ESSAYS IN ASSET PRICES AND MACROECONOMICS

BY

RAFAEL J. NIVIN VALDIVIEZO

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Economics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2018

Urbana, Illinois

Doctoral Committee:

Professor Anne Villamil, Chair
Professor Stefan Krasa
Professor Stephen Parente
Professor Nicholas C. Yannelis

Abstract

This dissertation is composed by two chapters relating asset prices and macroeconomic dynamics, the first one explores this relationship from a theoretical point of view while the second chapter is focused on a more empirical approach to use information from asset prices.

The first chapter contributes with the macro-finance literature of asset pricing in a general equilibrium framework. A Dynamic Stochastic General Equilibrium Model is developed aiming to replicate simultaneously the historical regularities for both macroeconomic and financial variables for the US economy. In a framework that includes households with recursive preferences for consumption goods and housing services in a two sector production economy that includes long run technological shocks, the model delivers several regularities about asset pricing behavior consistent with the US historical data and also some regularities about macroeconomic variables.

In particular, the model deliver a high equity premium, high volatility of equity returns and a low auto-correlation of equity returns. Moreover, by including an endogenously determination of housing supply, in combination with consumerś preference for housing services, the model also delivers a series of regularities about housing variables (risk premium, volatility and auto-correlation). In addition to, the model generates a more significant welfare cost of the business cycle in comparison with standard DSGE models, which is an important feature when trying to replicate asset pricing behavior in a general equilibrium framework.

The second chapter reviews the main methodologies that have been using in recent years to extract market expectations implicit in derivative prices. Through recovering the density functions of the price of the underlying asset on the maturity date of options negotiated on the market (called implied risk-neutral probability density functions or RNDs) it is possible to track how market expectations over a particular financial asset evolve over time, providing a useful tool to assess the risk of financial assets.

This chapter highlights the main difficulties that need to be faced when trying to estimate RNDs. A Monte Carlo analysis is implemented to check the robustness of the estimation method used here to obtain the RNDs from option price data, smoothing splines. An application for the Brazilian exchange rate is implemented to show the usefulness of this methodology, especially to identify changes on market expectations for emerging market's exchange rates in the wake of the US "taper tantrum".

To my son Sebastián, my wife Fiorella, my parents and siblings.

Acknowledgments

I would never have been able to finish my dissertation without the support of many people. First of all, I want to show my greatest gratitude to my advisor, Professor Anne Villamil, for her continuous support, encouragement and invaluable guidance. I am also sincerely grateful for my thesis committee members, Stefan Krasa, Stephen Parente and Nicholas C. Yannelis. Each of them has had an influence on my education and provided me with all the support needed to finish my dissertation. I also thank the faculty of the Economics Department at the University of Illinois, particularly, many thanks to Professors Anil Bera and Walter Sosa Escudero, for their many lessons that will endure over time. Thanks also goes to my classmates at the UIUC and my colleagues at the Central Bank of Peru.

And finally, a special thank to my son Sebastián and my wife Fiorella, for their unconditional love and support through this long journey. Thanks to my father Manuel and my mother Fidelina, and to my brothers and sisters. This thesis is dedicated to them.

Table of Contents

List of Tables

List of Figures

Chapter 1

Asset Pricing and Housing Supply in a Model with Recursive Preferences

1.1 Introduction

Modern asset pricing research has been devoted in explaining the main features of asset prices, in particular, the equity premium, its volatility, and its cyclical variation. There has been significant success in explaining the so called ”equity premium puzzle” (Mehra and Prescott, 1985) and other asset pricing anomalies using partial equilibrium models, containing alternative preferences and cash flow dynamics specified exogenously. For example, models that includes preferences with habit persistence (Constantinides, 1990; Campbell and Cochrane, 1999) and recursive preferences (Epstein and Zin, 1989, 1991; Bansal and Yaron, 2004) have been recurrent in the finance literature. Using these types of preferences in endowment economies where the dynamic of consumption and dividends are exogenously determined have been useful to explain the high equity premium, the high volatility of the equity returns and the low level and volatility of the risk free interest rate, which are clear characteristic of the US financial data.

However, explaining for the equity premium and other asset pricing regularities in General Equilibrium models where quantities such as consumption and dividends are endogenously determined has proven to be a more difficult challenge (Rouwenhorst, 1995; Jermann, 1998; Kaltenbrunner and Lochstoer, 2010). It has been showed that by endogenizing the dynamic of consumption and dividends, general equilibrium models often produce contradictory results when comparing the statistical moments for both macroeconomic and financial variables.

Currently, an important part of the macro-finance literature is trying to answer whether a general equilibrium model can match moments of both macroeconomic variables (aggregate quantities) and financial variables (asset returns).

This chapter contributes to the macro-finance literature of matching both macroeconomic and financial variables in a general equilibrium model. The model in this chapter is a general equilibrium model with recursive preferences similar to van Binsbergen et al. (2012) and An (2010) in the sense that like them we consider a real business cycle model with recursive preferences, labor supply, a production sector and stochastic technological growth. Different from those models, here housing supply is also considered. Jaccard (2011) and Iacoviello and Neri (2010) included a housing sector in a general equilibrium model but they considered habit persistence preferences, instead of recursive preferences as implemented here. The model considered in this paper is close related to Bernal-Verdugo (2011), which includes housing supply in a model with recursive preferences, however, the framework considered there is more restrictive than the one considered in this chapter, since it does not include labor supply and stochastic technological growth, which are key ingredients of the business cycle.

Recursive preferences (Epstein and Zin, 1989, 1991) are attractive in the finance literature for two main reasons. First, they allow to separate risk aversion and the intertemporal elasticity of substitution. Contrary to the use of standard expected utility preferences, where the coefficient of risk aversion and the elasticity of intertemporal substitution are close related (one is the inverse of the other), being able to separate these two parameter gives the model one more degree of freedom to match simultaneously the moments of risky assets and risk-free assets. Second, recursive preferences offer the intuitive appeal of having preference for an early or later resolution of uncertainty. Since Bansal and Yaron (2004) it has been argued that having households with preferences for an early resolution of uncertainty (under recursive preferences, the risk aversion coefficient being higher than the inverse of the elasticity of intertemporal substitution parameter), is a key feature to successfully match asset pricing behavior. Also, combining recursive preferences with other features such as long-run risk or stochastic volatility it is possible to account for many patterns in the data. Finally, it has been argued that using recursive preferences generates radically bigger welfare costs of the business cycle than those coming from standard expected utility (Croce, 2006; Tallarini, 2000; An, 2010).

A housing sector is included here since real estate is by far the largest component of household total wealth, therefore including a housing sector in a general equilibrium model that explains the joint dynamic of house prices and financial returns seems fairly relevant. Piazzesi et al. (2007) found that distinguishing housing consumption from from the consumption bundle can have interesting asset pricing implication. In a partial equilibrium setup, they found that the expenditure share of housing services affects the stochastic discount factor used to prices financial assets (composition risk). Moreover, Davis and Heathcote (2005) and Iacoviello and Neri (2010) showed the importance of including housing to model macroeconomic variables.

The rest of the chapter is organized as follows. Section 1.2 presents a detailed description of the model considered here. Section 1.3 derives the solution of the model. Section 1.4 shows the main results of the paper. Concluding remarks are offered in Section 1.5.

1.2 The model

There are two production sectors in this economy. The business sector uses labor and capital to produce a final good that can be consumed, invested in the same sector or used as input in the housing sector. The way how to include housing supply is adapted from Bernal-Verdugo (2011). New homes are produced in the housing sector using labor and inputs provided by the business sector. There is a representative household with recursive preferences over consumption goods, housing services and leisure. The representative household divides his time between leisure activities, hours worked in the business sector and hours worked in the housing sector. Both capital and housing stock face increasing adjustment costs reflecting financial constraints and building restrictions. Finally, there is only one source of exogenous disturbances which take the form of random shock to the productivity growth rate of labor in both sectors.

1.2.1 Producers:

There is two productive sectors in the economy, the business sector that produces non-housing consumption goods and the housing sector that provides of housing services to households.

Business sector

The business sector produces non-housing consumption goods using a standard Cobb-Douglas production function:

$$Y_t = K_{t-1}^{\alpha_g}(A_t l_{1t})^{1-\alpha_g}$$

where Y_t is aggregate output of consumption goods (excluding housing services), K_t is the end-of period stock of capital and l_{1t} is the quantity of labor demanded in this sector. Also, A_t represents an exogenous, labor-enhancing technology level.

This sector owns the stock of capital and each period maximizes dividends (D_1):

$$D_{1t} = Y_t - r_t^L l_{1t} - I_t$$

where r_t^L is rental rate of labor (wage rate), l_{1t} is the amount of labor demanded in the business sector and I_t is the investment in capital stock. The firm's investment will increase the stock of capital according to the following law of capital accumulation:

$$K_t = (1-\delta_K)K_{t-1} + G_K(\frac{I_t}{K_{t-1}})K_{t-1}$$

Where G_K is a concave function that represent costly capital adjustment and it has the functional form similar to Jermann (1998) and Kaltenbrunner and Lochstoer (2010).

Therefore, the representative firm in the business sector solves the following optimization problem:

$$\underset{\{I_t,K_t,l_{1t}\}}{Max} E_t\left[\sum_{s=0}^{\infty} Q_{t,t+s}\left\{D_{1,t+s} - q_{t+s}^K\left(K_{t+s} - (1-\delta_k)K_{t+s-1} - G_k\left(\frac{I_{t+s}}{K_{t+s-1}}\right)K_{t+s-1}\right)\right\}\right]$$

Where $Q_{t,t+1}$ is the stochastic discount factor and q_t^K is a Lagrange multiplier that represents the market price for installed physical capital.

Housing sector

The housing sector produces new homes according to standard Cobb-Douglas production function:

$$N_t = M_t^{\alpha_h}(A_t l_{2t})^{1-\alpha_h}$$

where N_t is aggregate output of new homes, M_t is the quantity of housing materials provided from the business sector and l_{2t} is the quantity of labor demanded in this sector. Also, A_t represents an exogenous, labor-enhancing technology level, which by assumption is similar as the business sector.
This sector owns the stock of houses and each period maximizes dividends (D_2):

$$D_{2t} = r_t^H H_{t-1} - r_t^L l_{2t} - M_t$$

where r_t^H is the rental rate of houses, H_{t-1} is the end of period stock of houses, l_{2t} is the amount of labor in the housing sector and M_t is the demand of housing materials from the non-housing sector.

The quantity of new houses will increase the stock of houses available to households according to the following law of housing accumulation:

$$H_t = (1-\delta_H)H_{t-1} + G_H(\frac{N_t}{H_{t-1}})H_{t-1}$$

where, similarly to the business sector, G_H is a concave function that represent costly adjustment in the stock of houses, which will be associated with building restrictions.

Then, the representative firm in the housing sector solves the following problem:

$$\underset{\{N_t,H_t,M_t,l_{2t}\}}{Max} E_t\left[\sum_{s=0}^{\infty} Q_{t,t+s}\left\{\begin{array}{c} D_{2,t+s} - p^N_{t+s}\left[N_{t+s} - M^{\alpha_h}_{t+s}(A_{t+s}l_{2,t+s})^{1-\alpha_h}\right] \\ -q^H_{t+s}\left(H_{t+s} - (1-\delta_H)H_{t+s-1} - G_H\left(\frac{N_{t+s}}{H_{t+s-1}}\right)H_{t+s-1}\right)\end{array}\right\}\right]$$

where p^N_t is the prices of new houses (housing investment) and q^H_t is a Lagrange multiplier that represents the market price for housing.

1.2.2 Adjustment costs:

Following Jermann (1998) and Kaltenbrunner and Lochstoer (2010), G_K is a function that represents costly capital adjustment:

$$G_K(\frac{I_t}{K_{t-1}}) = \frac{\alpha^K_1}{1-\frac{1}{\xi_K}}\left(\frac{I_t}{K_{t-1}}\right)^{(1-\frac{1}{\xi_K})} + \alpha^K_2$$

where ξ_K is the elasticity of the investment rate to Tobin's q. If ξ_K is low, capital adjustment costs are high; if $\xi_K = \infty$, capital adjustment costs are zero. The constants α^K_1 and α^K_2 are set such that there are no adjustment costs in the non-stochastic steady state. In particular, $\alpha^K_1 = [e^\gamma - (1-\delta)]^{\frac{1}{\xi_K}}$ and $\alpha^K_2 = \frac{e^\gamma-(1-\delta)}{1-\xi_K}$.

Similarly, the concave function G_H, representing costly adjustment in the stock of houses, is defined as

$$G_H(\frac{N_t}{H_{t-1}}) = \frac{\alpha^H_1}{1-\frac{1}{\xi_H}}\left(\frac{N_t}{H_{t-1}}\right)^{(1-\frac{1}{\xi_H})} + \alpha^H_2$$

where the constants α^H_1 and α^H_2 are set in similar fashion as the one considered in the law of motion for capital.

1.2.3 Technology:

I assume that the growth rate of technology follows a stochastic process of the form :

$$log(\frac{A_t}{A_{t-1}}) = \gamma + log(z_t)$$

$$log(z_t) = \rho_z log(z_{t-1}) + \sigma_z \epsilon_t$$

where the γ is the average growth rate of technology and z_t represents a shock to the growth rate of technology which follows an auto-regressive process with coefficient ρ and ϵ_t is $N(0,1)$, which is associated to the long run productivity risk (Croce, 2006).

1.2.4 Household

There is a representative household with Epstein-Zin Preferences (Epstein and Zin, 1989, 1991) over consumption of non-housing goods (G_t), housing services, which are proportional to the stock of homes (H_t), and leisure ($1 - l_{1t} - l_{2t}$) characterized by the following recursive function:

$$V_t = \underset{C_t, H_t, l_{1t}, l_{2t}}{Max} \left\{ (1-\beta)(C_t(1-l_{1t}-l_{2t})^{v})^{\frac{1-\rho}{1+v}} + \beta(E_t V_{t+1}^{1-\chi})^{\frac{1-\rho}{1-\chi}} \right\}^{\frac{1}{1-\rho}}$$

$$C_t = \left[G_t^{1-\theta} + \omega H_{t-1}^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

Note that H_t is the end of period stock of houses in the economy, and by assumption the housing services demanded for consumers are proportional to the stock of homes available at the start of the period.

The parameters in this preference representation include β, the discount factor, ν, which controls labor supply, χ, which controls risk aversion and ρ, the inverse of the elasticity of intertemporal substitution (EIS). Note that if $\chi = \rho$ the the model collapses to the CRRA utility case, where the inverse of the EIS and risk aversion coincide.

For better understanding, it is common to define the current utility (U_t) and the continuation value (W_t) as:

$$U_t = [C_t(1 - l_{1t} - l_{2t})^{v}]^{\frac{1}{1+\nu}}$$

$$W_t = (E_t V_{t+1}^{1-\chi})^{\frac{1}{1-\chi}}$$

Therefore, the representative household maximizes her lifetime utility characterized by a CES function that aggregates current utility and the continuation value. We can see that the continuation value is also a CES function which aggregates welfare across the states of the world.

The representative consumer then solves the following optimization problem:

$$V_t = \underset{G_t, H_t, l_{1t}, l_{2t}}{Max} \left\{ (1-\beta)(U_t)^{1-\rho} + \beta (W_t)^{1-\rho} \right\}^{\frac{1}{1-\rho}}$$

Subject to the following budget constrain:

$$G_t + r_t^H H_{t-1} + \frac{b_{t+1}}{R_t^f} = r_t^L (l_{1t} + l_{2t}) + D_{1t} + D_{2t} + b_t$$

where R_t^f is the risk free gross interest rate, b_{t+1} is the holding of a bond.

We can solve for the pricing kernel, Q_{t+1}:

$$Q_{t+1} = \frac{\partial V_t / \partial G_{t+1}}{\partial V_t / \partial G_t} = \beta \left(\frac{V_{t+1}}{W_t} \right)^{\rho-\chi} \left(\frac{U_{t+1}}{U_t} \right)^{1-\rho} \left(\frac{C_{t+1}}{C_t} \right)^{\theta-1} \left(\frac{G_{t+1}}{G_t} \right)^{-\theta}$$

which result in the following general equation for the stochastic discount factor with recursive preferences:

$$Q_{t+1} = \beta \left(\frac{V_{t+1}}{W_t} \right)^{\rho-\chi} \left(\frac{1 - l_{1,t+1} - l_{2,t+1}}{1 - l_{1t} - l_{2t}} \right)^{\frac{(1-\rho)\nu}{1+\nu}} \left(\frac{G_{t+1}}{G_t} \right)^{-\frac{\rho+\nu}{1+\nu}} \left[\frac{1 + \omega \left(\frac{H_t}{G_{t+1}} \right)^{1-\theta}}{1 + \omega \left(\frac{H_{t-1}}{G_t} \right)^{1-\theta}} \right]^{\frac{\theta(1+\nu)-(\rho+\nu)}{(1+\nu)(1-\theta)}}$$

1.2.5 Market equilibrium:

In equilibrium all produced consumption goods are either consumed, invested or supplied to the production of new houses:

$$G_t + I_t + M_t = Y_t$$

In addition to, labor supply is equal to labor demand. Finally the equilibrium in financial markets requires that consumers own all claims on both firms' dividends and that the other assets to have a zero net supply.

1.3 Model solution

Since the welfare theorems apply in this economy, it is better to solve the social planner problem and derive the first order conditions for this model.

1.3.1 Social Planner problem:

$$V_t = \underset{C_t,H_t,l_{1t},l_{2t}}{Max} \left\{(1-\beta)(U_t)^{1-\rho} + \beta(W_t)^{1-\rho}\right\}^{\frac{1}{1-\rho}}$$

where:

$$W_t = (E_t V_{t+1}^{1-\chi})^{\frac{1}{1-\chi}}$$

$$U_t = [C_t(1-l_{1t}-l_{2t})^{v}]^{\frac{1}{1+\nu}}$$

$$C_t = \left[G_t^{1-\theta} + \omega H_{t-1}^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

subject to:

$$G_t + I_t + M_t = K_{t-1}^{\alpha_g}(A_t l_{1t})^{1-\alpha_g}$$

$$N_t = M_t^{\alpha_h}(A_t l_{2t})^{1-\alpha_h}$$

$$K_t = (1-\delta_K)K_{t-1} + G_K(\frac{I_t}{K_{t-1}})K_{t-1}$$

$$H_t = (1-\delta_H)K_{t-1} + G_H(\frac{N_t}{H_{t-1}})H_{t-1}$$

$$log(\frac{A_t}{A_{t-1}}) = \gamma + log(z_t)$$

$$log(z_t) = \rho_z log(z_{t-1}) + \sigma_z \epsilon_t$$

Due to the growth rate of technology, the endogenous variables will be moving along a balance growth path, so in order to solve this model it is necessary to obtain a stationary form of this system.

1.3.2 Stationary form for the social planner problem:

Since the model is non-stationary we need to normalize the variables as follow: $x_t = \frac{X_t}{A_t}$ for $X \epsilon \{G, H, C, Y, I, K, S, M\}$ and $f_t = \frac{F_t}{A_t^{\frac{1}{1+\nu}}}$ for $F \epsilon \{V, U, W\}$ and therefore the stationary form of the social planner problem can be represented as:

$$v_t = max \left\{ (1-\beta)(c_t(1-l_{1t}-l_{2t})^{\upsilon})^{\frac{1-\rho}{1+\nu}} + \beta (w_t)^{1-\rho} \right\}^{\frac{1}{1-\rho}}$$

subject to:

$$c_t = \left[g_t^{1-\theta} + \omega (\frac{h_{t-1}}{z_t e^{\gamma}})^{1-\theta} \right]^{\frac{1}{1-\theta}}$$

$$g_t + i_t + m_t = \left(\frac{k_{t-1}}{z_t e^{\gamma}} \right)^{\alpha_g} (l_{1t})^{1-\alpha_g}$$

$$n_t = m_t^{\alpha_h} (l_{2t})^{1-\alpha_h}$$

$$k_t = \left[(1-\delta_K) + G_K \left(\frac{i_t}{k_{t-1}} z_t e^{\gamma} \right) \right] \frac{k_{t-1}}{z_t e^{\gamma}}$$

$$h_t = \left[(1-\delta_H) + G_H \left(\frac{n_t}{h_{t-1}} z_t e^{\gamma} \right) \right] \frac{h_{t-1}}{z_t e^{\gamma}}$$

where:

$$w_t = \left[E_t v_{t+1}^{1-\chi} (z_{t+1} e^{\gamma})^{\frac{1-\chi}{1+\nu}} \right]^{\frac{1}{1-\chi}}$$

Then the social planner needs to solve the following recursive problem is:

$$v_t = max \left\{ (1-\beta)(c_t(1-l_{1t}-l_{2t})^{\upsilon})^{\frac{1-\rho}{1+\nu}} + \beta (w_t)^{1-\rho} \right\}^{\frac{1}{1-\rho}} + \lambda_{1t} \left\{ \left[g_t^{1-\theta} + \omega (\frac{h_{t-1}}{z_t e^{\gamma}})^{1-\theta} \right]^{\frac{1}{1-\theta}} - c_t \right\}$$

$$+ \lambda_{2t} \left\{ \left(\frac{k_{t-1}}{z_t e^{\gamma}} \right)^{\alpha_g} (l_{1t})^{1-\alpha_g} - g_t - i_t - m_t \right\} + \lambda_{3t} \left\{ m_t^{\alpha_h} (l_{2t})^{1-\alpha_h} - n_t \right\}$$

$$+ \lambda_{4t} \left\{ \left[(1-\delta_K) + G_K \left(\frac{i_t}{k_{t-1}} z_t e^{\gamma} \right) \right] \frac{k_{t-1}}{z_t e^{\gamma}} - k_t \right\} + \lambda_{5t} \left\{ \left[(1-\delta_H) + G_H \left(\frac{n_t}{h_{t-1}} z_t e^{\gamma} \right) \right] \frac{h_{t-1}}{z_t e^{\gamma}} - h_t \right\}$$

The non-linear system of equations described in Appendix A represents the optimality conditions required to obtain the solution of the social planner problem.

1.4 Model results

1.4.1 Solution method

The solution method that will be considered here is Perturbation Methods. This method is a local solution method that has some advantages with respect to other solution methods. First at all, as was showed in Caldara et al. (2012), perturbation methods, Chebyshev polynomials and value function iteration provide a high level of accuracy; however, perturbation methods are more time efficient. Also, this method is the only method that performs well when the number of state variables increases (handles better the curse of dimensionality). Finally, another popular solution method, log-linearization, is a particular case of the more general perturbation method.

Based on the perturbation method, the set of optimality conditions can be represented by the following general form:

$$E_t F\left(y_{t+1}, y_t, x_{t+1}, x_t\right) = 0$$

where x_t is the set of endogenous and exogenous state variables and y_t is the set of co-state variables. That system requires a solution of the form:

$$y_t = g(x_t, \eta)$$

$$x_{t+1} = h(x_t, \eta) + \eta \Omega \varepsilon_{t+1}$$

Perturbation methods compute an n-order approximation for the function g and h around the steady state of the system. In particular, for the model presented here the solution was obtained using a second order approximation using DYNARE. A required step to implement this solution method is to obtain the steady state of the non-linear system, which is described in Appendix B.

1.4.2 Parameter selection

Table 1.1 shows the values of all parameters used in this model. They were selected using the following criteria:

Preference parameters

With recursive preferences, I assume the subjective discount factor to be 0.995, which is common in the finance literature (Croce (2006), among others). With respect to the elasticity of intertemporal substitution (EIS), I set it to be 1.5, which is similar to Kaltenbrunner and Lochstoer (2010), Kuehn (2008) and consistent with the empirical results in a similar setup found in van Binsbergen et al. (2012). The coefficient of risk aversion is set to 15 which is within the values used in the asset pricing literature (Kaltenbrunner and Lochstoer, 2010; Kuehn, 2008; Croce, 2006; An, 2010) and significantly lower to empirical findings found in van Binsbergen et al. (2012). Finally, similar to Caldara et al. (2012) I set the labor parameter to be consistent with a representative consumer that works one-third of his time.

Non-Housing sector

The constant capital share in the Cobb-Douglas production function, α is 0.33, which is common in the real business cycle literature. Following Davis and Heathcote (2005), the depreciation rate of capital ,δ_K, is set to 0.0139 while ξ_K, the elasticity of investment rate to Tobin's q, is set to 0.105 similar to Jaccard (2011).

Housing sector

Similar to the business sector, the constant capital share in the Cobb-Douglas production function, α is 0.30, to represent more labor intensity in the housing sector (Davis and Heathcote, 2005; Jaccard, 2011). Following Davis and Heathcote (2005), the depreciation rate of houses ,δ_H, is set to 0.0039 while ξ_H, representing the adjustment cost in the stock of houses, is set to 0.6 similar to Jaccard (2011).

Technology

The average quarterly growth rate of technology is set to 0.5 percent, which is similar to Jaccard (2011) and van Binsbergen et al. (2012). The stochastic part the growth rate of technology has a coefficient of auto-correlation of 0.8 which is similar to the one used in Croce (2006) and An (2010). Finally, shocks on the growth rate of technology have standard deviation of 0.01, which is within the values used in the literature about asset pricing in production economies (Kaltenbrunner and Lochstoer, 2010; Kuehn, 2008; van Binsbergen et al., 2012).

1.4.3 Results

Equity and housing risk premium

Table 1.2 shows the main statistics for asset returns from the calibrated model. The historical data are from Piazzesi et al. (2007). We can see that the model provides a excess return of capital of 4.20 percent, still one notch below the historical equity premium but higher than results from models using standard utility preferences. Housing risk premium predicted by the model is consistent with the historical data, which is less risky than return on capital. This lower housing risk premium is mainly explain by the lower volatility

Table 1.1: Parameters of the calibrated model

Parameter	Value	Description
β	0.995	discount factor
ρ	1/1.5	inverse of elasticity of intertemporal substitution
χ	15	risk aversion coefficient
γ	0.005	growth rate of technology
ν	1/0.357-1	labor supply coefficient
θ	0.877	(inverse) elasticity of substitution between G and H
ω	0.2	preference for housing consumption
α_g	0.33	capital share (non-housing sector)
α_h	0.30	capital share (housing sector)
δ_K	0.0139	depreciation rate (non-housing sector)
δ_H	0.0039	depreciation rate (housing sector)
ξ_K	0.105	capital adjustment cost coef. (non-housing sector)
ξ_H	0.6	capital adjustment cost coef. (housing sector)
ρ_z	0.8	correlation of technological growth rate
σ_z	0.01	SD of tech. shocks

of dividends in the housing sector, which is mainly determined by the low volatility of rents generated by the model. Business sector dividends are very volatile and pro-cyclical, and therefore has to be compensated by a higher risk premium. In addition to, the lower housing risk premium is also explained by the fact that the supply of housing is endogenously determined.

Even though the model doesn't match exactly the significantly small auto-correlation of equity returns, the model deliver a relatively small coefficient of auto-correlation. Moreover, the model also deliver a high auto-correlation of housing returns in line with the historical data, but volatility of housing returns are still too high with respect to historical data. Finally, consistent with the historical data, the model deliver that the expenditure share of non-housing services is around 80 percent, with low volatility and high auto-correlation as we can see in the data.

Business cycles statistics

Table 1.3 shows that the model delivers a volatility of the growth rate of output consistent with the historical data, around 1 percent. the volatility of consumption delivered from the model is 0.90 percent, which is higher than the historical data but it is consistent with the business cycle fact that consumption is less volatile than output. The model delivers a growth rate of investment of 0.63 percent, which is significantly lower than the historical data, which contradict the fact that in the data investment is more volatile than output. Croce (2006) argued that this results is due the the use of costly capital adjustment cost. While this is necessary to obtain a reasonable risk premium (since otherwise in a production economy, investment

Table 1.2: Asset returns

Moments	Variable	Model	Data
Mean	risk free rate	3.06	0.75
	excess return (capital)	4.20	6.19
	excess return (housing)	1.50	1.77
	expenditure share of non-housing	80.23	82.6
volatility (SD)	risk free rate	3.78	3.68
	return on capital	22.80	16.56
	return on housing	8.21	2.73
	expenditure share of non-housing	0.62	1.54
auto-correlation	risk free rate	0.80	0.73
	return on capital	0.11	-0.06
	return on housing	0.41	0.48
	expenditure share of non-housing	0.99	0.97

will be used as a tool to smooth consumption with the resulting low equity returns), adjustment cost will prevent investment from moving enough to be more volatile than output.

Even though the model does not replicate the exact amount for the auto-correlation of investment, output and consumption, the model generates a higher auto-correlation of consumption with respect to output. Finally, The model delivers a high cross correlation between consumption and output and a relatively lower cross correlation between investment and output.

Table 1.3: Business cycle statistics

Moments	Variable	Model	Data
volatility (SD)	output growth (g_Y)	1.05	0.85
	consumption growth (g_C)	0.90	0.52
	investment growth (g_I)	0.63	2.24
auto-correlation	output growth	0.77	0.36
	consumption growth	0.86	0.43
	investment growth	0.21	0.44
correlation wrt g_y	consumption growth	0.98	0.55
	investment growth	0.78	0.68

Welfare cost of the business cycle

Since Lucas (1987) measuring the welfare costs of business cycle fluctuations has been an recurrent challenge with important policy implications. If the cost of business cycles are high, devoting resources to stabilize consumption is a reasonable alternative. With a standard utility and serially uncorrelated consumption, Lucas (1987) found that the cost of the business cycles is as low as 0.1 percent of the lifetime consumption and therefore it is inefficient to devote resources to stabilize the business cycle. Since then, various researchers have revisited Lucas's calculation looking for new evidence of more significant welfare costs. A common approach to compute the welfare cost of the business cycles is to obtain the certainty equivalent measure of the sample path of consumption during the business cycles and then compare that certainty equivalence measure with the steady state level of consumption. With standard preferences this measure is not far from the results obtained in Lucas (1987), while with habit persistence preferences the cost of business cycles can reach 5 percent Jaccard (2011).

Tallarini (2000) and Croce (2006) argued that once the information about financial markets is taken into account general equilibrium models can imply large welfare cost, since the high risk premium observed in equity returns suggests that households can be extremely averse to even small fluctuations in consumption. With recursive preferences the computation of the welfare cost of the business cycles is straightforward, since one of the first order conditions, the one related to the stochastic discount factor, requires to solve for the value function of the representative household. By solving the model and obtaining the sample path for the macroeconomic quantities over the business cycle, the sample path of the value function will also be computed. By comparing the sample path of the value function over the business cycle with the value function in the steady state, it is possible to obtain a more clear measure of the welfare cost of the business cycle (An, 2010).

Let the welfare cost of the business cycle be defined as fraction of steady state consumption loss due to the business cycle. From the Appendix B we know that the steady state level of the value function is:

$$v = \phi_9 \left[c(1 - l_1 - l_2)^{\nu}\right]^{\frac{1}{1+\nu}}$$

where ϕ_9 is a function of the structural parameters in the model. Moreover, if we let λ_t be the measure of welfare cost of business cycles at time t, then the value function in time t can be represented as:

$$v_t = \phi_9 \left[(1 - \lambda_t)c(1 - l_1 - l_2)^{\nu}\right]^{\frac{1}{1+\nu}}$$

therefore, by using the sample path of the value function over the business cycle, it is possible to obtain a close form solution for the welfare cost of the business cycle over the sample path:

$$\lambda_t = 1 - e^{(1+\nu)dv_t}$$

where $dv_t = log(v_t) - Log(v)$, which cam be obtained from the sample path of the value function.

From the model the average welfare cost is 68%, which is higher than standard measures of welfare costs (i.e $\frac{E(c)-c}{c} = 29\%$) and significantly larger than results from models that do not include asset pricing information.

1.5 Conclusions

A general equilibrium asset pricing model with long run productivity shocks and endogenous housing supply can be useful to explain some of the asset prices puzzles found in previous research. By calibrating this model with parameters used before in the literature, the model delivers reasonable results for asset pricing, in particular the mean, volatility and auto-correlation of both housing and capital assets. The model also delivers consistent results about macroeconomic quantities (output, consumption and investment), even though the resulting low volatility of investment is still at odds with the US historical data. A way to deal with this low volatility of investment is to introduce financial leverage in the model as suggested by Croce (2006) or a more stylized DSGE model that includes financial intermediaries.

The model also delivers a significant high cost of the business cycles, which is consistent with the asset pricing literature that require consumers to be very sensitive about the volatility of consumption and therefore demanding a high compensation for bearing risky assets. This can be achieved by considering long run consumption risk with recursive preferences and costly adjustment cost.

Chapter 2

Extracting Risk Neutral Densities from Financial Derivatives

2.1 Introduction

Undoubtedly one of the most important topics of research in financial economics is the development of models that explain the evolution of financial asset prices, and through these models obtain a reliable forecast of the future evolution of those asset prices. Over the years there have been many attempts to find such models without obtaining acceptable forecasts of those prices.

However, the development of new financial instruments, specifically derivative contracts (options and futures) have brought a new and useful source of information that have renewed the interest among researchers for generating new techniques with more interesting results when comparing with the past attempts.

Over the last few years, there has been considerable interest among academics, market participants and policy-makers in extracting information of this kind from options prices. Different technique have been used, but a common way of displaying the information extracted comes in the form of an implied risk-neutral probability density function, or in short risk neutral density (RND) for the asset upon which the contract trades.

Since the appearance of this new source of information, data of observed option prices have been extensively used to estimate the implied risk neutral probability density function (RNDs). Since these RNDs represent forward-looking forecasts of the distributions of the prices of the underlying asset, they prove to be particularly useful for various applications. For example, they are used to price new complex financial derivatives, to test market rationality, to estimate risk preferences, among others. In particular, option implied RNDs have found an extensive use for monetary policy purposes by an increasing number of Central Banks, as they can use the estimation results to evaluate the risk of monetary policy decision. Also, it can be helpful for policy makers of commodity dependent countries for assessing the risk of future changes in the price of commodities.

This paper is organized as follows. Section 2.2 review the most popular methods to estimating risk neutral densities. Section 2.3 implement the estimation of the risk neutral density to the S&P 500 data using the smoothing splines method. Section 2.4 implements a Montecarlo analysis to assess how accurate is the smoothing splines methodology for estimating risk neutral densities. Section 2.5 performs an application of RND estimation to the Brazilian exchange rate. Section 2.6 concludes.

2.2 Methods of estimating risk neutral densities

Most methods start with the option-pricing relation, which states that the price of an option is the discounted risk-neutral expected value of the payoffs. In particular, for European type options this relationship is of the form:

$$C(t,T,K) = e^{-r(T-t)}E_t\left[max(S_T - K, 0)\right] = e^{-r(T-t)}\int_K^{\infty}(S_T - K)f(S_T)dS_T \ldots (1)$$

$$P(t,T,K) = e^{-r(T-t)}E_t\left[max(K - S_T, 0)\right] = e^{-r(T-t)}\int_{-\infty}^{K}(K - S_T)f(S_T)dS_T \ldots (2)$$

Where $C(t,T,K)$ and $P(t,T,K)$ are the prices of European calls and puts observed at time t having expiration date at time T and strike prices of K, r is the risk-less rate of interest, and $f(S_T)$ is the risk-neutral probability density function for the value of the underlying asset S at time T.

Different econometric and statistics methods have been developed for recovering the implied RNDs in this way, especially, parametric and nonparametric methods have been popular in this area. Parametric methods essentially rely on specific assumptions on the data-generating process, which depends on some unknown parameters. Nonparametric methods, in contrast, are flexible data-driven methods.

2.2.1 Parametric methods

The most popular approach of the parametric world is the mixture method, developed by Bahra (1997) and Melick and Thomas (1997). In this approach it is required to construct probability distributions as weighted averages by adding several simple and known probability distributions with different mixing probabilities. Since flexibility attained by adding different distributions comes at the cost of quickly increasing the number of parameters, the authors consider that the mixture of only two parametric distributions suffices to obtain a reliable estimation. the most common choices have been the lognormal distributions.

The double-lognormal method approximates this density function with a mixture of two log-normal density functions:

$$f(S_T) = \theta L(S_T|\mu_1, \sigma_1, S_t) + (1-\theta)L(S_T|\mu_2, \sigma_2, S_t) \dots (3)$$

$$L(S_T) = \frac{1}{S_T \sigma \sqrt{2\pi(T-t)}} \exp\left\{ \frac{-[\log S_T - \log S_t - (\mu - \frac{1}{2}\sigma^2)(T-t)]^2}{2\sigma^2(T-t)} \right\} \dots (4)$$

Where S_t is the current value of the underlying asset and $\{\mu_1, \sigma_1, \mu_2, \sigma_2, \theta\}$ are the unknown parameters that define the double-lognormal density functions; $\theta \in [0, 1]$. Thus, the fitted values for a call and put prices, given parameters $\{\mu_1, \sigma_1, \mu_2, \sigma_2, \theta\}$ are given by:

$$\widehat{C}_t(K|\mu_1, \sigma_1, \mu_2, \sigma_2, \theta) = e^{-r(T-t)} \int_K^{\infty} (S_T - K) \{\theta L(S_T|\mu_1, \sigma_1, S_t) + (1-\theta)L(S_T|\mu_2, \sigma_2, S_t)\} \, dS_T \dots (5)$$

$$\widehat{P}_t(K|\mu_1, \sigma_1, \mu_2, \sigma_2, \theta) = e^{-r(T-t)} \int_{-\infty}^{K} (K - S_T) \{\theta L(S_T|\mu_1, \sigma_1, S_t) + (1-\theta)L(S_T|\mu_2, \sigma_2, S_t)\} \, dS_T \dots (6)$$

Given observations of call and put prices, the parameters, $\{\mu_1, \sigma_1, \mu_2, \sigma_2, \theta\}$ of the implied double-lognormal PDF can be estimated using non-linear optimization methods to minimize the weighted sum of fitted price errors:

$$\underset{\mu_1,\sigma_1,\mu_2,\sigma_2,\theta}{Min}\left\{\sum_{i=1}^{N_c} w_i[C_t(K)-\widehat{C}_t(K|\mu_1,\sigma_1,\mu_2,\sigma_2,\theta)]^2+\sum_{j=1}^{N_p} w_j[P_t(K)-\widehat{P}_t(K|\mu_1,\sigma_1,\mu_2,\sigma_2,\theta)]^2\right\}\ldots(7)$$

subject to:

$$\sum_{i=1}^{N_c} w_i+\sum_{j=1}^{N_p} w_j=1$$

$$w_i, w_j \geq 0\ \forall i,j$$

Where N_c and N_p and the number of calls and put contracts in the estimation sample for a given pair of observations and expiry dates $\{t, T\}$ and the w_i, w_j are the weights placed on each option.

The main advantage of this method is that it's simple and easy to estimate without requiring a significant amount of computational capabilities. However, one of the main drawbacks of this method is that it tempts to over fitting the data that could result in an estimated risk-neutral densities exhibiting sharp spikes.

2.2.2 Non parametric methods

This approach follows the observation made by Breeden and Litzenberger (1978) that from differentiating the equations (1) and (2) we obtain the following equivalence for call options (an equivalent result applies for put options):

$$\frac{\partial C(t,T,K)}{\partial K}=e^{-r(T-t)}\left[F(S_T)-1\right]\ldots(8)$$

and

$$\frac{\partial^2 C(t,T,K)}{\partial K^2}=-e^{-r(T-t)}f(S_T)\ldots(9)$$

Therefore once obtained an estimation of the option price as a function of its strike price we can use that formula to recover the RND. Thus, the nonparametric methods differ in the approach to estimate the option price function.

We should notice that RNDs are not the same as real-world probabilities, since RNDs are influenced, perhaps heavily, by risk preferences. A change in risk-neutral probabilities can be due to changes in real-world probabilities, or risk preferences, or both.

We next review the two more popular nonparametric methods:

Kernel estimation

Kernel methods, used to fit the call price function and, at times, the implied volatility function, are related to nonlinear regressions. The main references about kernel methods are found in the work by Aït-Sahalia and Lo (1998) and Aït-Sahalia et al. (2001). These methods do not specify the linear form of a standard regression and instead, they are estimated starting from the concept that each data point suggests the center of a region through which the function passes. These methods try to estimate the option price formula as a function of its common arguments, mainly strike price, maturity date, interest rate. In order to perform such estimation it's required to make some assumption concerning the stability of the relationship between the option price and its arguments across time. These stability assumptions have been under debate since the episodes of crises in financial markets shows that the probability distribution can rapidly change over time.

Smoothing techniques

Smoothing techniques try to solve a nonparametric regression problem, requiring for this solution to be a smooth function. The most popular of such techniques are smoothing splines, which is the solution of a nonparametric regression problem with a penalty term and specifically, this nonparametric solution is a piecewise cubic polynomial. For a smooth spline the points in the X-axis correspond to each of the data points (defined as "knot points"). Between knot points the function is simply a cubic polynomial. However, the function is restricted in a way that its continuous in each knot point and also has continuous the first and second derivatives. The degree of smoothness of the spline is controlled by a smoothness penalty, λ, which multiply a measure of the degree of curvature in the function - the integral of the square of the second derivative of the function over its range. The objective function to be minimized is thus:

$$\widehat{f}(x) = \underset{f}{Argmin} \sum_{i=1}^{n} [y_i - f(x_i)]^2 + \lambda \int_{x_{\min}}^{x_{\max}} [f''(x)]^2 \, dx$$

Where $\widehat{f}(\cdot)$ is the function of interest, to be estimated with data on x and y. Note that the choice of the location of those knots is somewhat of an art and depends on specific circumstances: too many knots cause over fitting of the data and too few knots prevent the observations (y) from being matched.

It is important to mention that instead of obtaining directly an option price function, such techniques are primarily used to fit the volatility implied in the option prices with some smooth function. This smoothed implied volatility method was originally developed by Shimko (1993) and later improved by Malz (1997). Since this method relies on approximate a function to the volatility smile[1] rather than to the option price, it is necessary to convert the option prices to implied volatilities using the Black and Scholes formula[2]. Then, the estimated function is converted back to option price function and using the result from Breeden and Litzenberger (1978) we can obtain the RND for this option at a specific maturity.

Another key aspect of this nonparametric technique is the role of choosing λ, the smoothing parameter, which control the trade-off between fitting the data and how smooth the estimated curve will be. If $\lambda = 0$, the solution is the interpolant to the data while if $\lambda \to \infty$ we obtain a straight line, the least square estimator. Therefore, a large value of λ leads to a smooth curve but not so close to data and a small value of λ leads to a rough curve that follows the data closely. There several methods to chose λ, and the cross validation (CV) method is one of the most popular.

The cross validation (CV) criteria for choosing λ requires to select the parameter that minimizes the

[1] Volatility smile refers to the empirical evidence found in the volatility of option prices. At every maturity date when plotting the implied volatility of an option against its strike price it seem that the plot come from a convex function (a smile graph).

[2] Black and Scholes were the first ones in developing a model to price option contracts (Black and Scholes, 1973).

expected prediction error:

$$EPE(\lambda) = E(y^k - f_\lambda(x^k))$$

Where (x^k, y^k) are new data. Since additional data are not usually available, instead an estimator of EPE will be used, which is the cross validation function:

$$CV(\lambda) = \frac{1}{n}\sum_{1}^{n}(y_i - f_\lambda^{-i}(x_i))$$

Where f_λ^{-i} is the smoothing spline estimator fitted for all data except the observation i. Therefore a value of λ which minimizes the CV criteria would be an optimal choice for λ.

2.2.3 Comparison of the different techniques

As it has been noted, every estimation method has some advantages and disadvantages, and its use depends on the specific circumstances. However the high dependence of the parametric methods on assumptions about the underlying distribution makes them the less desired method nowadays. Moreover, the flexibility that nonparametric methods provide and the advance in computational capabilities for the estimation of these methods have produced an increase in the popularity of non parametric methods.

Concerning the nonparametric methods, since Kernel methods try to estimate an option price function that depends on many arguments (or at least two arguments, maturity and strike price, if stronger assumptions are made), it is necessary to have a significant amount of data in order to obtain a reliable estimation. Therefore, since the option markets even tough growing still represents a small market relative to the amount of data needed, Kernel estimation methods are still an ongoing project that could be more helpful in the future when the availability of data is big enough.

Finally, Smoothing techniques are getting popularity among academia due to its flexibility and high performance even with small amount of data. Moreover, comparisons of the stability and accuracy have been conducted by Bliss and Panigirtzoglou (2002) and Cooper (1999), who concluded that even though the three methods mentioned have similar results, the smoothing technique generally outperform the other two.

2.3 Estimating Risk Neutral Densities by Smoothing Splines

2.3.1 Data

As an application of the smoothing splines methodology to extract the RND from option prices Data on option prices were used, which are available at the Chicago Board of Trade (CBOT) and quote daily. These data refers to the ask and bid prices for the put and call options over the Standard & Poors 500 stock index. Option prices are quoted for a given maturity for several strike price (exercise price of the underlying asset when the option is executed at the maturity date). These data were used due to their high liquidity and because these options are European type (cancellation only possible at maturity), reducing the complexity of the estimation process. However, the methodology used here can be implemented, considering a few additional assumptions, using other type of derivatives and with different underlying assets.

From the collected data the average of ask and bid prices is used as the measure of option price. The use of ask-bid price has an advantage over the daily settlement price, which are also available and have been used extensively. Settlement prices are quoted for the exchange at the end of the day and since many of the strikes are traded with low frequency and with high volatility in the last trade of the day, the information available for the exchange which determines the settlement price is likely to be unsynchronous. In the other hand, the ask-bid prices, which show the intention of buyer and sellers to trade, are registered continuously and for all strikes (even though the trade is not performed) which make them a better approximation to the daily prices of the options, with more correspondence with the price of the underlying asset.

It is important to notice that there are two key issues that need to be faced when estimating the RND from option prices. Fist, there is numerous evidence of the different degrees of liquidity among options with different strikes and the same expiration day. Recognizing that it does not exist a model option price valuation that incorporates some measure of liquidity in the price formulation, this problem liquidity heterogeneity among strikes can be reduced if we do not include option prices far "out of the money"[3]. Put (Call) option prices are "out of the money" when the price is smaller (greater) than the current value of the underlying asset. Put (Call) option prices are "in of the money" when the prices is greater (smaller) than the current value of the underlying asset, which are the ones that register less liquidity.

[3] Options prices are "at the money" when the strike is equal to the current price of the underlying asset.

Secondly, the option prices provides information of the density of the underlying asset only at the strike prices and strictly speaking we can not say much what is happening between strikes. Moreover, even though the shape of the risk neutral density function between strikes can be constructed through smoothing, this estimation method is only valid for the range of prices between the biggest and lowest available strike. Therefore, the tail behavior is totally an assumption of the estimation method used. Hence, it is better to use as many strikes as possible in order to have the less dependence possible to assumptions over the shape of the tail distribution, also considering that the estimator of the high order moments, such as skewness and kurtosis, are sensitive to small changes in the tail distribution.

2.3.2 Estimation and results

Figure 2.1 shows data for put and ask prices (averaging ask-bid) for option over the S&P 500 stock price with maturity in December 2012, quoted on August, 2012. In line with the finance theory, those prices coincide in the current price of the underlying asset (or the option is "at the money"). Even though there is a temptation to use the smoothing spline in the strike-option price space, it is preferable to translate the smoothing technique to the space strike-volatility because smoothing in the strike-option price space does not restrict the density function to have non-negative values and unnecessary spikes.

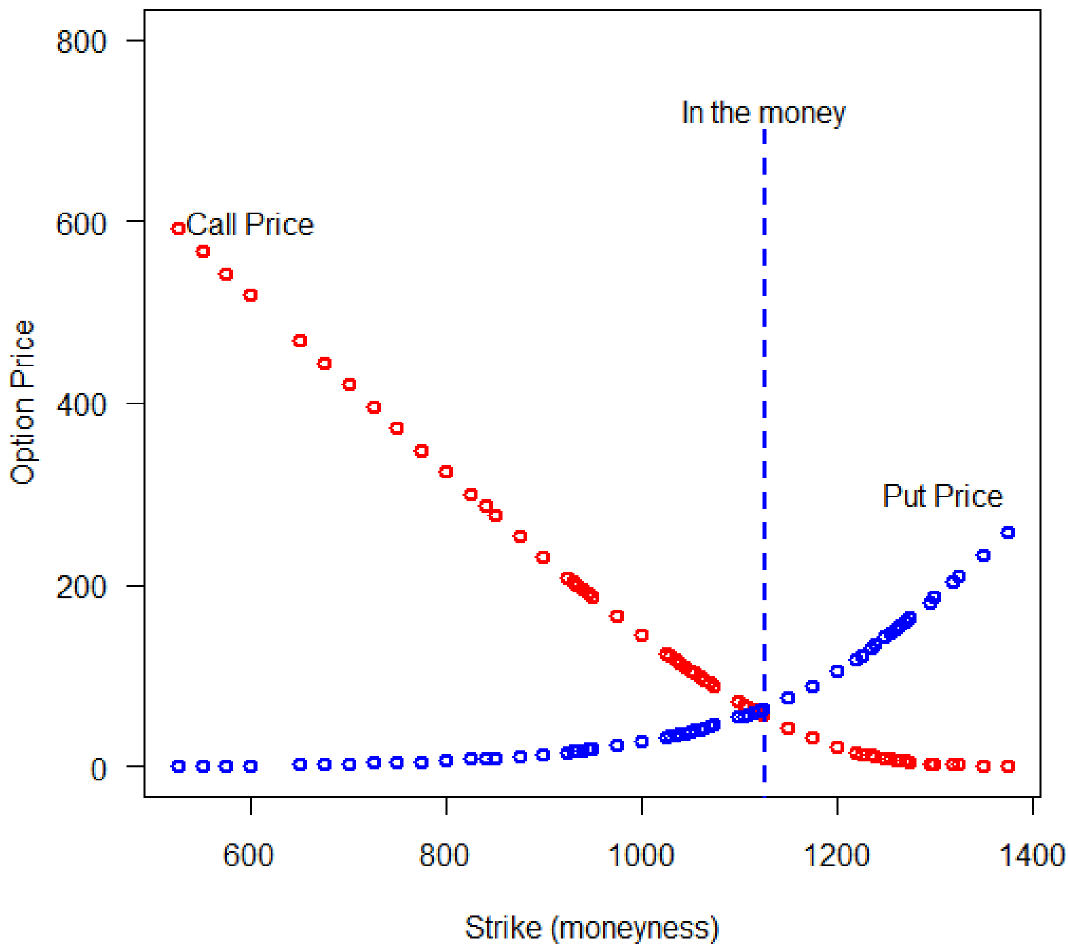

Figure 2.1: Option Prices for S&P 500 index

Shimko (1993) proposed transforming the option prices to their implied volatility before performing the smoothing technique by using the Black&Scholes formula and after implement the smoothing, re-transform back the estimated function to the option prices and then estimate the RND. This procedure does not assume that the Black-Scholes model is correct for the option prices (implying that the underlying asset follows a lognormal distribution), it only uses the Black and Scholes formula as a computational tool to transform the data to some space where the smoothing is more suitable. The Black and Scholes formula for a put option is as follows:

$$P(S_t, t) = Ke^{-r(T-t)}N(-d_2) - S_te^{-q(T-t)}N(-d_1)\ldots(10)$$

$$d_1 = \frac{\ln(S_t/K) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}} \ldots (11)$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)} \ldots (12)$$

Where K is the strike price, $(T-t)$ is the time to maturity, r is the risk free interest rate, q is the dividend yield, S_t is the current value of the stock index and σ is the volatility of the stock index. $N(\cdot)$ refers the cumulative normal distribution. Since all variables are known, including the option price, the goal is to solve for the volatility (that is the reason for calling it implied volatility).

Since the space strike-implied volatility only requires one volatility for each strike, and from call and put option prices we can extract two volatilities for each strike, the question is which one of the two option prices - call and put - should be used to perform the smoothing technique. For this case, since our main interest is to focus on the smoothing technique and in the estimation of the RND from this technique, the information contained in the put options is used. In figure 2.2 I show the implied volatility from the put options for each strike (this curve is well known as the volatility smile).

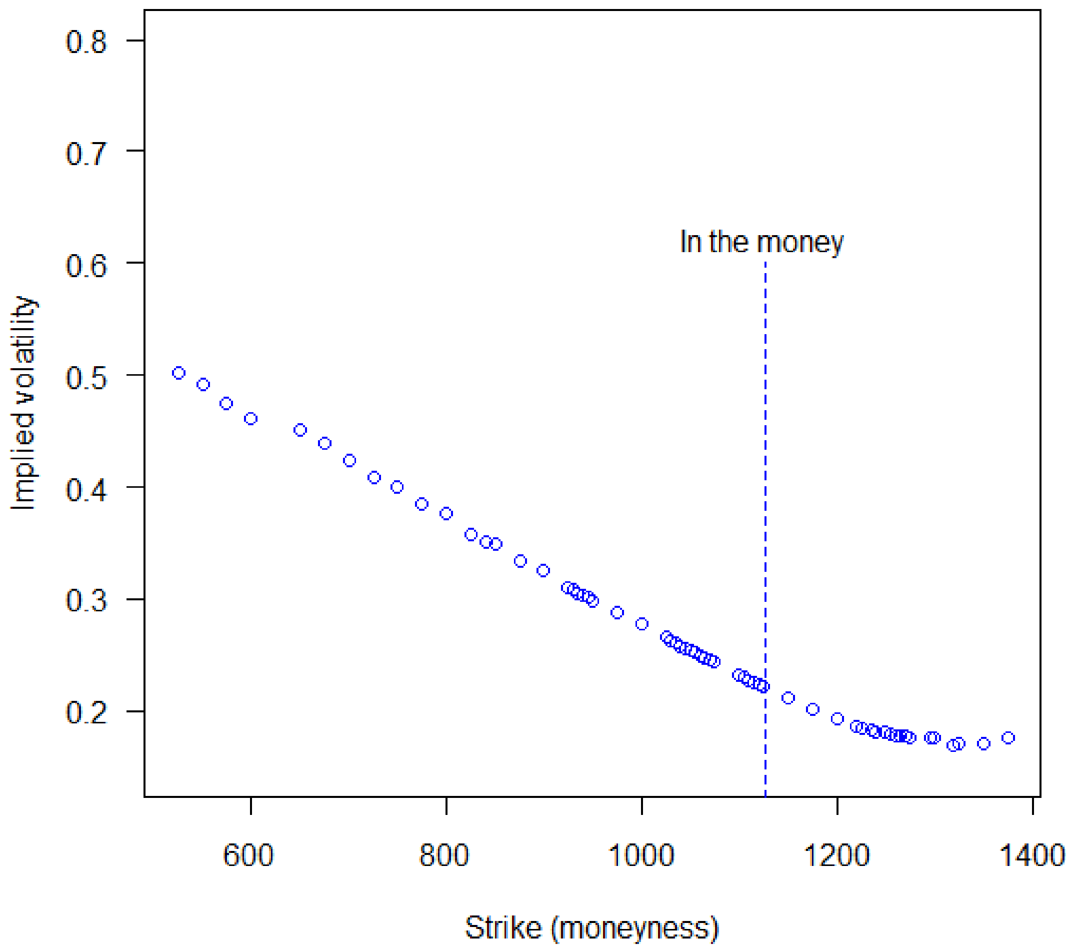

Figure 2.2: Implied Volatility from Put Prices

The choice of the put option rely on the argument of the ”put-call parity”, which indicates that since both put and call options reveal information about the same underlying asset (in this case the S&P 500 price index), then it will be the same to use either the put or the call option price. Under this argument, it is more suitable to use the put option prices because those prices register more liquidity.

Alternatively, it could be considered a mix of both types of option prices. Under this argument it is common to use both the call and the put prices when the options are well ”out of the money”; and a weighted average of both prices when the option is ”near the money” Options price are ”near the money” when the strike lies in a small interval around the current price of the underlying asset.

Smoothing splines is the suitable method to estimate a curve that ponders the effects of the market imperfections in the option prices. Figure 2.3 shows the estimated volatility smile resulting from applying the smoothing spline method to the observations of implied volatilities obtained from our original data on option prices.

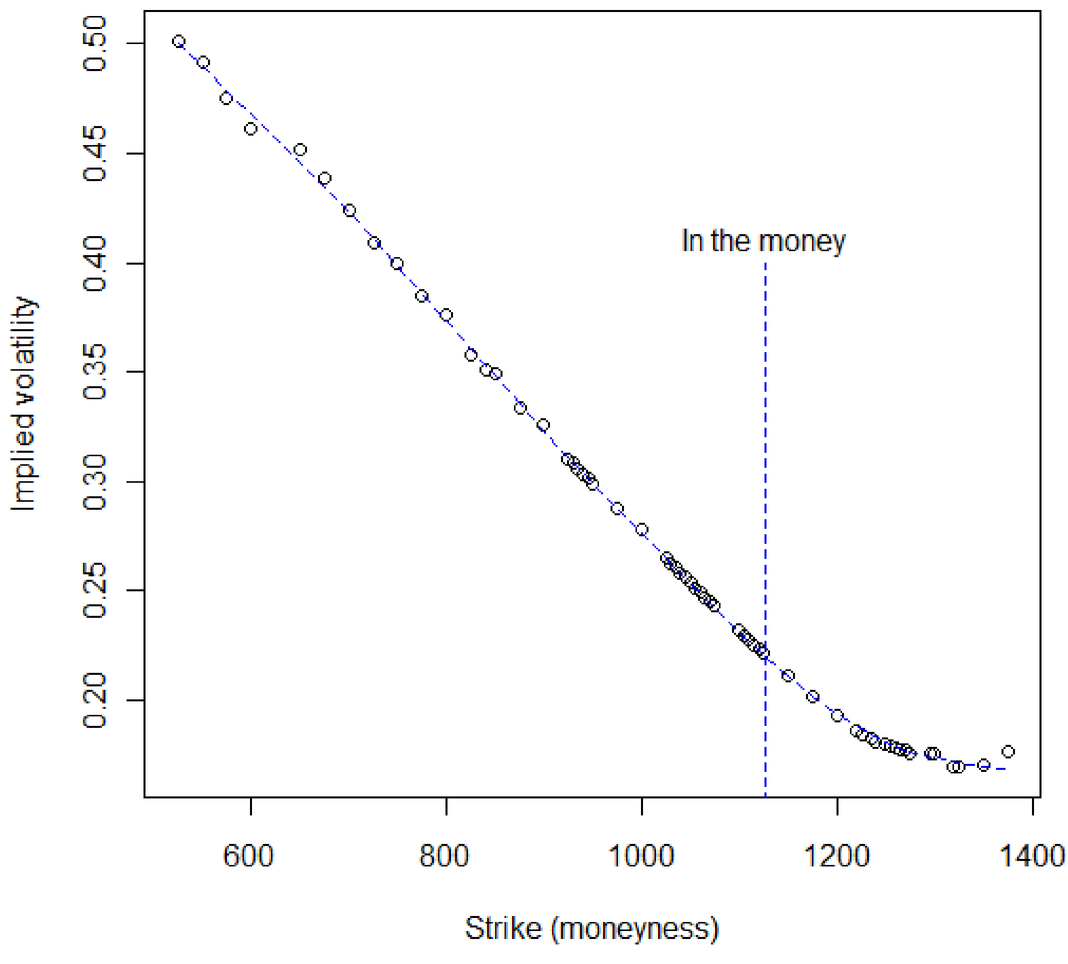

Figure 2.3: Implied volatility & Smoothing Splines

Implementing the smooth spline is both simple and computational efficient. Once the smooth spline is fitted, a large number of equally K-spaced points on the function are computed. These points are then converted into equally K-spaced values in the strike/option price space. These in turn are used to compute the RND using numerical methods.

Table 2.1 shows the results for the smoothing spline estimation showed in Figure 2.3. Due to the limited amount of observations (and one observation for each strike) the smoothing spline estimation using the cross validation criteria results in a curve that goes through all the available data points, therefore, in a volatility smile curve that implies a RND estimation with many sharp spikes. For this reason, a smoothing parameter (or equivalently, a degree of freedom) is selected to smooth the RND enough to eliminate sharp spikes or negative values, even if the fitted curve no longer go through all the data points. As, explained before, since the option prices obtained from CBOT could include some measurement errors and a liquidity bias, trying that the fitted curve pass through all data points seems as we are trying to over fit the Data.

Table 2.1: Smoothing Splines for option prices

Description	Value
Smoothing Parameter	413016.6
Equivalent Degrees of Freedom	7.008671
GCV Criterion	5.492472e-06
CV Criterion	7.903048e-06

Then, the volatility smile is estimated by smoothing splines for a large set of strikes. To accomplish this a sequence of 400 equally spaced strikes was created, taking into consideration that the amount of strikes need to be large enough to allow the resulting RNDs to add to one. Figure 2.4 shows the recovered option prices (put prices) for this estimated volatility smile using again the Black and Scholes formula to the volatility smile estimated by the smoothing spline.

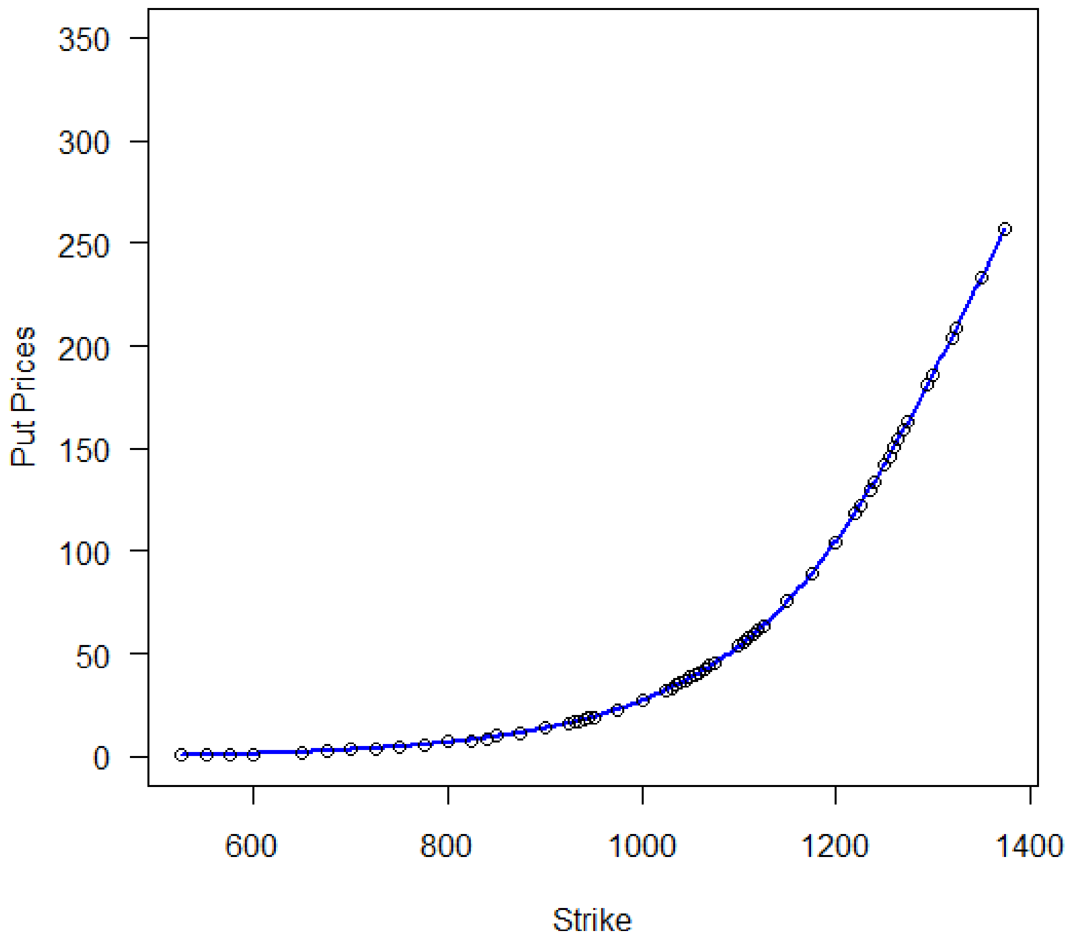

Figure 2.4: Put Prices from Smoothing Splines forecast

From this estimated option Price curve, It is straightforward to estimate the RND (both the cumulative and the density functions), which is showed in figure 2.5. These functions were obtained through numerical methods, using the following formulas:

$$F(S_n) = e^{r(\tau)} \left[\frac{P_{n+1} - P_{n-1}}{K_{n+1} - K_{n-1}} \right] \ldots (13)$$

$$f(S_n) = e^{r(\tau)} \left[\frac{P_{n+1} - 2P_n + P_{n-1}}{(0.5(K_{n+1} - K_{n-1}))^2} \right] \ldots (14)$$

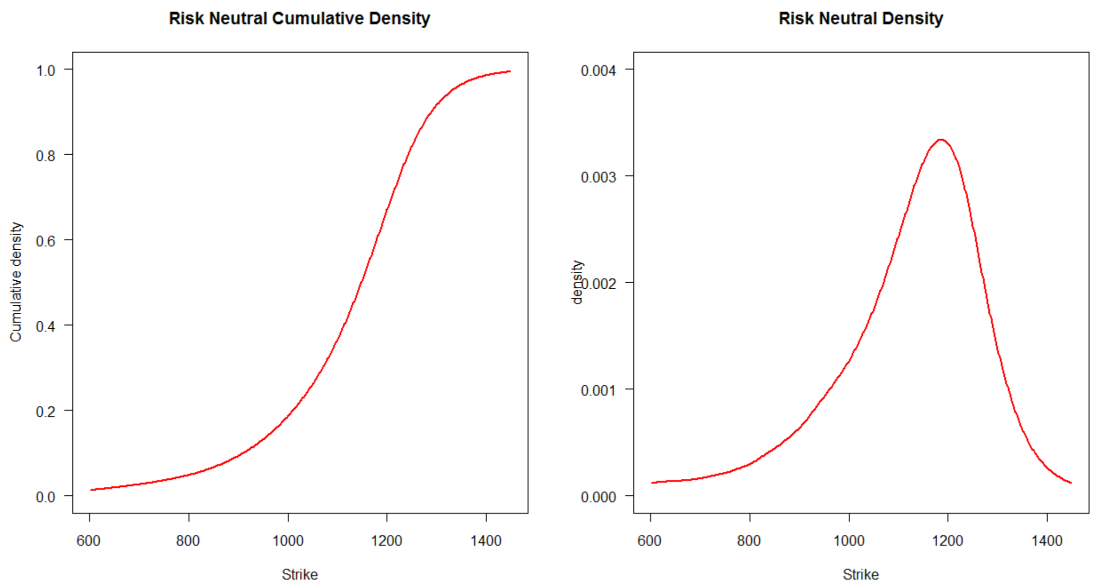

Figure 2.5: Risk Neutral Density by Smoothing Splines

It it important to mention that the smoothing splines assumes that the out the sample forecast of the estimated volatility smile is linear, and therefore, implicitly assuming that the tail behavior of the RND has a lognormal distribution. However, since the data on option prices used for the estimation cover a range large enough to only have a small mass left in the tails. Therefore, for the amount of data used for the effect of the assumption on the tail behavior is minimal.

The RND for the S&P 500 stock price index showed in the right graph of figure 2.5 appears to be asymmetric which is proved due to a skewness value of -1.11 (left sided). Its also straightforward to observe that the left tail of the RND is much fatter than the right tail, which implied that the likelihood of a sudden drop in the stock price index is higher than a sudden gain in the index.

Guidance on choosing the smoothing parameter

An important point to analyze is whether the chosen smoothing parameter is appropriate. In order to address this issue it is necessary to make clear that the degree of smoothness for the estimated curve can be settled in terms of the smoothing parameter λ or the degree of freedom (df), since there are a plain inverse relationship between those two parameters. Let $\{x_1, ..., x_n\}$ be the available dataset, then $1 < df \leq n$. If $df \to 1$ (which is equivalent to $\lambda \to \infty$) the estimated curve will be a straight line, the least square estimator for volatility smile[4]. On the other hand, when $df \to n$ (which is equivalent to $\lambda \to 0$) the estimated curve will be the interpolated function that pass through every data-point.

The most common criteria to select the degree of smoothness is the cross validation approach (CV), however in our case due to the small amount of data this criteria will tend to choose a curve that pass trough most data point (very high df or low λ). As we pointed out before a low value of the smoothing parameter will be associated with an estimated curve with some bumpy parts that will result in a RND with spikes and negative values, and therefore not suitable to represent a desirable density function.

To assess the suitability of the chosen smoothing parameter, the results of the baseline estimation ($df = 7$) is compared with the results from taking extreme values for the degree of freedom: df close to 1 ($\lambda \to \infty$) and df as high as 15 (lowest value of λ which results on a RND with values above zero). Figure 2.6 shows the comparison among the estimated implied volatility curves. With $df = 1$ the least square straight line is computed, while with $df = 15$ the estimated curve is a bumpy one that pass through most data points. The baseline smoothing spline estimation ($df = 7$) lays between those previous two. Figure 2.7 shows the estimated option price curve for the three alternatives, apparently having minimal differences.

[4] It is important to point out that even though in this case the resulting volatility smile is a straight line, the resulting RND will not be a flat curve (uniform distribution), since the RND results from taking the second derivative of the option prices, and since there is still a nonlinear relationship between the volatility smile and the option prices curve, option price curve will not be a straight line.

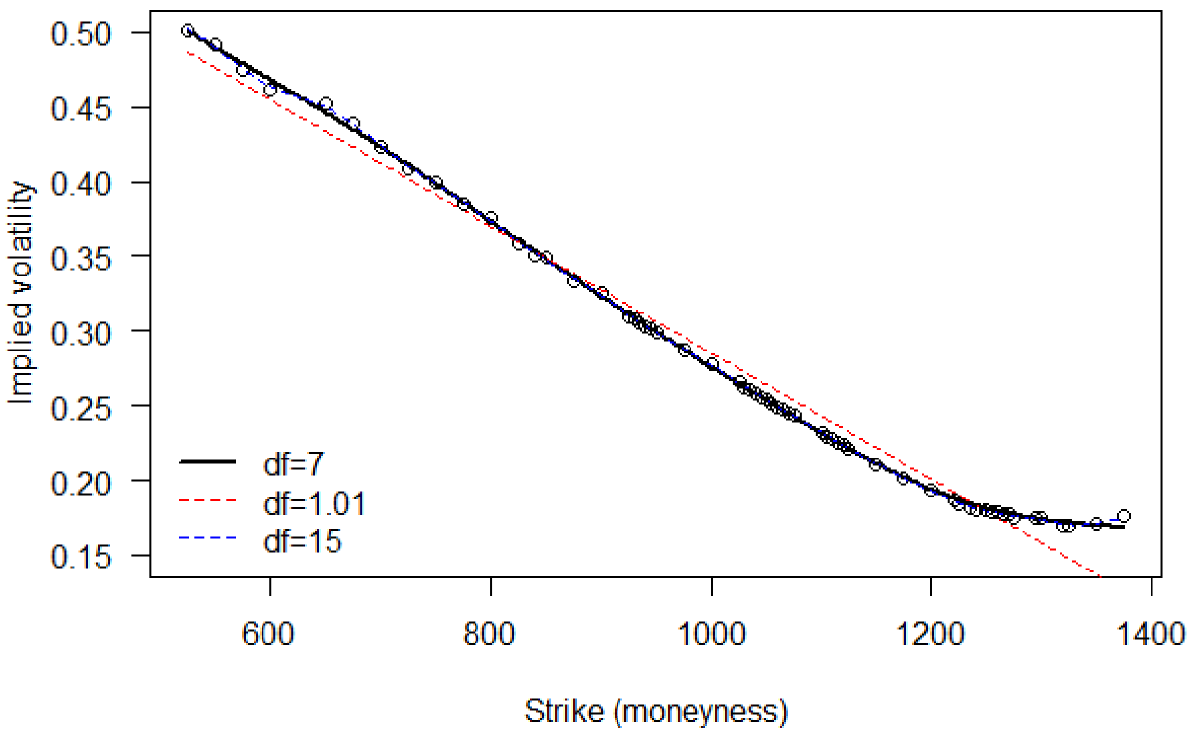

Figure 2.6: Volatility smile and degree of freedom (df)

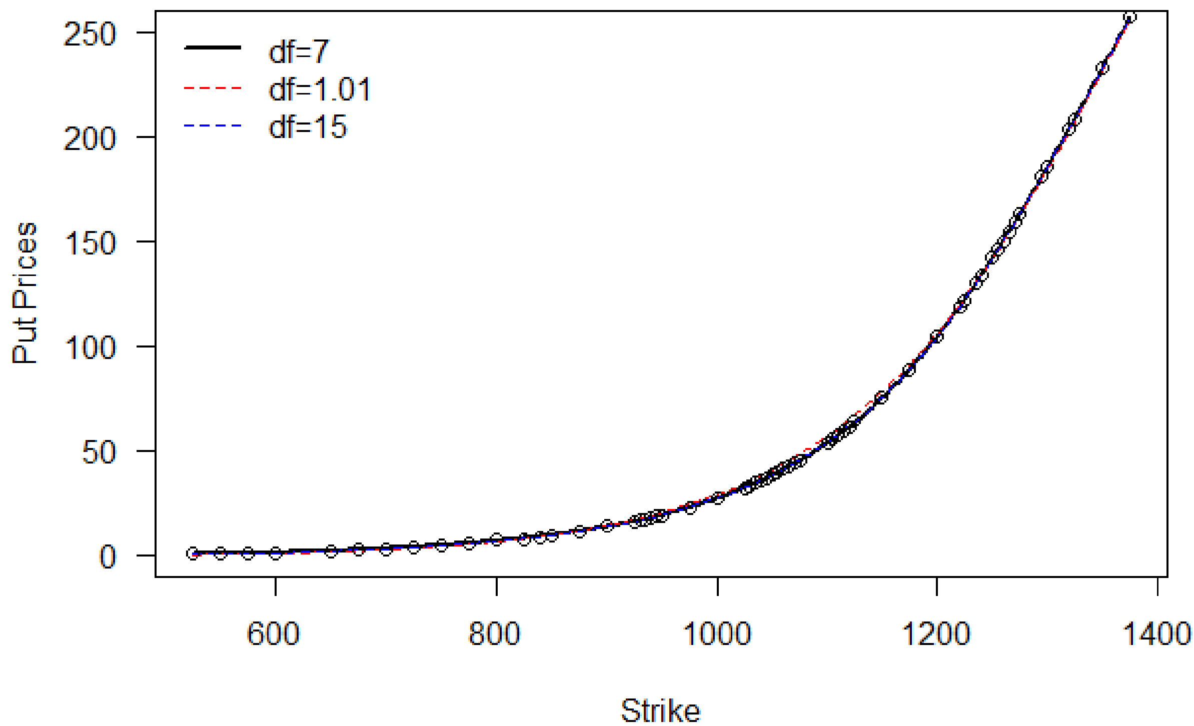

Figure 2.7: Option prices and degree of freedom (df)

Figure 2.8 shows the implicit risk neutral densities for the baseline estimation and the two extreme alternatives. Here we can notice that when the smoothing parameter has the largest value possible ($\lambda \to \infty$ or $df \to 1$), resulting in an straight line for the implied volatility curve, the estimated RND does not move far away from the baseline estimation. Moreover, since the cross validation selection criteria for this model prefers lower values of λ (high df) a RND with high degree of freedom is estimated ($df = 15$). We can notice that this RND displays several spikes and it seems to have a multimodal density function. However, we can not conclude that these characteristics are the real representation of the underlying density for this financial variable and instead can be attributed to a small sample problem. Alternatively, the baseline RND ($df = 7$) will approximately have the same properties as the estimated RND with $df = 15$ but without having the undesirable sharp spikes and the possible artificial multimodal behavior.

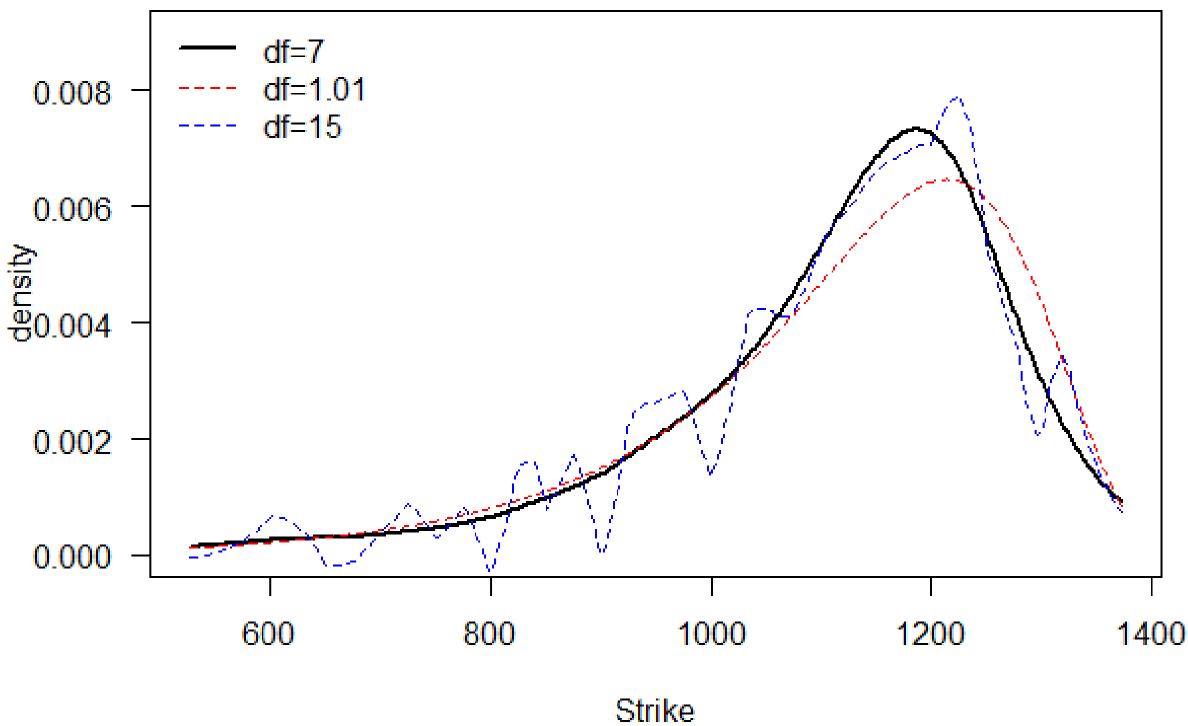

Figure 2.8: Risk neutral densities and degree of freedom (df)

2.4 Monte Carlo analysis

We propose two strategies to check the robustness of the Smoothing Splines method for extracting risk neutral densities. First, it is general knowledge that option prices are not quoted continuously, therefore the difference between the investor willingness to trade, which is considered to be continuous, and the prices at which they actually trade can generate a source of error in option pricing models, therefore we first address the robustness of this estimation method to this specific source of error. Second, I will allow for the possibility that some trades were quoted erroneously and therefore we should not account for them when estimating the risk neutral densities. For this matter, I will implement a bootstrap strategy to check the robustness of estimated RND to this different source of potential error involved in option prices. In order to quantify the effect of both sources of errors in the estimated RND I will compute confidence intervals in each strategy.

2.4.1 Confidence interval I: thick size effect

We consider here that the data on option prices were quoted in the exchange markets with some errors due to the discontinuity stipulated in the option contracts. It is common knowledge that every contract on options specify a minimum price change (called ”thick size”) allowed in the exchange markets. Therefore even if the investors’ willingness to buy or offer an option is continuous, option prices will only reflex that willingness at discrete amounts. This difference is a source of error that we are considering here. Particularly, for the options prices on the S&P 500 the thick size is 0.1, and therefore it is not possible to have a change in the option prices less that 0.1 even if investor are willing do so.

In order to address this potential source of error in the option prices, We will use the original data to simulate option prices that better represent continuous quotations. Specifically, I will perturb the original data to incorporate variations in option prices that are less than the thick size. Therefore, each option price will be perturbed by a uniform random variable with support from minus half thick size to half thick size. These uniform random errors will allow us to simulate continuity in the option prices.

Figure 2.9 shows the confidence intervals of 1000 simulated RNDs using this strategy. From the graph we can see that the confidence interval is considerably close to the RND from the original data. This narrow confidence interval implies that error due to the thick size is relatively small so we can use the data on option prices considering that they are allow to vary almost continuously and therefore we can consider that the resulting RND is representing significantly well investors’ expectations.

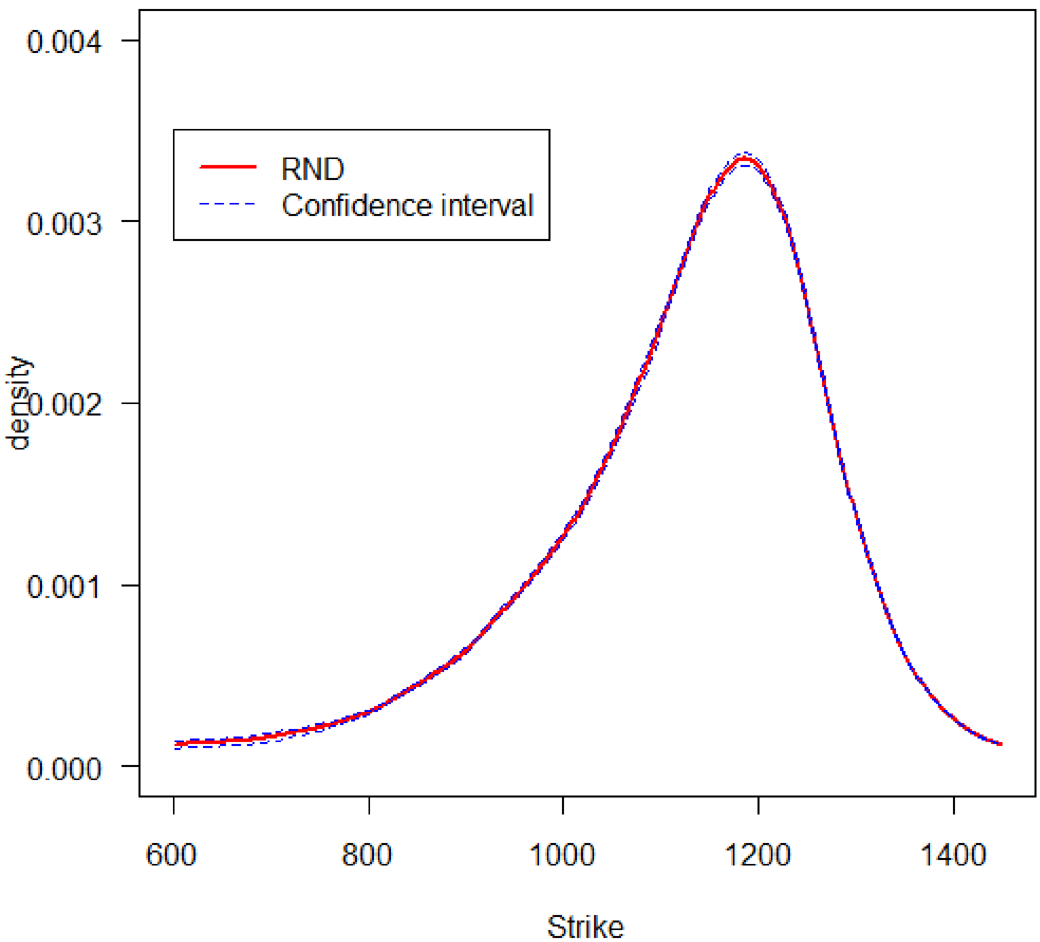

Figure 2.9: RND and Confidence Interval I

2.4.2 Confidence interval II: bootstrap analysis

In this section, we will follow a different approach to test the robustness of the RNDs estimated by smoothing splines. we will perform a bootstrap analysis to allow the possibility that some option prices were quoted erroneously. Then, considering that some prices were quoted erroneously implies that the RNDs estimated using those prices do not represent the true expectation of investors and therefore we should not consider those prices when estimating the RNDs.

we will extract different random samples from the original option price data and obtain RNDs from those random samples. Particularly, we will consider that 10 percent of the option price data was quoted erroneously and therefore should not be considered for estimating the RNDs. I will extract 1000 random samples that eliminated 10 percent of the original data in each sample and will construct confidence intervals from these simulated RNDs.

Figure 2.10 shows the confidence intervals from the simulated RNDs and includes the RND from the original data. We can see from the graph that we have now more volatility in the RNDs but still the RND from the original data does a good job in representing the option price data even when considering that 10 percent of the data were not part of the estimation. This result provides us with more support that the Smoothing Spline method is suitable for estimating RND from option prices.

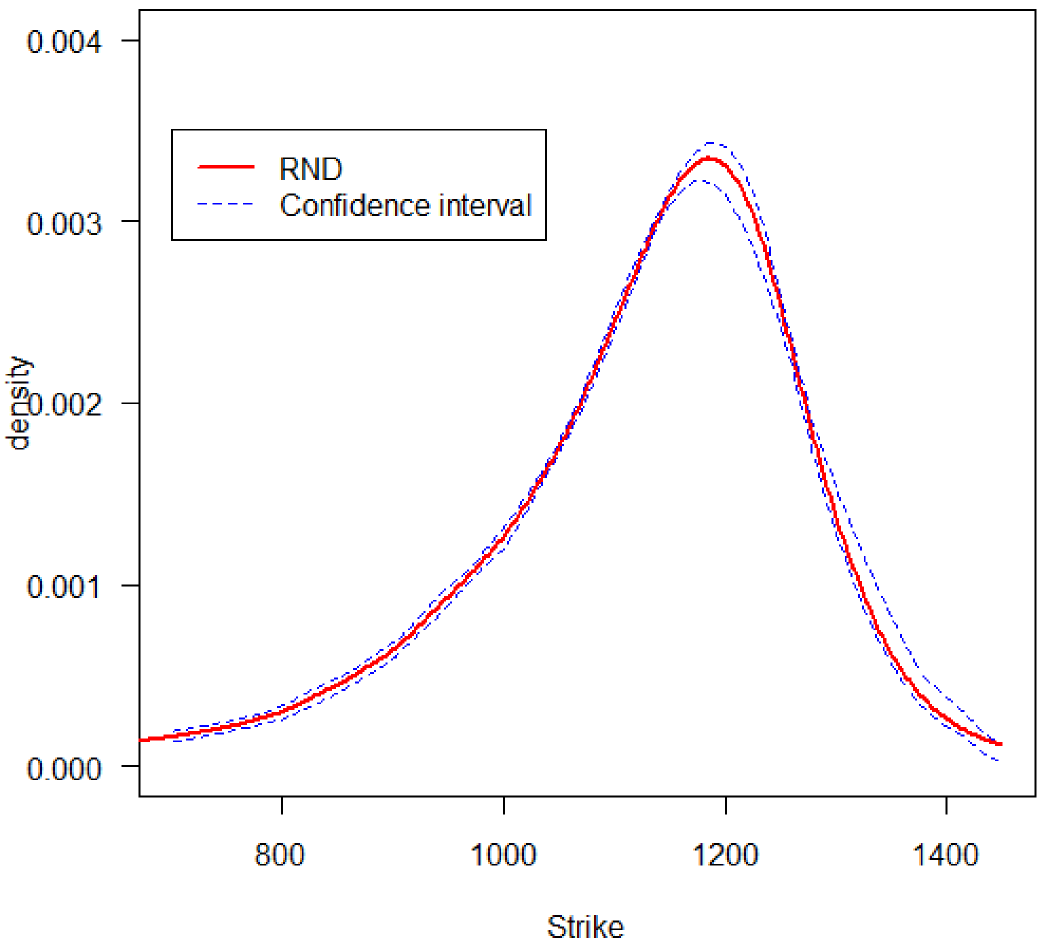

Figure 2.10: RND and Confidence Interval II

2.5 Application: Risk Neutral Density from currency options: The Brazilian case

Option contracts for currencies are commonly expressed by market participants as Black-Scholes implied volatilities. Moreover, the strike prices of the option contracts for a given day are typically expressed in terms of the Black-Scholes delta. therefore instead of working directly in the option price-volatility space, we will work in the option delta-volatility space, and using the Black-Scholes formula to translate from one space to the other.

Currencies options are typically traded in Bloomberg as combinations butterfly (BF) and risk reversals options (RR), which are combinations of out-of-the-money options. we can recover the price of individual options with specific deltas using the following formulas:

$$\sigma_{call,\delta} = \sigma_{ATM,\delta} + \sigma_{BF,\delta} + \frac{1}{2}\sigma_{RR}$$
$$\sigma_{put,\delta} = \sigma_{ATM,\delta} + \sigma_{BF,\delta} - \frac{1}{2}\sigma_{RR}$$

Moreover, the at-the-money (ATM) and the at-the-money-forward (ATMF) options have deltas close to, but not exactly, equal to 0.50. We obtain an option with a delta near 50 from the ATMF option, using the following formula:

$$\delta(.) = \frac{du}{dSt}C(S_t, T-t, K, \sigma, r_t, q_t)$$

Then, for this data structure it is more convenient to represent the volatility surface as a function $\sigma_t(\delta, T-t)$ for the delta rather than strike price. Computation of option prices in currency units can be done using the Black-Scholes formula.

We apply the methodology described above to the 1-month options on the USDBRL, the price of the dollar in terms of Brazilian real, from 2013 to 2016. This period is relevant due to it contains a period of high volatility in Latinamerican exchange rate markets after the announcement of end of the monetary policy easing implemented by the US Fed (”*taper tantrum*”). Once the input data has been prepared in terms of deltas and volatilities - (δ, σ)-space, the volatility smile can be interpolated using a clamped cubic spline (as described in section 2).

This approach is illustrated in Figure 2.11 for two dates, March 29, 2013 and August 30, 2013. These dates are selected to show how the *taper tantrum* announced by the Fed at the end of May, 2013 increased the volatility in the emerging markets exchange rates. We can note from the graph, that in the later date the implied volatility smile shows an increase in the volatility and a more disperse volatility surface with respect to the earlier date.

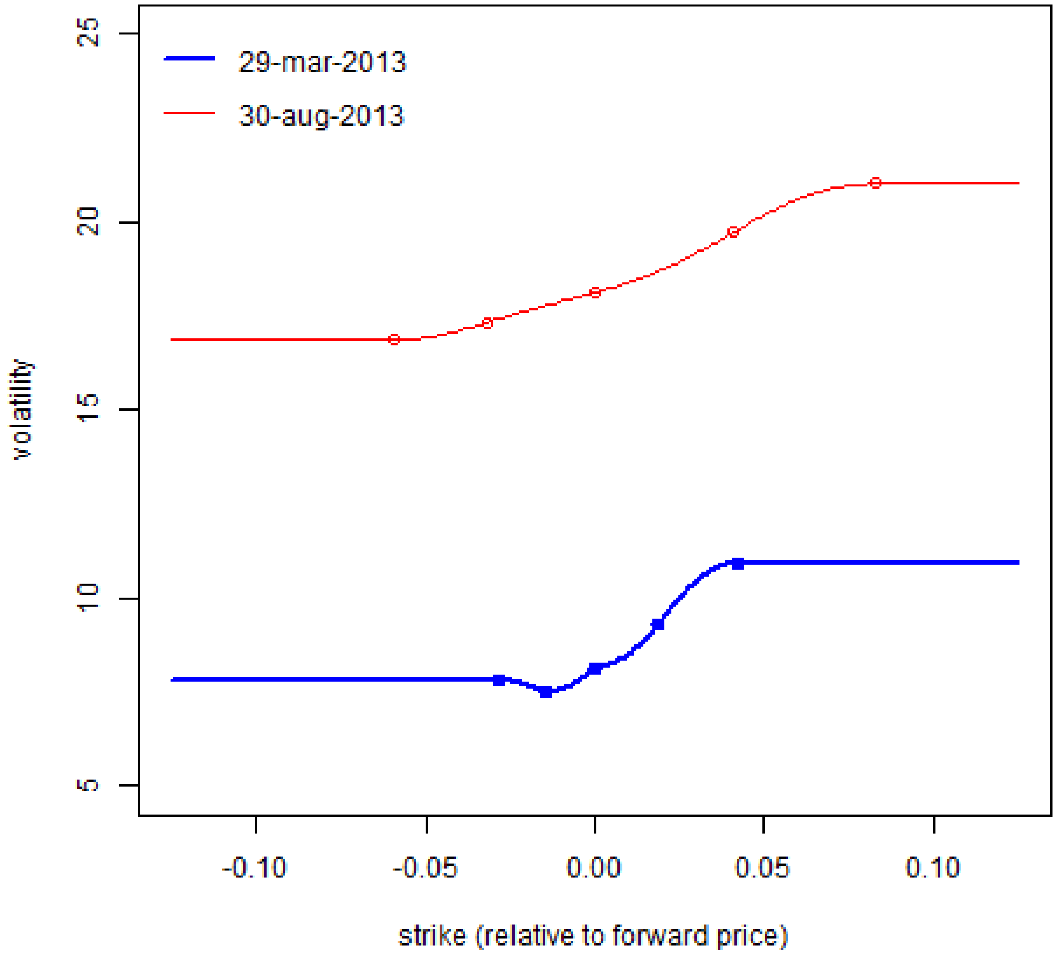

Figure 2.11: Brazilian Real: implied volatility smile

Computations using these data are illustrated in Figure 2.12 and 2.13 for the same two dates as in figure 2.11. The x-axis is expressed as the proportional difference from the 1-month forward rate (BRL per USD). The RND are computed using $\Delta = 0.005$ (as a fraction of the forward rate). Option prices from USD-BRL, forward exchange rates and the diagnostics in Table 2.2 are calculated using 1-month Brazilian and US interest rates as the financing and underlying cash flow rates.

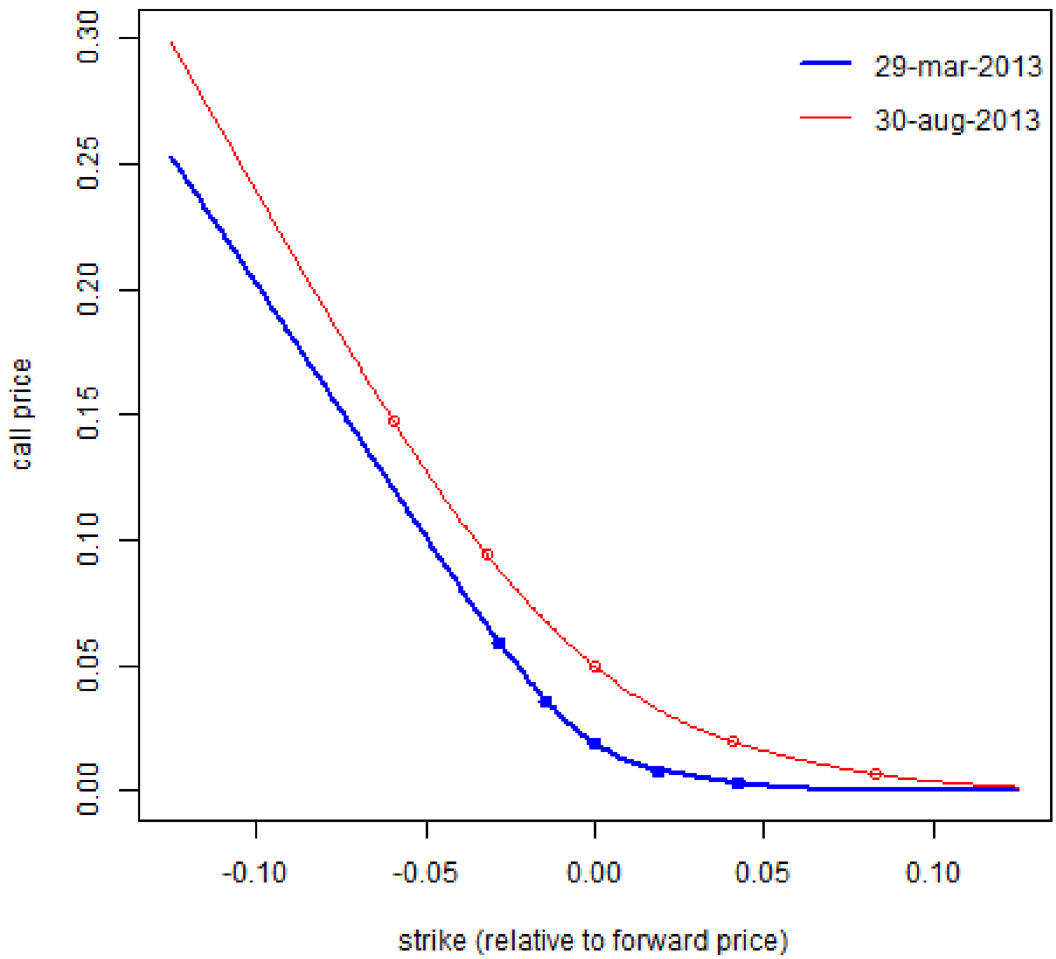

Figure 2.12: Brazilian Real: call prices using smoothing splines

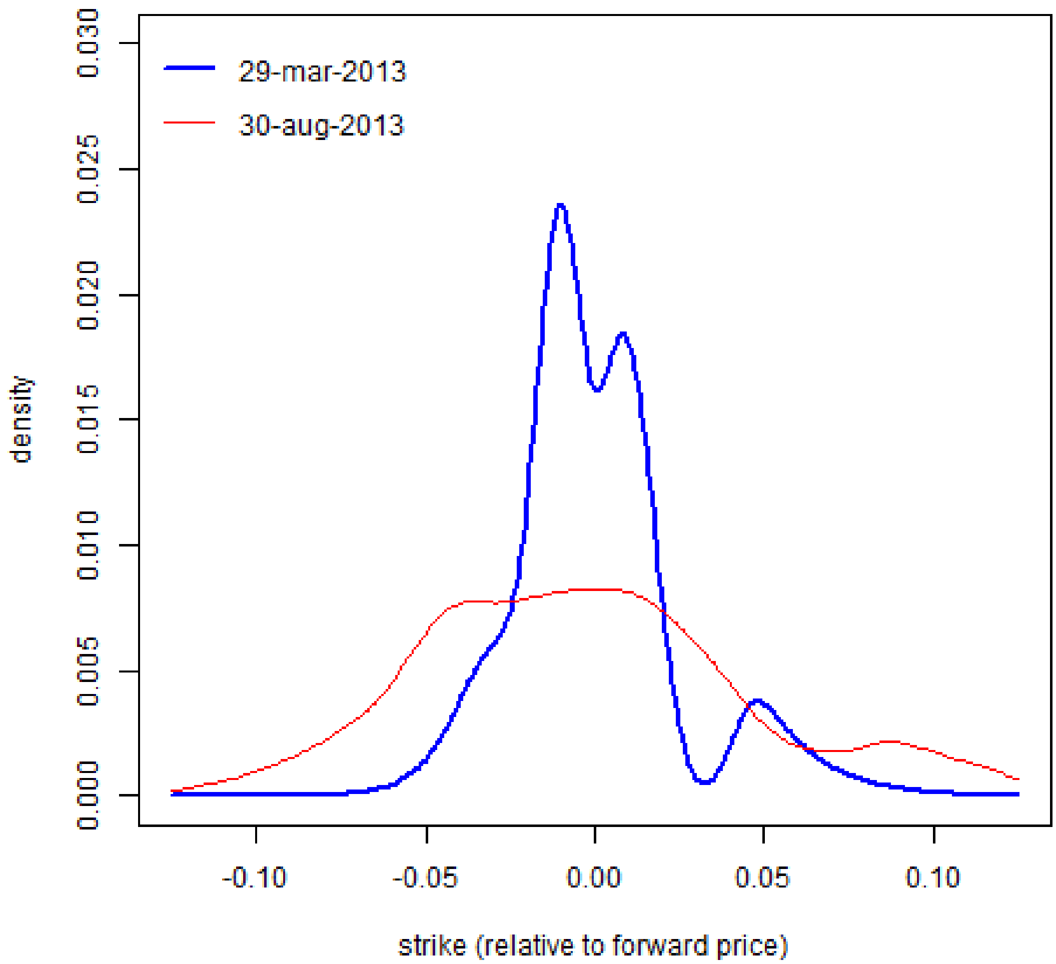

Figure 2.13: Brazilian Real: risk neutral densities

Table 2.2: Data and diagnostics for Brazilian exchange rate

29 − Mar − 2013

X	$\frac{\mathbf{X}}{\mathbf{F}} - \mathbf{1}$	**Volatility**	**Call Value**	**Delta**	**Lower Bound**	**Upper Bound**	$\mathbf{\Pi(X)}$
1.9708	-0.0281	7.8125	0.05930	0.9000	0.0000	0.1576	0.1040
1.9991	-0.0142	7.5233	0.03556	0.7500	0.1576	0.4221	0.2570
2.0278	0.0000	8.1000	0.01901	0.5072	0.4221	0.7110	0.5020
2.0660	0.0188	9.3217	0.00801	0.2500	0.7110	0.8927	0.7584
2.1129	0.0420	10.943	0.00298	0.1000	0.8927	1.0000	0.9054

30 − Aug − 2013

X	$\frac{\mathbf{X}}{\mathbf{F}} - \mathbf{1}$	**Volatility**	**Call Value**	**Delta**	**Lower Bound**	**Upper Bound**	$\mathbf{\Pi(X)}$
2.2603	-0.0594	16.8830	0.14744	0.9000	0.0000	0.1930	0.1087
2.3264	-0.0320	17.3370	0.09457	0.7500	0.1930	0.4131	0.2661
2.4032	0.0000	18.1250	0.04981	0.5105	0.4131	0.6907	0.5103
2.5013	0.0408	19.7120	0.01968	0.2500	0.6907	0.8704	0.7677
2.6024	0.0829	21.0230	0.00669	0.1000	0.8704	1.0000	0.9102

From figure 2.13, we can note that the two dates display a sharp contrast in the direction and skewness of the risk-neutral distribution. On the earlier date, there is a concentration in the distribution and skewed to the right (risk of depreciation), while on the later date the distribution has increased its variance and also shows fatter tails.Diagnostics for the data and the computations are showed in Table 2.2.

Using the entire data we estimate the time series for the 1-month risk neutral density standard deviation as showed in figure 2.14. We can note from that graph that the first months of the 2013 the Brazilian real experience low volatility, but from end of May (following the *taper tantrum*) the volatility increased until earlier 2014. Moreover, since mid-2014 volatility started to increase significantly due to the political turmoil related to the *lava jato* corruption scandal which eventually resulted in the impeachment of the President in May-2016. after that, volatility decreased but without reaching the levels registered prior to the political turmoil.

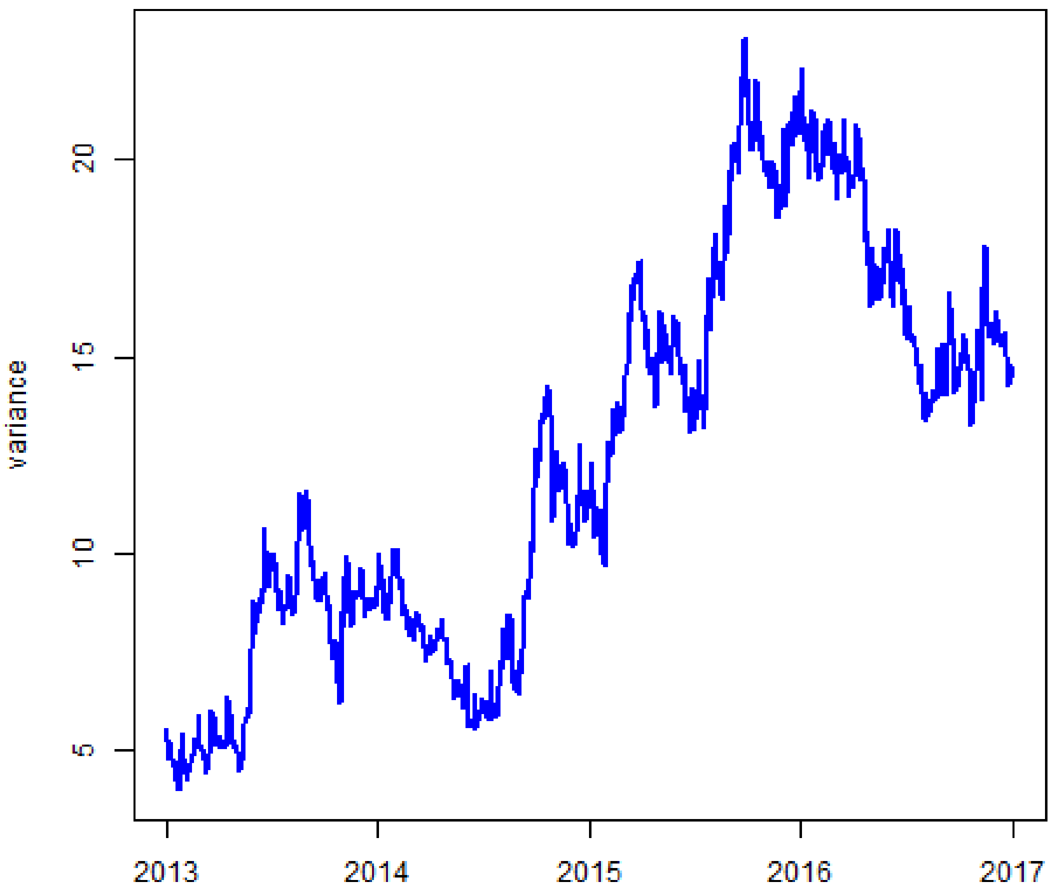

Figure 2.14: Brazilian Real: variance times series implied from option prices

To assess how effective is the variance measured using the risk neutral density, we proceed to compare this result with a estimated conditional variance from a GARCH model, which is a econometric model commonly used to represent financial assets with high frequency data. Thus, we estimate the GARCH model that fits best to the return of the Brazilian exchange rate, which turn out to be a GARCH(1,1) with zero mean. Figure 13 compare the RND standard deviation with the conditional standard deviation from the GARCH(1,1) model. We can note from the graph that both measures show similar patters most of the time. However, there are some differences to highlight. First, it seems that the RND volatility precedes the GARCH volatility, in both during an upturn and during a downturn. Since the GARCH models is a backward looking measure while the RND measure is a forward looking measure, this difference illustrated the importance using risk neutral densities to assess the risk associated to financial assets such as the exchange rates. Second, we can observe from figure 2.15 that both measured differ during the beginning of 2016. While the GARCH models implied that the volatility decreased significantly at the end of 2015 and increased again at the beginning of 2016, the RND measure shows that the volatility stay high during that period. While GARCH model reverse can be explained by a series of returns of the same low magnitude, risk neutral density shows that even in the case the exchange rate level did not move much those days, the risk associate to the exchange rate these days were indeed high instead of low as implied by the GARCH model.

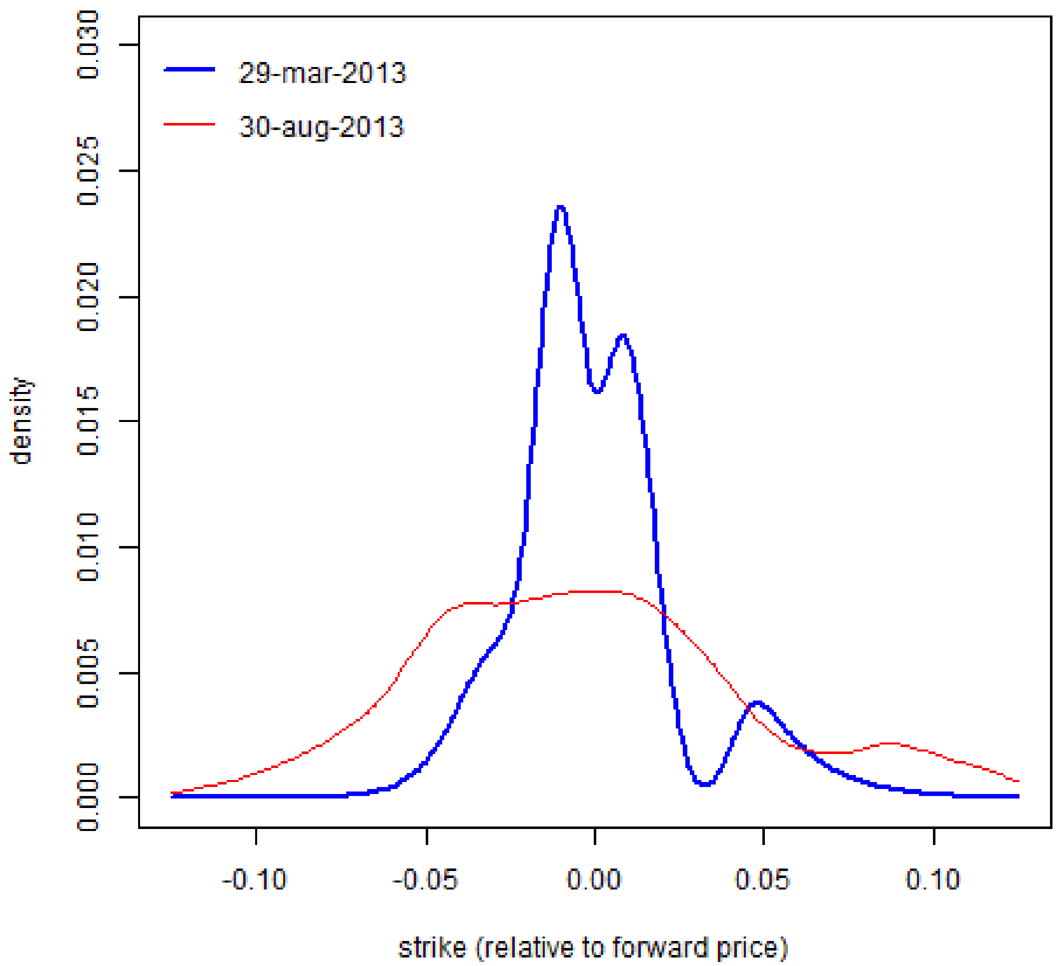

Figure 2.15: Brazilian Real: variance times series implied from option prices

2.6 Conclusions

Risk Neutral Densities from option prices have become a useful tool for measuring market expectations. In this chapter we overview the main methods used to estimate these RNDs, recognizing that smoothing splines methods have gained popularity among researchers. Moreover, we applied the smoothing spline methods for extracting the RND for the SP&500 stock price index. In doing so, a Montecarlo analysis was implemented to highlight that the smoothing splines method is very accurate. Moreover, the smoothing splines methodology was performed to estimate the volatility of the exchange rate for Brazil to show how useful this measure can be to identify risks associated to exchange rates, a very important asset from the point of view of a policy maker from an emerging market economy. Therefore, RND can be considered as a useful tool for showing changes in investor's expectations.

References

Aït-Sahalia, Y. and Lo, A. (1998). Nonparametric estimation of state-price densities implicit in financial asset prices. *The Journal of Finance*, 53(2):499–547.

Aït-Sahalia, Y., Wang, Y., and Yared, F. (2001). Do option markets correctly price the probabilities of movement of the underlying asset? *Journal of Econometrics*, 102(1):67–110.

An, S. (2010). Optimal Monetary Policy in a Model with Recursive Preferences. *Research Collection School of Economics*, 1259.

Bahra, B. (1997). Implied risk-neutral probability density functions from option prices: theory and application.

Bansal, R. and Yaron, A. (2004). Risks for the Long Run: A Potential Resolution of Asset. *The Journal of Finance*, LIX(4):1481–1509.

Bernal-Verdugo, L. (2011). Housing and Asset Prices in a Production.

Black, F. and Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81(3):637–654.

Bliss, R. R. and Panigirtzoglou, N. (2002). Testing the stability of implied probability density functions. *Journal of Banking & Finance*, 26(2-3):381–422.

Breeden, D. and Litzenberger, R. (1978). Prices of state-contingent claims implicit in option prices. *Journal of business*, 51(4):621–651.

Caldara, D., Fernández-Villaverde, J., Rubio-Ramírez, J. F., and Yao, W. (2012). Computing DSGE models with recursive preferences and stochastic volatility. *Review of Economic Dynamics*, 15(2):188–206.

Campbell, J. Y. and Cochrane, J. H. (1999). By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior. *Journal of Political Economy*, 107(2):205.

Constantinides, G. (1990). Habit formation: A resolution of the equity premium puzzle. *Journal of political Economy*, 98(3):519–543.

Cooper, N. (1999). Testing Techniques for Estimating Implied RNDs from the Prices of European-Style Options.

Croce, M. M. (2006). *Welfare Costs, Long Run Consumption Risk, and a Production Economy.* PhD thesis, NYU Stern.

Davis, M. a. and Heathcote, J. (2005). Housing and the business cycle. *International Economic Review*, 46(3):751–784.

Epstein, L. G. and Zin, S. E. (1989). Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica*, 57(4):937–69.

Epstein, L. G. and Zin, S. E. (1991). Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: An Empirical Analysis. *Journal of Political Economy*, 99(2):263–86.

Iacoviello, M. and Neri, S. (2010). Housing Market Spillovers: Evidence from an Estimated DSGE Model. *American Economic Journal: Macroeconomics*, 2(2):125–164.

Jaccard, I. (2011). Asset Pricing and Housing Supply in a Production Economy. *The BE Journal of Macroeconomics*, 11(1).

Jermann, U. J. (1998). Asset pricing in production economies. *Journal of Monetary Economics*, 41(2):257–275.

Kaltenbrunner, G. and Lochstoer, L. (2010). Long-run risk through consumption smoothing. *Review of Financial Studies*, 23(8):3190–3224.

Kuehn, L.-a. (2008). Asset pricing with real investment commitment.

Lucas, R. (1987). *Models of Business Cycles*. Oxford: Basil Blackwell.

Malz, A. M. (1997). Option-Implied Probability Distributions and Currency Excess Returns. Technical report, Staff Reports, Federal Reserve Bank of New York.

Mehra, R. and Prescott, E. C. (1985). The equity premium: A puzzle. *Journal of Monetary Economics*, 15(2):145–161.

Melick, W. and Thomas, C. (1997). Recovering an asset's implied PDF from option prices: an application to crude oil during the Gulf crisis. *Journal of Financial and Quantitative Analysis*, 32(1):91–115.

Piazzesi, M., Schneider, M., and Tuzel, S. (2007). Housing, consumption and asset pricing. *Journal of Financial Economics*, 83(3):531–569.

Rouwenhorst, K. G. (1995). Asset pricing implications of equilibrium business cycle models. In Cooley, T. F., editor, *Frontiers of business cycle research*, pages 293–330. Princeton University Press.

Shimko, D. (1993). Bounds of probability. *Risk*, (6):33–37.

Tallarini, T. (2000). Risk-sensitive real business cycles. *Journal of Monetary Economics*, 45:507–532.

van Binsbergen, J. H., Fernández-Villaverde, J., Koijen, R. S., and Rubio-Ramírez, J. (2012). The term structure of interest rates in a DSGE model with recursive preferences. *Journal of Monetary Economics*, 59(7):634–648.

Appendix A

Optimality conditions

1. Value Function:

$$v_t = \left\{(1-\beta)(c_t(1-l_{1t}-l_{2t})^{v})^{\frac{1-\rho}{1+\nu}} + \beta(w_t)^{1-\rho}\right\}^{\frac{1}{1-\rho}}$$

2. Auxiliary equation:

$$w_t = \left[E_t v_{t+1}^{1-\chi}(z_{t+1}e^{\gamma})^{\frac{1-\chi}{1+\nu}}\right]^{\frac{1}{1-\chi}}$$

3. Aggregate consumption:

$$c_t = \left[g_t^{1-\theta} + \omega\left(\frac{h_{t-1}}{z_t e^{\gamma}}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

4. labor decision:

$$\nu\frac{c_t^{1-\theta}g_t^{\theta}}{1-l_{1t}-l_{2t}} = (1-\alpha_g)\frac{y_t}{l_{1t}}$$

$$\nu\frac{c_t^{1-\theta}g_t^{\theta}}{1-l_{1t}-l_{2t}} = \frac{1-\alpha_h}{\alpha_h}\frac{m_t}{l_{2t}}$$

5. non-housing sector production function:

$$y_t = \left(\frac{k_{t-1}}{z_t e^{\gamma}}\right)^{\alpha_g}(l_{1t})^{1-\alpha_g}$$

6. housing sector production function:

$$n_t = m_t^{\alpha_h}(l_{2t})^{1-\alpha_h}$$

7. market clearing condition:

$$g_t + i_t + m_t = y_t$$

8. Stochastic discount factor:

$$Q_t = \beta \left(z_t e^{\gamma}\right)^{-\frac{\chi+\nu}{1+\nu}} \left(\frac{v_t}{w_{t-1}}\right)^{\rho-\chi} \left(\frac{1-l_{1t}-l_{2t}}{1-l_{1t-1}-l_{2t-1}}\right)^{\frac{(1-\rho)\nu}{1+\nu}} \left(\frac{g_t}{g_{t-1}}\right)^{-\frac{\rho-\nu}{1+\nu}} \left[\frac{1+\omega\left(\frac{h_{t-1}}{g_t z_t e^{\gamma}}\right)^{1-\theta}}{1+\omega\left(\frac{h_{t-2}}{g_{t-1} z_{t-1} e^{\gamma}}\right)^{1-\theta}}\right]^{\frac{\theta(1+\nu)-(\rho+\nu)}{(1+\nu)(1-\theta)}}$$

9. capital asset pricing:

$$q_t^K = \frac{1}{G_K'(t)} = \frac{1}{a_1^K}\left(\frac{i_t}{k_{t-1}} z_t e^{\gamma}\right)^{\frac{1}{\xi_K}}$$

$$r_t^K = \alpha_g \frac{y_t}{k_{t-1}} z_t e^{\gamma}$$

$$R_t^K = \frac{r_t^K + q_t^K\left[\frac{k_t}{k_{t-1}} z_t e^{\gamma}\right] - \frac{i_t}{k_{t-1}} z_t e^{\gamma}}{q_{t-1}^K}$$

$$E_t\left\{Q_{t+1} R_{t+1}^K\right\} = 1$$

10. Housing asset pricing:

$$p_t^N = \frac{1}{\alpha_h\left(\frac{n}{m}\right)}$$

$$q_t^H = \frac{p_t^N}{G_H'(t)} = \frac{p_t^N}{a_1^H}\left(\frac{n_t}{h_{t-1}} z_t e^{\gamma}\right)^{\frac{1}{\xi_H}}$$

$$r_t^H = \omega\left(\frac{h_{t-1}}{g_t z_t e^{\gamma}}\right)^{-\theta}$$

$$R_t^H = \frac{r_t^H + q_t^H\left[\frac{h_t}{h_{t-1}} z_t e^{\gamma}\right] - p_t^N \frac{n_t}{h_{t-1}} z_t e^{\gamma}}{q_{t-1}^H}$$

$$E_t\left\{Q_{t+1} R_{t+1}^H\right\} = 1$$

11. Risk free interest rate:

$$\frac{1}{r_{t+1}^f} = E_t\left[Q_{t+1}\right]$$

12. capital accumulation:

$$k_t = \left[(1-\delta_K) + \frac{\alpha_1^K}{1-\frac{1}{\xi_K}} \left(\frac{i_t}{k_{t-1}} z_t e^{\gamma} \right)^{(1-\frac{1}{\xi_K})} + \alpha_2^K \right] \frac{k_{t-1}}{z_t e^{\gamma}}$$

13. Housing accumulation:

$$h_t = \left[(1-\delta_H) + \frac{\alpha_1^H}{1-\frac{1}{\xi_H}} \left(\frac{n_t}{h_{t-1}} z_t e^{\gamma} \right)^{(1-\frac{1}{\xi_H})} + \alpha_2^H \right] \frac{h_{t-1}}{z_t e^{\gamma}}$$

14. technology dynamics:

$$log(z_t) = \rho_z log(z_{t-1}) + \sigma_\epsilon \epsilon_t$$

Appendix B

Steady State

$$z = 1$$

$$Q = \beta e^{-\frac{\gamma(\rho+\nu)}{1+\nu}}$$

$$R^f = R^K = R^H = \frac{1}{Q}$$

$$l_1 = \frac{S_3F_2 - S_2F_3}{S_1F_2 - S_2F_1}$$
$$l_2 = \frac{S_1F_3 - S_3F_1}{S_1F_2 - S_2F_1}$$

$$y = \frac{\phi_2}{\phi_3} l_1$$

$$m = \phi_5 l_2$$

$$n = \phi_5^{\alpha_h - 1} m$$

$$g = \frac{\phi_7 \phi_2}{\phi_3} \left(1 - l_1 - l_2\right)$$

$$h = \phi_6 g$$

$$k = \frac{y}{\phi_2}$$

$$i = \phi_0 k$$

$$q^K = 1$$

$$r^K = \frac{1}{Q} - e^{\gamma} \left(1 - \phi_0\right)$$

$$p^N = \frac{\phi_5^{1-\alpha_h}}{\alpha_h}$$

$$q^H = p^N$$

$$r^H = p^S \left[\frac{1}{Q} - e^{\gamma} \left(1 - \phi_8 \right) \right]$$

$$c = \left[g^{1-\theta} + \omega \left(\frac{h}{e^{\gamma}} \right)^{1-\theta} \right]^{\frac{1}{1-\theta}}$$

$$v = \phi_9 \left[c(1 - l_1 - l_2)^{\nu} \right]^{\frac{1}{1+\nu}}$$

$$w = v e^{\frac{\gamma}{1+\nu}}$$

where all the constants $\phi_{\{1,\ldots,9\}}$, $S_{\{1,2,3\}}$ and $F_{\{1,2,3\}}$ are functions of the structural parameters from the model.