree.

Suppose T' is the tree produced from $[\dots[[B_1, B_2], B_3], \dots B_s]$ as in the definition of f. Suppose T is a linear I-tree containing the same coordinate set as T' and has the same number of rectangle vertices as T'. As mentioned previously, each rectangle vertex of T corresponds to a single curly bracket. If the ordering of rectangle vertices in T' and T are different, then again there exists a representative of $[\alpha]$ that does not intersect the chain corresponding to T. Similarly, if the circles attached to a rectangle vertex in T do not match those in T', then there exists a change of orientation of edges of T such that α does not intersect the the chain corresponding to this new tree. Which of the elements from the curly bracket is not in the rectangle vertex is determined uniquely based on the conditions for a tree being a linear I-tree. Thus, if the intersection product of α and the chain corresponding to T is nonzero, then $T = T'$. □

Corollary 2.11. *$\mathcal{H}$ and $\mathcal{C}$ are bases for $H_*\mathcal{M}_{I,d}(\vec{n})$ and $H^*\mathcal{M}_{I,d}(\vec{n})$, respectively.*

For $d = 1$, there is a slightly difference basis. The reason we need a different basis is because changing the direction of edges does not just produce a sign as in the $d > 1$ case. If we remove the condition requiring the minimal N_1 element to be in B_1, then a very similar to the proof of Theorem 2.10 would show that products of singleton circles and a *single* linear I-tree forms a basis for $H^*\mathcal{M}_{I,1}(\vec{n})$. This basis is not dual to the basis of $H_*\mathcal{M}_{I,1}(\vec{n})$ given in Theorem 2.8, but with a suitable ordering, the intersection pairing matrix is upper triangular.

Multiplicative Structure

Definition. *Let $T_1, T_2 \in H^*\mathcal{M}_{I,d}(n_1, n_2)$ be two admissible forests. Suppose for all rectangles A in T_1 and B in T_2, $A \cap B = \emptyset$. Let $T_1 \cup T_2$ be the tree defined as follows:*

- *if i, j are in a common rectangle in T_1 or T_2, then i, j are in a common rectangle in $T_1 \cup T_2$*
- *if i is in a circle in both T_1 and T_2, then i is in a circle in $T_1 \cup T_2$*
- *if $i \in A, j \in B$ in T_k and there exists an edge from A to B in T_k, then there exists an edge from the vertex containing i to the vertex containing j in $T_1 \cup T_2$*

Theorem 2.12. *Let $T_1, T_2 \in H^*\mathcal{M}_{I,d}(n_1, n_2)$ be two admissible forests. The product of T_1 and T_2, $T_1 \cdot T_2$, is given as follows:*

1. *If there exists rectangles A in T_1 and B in T_2 such that $A \cap B \neq \emptyset$, then $T_1 \cdot T_2 = 0$.*

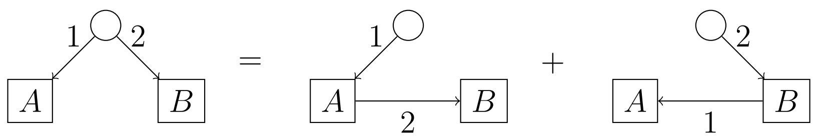
2. *If there exists two indices that are in a common tree in both T_1 and T_2, then $T_1 \cdot T_2 = 0$*

3. *If $T_1 \cup T_2$ has a cycle, then $T_1 \cdot T_2 = 0$.*

4. *If $T_1 \cup T_2$ has a rectangle with no circles attached to it, then $T_1 \cdot T_2 = 0$*

5. *If $T_1 \cup T_2$ is an admissible forest, then $T_1 \cdot T_2 = T_1 \cup T_2$ with orientation set given by concatenation.*

6. *If $T_1 \cup T_2$ satisfies none of the above, then use the following relation to make $T_1 \cup T_2$ admissible*



Proof. 1. If $A \neq B$, then the corresponding chains to T_1 and T_2 do not intersect in $\mathcal{M}_{I,d}(\vec{n})$. If $A = B$, then one can perturb the chains slightly so that they do not intersect.

2. There exist orientations of edges such that the two corresponding chains do not intersect in $\mathcal{M}_{I,d}(\vec{n})$.

3. There exist orientations of edges such that the two corresponding chains do not intersect in $\mathcal{M}_{I,d}(\vec{n})$.

4. This is relation 4 from Lemma 2.9 with $r = 0$.

5. The chains corresponding to T_1 and T_2 are transversal and their intersection is the chain corresponding to $T_1 \cup T_2$.

6. Combine the proofs of (5) and relation 2 from Lemma 2.9. □

A similar construction works if we remove the condition $\sum n_i \leq 2$ implies $\vec{n} \in I$. When it comes to homology, the only additional complication arises in the proof of Theorem 2.7. In the second subcase of case 3, $[x^k_{i_{k,s_k}}, x^j_i]$ may be nonzero. If this is the case, it can be replaced by $\{x^k_{i_{k,s_k}}, x^j_i\}$. This can then be written in the desired form using previous cases.

The majority of complications arise in cohomology. This is because rectangles may have a single coordinate in them. When it comes to the corresponding chain, there is no difference between rectangles containing one element and circles. Thus, we may allow rectangles to have no circles attached to them and add the relation that if a rectangle with one coordinate in it is attached to at most one rectangle and nothing else, it may be turned into a circle. Similarly, we may turn a circle into a rectangle provided such rectangles are allowed. With these changes, linear I-trees still form a basis . The only change in multiplication is condition 1 from Theorem 2.12 must additionally assume that A and B both have weight at least 2.

2.4 Bicolored PCS

As in the previous section, we will assume that for all $\vec{n} = (n_1, n_2)$ with $n_1 + n_2 \leq 2$, we have $\vec{n} \in I$. In this section, the complications that arise are worse than those in the previous section. At the end of this section, we will again comment on these complications. Throughout this section, because we only have two colors, we will write x_i for x_i^1 and y_i for x_i^2.

2.4.1 Homology of Bicolored PCS

Again, we will be concerned with homology with $\mathbb{Z}_2$ coefficients, ignoring the orientations of homology representatives. A generalization to $\mathbb{Z}$ coefficients is straightforward if one is careful with signs.

As is evident in the statement of Theorem 2.3, there are two non-trivial building blocks for $H_*\mathcal{M}_{I,d}$:

- elements from $H_{(w_{(\ell_1,\ell_2)}-1)d-1}\mathcal{M}_{I,d}(\ell_1, \ell_2)$ for $(\ell_1, \ell_2) \in \mathcal{C}_I$

- elements from $H_{(w_{(\ell_1,\ell_2)}+1)d-2}\mathcal{M}_{I,d}(\ell_1+1,\ell_2+1)$ for $(\ell_1,\ell_2)\in\mathcal{D}_I$

The first case was discussed in section 2.3.1. Unlike the first case, determining the homotopy type of $H_{(w_{(\ell_1,\ell_2)}+1)d-2}\mathcal{M}_{I,d}(\ell_1+1,\ell_2+1)$ for $(\ell_1,\ell_2)\in\mathcal{D}_I$ requires more information about I than just knowing (ℓ_1,ℓ_2) is in $\mathcal{D}_I$. For instance, in the case that neither $(\ell_1+1,0)$ nor $(0,\ell_2+1)$ is in I, then $\mathcal{M}_{I,d}(\ell_1+1,\ell_2+1)$ is a product of a no (ℓ_1+1)-equal space and a no (ℓ_2+1)-equal space. In this case, $H_{(w_{(\ell_1,\ell_2)}+1)d-2}\mathcal{M}_{I,d}(\ell_1+1,\ell_2+1)=0$. However, if at least one of $(\ell_1+1,0)$ or $(0,\ell_2+1)$ is in I (which is what we assume when we assume I is not rectangular), then $H_{(w_{(\ell_1,\ell_2)}+1)d-2}\mathcal{M}_{I,d}(\ell_1+1,\ell_2+1)\neq 0$. In the case that $(\ell_1+1,0)\in I$, a non-trivial class in $H_{(w_{(\ell_1,\ell_2)}+1)d-2}\mathcal{M}_{I,d}(\ell_1+1,\ell_2+1)$ is given by a product of spheres satisfying the equations:

$$c+\sum_{i=1}^{\ell_1+1}x_i=0 \qquad |c|^2+\sum_{i=1}^{\ell_1+1}|x_i|^2=1$$
$$\sum_{j=1}^{\ell_2+1}(y_j-c)=0 \qquad \sum_{j=1}^{\ell_2+1}|y_j-c|^2=\epsilon$$

Definition. *Denote the above product of spheres by* $\{x_1,\ldots,x_{\ell_1+1},\{y_1,\ldots,y_{\ell_2+1}\}\}$.

To see that $\{x_1,\ldots,x_{\ell_1+1},\{y_1,\ldots,y_{\ell_2+1}\}\}$ is non-zero, we will again consider a chain in $\mathcal{M}_{I,d}(\ell_1+1,\ell_2+1)$. Consider the chain given by the following equations:

$$x_1=\ldots=x_{\ell_1}=y_1=\ldots y_{\ell_2}$$
$$(x_1)_1<(x_{\ell_1+1})_1$$
$$(x_1)_\ell=(x_{\ell_1+1})_\ell \text{ for all } \ell>1$$
$$(x_1)_1<(y_{\ell_2+1})_1$$
$$(x_1)_\ell=(y_{\ell_2+1})_\ell \text{ for all } \ell>1$$

The boundary of this chain is in the complement to $\mathcal{M}_{I,d}(\ell_1+1,\ell_2+1)$. Thus, it represents a class in $H^*(\mathcal{M}_{I,d}(\ell_1+1,\ell_2+1),\mathbb{Z}_2)$. The intersection pairing between this class and $\{x_1,\ldots,x_{\ell_1+1},\{y_1,\ldots,y_{\ell_2+1}\}\}$ is non-zero.

Definition. *Define* local classes *to be classes of one of the following forms:*

- $\{x_1, \ldots, x_{\ell_1}, y_1, \ldots, y_{\ell_2}\} \in H_{(w_{(\ell_1,\ell_2)}-1)d-1}\mathcal{M}_{I,d}(\ell_1, \ell_2)$ *for* $(\ell_1, \ell_2) \in \mathcal{C}_I$
- $\{x_1, \ldots, x_{\ell_1+1}, \{y_1, \ldots, y_{\ell_2+1}\}\} \in H_{(w_{(\ell_1,\ell_2)}+1)d-2}\mathcal{M}_{I,d}(\ell_1+1, \ell_2+1)$ *for* $(\ell_1, \ell_2) \in \mathcal{D}_I$
- $x_1 \in H_0\mathcal{M}_{I,d}(1, 0)$
- $y_1 \in H_0\mathcal{M}_{I,d}(0, 1)$

We will show that all homology classes of $H_*\mathcal{M}_{I,d}$ can be built up using the left action of $\mathcal{M}_d$ on local classes. Again, our proof will use a more general space. We will again consider the homotopy of an s-chain moving one of the points away from the others. The $(s+1)$-chain produced from this homotopy may intersect forbidden subspaces. In the decreasing setting, these subspaces do not intersect. In the bicolored setting, these subspaces may intersect. This is what produces the products of spheres mentioned above and what forces us to introduce more than one type of auxiliary point.

Let $\ell_I = |\mathcal{D}_I|$. We will consider a system of points colored with $\ell_I + 3$ colors. Throughout the proof, we will assume that for all $(\ell_1, \ell_2) \in \mathcal{D}_I$, we have $(\ell_1 + 1, 0) \in I$. At the end of this section, we will comment what changes occur if this is not the case. To emphasize the importance of the added colors, we will refer to color 3 points by z_i and color $i + 3$ points by ${}^iw$ for $1 \leq i \leq m - 3$.

Let (α_i, β_i) be an enumeration of the elements of $\mathcal{D}_I$. Consider

$$I' = \{(a, b, 0, \ldots, 0) \mid (a, b) \in I\} \cup \{(0, 0, 1, 0, \ldots, 0)\} \cup \{(a, 0, 0, \vec{1}_i) \mid a \leq \alpha_i\} \subset \mathbb{N}^{\ell_I+3}$$

where $\vec{1}_i$ has all zeros except for a 1 in the i^{th} coordinate. We will be concerned with the polychromatic configuration space $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$. In addition to classes previously mentioned, we will also consider classes of the form $\{x_1, \ldots, x_{\alpha_i+1}, {}^iw_1\}$. These are defined analogously to previous local classes.

Definition. *Define* augmented local classes *to be classes of one of the following forms:*

- $\{x_1, \ldots, x_{\ell_1}, y_1, \ldots, y_{\ell_2}\}$ *for* $(\ell_1, \ell_2) \in \mathcal{C}_I$
- $\{x_1, \ldots, x_{\ell_1+1}, \{y_1, \ldots, y_{\ell_2+1}\}\}$ *for* $(\ell_1, \ell_2) \in \mathcal{D}_I$

- $\{x_1, \ldots, x_{\alpha_i+1}, {}^i w_1\}$ *for* $(\alpha_i, \beta_i) \in \mathcal{D}_I$
- *An element of* $H_0(\vec{e}_j)$ *for some* $j \leq \ell_I + 3$

Theorem 2.13. *The left module* $H_*\mathcal{M}_{I',d}(\cdot, \ldots, \cdot)$ *is generated by augmented local classes.*

As a corollary of this theorem, we get Theorem 2.3.

Definition. *Call a class* organized *if it can be written as a sum of products of augmented local classes.*

The main idea in the proof is that for each closed s-chain, γ, we write γ as the sum of two closed s-chains: one that is organized and one that is less complex for some suitable measure of complexity.

Definition. *For* $n_2 > 0$*, let*

$$g_0(\gamma) = \sup\{k \mid \exists \text{ distinct } j_1, \ldots, j_k \text{ such that } \gamma \cap \{y_{j_1} = \ldots = y_{j_k}\} \neq \emptyset\}$$
$$g_1(\gamma) = \sup\{k \mid \exists\, i \text{ and distinct } j_1, \ldots, j_k \text{ such that } \gamma \cap \{x_i = y_{j_1} = \ldots = y_{j_k}\} \neq \emptyset\}$$

In the case $n_2 = 0$*, define* $g_0(\gamma)$ *to be* 0*.*

Ideally g_0 would be our measure of complexity; however, this is difficult to achieve, so we settle for g_1. Nonetheless, while we decrease g_1, we still want to control g_0. In our proof, we write each closed s-chain as a sum of two chains: one that is organized and one with lesser g_1 and suitably bounded g_0. Eventually, g_1 cannot get any smaller; we show that this chain is organized.

Definition. *Let* $f_I : \mathbb{N} \to \mathbb{N} \cup \{-\infty, \infty\}$ *be defined by* $f_I(n) = \sup\{m \mid (n, m) \in I\}$*.*

In the statement of the following lemma, the ordering of elements of $\mathbb{N}^2$ is the lexicographic ordering.

Lemma 2.14. *Suppose* $\tilde{n}_1 > 0$ *and organized classes span* $H_*\mathcal{M}_{I',d}(\tilde{n}_1, \ldots, \tilde{n}_{\ell_I+3})$ *for all* $(\tilde{n}_1, \tilde{n}_2) < (n_1, n_2)$*. Let* γ *be any closed s-chain in* $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$*. For all* $a \in \mathbb{N}$*,* $[\gamma] = [\gamma_1^a] + [\gamma_2^a]$ *where* $[\gamma_1^a]$ *is organized and* γ_2^a *satisfies one the following:*

- $[\gamma_2^a] = 0$

- *$g_0(\gamma_2^a) \leq f_I(1) + 1$ and $g_1(\gamma_2^a) \leq f_I(a)$*

The proof of this lemma involves four lemmas and is left to the its own section at the end of this chapter. Given any $a \in \mathbb{N}$ and any closed s-chain γ such that $g_0(\gamma) \leq f_I(1) + 1$ and $g_1(\gamma) \leq f(a)$, these lemmas allow us to write $[\gamma] = [\gamma_1] + [\gamma_2]$ where $[\gamma_1]$ is organized and either $[\gamma_2] = 0$ or $g_0(\gamma_2) \leq f_I(1) + 1$ and $g_1(\gamma_2) \leq f_I(a + 1)$.

Before proving Theorem 2.13, we recall notation that we will use.

Definition. *For any $N \in H_*\mathcal{M}_{I,d}(\vec{n})$, let $N|_{a=A}$ be the class in $H_*\mathcal{M}_{I,d}(\vec{n}')$ given by substituting A for a where a is some z or w coordinate and A is some s-chain $\mathcal{M}_{I,d}$.*

Note: such a substitution does not always produce a class in $H_*\mathcal{M}_{I,d}(\vec{n}')$. However, we will be sure to only make substitutions that do.

We now prove Theorem 2.13.

Proof. We will show organized classes span $H_*\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$ for all $(n_1, \ldots, n_{\ell_I+3})$. This will be done by induction on (n_1, n_2). First suppose $n_1 = 0$. Because no ${}^iw$ may equal any y or z coordinate, we may treat them as z coordinates and apply Theorem 2.5.

Now suppose $n_1 > 0$ and $H_*\mathcal{M}_{I',d}(\tilde{n}_1, \ldots, \tilde{n}_{\ell_I+3})$ is spanned by organized classes whenever $(\tilde{n}_1, \tilde{n}_2) < (n_1, n_2)$. Let γ be any closed s-chain in $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$. Additionally, let $a = \min\{b \mid f_I(b) = f_I(n) \; \forall \; n > b\}$.

By Lemma 2.14, $[\gamma] = [\gamma_1^a] + [\gamma_2^a]$ where $[\gamma_1^a]$ is organized and γ_2^a satisfies one the following:

- $[\gamma_2^a] = 0$

- $g_0(\gamma_2^a) \leq f_I(1) + 1$ and $g_1(\gamma_2^a) \leq f_I(a)$

In the first instance, $[\gamma]$ is an organized class. Thus, assume the second case holds. We cannot have $g_1(\gamma_2^a) \leq -\infty$; thus, it must be the case that $f_I(a) \geq 0$. Consider the homotopy of γ_2^a affecting only the first x coordinate, $\gamma_t = \gamma + v \cdot t$ where v is a vector that is non-zero only in the x_1 coordinate. Let Γ be the $(s+1)$-chain given by the homotopy. The only forbidden subspaces it may intersect are:

$$x_1 = y_{j_1} = \ldots = y_{j_{f_I(1)+1}}$$
$$x_1 = z_j$$
$$x_1 = x_{i_2} = \ldots = x_{i_{f_I(\alpha_\ell)+1}} = {}^\ell w_j$$

Remove sufficiently small tubular neighborhoods of each of these subspaces. Intersecting Γ with tubular neighborhoods of subspaces of the first type produces classes of the form $N|_{z_{n_3+1}=\{x_1,y_{j_1},\ldots,y_{j_{f_I(1)+1}}\}}$ for some $N \in H_*\mathcal{M}_{I',d}(n_1-1, n_2-(f_I(1)+1), n_3+1, \ldots, n_{\ell_I+3})$. Similarly, intersections from subspaces of the second type produces classes of the form $N|_{z_j=[x_1,z_j]}$ where $N \in H_*\mathcal{M}_{I',d}(n_1-1, n_2, n_3, \ldots, n_{\ell_I+3})$. The third case produces classes of the form $N|_{z_{n_3+1}=\{x_1,x_{i_2},\ldots,{}^\ell w_j\}}$ where $N \in H_*\mathcal{M}_{I',d}(n_1-(f_I(\alpha_\ell)+1), n_2, n_3+1, \ldots, n_{\ell+3}-1, \ldots, n_{\ell_I+3})$. For $t = M$, we get $N \cdot x_1$ where $N \in H_*\mathcal{M}_{I',d}(n_1-1, n_2, n_3, \ldots, n_{\ell_I+3})$. Inductively, all of these are organized classes. Thus, Γ with its intersection with these tubular neighborhoods removed allows us to write $[\gamma]$ as a sum of organized classes. Thus, for all $(n_1, \ldots, n_{\ell_I+3})$, organized classes span $H_*\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$. □

Theorem 2.13 produces a generating set for $H_*\mathcal{M}_{I,d}(n_1, n_2)$; we would like a basis. For this, relations between various elements in the generating set are needed. The following relations are for $d > 1$. Some of these relations involve elements that are not organized, but their meanings should be apparent. Additionally, some of the terms shown may be zero depending on I. Let $X = \{x_i \mid i \leq n_1\}, Y = \{y_i \mid i \leq n_2\}$

Lemma 2.15. *Whenever $d > 1$, the elements of $H_*(\mathcal{M}_{I,d}(n_1, n_2), \mathbb{Z}_2)$ satisfy the following relations:*

1. *If $(n-1, m), (n, m-1) \in \mathcal{C}_I$, then*

$$\sum_{i=1}^{n+m} [\{z_1, \ldots, \hat{z}_i, \ldots, z_{n+m}\}, z_i] = 0$$

where $\{z_1, \ldots, z_n\} \subset X, \{z_{n+1}, \ldots, z_{n+m}\} \subset Y$.

2. (a) *If $(n-1, m) \in \mathcal{C}_I, (n, m-1) \in I$, then*

$$\sum_{i=1}^{n} [\{x_1, \ldots, \hat{x}_i, \ldots, x_n, y_1, \ldots, y_m\}, x_i] = 0$$

(b) *Similar relation for* $(n, m-1) \in \mathcal{C}_I, (n-1, m) \in I$

3. *If* $(n, m) \in \mathcal{C}_I,\ (n+1, m-2) \notin I,$ *then*

   (a) $\sum_{i=1}^{n+1}[\{x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}, y_1, \ldots, y_m\}, x_i] = \{y_1, \ldots, y_m, \{x_1, \ldots, x_{n+1}\}\}$

   (b) $\sum_{i=1}^{m+1}[\{y_1, \ldots, \hat{y}_i, \ldots, y_{m+1}, \{x_1, \ldots, x_{n+1}\}\}, y_i] = [\{y_1, \ldots, y_{m+1}\}, \{x_1, \ldots, x_{n+1}\}]$

   (Similar relations for $(n, m) \in \mathcal{C}_I, (n-2, m+1) \notin I$*)*

4. *If* $(n-1, m-1) \in \mathcal{D}_I, (n, 0) \in I,$ *then*

$$\sum_{i=1}^{n+1}[\{x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}, \{y_1, \ldots, y_m\}\}, x_i] = [\{x_1, \ldots, x_{n+1}\}, \{y_1, \ldots, y_m\}]$$

   (Similar relation for $(n-1, m-1) \in \mathcal{D}_I, (0, m) \in I$*)*

5. *If* $(n-1, m-1) \in \mathcal{D}_I, (n, 0) \in I,$ *then*

$$[\{x_1, \ldots, x_n, \{y_1, \ldots, y_m\}\}, y_{m+1}] + [\{x_1, \ldots, x_n, y_{m+1}\}, \{y_1, \ldots, y_m\}] \\ = \{x_1, \ldots, x_n, [\{y_1, \ldots, y_m\}, y_{m+1}]\}$$

   (Similar relation for $(n-1, m-1) \in \mathcal{D}_I, (0, m) \in I$*)*

6. *If* $(n-1, m-1) \in \mathcal{D}_I, (n, 0) \in I,$ *then*

$$\sum_{i=1}^{m+1}\{x_1, \ldots, x_n, [\{y_1, \ldots, \hat{y}_i, \ldots, y_{m+1}\}, y_i]\} = 0$$

Proof. 1. Consider the sphere, S, given by the following equations:

$$\sum_{i=1}^{n+m} z_i = 0$$

$$\sum_{i=1}^{n+m} |z_i|^2 = 1$$

Remove from S tubular neighborhoods of points on S that are not in $\mathcal{M}_{I,d}$. This gives the above relation.

2. Same proof as above; it is just a different set of points removed.

3. (a) Consider the sphere, S, given by the following equations:

$$\sum_{i=1}^{n+1} x_i + \sum_{j=1}^{m} y_j = 0$$
$$\sum_{i=1}^{n+1} |x_i|^2 + \sum_{j=1}^{m} |y_j|^2 = 1$$

Remove from S tubular neighborhoods of its intersection with the following:

$$x_1 = \ldots = \hat{x}_i = \ldots = x_{n+1} = y_1 = \ldots = y_m$$
$$x_1 = \ldots = x_{n+1}$$

What is left is a chain that gives the above relation.

(b) Consider the following augmented arrangement: there exists a w coordinate that is not allowed to be equal to any x coordinates and at most m y coordinates. Consider the sphere, S, given by the following equations:

$$w + \sum_{j=1}^{m+1} y_j = 0$$
$$|w|^2 + \sum_{j=1}^{m+1} |y_j|^2 = 1$$

Remove from S tubular neighborhoods of its intersection with the following:

$$w = y_1 = \ldots = \hat{y}_j = \ldots = y_{m+1}$$
$$y_1 = \ldots = y_{m+1}$$

This gives the relation:

$$\sum_{i=1}^{m+1} [\{y_1, \ldots, \hat{y}_i, \ldots, y_m, w\}, y_i] = [\{y_1, \ldots, y_{m+1}\}, w].$$

Because of the limited interactions allowed for w, we can consider the same chain but with $\{x_1, \ldots, x_{n+1}\}$ substituted for w. This gives the relation.

4. Similar proof as 3(b).

5. Consider an augmented space in which a w coordinate is added. This coordinate is allowed to be equal to no y coordinates and up to $n-1$ x coordinates. Consider the sphere, S given by the following:

$$w + \sum_{i=1}^{n} x_i + y_{m+1} = 0$$
$$|w|^2 + \sum_{i=1}^{n} |x_i|^2 + |y_{m+1}|^2 = 1$$

Remove from S tubular neighborhoods of its intersection with the following:

$$x_1 = \ldots = x_n = w$$
$$x_1 = \ldots = x_n = y_{m+1}$$
$$w = y_{m+1}$$

This gives the relation:

$$[\{x_1, \ldots, x_n, w\}, y_{m+1}] + [\{x_1, \ldots, x_n, y_{m+1}\}, w] = \{x_1, \ldots, x_n, [w, y_{m+1}]\}$$

In the above chain, we can substitute $\{y_1, \ldots, y_m\}$ for w to get the desired relation.

6. Note that on the chain used to prove

$$\sum_{i=1}^{m+1} [\{y_1, \ldots, \hat{y}_i, \ldots, y_{m+1}\}, y_i] = 0$$

there were never m y coordinates all equal. Thus, we may substitute it in for w in $\{x_1, \ldots, x_n, w\}$ to get

$$\{x_1, \ldots, x_n, \sum_{i=1}^{m+1} [\{y_1, \ldots, \hat{y}_i, \ldots, y_{m+1}\}, y_i]\} = 0$$

Since we can move the sum out of the brackets, we get the desired relation. □

Using these relations, along with the Jacobi and anti-symmetry relations from $H_*\mathcal{M}_d$, we can find a smaller generating set for $H_*\mathcal{M}_{I,d}(n_1, n_2)$.

Theorem 2.16. *For all* $d > 1, (n_1, n_2) \in \mathbb{N}^2$*, let* S *be the set of elements of* $H_*\mathcal{M}_{I,d}(n_1, n_2)$ *that can be written as a product where each factor is an* x_i*, a* y_j *or of the form:*

$$[\dots[[B_1, B_2], B_3] \dots B_\ell], \ell \geq 1 \tag{2.2}$$

where each B_s *is of one of the following forms:*

$$[\dots[[\dots[\{x_{i_1}, \dots, x_{i_n}, y_{j_1}, \dots, y_{j_m}\}, x_{i'_1}] \dots x_{i'_a}], y_{j'_1}] \dots y_{j'_b}]$$

$$[\dots[[\dots[\{x_{i_1}, \dots, x_{i_n}, \{y_{j_1}, \dots, y_{j_m}\}\}, x_{i'_1}] \dots x_{i'_a}], y_{j'_1}] \dots y_{j'_b}]$$

In the first case, $(n, m) \in \mathcal{C}_I, i_1 < \dots < i_n, j_1 < \dots < j_m, a, b \geq 0, i'_1 < \dots < i'_a$*, and* $j'_1 < \dots < j'_b$*. Furthermore, we require the following:*

- *if* $(n + 1, m - 2) \in I$*, then* $i_n > i'_a$*.*
- *if* $(n - 1, m + 1) \in I$*, then* $j_m > j'_b$*.*
- *if* $(n + 1, m - 2) \notin I$ *and* $(n - 1, m + 1) \notin I$*, then* $i_n > i'_a$ *or* $j_m > j'_b$*.*

In the second case, $(n - 1, m - 1) \in \mathcal{D}_I, i_1 < \dots < i_n, j_1 < \dots < j_m, a, b \geq 0, i'_1 < \dots < i'_a$*, and* $j'_1 < \dots < j'_b$*. Furthermore, we require* $i_n > i'_a$ *and* $j_m > j'_b$*.*

Additionally, we require the smallest x *index in* $B_1, \dots, B_\ell$ *to be in* B_1*. Then* S *is a generating set for* $H_*\mathcal{M}_{I,d}(n_1, n_2)$*.*

The proof follows in the same manner as the proof to Theorem 2.7 in the previous section. Again, in the case $d = 1$, there are similar relations to those from Lemma 2.15. The only difference is any mention of $[B_1, B_2]$ is replaced with $B_1 \cdot B_2 + B_2 \cdot B_1$. Using these relations, we get the $d = 1$ analogue to Theorem 2.7.

Theorem 2.17. *Let $d = 1$, $(n_1, n_2) \in \mathbb{N}^2$, let S be the set of elements of $H_*\mathcal{M}_{I,d}(n_1, n_2)$ that can be written in the form:*

$$A_{I_0} \cdot B_{J_1} \cdot A_{I_1} \cdot \ldots \cdot B_{J_\ell} \cdot A_{I_\ell}$$

Here, $I_0, J_1, \ldots, J_\ell, I_\ell$ is a partition of $X \cup Y$. $A_{I_s} = x_{i'_{1,s}} \cdot \ldots \cdot x_{i'_{a_s,s}} \cdot y_{j'_{1,s}} \cdot \ldots \cdot y_{j'_{b_s,s}}$ where $I_s = \{x_{i'_{1,s}}, \ldots, x_{i'_{a_s,s}}, y_{j'_{1,s}}, \ldots, y_{j'_{b_s,s}}\}$ with $i'_{1,s} < \ldots < i'_{a_s,s}$ and $j'_{1,s} < \ldots < j'_{b_s,s}$. B_{J_s} is of one of the two forms:

- $\{x_{i_{1,s}}, \ldots, x_{i_{n_s,s}}, y_{j_{1,s}}, \ldots, y_{j_{m_s,s}}\}$ *where J_s is the set $\{x_{i_{1,s}}, \ldots, x_{i_{n_s,s}}, y_{j_{1,s}}, \ldots, y_{j_{m_s,s}}\}$ for some $(n_s, m_s) \in \mathcal{C}_I$.*
- $\{x_{i_{1,s}}, \ldots, x_{i_{n_s,s}}, \{y_{j_{1,s}}, \ldots, y_{j_{m_s,s}}\}\}$ *where J_s is the set $\{x_{i_{1,s}}, \ldots, x_{i_{n_s,s}}, y_{j_{1,s}}, \ldots, y_{j_{m_s,s}}\}$ for some $(n_s - 1, m_s - 1) \in \mathcal{D}_I$*

In either case we have $i_{1,s} < \ldots < i_{a_s,s}$ and $j_{1,s} < \ldots < j_{b_s,s}$. Furthermore, if B_{J_s} is of the first type, we require the following:

- *if $(n+1, m-2) \in I$, then $i_{n_s,s} > i'_{a_s,s}$.*
- *if $(n-1, m+1) \in I$, then $j_{m_s,s} > j'_{b_s,s}$.*
- *if $(n+1, m-2) \notin I$ and $(n-1, m+1) \notin I$, then $i_{n_s,s} > i'_{a_s,s}$ or $j_{m_s,s} > j'_{b_s,s}$.*

If B_{J_s} is of the second type, we have $i_{n_s,s} > i'_{a_s,s}$ and $j_{m_s,s} > j'_{b_s,s}$. Then S is a generating set for $H_*\mathcal{M}_{I,d}(n_1, n_2)$.

The proof follows similar to past proofs.

In the next section, we show that the generating sets given in Theorems 2.16 and 2.17 are actually bases. As previously mentioned, there is a slight modification if there exists $(n-1, m-1) \in \mathcal{D}_I$ such that $(n, 0) \notin I$. In this case, $\{x_1, \ldots, x_n, \{y_1, \ldots, y_m\}\}$ does not live in $\mathcal{M}_{I,d}(n_1, n_2)$. If in addition $(0, m) \notin I$, then, as mentioned earlier, $\mathcal{M}_{I,d}(n_1, n_2)$ is a product of two no-k-equal spaces. The homology of $\mathcal{M}_{I,d}(n_1, n_2)$ can be determined using this structure. In this case, there is a generating set similar to that in Theorem 2.16 where the second type of B_s is never present (similarly for 2.17 and the second type of B_{J_s}). In the event $(0, m) \in I$, then $\{y_1, \ldots, y_m, \{x_1, \ldots, x_n\}\}$ is a chain in $\mathcal{M}_{I,d}(n_1, n_2)$. We replace any mention of $\{x_1, \ldots, x_n, \{y_1, \ldots, y_m\}\}$ with $\{y_1, \ldots, y_m, \{x_1, \ldots, x_n\}\}$. The proof of this follows in a slightly modified but straightforward manner.

2.4.2 Cohomology of Bicolored PCS

The cohomology ring of $\mathcal{M}_{I,d}(n_1, n_2)$ can be described by a set of forests. We will be computing cohomology with integer coefficients. We will still be working under the assumption that I is not rectangular. Again, there are distinguished elements of $\mathbb{N}^2$

Definition. *Let $I \subset \mathbb{N}^2$ be an ideal. Let $\mathcal{C}'_I$ and $\mathcal{D}'_I$ be defined as follows:*

$$\mathcal{C}'_I = \{(n, m) \in I : (n+1, m), (n, m+1) \notin I\}$$

$$\mathcal{D}'_I = \{(n, m) \in \mathcal{C}_I : \{(n-1, m+1), (n+1, m-1)\} \cap \mathbb{N}^2 \subset I\}$$

Definition. *An* admissible forest *is a forest satisfying the following: it has three types of vertices: rectangles, circles, and diamonds. Each circle and diamond contains exactly one element of $X \cup Y$. Each circle is connected to at most one rectangle and nothing else. Each diamond is connected to exactly one rectangle and nothing else. Each rectangle is connected to at least one circle. Each rectangle satisfies one of the following:*

1. *it contains n elements from X and m elements from Y for some $(n, m) \in \mathcal{C}'_I$*
2. *it contains $n-1$ elements from X and $m-1$ elements from Y for some $(n, m) \in \mathcal{D}'_I$*
3. *it contains $n-1$ elements from X for $(n, 0) \in \mathcal{D}'_I$*
4. *it contains $m-1$ elements from Y for $(0, m) \in \mathcal{D}'_I$*

In case 1, there are no diamonds attached to the rectangle. In case 2, either all circles attached to the rectangle contain elements of Y and all diamonds attached to it contain elements from X or all circles attached to the rectangle contain elements from X and all diamonds attached to it contain elements from Y. In case 3, there are no diamonds attached *to it, and all circles attached to it contain elements from X. In case 4, there are no diamonds attached to it, and all circles attached to it contain elements from Y. Finally, each element of $X \cup Y$ must be in exactly one vertex.*

An orientation *of an admissible forest is:*

- *an orientation of each edge*
- *an ordering of elements within each rectangle*
- *an ordering of the set of rectangles and edges*

For a rectangle, if it has no diamonds attached to it, we say its weight *is the number of elements it contains. Otherwise, its weight is one more than the number of elements it contains.*

Notice, because of the assumption that $(2,0),(1,1),(0,2) \in I$, all rectangles have weight at least 2.

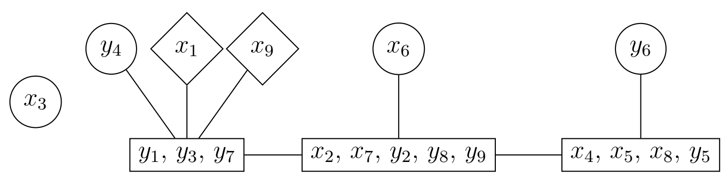


Figure 2.3: An example of an unoriented admissible forest for an ideal $I \subset \mathbb{N}^2$ such that $\{(2,3),(1,4)\} \subset \mathcal{C}'_I$ and $(1,4) \in \mathcal{D}'_I$.

To each admissible forest, we associate a chain in $\mathbb{R}^{d(n+m)}$ whose boundary lies in the complement of $\mathcal{M}_{I,d}$. Thus, each forest will represent a cocycle in $H^*\mathcal{M}_{I,d}(n,m)$. The chain associated to a forest is the set of all points satisfying the following:

- for each rectangle A and each $i,j \in A$, $i = j$
- if there is an edge from A to B and $i \in A, j \in B$, then $(i)_1 \leq (j)_1$ and $(i)_\ell = (j)_\ell$ for all $\ell > 1$ where $(i)_\ell$ is the ℓ^{th} coordinate of i
- for each rectangle A with diamonds B attached to it, if $j \in A$, then $\exists\, i \in B$ such that $i = j$.

The rest of the orientation data is used to coorient the chain. We coorient the chain by giving an explicit basis for the normal bundle. Suppose there exists an edge from vertex A to vertex B. Suppose i is the first element in vertex A and j is the first element in vertex B. Then this edge contributes:

$$\partial(j)_2 - \partial(i)_2, \ldots, \partial(j)_d - \partial(i)_d$$

Suppose there exists a rectangle vertex with ordered elements $(i_1, \ldots, i_\ell)$. This rectangle vertex contributes:

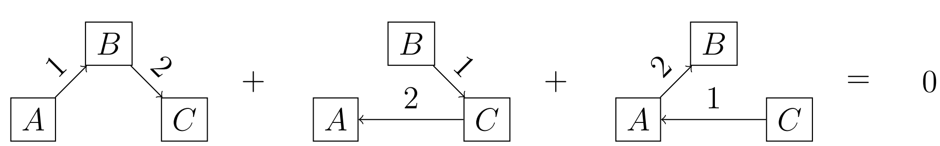
$$\partial(i_2)_1 - \partial(i_1)_1, \ldots, (i_2)_d - \partial(i_1)_d, \ldots, \partial(i_\ell)_d - \partial(i_1)_d$$

If this rectangle vertex has diamonds attached to it, then for the subspace where j behaves like an element in the rectangle, add $\partial(j)_d - \partial(i_1)_d$ to the end of the rectangle's contribution.

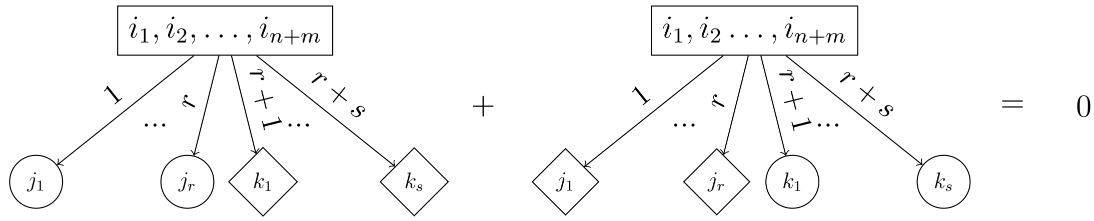
We now give relations between forests:

Lemma 2.18. *For $d > 1$, the cohomology classes given by admissible forests have the following relations:*

1. *Orientation Relations:*
   (a) *Changing the order of the orientation set produces the Koszul sign of the permutation*
   (b) *A permutation $\sigma \in S_n$ of elements inside a rectangle vertex produces a sign $(-1)^{|\sigma| d}$.*
   (c) *Changing the orientation of an edge produces the sign $(-1)^d$.*
2. *3-term relation:*



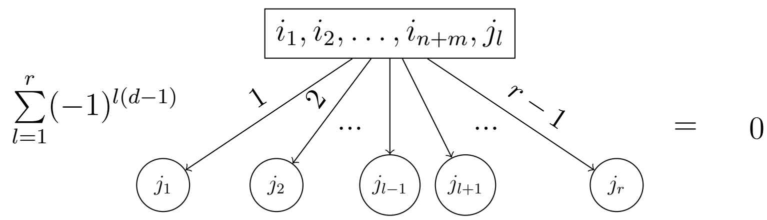
3. *Switching which color are diamonds: If $(n+1, m+1) \in \mathcal{D}'_I$, then*



where $\{i_1, \dots, i_n, j_1, \dots, j_r\} \subset X, \{i_{n+1}, \dots, i_{n+m}, k_1, \dots, k_s\} \subset Y.$

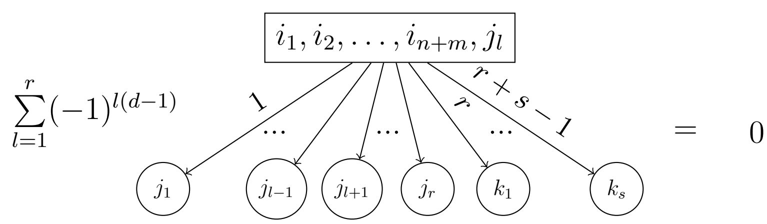
4. *Relations exchanging values in rectangles:*

   (a) *If* $(n+1, m), (n, m+1) \in \mathcal{C}'_I$, *then*



   where $\{i_1, \dots, i_n\} \subset X, \{i_{n+1}, \dots, i_{n+m}\} \subset Y$, *and* $\{j_1, \dots, j_r\} \subset X \cup Y$.

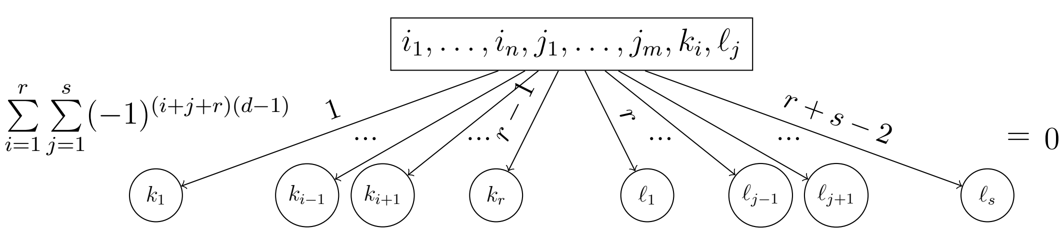
   (b) *If* $(n+1, m) \in \mathcal{C}'_I, (n, m+1) \notin I$, *then*



   where $\{i_1, \dots, i_n, j_1, \dots, j_r\} \subset X, \{i_{n+1}, \dots, i_{n+m}, k_1, \dots, k_s\} \subset Y.$

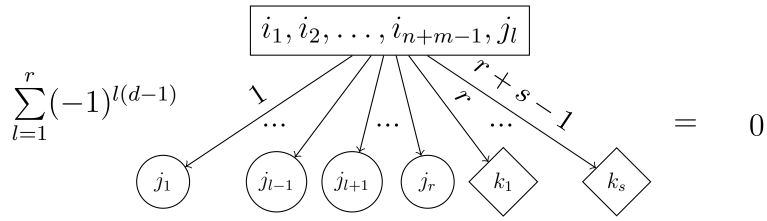
   (c) *Similar relation if* $(n, m+1) \in \mathcal{C}'_I, (n+1, m) \notin I$

   (d) *For any* $(n+1, m+1) \in \mathcal{C}'_I$,



   where $\{i_1, \dots, i_n, k_1, \dots, k_r\} \subset X, \{j_1, \dots, j_m, \ell_1, \dots, \ell_s\} \subset Y$

   (e) *For any* $(n+1, m+1) \in \mathcal{D}'_I$,

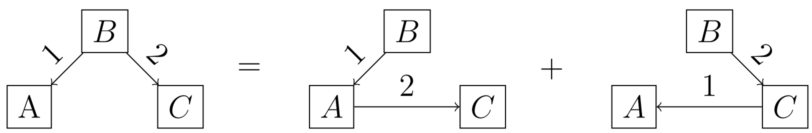


where $\{i_1, \dots, i_{n-1}, j_1, \dots, j_r\} \subset X, \{i_n, \dots, i_{n+m-1}, k_1, \dots, k_s\} \subset Y$.

In relations 2-4, the rectangles may be attached to other rectangles.

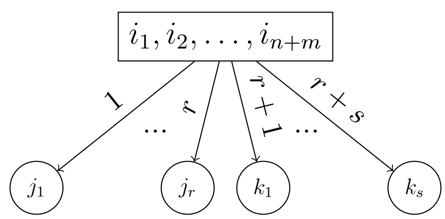
Proof. Relations 1(a) and 1(b) come from changing the coorientation. For 1(c), the inequality changes from $x_i^1 \leq x_j^1$ to $x_i^1 \geq x_j^1$. To see these are homologous (up to a sign), consider the chain given by the the inequality $x_i^2 < x_j^2$. Its boundary is a sum of the two chains in question.

Relation 2 is equivalent to

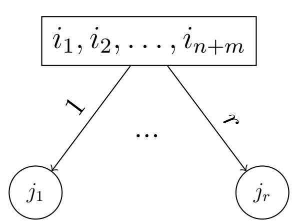


The chain corresponding to the left hand side is the union of the chains corresponding to the trees on the right hand size.

Relation 3 comes from looking at the boundary of the chain corresponding to:

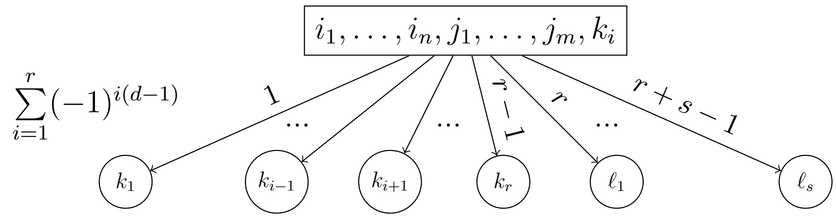


Relation 4(a) comes from looking at the boundary of the chain corresponding to:

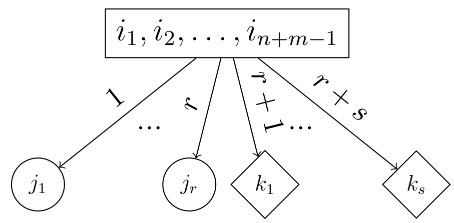


Relations 4(b) and 4(c) comes from looking at similar chains.

Relation 4(d) comes from looking at the boundary of the chain corresponding to:

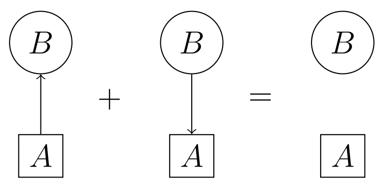


Relation 4(e) comes from looking at the boundary of the chain corresponding to:



□

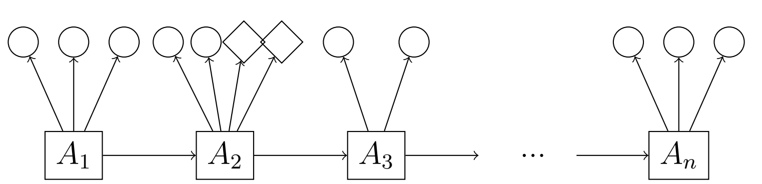
The only relation that does not work when $d = 1$ is relation 1(c). There is a substitute for 1(c) in the case that $d = 1$:



Here, the circle may be replaced with a rectangle. One may also replace the circles on the left hand side of the equation with diamonds. In the event that this causes a rectangle that should have diamonds attached to it to no longer have any diamonds attached to it, the right hand side is zero.

Using these relations, we produce bases for cohomology similar to that from section 2.3.2.

Definition. *Define a* linear I-tree *to be an admissible tree of the following form:*



such that

- *The elements in A_i appear elements from X first, then Y, all in their linear order*
- *The circles and diamonds attached to each rectangle are ordered similarly*
- *All elements in diamonds are from Y*
- *The minimal X element in the tree is in B_1*
- *For each i, let ℓ_i^1 be the maximum element from X in B_i; let ℓ_i^2 be the maximum element from Y in B_i*
  - *If A_i contains n elements from X and m elements from Y for some $(n, m) \in \mathcal{C}'_I$:*
    - *If $(n-1, m+2) \in I$, then $\ell_i^1 \notin A_i$*
    - *If $(n+1, m-1) \in I$, then $\ell_i^2 \notin A_i$*
    - *If $(n-1, m+2), (n+1, m-1) \notin I$, then $\ell_i^1 \notin A_i$ or $\ell_i^2 \notin A_i$*
  - *If A_i contains n elements from X, m elements from Y for some $(n+1, m+1) \in \mathcal{D}'_I$:*
    - *If $n = 0$, then $\ell_i^1 \notin A_i$*
    - *If $m = 0$, then $\ell_i^2 \notin A_i$*
    - *If $n, m \neq 0$, then $\ell_i^1 \notin A_i$ and $\ell_i^2 \notin A_i$*

where B_i is the set of elements in A_i, circles attached to A_i, and diamonds attached to A_i.

Using the relations from Lemma 2.18, any admissible forest can be written as a forest whose components are linear I-trees and singleton circles. We will show that this is a basis for $H^*\mathcal{M}_{I,d}(n_1, n_2)$. For $d > 1$, this basis will be dual to the generating set for homology from Theorem 2.16.

Definition. *Let $\mathcal{H}$ be the set of generators given in Theorem 2.16. Let $\mathcal{C}$ be the set of cohomology classes represented by products of linear I-trees and singleton circles. Define $f : \mathcal{H} \to \mathcal{C}$ as follows:*

Let $A \in \mathcal{H}$. Let $f(A)$ be the forest satisfying the following:

- *For each x_i factor in A, $f(A)$ has a singleton circle containing x_i*

- *For each y_i factor in A, $f(A)$ has a singleton circle containing y_i*
- *Each other factor in A has a corresponding linear I-tree as follows: Suppose the factor is given by $[\dots[[B_1, B_2], B_3]\dots B_\ell]$, then each B_i has a corresponding rectangle vertex, A_i. These rectangle vertices form a path from A_1 to A_ℓ. For each B_i, the corresponding rectangle vertex has the following form:*
  - *if $B_i = [\dots[[\dots[\{x_{i_1}, \dots, x_{i_n}, y_{j_1}, \dots, y_{j_m}\}, x_{i'_1}]\dots x_{i'_a}], y_{j'_1}]\dots y_{j'_b}]$, $(n, m) \in \mathcal{C}_I$:*
    - *if $(n-1, m) \in \mathcal{C}'_I$ and $i_n > i'_a$, then A_i contains $x_{i_1}, \dots, x_{i_{n-1}}, y_{j_1}, \dots, y_{j_m}$, all other elements in B_i correspond to circles attached to A_i*
    - *if the previous condition does not hold and $(n, m-1) \in \mathcal{C}'_I$, then A_i contains $x_{i_1}, \dots, x_{i_n}, y_{j_1}, \dots, y_{j_{m-1}}$, all other elements in B_i correspond to circles attached to A_i*
    - *if $(n, m) \in \mathcal{D}'_I$, then A_i contains $x_{i_1}, \dots, x_{i_{n-1}}, y_{j_1}, \dots, y_{j_{m-1}}$, all other x_j in B_i correspond to circles attached to A_i, all other y_j in B_i correspond to diamonds attached to A_i*
  - *if $B_i = [\dots[[\dots[\{x_{i_1}, \dots, x_{i_n}, \{y_{j_1}, \dots, y_{j_m}\}\}, x_{i'_1}]\dots x_{i'_a}], y_{j'_1}]\dots y_{j'_b}]$ for some n, m with $(n-1, m-1) \in \mathcal{D}_I$, then A_i contains $x_{i_1}, \dots, x_{i_{n-1}}, y_{j_1}, \dots, y_{j_{m-1}}$, all other elements in B_i correspond to circles attached to A_i.*

It is again a (somewhat tedious) exercise to check f is a bijection.

Order $\mathcal{H}$ such that if A has more rectangles than B, then A comes after B. Order $\mathcal{C}$ according to the ordering of corresponding elements in $\mathcal{H}$.

Theorem 2.19. *With this ordering, the intersection pairing matrix is diagonal with ± 1 on the diagonal.*

The proof of Theorem 2.19 is essentially the same as that of Theorem 2.10.

Corollary 2.20. *$\mathcal{H}$ and $\mathcal{C}$ are bases for $H_*\mathcal{M}_{I,d}(n_1, n_2)$ and $H^*\mathcal{M}_{I,d}(n_1, n_2)$, respectively.*

Multiplicative Structure

The addition of diamonds causes us to have additional rules for multiplication not in the previous section.

Definition. *Let $T_1, T_2 \in H^*\mathcal{M}_{I,d}(n_1, n_2)$ be two admissible forests. Suppose for all rectangles A in T_1 and B in T_2, $A \cap B = \emptyset$. Let $T_1 \cup T_2$ be the tree defined as follows:*

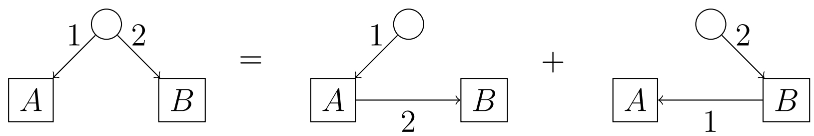
- *if i, j are in a common rectangle in T_1 or T_2, then i, j are in a common rectangle in $T_1 \cup T_2$*
- *if i is in a circle in both T_1 and T_2, then i is in a circle in $T_1 \cup T_2$*
- *if i is in a diamond in both T_1 and T_2, then i is in a diamond in $T_1 \cup T_2$*
- *if i is in a diamond in T_1 and is in a circle attached to nothing in T_2, then i is in a diamond in $T_1 \cup T_2$ (likewise switching T_1 and T_2)*
- *if i is in a diamond in T_1 and is in a circle attached to a rectangle in T_2, then i is in a star in $T_1 \cup T_2$*
- *if $i \in A, j \in B$ in T_k and there exists an edge from A to B in T_k, then there exists an edge from the vertex containing i to the vertex containing j in $T_1 \cup T_2$*

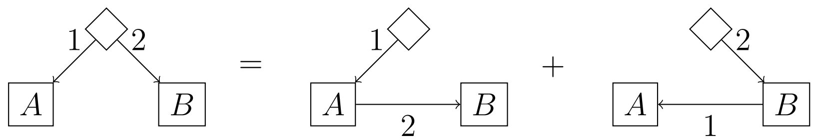
Theorem 2.21. *Let $T_1, T_2 \in H^*\mathcal{M}_{I,d}(n_1, n_2)$ be two admissible forests. The product of T_1 and T_2, $T_1 \cdot T_2$, is given as follows:*

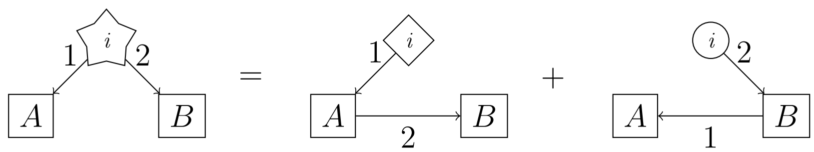
1. *If there exists rectangles A in T_1 and B in T_2 such that $A \cap B \neq \emptyset$, then $T_1 \cdot T_2 = 0$.*
2. *If there exists two indices that are in a common tree in both T_1 and T_2, then $T_1 \cdot T_2 = 0$*
3. *If $T_1 \cup T_2$ has a cycle, then $T_1 \cdot T_2 = 0$.*
4. *If $T_1 \cup T_2$ has a rectangle with no circles attached to it, then $T_1 \cdot T_2 = 0$*
5. *If $T_1 \cup T_2$ has a rectangle that should have diamonds attached to it but doesn't, then $T_1 \cdot T_2 = 0$.*

6. *If $T_1 \cup T_2$ is an admissible forest, then $T_1 \cdot T_2 = T_1 \cup T_2$ with orientation set given by concatenation.*

7. *If $T_1 \cup T_2$ satisfies none of the above, then use the following relations to make $T_1 \cup T_2$ admissible*

(a) 

(b) 

(c) 

where A is the vertex that i is attached to as a diamond in T_1 or T_2 and B is the vertex that i is attached to as a circle in T_1 or T_2.

Proof. 1. If $A \neq B$, then the corresponding chains to T_1 and T_2 do not intersect in $\mathcal{M}_{I,d}(n_1, n_2)$. If $A = B$, then one can perturb the chains slightly so that they do not intersect.

2. There exist orientations of edges such that the two corresponding chains do not intersect in $\mathcal{M}_{I,d}(n_1, n_2)$.

3. There exist orientations of edges such that the two corresponding chains do not intersect in $\mathcal{M}_{I,d}(n_1, n_2)$.

4. This is relation 4 from Lemma 2.18 with $r = 0$ or $r = s = 0$.

5. In this case, the two corresponding chains do not intersect in $\mathcal{M}_{I,d}(n_1, n_2)$.

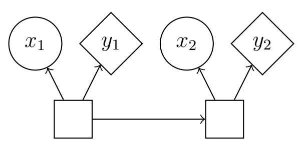
6. The chains corresponding to T_1 and T_2 are transversal and their intersection is the chain corresponding to $T_1 \cup T_2$.

7. Combine the proofs of (6) and relation 2 from Lemma 2.18. □

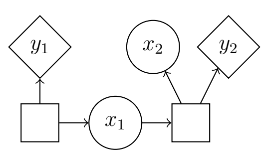
I not a rectangle and $\{(2,0),(1,1),(0,2)\} \not\subset I$

The difficulty in applying the same definition of admissible forests to this case is the fact that there exist weight one rectangles. This case splits into two sub-cases: whether $(1,1)$ is in $\mathcal{D}'_I$ or not. In the case that $(1,1) \notin \mathcal{D}'_I$, the above construction can be altered so that it works with weight one rectangles. We keep the same space of forests as in the general case except that we now allow rectangles to have no circles attached to them. In addition, because when it comes to the corresponding chain, there is no difference between rectangles containing one element and circles, we add the relation that if a weight one rectangle is attached to at most one rectangle and nothing else, it may be turned into a circle. Similarly, we may turn a circle containing an element of M to a rectangle, provided such rectangles are allowed (similarly for N). With these changes, our basis consisting of linear I-trees and singletons is also a basis in this scenario. The only change in multiplication is condition 1 from Theorem 2.21 must additionally assume that A and B both have weight at least 2.

In the case $(1,1) \notin \mathcal{D}'_I$, this construction does not work. An enlightening example is the following tree:

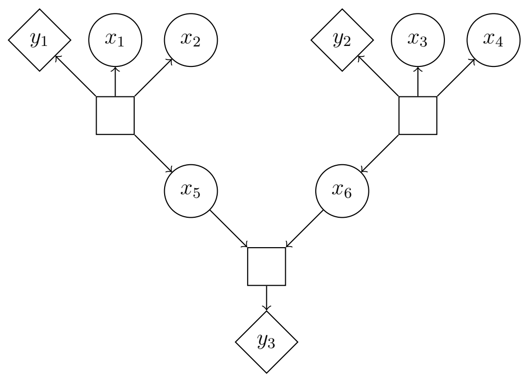


If the previous space of forests were applicable to the case $(1,1) \in \mathcal{D}'_I, d > 1$, then this tree should be dual to the homology element $[\{x_1, y_1\}, \{x_2, y_2\}]$. The problem is the boundary of the chain corresponding to this tree does not live in the complement of $\mathcal{M}_{I,d}(2,2)$. One way to remedy this is to allow for circle vertices to be connected to two rectangles. Instead of the above tree, we could take the following tree:

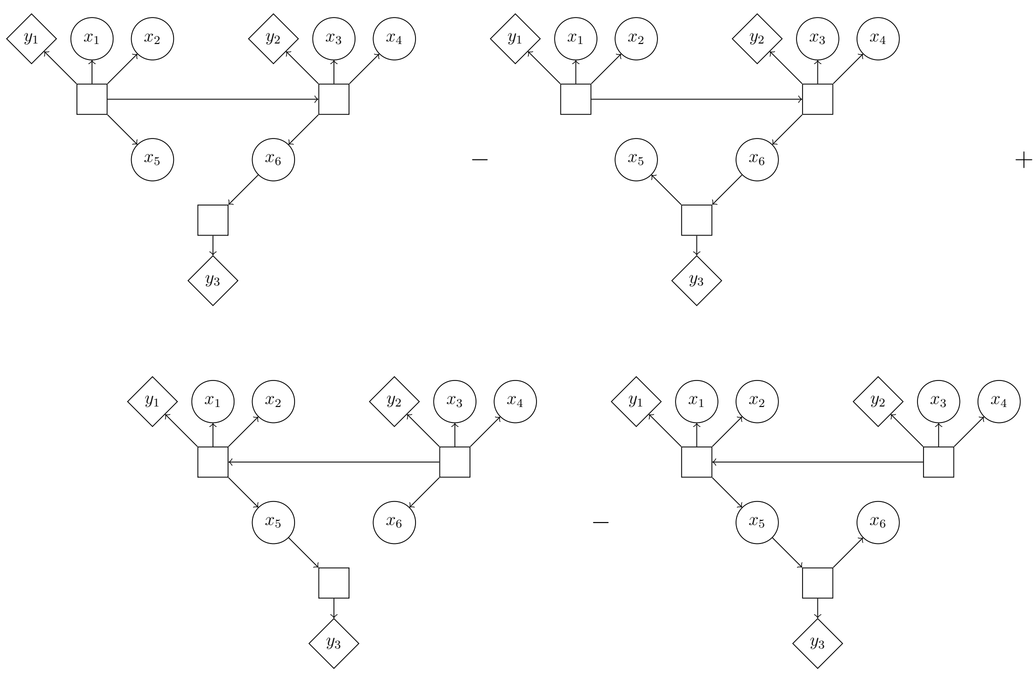


In fact, consider linear I-trees with the following change: for each weight one rectangle that is not last in the chain of rectangles, the maximum circle attached to it is between it and the next rectangle in the chain. Then our basis of products of linear I-trees and singleton circles becomes a basis in this situation. This raises a few questions: what is the full space of forests analogous to the previous situations? How does multiplication behave?

The first question does not have a clear answer. One possibility is that each weight one rectangle should have a circle between it and any other rectangle. Alternatively, one could restrict so that this only need be true between two weight one rectangles. In either case, we want to write any tree as a sum of linear I-trees; that is, we want to be able to decrease the degree of rectangles. We consider an example:



We may have a tree that has this as a subtree in it and has the bottom empty rectangle attached to other rectangles. Following the same idea in the proof of the three term relation in Lemma 2.9, we get this tree is equal to the following:



These are not in the space of allowable trees; in each of the four summands, there are weight one rectangles adjacent to each other. Thus, we need to "simplify" these more. This is possible to do, but it depends on what other rectangles the upper two rectangles are connected to. In each step in the simplification, there are trees not in the space of admissible trees. It may be possible to redefine admissible trees so that these intermediate trees are admissible. The trouble with that is the chains corresponding to each individual tree does not have boundary in the complement to $\mathcal{M}_{I,d}$. It is only when considered together that their collective boundary is in the complement.

2.5 General PCS

Recall the main difference between the homology of decreasing polychromatic configuration spaces and the homology of bicolored configuration spaces: the homology of decreasing polychromatic configuration spaces is generated as an $\mathcal{M}_d$ module by the homology of $\mathcal{M}_{I,d}(\vec{n})$ for $\vec{n} \in \mathcal{C}_I$ while the homology of bicolored configuration spaces is generated as an $\mathcal{M}_d$ module by the homology of $\mathcal{M}_{I,d}(n_1, n_2)$ for $(n_1, n_2) \in \mathcal{C}_I$ and $\mathcal{M}_{I,d}(n_1 + 1, n_2 + 1)$ for $(n_1, n_2) \in \mathcal{D}_I$. That is, there exists a new type of class in the bicolored setting that is not present in the decreasing setting. The obvious question to ask next is what happens for higher m. For this, we will focus on $m = 3$. However, before doing so, we will discuss a couple examples in $m = 2$.

Consider the following ideal in $\mathbb{N}^2$: $I = \{(\ell_1, \ell_2) \mid 0 \leq \ell_i \leq 2 \ \forall \ i\}$. Also, consider $I' = I \cup \{3\vec{e_1}, 3\vec{e_2}\}$. That is I' is I with one 2-tuple added along each axis. Now, $H_3\mathcal{M}_{I',d}(3, 3) = \mathbb{Z}$ while $H_3\mathcal{M}_{I,d}(3, 3) = 0$. A generator for this additional class that appears is $\{x_1, x_2, x_3, \{y_1, y_2, y_3\}\}$.

We will now consider the analogous construction in $m = 3$. We consider the ideals $I = \{(\ell_1, \ell_2, \ell_3) \mid 0 \leq \ell_i \leq 2 \ \forall \ i\}$ and $I' = I \cup \{3\vec{e_1}, 3\vec{e_2}, 3\vec{e_3}\}$. Since $\mathcal{M}_{I',d}(3, 3, 3)$ is the complement to a subspace arrangement, we can use the formula of Goresky-MacPherson to compute its cohomology groups, and, thus, homology groups [14]. Using this formula, we get $H_5\mathcal{M}_{I',d}(3, 3, 3) = \mathbb{Z}$ while $H_5(\mathcal{M}_{I,d}(3, 3, 3)) = 0$. What is this extra class?

It is not the class $\{x_1^1, x_2^1, x_3^1, \{x_1^2, x_2^2, x_3^2, \{x_1^3, x_2^3, x_3^3\}\}\}$ as this chain does not live in $\mathcal{M}_{I',d}$

for it contains points where $x_1^1 = x_2^1 = x_3^1 = x_1^2 = x_2^2$. The same argument disqualifies $\{x_1^1, x_2^1, x_3^1, \{x_1^2, x_2^2, x_3^2\}, \{x_1^3, x_2^3, x_3^3\}\}$.

Another asymmetry between the two above cases is the following. When $m = 2$, we can instead consider $I' = I \cup \{3\vec{e_1}\}$. Again, we have $H_3\mathcal{M}_{I',d}(3,3) = \mathbb{Z}$. In the $m = 3$ case, we could either consider $I' = I \cup \{3\vec{e_1}\}$ or $I' = I \cup \{3\vec{e_1}, 3\vec{e_2}\}$. In either case $H_5\mathcal{M}_{I',d}(3,3,3) = 0$. Thus, there is an asymmetry in when the new classes appear. Furthermore, one can show it cannot be expressed as iterated curly brackets in the sense of the new class for $m = 2$ was. Thus, a new type of bracket must be introduced.

2.6 Proof of Lemma 2.14

In order to prove Lemma 2.14, we will first prove a few technical lemmas. These lemmas will decrease $g_1(\gamma)$. For $(n-1, m-1) \in \mathcal{D}_I$, two of the lemmas will involve the following:

Definition. *For all $a \in \mathbb{N}$, let $m_a = \min\{k \mid f_I(k) = f_I(a)\}$.*

The proof involves removing from chains their intersections with tubular neighborhoods of subspaces. Many times, these subspaces lie in the complement of $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$. However, there are cases where they do not. For these, we only want to remove tubular neighborhoods of subsets of these subspaces. We restrict by using distances between points. Let $\underline{n}$ denote $\{1, 2, \ldots, n\}$.

Definition. *For any set $A = \{j_1, \ldots, j_k\} \subset \underline{n_2}$, let*

$$\tilde{A} = (\mathbb{R}^d)^{(n_1+\ldots+n_{\ell_I+3})} \cap \{y_{j_1} = \ldots = y_{j_k}\}$$

For any point $\bar{x} \in \gamma \cap \tilde{A}$, let

$$d_{\gamma,A}^{(b,c)}(\bar{x}) = \min_K \max_{k \in K}\{d(y_{j_1}, k)\}$$

where the minimum ranges over all K containing b distinct x coordinates and c distinct y coordinates not in $\{y_j : j \in A\}$.

That is, $d^{(b,c)}_{\gamma,A}(\bar{x})$ is the minimum radius, r, such that the ball of radius r centered at y_{j_1} contains b points of color one and c points of color two not labeled by an element of A.

Lemma 2.22. *Let $(n_1, n_2) \in \mathbb{N}^2$, $n_1 > 0$. Suppose $\mathcal{M}_{I',d}(\tilde{n}_1, \ldots, \tilde{n}_{\ell_I+3})$ is spanned by organized classes for all $(\tilde{n}_1, \tilde{n}_2) < (n_1, n_2)$. Let $a \in N$. Let γ be a closed s-chain in $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$ such that $g_0(\gamma) \leq f_I(1) + 1$ and $g_1(\gamma) \leq f_I(a)$. Furthermore, suppose $f_I(a+1) < f_I(a) - 1$ and $f_I(a-1) = f_I(a)$. Then $[\gamma] = [\gamma_1] + [\gamma_2]$ where $[\gamma_1]$ is organized and γ_2 satisfies one the following:*

- $[\gamma_2] = 0$
- *there exist $0 < \epsilon_1 < \epsilon_2$ such that γ_2 satisfies the following:*
  - $g_0(\gamma_2) \leq f_I(1) + 1$
  - $g_1(\gamma_2) \leq f_I(a)$
  - *for any A with $|A| = f_I(a) + 1$ and any $\bar{x} \in \gamma_2 \cap \tilde{A}$, $d^{(m_a,0)}_{\gamma_1,A}(\bar{x}) > \epsilon_2$*
  - *for any A with $|A| = f_I(a)$ and any $\bar{x} \in \gamma_2 \cap \tilde{A}$, $d^{(m_a,1)}_{\gamma_1,A}(\bar{x}) > \epsilon_2$*
  - *for any A with $|A| = f_I(a)$ and any $\bar{x} \in \gamma_2 \cap \tilde{A}$, $d^{(m_a,0)}_{\gamma_1,A}(\bar{x}) \in [0, \epsilon_1) \cup (\epsilon_2, \infty)$*

Proof. In this case, $(a, f_I(a)) \in \mathcal{D}_I$; suppose $a = \alpha_k$. We will use induction to prove a more general statement. We will show by induction that for all $q \leq n_1$, $[\gamma] = [\gamma^q_1] + [\gamma^q_2]$ where $[\gamma^q_1]$ is organized and γ^q_2 satisfies one of the following:

- $[\gamma^q_2] = 0$
- there exist $0 < \epsilon^q_1 < \epsilon^q_2$ such that γ^q_2 satisfies the following:
  - $g(\gamma^q_2) \leq f_I(1) + 1$
  - $g_1(\gamma^q_2) \leq f_I(a)$
  - for any A with $|A| = f_I(a) + 1$ and any $\bar{x} \in \gamma^q_2 \cap \tilde{A}$, $d^{(m_a,0)}_{\gamma^q_2,A}(\bar{x}) > \epsilon^q_2$
  - for any A with $|A| = f_I(a)$ and any $\bar{x} \in \gamma^q_2 \cap \tilde{A}$, $d^{(m_a,1)}_{\gamma^q_2,A}(\bar{x}) > \epsilon^q_2$
  - for any A with $|A| = f_I(a)$ and any $\bar{x} \in \gamma^q_2 \cap \tilde{A}$, $d(y_j, x_i) \in [0, \epsilon^q_1) \cup (\epsilon^q_2, \infty)$ for all $i \leq q$ and $j \in A$.

Suppose γ is a closed s-chain in $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$ such that $g_1(\gamma) \leq f_I(a)$ and $g_0(\gamma) \leq f_I(1) + 1$. Let $\gamma_2^0 = \gamma$. The first and second conditions are true by assumption. The fifth condition is vacuously true. To get the third condition, notice that for all A with $|A| = f_I(a) + 1$ and any $\bar{x} \in \gamma_2^q \cap \tilde{A}$, we have $d^{(m_a,0)}_{\gamma_2^q,A}(\bar{x}) > 0$. Because γ is compact, there exists $\delta_1 > 0$ such that $d^{(m_a,0)}_{\gamma_2^q,A}(\bar{x}) > \delta_1$. Similarly, there exists $\delta_2 > 0$ such that for any A with $|A| = f_I(a)$ and any $\bar{x} \in \gamma_2^q \cap \tilde{A}$, $d^{(m_a,1)}_{\gamma_2^q,A}(\bar{x}) > \delta_2$. Letting $\epsilon_2^0 = \min\{\delta_1, \delta_2\}$ and $\epsilon_1^0 = \epsilon_2^0/2$ gives the claim for $q = 0$.

Now, suppose $0 < q \leq n_1$ and the claim holds for all $\tilde{q} < q$. Consider the homotopy of γ_2^{q-1} affecting only the x_q coordinate, $\gamma_t = \gamma_2^{q-1} + v_q \cdot t$ where v_q is a vector that is non-zero only in the x_q coordinate. For large enough t, say $t = M$, the x_q coordinate is always far away from all other points. Call the $(s+1)$-chain given by this homotopy Γ. Γ may not be a chain in $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$. It may intersect forbidden subspaces of the forms:

$$x_q = y_{j_1} = \ldots = y_{j_{f_I(1)+1}}$$

$$x_q = z_j$$

$$x_q = x_{i_2} = \ldots = x_{i_{u+1}}$$

$$x_q = x_{i_2} = \ldots = x_{i_{\alpha_m+1}} = {}^m w_j$$

$$x_q = x_{i_2} = \ldots = x_{i_b} = y_{j_1} = \ldots = y_{j_c} \text{ where } b > a, 1 \leq c \leq f_I(a)$$

In the first case, remove a sufficiently small tubular neighborhood. The intersection of Γ with the boundary of this neghborhood is a class of the form $N|_{z_{n_3+1}=\{x_q,y_{j_1},\ldots,y_{j_{f_I(1)+1}}\}}$ for some $N \in H_*\mathcal{M}_{I',d}(n_1 - 1, n_2 - (f_I(1)+1), n_3 + 1, \ldots, n_{\ell_I+3})$.

In the second case, again remove a sufficiently small tubular neighborhood. The intersection of Γ with the boundary of this neighborhood produces a class $N|_{z_j=[x_q,z_j]}$ where $N \in H_*\mathcal{M}_{I',d}(n_1 - 1, n_2, n_3, \ldots, n_{\ell_I+3})$.

The third case only occurs if $(u+1, 0) \notin I$. If so, proceed as in the two preceding cases. Remove a small tubular neighborhood. The intersection of Γ with the boundary of this neighborhood is $N|_{z_{n_3+1}=\{x_q,\ldots,x_{i_{u+1}}\}}$ where $N \in H_*\mathcal{M}_{I',d}(n_1-(u+1), n_2, n_3+1, n_4, \ldots, n_{\ell_I+3})$.

For the fourth case, again remove a sufficiently small tubular neighborhood. The intersection of Γ with the boundary of this tubular neighborhood produces $N|_{z_{n_3+1}=\{x_q,\ldots,{}^m w_j\}}$ where $N \in H_*\mathcal{M}_{I',d}(n_1 - (\alpha_m+1), n_2, n_3+1, n_4, \ldots, n_{m+3} - 1 \ldots, n_{\ell_I+3})$.

The fifth case must be treated differently because various subspaces of this form are connected, making it impossible to find disjoint tubular neighborhoods. However, as before, we want to remove tubular neighborhoods of all of these, say of radius r. Let γ_2^q be the intersection of Γ with the boundary of the unions of these tubular neighborhoods. The radii of these tubular neighborhoods can be chosen sufficiently small so that we still have $g_0(\gamma_2^q) \leq f_I(1) + 1$ and $g_1(\gamma_2^q) \leq f_I(a)$. Thus, conditions 1 and 2 still hold.

Let $A \subset \underline{n_2}$ with $|A| = f_I(a) + 1$ and $\gamma_2^q \cap \tilde{A} \neq \emptyset$. Let $\bar{x} \in \gamma_2^q \cap \tilde{A}$. Then $\bar{x}$ comes from a point on Γ with the x-coordinates and y-coordinates corresponding to a forbidden subspace perturbed. We can choose r small enough so that $A \cap \{j_1, \ldots, j_c\} = \emptyset$. Let $j \in A$. Note that each forbidden subspace that we removed tubular neighborhoods of in the fifth case involved at least a x-coordinates and at least 1 y-coordinate. Thus, it comes from a point on γ_2^{q-1} where at least $a-1$ x-coordinates and 1 y-coordinate were equal. Also note that $a - 1 \geq m_a$. Thus, at $\bar{x}$, $d(y_j, x_q) > \epsilon_2^{q-1} - r$. The only other coordinates that were moved were moved by at most r. Thus, $d_{\gamma_2^q, A}^{(m_a,0)}(\bar{x}) > \epsilon_2^{q-1} - r$.

Now let $A \subset \underline{n_2}$ with $|A| = f_I(a)$ and $\gamma_2^q \cap \tilde{A} \neq \emptyset$. Let $\bar{x} \in \gamma_2^q \cap \tilde{A}$. By choosing r small enough, we can restrict to two cases: either A is disjoint from $\{j_1, \ldots, j_c\}$ or $A = \{j_1, \ldots, j_c\}$. The first case follows in a very similar manner to the previous argument. In the second case, prior to the x_q-coordinate being equal to y_{j_i}, there were already m x coordinates equal to these $f_I(a)$ y coordinates. Thus, the next closest y coordinate had to be at least ϵ_2^{q-1} far away. Thus, in either case, $d_{\gamma_2^q, A}^{(m_a,1)}(\bar{x}) > \epsilon_2^{q-1} - r$.

Again let $A \subset \underline{n_2}$ with $|A| = f_I(a)$ and $\gamma_2^q \cap \tilde{A} \neq \emptyset$. Let $\bar{x} \in \gamma_2^q \cap \tilde{A}$. As before, there are two cases: the y-coordinates in A come from one of the forbidden subspaces or not. Suppose $j \in A$. In the first case, $d(y_j, x_q) \in [0, 2r)$. In the second case, $d(y_j, x_q) > \epsilon_2^{q-1} - r$. Because we're only changing coordinates other than x_q by at most a distance of r from γ_2^{q-1}, for all $i < q$, we have $d(y_j, x_i) \in [0, \epsilon_1^{q-1} + r) \cup (\epsilon_2^{q-1} - r, \infty)$. Thus, assuming r has been chosen sufficiently small, for all $i \leq q$, we have $d(y_j, x_i) \in [0, \epsilon_1^{q-1} + r) \cup (\epsilon_2^{q-1} - r, \infty)$.

For $t = M$, we have a class $N \cdot x_q$ where $N \in H_*\mathcal{M}_{I',d}(n_1 - 1, n_2, n_3, \ldots, n_{\ell_I+3})$.

Thus, Γ with its intersection with the above tubular neighborhoods removed allows us to write $[\gamma] = [\gamma_1^q] + [\gamma_2^q]$ where $[\gamma_1^q]$ is organized. Assuming r has been chosen sufficiently small, $[\gamma_2^q]$ satisfies the above conditions. This proves the claim. $\square$

Lemma 2.23. *Let* $(n_1, n_2) \in \mathbb{N}^2$, $n_1 > 0$. *Suppose* $\mathcal{M}_{I',d}(\tilde{n}_1, \ldots, \tilde{n}_{\ell_I+3})$ *is spanned by organized classes for all* $(\tilde{n}_1, \tilde{n}_2) < (n_1, n_2)$. *Let* $a \in N$ *be such that* $f_I(a+1) < f_I(a) - 1$ *and* $f_I(a-1) = f_I(a)$. *Let* γ *be a closed s-chain in* $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$ *such that* $g_0(\gamma) \le f_I(1)+1$, $g_1(\gamma) \le f_I(a)$, *and there exist* $0 < \epsilon_1 < \epsilon_2$ *as in Lemma 2.22. Then* $[\gamma] = [\gamma_1] + [\gamma_2]$ *where* $[\gamma_1]$ *is organized and* γ_2 *satisfies one of the following:*

- $[\gamma_2] = 0$
- $g_0(\gamma_2) \le f_I(1) + 1$ *and* $g_1(\gamma_2) \le f_I(a+1) + 1$

Proof. If $n_2 = 0$, then $g_1(\gamma) \le f_I(a+1)$ and the claim holds. Thus, suppose $n_2 > 0$. We will show by induction that for all $q \le n_2$, $[\gamma] = [\gamma_1^q] + [\gamma_2^q]$ where $[\gamma_1^q]$ is organized and γ_2^q satisfies one of the following:

- $[\gamma_2^q] = 0$
- there exists $0 < \epsilon_1^q < \epsilon_2^q$ such that γ_2^q satisfies the following
  - $g_0(\gamma_2^q) \le f_I(1) + 1$
  - $g_1(\gamma_2^q) \le f_I(a)$
  - for any A with $|A| = f_I(a) + 1$ and any $\bar{x} \in \gamma_2^q \cap \tilde{A}$, $d^{(m_a,0)}_{\gamma_2^q,A}(\bar{x}) > \epsilon_2^q$
  - for any A with $|A| = f_I(a)$ and any $\bar{x} \in \gamma_2^q \cap \tilde{A}$, $d(y_j, x_i) \in [0, \epsilon_1^q) \cup (\epsilon_2^q, \infty)$ for all $i \le q$ and $j \in A$.
  - for all $\tilde{q} \le q$, there does not exist i and distinct $j_2, \ldots, j_{f_I(a+1)+2}$ such that $\gamma_2^q \cap \{x_i = y_{\tilde{q}} = y_{j_2} = \ldots = y_{j_{f_I(a+1)+2}}\} \neq \emptyset$

For $q = 0$, the claim is assumed. Thus, suppose $0 < q \le n_2$ and the claim holds for all $\tilde{q} < q$. Consider the homotopy of γ_2^{q-1} affecting only the y_q coordinate, $\gamma_t = \gamma_2^{q-1} + v_q \cdot t$ where v_q is a vector that is non-zero only in the y_q coordinate. For large enough t, say $t = M$, the y_q coordinate is always far away from all other points. Call the $(s+1)$-chain given by this homotopy Γ. Γ may not be a chain in $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$. It may intersect forbidden subspaces of the forms:

$$y_q = y_{j_2} = \ldots = y_{j_{f_I(0)+1}} \text{ (if } f_I(0) = f_I(1)+1)$$

$$y_q = z_j$$

$$y_q =^m w_j$$

$$y_q = x_{i_1} = \ldots = x_{i_b} = y_{j_2} = \ldots = y_{j_{f_I(a)+1}} \text{ where } m_a \leq b \leq a$$

$$y_q = x_{i_1} = \ldots = x_{i_b} = y_{j_2} = \ldots = y_{j_c} \text{ where } b \geq a+1, 1 \leq c \leq f_I(a+1)+1$$

In each of the first three cases, we may proceed as in the previous lemma. We remove a sufficiently small tubular neighborhood. The intersection of Γ with the boundaries of these neighborhoods will produce organized classes.

For the fourth case, remove a small tubular neighborhood of the set:

$$T = \{y_q = y_{i_2} = \ldots = y_{i_{f_I(a)+1}} : m_a^{th} \text{ closest } x\text{-coordinate is less than } \epsilon_2^{q-1} \text{ away } \}$$

The intersection in question lives within this set. Additionally, by the third and fifth conditions for γ_2^{q-1}, we have $T \cap \gamma_2^{q-1} = \emptyset$ and $\Gamma \cap \partial T = \emptyset$, respectively. The intersection of γ_2^{q-1} and the boundary of the tubular neighborhood of T is a class $N|_{^iw_{n_{i+3}+1} = \{y_q, y_{j_2}, \ldots, y_{j_{f_I(a)+1}}\}}$ for some $N \in H_*\mathcal{M}_{I',d}(n_1, n_2 - (f_I(a+1)+1), n_3, \ldots, n_{i+3}+1, \ldots, n_{\ell_I+3})$. In the case that $a \neq 0$ and this $^iw_{n_{i+3}+1}$ is not in some $\{x_{i_1}, \ldots, x_{i_{a+1}}, {^iw_{n_{i+3}+1}}\}$, then this class is null homologous. Otherwise, this class is organized.

For the fifth case, remove tubular neighborhoods of all of these, say of radius r. Let γ_2^q be the intersection of Γ with the boundary of the unions of these tubular neighborhoods. The radii of these tubular neighborhoods can be made sufficiently small so $g_0(\gamma_2^q) \leq f_I(1)+1$ and $g_1(\gamma_2^q) \leq f_I(a)$. For this case we have $c \leq f_I(a+1)+1 < f_I(a)$, so we can choose r small enough so that conditions three and four hold for $\epsilon_1^q = \epsilon_1^{q-1}+r$ and $\epsilon_2^q = \epsilon_2^{q-1}-r$. Furthermore, r can be chosen small enough so that there does not exist i and distinct $j_2, \ldots, j_{f_I(a+1)+1}$ such that $\gamma_2^q \cap \{x_i = y_q = y_{j_2} = \ldots = y_{j_{f_I(a+1)+1}}\} \neq \emptyset$ and such that this property still holds for all $\tilde{q} < q$.

For $t = M$, we have a class $N \cdot y_q$ where $N \in H_*\mathcal{M}_{I',d}(n_1, n_2 - 1, n_3, \ldots, n_{\ell_I+3})$.

Thus, Γ with its intersection with the above tubular neighborhoods removed allows us to write $[\gamma] = [\gamma_1^q] + [\gamma_2^q]$ where $[\gamma_1]$ is organized and $[\gamma_2^q]$ satisfies the required conditions. Thus, the claim holds for all $q \leq n_2$. The lemma is the case where $q = n_2$. □

Lemma 2.24. *Let $(n_1, n_2) \in \mathbb{N}^2$, $n_1 > 0$. Suppose $\mathcal{M}_{I',d}(\tilde{n}_1, \ldots, \tilde{n}_{\ell_I+3})$ is spanned by organized classes for all $(\tilde{n}_1, \tilde{n}_2) < (n_1, n_2)$. Let $a \in N$ be such that $f_I(a+1) < f_I(a)$ and $f_I(a-1) \neq f_I(a)$. Let γ be a closed s-chain in $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$ such that $g_0(\gamma) \leq f_I(1)+1$ and $g_1(\gamma) \leq f_I(a)$. Then $[\gamma] = [\gamma_1]+[\gamma_2]$ where $[\gamma_1]$ is organized γ_2 satisfies one the following:*

- $[\gamma_2] = 0$
- $g_0(\gamma_2) \leq f_I(1) + 1$ *and* $g_1(\gamma_2) \leq f_I(a+1) + 1$

Proof. As in the proof of the previous lemma, if $n_2 = 0$, the claim holds. Thus, suppose $n_2 > 0$. We will show by induction that for all $q \leq n_2$, $[\gamma] = [\gamma_1^q]+[\gamma_2^q]$ where $[\gamma_1^q]$ is organized and γ_2^q satisfies one of the following:

- $[\gamma_2^q] = 0$
- $g_0(\gamma_2^q) \leq f_I(1)+1$, $g_1(\gamma_2^q) \leq f_I(a)$, and for all $\tilde{q} \leq q$, there does not exist i and distinct $j_2, \ldots, j_{f_I(a+1)+2}$ such that $\gamma_2^q \cap \{x_i = y_{\tilde{q}} = y_{j_2} = \ldots = y_{j_{f_I(a+1)+2}}\} \neq \emptyset$

For $q = 0$, the claim is assumed. Thus, suppose $0 < q \leq n_2$ and the claim holds for all $\tilde{q} < q$. Consider the homotopy of γ_2^{q-1} affecting only the y_q coordinate, $\gamma_t = \gamma_2^{q-1} + v_q \cdot t$ where v_q is a vector that is non-zero only in the y_q coordinate. For large enough t, say $t = M$, the y_q coordinate is always far away from all other points. Call the $(s+1)$-chain given by this homotopy Γ. Γ may not be a chain in $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$. It may intersect forbidden subspaces of the forms:

$$y_q = y_{j_2} = \ldots = y_{f_I(0)+1} \text{ (if } f_I(0) = f_I(1)+1)$$
$$y_q = z_j$$
$$y_q =^m w_j$$
$$y_q = x_{i_1} = \ldots = x_{i_a} = y_{j_2} = \ldots = y_{j_{f_I(a)+1}}$$
$$y_q = x_{i_1} = \ldots = x_{i_b} = y_{j_2} = \ldots = y_{j_c} \text{ where } b \geq a+1, 1 \leq c \leq f_I(a+1)+1$$

In each of the first four cases, we may proceed as we have done so previously. We remove sufficiently small tubular neighborhoods. The intersection of Γ with these boundaries will produce organized classes.

For the fifth case, remove tubular neighborhoods of all of these, say of radius r. Let γ_2^q be the intersection of Γ with the boundary of the unions of these tubular neighborhoods. The radii of these tubular neighborhoods can be made sufficiently small so that $g_0(\gamma_2^q) \leq f_I(1)+1$ and $g_1(\gamma_2^q) \leq f_I(a)$. Furthermore, they can be chosen small enough so that there does not exist i and distinct $j_2, \ldots, j_{f_I(a+1)+2}$ such that $\gamma_2^q \cap \{x_i = y_q = y_{j_2} = \ldots = y_{j_{f_I(a+1)+2}}\} \neq \emptyset$. They can also be chosen small enough to ensure this property still holds for all $\tilde{q} < q$.

For $t = M$, we have a class $N \cdot y_q$ where $N \in H_*\mathcal{M}_{I',d}(n_1, n_2 - 1, n_3, \ldots, n_{\ell_I+3})$.

Thus, Γ with its intersection with the above tubular neighborhoods removed allows us to write $[\gamma] = [\gamma_1^q] + [\gamma_2^q]$ where $[\gamma_1^q]$ is organized and $[\gamma_2^q]$ satisfies the above conditions. Thus, the claim holds for all $q \leq n_2$. The lemma is the case where $q = n_2$. □

Lemma 2.25. *Let $(n_1, n_2) \in \mathbb{N}^2$, $n_1 > 0$. Suppose $\mathcal{M}_{I',d}(\tilde{n}_1, \ldots, \tilde{n}_{\ell_I+3})$ is spanned by organized classes for all $(\tilde{n}_1, \tilde{n}_2) < (n_1, n_2)$. Let $a \in \mathbb{N}$. Let γ be a closed s-chain in $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$ such that $g_0(\gamma) \leq f_I(1) + 1$ and $g_1(\gamma) \leq f_I(a+1) + 1$. Then $[\gamma] = [\gamma_1] + [\gamma_2]$ where $[\gamma_1]$ is organized and γ_2 satisfies one the following:*

- $[\gamma_2] = 0$
- $g_0(\gamma_2) \leq f_I(1) + 1$ *and* $g_1(\gamma_2) \leq f_I(a+1)$

Proof. If $n_1 = 0$, then $g_1(\gamma) \leq f_I(a+1)$ and the claim holds. Thus, suppose $n_1 > 0$. We will show by induction that for all $q \leq n_1$, $[\gamma] = [\gamma_1^q] + [\gamma_2^q]$ where $[\gamma_1^q]$ is organized and γ_2^q satisfies one the following:

- $[\gamma_2^q] = 0$
- $g_0(\gamma_2^q) \leq f_I(1) + 1$, $g_1(\gamma_2^q) \leq f_I(a+1) + 1$, and for all $\tilde{q} \leq q$, there does not exist distinct $j_1, \ldots, j_{f_I(a+1)+1}$ such that $\gamma_2^q \cap \{x_{\tilde{q}} = y_{j_1} = \ldots = y_{j_{f_I(a+1)+1}}\} \neq \emptyset$

For $q = 0$, the claim is assumed. Thus, suppose $0 < q \leq n_1$ and the claim holds for all $\tilde{q} < q$. Consider the homotopy of γ_2^{q-1} affecting only the x_q coordinate, $\gamma_t = \gamma_2^{q-1} + v_q \cdot t$ where v_q is a vector that is non-zero only in the x_q coordinate. For large enough t, say $t = M$, the x_q coordinate is always far away from all other points. Call the $(s+1)$-chain given by this homotopy Γ. Γ may not be a chain in $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$. It may intersect forbidden subspaces of the forms:

$$x_q = y_{j_1} = \ldots = y_{j_{f_I(1)+1}}$$
$$x_q = z_j$$
$$x_q = x_{i_2} = \ldots = x_{i_{u+1}}$$
$$x_q = x_{i_2} = \ldots = x_{i_{\alpha m+1}} = {}^m w_j$$
$$x_q = x_{i_2} = \ldots = x_{i_{a+1}} = y_{j_1} = \ldots = y_{j_{f_I(a+1)+1}}$$
$$x_q = x_{i_2} = \ldots = x_{i_b} = y_{j_1} = \ldots = y_{j_c} \text{ where } b > a+1, 1 \leq c \leq f_I(a+1)$$

In each of the first five cases, we may proceed as we have done so previously. We remove sufficiently small tubular neighborhoods. The intersection of Γ with the boundaries of these neighborhoods will produce organized classes.

For the sixth case, remove tubular neighborhoods of all of these, say of radius r. Let γ_2^q be the intersection of Γ with the boundary of the unions of these tubular neighborhoods. The radii of these tubular neighborhoods can be chosen sufficiently small so that $g_0(\gamma_2^q) \leq f_I(1)+1$ and $g_1(\gamma_2^q) \leq f_I(a+1)+1$. Furthermore, they can be chosen small enough so that there does not exist distinct $j_1, \ldots, j_{f_I(a+1)+1}$ such that $\gamma_2^q \cap \{x_q = y_{j_1} = \ldots = y_{j_{f_I(a+1)+1}}\} \neq \emptyset$. They can also be small enough to ensure this property still holds for all $\tilde{q} < q$.

For $t = M$, we have a class $N \cdot x_q$ where $N \in H_*\mathcal{M}_{I',d}(n_1 - 1, n_2, n_3, \ldots, n_{\ell_I+3})$.

Thus, Γ with its intersection with the above tubular neighborhoods removed allows us to write $[\gamma] = [\gamma_1^q] + [\gamma_2^q]$ where $[\gamma_1^q]$ is organized and γ_2^q satisfies the above conditions. Thus, the claim holds for all $q \leq n_1$. The lemma is the case where $q = n_1$. ☐

We are now ready to prove Lemma 2.14.

Lemma 2.14. *Suppose $\tilde{n}_1 > 0$ and organized classes span $H_*\mathcal{M}_{I',d}(\tilde{n}_1, \ldots, \tilde{n}_{\ell_I+3})$ for all $(\tilde{n}_1, \tilde{n}_2) < (n_1, n_2)$. Let γ be any closed s-chain in $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$. For all $a \in \mathbb{N}$, $[\gamma] = [\gamma_1^a] + [\gamma_2^a]$ where $[\gamma_1^a]$ is organized and γ_2^a satisfies one the following:*

- $[\gamma_2^a] = 0$
- $g_0(\gamma_2^a) \leq f_I(1) + 1$ *and* $g_1(\gamma_2^a) \leq f_I(a)$

Proof. Let $(n_1, n_2) \in \mathbb{N}^2$ with $n_1 > 0$. Suppose organized classes span $\mathcal{M}_{I',d}(\tilde{n}_1, \ldots, \tilde{n}_{\ell_I+3})$ whenever $(\tilde{n}_1, \tilde{n}_2) < (n_1, n_2)$. Let γ be a closed s-chain in $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$. Let $a = 0$. If

$n_2 = 0$, then $g_0(\gamma) = g_1(\gamma) = 0$ and the claim holds. Thus, suppose $n_2 > 0$. It is clear that $g_1(\gamma) \leq f_I(0)$. There are two cases. First, suppose $f_I(0) = f_I(1)$. Then $g_0(\gamma) < f_I(0) + 1 = f_I(1) + 1$, and the claim holds. Second, suppose $f_I(0) > f_I(1)$. We will prove by induction that for all $q \leq n_2$, we can write $[\gamma] = [\gamma_1^q] + [\gamma_2^q]$ where $[\gamma_1^q]$ is organized and γ_2^q satisfies one the following:

- $[\gamma_2^q] = 0$
- for all $\tilde{q} \leq q$, $\tilde{q} \in A \subset \underline{n_2}$, $|A| = f_I(1) + 2$, we have $\gamma_2^q \cap \tilde{A} = \emptyset$

For $q = 0$, the claim is trivial. Thus, suppose $0 < q \leq n_2$ and the claim holds for all $\tilde{q} < q$. Consider the homotopy of γ_2^{q-1} affecting only the y_q coordinate, $\gamma_t = \gamma_2^{q-1} + v_q \cdot t$ where v_q is a vector that is non-zero only in the y_q coordinate. For large enough t, say $t = M$, the y_q coordinate is always far away from all other points. Call the $(s+1)$-chain given by this homotopy Γ. Γ may not be a chain in $\mathcal{M}_{I',d}(n_1, \ldots, n_{\ell_I+3})$. It may intersect forbidden subspaces of the forms:

$$y_q = y_{j_2} = \ldots = y_{j_{f_I(0)+1}}$$
$$y_q = z_j$$
$$y_q =^m w_j$$
$$y_q = x_{j_1} = \ldots = x_{j_b} = y_{i_2} = \ldots = y_{i_c} \text{ where } b \geq 1, 1 \leq c \leq f_I(1) + 1$$

In each of the first three cases, we proceed as we have done so previously. We remove sufficiently small tubular neighborhoods. The intersection of Γ with the boundaries of these neighborhoods will produce organized classes.

For the fourth case, remove tubular neighborhoods of all of these, say of radius r. We can choose r arbitrarily small. Let γ_2^q be the intersection of Γ with the boundary of the unions of these tubular neighborhoods. The radii of these tubular neighborhoods can be chosen sufficiently small so that for all $A \subset \underline{n_2}, |A| = f_I(1) + 2$ with $q \in A$, we have $\gamma_2^q \cap \tilde{A} = \emptyset$. Furthermore, the radii can be chosen small enough so that this property still holds for all $A \subset \underline{n_2}$ of size $f_I(1) + 2$ containing $\tilde{q}$ for all $\tilde{q} < q$.

For $t = M$, we have a class $N \cdot y_q$ where $N \in H_*\mathcal{M}_{I',d}(n_1, n_2 - 1, n_3, \ldots, n_{\ell_I+3})$.

Thus, Γ with its intersection with the above tubular neighborhoods removed allows us to write $[\gamma] = [\gamma_1^q] + [\gamma_2^q]$ where $[\gamma_1^q]$ is organized and $[\gamma_2^q]$ satisfies the above conditions. Thus, the claim holds for all $q \leq n_2$.

Thus, we may write $[\gamma]$ as a sum $[\gamma_1^0] + [\gamma_2^0]$ where $[\gamma_1^0]$ is organized and either:

- $[\gamma_2^0] = 0$
- $g_0(\gamma_2^0) \leq f_I(1) + 1$ and $g_1(\gamma_2^0) \leq f_I(0)$

All that is left to show is that if this property holds for some $a \geq 0$, then it also holds for $a + 1$. There are four cases.

Case I: $f_I(a+1) = f_I(a)$: This case is trivial.

Case II: $f_I(a+1) = f_I(a) - 1$: Use Lemma 2.25

Case III: $f_I(a+1) < f_I(a) - 1$ and $f_I(a-1) \neq f_I(a)$: Use Lemma 2.24 followed by Lemma 2.25.

Case IV: $f_I(a+1) < f_I(a) - 1$ and $f_I(a-1) = f_I(a)$: Use Lemma 2.22 followed by Lemma 2.23 followed by Lemma 2.25. □

Chapter 3

RELATION TO K-TREES

This chapter is joint work with Yuliy Baryshnikov and Caroline Klivans.

3.1 Preliminaries

In this section we introduce d-dimensional spanning trees for d-dimensional cell complexes. For any topological space, X, we denote the rank of the i^{th} homology group of X by $\beta_i(X)$. For any cell complex Σ, we refer to the cells of Σ as *faces* and write $f_\ell(\Sigma)$ for the number of ℓ-dimensional faces in Σ. The *k-skeleton* of Σ is the collection of all faces of dimension k or less and will be denoted by Σ_k. A *facet* is any face of maximal dimension.

The following definition is not the most general notion of a higher dimensional tree but is sufficiently general for our purposes and avoids unnecessary technical complications, see [12] for more details.

Definition. *Let Σ be a d-dimensional cell complex such that $\beta_{d-1}(\Sigma) = 0$. A subcomplex $T \subset \Sigma$ such that $T_{d-1} = \Sigma_{d-1}$ is a d-*spanning tree *if*

$$H_d(T, \mathbb{Z}) = 0, \tag{3.1a}$$

$$|H_{d-1}(T, \mathbb{Z})| < \infty, \quad \textit{and} \tag{3.1b}$$

$$f_d(T) = f_d(\Sigma) - \beta_d(\Sigma). \tag{3.1c}$$

The initial condition that the $d-1$ skeleta are equal is the spanning condition. The other three homological conditions are analogues to the graphical conditions for a tree on n vertices: acyclicity, connectedness and having $n-1$ edges.

Spaces which are themselves spanning trees include any triangulation of a disk, but also any triangulation of $\mathbb{R}P^2$. If Σ is the boundary of a convex polytope in $\mathbb{R}^d$, then any collection of all but one facet gives a d-dimensional spanning tree. More generally, cellulated spheres are the higher dimensional analogue of cycle graphs where the removal of any one edge yields a spanning tree.

We will be primarily concerned with spanning trees of skeleta of cubes. Let Cube_n denote the n-dimensional hypercube, thought of either as a geometric convex polytope or a combinatorial cell complex. As a geometric object Cube_n is the convex hull of the 2^n points in $\mathbb{R}^n$ whose coordinates are all 0 or 1. Combinatorially, the face lattice consists of all ordered n-tuples $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, where $\sigma_i \in \{0, 1, *\}$. A face σ is contained in a face τ if $\sigma_i \leq \tau_i \, \forall \, i$, where the digits are ordered $0 < *, 1 < *$, and $0, 1$ are incomparable. With this encoding, the dimension of a face is simply the number of $*$s in its string. Let $\mathrm{Cube}_{n,k}$ denote the k-skeleton of the n-cube, then the facets of $\mathrm{Cube}_{n,k}$ are all $\{0, 1, *\}$ strings of length n with exactly k $*$s.

Let $T \subset \mathrm{Cube}_{n,k}$ be a cellular spanning tree of $\mathrm{Cube}_{n,k}$, see [11] for a detailed study of spanning trees of cubical complexes. The size of T, i.e. the number of facets of T, or equivalently, the kth entry of the f-vector $f_k(T)$, is:

$$|T| = f_k(T) = \sum_{i=k}^{n} \binom{n}{i} \binom{i-1}{k-1}.$$

The hypercube Cube_n is dual to the n-dimensional cross polytope, Cross_n. Namely, there is an inclusion reversing bijection from the cells of Cube_n to the cells of Cross_n. Moreover, as algebraic cell complexes, the boundary maps of Cube_n equal the coboundary maps of Cross_n. The cross polytope is realized as the convex hull of the n standard basis vectors of $\mathbb{R}^n$ and their opposites:

$$\mathrm{Cross}_n = \{x \in \mathbb{R}^n : |x_1| + |x_2| + \ldots + |x_d| \leq 1\}.$$

The hypercube is a simple polytope, each vertex of Cube_n is contained in precisely n facets. Dually, the crosspolytope Cross_n is a simplicial polytope, each facet of Cross_n contains precisely n vertices.

As mentioned in the introduction, the n^{th} *no k-equal space of X* consists of the collection of all sets of n points on X such that no k of them are equal.

Definition. *The* k-equal arrangement *of $\mathbb{R}^n$ is the subspace arrangement consisting of all subspaces of the form $\{x_{i_1} = \cdots = x_{i_k}\}$ for $1 \leq i_1 < \ldots < i_k \leq n$. We will denote it by $\mathcal{A}_{n,k}$.*

Definition. *The* n^{th} no k-equal space *of $\mathbb{R}$ is the complement in $\mathbb{R}^n$ of the k-equal arrangement,*

$$\mathcal{M}_{n,k} = \mathbb{R}^n \setminus \mathcal{A}_{n,k}.$$

Björner and Welker were the first to explicitly compute the Betti numbers of $\mathcal{M}_{n,k}$.

Theorem 3.1 (Theorem 1.1 of [6])**.** *The cohomology groups of $\mathcal{M}_{n,k}$ are free. Furthermore,*

$$\text{rank } H^{k-2}(\mathcal{M}_{n,k}) = \sum_{i=k}^{n} \binom{n}{i}\binom{i-1}{k-1}, \text{ if } k \geq 3.$$

The proof of this theorem uses a theorem of Goresky-MacPherson. They give a method to compute the above cohomology combinatorially. More specifically, the theorem of Goresky-MacPherson gives the cohomology of the complement of a subspace arrangement in terms of the homology groups of order complexes formed from the intersection lattice of the arrangement [14].

3.1.1 Simplicial Resolutions

The last bit of background information that concerns us is an elementary construction of a kind of "simplicial resolution". For a finite set of points $S \subset \mathbb{R}^n$, let $\text{conv}(S)$ denote the convex hull of S.

We say that a (compact) set $X \subset \mathbb{R}^n$ is *m-avoiding* if for any $2m$-tuple of distinct points $\{x_1, \ldots, x_m, x'_1, \ldots, x'_m\}, x_k, x'_k \in X, 1 \leq k \leq m$, the convex hulls $\text{conv}(x_1, \ldots, x_m)$, $\text{conv}(x'_1, \ldots, x'_m)$ of these tuples do not intersect.

The following Lemma, which is an immediate corollary of the Thom Transversality Theorem (see e.g. [15, Chapter 3] or [25, Chapter 4]) shows that any subset X of $\mathbb{R}^n$ can be embedded as an m-avoiding subset:

Lemma 3.2. *For any m and large enough N, a generic polynomial embedding of $\mathbb{R}^n$ into $\mathbb{R}^N$ is m-avoiding.*

This will be useful in the following situation which we will encounter later on:

Definition. *Let $f : X \to Y$ be a continuous surjective map such that $|f^{-1}(y)| \leq m$ for all $y \in Y$. Let $i : X \to \mathbb{R}^n$ be an m-avoiding embedding. Define X^Δ by:*

$$X^\Delta = \{(y, z) \in Y \times \mathbb{R}^n : z \in conv(i(f^{-1}(y)))\}.$$

The extension of f to X^Δ is well-defined because of the m-avoiding condition. Denote this extension as f^Δ.

The simplicial resolution *of (f, i) is the pair (X^Δ, f^Δ).*

Note that if X is a compact subset of $\mathbb{R}^n$, then so is X^Δ. The property of simplicial resolutions that we will be most concerned with is the following:

Proposition 3.3. *For a simplicial mapping between simplicial complexes $f : X \to Y$, its simplicial resolution*

$$f^\Delta : X^\Delta \to Y$$

is a homotopy equivalence.

Proof. Indeed, in this situation, the mapping is a fibration with contractible fibers. □

3.2 Proof of Theorem 3.6

The final observation we will need concerns the relative sizes of trees across dimension and duality. First, an Alexander duality for trees.

Proposition 3.4. *[11, Proposition 6.1] Let X and Y be dual d-dimensional complexes and f^* be the inclusion reversing bijection from cells of X to cells of Y. Furthermore let $T \subseteq X_i$ and $U = \{f^* \mid f \in X_i \backslash T\}$. Then T is an i-tree of X if and only if U is a $(d - i)$-tree of Y.*

Second, spanning trees of a complex Σ in adjacent dimensions Σ_i, Σ_{i+1} have complementary size. This result appears, e.g., as Proposition 2.6 of [11]. There the proof is formulated in terms of the long exact sequence for relative homology. We give an alternative argument here for polytopes that relates more directly to our proof of the main theorem.

Proposition 3.5. *Let P be a convex polytope in $\mathbb{R}^n$, P_k its k-skeleton and T a k-dimensional spanning tree of P_k. Then $f_k(T) = \beta_{k-1}(P_{k-1})$.*

Proof. By definition, we have

$$f_k(T) = f_k(P_k) - \beta_k(P_k).$$

Because P_k is shellable, it is homotopy equivalent to a wedge of spheres. Thus, its Euler characteristics is

$$\chi(P_k) = 1 + (-1)^k \beta_k(P_k).$$

We may also express the Euler characteristic as an alternating sum of the numbers of faces in each dimension:

$$\chi(P_k) = \sum_{i=0}^{k} (-1)^i f_i(P_k).$$

Using the same relations for P_{k-1} and the fact that $\chi(P_k) = \chi(P_{k-1}) + (-1)^k f_k(P)$, one gets the desired result. □

Specializing to the case of the cube, we conclude that the following are equinumerous:

- the size of a k-dimensional tree of $\mathrm{Cube}_{n,k}$
- the size of a $(n-k)$-dimensional tree of $\mathrm{Cross}_{n,n-k}$
- the size of the complement of a $(k-1)$-dimensional tree of $\mathrm{Cube}_{n,k-1}$
- the size of the complement of a $(n-k-1)$-dimensional tree of $\mathrm{Cross}_{n,n-k-1}$

where $\text{Cross}_{n,k}$ denotes the k-dimensional skeleton of the n-dimensional cross-polytope and the complements are all taken within the appropriate skeletons. Numerically, this gives:

$$\begin{aligned} f_k(T(\text{Cube}_{n,k})) &= f_{n-k}(T(\text{Cross}_{n,n-k})) \\ &= \binom{n}{k-1}2^{n-k+1} - f_{k-1}(T(\text{Cube}_{n,k-1})) \\ &= \binom{n}{k+1}2^{n-k-1} - f_{n-k-1}(T(\text{Cross}_{n,n-k-1}). \end{aligned}$$

We are now ready to prove our main result:

Theorem 3.6. *The rank of the $(k-2)$-dimensional homology group of the no k-equal subspace of $\mathbb{R}$ is equal to the number of facets in a k-dimensional spanning tree of the k-skeleton of the n-dimensional hypercube.*

Proof. First, assume $k < n$.

As discussed above, by Alexander duality, we have:

$$\beta_{k-1}(\text{Cube}_{n,k-1}) = \beta_{n-k-1}(\text{Cross}_{n,n-k-1})$$

The $(n-k-1)$-skeleton of Cross_n consists of simplices that are convex hulls of $(n-k)$ of its vertices. These simplices can be defined explicitly as follows. For any $I = \{i_1, \ldots, i_k \mid 1 \leq i_1 < \cdots < i_k \leq n\}$, let L_I denote the subspace:

$$L_I = \{x_{i_1} = \cdots = x_{i_k} = 0\}.$$

The faces of the $(n-k-1)$ skeleton of Cross_n are intersections of the L^1-sphere with subspaces of the form L_I. We will denote the union of all such L_I by Coor_k, the codim-k coordinate arrangement. Now, consider the suspension of the intersection of the L^1-sphere and Coor_k. The suspension is homeomorphic to the one point compactification of Coor_k, ${\text{Coor}_k}^*$. Thus, $\beta_{n-k-1}(\text{Cross}_{n,n-k-1}) = \beta_{n-k}({\text{Coor}_k}^*)$.

Let $S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum x_i = 0\}$ and let

$$\pi : \mathrm{Coor}_k \to S$$

be the projection of the coordinate arrangement to S along the diagonal. Note that the image $\pi(\mathrm{Coor}_k)$ lands inside $\mathcal{A}_{n,k}$. Furthermore, this extends continuously to one point compactifications. Slightly abusing notation, we use π to refer to this extension.

We are now in the situation of the definition of simplicial resolutions – we may safely assume that the one-point compactifications of our arrangements are triangulated subsets of spheres in Euclidean space.

In the case that $n < 2k$, π is a homeomorphism. However, when $n \geq 2k$, it is not: the point where several k-diagonals intersect has multiple preimages. The number of preimages is bounded from above by $m = \lfloor n/k \rfloor$.

Consider the simplicial resolution of (π, i), $(\mathrm{Coor}_k^*)^\Delta$. Using Theorem 3.3, π is a homotopy equivalence. Thus, $\beta_{n-k}((\mathrm{Coor}_k^*)^\Delta) = \beta_{n-k}(\mathcal{A}_{n,k}^*)$. All simplices added while taking the simplicial resolution are of dimension at most $n - k - 2$: indeed, the dimension of the cells glued over the preimages of l-fold intersections of the k-diagonals is equal to

$$n - l(k-1) + (l-1)$$

(the first summand is the dimension of the l-fold intersection; the second, of the simplices over each point of the self-intersection). As $l \geq 2$ and $k \geq 3$, we obtain the desired bound.

Therefore, the cells added to Coor_k^* to obtain the simplicial resolution do not affect homology in dimension $n - k$. Therefore, $\beta_{n-k}(\mathrm{Coor}_k^*) = \beta_{n-k}(\mathcal{A}_{n,k}^*)$. Finally, by Alexander duality, $\beta_{n-k}(\mathcal{A}_{n,k}^*) = \beta_{k-2}(\mathcal{M}_{n,k})$ and $f_k(T) = \beta_{k-2}(\mathcal{M}_{n,k})$ as desired.

For $k = n$, the n-dimensional hypercube is an n-dimensional spanning tree of itself; $f_n(T) = 1$. The n^{th} no n-equal space of $\mathbb{R}$ is homotopy equivalent to an $(n-2)$-dimensional sphere, so $\beta_{n-2}(\mathcal{M}_{n,n}) = 1$. Thus, the claim holds for all $k \leq n$. $\square$

3.3 Generalization to Comb Arrangement

The identity in Theorem 3.6 can be generalized to the following situation. Consider the comb no k-equal subspace arrangement defined as follows:

Definition. *Let $A_j \subset \mathbb{R}, j = 1, \ldots, n$ be finite non-empty subsets of the $\mathbb{R}$.*

The A-comb k-equal arrangement *of $\mathbb{R}^n$ consists of all subspaces of the form*

$$\{x_{i_1} - a_{i_1} = \cdots = x_{i_k} - a_{i_k}\}$$

for $1 \leq i_1 < \ldots < i_k \leq n$ and $a_{i_j} \in A_j$.

The A-comb no k-equal space *of $\mathbb{R}$ is the complement in $\mathbb{R}^n$ of the A-comb k-equal arrangement. We will denote this aforementioned arrangement as $\Delta_k^A \subset \mathbb{R}^{n-1}$, and its complement as M_k^A.*

Notice that we recover the no k-equal arrangement when all the A_js are $\{0\}$.

Define a *k-dependence* between the sets A_j as a collection of k distinct pairs $\{x_{j_1}, x'_{j_1}\} \subset A_{j_1}, \ldots, \{x_{j_k}, x'_{j_k}\} \subset A_{j_k}$ such that $x_{j_i} - x'_{j_i}$ coincide for all $i = 1, \ldots, k$.

Definition. *A* pile of cubes of size $\prod_{j=1}^{n} N_j$ *is the (cubical) CW complex consisting of the parallelogram $[0, N_1] \times [0, N_2] \times \cdots \times [0, N_n]$ naturally stratified by the integer grid.*

Theorem 3.7. *Assuming that the there are no k-dependences between the A_js, the rank of the $(k-2)-$dimensional homology of M_k^A is equal to the number of facets in a k-dimensional spanning tree of the k-skeleton of the pile of cubes of size $\prod_{j=1}^{n} n_j$.*

The key component of the proof is the following result:

Proposition 3.8. *The rank of the $(n-k)$-th integer homology of the one-point compactification of the arrangement Δ_k^A equals the rank of the $(k-1)$-st integer homology of the $(k-1)$-st skeleton of the pile of cubes of size $\prod_{j=1}^{n} n_j$.*

Proof. We start with a construction of a pile of cubes in $\mathbb{R}^n$: pick one point in the interior of the n_j open intervals into which A_j partitions $\mathbb{R}$. We will denote this subset as B_j. The product of the collections of the n_j closed intervals in the j-th factor of $\mathbb{R}^n$ defines a pile of cubes B of size $\prod_{j=1}^{n} n_j$.

We consider our Euclidean n-space $\mathbb{R}^n \subset S^n$ as an open subset of its one-point compactification. Adding the large open cell at infinity to the pile of cubes B defines a (cubical) regular CW complex structure on the n-sphere.

On the other hand, we have a natural CW complex obtained by taking the products of the points of the A_js and the intervals into which A_js split the real line. This CW complex can be compactified into a finite regular CW complex by adding a point at infinity; we will denote this complex as A. Both A and B are homeomorphic to the n-sphere.

Importantly, these two CW complexes are dual: for each k cell of one there exists exactly one $(n-k)$ cell of the other, intersecting at a unique point, and the boundary operators on these two complexes are automatically dual to each other.

This implies that the k-th homology of the k-skeleton of one of these CW-complexes is isomorphic to the $(n-k-1)$-st homology of the $(n-k-1)$-skeleton of the other. Thus, the $(k-1)$-st homology of the $(k-1)$-skeleton of B is isomorphic to the $(n-k)$-th homology of the $(n-k)$-skeleton of A.

Analogous to the proof of Theorem 3.6, we consider the projection of A into the space $S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum x_i = 0\}$. The image of this projection lives in Δ_k^A. We may once again extend this to a one point compactification. Once more consider the simplicial resolution of this projection. The fact that there are no k-dependences between the A_js ensures that the dimension of the cells added in the construction of the simplicial resolution are at most $n-k-2$. Thus, the $(n-k)$-th homology of the $(n-k)$-skeleton of A is isomorphic to the $(n-k)$-th homology of the one point compactification of Δ_k^A. □

The rest of the proof of Theorem 3.7 follows from Proposition 3.5 at the beginning and Alexander duality at the end.

Corollary 3.9. *Assuming that the there are no k-dependences between the A_js, the rank of* *the* $(k-2)-$ *dimensional homology of* M_k^A, β_{k-2}, *satisfies the following:*

$$1 + (-1)^{k-1}\beta_{k-2} = \prod_{j=1}^{n}(n_j+1)\left(\sum_{\ell=0}^{k-1}(-1)^{\ell}\sum_{|I|=\ell}\prod_{i\in I}\frac{n_i}{n_i+1}\right)$$

where the I *are subsets of* $\{1,\ldots,n\}$.

Proof. By Theorem 3.7, β_{k-2} equals the number of facets in a k-dimensional spanning tree of the k-skeleton of the pile of cubes of size $\prod_{j=1}^{n} n_j$. Let P_ℓ denote the ℓ-skeleton of this pile of cubes. By Proposition 3.5, the number of facets in a k-dimensional spanning tree of P_k is equal to $\beta_{k-1}(P_{k-1})$. $\beta_{k-1}(P_{k-1})$ satisfies

$$1 + (-1)^{k-1}\beta_{k-1}(P_{k-1}) = \prod_{j=1}^{n}(n_j+1)\left(\sum_{\ell=0}^{k-1}(-1)^{\ell}\sum_{|I|=\ell}\prod_{i\in I}\frac{n_i}{n_i+1}\right).$$

The left hand side is the Euler characteristic of P_{k-1} computed using the fact that P_{k-1} is homotopy equivalent to a wedge of spheres. The right hand side is the Euler characteristic computed as an alternating sum of the number of cells in each dimension. $\square$

Chapter 4

NO ℓ-INTERSECTING HYPERPLANES

4.1 Preliminaries

4.1.1 (Codimension c) Discriminantal Arrangements

Discriminantal arrangements were introduced by Manin and Schechtman as a way to generalize braid arrangements [20]. We begin by giving two equivalent definitions of discriminantal arrangements. The first definition is in terms of determinants of matrices built up from the normal vectors to a collection of hyperplanes.

Definition. *A hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^d$ is* essential *if their normal vectors span $\mathbb{R}^d$.*

Definition. *Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an essential hyperplane arrangement in $\mathbb{R}^d$ with normal vectors $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$. The* discriminantal arrangement based on $\mathcal{A}$ *is the arrangement of hyperplanes in $\mathbb{R}^n$ with normal vectors the distinct, nonzero vectors of the form*

$$\boldsymbol{\alpha}_{\mathcal{S}} = \sum_{i=1}^{d+1} (-1)^i \det(\boldsymbol{\alpha}_{s_1}, \ldots, \hat{\boldsymbol{\alpha}}_{s_i}, \ldots, \boldsymbol{\alpha}_{s_{d+1}}) \cdot \mathbf{e}_{s_i}$$

where $\mathcal{S} = \{s_1, \ldots, s_{d+1}\}$ ranges over all $d+1$ subsets of $\{1, \ldots, n\}$.

The above definition does not make clear what this space represents in terms of the hyperplanes in $\mathcal{A}$. The second definition will make this more clear.

Let $\mathcal{A}$ be an essential hyperplane arrangement in $\mathbb{R}^d$ with n hyperplanes. Let $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$ be the normal vectors to hyperplanes in $\mathcal{A}$. To each $\mathbf{b} \in \mathbb{R}^n$, we associate an arrangement of translations of the hyperplanes in $\mathcal{A}$ as follows:

Definition. *For $\mathbf{b} \in \mathbb{R}^n$, let $\mathcal{A}_{\mathbf{b}}$ denote the set of hyperplanes $\{\boldsymbol{\alpha}_i \cdot \mathbf{x} = b_i \mid 1 \leq i \leq n\}$*

Definition. *We will say the hyperplanes* $\mathcal{A}_{\mathbf{b}} = \{H_1, \ldots, H_n\}$ *are* no ℓ-intersecting *if*

$$\bigcap_{i \in S} H_i = \emptyset \textit{ for all subsets } S \subset \underline{n} \textit{ with } |S| = \ell.$$

Let $\mathcal{M}_\ell(\mathcal{A}) = \{\mathbf{b} \in \mathbb{R}^n \mid \mathcal{A}_{\mathbf{b}} \textit{ is no } \ell\textit{-intersecting}\}$.

Definition. *The* discriminantal arrangement based on $\mathcal{A}$ *is* $\mathbb{R}^n - \mathcal{M}_{d+1}(\mathcal{A})$.

The equivalence of these two definitions was shown by Bayer and Brandt [3]. We will be working with no ℓ-intersecting translates of $\mathcal{A}$ for $\ell > d+1$. As with discriminantal arrangements, we will have an equivalent definition using determinants of matrices. However, we will be more restrictive of the arrangement $\mathcal{A}$. The following definition was introduced by Athanasiadis in his work on the intersection lattice of discriminantal arrangements.

Definition (Athanasiadis [1])**.** *Let* $\mathcal{A}$ *be an essential hyperplane arrangement in* $\mathbb{R}^d$ *with normal vectors* $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$. *Suppose* $\boldsymbol{\alpha}_i = (\alpha_{ij})_{j=1}^d$. *For each set* $\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_m\}$ *of* $(d+1)$*-subsets of* $\underline{n}$ *with* $m \leq n$, *denote by* $A_{\mathcal{S}}$ *the* $m \times n$ *matrix whose* (r, j) *entry is the* j^{th} *coordinate of* $\boldsymbol{\alpha}_{\mathcal{S}_r}$. *Let* $p_{\mathcal{S}}$ *be the sum of the squares of the* $m \times m$ *minors of* $A_{\mathcal{S}}$, *considered as a polynomial in the indeterminants* α_{ij}. *Then* $\mathcal{A}$ *is called* sufficiently general *if* $p_{\mathcal{S}}(\alpha_{ij}) \neq 0$ *for all* $\mathcal{S}$ *such that* $p_{\mathcal{S}}$ *is not identically zero.*

Definition. *Let* $\mathcal{A} = \{H_1, \ldots, H_n\}$ *be a sufficiently general hyperplane arrangement in* $\mathbb{R}^d$ *with normal vectors* $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$. *The* codimension-$c$ discriminantal arrangement based on $\mathcal{A}$ *is the arrangement of codimension-*c *subspaces in* $\mathbb{R}^n$ *that are nullspaces of matrices* M_S *where* M_S *is the* $c \times n$ *matrix whose* j^{th} *row is:*

$$\sum_{i=1}^{d+1} (-1)^i \det(\boldsymbol{\alpha}_{s_j}, \ldots, \hat{\boldsymbol{\alpha}}_{s_{j+i-1}}, \ldots, \boldsymbol{\alpha}_{s_{j+d}}) \cdot \mathbf{e}_{s_{j+i-1}}$$

where $S = \{s_1, \ldots, s_{d+c}\}$ *ranges over all* $d+c$ *subsets of* $\{1, \ldots, n\}$.

The proof that the codimension-c discriminantal arrangement based on $\mathcal{A}$ is equivalent to the complement of the no $(d+c)$-intersecting translations of $\mathcal{A}$ follows very similar to Bayer and Brandt's proof of this fact when $c = 1$.

4.1.2 Intersection Lattice of $\mathcal{M}_{d+1}(\mathcal{A})$

The intersection lattice of $\mathcal{M}_{d+1}(\mathcal{A})$ will be of interest for us. Ideally, this would only depend on n and d; however, it was shown by Falk that this is not the case [13]. Manin and Schechtman themselves did not make the claim that the topology of $\mathcal{M}_{d+1}(\mathcal{A})$ is dependent only on n and d; instead, they were "concerned mostly with its combinatorial invariants which are constant on an open Zariski dense subset of all n-arrangements" [20]. Bayer and Brandt conjectured that on this open subset, there exists an isomorphism between the intersection lattice of $\mathcal{M}_{d+1}(\mathcal{A})$ and a poset they described. This Zariski open subset is the set of sufficiently general arrangements defined by Athanasiadis mentioned above. Further, Athanasiadis proved the conjecture by Bayer and Brandt [1].

Definition. *Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be subsets of $\underline{n}$. We say $\mathcal{S}$ is an* anti-chain *if $S_i \notin S_j$ for all $i \neq j$. For any two anti-chains, $\mathcal{S} = \{S_1, \ldots, S_m\}$ and $\mathcal{T} = \{T_1, \ldots, T_\ell\}$, we say $\mathcal{S} \leq \mathcal{T}$ if for all i, there exists j, such that $S_i \subset T_j$.*

Definition. *Let $n, d \in \mathbb{N}$. $P(n, d)$ is the poset consisting of anti-chains $\mathcal{S} = \{S_1, \ldots, S_m\}$ such that $|S_i| \geq d$ for all i and for all $\mathcal{F} \subset \mathcal{S}$ such that $|\mathcal{F}| \geq 2$, we have*

$$|\bigcup_{F \in \mathcal{F}} F| - d + \sum_{F \in \mathcal{F}} (|F| - d) > 0$$

Theorem 4.1 (Athansiadis [1]). *The intersection lattice of the discriminantal arrangement of a sufficiently general hyperplane arrangement is isomorphic to $P(n, d)$.*

The anti-chain $\mathcal{S} = \{S_1, \ldots, S_m\}$ in $P(n, d)$ corresponds to the subspace of $R^n - \mathcal{M}_{d+1}(\mathcal{A})$ where for each i, the hyperplanes from S_i all intersect in a single point. Moreover, if x being in the subspace corresponding to $\mathcal{S}$ implies that hyperplanes $\ell_1, \ldots, \ell_j$ all intersect in a single point, then $\{\ell_1, \ldots, \ell_j\} \subset S_i$ for some i.

4.1.3 Oriented Matroids

One concept that we will use in our discussion of the homology and cohomology of no $(d+c)$-intersecting hyperplanes is oriented matroids. For more background on oriented matroids, see a chapter by Richter-Gebert and Ziegler [23]. Using the covector definition, oriented matroids are subsets of $\{-, 0, +\}^B$ for some base space B that satisfy certain axioms.

Definition. *For any two $C, D \in \{-, 0, +\}^B$, define $C \circ D$ by*

$$(C \circ D)_b = \begin{cases} C_b & \text{if } C_b \neq 0 \\ D_b & \text{otherwise} \end{cases}$$

Definition. *For any two $C, D \in \{-, 0, +\}^B$, let $S(C, D)$ be defined by*

$$S(C, D) = \{b \in B \mid C_b = -D_b \neq 0\}$$

Definition. *An* oriented matroid *is a pair $(B, \mathcal{C})$ where $\mathcal{C} \subset \{-, 0, +\}^B$ satisfying:*

1. $0 \in \mathcal{C}$
2. *If $C \in \mathcal{C}$, then $-C \in \mathcal{C}$*
3. *If $C, D \in \mathcal{C}$, then $C \circ D \in \mathcal{C}$*
4. *If $C, D \in \mathcal{C}, b \in S(C, D)$, then there exists $Z \in \mathcal{C}$ such that $Z_b = 0$ and $Z_{b'} = (C \circ D)_{b'}$ for all $b' \in B - S(C, D)$*

Example. *Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an essential hyperplane arrangement in $\mathbb{R}^d$. For each i, let $\boldsymbol{\alpha}_i$ be a normal vector for H_i. For $x \in \mathbb{R}^d$, let $C_x \in \{-, 0, +\}^{\underline{n}}$ be defined by $(C_x)_i = sgn(\boldsymbol{\alpha}_i \cdot x)$. $\mathcal{C}_{\mathcal{A}} = \{C_x \mid x \in \mathbb{R}^d\}$ is an oriented matroid.*

Consider the oriented matroid $\mathcal{C}_{\mathcal{H}}$ where $\mathcal{H}$ is the (codimension-1) discriminantal arrangement based on $\mathcal{A}$. In this setting, each hyperplane in H is indexed by $d + 1$ hyperplanes from $\mathcal{A}$. For each $f \in \mathcal{C}_{\mathcal{H}}$, $f_{\{s_1, \ldots, s_{d+1}\}}$ gives information regarding where the intersection of hyperplanes $s_1, \ldots, s_d$ is relative to hyperplane s_{d+1}. However, we prefer this information be made more explicit. For this reason, we will work with a slightly different oriented matroid. The oriented matroid we will be working with uses $\binom{\underline{n}}{d} \times \underline{n}$ as its base space where $\binom{X}{d}$ denotes d element subsets of X.

Definition. *Let $\mathcal{A}$ be a sufficiently general central hyperplane arrangement in $\mathbb{R}^d$ with n hyperplanes with normal vectors $\{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n\} \subset \mathbb{R}^d$. Let $\mathbf{b} \in \mathbb{R}^n$, $S \in \binom{n}{d}, r \in \underline{n}$. Let*

$$x = \bigcap_{s \in S} \{\mathbf{x} \mid \boldsymbol{\alpha}_s \cdot \mathbf{x} = b_s\} \qquad\qquad y \in \{\mathbf{x} \mid \boldsymbol{\alpha}_r \cdot \mathbf{x} = b_r\}$$

Define $C^{\mathbf{b}} : \binom{n}{d} \times \underline{n} \to \{-, 0, +\}$ by $C^{\mathbf{b}}(S, r) = sign((x - y) \cdot \boldsymbol{\alpha}_r)$. Let $\mathcal{O}_{\mathcal{A}} = \{C^{\mathbf{b}} \mid \mathbf{b} \in \mathbb{R}^n\}$.

With this definition, $C^{\mathbf{b}}(S, r)$ gives explicit information of where the point of intersection of hyperplanes from S is relative to hyperplane r.

Definition. *For any $f \in \mathcal{O}_{\mathcal{A}}$. We will say $\mathbf{b} \in \mathbb{R}^n$ is a* realization *of f if $C^{\mathbf{b}} = f$.*

Proposition 4.2. *$\mathcal{O}_{\mathcal{A}}$ is an oriented matroid.*

Proof. We will check each condition of being an oriented matroid.

1. $\mathbf{0}$ is a realization for the 0 function on $\binom{n}{d} \times \underline{n}$.

2. If $\mathbf{b}$ is a realization for C, then $-\mathbf{b}$ is a realization for $-C$.

3. If $\mathbf{b_1}, \mathbf{b_2}$ are realizations for C, D respectively, then for some $\varepsilon > 0, \mathbf{b_1} + \varepsilon \mathbf{b_2}$ is a realization for $C \circ D$.

4. If $\mathbf{b_1}, \mathbf{b_2}$ are realizations for C, D respectively, then for some $0 < t < 1, \mathbf{b_1} + t(\mathbf{b_2} - \mathbf{b_1})$ is a realization for Z □

We will not always be working with functions $f : \binom{n}{d} \times \underline{n} \to \{-, 0, +\}$. Sometimes, we will only need this function to be defined on a subset of $\binom{n}{d} \times \underline{n}$. In order to do this, we add an element to the range and instead discuss these as functions $f : \binom{n}{d} \times \underline{n} \to \{-, 0, +, *\}$. In a similar vein to the above definition:

Definition. *Let $f : \binom{n}{d} \times \underline{n} \to \{-, 0, +, *\}$. Let $X = f^{-1}(\{-, 0, +\})$. We will say that $\mathbf{b} \in \mathbb{R}^n$ is a* realization *of f if $C^{\mathbf{b}}(S, r) = f(S, r)$ for all $(S, r) \in X$. We will denote by V_f the set of all realizations of f.*

To help with the description of homology and cohomology classes, we will care about functions $f : \binom{n}{d} \times \underline{n} \to \{-, 0, +, *\}$ that have a particular form.

Definition. *Let* $\mathcal{F}_{\mathcal{A},m}$ *be the functions* $f : \binom{n}{d} \times \underline{n} \to \{-, 0, +, *\}$ *such that:*

- *There exists* $\{X_1, \ldots X_m\} \in P(n, d)$ *such that:*
  - $f(S, r) = 0$ *for all* $(S, r) \in \binom{X_i}{d} \times X_i$
  - $f(S, r) \in \{+, -\}$ *for all* $S \in \binom{X_i}{d}, r \notin X_i$
  - $f(S, r) = *$ *for all other* (S, r)
- $V_f \neq \emptyset$

For such a function f*, the set* $\{X_1, \ldots, X_m\}$ *will be called the* intersection sets *of* f *and will be denoted by* Z_f*.*

Lemma 4.3. *Let* $f \in \mathcal{F}_{\mathcal{A},m}$*, then there exists* $\mathbf{b} \in V_f$ *such that* $C^{\mathbf{b}}(S, r) \neq 0$ *whenever* $f(S, r) = *$*.*

Proof. Let $\mathbf{v} \in V_f$. Let $\mathcal{S} \in P(n, d)$ be the maximal flat in the intersection lattice of the discriminantal arrangement based on $\mathcal{A}$ that $\mathbf{v}$ is in. Then $Z_f \subset \mathcal{S}$. Let $\mathbf{v}'$ be in the flat corresponding to Z_f. Then for small enough $\varepsilon > 0$, $\mathbf{v}+\varepsilon\mathbf{v}'$ satisfies the desired conditions. □

To each such function, we will associate a few sets and functions.

Definition. *Let* $f \in \mathcal{F}_{\mathcal{A},m}$*,* $X \in Z_f$ *with* $|X| > d, k \in X$*. Define* $f^{X,\pm k}$ *to be the function defined as follows:*

- $f^{X,\pm k}(S, r) = 0$ *for all* $(S, r) \in \binom{Y}{d} \times Y$ *for* $Y \in Z_f, Y \neq X$
- $f^{X,\pm k}(S, r) = 0$ *for all* $(S, r) \in \binom{X-\{k\}}{d} \times X - \{k\}$
- $f^{X,\pm k}(S, k) = \pm$ *for all* $S \in \binom{X-\{k\}}{d}$*.*
- $f^{X,\pm k}(S, r) = f_{S,r}$ *for all other* S, r

Notice that $f^{X,\pm k} \in \mathcal{F}_{\mathcal{A},m}$ with intersection set equal to $\{Y \in Z_f \mid Y \neq X\} \cup \{X - \{k\}\}$. Using the description of the intersection lattice mentioned earlier, one can show that $V_{f^{X,\pm k}}$ is non empty.

Definition. *Let $f \in \mathcal{F}_{\mathcal{A},m}$, $X \in Z_f$. Define $P_{f,X}$ and $N_{f,X}$ as follows:*

$$P_{f,X} = \{j \in \underline{n} \mid \exists\, g \in \mathcal{F}_{\mathcal{A},m} \text{ such that } f = g^{X\cup\{j\},+j}\}$$
$$N_{f,X} = \{j \in \underline{n} \mid \exists\, g \in \mathcal{F}_{\mathcal{A},m} \text{ such that } f = g^{X\cup\{j\},-j}\}$$

Definition. *Let $f \in \mathcal{F}_{\mathcal{A},m}, X \in Z_f$. For $j \in P_{f,X}$, let $f_{X,+j}$ denote the function such that $(f_{X,+j})^{X\cup\{j\},+j} = f$. Similarly, for $j \in N_{f,X}$, let $f_{X,-j}$ denote the function such that $(f_{X,-j})^{X\cup\{j\},-j} = f$.*

4.2 Homology

4.2.1 Homology Class Descriptions

Before proving our main theorem, we will define the building blocks for the homology elements we will be discussing.

Let $\mathcal{A}$ be be a sufficiently general collection of $d + c$ hyperplanes in $\mathbb{R}^d$, then $\mathcal{M}_{d+c}(\mathcal{A})$ is homotopy equivalent to S^{c-1}. A homotopy equivalence can be given by retracting $\mathcal{M}_{d+c}(\mathcal{A})$ onto the sphere given by equations:

$$\sum_{i=1}^{d+c} b_i \boldsymbol{\alpha}_i = \mathbf{0} \qquad \sum_{i=1}^{d+c} |b_i|^2 = 1$$

When $\mathcal{A}$ contains more than n hyperplanes, generators of $H_*\mathcal{M}_{d+c}(\mathcal{A})$ will be associated to particular elements of $\mathcal{O}_{\mathcal{A}}$. In the situation above, a choice of generator for the above sphere will be associated to $f \in \mathcal{O}_{\mathcal{A}}$ defined by $f(S, r) = 0$ for all $(S, r) \in \binom{d+c}{d} \times \underline{d + c}$.

In the $d = 1$ setting, one is able to build up higher dimension homology classes from a single generator because there exists an action of the configuration spaces of $\mathbb{R}$ on the no k-equal spaces of $\mathbb{R}$ (see [9]). One replaces points by sufficiently small generators of $H_{c-1}\mathcal{M}_{c+1}(\mathcal{A})$. If $d > 1$, this action no longer exists. However, we still want all of our higher dimension generators to, in some sense, be built up from this one generator.

Let $\mathcal{A}$ be a sufficiently general collection of n hyperplanes in $\mathbb{R}^d$. We will associate generators of $H_*\mathcal{M}_{d+c}(\mathcal{A})$ with elements of $\mathcal{O}_{\mathcal{A}}$. For this, we want specific elements of $\mathcal{O}_{\mathcal{A}}$.

Definition. *Let $\ell, m \in \mathbb{N}$. Let $\mathcal{O}_{\mathcal{A},\ell,m} \subset \mathcal{O}_{\mathcal{A}}$ be the elements, f, with the following properties:*

- *There exists $\{X_1, \ldots, X_m\} \in P(n,d)$ such that $|X_i| = d + \ell$ and $f(S,r) = 0$ for all $(S,r) \in \binom{X_i}{d} \times X_i$*
- *For all $S \subset \underline{n}$ such that $|S| = d + \ell, S \neq X_i$ for any i, there exists $(S', r) \in \binom{S}{d} \times S$ such that $f(S', r) \neq 0$*
- *There does not exist X such that $|X| = d+\ell+1$ and $f(S,r) = 0$ for all $(S,r) \in \binom{X}{d} \times X$*

Again, we will let $Z_f = \{X_1, \ldots, X_m\}$.

In terms of the hyperplanes, $\mathcal{O}_{\mathcal{A},\ell,m}$ corresponds to translates of $\mathcal{A}$ such that exactly m sets of $d + \ell$ hyperplanes intersect and no collection of $d + \ell + 1$ hyperplanes intersect. To each element of $\mathcal{O}_{\mathcal{A},c,m}$, we can associate an element of $H_{m(c-1)}\mathcal{M}_{d+c}(\mathcal{A})$. Let $f \in \mathcal{O}_{\mathcal{A},c,m}$. Let $\mathbf{b} \in \mathbb{R}^n$ be a realization of f, then near $\mathbf{b}$, $\mathcal{M}_{d+c}(\mathcal{A})$ is a product of m punctured $\mathbb{R}^c$'s and a linear space. We denote by f_* the homology class that is a product of the m $(c-1)$-spheres about the punctured $\mathbb{R}^c$'s. Any choice of realization will produce homologous chains. Furthermore, multiple elements of $\mathcal{O}_{\mathcal{A},c,m}$ will produce homologous chains if they only differ on particular inputs.

Proposition 4.4. *Let $f, g \in \mathcal{O}_{\mathcal{A},c,m}$. Suppose $Z_f = Z_g$. Let $Z_f = \{X_1, \ldots, X_m\}$. If $f(S,r) = g(S,r)$ for all $S \in \binom{X_i}{d}, r \notin X_i$, then $f_* = g_*$.*

Proof. Let $X = \bigcup \binom{X_i}{d} \times \underline{n}$. Let $h : \binom{n}{d} \times \underline{n} \to \{-, 0, +, *\}$ be defined by

$$h(S,r) = \begin{cases} f(S,r) & (S,r) \in X \\ * & (S,r) \notin X \end{cases}$$

Let $Y = \{S \in \binom{n}{d+c} \mid S \neq X_i \text{ for all } i\}$. For each $S \in Y$, let h_S be defined by

$$h_S(S', r) = \begin{cases} 0 & (S', r) \in \binom{S}{d} \times S \\ * & \text{else} \end{cases}$$

Let b_f and b_g be realizations of f and g, respectively. Consider the straight line between b_f and b_g. It may intersect V_{h_S} for some $S \in Y$. Because $c > 1$, $V_{h_S} \cap V_h$ is codimension at least 2 in V_h. Thus, we may perturb this path so that it avoids all subspaces of the form V_{h_S} while still remaining in V_h. For each point on this path, $\mathbf{b}$, we have if $|X'| = d + c$ and $C^{\mathbf{b}}(S, r) = 0$ for all $(S, r) \in \binom{X'}{d} \times X'$, then $X' = X_i$ for some i. Thus, as above, near $\mathbf{b}$, $\mathcal{M}_{d+c}(\mathcal{A})$ is a product of m punctured $\mathbb{R}^c$'s and a linear space. Taking a product of small enough spheres about the punctured $\mathbb{R}^c$'s for each point on this path in a continuous manner gives the desired relation. □

This proposition shows that we don't actually need to be working with $\mathcal{O}_{\mathcal{A},c,m}$.

Definition. *Let $\ell, m \in \mathbb{N}$. Let $\mathcal{P}_{\mathcal{A},\ell,m} \subset \mathcal{F}_{\mathcal{A},m}$ be the elements, f, such that $|X| = d + \ell$ for all $X \in Z_f$.*

To each element of $f \in \mathcal{P}_{\mathcal{A},c,m}$, we associate an element of $H_{m(c-1)}\mathcal{M}_{d+c}(\mathcal{A})$ by picking some $g \in \mathcal{O}_{\mathcal{A},c,m}$ that agrees with f on $f^{-1}(\{-, 0, +\})$. Such a g exists by Lemma 4.3. We then define f_* to be g_*. We will denote by $\mathcal{B}_{\mathcal{A},c,m}$ the set of all such f_*.

In the following section, we will use relations to rewrite these classes as sums of a smaller set of classes. When doing this, we will want some value associated to each class that we decrease. We will call this value a class's *positivity vector*, or *p-vector* for short.

Definition. *Let $f \in \mathcal{P}_{\mathcal{A},c,m}$. Let $Z_f = \{X_1, \dots, X_m\}$. For each i, let $S_i \in \binom{X_i}{d}$. Then f's* p-vector, *$\vec{v}$ is defined by*

$$v_j = |\{i \mid f(S_i, j) = +\}|$$

That is, the j^{th} entry is the number of points of intersections corresponding to some X_i that are on the positive side of the j^{th} hyperplane.

4.2.2 Homology

Throughout this section, homology groups will refer to reduced homology. Furthermore, we will be concerned with homology with $\mathbb{Z}_2$ coefficients. A generalization to integral homology is possible as long as one is careful with orientations. Furthermore, we will fix c to be an integer greater than 1.

Questions. *Let $\mathcal{A}$ be a sufficiently general collection of n hyperplanes in $\mathbb{R}^d$:*

1. *Is $\mathcal{B}_{\mathcal{A},c,m}$ a generating set for $H_{m(c-1)}\mathcal{M}_{d+c}(\mathcal{A})$?*
2. *Is $H_\ell\mathcal{M}_{d+c}(\mathcal{A}) = 0$ for ℓ not divisible by $c-1$?*

The work of Baryshnikov answers these questions positively in the case $d = 1$. We will show the answer the first question is yes in the case $m \leq 2$ and the answer to the second questions is yes whenever $\ell < 2(c-1)$.

The proof by Baryshnikov uses the fact that the subspace of a no k-equal spaces where there exists a coordinate such that all other coordinates are either strictly greater than or strictly less than this coordinate splits into product of two smaller no k-equal spaces. Unfortunately, that is not that case here as hyperplanes intersecting is no longer a local condition. The fact that we can't split our no $(d+c)$-intersecting hyperplanes spaces into products of two smaller ones forces us to consider the homology of an auxiliary space.

Let $X \subset \underline{n}$ be of size $d+c$. Let $\mathcal{M}_{d+c}(\mathcal{A}, X)$ be the space of no $(d+c)$-intersecting translates of $\mathcal{A}$ with the exception that the hyperplanes indexed by elements of X must intersect. To each element $f_* \in \mathcal{B}_{\mathcal{A},c,2}$ such that $X \in Z_f$, we can associate an element of $H_{c-1}\mathcal{M}_{d+c}(\mathcal{A}, X)$ similarly to the way in the previous section. We will denote said class by f_*^X. We will denote the set of all such classes by $\mathcal{B}^X_{\mathcal{A},c,2}$.

Lemma 4.5. *Let $d > 1$. Let $\mathcal{A}$ be a sufficiently general collection of n hyperplanes in $\mathbb{R}^d$. Let $X \subset \underline{n}$ be of size $d+c$. Then*

1. *$\mathcal{B}^X_{\mathcal{A},c,2}$ is a generating set for $H_{c-1}\mathcal{M}_{d+c}(\mathcal{A}, X)$.*
2. *$H_\ell\mathcal{M}_{d+c}(\mathcal{A}, X) = 0$ for $\ell < c-1$.*

Proof. The proof will be by induction of n. First, if $n = d+c$, the claim is trivial as $\mathcal{M}_{d+c}(\mathcal{A}, X)$ is a contractible space. Thus, suppose $n > d+c$ and that the claim holds for all $n' < n$.

Let γ be a closed $(c-1)$-chain. Let $m \in \underline{n} - X$. We will assume that at all points in the image of γ, the point of intersection of the hyperplanes indexed by elements from X are on the negative side of m. A similar argument can be made for if it is on the positive side of m.

If there exists points in the image of γ of both types, one can instead consider the chains of each type separately (which will each also be closed).

We may perturb γ so that it is transversal to all subspaces of the following form:

> There exist $Y, Z \not\subset X, |Y| = |Z| = d + c - 1, m \notin Y, Z$ such that the hyperplanes indexed by elements from Y all intersect (in a single point), the hyperplanes indexed by elements from Z all intersect (in a single point), and the vector between these two points is contained in hyperplane m.

Notice that these subspaces have codimension at least c. Thus γ intersecting these subspaces transversally implies that γ does not intersect subspaces of these forms.

Consider the homotopy of γ only affecting the m^{th} coordinate of γ, $\gamma_t = \gamma + t\vec{e}_m$. For some large value t, say M, the hyperplane m does not interact with any points of intersection of d other hyperplanes. Let Γ be the c-chain given by this homotopy. This homotopy may intersect subspaces not in $\mathcal{M}_{d+c}(\mathcal{A}, X)$. These spaces are of the following form:

> There exists $Y \neq X, |Y| = d + c, m \in Y$ such that the hyperplanes indexed by elements from Y all intersect in a single point.

Consider Γ with tubular neighborhoods of these subspaces removed. The resultant c-chain allows us to write γ as a sum of terms arising from the boundaries of these tubular neighborhoods and the class when $t = M$. The former are of the form f_*^X where $f_*^X \in \mathcal{B}_{\mathcal{A},c,2}^X$. Inductively, the latter will be of the form f_*^X for some $f_*^X \in \mathcal{B}_{\mathcal{A}-m,c,2}^X$ with the line m added far away. This, too, is of the form f_*^X for some $f_*^X \in \mathcal{B}_{\mathcal{A},c,2}^X$.

For part (2), Let γ be a closed s-chain where $s < c - 1$. We can perturb γ so that it is transversal to all subspaces where $d + c - 1$ hyperplanes not in X intersect. These subspaces have codimension $c - 1$. Transversality will again imply that γ does not intersect any of these subspaces. Thus, this time when considering the analogous homotopy Γ, it will not intersect any forbidden subspaces. Thus, γ will be written of the form $N \in H_s\mathcal{M}_{c+d}(\mathcal{A} - m, X)$ with the line m added far away. Inductively, $H_s\mathcal{M}_{c+d}(\mathcal{A} - m, X)$ is trivial. Thus, $H_s\mathcal{M}_{c+d}(\mathcal{A}, X)$ will be too. □

With this lemma, we are now able to prove the aforementioned results.

Proposition 4.6. *Let $d > 1$. Let $\mathcal{A}$ be a sufficiently general collection of n hyperplanes in $\mathbb{R}^d$:*

1. *$\mathcal{B}_{\mathcal{A},c,m}$ is a generating set for $H_{m(c-1)}\mathcal{M}_{d+c}(\mathcal{A})$ for $m \leq 2$.*
2. *For all other $\ell < 2(c-2)$, $H_\ell \mathcal{M}_{d+c}(\mathcal{A}) = 0$.*

Proof. Same argument as lemma but with minor variations. □

The elements of $\mathcal{B}_{\mathcal{A},c,m}$ are not linearly independent; they satisfy the following relations:

Theorem 4.7. *Let $f \in \mathcal{F}_{\mathcal{A},m}$. Suppose $X \in Z_f$, $|X| = d + c + 1$, and $|Y| = d + c$ for all $Y \in Z_f, Y \neq X$, then*

$$\sum_{j \in X} f_*^{X,+j} + f_*^{X,-j} = 0$$

Proof. First consider the case where $m = 1$ and $n = d + c + 1$. Consider the sphere given by the equations:

$$\sum_{i=1}^{d+c+1} b_i \boldsymbol{\alpha}_i = \mathbf{0} \qquad \sum_{i=1}^{d+c+1} |b_i|^2 = 1$$

Remove from this sphere small tubular neighborhoods of points not contained in $M_{d+c}(\mathcal{A})$. The resultant chain gives the above relation.

For the general case, consider the analogous chain near any generic point of V_f. □

Using these, we can find a smaller set that generate the same classes as $\mathcal{B}_{\mathcal{A},c,m}$.

Definition. *Let $\mathcal{H}_{\mathcal{A},c,m} = \{f_* \in \mathcal{B}_{\mathcal{A},c,m} \mid \max(P_{f,X} \cup X) \in X \ \forall \ X \in Z_f\}$.*

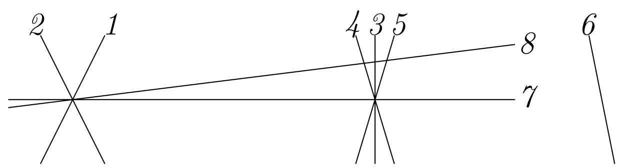
Proposition 4.8. *Let $\mathcal{A}$ be a sufficiently general hyperplane arrangement, $c > 1$, and $m \geq 0$. The group generated by $\mathcal{H}_{\mathcal{A},c,m}$ is equal to the group generated by $\mathcal{B}_{\mathcal{A},c,m}$.*

Proof. Let A be the group generated by $\mathcal{H}_{\mathcal{A},c,m}$, B the group generated by $\mathcal{B}_{\mathcal{A},c,m}$. The proof will use p-vectors of classes. We will order p-vectors lexicographically when read right to left.

Consider $f_* \in \mathcal{B}_{\mathcal{A},c,m}$. Suppose there exists $X \in Z_f$ such that $\max(P_{f,X} \cup X) \in P_{f,X}$. Let $m = \max P_{f,X}$. Using the relation from Theorem 4.7 with $(f_{X \cup \{m\},+m})_*$, we may write f as a sum of elements with lesser p-vector. Since there is a limit to how small a classes p-vector can get, f must be expressible as a sum of elements from $\mathcal{H}_{\mathcal{A},c,m}$. Thus, $f_* \in A$, so $B \subset A$. It is obvious that $A \subset B$. Thus, $A = B$ as desired. $\square$

In the following section, we will show that for $m = 1$, $\mathcal{H}_{\mathcal{A},c,m}$ is a basis. Furthermore, if we use a particular choice of orientations and a particular enumeration of the hyperplanes, then $\mathcal{H}_{\mathcal{A},c,m}$ is also a basis when $d = m = 2$. However, for general orderings and choices of orientations, $\mathcal{H}_{\mathcal{A},c,m}$ is not a basis even for $m = d = 2$.

Example. *Suppose* $\mathbf{b} \in \mathbb{R}^8$ *corresponds to the following arrangement of lines.*



Note that $\mathbf{b} \in V_f$ *for some* $f_* \in \mathcal{H}_{\mathcal{A},2,2}$. *Suppose line* 6 *has normal vector pointing left, line* 7 *has normal vector pointing up, and line* 8 *has normal vector pointing down. Using the relation from Theorem 4.7 twice allows us to write* f_* *as a sum of elements in* $\mathcal{H}_{\mathcal{A},2,2}$ *each with lesser p-vector.*

In general, one can show that the group generated by $\mathcal{H}_{\mathcal{A},c,m}$ is also equivalent to the group generated by the following:

$$\mathcal{H}'_{\mathcal{A},c,m} = \{f_* \in \mathcal{B}_{\mathcal{A},c,m} \mid \forall\, X \in Z_f \text{ either } P_{f,X} = \emptyset \text{ or } \exists\, j \in X \text{ with } \max P_{f^{X,+j},X-\{j\}} = j\}$$

In the above example, suppose $X = \{3, 4, 5, 7\}$. Notice that f_* in the above example is not in $\mathcal{H}'_{\mathcal{A},2,2,}$ as $6 \in P_{f,X}$, $8 \in P_{f^{X,+7},X-\{7\}}$, and $6 \in P_{f^{X,+j},X-\{j\}}$ for $j \in \{3, 4, 5\}$.

For $m = 1$, $\mathcal{H}_{\mathcal{A},c,m}$ is the same as $\mathcal{H}'_{\mathcal{A},c,m}$. For $m > 1$, this is not necessarily the case; however, it is for a particular choice of orientations and enumeration of hyperplanes when $d = m = 2$.

4.3 Cohomology

As with homology, the cohomology of $\mathcal{M}_{d+c}(\mathcal{A})$ will be described with the help of the oriented matroid defined earlier $\mathcal{O}_{\mathcal{A}}$. For each element $f \in \mathcal{P}_{\mathcal{A},c-1,m}$, its associated set of realizations, V_f, is an $m(c-1)$-dimensional change whose boundary lies in the complement to $\mathcal{M}_{d+c}(\mathcal{A})$. Thus, it represents an element of $H^{m(c-1)}(\mathcal{M}_{d+c}(\mathcal{A}), \mathbb{Z}_2)$ which we will denote by f^*. We will denote the set of all such f^* by $\mathcal{C}_{\mathcal{A},c,m}$.

These satisfy the following relations:

Theorem 4.9. *Let $f \in \mathcal{F}_{\mathcal{A},m}, X \in Z_f$ with $|X| = d+c-1$. Suppose $|Y| = d+c$ for all $Y \in Z_f, Y \neq X$, then*

$$\sum_{j \in P_{X,+j}} f_{X,+j} + \sum_{j \in N_{X,k}} f_{X,-j} = 0$$

Proof.

$$\partial V_f = \sum_{Y \in Z_f} (\sum_{j \in P_{Y,+j}} V_{f_{Y,+j}} + \sum_{j \in N_{f,Y}} V_{f_{Y,-j}})$$

All of the realizations for summands with $Y \neq X$ are not in $\mathcal{M}_{d+c}(\mathcal{A})$. Thus, they correspond to the zero cohomology class. Hence, we get the desired relation. □

Using these, we want to write any cohomology class in a more organized manner. In the $m = 1$ or $d = 1$ setting, we could use the following set as a generating set:

$$\mathcal{CH}_{\mathcal{A},c,m} = \{f^* \in \mathcal{C}_{\mathcal{A},c,m} \mid \max(P_{f,X} \cup X) \in P_{f,X} \; \forall X \in Z_f\}$$

However, for $m, d > 1$, this set is not enough. This is due to the fact that for $m, d > 1$, removing a hyperplane from some X may affect which other hyperplanes can then reach the resultant intersection point, which is not the case when $m = 1$ or $d = 1$. Thus, we define the following:

$$\mathcal{CH}'_{\mathcal{A},c,m} = \{f^* \in \mathcal{C}_{\mathcal{A},c,m} \mid \max P_{f^{X,+m}, X-\{m\}} > m \; \forall \, m \in X$$
$$\text{such that } \max(P_{f,X} \cup \{m\}) = m \; \forall X \in Z_f\}$$

Proposition 4.10. *The group of cohomology classes generated by $\mathcal{CH}'_{\mathcal{A},c,m}$ is equal to the group of cohomology classes generated by $\mathcal{C}_{\mathcal{A},c,m}$*

Proof. Let A be the group generated by $\mathcal{CH}'_{\mathcal{A},c,m}$, B the group generated by $\mathcal{C}_{\mathcal{A},c,m}$.

Consider $f^* \in \mathcal{C}_{\mathcal{A},c,m}$. Suppose there exists $X \in Z_f$ with $\ell \in X$, $\max(P_{f,X} \cup \{\ell\}) = \ell$ such that $\max P_{f^{X,+\ell},X-\{\ell\}} = \ell$. Using the relation from Theorem 4.9 for $f^{X,+\ell}$ allows us to write f as a sum of elements all with greater p-vector. Since there is a limit to how large a p-vector can be, continuing this process will eventually terminate when f is expressed as a sum of elements from $\mathcal{CH}'_{\mathcal{A},c,m}$. Thus, $f^* \in A$, so $B \subset A$. It is obvious that $A \subset B$. Thus, $A = B$ as desired. □

4.3.1 Multiplication

Definition. *Let $f \in \mathcal{P}_{\mathcal{A},c,m_1}, g \in \mathcal{P}_{\mathcal{A},c,m_2}$. Suppose $Z_f \cap Z_g = \emptyset$ and $Z_f \cup Z_g \in P(n,d)$. Define $f \cup g$ to be the function such that:*

- $(f \cup g)(S,r) = 0$ *for all* $(S,r) \in \binom{X}{d} \times X$ *for all* $X \in Z_f \cup Z_g$
- $(f \cup g)(S,r) = f(S,r)$ *for all* $S \in \binom{X}{d}, r \notin X$ *for all* $X \in Z_f \cup Z_g$
- $(f \cup g)(S,r) = *$ *for all* $S \not\subset X$ *for any* $X \in Z_f \cup Z_g$

Notice that $f \cup g$ is in $\mathcal{P}_{\mathcal{A},c,m_1+m_2}$ as long as $V_{f\cup g} \neq \emptyset$. Using this, we can describe multiplication:

Theorem 4.11. *Let $f^* \in \mathcal{C}_{\mathcal{A},c,m_1}, g^* \in \mathcal{C}_{\mathcal{A},c,m_2}$. Then $f^* \cup g^*$ is 0 if any of the following hold:*

- $Z_f \cap Z_g \neq \emptyset$
- $Z_f \cup Z_g \notin P(n,d)$
- $V_{f\cup g} = \emptyset$

Otherwise $f^* \cup g^* = (f \cup g)^*$

Proof. In the first case, we may perturb the chains corresponding to f and g so that they do not intersect. In the case the first one does not hold and either the second or third does, then the chains corresponding to f and g do not intersect in $\mathcal{M}_{d+c}(\mathcal{A})$. If none of those three hold, then the chains corresponding to f and g intersect transversally. Their intersection is the chain corresponding to $(f \cup g)^*$ □

4.4 Some Betti Numbers

Corollary 4.12. *For any sufficiently general arrangement $\mathcal{A}$,*

$$\beta_{c-1}\mathcal{M}_{c+d}(\mathcal{A}) = \sum_k \binom{n}{k}\binom{k-1}{d+c-1}.$$

Proof. Consider the map $\varphi : \mathcal{CH}_{\mathcal{A},c,1} \to \mathcal{H}_{\mathcal{A},c,1}$ defined by

$$\varphi(f^*) = (f_{X,+\max P_{f,X}})_*$$

where X is the lone element of Z_f. This map is a bijection between $\mathcal{CH}_{\mathcal{A},c,1}$ and $\mathcal{H}_{\mathcal{A},c,1}$. Order $\mathcal{CH}_{\mathcal{A},c,1}$ by p-vectors. Use this ordering and φ to order $\mathcal{H}_{\mathcal{A},c,1}$. With this ordering, the intersection pairing matrix between $\mathcal{CH}_{\mathcal{A},c,1}$ and $\mathcal{H}_{\mathcal{A},c,1}$ is upper triangular. Thus, $\mathcal{H}_{\mathcal{A},c,1}$ is a linearly independent generating set for $H_{c-1}\mathcal{M}_{c+d}(\mathcal{A})$.

To count the elements of $\mathcal{H}_{\mathcal{A},c,1}$, we partition them by $|X \cup P_{f,X}|$. Suppose $|X \cup P_{f,X}| = k$. There are $\binom{n}{k}$ ways to choose these k elements. The only restriction is the maximum of this set must be in X. There are $\binom{k-1}{c+d-1}$ ways to pick the non maximum elements of X. Thus, there are $\binom{n}{k}\binom{k-1}{d+c-1}$ elements with $|X \cup P_{f,X}| = k$. Thus,

$$\beta_{c-1}\mathcal{M}_{c+d}(\mathcal{A}) = |\mathcal{H}_{\mathcal{A},c,1}| = \sum_k \binom{n}{k}\binom{k-1}{d+c-1}$$ □

This finishes the cases when $m = 1$ or $d = 1$. Thus, we will now move our focus to the next simplest situation $m = d = 2$. For this, we will work with any sufficiently general arrangement of lines $\mathcal{A}$; however, we will work with particular choices of normal vectors and a particular enumeration of the lines. Before proceeding to finding $\beta_{2(c-1)}\mathcal{M}_{c+2}(\mathcal{A})$, we will give a full description of elements in $\mathcal{F}_{\mathcal{A},2}$. For $f \in \mathcal{F}_{\mathcal{A},2}$, we will denote the two elements of Z_f by $X_{f,1}$ and $X_{f,2}$. When it's clear which f we are referring to, we will sometimes omit the f subscript.

Definition. *Let $\mathcal{A}$ be a sufficiently general collection of lines in $\mathbb{R}^2$ such that none are horizontal. Let $\{\mathbf{v}_i\}$ be a collection of direction vectors each with positive y-coordinate. For each line, let θ_i be the angle between $\langle -1, 0\rangle$ and $\mathbf{v}_i$. We will say $\mathcal{A}$ are* suitably ordered *if $i < j$ implies $\theta_i < \theta_j$*

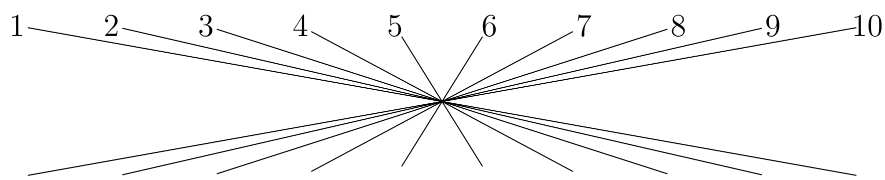


Figure 4.1: An example of suitably ordered collection of ten lines in $\mathbb{R}^2$

Let $f \in \mathcal{F}_{\mathcal{A},2}$, $S_1 \in \binom{X_1}{d}$, and $S_2 \in \binom{X_2}{d}$. The following sets and definitions will be useful:

- $I_f = \{\ell \mid f_{S_1,\ell} = f_{S_2,\ell} = -\}$
- $J_f = \{\ell \notin X_1 \cup X_2 \mid f_{S_1,\ell} \neq f_{S_2,\ell}\}$
- $J'_f = J_f \cup X_1 \cup X_2$
- $K_f = \{\ell \mid f_{S_1,\ell} = f_{S_2,\ell} = +\}$

Again, when it's clear which f we are working with, we will sometimes omit the subscripts.

Definition. *Let $\mathcal{A}$ be a sufficiently general collection of lines in $\mathbb{R}^2$ with normal vectors $\{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\}$. Suppose $\mathcal{A}$ is suitably ordered. For $f \in \mathcal{F}_{\mathcal{A},2}$, $\mathbf{b} \in V_f$, let*

$$\mathbf{x}_{\mathbf{b},i} = \bigcap_{i \in X_i} \{\boldsymbol{\alpha}_i \cdot \mathbf{x} = b_i\}$$

Without loss of generality, assume the y-coordinate of $\mathbf{x}_{\mathbf{b},2}$ is not less than that of $\mathbf{x}_{\mathbf{b},1}$. If they are equal, assume the x-coordinate of $\mathbf{x}_{\mathbf{b},1}$ is less than that of $\mathbf{x}_{\mathbf{b},1}$. Furthermore, let

$\mathbf{v_b} = \mathbf{x}_{\mathbf{b},2} - \mathbf{x}_{\mathbf{b},1}$ *and* $\theta_{\mathbf{b}}$ *be the angle between* $\langle -1, 0\rangle$ *and* $\mathbf{v_b}$*. The above assumptions imply* $0 \leq \theta_{\mathbf{b}} < \pi$*. Let* $\theta_0 = 0$ *and* $\theta_\infty = \pi$*. Let*

$$\theta_f^< = \max_{\mathbf{b}\in V_f} \max\{\theta_j \mid \theta_j \leq \theta_{\mathbf{b}}, j \in J' \cup \{0, \infty\}\}$$

$$\theta_f^> = \min_{\mathbf{b}\in V_f} \min\{\theta_j \mid \theta_j > \theta_{\mathbf{b}}, j \in J' \cup \{0, \infty\}\}$$

If $\theta_f^< < \theta_f^>$, then $\theta_f^< \leq \theta_{\mathbf{b}} < \theta_f^>$ for all $\mathbf{b} \in V_f$. The one case where realization does matter is when $\theta_f^< > \theta_f^>$. This only occurs if $\theta_f^< = \max J'$ and $\theta_f^> = \min J'$. We partition $\mathcal{F}_{\mathcal{A}}$ into the three sets defined as follows:

Definition. *Let* $\mathcal{A}$ *be a sufficiently general collection of lines in* $\mathbb{R}^2$*. Suppose* $\mathcal{A}$ *is suitably ordered. Define* $\mathcal{F}^i_{\mathcal{A},2}$ *as follows:*

$$\begin{aligned}
\mathcal{F}^1_{\mathcal{A},2} &= \{f \in \mathcal{F}_{\mathcal{A},2} : |X_{f,1} \cap X_{f,2}| = 1\} \\
\mathcal{F}^2_{\mathcal{A},2} &= \{f \in \mathcal{F}_{\mathcal{A},2} : X_{f,1} \cap X_{f,2} = \emptyset \text{ and } \theta_f^< \neq \theta_{\max J'}\} \\
\mathcal{F}^3_{\mathcal{A},2} &= \{f \in \mathcal{F}_{\mathcal{A},2} : X_{f,1} \cap X_{f,2} = \emptyset \text{ and } \theta_f^< = \theta_{\max J'}\}
\end{aligned}$$

In the following description, we will assume that $\mathcal{A}$ is suitably ordered at that each normal vector chosen has positive x-coordinate.

(1) $f \in \mathcal{F}^1_{\mathcal{A},2}$

For this situation, suppose the point of intersection of the lines indexed by X_1 has lesser y-coordinate than the point of intersection of the lines indexed by X_2.

Let $\{j\} = X_1 \cap X_2$. Notice that the information given so far is enough to determine $f(S_i, \ell)$ for all $(S_i, \ell) \in \binom{X_i}{d} \times X_j$. Each $\ell \notin X_1 \cup X_2$ is in exactly one of I, J, or K. Which hyperplanes are in X_1, X_2, I, J, and K is enough to determine exactly P_{f,X_i} and N_{f,X_i}:

$$P_{f,X_1} = \{J_{>j} \cup K_{<j}\} - X_1 \qquad N_{f,X_1} = \{I_{>j} \cup J_{<j}\}$$

$$P_{f,X_2} = \{J_{<j} \cup K_{>j}\} - X_2 \qquad N_{f,X_2} = \{I_{<j} \cup J_{>j}\}$$

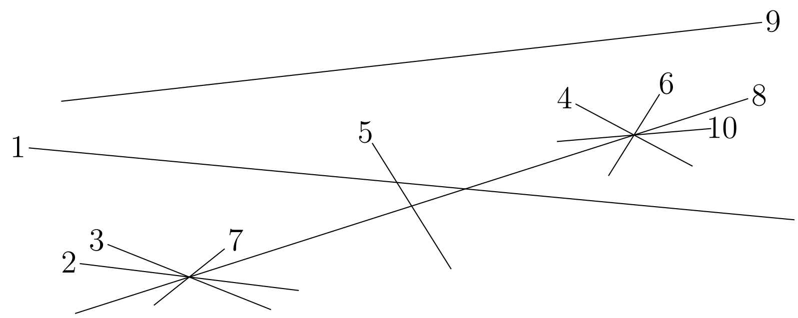


Figure 4.2: An example of a realization of an f of type (1) where $X_1 = \{2,3,7,8\}$, $X_2 = \{4,6,8,10\}, I = \emptyset, J = \{1,5\}, K = \{9\}$.

(2) $f \in \mathcal{F}^2_{\mathcal{A},2}$

For this situation, suppose the point of intersection of the lines indexed by X_1 has lesser y-coordinate than the point of intersection of the lines indexed by X_2. Let $\{a_1 < \cdots < a_m\}$ be an enumeration of J'. In this case, there exists $j < m$ such that $\theta_f^< = \theta_{a_j}$ and $\theta_f^> = \theta_{a_{j+1}}$. From this, we can determine P_{f,X_i} and N_{f,X_i}:

$$P_{f,X_1} = \{J_{>a_j} \cup K_{<a_{j+1}} \cup \{a_{j+1}\}\} - X_1$$
$$N_{f,X_1} = \{I_{>a_j} \cup J_{\leq a_j} \cup \{a_j\}\} - X_1$$
$$P_{f,X_2} = \{J_{\leq a_j} \cup K_{>a_j} \cup \{a_j\}\} - X_2$$
$$N_{f,X_2} = \{I_{<a_{j+1}} \cup J_{>a_j} \cup \{a_{j+1}\}\} - X_2$$

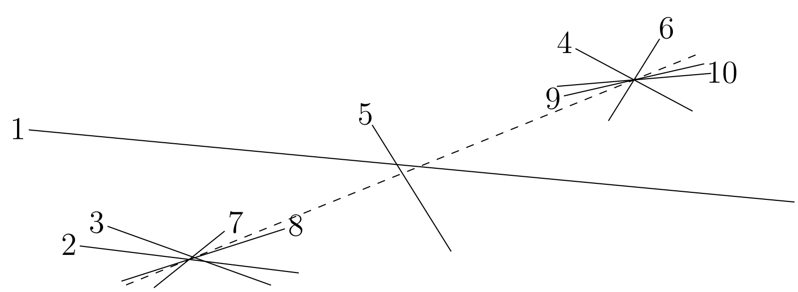


Figure 4.3: An example of a realization of an f of type (2) where $a_j = 7$, $X_1 = \{2,3,7,8\}$, $X_2 = \{4,6,9,10\}, I = \emptyset, J = \{1,5\}, K = \emptyset$.

(3) $f \in \mathcal{F}^3_{\mathcal{A},2}$

For this situation, suppose the point of intersection of the lines indexed by X_1 has lesser x-coordinate than the point of intersection of the lines indexed by X_2. Let $\{a_1 < \cdots < a_m\}$ be an enumeration of J'. In this situation $\theta^<_f = \theta_{a_m}$ and $\theta^>_f = \theta_{a_1}$. From this information, we can determine exactly P_{f,X_i} and N_{f,X_i}:

$$P_{f,X_1} = K \qquad N_{f,X_1} = \{I_{<a_1} \cup I_{>a_m} \cup J \cup \{a_1, a_m\}\} - X_1$$

$$N_{f,X_2} = I \qquad P_{f,X_2} = \{K_{<a_1} \cup K_{>a_m} \cup J \cup \{a_1, a_m\}\} - X_2$$

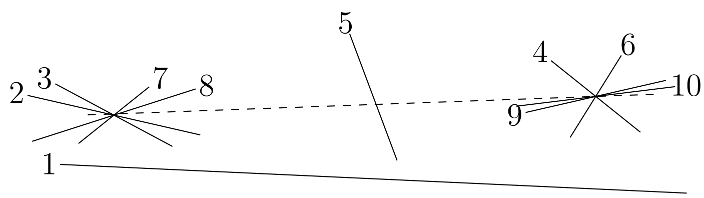


Figure 4.4: An example of a realization of an f of type (3) where $X_1 = \{2, 3, 7, 8\}$, $X_2 = \{4, 6, 9, 10\}, I = \emptyset, J = \{5\}, K = \{1\}$.

Definition. *Let $\mathcal{A}$ be a sufficiently general arrangement of lines in $\mathbb{R}^2$. Suppose $\mathcal{A}$ is ordered suitably. Define $\mathcal{P}_1, \mathcal{P}_2$, and $\mathcal{P}_3$ as follows:*

$$\mathcal{P}_i = \{f \in \mathcal{F}^i_{\mathcal{A},2} \mid f_* \in \mathcal{H}_{\mathcal{A},c,2}\}$$

Before showing that $\mathcal{H}_{\mathcal{A},c,2}$ and $\mathcal{CH}_{\mathcal{A},c,2}$ are bases in the situation above, we will prove a couple lemmas.

Lemma 4.13. *Let $\mathcal{A}$ be a sufficiently general arrangement in $\mathbb{R}^d$. Let $f^* \in \mathcal{CH}_{\mathcal{A},c,2}$. Let $a_i = \max P_{f,X_i}$. Then $a_2 \in P_{f_{X_1,+a_1},X_2}$. Moreover, $((f_{X_1,+a_1})_{X_2,+a_2})_* \in \mathcal{H}_{\mathcal{A},c,2}$.*

Proof. First notice that $a_2 \in P_{f_{X_1,+a_1},X_2}$ if and only if $a_1 \in P_{f_{X_2,+a_2},X_1}$. There are two cases: $a_1 \neq a_2$ or $a_1 = a_2$.

Case 1: $a_1 \neq a_2$

Without loss of generality, we will assume that $a_2 > a_1$. Because $a_2 \in P_{f,X_2}$, we have $f(S_2, a_2) = +$ for all $S_2 \in \binom{X_2}{d}$. Thus, $f_{X_1,+a_1}(S_2, a_2) = +$ for all such S_2. Let $\mathbf{b}$ be any realization of $f_{X_1,+a_1}$. Consider the map $\vec{r}(t) = \vec{v} + t\vec{e}_{a_2}$. Because $f_{X_1,+a_1}(S_2, a_2) = +$, there exists t, call it M, such that $\mathbf{x}_{\mathbf{b},2}$ lies on the hyperplane a_2.

Suppose there exists $t \leq M$ such that $\mathbf{x}_{\mathbf{b},1}$ lies on the hyperplane a_2. In this case, we have $a_2 \in P_{f_{X_1,+a_1},X_1 \cup \{a_1\}}$. This would imply that $a_2 \in P_{f,X_1}$. However, $\max P_{f,X_1} = a_1 < a_2$. Thus, no such t can exist. Because there does not exist such a t, $\vec{r}(M)$ is a realization of $(f_{X_1,+a_1})_{X_2,+a_2}$. Thus, $a_2 \in P_{f_{X_1,+a_1}}$.

Case 2: $a_1 = a_2$

Let $\mathbf{b_i}$ be a realization of $f_{X_i,+a_i}$. Let $\mathbf{y}_i$ be any point on hyperplane $a_1 = a_2$ in $\mathbf{b_i}$. Then we have the following:

$$(\mathbf{x}_{\mathbf{b}_1,1} - \mathbf{y}_1) \cdot \boldsymbol{\alpha}_{a_1} < (\mathbf{x}_{\mathbf{b}_1,2} - \mathbf{y}_1) \cdot \boldsymbol{\alpha}_{a_1} \qquad (\mathbf{x}_{\mathbf{b}_2,1} - \mathbf{y}_2) \cdot \boldsymbol{\alpha}_{a_1} > (\mathbf{x}_{\mathbf{b}_2,2} - \mathbf{y}_2) \cdot \boldsymbol{\alpha}_{a_1}$$

Thus, on the line connecting $\mathbf{b}_1$ and $\mathbf{b}_2$, there exists a point, $\mathbf{b_3}$, such that

$$(\mathbf{x}_{\mathbf{b}_3,1} - \mathbf{y}_3) \cdot \boldsymbol{\alpha}_{a_1} = (\mathbf{x}_{\mathbf{b}_3,2} - \mathbf{y}_3) \cdot \boldsymbol{\alpha}_{a_1}$$

where $\boldsymbol{\alpha}_{3,i}$ and $\mathbf{y}_3$ are defined analogously as above. Then $\mathbf{b_3} + ((\mathbf{x}_{\mathbf{b}_3,1} - \mathbf{y}_3) \cdot \boldsymbol{\alpha}_{a_1})\vec{e}_{a_1}$ gives a realization for $((f_{X_1,+a_1})_{X_2,+a_2})$. Thus, $a_2 \in P_{f_{X_1,+a_1}}$.

Let $g = (f_{X_1,+a_1})_{X_2,+a_2}$. Let $X_i' = X_i \cup \{a_i\}$. The fact that $g_* \in \mathcal{H}_{\mathcal{A},c,2}$ follows from observing $\max P_{f,X_i} \geq \max P_{g,X_i'}$. Thus, $a_i \geq \max P_{g,X_i'}$. Since $a_i \in X_i'$, it can't be equal to $\max P_{g,X_i'}$. Thus, $a_i > \max P_{g,X_i'}$, as desired. □

Lemma 4.14. *Let $\mathcal{A}$ be a sufficiently general arrangement of lines in $\mathbb{R}^2$. Suppose $\mathcal{A}$ is ordered suitably and all normal vectors have positive x-coordinate. Let $f_* \in \mathcal{H}_{\mathcal{A},c,2}$. Let $a_i = \max X_i$, then $g^* = ((f^{X_1,+a_1})^{X_2,+a_2})^* \in \mathcal{CH}_{\mathcal{A},c-1,2}$. Moreover,* $\max P_{g,X_i - \{a_i\}} = a_i$.

Proof. The fact that $g^* \in \mathcal{CH}_{\mathcal{A},c-1,2}$ follows from $\max P_{g,X_i - \{a_i\}} \geq a_i > \max X_{g,X_i - \{a_i\}}$. The fact that $a_i = \max P_{g,X_i - \{a_i\}}$ follows from a careful case analysis using the above descriptions of $\mathcal{F}_{\mathcal{A},2}$. □

With these lemmas, we will now show that $\mathcal{H}_{\mathcal{A},c,2}$ and $\mathcal{CH}_{\mathcal{A},c,2}$ are bases in the situation above.

Theorem 4.15. *Let $\mathcal{A}$ be a sufficiently general arrangement of lines in $\mathbb{R}^2$ that is ordered suitably. Suppose that all chosen normal vectors have positive x-coordinate. Let $c > 1$. Then $\mathcal{H}_{\mathcal{A},c,2}$ and $\mathcal{CH}_{\mathcal{A},c,2}$ are bases for $H_{2(c-1)}\mathcal{M}_{c+2}(\mathcal{A})$ and $H^{2(c-1)}\mathcal{M}_{c+2}(\mathcal{A})$, respectively.*

Proof. Consider the map $\psi : \mathcal{H}_{\mathcal{A},c,2} \to \mathcal{CH}_{\mathcal{A},c,2}$ defined by $\psi(f_*) = ((f^{X_1,+a_1})^{X_2,+a_2})^*$ where $a_i = \max X_{f,X_i}$. This map has an inverse, φ defined by $\varphi(f^*) = ((f_{X_1,+a_1})_{X_2,+a_2})_*$ where $a_i = \max P_{f,X_i}$. The previous two lemmas show that φ and ψ both map to the desired range. Furthermore, they are inverses. Thus, they give a bijection between $\mathcal{H}_{\mathcal{A},c,2}$ and $\mathcal{CH}_{\mathcal{A},c,2}$.

Order the elements of $\mathcal{CH}_{\mathcal{A},c,2}$ such that their p-vectors are decreasing. Order the elements of $\mathcal{H}_{\mathcal{A},c,2}$ so that $f_* < g_*$ whenever $\psi(f_*) < \psi(g_*)$. Considering the intersection pairing matrix with this ordering, M. Each class f_* intersects $\psi(f_*)$ in a single point. All other elements of $\mathcal{CH}_{\mathcal{A},c,2}$ that it intersects have lesser p-vector. Thus, M is triangular with ones along the diagonal. Thus, $\mathcal{H}_{\mathcal{A},c,2}$ and $\mathcal{CH}_{\mathcal{A},c,2}$ are bases for homology and cohomology, respectively. □

With this basis in hand, we now count the number of elements in $\mathcal{H}_{\mathcal{A},c,2}$. We will show

$$\beta_{2(c-1)}\mathcal{M}_{c+2}(\mathcal{A}) = n\binom{n-1}{c+1}\binom{n-c-2}{c+1}3^{n-2c-3}$$

We will do this by producing a bijection between elements of $\mathcal{H}_{\mathcal{A},c,2}$ and functions $h : \underline{n} \to \underline{6}$ such that

- $|h^{-1}(1)| = 1$
- $|h^{-1}(3)| = c + 1$
- $|h^{-1}(5)| = c + 1$

We will denote the set of such functions by C. The number above can be seen as the number of such functions by first picking which numbers map to one, then which numbers map to three, then which numbers map to 5. Each remaining number can be mapped to one of three possibilities: two, four, or six.

We can separate these functions into three classes. The first class, denoted by C_1, consists of functions that satisfy:

- $h(\max h^{-1}(\{1,3,4\})) \neq 4$
- $h(\max h^{-1}(\{1,5,6\})) \neq 6$

The second class, denoted by C_2, consists of functions that satisfy:

- $h(\max h^{-1}(\{1,3,4\})) = 4$ or $h(\max h^{-1}(\{1,5,6\})) = 6$
- $h(\max h^{-1}(\{1,3,4\})) \neq 1$

The third class, denoted by C_3, consists of functions that satisfy:

- $h(\max h^{-1}(\{1,5,6\})) = 6$
- $h(\max h^{-1}(\{1,3,4\})) = 1$

Notice that each h belongs to exactly one class. We will create the aforementioned bijection by finding bijections between C_i and $\mathcal{P}_i$ for each i. In the following, we will use the numbering conventions for X_1 and X_2 introduced in the descriptions of $\mathcal{P}_i$. Furthermore, we will write $P_{f,i}$ for $P_{f,X_{f,i}}$.

Lemma 4.16. *There exists a bijection between C_1 and $\mathcal{P}_1$*

Proof. Let $\psi : C_1 \to \mathcal{P}_1$ be defined by $\psi(h) = f$ where f satisfies:

$$X_{f,1} = h^{-1}(\{1,3\}) \qquad X_{f,2} = h^{-1}(\{1,5\}) \qquad I_f = h^{-1}(2)$$
$$P_{f,1} = h^{-1}(4) \qquad P_{f,2} = h^{-1}(6)$$

Such an f exists and is uniquely determined by this information; however, we need to check that it is in $\mathcal{P}_1$. The conditions to be in C_1 ensure that $\max(X_{f,i} \cup P_{f,i}) \in X_{f,i}$.

Consider also the function $\varphi : \mathcal{P}_1 \to C_1$ defined by:

$$(\varphi(f))(i) = \begin{cases} 1 & i \in X_{f,1} \cap X_{f,2} \\ 2 & i \in I \\ 3 & i \in X_{f,1} - X_{f,2} \\ 4 & i \in P_{f,1} \\ 5 & i \in X_{f,2} - X_{f,1} \\ 6 & i \in P_{f,2} \end{cases}$$

Notice first that this does describe the behavior of each $\ell \in \underline{n}$ as if $\ell \notin X_{f,1} \cup X_{f,2} \cup I_f$, then $\ell \in P_{f,i}$ for some i. The conditions to be in $\mathcal{P}_1$ ensure that $\varphi(f)$ is in C_1.

Let $h \in C_1$. Let $g = \varphi(\psi(h))$. Then

$$\begin{aligned} g^{-1}(1) &= X_{\psi(h),1} \cap X_{\psi(h),2} &&= h^{-1}(1) \\ g^{-1}(2) &= I_{\psi(h)} &&= h^{-1}(2) \\ g^{-1}(3) &= X_{\psi(h),1} - X_{\psi(h),2} &&= h^{-1}(3) \\ g^{-1}(4) &= P_{\psi(h),1} &&= h^{-1}(4) \\ g^{-1}(5) &= X_{\psi(h),2} - X_{\psi(h),1} &&= h^{-1}(5) \\ g^{-1}(6) &= P_{\psi(h),2} &&= h^{-1}(6) \end{aligned}$$

Thus, $\varphi(\psi(h)) = h$.

Now let $f \in \mathcal{P}_1$. Let $g = \psi(\varphi(f))$. Then

$$\begin{aligned} X_{g,1} &= (\varphi(f))^{-1}(\{1,3\}) &&= X_{f,1} \\ X_{g,2} &= (\varphi(f))^{-1}(\{1,5\}) &&= X_{f,2} \\ I_g &= (\varphi(f))^{-1}(2) &&= I_f \\ P_{g,1} &= (\varphi(f))^{-1}(4) &&= P_{f,1} \\ P_{g,2} &= (\varphi(f))^{-1}(6) &&= P_{f,2} \end{aligned}$$

In $\mathcal{P}_1$, this is enough to determine that $f = g$. Thus, $f = \psi(\varphi(f))$. Since φ and ψ are inverses, they are both bijections.

$\square$

Lemma 4.17. *There exists a bijection between C_2 and $\mathcal{P}_2$.*

Proof. First, we will construct a function $\psi : C_2 \to \mathcal{P}_2$. Let $h \in C_2$. Let

$$a = \begin{cases} \max h^{-1}(4) & \text{if } h(\max h^{-1}(\{1,3,4\})) = 4 \\ h^{-1}(1) & \text{else} \end{cases}$$

$$b = \begin{cases} \max h^{-1}(6) & \text{if } h(\max h^{-1}(\{1,5,6\})) = 6 \\ h^{-1}(1) & \text{else} \end{cases}$$

Let $\psi(h) = f$ where f satisfies the following:

$$X_{f,1} = h^{-1}(3) \cup \{a\} \qquad X_{f,2} = h^{-1}(5) \cup \{b\}$$

$$I_f = h^{-1}(2) \qquad J_f = h^{-1}(4) - \{a\}$$

$$K_f = h^{-1}(6) - \{b\} \qquad \theta_f^< = \theta_{h^{-1}(1)}$$

Let $\{a_1 < \cdots < a_m\}$ be an enumeration of J'. Because h is in C_2, at least one of $h(\max h^{-1}(\{1,3,4\})) = 4$ or $h(\max h^{-1}(\{1,5,6\})) = 6$ is satisfied. Thus, at least one of a or b is greater than $h^{-1}(1)$ and in J'_f. Thus, $\theta_f^< \neq \theta_{a_m}$. This implies that $\theta_f^< = \theta_{a_j} < \theta_{a_{j+1}} = \theta_f^>$ for some $j < m$.

As shown above

$$P_{f,1} = \{J_{\geq a_j} \cup K_{<a_{j+1}} \cup \{a_{j+1}\}\} - X_{f,1}$$

$$P_{f,2} = \{J_{\leq a_j} \cup K_{>a_j} \cup \{a_j\}\} - X_{f,2}$$

Thus, to show $f \in \mathcal{P}_2$, it is sufficient to show the following:

$$\max(X_{f,1} \cup J_f) \in X_{f,1} \qquad \max X_{f,1} \geq a_{j+1}$$
$$\max(X_{f,2} \cup K_f) \in X_{f,2} \qquad \max X_{f,2} \geq a_j$$

If $h(\max h^{-1}(\{1,3,4\})) = 4$, then $\max X_{f,1} = \max h^{-1}(4)$. By choice of a, we have $\max(X_{f,1} \cup J_f) \in X_{f,1}$. Furthermore, $\max X_{f,1} > h^{-1}(1) = a_j$. Thus, $\max X_{f,1} \geq a_{j+1}$.

Alternatively, if $h(\max h^{-1}(\{1,3,4\})) \neq 4$, then $(\max X_{f,1} \cup J_f) \in X_{f,1}$. Since in addition we have $h(\max h^{-1}(\{1,3,4\})) \neq 1$, it must be the case that $h(\max h^{-1}(\{1,3,4\})) = 3$. Thus, $\max X_{f,1} > a_j$, so $\max X_{f,1} \geq a_{j+1}$.

If $h(\max h^{-1}(\{1,5,6\})) = 6$. Then $\max X_{f,2} = \max h^{-1}(6)$. By choice of b, we have $\max(X_{f,2} \cup K_f) \in X_{f,2}$. Furthermore, $\max X_{f,2} > h^{-1}(1) = a_j$.

Alternatively, if $h(\max h^{-1}(\{1,5,6\})) \neq 6$. Then $\max X_{f,2} > \max h^{-1}(6)$, so once again $\max(X_{f,2} \cup K_f) \in X_{f,2}$. Furthermore, $\max X_{f,2} \geq h^{-1}(1) = a_j$.

Thus, $\psi(h) \in \mathcal{P}_2$.

We will now construct a function $\varphi : \mathcal{P}_2 \to C_2$. Let $f \in \mathcal{P}_2$. Suppose $J'_f = \{a_1 < \cdots < a_m\}$ as defined above. Suppose $\theta_f^< = \theta_{a_j} < \theta_{a_{j+1}} = \theta_f^>$.

Let

$$a = \begin{cases} a_j & \text{if } a_j \in X_{f,1} \\ \max X_{f,1} & \text{else} \end{cases}$$

$$b = \begin{cases} a_j & \text{if } a_j \in X_{f,2} \\ \max X_{f,2} & \text{else} \end{cases}$$

Let

$$J^* = \begin{cases} J & \text{if } a_j \in X_{f,1} \\ J \cup \{\max X_{f,1}\} & \text{else} \end{cases}$$

$$K^* = \begin{cases} K & \text{if } a_j \in X_{f,2} \\ K \cup \{\max X_{f,2}\} & \text{else} \end{cases}$$

Define $\varphi(f)$ as follows:

$$\varphi(f)(i) = \begin{cases} 1 & \text{if } i = a_j \\ 2 & \text{if } i \in I \\ 3 & \text{if } i \in X_{f,1} - \{a\} \\ 4 & \text{if } i \in J^* \\ 5 & \text{if } i \in X_{f,2} - \{b\} \\ 6 & \text{if } i \in K^* \end{cases}$$

Since $a_j \notin X_{f,1} \cap X_{f,2}$ at least one of the else conditions is used in selecting a and b. Thus at least one of $\varphi(f)(\max \varphi(f)^{-1}(\{1,3,4\})) = 4$ or $\varphi(f)(\max \varphi(f)^{-1}(\{1,5,6\})) = 6$ holds. Since $f \in \mathcal{P}_2$, $\max X_{f,1} \geq a_{j+1} > a_j$, we have $\varphi(f)(\max \varphi(f)^{-1}(\{1,3,4\})) \neq 1$. Thus, $\varphi(f) \in C_2$.

Let $f \in \mathcal{P}_2, h = \varphi(f)$, and $g = \psi(\varphi(f))$. We will now show $f = g$. It is sufficient to show all of the following:

$$X_{f,1} = X_{g,1} \qquad X_{f,2} = X_{f,2} \qquad I_f = I_g$$
$$J_f = J_g \qquad K_f = K_g \qquad \theta_f^< = \theta_g^<$$

First notice that $\theta_f^< = \theta_{h^{-1}(1)} = \theta_g^<$. Secondly, $I_g = h^{-1}(2) = I_f$.

Suppose $a_j \in X_{f,1}$. First notice that $\max(X_{f,1} \cup P_{f,1}) \in X_{f,1}$ and $J_{\geq a_j} \subset P_{f,1}$. From this, we deduce $h(\max h^{-1}(\{1,3,4\})) \neq 4$. Thus,

$$X_{g,1} = h^{-1}(3) \cup \{h^{-1}(1)\} = (X_{f,1} - a_j) \cup \{a_j\} = X_{f,1}$$

$$J_g = h^{-1}(4) - h^{-1}(1) = J_f - \{a_j\} = J_f$$

Suppose instead $a_j \notin X_{f,1}$. First notice that $\max P_{f,1} \geq a_{j+1} > a_j$. Thus, we have $\max h^{-1}(4) > h^{-1}(1)$. Then, because $\max(X_{f,1} \cup P_{f,1}) \in X_{f,1}$ and $J_{\geq a_j} \subset P_{f,1}$, we have, we have $\max h^{-1}(4) = \max X_{f,1} > \max h^{-1}(3)$. Thus,

$$X_{g,1} = h^{-1}(3) \cup \{\max h^{-1}(4)\} = (X_{f,1} - \{\max X_{f,1}\}) \cup \{\max X_{f,1}\} = X_{f,1}$$

$$J_g = h^{-1}(4) - \{\max h^{-1}(4)\} = J \cup \{\max X_{f,1}\} - \{\max X_{f,1}\} = J_f$$

Similar arguments to those for $X_{g,1}$ and J_g will show that $X_{g,2} = X_{f,2}$ and $K_g = K_f$. Thus, $g = f$ as desired.

Let $h \in C_2$, $f = \psi(h)$, and $g = \varphi(\psi(h))$. Then

$$\theta_{g^{-1}(1)} = \theta_f^{<} = \theta_{h^{-1}(1)}$$

$$g^{-1}(2) = I_f = h^{-1}(2)$$

Suppose first that $h(\max h^{-1}(\{1,3,4\})) = 4$. Then $X_{f,1} = h^{-1}(3) \cup \{\max h^{-1}(4)\}$. In this case, $h^{-1}(1)$ is not in $X_{f,1}$. Thus, in the computation of φ, $a = \max X_{f,1} = \max h^{-1}(4)$. This gives

$$g^{-1}(3) = X_{f,1} - \{\max X_{f,1}\} = h^{-1}(3) - \{\max h^{-1}(4)\} = h^{-1}(3)$$

$$g^{-1}(4) = J_f \cup \{\max X_{f,1}\} = (h^{-1}(4) - \{\max h^{-1}(4)\}) \cup \{\max h^{-1}(4)\} = h^{-1}(4)$$

.

On the other hand, if $h(\max h^{-1}(\{1,3,4\})) \neq 4$, then $X_{f,1} = h^{-1}(\{1,3\})$. Then

$$g^{-1}(3) = X_{f,1} - h^{-1}(1) = h^{-1}(\{1,3\}) - h^{-1}(1) = h^{-1}(3)$$

$$g^{-1}(4) = J_f = h^{-1}(4) - h^{-1}(1) = h^{-1}(4)$$

Similar arguments show $g^{-1}(5) = h^{-1}(5)$ and $g^{-1}(6) = h^{-1}(6)$. Thus, $g = h$ as desired. Since φ and ψ are inverses of one another, they are both bijections.

□

Lemma 4.18. *There exists a bijection between C_3 and $\mathcal{P}_3$.*

Proof. First, we will construct a function $\psi : C_3 \to \mathcal{P}_3$. Let $h \in C_3$. Let

Let $\psi(h) = f$ where f satisfies the following:

$$X_{f,1} = h^{-1}(1,3) \qquad X_{f,2} = h^{-1}(5) \cup \{\max h^{-1}(6)\}$$
$$I = h^{-1}(2) \qquad J = h^{-1}(6) - \{\max h^{-1}(6)\}$$
$$K = h^{-1}(4) \qquad \theta_f^{<} = \theta_{\max h^{-1}(\{1,3,5,6\})}$$

The above information is enough to completely determine f. By choice of $\theta_f^{<}$, we have $\theta_f^{<} = \theta_{\max J'}$ as desired.

As described above

$$P_{f,1} = K$$

$$P_{f,2} = \{J \cup K_{<a_1} \cup K_{>a_m} \cup \{a_1\}\} - X_{f,2}$$

By assumption $h(\max h^{-1}(\{1,3,4\})) = 1$, so $\max(X_{f,1} \cup P_{f,1}) \in X_{f,1}$. Also by assumption, we must have $h(\max h^{-1}(\{1,5,6\})) = 6$. Thus, $h(\max h^{-1}(\{4,5,6\})) = 6$. Hence, we have $\max(X_{f,2} \cup K) \in X_{f,2}$. By construction of $X_{f,2}$, we have $\max(X_{f,2} \cup J) \in X_{f,2}$. Thus, $\max(X_{f,2} \cup P_{f,2}) \in X_{f,2}$ as desired. Hence, $f \in \mathcal{P}_2$.

Define $\varphi : \mathcal{P}_3 \to C_3$ as follows:

$$\varphi(f)(i) = \begin{cases} 1 & \text{if } i = \max X_{f,1} \\ 2 & \text{if } i \in I \\ 3 & \text{if } i \in X_{f,1} - \{\max X_{f,1}\} \\ 4 & \text{if } i \in K \\ 5 & \text{if } i \in X_{f,2} - \{\max X_{f,2}\} \\ 6 & \text{if } i \in J \cup \{\max X_{f,2}\} \end{cases}$$

By construction, we have $h(\max h^{-1}(\{1,5,6\})) = 6$ and $h(\max h^{-1}(\{1,3\})) = 1$. Because $f \in \mathcal{P}_2$, $\max(X_{f,1} \cup K) \in X_{f,1}$, so $h(\max h^{-1}(\{1,3,4\})) = 1$. Thus, $\varphi \in C_3$.

Showing that φ and ψ are inverses follows similarly as the analogous fact in the previous lemma. □

Corollary 4.19. *Let $\mathcal{A}$ be a sufficiently general hyperplane arrangement in $\mathbb{R}^2$, then*

$$\beta_{2(c-1)}\mathcal{M}_{c+2}(\mathcal{A}) = n\binom{n-1}{c+1}\binom{n-c-2}{c+1}3^{n-2c-3}$$

Proof. Any sufficiently general hyperplane arrangement in $\mathbb{R}^2$ is homeomorphic to one that is suitably ordered and with the choice of normal vectors all with positive x-coordinate. Combining this with the three preceding lemmas gives the result. □

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