THREE ESSAYS ON DEBT PRICING

BY

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DISSERTATION
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ABSTRACT

This dissertation contains three chapters that study risky debt pricing. The first chapter studies defaultable consumer debt in general equilibrium. The second and third chapters study corporate debt in partial equilibrium. Below are the individual abstracts for each chapter.

Chapter 1: What Drives the Consumer Credit Spread? An Explanation Based on Rare Event Risk and Belief Dispersion.

What drives consumers borrowing/lending and the credit spread over their debt? This paper offers a novel explanation based on rare event risk and belief dispersion in a dynamic general equilibrium model. Heterogeneous beliefs drive consumers to borrow, but the market is incomplete and subject to rare event risk and thus default endogenously occurs in equilibrium. The paper derives the credit spread in closed form and yields a credit spread similar to real data when the model is calibrated. The model also well captures the relationship between belief dispersion, risk-free rate and credit spread. It shows that belief dispersion, rare event risk and wealth distribution together drive both credit spread and risk-free rate. An increase in either rare event risk or belief dispersion leads to a higher credit spread and a lower risk-free rate. However, the underlying mechanisms are quite different, as the former (rare event risk) is due to substitution effect while the latter (belief dispersion) is due to wealth effect. The paper also makes a contribution to the literature on rare disaster by endogenizing default and augments Barro’s argument on the countervailing effects of rare disasters on interest rates.

Chapter 2: Scooping Up Own Debt On the Cheap: The Effect Of Corporate Bonds Buy-
back on Firm’s Credit Condition

The paper constructs a structural model to study the effect of corporate bonds buyback on the firm’s credit conditions. The model implies that the firm strategically choose how much debt to buy back and the buyback reduces the firm’s probability of default. In contrast to commonly perceived deleverage channel, the model highlights a novel channel that buying back bonds on the cheap transfers value from bondholders to equity holders and incentivizes the equity holders to choose a much lower assets value to declare default. The lowered default boundary furthermore reduces debt overhang and increases return to equity. The virtuous cycle does not stop until the marginal benefit of bonds buyback equals its marginal cost. The model also implies that when bonds market liquidity dries up, the firm should buy back more bonds, as the shortage of liquidity is independent of the firm’s fundamental but depresses the market price of bonds. The paper also provides empirical evidences for the implications.

Chapter 3: How to roll over debt? The Effect of Risky Bond Yield Curve on Optimal Rollover Strategy

This chapter studies how firms can exploit the risky bond yield curve (treasury yield curve plus term structure of credit spread) to manage the maturity profile of new debt issuance. In contrast to existing literature, I shut down the rollover frequency channel, but highlight the clean effect of the maturity profile of new debt issuance on firm’s credit risk and value. A better rollover strategy takes into account of the curvature, level as well as the sensitivity (market depth) of the risky bond yield curve. The model implies that the firms should disperse maturity dates of new debt issuance when the risky bond yield curve
is concave; they also should trade off the level against the sensitivity of the risky bond yield curve when issuing new debt. The consequent rollover strategy assuages both adverse assets shocks and liquidity shocks better. The model derives multiple testable implications and accentuate the endogenous interaction between rollover strategy and risky bond yield curve.
To My Mom, and My Late Dad
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Chapter 1

What Drives the Consumer Credit Spread? An Explanation Based on Rare Event Risk and Belief Dispersion

1.1 Introduction

Recent years have witnessed rapid increase in consumer debt. In fact, the total outstanding consumer debt is significantly more than corporate debt (see Figure 1.1). Given the economical importance of consumer debt as a macroeconomic variable, it is critical to understand the cost of consumer borrowing and the economic factors that drive it.

Compared to the vast literature on corporate bonds credit spread, the research on consumer credit spread is thin. Corporate and consumer credit spread share many similarities, but they do differ and one cannot simply transplant the theory of corporate bonds credit spread to households and consumers. For example, although corporate and consumer credit spread is mostly positive correlated, the correlation is far less than perfect. Table 1.1 shows the correlation coefficients between different measures of consumer credit spread and corporate bonds credit spread. Furthermore, from the perspective of economic theory, while
capital structure is deemed as the major determinant of corporate bonds credit spread, a
theory of consumer credit spread (price) is supposed to take into account household con-
sumption/saving decision (quantity), portfolio choices and risk aversion.

In this paper, we build up a dynamic general equilibrium model to explain the factors that
drive the consumer credit spread qualitatively and quantitatively. We model the consumer
as an agent who makes decisions on consumption and portfolio choices. The model features
two types of agents with heterogeneous beliefs (the optimist vs. the pessimist) and rare
event (jump) risk in an incomplete market. The heterogeneous beliefs cause the agents to
trade endogenously while the rare event risk, which occurs infrequently but drastically affects
consumption, generates significant credit spread. The securities space is not complete to fully
hedge the rare event risk, leaving room for default to happen in the equilibrium. We calibrate
our model to match the data on lending risk premium, the difference between prime rate \(^1\)
and treasury bill rate. The lending risk premium is a key component of borrowing cost for
all consumers and compensates the systematic risk, which accurately interprets the credit
spread in our model. Table \(1.2\) summarizes the lending risk premium and the delinquency
rate of consumers who can borrow at the prime rate. To our knowledge, our paper is the
first asset pricing model that studies as well as quantifies safe debt and risky household debt
and their prices in general equilibrium. In particular, our model features the relationship
between systematic risk and risky household debt.

\(^1\)The prime rate, is a “reference or base rate” that banks use to set the price or interest rate on many
of their commercial loans and some of their consumer loan products (See What is the prime rate, and
does the Federal Reserve set the prime rate? Board of Governors of the Federal Reserve System, https:
//www.federalreserve.gov/faqs/credit_12846.htm). It is also the rate that banks charge the customer
with the highest credit ratings for short-term credit (Booth (1994)).
Since the market in the paper is incomplete, there is no guarantee that equilibrium exists. Hence, some discussions regarding the existence of the equilibrium are necessary meanwhile assets prices are derived in equilibrium, if any. We first start with a simple example to illustrate the main idea of the paper, considering a typical problem in the portfolio choice literature. An infinite-horizon representative agent has logarithmic utility and tries to maximize the expected utility by choosing consumption and adjusting investment portfolios. There are two assets in the market: risk-free debt and risky asset, e.g. stock and/or housing, that subjects to jump risk. The example shows that the agent’s position on risky asset is highly constrained by the potential jump size. Given the potential jump size is sufficiently large, even if such a jump only realizes with a tiny probability, the agent is reluctant to take any leverage! The partial equilibrium simple example poses difficulty in risk sharing when the market only features risk-free debt and risky asset which subjects to rare event risk. The problem lies in the fact that a risk-free debt contract is too much to ask under such circumstance. With the intuition, we formally show that the same holds in general equilibrium: there is no such interest rate that clears the risk-free debt market.

In this sense, the defaultable bonds endogenously emerge in the market and therefore we introduce defaultable debt securities in the model with one being against large downward jump and the other one being against positive jump. When belief dispersion is mild, the defaultable bond against downward jump can restore the competitive equilibrium. In the equilibrium, the optimistic agents will issue risky bonds to finance the long position in risky asset. The optimist’s position in risky asset and the downward jump size that just triggers
default are closely related and will be jointly determined in equilibrium. If a substantial downward jump occurs out of a sudden, the optimists will default and part of the risky debt are written down. As a consequence, the pessimists, who are the buyers of the risky bonds, will suffer a loss and therefore require a risk premium ex-ante.

However, when belief dispersion is large, the defaultable bond against upward jump is also indispensable to establish equilibrium. The upward jump echoes “rare boom” in Tsai and Wachter (2015b). When disagreement between agents is significant, the optimists would perceive the total return to risky asset is much higher than the financing cost of leverage and thus would like to take an aggressive leverage position. Nevertheless, the supply of risky asset is finite (normalized to “1”) and hence the pessimists need to short sell some shares to “create” the supply. Yet, given the possibility of large positive jump, the pessimists will opt out of short-selling without the defaultable bonds against upward jump. When short selling, the pessimists need to issue risky bonds to cover the short position. In the case that an upward jump realizes, the pessimists will deliver the short shares and default on the risky bonds. We quantify the upper bound of the belief dispersion above which the equilibrium fails to exist without the risky bonds against positive jump. By doing so, the model also provides a framework to evaluate to what extent regulations on short selling limits risk sharing among agents.

The model is built on time-varying belief dispersion and rare event risk (Gabaix (2012), Wachter (2013)). The agents are able to learn from various sources and update their beliefs about the unknown parameters but are subject to behavioral bias (Scheinkman and Xiong)
(2003), Pastor and Veronesi (2009)), resulting in time-varying belief dispersion. By assuming the jump size follows a generalized logistic distribution, this paper manages to achieve credit spread in closed-form. Finally, the calibrated model yields a time series of household credit spread, belief dispersion and risk-free rate comparable to real data.

One main contribution of the model is that it generates a correlation between credit spread, belief dispersion as well as risk-free rate similar to what is observed in the data. Our model implies that asset prices, including credit spread and risk-free rate, are driven by belief dispersion, rare event risk as well as the relative wealth between the two types of agents. To better understand the relationship between asset prices and the three fundamental economic variables, we run regression of credit spread as well as risk-free rate on belief dispersion, rare event risk and the relative wealth ratio. The regression indicates that both greater belief dispersion and rare event risk tend to increase the credit spread and lower the risk-free rate. However, the underlying mechanism is quite different. When the rare event risk increases, it is more likely for the rare event to occur and trigger the default on defaultable bonds. The risky bonds holders thus would require a higher premium compensation for the increased risk. Therefore, it is more costly for the optimist to leverage up and consequently they re-balance their portfolios toward more risk-free bonds holding, pushing down the return on the risk-free bonds. In essence, the increased rare event risk accentuates the substitution between risky asset and risk-free bonds.

On the other hand, if belief dispersion between the agents gets wider, the optimist would regard the cost of leverage is cheaper with respect to the return on risky asset and there-
fore would like to borrow more via issuing more defaultable bonds. This translates into a higher default risk and higher credit spread; However, given more resources at disposal from borrowing and wider belief dispersion, there is also a “wealth effect”: the optimist not only purchases more shares of risky asset, but also more risk-free bonds to optimize his portfolios. The increased demand on safe bonds would also push down the risk-free rate. This insight is novel and different from other similar models in the early literature, which typically features risky asset and risk-free bonds in a complete market. In those models, risky asset and risk-free debt are always substitutes in the sense that increasing risky asset holdings decreases risk-free bonds holdings given wealth, as the agent has no other financing vehicle. In contrast, the optimist in our model invests in both risky asset and risk-free bonds by borrowing via risky bonds; the pessimist, on the other hand, would mainly invest in risky bonds and risk-free bonds instead of risky asset.

The relationship between corporate bonds credit spread, risk-free rate and belief dispersion have been empirically documented extensively in the literature. For example, Buraschi, Trojani, and Vedolin (2013) found that the belief dispersion has a time-varying systematic counter-cyclical component and peaks near the Great Recessions. Güntay and Hackbarth (2010) reached a same conclusion on disagreement and credit spread, showing the forecast dispersion can explain about 23% of the cross sectional variation in credit spreads; Albagli, Hellwig, and Tsyvinski (2014) developed a nonlinear and noisy rational expectation equilibrium model and found that the belief dispersion can explain 16% to 42% and 35% to 46% of the observed credit spread over 4-year and 10-year investment grade bonds, respectively.
Buraschi and Whelan (2013) found that short rate is negatively related to disagreement; Xiong and Yan (2010) found that the belief dispersion between agents can explain the term structure of risk-free rate; Among others, Collin-Dufresne, Goldstein, and Martin (2001), Duffee (1998), Morris et al. (1998) found that credit spread is inversely related to risk-free rate. Apparently, the three variables are underlyingly related. However, none of the studies accounts for them at the same time. Our study not only empirically documents the relationship between the consumer debt credit spread, risk-free rate and belief dispersion but also provides a novel explanation based on rare event risk and heterogeneous beliefs in a single model.

This paper has several other contributions. One important theoretical contribution of the paper is to endogenize default in the rare event risk model. Earlier research on rare event risk \(^2\) takes an ad hoc approach in modeling default, assuming that default occurs with an exogenous probability during disaster time and some fraction of the gross return on debt is wiped out. As noted in Tsai and Wachter (2015a),

“In this case, the assumption of complete markets still implies that there exists a risk-free rate; it just is not comparable to the government bill rate. What happens if markets are incomplete and there is no risk-free rate? This is a hard question to answer because the representative investor framework no longer applies, and we are not aware of any work that addresses it.”

---

In this paper, we precisely address the problem. Default on debt occurs endogenously in equilibrium. The probability of default as well as exposure at default is time-varying and strongly correlated with rare-event risk and belief dispersion. Endogenizing default is important in that it allows to analyze the channel through which the default on debt occurs. For example, although the effects of rare event risk and belief dispersion on credit spread and risk-free rate are similar, the underlying channels are quite different. Moreover, by endogenizing default, we are able to elaborate on Barro’s argument on the effect of rare event risk on the interest rate. Barro (2006) argues that on one hand, an increase in likelihood of a disaster would lower the agent’s expectation and decrease interest rate; On the other hand, the probability of default due to disaster also rises and hence increases the interest rate. The net effect is ambiguous. In the current paper, we show that both effects exist but they affect different interest rates on different bonds.

Closest to the current paper is Chen, Joslin, and Tran (2012). They employed a consumption-based asset pricing model and focused on the effect of disagreement on the likelihood and severity of rare disasters between agents on the disaster risk premium and risk sharing. Our paper differs from theirs in several ways. First, while they focused on the disagreement on the likelihood and severity of rare disasters, we emphasize the belief dispersion on the expected growth rate; Second, while they discussed extensively the effect of disagreement and rare disasters on the stock market, we concentrate on the riskless and risky debt market. Last, but certainly not least, they extended Bates (2008) and introduced a continuum of
contingent claims for the agents to fully hedge the disaster risk\footnote{Alternatively, \cite{Jones1984} shows that if an underlying asset’s jump size has a finite state distribution, a sufficient number of different contingent claims written on this asset can help to fully hedge jump risk and complete the market.} whereas in our model, the agents can only trade two risky debt securities in addition to stock and safe debt, leaving the market incomplete and making default on debt in equilibrium possible.

Chatterjee et al. (2007b) studied unsecured consumer loans and default in a general equilibrium model based on Imrohoroglu (1989), Huggett (1993) and Aiyagari (1994). They assumed that households smooth consumption by means of a riskless asset and unsecured loans and linked the household idiosyncratic risk, e.g. large medical bills, to consumer debt default. In contrast, our model features the role of financial assets such as stock/housing, risk-free and defaultable debt, and highlights the link between systematic risk (or aggregate shock) and consumer debt default. Therefore, our model can be regarded as complimentary to Chatterjee et al. (2007b).

This paper also provides a framework to evaluate the welfare effect of debt default. In a partial equilibrium model, Zame (1993) argues that default can help to improve welfare in an incomplete market. Whether the conclusion would still remain in general equilibrium is not clear. Our model provides a clear-cut answer to the question. The defaultable bonds facilitate risk sharing and trading between agents, helps to complete the securities space in an incomplete market and can thus increase welfare. In particular, we show that without defaultable bonds, there would be no trading in an economy where the agents are exposed to rare event risk. Recently, Brunnermeier, Simsek, and Xiong (2014) provides a general
welfare criterion for models with distorted beliefs. Our paper, jointly with theirs, can lay a groundwork for discussion of optimal defaultable securities design.

The remainder of this paper proceeds as follows. In Section 1.2, we start with one simple example to illustrate the main idea of the paper. In Section 1.3, we introduce the model, present equilibrium asset prices and discuss the model implications, and Section 1.4 concludes. All proofs are provided in the appendix.

1.2 One Simple Example

To illustrate the main idea of this paper, we first consider a typical portfolio choice problem when the risky asset is subject to jump risk. The example shows the necessity of risky bonds in an economy featuring rare event risk. It also sheds light on how we introduce endogenous default and generate credit spread in a classical consumption-based asset pricing model.

Agents

An infinite-horizon representative agent is endowed with initial wealth $W_0$, has time preference $\rho$ and derives utility from logarithmic consumption.

Assets Market
There are two types of securities in the market: stock\(^4\) and risk-free debt. The stock price \(S_t\) follows a jump-diffusion process given by

\[
\frac{dS}{S} = \mu dt + \sigma dz_t + kdN_t \tag{1.1}
\]

where \(\{z_t\}\) is a standard Brownian motion, \(\{N_t\}\) is a Poisson process with constant intensity \(\lambda\). \(k\) for now is considered to be constant. To avoid the trivial case of jump-to-ruin, we assume \(k > -1\).

There is also an instantaneous zero-coupon bond \(B_t\) that follows

\[
\frac{dB}{B} = r dt
\]

In the partial equilibrium simple example here, \(r\) is assumed to be constant.

Now we can formulate the portfolio choice problem. An agent chooses portfolio weight on risky asset \(\{\theta_t\}\) and intermediate consumption \(\{C_t\}\) to maximize expected utility.

**Problem 1.**

\[
\max_{\theta_t,C_t} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \log(C_t) dt \right\} \tag{1.2}
\]

subject to

\[
\frac{dW}{W} = \theta_t \frac{dS}{S} + (1 - \theta_t) \frac{dB}{B} - c_t dt \tag{1.3}
\]

where \(c_t := \frac{C_t}{W_t}\) is the consumption wealth ratio.

The problem is well studied in the portfolio choice literature, for example, Aït-Sahalia, Cacho-Diaz, and Hurd (2009) and Jin and Zhang (2012). Proposition 1 characterizes the

---

\[^4\] Throughout the paper, we follow the convention in the literature and interpret the risky asset as stock. However, similar to Piazzesi, Schneider, and Tuzel (2007), it can also be interpreted as housing service. In section 1.3.7.1, we will explicitly reconcile our model with 2008 housing crisis and demonstrate that our model can capture the dynamics of key macroeconomic variables pre and post crisis.
solution to Problem 1

Proposition 1.

c = \rho

optimal \( \theta \) solves the following (quadratic) equation

\[- \sigma^2 k \theta^2 + (\mu - r)k - \sigma^2 \theta + \mu - r + \lambda k = 0\] (1.4)

(1.4) has two solutions, one is greater than \(-\frac{1}{k}\) and the other one is less than \(-\frac{1}{k}\).

Proof. See Appendix (1.5.1) \qed

If \( k \) is positive, the representative agent would choose the \( \theta \) greater than \(-\frac{1}{k}\); if \( k \) is negative, he would choose the one less than \(-\frac{1}{k}\). Both cases imply that the agent fully incorporates the jump risk in the portfolio choice and the risk sharing is limited by the potential jump size of the risky asset. The reason can be seen from the dynamics of wealth (1.3),

\[
\frac{dW}{W} = [\theta_t \mu + (1 - \theta_t) r - c_t] \ dt + \theta_t \sigma d z_t + \theta_t k dN_t
\] (1.5)

Given \( k \) is negative, if the agent’s leverage is too high, i.e. \( \theta_t \geq -\frac{1}{k} \), his marginal utility becomes too low \((-\infty)\) upon arrival of Poisson jump; The same happens when \( k \) is positive and the agent’s short selling strategy is too aggressive, i.e. \( \theta_t \leq -\frac{1}{k} \). In either case, the disutility ex post is so severe that agent’s risky asset holding is strictly limited by the bound of \(-\frac{1}{k}\).

This poses difficulty in risk sharing and trading when the agents in the economy are, for example, heterogeneous in beliefs and have motive to trade. To see this, let \( \lambda \to 0 \) and \( k \to -1 \). \( \lambda \to 0 \) means that in most of the time, the jump risk does not realize and trading is expected to occur in the market between agents holding different beliefs: the more
optimistic ones would like to borrow from the pessimistic ones to purchase risky assets and
the incentive would be stronger when the interest rate is low; However, \( k \rightarrow -1 \) implies that
\( \theta \leq 1 \) and the trading is highly restricted by the very rarely realized risk. It also raises the
question regarding the existence of the equilibrium, as market clearing condition asks for the
agents to hold all the outstanding shares.

Not only the predicament arises when the jump amplitude \( k \) is a constant, it also happens
when \( k \) follows a distribution, for example, on \((-1, \infty)\). The problem lies in the fact that
a safe bond without other financial instruments (in addition to stock) is too much to ask
in such an economy. In Section 1.3 we introduce defaultable debt securities as well as safe
bond and stock for agents holding heterogeneous beliefs to trade, and derive assets prices in
equilibrium. Meanwhile, to motivate the launch of defaultable debt securities, we also show
that there does not exist an equilibrium if the economy only features safe bond and stock in
Section 1.3.3.

1.3 Model

1.3.1 Model Setup

In this section, we lay out the basic set-up for the model.

1.3.1.1 Aggregate Endowment

The model is a version of Lucas Jr (1978) with an exogenous endowment. Time is continuous.
The aggregate endowment \( \mathcal{E}_t \) follows the stochastic process:
\[
\frac{d \varepsilon_t}{\varepsilon_t} = \left( \mu_t - \lambda_t \mathbb{E} [e^Y - 1] \right) dt + \sigma d\zeta_t + (e^Y - 1) dN_t
\]  \hspace{1cm} (1.6)

or equivalently,

\[
\varepsilon_t = \varepsilon_0 \exp \left( \int_0^t \mu_s ds + \sigma \zeta_s - \frac{\sigma^2}{2} t - \int_0^t \lambda_s \mathbb{E} [e^Y - 1] \, ds \right) \prod_{i=1}^{N_t} e^{Y_i} \]  \hspace{1cm} (1.7)

where \( \mu_t \) is the time-varying expected growth rate of the aggregate endowment. \( \sigma \) is the constant volatility. \( \zeta_t \) is a standard Brownian motion with \( \zeta_0 = 0 \) and \( N_t \) is a Poisson process with intensity \( \lambda_t \). \( k := e^Y - 1 \) is the stochastic jump amplitude. We will elaborate \( \lambda_t, k := e^Y - 1 \) and \( \mu_t \) in details below.

**Jump intensity** \( \lambda_t \) follows a CIR-type stochastic process

\[
d\lambda_t = \alpha_\lambda (\bar{\lambda} - \lambda_t) \, dt + \sigma_\lambda \sqrt{\lambda_t} \, dz^\lambda_t
\]  \hspace{1cm} (1.8)

with unconditional mean \( \bar{\lambda} \) and stationary variance \( \frac{\bar{\lambda} \sigma_\lambda^2}{2 \alpha_\lambda} \). \( \alpha_\lambda \) is the mean-reversion parameter, \( \sigma_\lambda \) is the volatility parameter, \( z_t^\lambda \) is a standard Brownian motion independent of \( \zeta_t \). We impose a standard technical condition \( 2 \alpha_\lambda \bar{\lambda} \geq \sigma_\lambda^2 \) to prelude \( \lambda_t \) ever being zero.

**Jump amplitude** \( k := e^Y - 1 \) represents the instantaneous drop or boom in aggregate endowment upon arrival of the rare event. \( Y_i \) are independent and identically distributed random variables and follow a generalized logistic distribution on the real line with probability density function (p.d.f) given by

\[
p_Y(y) = \frac{1}{B(2, 2)} \frac{e^{-2y}}{(1 + e^{-y})^4}, \quad y \in (-\infty, \infty)
\]  \hspace{1cm} (1.9)
where \( \mathcal{B} \) is the Beta function\(^5\)

One might wonder that the standard logistic distribution seems to be a more natural choice\(^6\). However, \( \mathbb{E}(e^Y) \) does not exist when \( Y \) follows a standard logistic distribution, implying a heavy tail. The generalized logistic distribution has a thinner tail compared to the standard one and, as shown in Section 1.3.4, provides closed-form characterization of equilibrium credit spread compared to normal distribution (Backus, Chernov, and Martin (2011)). Figure 1.2 compares standard normal, standard logistic and generalized logistic distribution.

**Time-varying expected growth rate of the aggregate endowment** \( \mu_t \) follows a mean-reverting process whose dynamics is given by

\[
d\mu_t = \alpha_\mu (\bar{\mu} - \mu_t) \, dt + \sigma_\mu \, d\mu_t
\]

(1.10)

\( \alpha_\mu, \sigma_\mu \) are mean-reversion and volatility parameters, respectively. \( \bar{\mu} \) is the unconditional mean of \( \mu_t \) and \( z_t^\mu \) is a standard Brownian motion independent of \( \{z_t^?, z_t^A\} \). \( \mu_t \) is unknown to

---

\(^5\)The p.d.f of the generalized logistic distribution in general form is

\[
f(x; \alpha) = \frac{1}{\mathcal{B}(\alpha, \alpha)} \frac{e^{-\alpha x}}{(1 + e^{-x})^{2\alpha}}, \ x \in (-\infty, \infty)
\]

(Johnson and Samuel Kotz (1995)). As \( \alpha \) increases, the tail becomes thinner. For \( \mathbb{E}(e^Y) \) to be well defined as well as simplicity, we therefore choose \( \alpha = 2 \).

\(^6\)Power law distribution is also commonly assumed for rare disaster (Barro and Jin (2011)) and rare boom (Tsai and Wachter (2015b)). The problem with power law distribution is that it is not well defined at zero, while we wish for a single continuous distribution for both disaster and room. Also, it becomes clear later that the left tail (disaster) matters more than the right tail (boom) for the quantitative result and our assumption is conservative about rare disaster. In particular, the average disaster size implied by the distribution is

\[
\int_{-\infty}^{0} (e^y - 1)p_Y(y) = -0.25,
\]

similar to 23% in Barro and Ursúa (2008) based on the international data on large consumption declines.
the agents but all other parameters are public information. However, the agents can learn \( \mu_t \) from (1.6) and (1.10). Nevertheless, as shown below, the agents display behavioral bias during learning and thus generate time-varying belief dispersion endogenously.

### 1.3.1.2 Agents

Since \( \mu_t \) is unknown, agents have to learn and make an inference about the true underlying parameter \( \mu_t \). However, their learning could be influenced by behavioral bias such as over-confidence, leading to heterogeneity in beliefs and trading between different agents (Harrison and Kreps (1978), Basak (2000), Basak (2005)). We follow Scheinkman and Xiong (2003) to model the learning process\(^7\). Specifically, we assume that there are two types of agents in the market, \( A \) and \( B \). They both have time preference \( \rho \), derive utility from logarithmic consumption \( \log(C) \) and are infinitely lived. In other words, their objective functions are given by

\[
\mathbb{E}^i \left[ \int_0^\infty e^{-\rho t} \log(C^i_t) \, dt \right], \quad i \in \{A, B\}
\]

(1.11)

\( \mathbb{E}^i \) is the expectation with respect to the belief of each type of agents. Agents form their beliefs from learning the information. In addition to the public information described in section 1.3.1.1, they each respectively also receive a signal regarding \( \mu_t \), \( s^A_t \) and \( s^B_t \), which follows

\[
\begin{align*}
    ds^A_t &= \mu_t \, dt + \sigma_s \, dz^A_t \\
    ds^B_t &= \mu_t \, dt + \sigma_s \, dz^B_t
\end{align*}
\]

(1.12) (1.13)

\(^7\)Alternative learning process may also suffice as long as it introduces heterogeneity in beliefs, for example Buraschi and Jiltsov (2006)
Without loss of generality, we assume \( \{z_t^\varphi, z_t^A, z_t^B, z_t^\lambda, z_t^\mu\} \) are Brownian motions independent of each other. Both types of agents know each other’s signals. Nevertheless, either type exaggerates his own signal and displays overconfidence towards it when learning. In specific, agent A perceives \( s_t^A \) as

\[
ds_t^A = \mu_t \, dt + \phi \sigma \, dz_t^\mu + (1 - \phi) \sigma \, dz_t^A
\]

meaning that agent A (falsely) believes the innovation of \( s_t^A \) is correlated with the innovation of \( \mu_t \).

And the similar bias occurs to \( B \), too. Agent B perceives \( s_t^B \) as

\[
ds_t^B = \mu_t \, dt + \phi \sigma \, dz_t^\mu + (1 - \phi) \sigma \, dz_t^B
\]

Note the two types of agents display symmetric behavioral biases. The behavioral biases would generate time-varying belief dispersion and trading between the agents. And the symmetry helps to keep the model dynamics stationary. In the early literature presenting belief dispersion, the agent whose belief is relatively closer to the true underlying value will dominate the wealth share in the long run and the other agent will be driven out of the market, leading to non-stationary wealth dynamics (Kogan et al. (2006), Yan (2008), Kogan et al. (2009)). As neither of the agents has relative advantage over learning in our model, we will see that the wealth ratio \( \frac{W_A}{W_B} \) between the two agents fluctuates around 1.
1.3.1.3 Learning and Inference

In this section, we discuss the learning and inference problem of the agents. As jump size $Y$ (or $k$) is observable by assumption, we first define

$$\tilde{E}_t := \mathcal{E}_t \exp \left( \int_0^t \left( \frac{\sigma^2}{2} + \lambda_s E[k] \right) ds \right) \prod_{i=1}^{N_t} e^{Y_i}$$

Therefore

$$d \ln(\tilde{E}_t) = \mu_t dt + \sigma dz^\varphi$$

Note $\ln(\tilde{E}_t)$ is a diffusion process without jump. Thus, the learning problem falls into optimal filtering problems that have been studied extensively in the literature (Liptser and Shiryaev (2001)). The agent’s posterior distribution about $\mu_t$ conditional on all information $I_t$ up to time $t$ follows a normal distribution

$$\mu_t | I_t \sim N(\tilde{\mu}_i^t, \nu_t^i), i \in \{A, B\}$$

where $I_t^A$ includes \{1.10, 1.13, 1.14, 1.17\} and $I_t^B$ includes \{1.10, 1.12, 1.15, 1.17\}.

Since $\mu_t$ is a time-varying process, in general each type of agents will never learn the true value perfectly, and thus there exists a steady state for $\nu_t^i, i \in \{A, B\}$.

Similar to Scheinkman and Xiong (2003), we can derive the stationary variance $v^*$ and the dynamics of $\tilde{\mu}_t^i$,

$$v^* = \sqrt{\left[ \alpha_{\mu} + \left( \phi \sigma_{\mu} \sigma_s \right) \right]^2 + \left(1 - \phi^2\right) \left[ \frac{\sigma_{\mu}^2}{\sigma_s^2} + \frac{\sigma_{\nu}^2}{\sigma_t^2} \right] - \left[ \alpha_{\mu} + \left( \frac{\phi \sigma_{\mu}}{\sigma_s} \right) \right]}$$

$$\left( \frac{1}{\sigma} \right)^2 + \frac{2}{\sigma_t^2}$$

(1.19)
The mean $\tilde{\mu}_A$ of agent A follows

$$
\frac{d\tilde{\mu}_A}{\tilde{\mu}_A} = -\alpha \left( \tilde{\mu}_A - \tilde{\mu} \right) dt + \frac{\phi \sigma^2 \sigma_\mu + v^*}{\sigma^2} \left( ds^A - \tilde{\mu}_A dt \right) \\
+ \frac{v^*}{\sigma^2} \left( ds^B - \tilde{\mu}_B dt \right) + \frac{v^*}{\sigma^2} \left( d \ln \left( \tilde{\phi}_t \right) - \tilde{\mu}_A dt \right)
$$

(1.20)

The mean $\tilde{\mu}_B$ of agent B follows an isomorphic process.

As we focus on $\tilde{\mu}_A$ and $\tilde{\mu}_B$, we shall assume that the agents start with stationary variance $v_0 = v^*$. And to facilitate our discussion, we shall use “optimists” to describe the type of agents with greater mean belief $\tilde{\mu}_t$ and “pessimists” to describe the other type of agents. Correspondingly, we let

$$
\mu^o = \max\{\tilde{\mu}_A, \tilde{\mu}_B\} \\
\mu^p = \min\{\tilde{\mu}_A, \tilde{\mu}_B\}
$$

(1.21) (1.22)

Depending on the posterior belief $\tilde{\mu}_A$ and $\tilde{\mu}_B$, the roles of optimists and pessimists are not fixed but could flip over time, i.e. sometimes A is the optimist while other times B is the optimist. In the subsequent sections, we refer to the agents as optimists/pessimists instead of A/B.

1.3.1.4 Assets Market Structure

Default would never occur in general equilibrium with complete market (Dubey, Geanakoplos, and Shubik (2005)). With incomplete market, the structure of the assets market and availability of financial instruments are critical in assets pricing and risk sharing between agents. We consider four different kinds of financial securities: stock, safe debt and two
different kinds of defaultable (risky) debt. We will elaborate why we need these securities in Section [1,3.3]

**Stock** $S$ is the claim to aggregate endowment. The total outstanding share of stock is normalized to 1. Let $S$ denote the price of stock, and given logarithmic utility functions, we conjecture that it follows:

$$\frac{dS}{S_t} = \mu^s \, dt + \sigma^s \, dz_t + k^s \, dN_t$$

(1.23)

The superscript $s$ means “stock”. The dividend-price ratio is $\frac{\varepsilon}{S}$. Therefore, the total return of holding stock is

$$\frac{dS}{S} + \frac{\varepsilon}{S}$$

(1.24)

The other three securities belong to the class of debt instruments.

**Safe Debt** $B^f$ is an instantaneous risk-free zero-coupon bond. A borrower borrows $B^f_{t-}$ at $t-$ and promises to repay principal $B^f_{t-}$ and interest $B^f_{t-} r^f_{t-}$ at $t + dt$. Put in differential form,

$$\frac{dB^f_t}{B^f_{t-}} = r^f_{t-} \, dt$$

(1.25)

where $r^f_{t-}$ is the risk free interest rate written in the contract at $t-$. Note that since the bond $B^f$ is absolutely riskless, it does not allow any form of default. The bond can be regarded as the “safe asset” in Barro and Mollerus (2014).

**Defaultable Debt** $B^d$ is an instantaneous zero-coupon but risky bond, meaning a bond issuer can default on the debt if he is not able to repay the principal or interest whenever
there is a large downward jump. A bond issuer borrows $B_{t-}^d$ at $t-$ and promises to repay principal $B_{t-}^d$ and interest $B_{t-}^d r_{t-}^d$ at $t + dt$, if he does not default at $t + dt$. Precisely,

$$
\frac{dB_t^d}{B_t^d} = r_{t-}^d dt + k_t^d dN_t
$$

(1.26)

where $r_{t-}^d$ is the interest rate of the risky bonds written in the contract at $t-$ and $k_t^d$ is the writedown when default is triggered.

To complete the defaultable bond characterization, we make the following assumptions.

**Assumption 1.** The default occurs when the issuer’s wealth (i.e. net worth) drops no less than $|\gamma|$, $-1 < \gamma < 0$.

The assumption echoes Black and Cox (1976) and Longstaff and Schwartz (1995), resembling the lenders “mark” default on borrowers who they think are not able to repay the debt. In the language of consumer finance, $\gamma$ can be regarded as the net worth shock that triggers the consumer debt default.

**Assumption 2.** The writedown $k^d = k$ when default is triggered.

The assumption says that the writedown co-moves with the market when debt defaults. It is a legitimate assumption also adopted by Barro (2006). Since the default can only possibly occur when the rare event $N_t$ happens, a high writedown $k^d$ reflects the difficulty recovering bond value when the market plunges deep. This approach essentially models a stochastic recovery of face value of the bonds, similar to Duffie and Singleton (1999).

---

It is implicitly assumed that debt default does not involve dead-weight cost that goes to a third party, such as court or liquidator, i.e. the loss of the creditor is the gain of the debtor.
The defaultable debt contract is written on the state $k$. We focus on finding a “threshold equilibrium”:

$$k^d = \begin{cases} 
0, & \text{if } k > \bar{k} \\
 k, & \text{if } k \leq \bar{k} 
\end{cases} \quad (1.27)$$

Occurrence of rare event does not necessarily trigger default; default happens only when $k = e^Y - 1$ lower than a threshold $\bar{k}$, i.e. the downward jump is sufficiently large. And it is not hard to see that $\bar{k}$ is closely related to the agent’s risky assets and leverage position.

**Defaultable Debt** $B^d$ is similar to the defaultable debt $B^d$ and follows

$$\frac{dB^d}{B^d_t} = r^d_t dt - k^d_t dN_t \quad (1.28)$$

and the default now is triggered when a sizable rare boom (i.e. positive jump) realizes in the economy. Similar to Assumptions 1 and 2, we have the following ones with regard to defaultable bond $B^d$

**Assumption 3.** The default occurs when the issuer’s wealth (i.e. net worth) drops no less than $|\zeta|, -1 < \zeta < 0$.

**Assumption 4.** The writedown $k^d = k$ when default is triggered.

And we are looking for a “threshold equilibrium” in the following form

$$k^\tilde{d} = \begin{cases} 
0, & \text{if } k < \tilde{k} \\
k, & \text{if } k \geq \tilde{k} 
\end{cases} \quad (1.29)$$

where $\tilde{k}$ is endogenously determined §

---

§ Given the positive jump ranges in $(0, \infty)$ in equation $1.9$, the writedown can possibly be greater than 1. Therefore, the debt contract $B^d$ can also be interpreted as an (incomplete) insurance contract.
In contrast to the early literature (for example, Merton (1973) and Cox, Ingersoll Jr, and Ross (1985) among others), our model features two additional defaultable debt securities $B^d$ and $B^d$. As it becomes clear later, the optimists would finance his leverage position by issuing defaultable bond $B^d$; the pessimists might need to issue defaultable bond $B^d$, depending on whether he is engaged in short selling or not. They are both important to establish the competitive equilibrium. We will discuss their roles in risk sharing in more detail in Section 1.3.3.

### 1.3.2 The Problems of Agents

In this section, we will state the consumption and portfolio choice problem for the agents. As the pessimist’s problem is similar to the optimist’s, I will mainly focus on analyzing the optimist’s problem without loss of generality\(^{10}\). The optimist chooses consumption $C^o$ and the portfolio weights $\{\theta^o, \theta^d,o, \theta^d,o\}$ on stock and defaultable debt securities respectively to maximize the expected utility. His problem is

**Problem 2.**

$$\max_{\theta^o, \theta^d,o, \theta^d,o, C^o} \mathbb{E}_0^c \left\{ \int_0^\infty e^{-\rho t} \log(C^o) dt \right\}$$

subject to

$$\frac{dW^o}{W^o} = \theta^o \left( \frac{dS}{S} + \frac{\delta}{S} \right) + \theta^d,o \frac{dB^d}{B^d} + \theta^d,o \frac{dB^d}{B^d} + (1 - \theta^o - \theta^d,o - \theta^d,o) \frac{dB^f}{B^f} - c^o$$

where $\frac{dS}{S}$, $\frac{dB^d}{B^d}$, $\frac{dB^d}{B^d}$ and $\frac{dB^f}{B^f}$ are given by (1.23), (1.26), (1.28) and (1.25). $\theta^o$, $\theta^d,o$ and $\theta^d,o$ are optimist’s positions on stock and risky debt.

Definition \([\text{1}]\) defines the competitive equilibrium we are about to characterize in Section 1.3.4.

\(^{10}\) The $t$ subscript on the variables will be omitted when not essential for clarity
Definition 1. A competitive equilibrium is composed of \(\{\theta^o, \theta^{d,o}, \theta^{d,o}, \theta^p, \theta^{d,p}, c^o, c^p\}\), prices \(\{r^d, r^d, r^f, S\}\) and debt contract variables \(\{\bar{k}, \tilde{k}\}\) such that

1. Given prices \(\{r^d, r^d, r^f, S\}\) and \(\{\bar{k}, \tilde{k}\}\), \(\{\theta^o, \theta^{d,o}, \theta^{d,o}, c^o\}\) solve the optimist’s problem 

2. Given prices \(\{r^d, r^d, r^f, S\}\) and \(\{\bar{k}, \tilde{k}\}\), \(\{\theta^p, \theta^{d,p}, \theta^{d,p}, c^p\}\) solve the pessimist’s problem similar to 2.

3. The optimists default when their net worth suddenly drops no less than 100\(|\gamma|\)%, i.e. \(\theta^o k^s \leq \gamma \) or \(k^s \leq \bar{k}\); If default occurs, the writedown \(k^d = \bar{k}\), otherwise \(k^d = 0\).

4. The pessimists default when their net worth suddenly drops no less than 100\(|\zeta|\)%, i.e. \(\theta^p k^s \leq \zeta \) or \(k^s \geq \tilde{k}\); If default occurs, the writedown \(k^d = \tilde{k}\), otherwise \(k^d = 0\).

5. Given \(\{\bar{k}, \tilde{k}\}\), all the markets clear, i.e.
   
   (a) \(c^o W^o + c^p W^p = \epsilon\)
   
   (b) \(\theta^o W^o + \theta^p W^p = S\)
   
   (c) \(\theta^{d,o} W^o + \theta^{d,p} W^p = 0\)
   
   (d) \(\theta^{d,o} W^o + \theta^{d,p} W^p = 0\)

6. The optimists and pessimists are engaged in the non-cooperative, fully decentralized bargaining with no cost and determine \(\{\bar{k}, \tilde{k}\}\) in equilibrium.

It is worth of noting that \((\theta^o, \bar{k})\) are jointly determined in the equilibrium and so do \((\theta^p, \tilde{k})\). To see why, we will take the optimists as an example. On one hand, the optimists have to declare default when the jump size \(k^s \leq \frac{\gamma}{\theta^o}\) and therefore \(\bar{k} \geq \frac{\gamma}{\theta^o}\). On the other hand, given their position on risky asset \(\theta^o\), they do not prefer contract with \(\bar{k} > \frac{\gamma}{\theta^o}\) as the contract allows them to borrow more than they actually need with a higher interest cost.

The defaultable debt markets will clear via non-cooperative, fully decentralized bargaining with no cost. In equilibrium, the market determines the contract traded: the optimists would like to pay \(r^d\) and get a loan with default triggered at \(\bar{k}\); the pessimists would like to receive \(r^d\) and underwrite such a contract \footnote{Alternatively, similar to Geanakoplos (2010), Simsek (2013) and Walsh (2014), we can}.
1.3.3 Assets Market Structure: Revisit

Before we characterize the competitive equilibrium in Section [1.3.4], we shall revisit the assets market structure in Section [1.3.1.4]. As aforementioned, in contrast to early literature, our model features two additional defaultable debt securities in addition to standard stock asset and safe debt. In this section, we highlight the importance of defaultable debt securities in risk sharing and establishing equilibrium by addressing two questions.

The first question is why defaultable debt securities are needed. Suppose the market is “safe-debt-only” instead, i.e. it only features stock and safe debt. Correspondingly, we can modify Problem [2] by imposing \( \theta^{d,o} = \theta^{d,o} = 0 \) and equilibrium definition [1] by excluding two defaultable debt securities. It turns out, as Proposition [2] shows, there is no equilibrium in the “safe-debt-only” market. The equilibrium does not exist in the sense that no \( r^f \in (-\infty, \infty) \) can bridge between the agents holding heterogeneous beliefs. The intuition behind Proposition [2] is that “safe” debt security makes no room for trade. Note \( k_t \) in (1.6) has support on \((-1, \infty)\). The aggregate endowment has a risk of dropping to a positive yet arbitrarily small amount. The stock price will fluctuate (or, co-move) with the aggregate endowment. As a result, any nontrivial leverage position in the stock market (i.e. the optimist borrows a bit to purchase stock) would result in negative wealth with positive probability (i.e. \( P(k^s < \frac{-1}{\theta^o}) > 0 \)), which is inadmissible in the log utility case. In this introduce a continuum of defaultable debt contracts with different thresholds \( \tilde{k} \) and \( \bar{k} \), \( \{ (B^d(\tilde{k}), B^d(\bar{k})) | \tilde{k} \in (-1, 0), \bar{k} \in (0, \infty) \} \), and let the agents determine which contract to trade in equilibrium. The new definition of the equilibrium needs to be modified to incorporate the market clearing condition for each possible contract. However, it turns out that the equilibrium remains the same: in the equilibrium the agents would trade only one debt contract. The intuition is that the agents always would like to choose an optimal position on risky asset first and then pick a defaultable debt contract that is least costly yet provides enough insurance against adverse rare events, given \( \gamma \) and \( \zeta \).
sense, defaultable bonds emerge endogenously in the economy and the optimists would issue
defaultable bonds $B^d$ to finance his leveraged position.

**Proposition 2.** There is no equilibrium in the “safe-debt-only” market.

*Proof.* See Appendix (1.5.2) □

The second question would be why two defaultable debt securities instead of one are
needed. Correspondingly, we can modify Problem 2 by imposing $\theta^{d,o} = 0$ and equilibrium
definition by excluding the defaultable debt security $B^d$. It turns out that the defaultable
bond $B^d$ facilitates risk sharing only when belief dispersion is mild. When disagreement
between agents gets above some threshold, the optimist would think the borrowing cost $r^d$
is very low, stock return $\mu^o$ is high and would like to take a significant leverage position.
However, the supply of the risky asset is finite (“1”) and hence short selling is needed to
“create” more supply. Notwithstanding, the economy (as well as the stock price) also subjects
to positive jump and thus short selling position would result in negative wealth when the
positive jump is realized. Therefore, the pessimists would like to issue the other defaultable
bonds $B^d$ to protect themselves from negative wealth. Should the stock price appreciate,
the pessimists would default on the risky bonds but fulfill the short position.

**Proposition 3.** Without $B^d$, the equilibrium exists when

\[ \mu^o - \mu^p \leq \max_{\theta^o \in [1,1+\frac{1}{\omega}]} \mathcal{Q}(\theta^o;\omega,\gamma), \]

where $\mathcal{Q}(\theta^o;\omega,\gamma)$ is a continuous function on $\theta^o \in [1,1+\frac{1}{\omega}]$ and $\omega := \frac{W^o}{W^p}$ is the relative
wealth share.

*Proof.* See Appendix (1.5.3) □
1.3.4 Equilibrium

In this section, we establish and characterize the equilibrium in Definition 1. Again, as the problems for the two types of agents are isomorphic, we focus on the optimists. We first look at the First Order Conditions (FOCs) for Problem 2:

\[ \mu^o + \rho - \lambda \mathbb{E}[k] - r^f - \theta^o \sigma^2 + \lambda \mathbb{E} \left[ \frac{k}{1 + \theta^o k + \theta^{d,o} k_d + \theta^{\bar{d},o} k_{\bar{d}}} \right] = 0 \] (1.32)

\[ r^d - r^f + \lambda \mathbb{E} \left[ \frac{k^d}{1 + \theta^o k + \theta^{d,o} k_d + \theta^{\bar{d},o} k_{\bar{d}}} \right] = 0 \] (1.33)

\[ r^{\bar{d}} - r^f - \lambda \mathbb{E} \left[ \frac{k^{\bar{d}}}{1 + \theta^o k + \theta^{d,o} k_d + \theta^{\bar{d},o} k_{\bar{d}}} \right] = 0 \] (1.34)

\[ c^o - \rho = 0 \] (1.35)

Given the class of defaultable bonds we are considering ((1.26), (3.48)) and ((1.28), (1.29)), the first order conditions can be simplified as:

\[ r^d + r^{\bar{d}} - r^f = \mu^o + \rho - \lambda \mathbb{E}[k] - \theta^o \sigma^2 + \lambda \mathbb{E} \left[ \frac{k}{1 + \theta^o k} \left| \bar{k} < k < \tilde{k} \right. \right] \mathbb{P} \left( \bar{k} < k < \tilde{k} \right) \] (1.36)

\[ r^d - r^f = -\frac{\lambda}{\text{Probability of jump}} \mathbb{P} \left( k \leq \bar{k} \right) - \frac{\lambda}{\text{Probability of severe jump}} \mathbb{E} \left[ \frac{k}{1 + \left( \theta^o + \theta^{d,o} \right) k} \left| k \leq \bar{k} \right. \right] \] (1.37)

\[ r^{\bar{d}} - r^f = \lambda \mathbb{P} \left( k \geq \tilde{k} \right) \mathbb{E} \left[ \frac{k}{1 + \left( \theta^o + \theta^{\bar{d},o} \right) k} \left| k \geq \tilde{k} \right. \right] \] (1.38)

Note \( \frac{1}{1 + \theta^o k} \) is the marginal rate of substitution (kernel) conditional on that rare event occurs at \( t \) and \( \bar{k} < k < \tilde{k} \). \( \frac{1}{1 + (\theta^o + \theta^{d,o}) k} \) and \( \frac{1}{1 + (\theta^o + \theta^{\bar{d},o}) k} \) have similar interpretations as pricing kernels.
Equation (1.37) and (1.38) show the credit spread on the different defaultable debt securities. However, they have the same economic interpretations that the credit spread is the product of three components: the probability of a disaster occurring, the probability of the disaster triggering default and the loss given default under risk-neutral probability.

Theorem 1 establishes the equilibrium result.

**Theorem 1.** The competitive equilibrium exists. The optimist’s position on stock is \( \theta_o^* \) which is the solution to the equation \( \mu^o - \mu^p = Q (\theta_o^*; \omega, \gamma, \zeta) \) in \((1, \infty)\), where \( \omega := \frac{W_o}{W_p} \) is the relative wealth share. The pessimist’s position on stock \( \theta_p^* = 1 + \omega - \omega \theta_o^* \). Safe Bonds \( B^f \) is in zero supply. The credit spread on \( B^d \) is

\[
r^d - r^f = -\lambda_t \left( \frac{4}{(k + 2)^3} - \frac{3}{(k + 2)^2} - 1 \right)
\]

(1.39)

The credit spread on \( B^\tilde{d} \) is

\[
r^\tilde{d} - r^f = \lambda_t \left( \frac{2 + 3\tilde{k}}{(2 + \tilde{k})^3} \right)
\]

(1.40)

where \( \tilde{k} = \frac{\gamma}{\theta_o^*} \) and \( \bar{k} = \frac{\zeta}{\theta_p^*} \)

**Proof.** See Appendix (1.5.4) \qquad \square

In appendix, we show

\[
Q (\theta^o; \omega, \gamma, \zeta) = 1_{\theta^o \in (1, \frac{1}{\omega} + 1]} Q (\theta^o; \omega, \gamma) + 1_{\theta^o \in (\frac{1}{\omega} + 1, \infty)} \tilde{Q} (\theta^o; \omega, \gamma, \zeta)
\]

(1.41)

\( Q (\theta^o; \omega, \gamma, \zeta) \) consists of two parts: \( 1_{\theta^o \in (1, \frac{1}{\omega} + 1]} Q (\theta^o; \omega, \gamma) \), when belief dispersion is mild and risky bond for positive jump is not needed for trade to happen between agents; and \( 1_{\theta^o \in (\frac{1}{\omega} + 1, \infty)} \tilde{Q} (\theta^o; \omega, \gamma, \zeta) \), when belief dispersion is large, risky bonds for positive jump are issued by short sellers to cover the position on stock. We separately plot \( Q (\theta^o; \omega, \gamma) \) and \( \tilde{Q} (\theta^o; \omega, \gamma, \zeta) \) in Figure 1.3 (a) and 1.3 (b). Note that the upper bounds of \( Q (\theta^o; \omega, \gamma) \)

28
are the starting points for $\tilde{Q}(\theta^o; \omega, \gamma, \zeta)$, i.e. the two functions can be “glued” together seamlessly, indicating $Q(\theta^o; \omega, \gamma, \zeta)$ is a continuous function on $\theta^o \in (1, \infty)$.

So, what happens when some rare event occurs? Let's take the downward jump and risky bonds issued by the optimists as an example. When the rare event realizes at $t+$ and the actual decline of the endowment $k_{t+}$ is less severe than $\bar{k}_{t-}$ written in contract at $t-$, the default is not triggered. Under that circumstance, the optimist’s wealth changes by $\theta^o k_{t+}$, yet they are able to pay off the risky bonds in the amount of $|\theta^{d,o}| W^o$ in addition to the interest. In contrast, when $k_{t+} \leq \bar{k}_{t-}$, the default is triggered. Their wealth changes by $\theta^o k_{t+}$, which could possibly have led to negative wealth. Nevertheless, the writedown terms in the debt contract would exempt part of the debt repayment. In specific, the debt in the amount of $|\theta^{d,o}| W^o k_{t+}$ is exempt from the optimists and the net change of their net worth is $(\theta^o + \theta^{d,o}) W^o k_{t}$. In one word, when rare event and default happens, the change of the aggregate wealth $W^o + W^p$ is proportional to the total endowment $\mathcal{E}$; the risky bonds and writedown terms alter the allocation between the two types of agents. In essence, the risky bonds and the embedded writedown terms provide an insurance for the optimists to take high leverage and protect them from non-positive net worth. Meanwhile, the pessimists, i.e. the buyer of the risky bonds, would like to buy as they earn the credit spread as a premium.

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12 Similar to Livshits, MacGee, and Tertilt (2007), in the context of our model, the debtor is able to re-enter the market and trade upon default. As noted by Musto (1999), a household usually has difficulty accessing credit market for a period of time post-bankruptcy filing of Chapter 7 and Chapter 13. Our model can be modified to capture the feature by adding a period of “autarky”. The period of no trade essentially increases the cost of default.
1.3.5 Model Calibration

In this subsection, we follow several pieces of literature to calibrate the parameters used in the model. Consistent with Brennan and Xia (2001), we set $\sigma = 3.44\%$, $\sigma_\mu = 1.1\%$, $\alpha_\mu = 0.05$ and $\bar{\mu} = 1.55\%$; we set time preference parameter $\rho = 0.03$, consistent with Barro (2006). The value is also often used in the saving literature, such as Hubbard, Skinner, and Zeldes (1995); Gabaix (2012) set $\bar{\lambda} = 3.63\%$, which is based on Barro and Ursúa (2008). In a study of time-varying disaster risk, Tsai and Wachter (2015b) set $\alpha_\lambda = 0.11$, $\sigma_\lambda = 0.081$; Scheinkman and Xiong (2003) set $\sigma_\mu / \sigma_s = 2$ in their numerical example, which leads us to $\sigma_s = 0.55\%$.

Using Measures of Forecast Dispersion for the Survey of Professional Forecasters from Philadelphia Fed, we compute the annual mean and volatility of belief dispersion which are 1.23\% and 0.71\%, respectively. We calibrate $\phi = 9.7$ to match the mean of belief dispersion, which also gives us the volatility of belief dispersion of the same magnitude as the real data; We set $\gamma = \zeta = -0.7$, i.e. default is triggered when leveraged agents lose 70\% of their net worth, which is rather conservative. Admittedly, it is challenging to determine the magnitude of the loss that triggers the leveraged household to default without micro-data. We calibrate the parameter $\gamma$ (or $\zeta$) to match the average of correlation coefficients between credit spread, risk-free rate and belief dispersion. Later, we will conduct comparative statics to study the effect of $\gamma$ (or $\zeta$) on the credit spread. Table 1.3 summarizes the parameters used in the baseline model.

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13 The signal $s$ can be interpreted as inflation rate, for example. Exclusive of Era of Stagnation from 1964 to 1985, the annual volatility of inflation expectation is 0.421\%
1.3.6 Model Result

The model generates belief dispersion, risk-free rate as well as credit spread compared to real data. Table 1.4 shows the summary statistics of belief dispersion, credit spread and risk-free rate from the model and data. The model generates an average credit spread of 224.52 basis points (bps) with average probability of default around 0.59%, approximately 90% of the average credit spread 249.53 bps in the data. The model also generates a low risk-free rate 0.57% and helps to explain the “risk-free rate puzzle”. Table 1.5 (a) and Table 1.5 (b) compare the correlation between credit spread, belief dispersion and risk-free rate from the data and the model and justify our choice on \( \gamma = -70\% \). Of course, the magnitude of the credit spread and risk-free rate generated by the model depends on \( \gamma \), i.e. the net worth shock that triggers default. In Section 1.3.7.2, we will look at the effect of \( \gamma \) on the assets prices.

There are three fundamental variables that drive the entire economy: belief dispersion, time-varying rare event risk intensity \( \lambda_t \) and the relative wealth ratio which essentially determines whose belief the average belief of the economy will be toward. The three variables affect the equilibrium risk-free rate and credit spread through agents’ trading in the market.

To see the effect of each individual variable on credit spread as well as risk-free rate, I run regressions of credit spread and risk-free rate on the covariates using model-generated data.

---

14 In the calibrated model, the credit spread is small on risky bonds \( B_d \) that would default whenever there is a large positive jump. Therefore, we only calculate the credit spread on risky bonds issued by the optimist, i.e. the bonds which would default whenever there is a large downward jump.

15 We also considered a version of model with disaster risk intensity \( \lambda \) being constant and fixed at 3.63%. The model simulated moments still well match the data except the volatility of the credit spread. Hence the time-varying jump intensity is essential for the model to match the credit spread volatility.
and the results are in Table 1.6 and Table 1.7.

Although both increase in rare event risk and belief dispersion raises credit spread and pushes down risk-free rate, the underlying mechanism is quite different. To see this, in Table 1.8 we calculate the correlation between $\lambda$, $\mu^o - \mu^p$ and the endogenous default threshold $\bar{k}$, i.e. the jump amplitude that just triggers the default given leverage. The correlations have different signs, showing an increase in $\lambda$ lowers $\bar{k}$ while $\mu^o - \mu^p$ behaves in the opposite way. What are the underlying channels, respectively? From equation (1.39), an increase in the rare event risk intensity ($\lambda_t$) would increase the credit spread ceteris paribus. As a consequence, it is more costly for the optimist to leverage up and purchase risky assets. Additionally, they have to re-balance their portfolios. Therefore, they reduce their position on stock, pushing down $\bar{k}$ (i.e. a downward jump of larger size to trigger default) and increase the demand for risk-free bonds, causing the required return on the safe-bonds to fall. This echoes the so-called “flight-to-quality” phenomenon. In essence, our model characterizes risky asset and safe bonds as two investment substitutes. When rare event risk becomes more likely, investors will substitute safe-bonds for risky asset.

On the contrary, Table 1.8 also shows that belief dispersion $\mu^o - \mu^p$ and $\bar{k}$ are positively correlated. As belief dispersion gets wider, the optimist deems the required return on risky bonds and the cost of leverage is cheaper and therefore would like to borrow more via issuing more risky bonds. The higher leverage translates into a higher $\bar{k}$ (i.e. a downward jump of smaller size to trigger default) and pushes up the credit spread, as in (1.39). Yet, given greater belief dispersion and more resources at disposal from borrowing, not only does the
optimist purchase more shares of stock, he would also purchase more risk-free bonds to optimize the portfolio, pushing down the required return on the risk-free bonds. This is the “wealth effect”.

To clearly examine the “wealth effect”, we employ a numerical example to study comparative statics of belief dispersion on assets holdings. We set set $r^f = 1.07\%, r^d = 3.57\%, \lambda = 3.63\%, \gamma = -70\%, \omega = 1.0$. Without loss of generality, we focus on the assets holdings by the optimist when the belief dispersion is mild, i.e. no short-selling in the market. Figure 1.4 (a) plots the portfolio weight on stocks $\theta_o$, default boundary $\bar{k}$ and belief dispersion. Figure 1.4 (b) plots the relationship between belief dispersion and the demand for risk-free asset $1 - \theta_o - \theta^{d,o}$. It is clearly shows that an increase in belief dispersion would push up the leverage and the default boundary $\bar{k}$, which leads to a higher demand for risk-free bonds, too. As a consequence, in the equilibrium, the risk-free rate has to drop to clear the market.

This insight is utterly different from the similar models in the early literature. Those models typically feature only stock and risk-free bonds in a complete market. By design, they are always substitutes in the sense that increasing stock holdings requires decreasing risk-free debt holdings (or becoming net borrowers) given wealth. Nevertheless, in our model, the optimist invests in both stocks and risk-free bonds by borrowing via issuing risky bonds; the pessimist would mainly invest in risky bonds and risk-free bonds instead of stock in the sense that as the belief dispersion becomes greater, the pessimist starts to sell shares of stock short instead of holding them.
Krishnamurthy and Vissing-Jorgensen (2012) has found that corporate bonds spread is negatively correlated with US government debt over GDP ratio, suggesting substitution between safe bonds and corporate bonds from the perspective of investors. Nevertheless, due to the two effects mentioned above, the evidence from data on the relationship between risky consumer debt and safe bonds is mixed. We take a slightly different approach from Krishnamurthy and Vissing-Jorgensen (2012), as the risk-free rate is a “shadow” price in our model. Figure 1.5(a) plots risk-free rate and consumer debt over GDP ratio. It shows the two time-series positively co-move at certain times while not at other times. The overall correlation is -0.075 but insignificant. Table 1.9(b) shows that the simulated model generates an insignificant correlation coefficient of 0.0009 between risk-free rate $r^f$ and consumer debt over GDP ratio $\frac{\theta_{d,o} W_o}{W_o + W_p}$. The insignificance of the correlation coefficients highlights two countervailing forces: we would observe positive co-movement if the two series are driven by rare event risk; reverse co-movement if they are driven by belief dispersion. As a result, when taken together, the two effects would offset each other and leave an insignificant correlation coefficient, as shown both in the data and model. However, if we decompose the risk-free rate into the rare event risk component and belief dispersion component respectively as indicated by the regression in Table 1.7, it clearly shows that the rare event risk part strongly positively co-moves with Debt over GDP ratio while belief dispersion strongly and reversely co-moves with Debt over GDP ratio.

Similar pattern shows up in the relationship between borrower’s leverage and risk-free rate. As for the data, We use FODSP, Household Financial Obligations as a percent of
Disposable Personal Income, from FRED to measure the household leverage. Figure 1.5(b) plots the FODSP and risk-free rate over time. The correlation is 0.033 but not significant. In the model, $\theta^{d,o}$ can be interpreted as household leverage and the simulation shows the correlation between $\theta^{d,o}$ and risk-free rate is -0.0003 and insignificant. Nevertheless, if we compute the correlation between household leverage and rare event risk component and belief dispersion component of the risk-free rate respectively, we see a strong positive correlation between leverage and rare event risk component while a strong negative one between leverage and belief dispersion component.

Last but not least, the regression results in Table 1.6 and Table 1.7 indicate that the credit spread decreases and the risk-free rate increases with the relative wealth share. To clearly see the effect, we fix belief dispersion and rare event intensity and plots the credit spread and relative wealth share in Figure 1.6. In this way, the credit spread variation is only driven by the endogenous variation of the relative wealth share. This relationship is very intuitive. As the wealth share increases, the optimists in the economy possess most of the wealth and the average (wealth-weighted) beliefs of the market will stand closer to their belief. The asset prices will reflect such an average belief. This is in the same spirit of Xiong and Yan (2010): they showed that to replicate the heterogeneous beliefs economy in a complete market, a representative agent should have the wealth-weighted belief. As a consequence, the risk-free rate rises and it is more costly for the optimists to issue debt. They reduce their leverage positions and credit spread falls. This implication is also consistent with the empirical fact: the optimists tend to possess more wealth in the good times when the credit spread is low.
If we go extreme and let the wealth share $\omega \rightarrow \infty$, i.e. the optimists dominate in the market, the model economy reduces to a representative agent economy. In that case, there is only one debt whose rate $r^f$ is determined by the representative agent, and the credit spread $r^d - r^f$ shrinks to zero.

1.3.7 Discussion

1.3.7.1 Link to 2008 financial crisis

Household debt is one precipitating factor of 2008 financial crisis\textsuperscript{16}. Our model features three fundamental state variables: belief dispersion, rare event risk and relative wealth distribution, and they can capture dynamics of several macroeconomic variables both pre and post-crisis.

Before the crisis, belief dispersion increases over time (Buraschi, Trojani, and Vedolin (2013)) and optimism drives households to take more leverage. Although we follow the convention in the literature to call the risky asset “stock”, it can also be interpreted as housing services, after all “real estate is an important asset that pays off housing services, a major consumption good” (Piazzesi, Schneider, and Tuzel (2007)). Cheng, Raina, and Xiong (2014) also emphasizes the role of distorted belief on the recent crisis and provides direct empirical evidence on beliefs and household leverage\textsuperscript{17}. They found that mid-level managers in the mortgage-securitisation business increased leverage and housing exposure

\textsuperscript{16}Paul Krugman wrote in \textit{New York Times} on Dec 12, 2010: “The root of our current troubles lies in the debt American families ran up during the Bush-era housing bubble. Twenty years ago, the average American household’s debt was 83 percent of its income; by a decade ago, that had crept up to 92 percent; but by late 2007, debts were 130 percent of income.”

\textsuperscript{17}Other literature that debate the possibility of distorted beliefs influencing house prices include Himmelberg, Mayer, and Sinai (2005), Smith and Smith (2006) and Shiller (2007).
during the boom period by purchasing second home and swapping into more expensive homes and performed worse than some control group, suggesting that they wishfully believed the housing price would soar and were unaware of the potential housing market crash risk, though they arguably had more private information regarding housing market and price. Nevertheless, although belief dispersion got wider over time, the credit spread does not become extremely high. It is rooted in low rare event risk. The rare event risk before the crisis is relatively low. Using far-out-of-money put option prices with maturity ranging from one month to six months, Barro and Liao (2016) estimated rare disaster probability and showed that the rare event risk is low between 2004 to end of 2007. Siriwardane (2015) also reached a similar conclusion based option prices data. Moreover, although it is challenging to directly measure the relative wealth ratio between the optimists and pessimists, it is not hard to see that the wealth-weighted belief does not tilt toward the pessimist and hence also contributes to low credit spread.

After the crisis, belief dispersion decreases and therefore less borrowing and lending are expected to occur in the market. Yet, the rare event risk peaking around 2008-2009 and the associated flight-to-quality causes the credit spread to increase and risk-free rate to decrease at the same time. Moreover the wealth loss of the optimists during the crisis shifts the average belief toward the pessimists and thus exacerbate the increased credit spread and decreased risk-free rate.
1.3.7.2 Default Trigger $\gamma$

One critical element in our model is $\gamma$. $\gamma$ represents how the lenders and borrowers define “default”, i.e. the default happens when the net worth of the borrowers suddenly changes by $\gamma$. Apparently, different $\gamma$ would affect the optimist’s choice on $\theta^o$, $\theta^{d,o}$ and subsequently on $\bar{k}$, which is directly linked to credit spread. In the model, we calibrated $\gamma$ to match the correlation coefficient in Table 1.5. Yet, it is necessary to study the effect of tightening or loosening “$\gamma$” on the credit spread.

We conduct such comparative statics study in Table 1.11. Athreya and Neelakantan (2011) roughly estimated 40% as an upper bound for household net worth shock in normal times. Given the two features of the rare event risk: scarce and catastrophic, we consider $\gamma$ in the range of $[-70\%, -40\%]$. As it shows in the table, the credit spread keeps decreasing as we relax $\gamma$. As $\gamma$ gets smaller, it requires a larger net worth shock to set off the default and writedown terms in the debt contract and therefore the required return on risky bonds would also become smaller. This is also demonstrated by the default boundary $\bar{k}$ which becomes smaller with $\gamma$, meaning a larger endowment contraction to trigger the default when $\gamma$ is smaller. Note that the average position on stock $\theta^o$ also decreases as $\gamma$ gets smaller. This is because that, as $\gamma$ gets smaller, the optimist is not “insured” against less severe jumps any more and therefore becomes increasingly wary.
1.4 Conclusion

In this paper, we present a dynamic general equilibrium model to study the credit spread over consumer debt, in particular the link between systematic risk and consumer debt. The model generates credit spread and risk-free rate comparable to real data and it particularly implies that rare event risk, belief dispersion as well as relative wealth distribution jointly determine the credit spread and risk-free rate. Although previous empirical studies have documented their relationship pairwise, our model is the first one to account for belief dispersion, credit spread and risk-free rate at the same time and point out the underlying economic mechanisms.

One major theory contribution of our paper is to introduce endogenous default in rare disaster models. The endogenous default set-up allows us to discover the links between debt default and belief dispersion as well as rare event risk that are missing in existing rare disaster literature. Moreover, by assuming jump size follows a generalized logistic distribution, we are able to derive the credit spread in closed-form and discuss comparative statics analytically.

There are two important questions that our model does not cover. First, the model does not explicitly characterize labor income. The agent faces undiversifiable labor income risk and a significant unanticipated adverse labor income shock is one reason why the agent defaults on debt (Lopes (2008), Chatterjee et al. (2007a)). Second, all the debt contracts in the model are short-term. In fact, they are all instantaneous bonds. In reality, bonds of various of maturities are traded in the market. However, given the market is incomplete in

\footnote{For example, an economic disaster can also cause massive unemployment.}
our model, the availability of new securities might change the equilibrium. How the bonds of longer maturities may affect the market equilibrium is not clear. We leave these questions for future research.

1.5 Proofs

1.5.1 Proof of Proposition 1

Let \( J \) be the value function of Problem (1). The Bellman Equation is

\[
0 = \sup_{\theta, c} \left\{ J_t + J_W (\theta \mu + (1 - \theta)r - c) W + \frac{J_{WW}}{2} \theta^2 \sigma^2 W^2 + \lambda (J ((1 + \theta k) W) - J(W)) + e^{-\rho t} \log(cW) \right\}
\]

(1.42)

Conjecture \( J = e^{-\rho t} \left( \frac{\log(W)}{\rho} + \mathcal{G} \right) \) where \( \mathcal{G} \) is constant. (1.42) reduces to

\[
-\rho \left( \frac{1}{\rho} \log(W) + \mathcal{G} \right) + \frac{\theta \mu + (1 - \theta)r - c}{\rho} - \frac{\theta^2 \sigma^2}{2\rho} + \lambda \frac{\log(1 + \theta k)}{\rho} + \log(cW) = 0
\]

(1.43)

First Order Conditions:

\[
\frac{1}{c_*} - \frac{1}{\rho} = 0
\]

(1.44)

\[
\frac{\mu - r}{\rho} - \frac{\sigma^2}{\rho} \theta_* + \lambda \frac{k}{\theta_* (1 + \theta_* k)} = 0
\]

(1.45)
\( \mathcal{G} \) is given by

\[
\mathcal{G} = \frac{1}{\rho} \left( \frac{\theta_* \mu + (1 - \theta_*)r}{\rho} - 1 - \frac{\theta^2 \sigma^2}{\rho} + \frac{\lambda}{\rho} \log (1 + \theta_* k) + \log \rho \right)
\]  

(1.46)

After some algebra, (1.45) becomes

\[
-\sigma^2 k \theta_*^2 + \left( (\mu - r) k - \sigma^2 \right) \theta_* + \mu - r + \lambda k = 0
\]

(1.47)

The discriminant, \( \triangle \), is

\[
\triangle = \left( (\mu - r) k - \sigma^2 \right)^2 + 4\sigma^2 k (\mu - r + \lambda k)
\]

\[
= \left( (\mu - r) k + \sigma^2 \right)^2 + 4\sigma^2 k^2 \lambda > 0
\]

(1.48)

implying the equation (1.47) has two distinct roots, \( \theta_1 \) and \( \theta_2 \). Based on the relationship between roots and coefficients for a quadratic equation,

\[
\left( \theta_1 + \frac{1}{k} \right)\left( \theta_2 + \frac{1}{k} \right) = \theta_1 \theta_2 + \frac{\theta_1 + \theta_2}{k} + \frac{1}{k^2}
\]

\[
= \frac{\mu - r + \lambda k}{-\sigma^2 k} + \frac{1}{k} \frac{(\mu - r) k - \sigma^2}{\sigma^2 k} + \frac{1}{k^2}
\]

\[
= \frac{-\lambda}{\sigma^2} < 0
\]

(1.49)

meaning between \( \theta_1 \) and \( \theta_2 \), one is greater than \( -\frac{1}{k} \), the other one is less than \( -\frac{1}{k} \).
1.5.2 Proof of Proposition 2

In the proof, we assume that \( \lambda, \mu^o \) and \( \mu^p \) are constants. This can be regarded as a special case of the model in Section 1.3. The assumption is innocuous: agents with logarithmic utility functions are “myopic” in the sense that their decisions today just depend on today’s state variables without looking forward in the future. The advantage of treating \( \lambda, \mu^o \) and \( \mu^p \) constants is to reduce state variables for Bellman Equation and save the space.

The proof follows 3 steps:

1. Conjecture the value function and derive the first order conditions

2. Construct a series of discrete random variables \( k^{(n)} \) (or equivalently, \( Y^{(n)} \)) to approximate \( k \) (or \( Y \)). And solve the consumption and portfolio choice under the discrete approximation, prove the equilibrium exists and verify the conjectures in step (1).

3. Let \( n \to \infty \), \( k^{(n)} \to k \) and \( Y^{(n)} \to Y \) but portfolio choice \( \theta^o \to 1 \). Hence, no trade occurs in the market.

To be clear, we re-state the agent’s problem and equilibrium definition in the “safe-debt-only” market. They are the special case of what is stated in Section 1.3.2.

Problem 3.

\[
\max_{\theta^o_t, C^o_t} \mathbb{E}_0^o \left\{ \int_0^\infty e^{-\rho t} \log(C_t^o) dt \right\} \quad (1.50)
\]

subject to

\[
\frac{dW_t^o}{W_t^o} = \theta_{t-}^o \left( \frac{dS_t}{S_t} + \frac{\delta_{t-}}{S_{t-}} \right) + (1 - \theta_{t-}^o) \frac{dB_t^f}{B_{t-}^f} - c_{t-}^o \quad (1.51)
\]
where $\mathbb{E}^o$ denotes the expectation under the optimist’s belief and $c_{t-}^o := \frac{c_{t-}}{W_{t-}}$ is the consumption-wealth ratio.

**Definition 2.** A competitive equilibrium of the “safe-debt-only” market is composed of \( \{\theta^o, \theta^p, c^o, c^p\} \) and prices \( \{r^f, S\} \) such that

1. Given prices \( \{r^f, S\} \), \( \theta^o, c^o \) solve the optimist’s problem.
2. Given prices \( \{r^f, S\} \), \( \theta^p, c^p \) solve the pessimist’s problem similar to 3.
3. Market Clears, i.e.
   \[
   \begin{align*}
   (a) \quad & c^o W^o + c^p W^p = \mathcal{E} \\
   (b) \quad & \theta^o W^o + \theta^p W^p = S \\
   (c) \quad & (1 - \theta^o) W^o + (1 - \theta^p) W^p = 0
   \end{align*}
   \]

**Step 1**

Denote \( \mathcal{J} \) the value function of a representative optimistic agent. It is not hard to see that \( \mathcal{J} \) is a function of two state variables: \( W^o \) and endogenous state wealth ratio \( \omega_{t-} := \frac{W^o}{W_{t-}} \).

We conjecture the dynamics of \( \omega \) under objective probability measure can be written as

**Conjecture 1.**

\[
\frac{d\omega_{t-}}{\omega_{t-}} = f_1(\omega_{t-}) dt + f_2(\omega_{t-}) dz_t + f_3(\omega_{t-}) dN_t 
\]
where \( f_1, f_2, f_3 \) are some function of \( \omega_t \) satisfying regular conditions.

The conjecture \([1]\) says the dynamics of \( \omega \) is autonomous and is not affected by \( W_i, i \in \{o, p\} \). And this will be confirmed later.

Under the optimist’s belief, (1.52) follows:

\[
\frac{d\omega_o^\epsilon}{\omega_o^\epsilon} = \left( f_1(\omega_{t-}) + \frac{f_2(\omega_{t-})}{\sigma} (\mu^o - \mu) \right) dt + f_2(\omega_{t-}) d\omega_o^\epsilon + f_3(\omega_{t-}) dN_t
\] (1.53)

Conjecture 2.

\[
\mu_s = \mu - \lambda \mathbb{E}[k], \sigma_s = \sigma, k_s = k, \frac{\delta}{S} = \rho
\] (1.54)

Hence, the total return to stock is \( \frac{dS}{S} + \frac{\delta}{S} = \mu + \rho - \lambda \mathbb{E}[k] \). All conjectures will be confirmed below.

Applying Ito’s Lemma, the value function \( \mathcal{J}(t, W^o, \omega) \) satisfies the following PDE

\[
0 = \sup_{\theta, \omega} \left\{ \mathcal{J}_t + \mathcal{J}_{W^o} (\theta^o (\mu^o - \lambda \mathbb{E}[k] + \rho) + (1 - \theta^o) r^f - c^o) W^o + \frac{1}{2} \mathcal{J}_{W^o} (W^o)^2 (\theta^o)^2 \sigma^2 + \mathcal{J}_{\omega} \left( f_1(\omega_{t-}) + \frac{f_2(\omega_{t-})}{\sigma} (\mu^o - \mu) \right) \omega + \frac{1}{2} \mathcal{J}_{\omega \omega} f_2^2(\omega) \omega^2 + \lambda (\mathcal{J}(t, (1 + \theta^o k) W^o, (1 + f_3) \omega) - \mathcal{J}(t, W^o, \omega)) + \mathcal{J}_{W^o \omega} \theta^o W^o \sigma f_2(\omega) + e^{-\rho t} \log (c^o W^o) \right\}
\] (1.55)
Conjecture 3.

\[ \mathcal{J} = e^{-\rho t} \left[ \frac{\log(W^o)}{\rho} + \mathcal{G}(\omega) \right] \]  

(1.56)

With (1.55) and (1.56), we have

\[
0 = \sup_{\theta^o, c^o} \left\{ -\log(W^o) - \rho \mathcal{G}(\omega) + \frac{\theta^o (\mu^o - \lambda \mathbb{E}[k] + \rho) + (1 - \theta^o) r^f - c^o}{\rho} - \frac{(\theta^o \sigma)^2}{2\rho} \right.

+ \mathcal{G}(f_1(\omega t) + f_2(\omega t - \mu \sigma) - (\mu^o - \mu)) \omega + \frac{1}{2} \mathcal{G}(f_2(\omega t) \omega^2) + \lambda \mathbb{E} \left[ \frac{\log(1 + \theta^o k)}{\rho} + \mathcal{G}(1 + f_3 \omega) - \mathcal{G}(\omega) \right]

+ \log(c^o W^o) \left\} \right.

(1.57)

From first order conditions,

\[ 0 = \mu^o - \lambda \mathbb{E}[k] + \rho - r^f - \sigma^2 + \lambda \mathbb{E} \left[ \frac{k}{1 + \theta^o k} \right] \]  

(1.58)

\[ 0 = \frac{1}{c^o} - \frac{1}{\rho} \]  

(1.59)

Similar to (1.58) and (1.59), we can derive the consumption and portfolio choice by the pessimistic agent.

\[ 0 = \mu^p - \lambda \mathbb{E}[k] + \rho - r^f - \sigma^2 + \lambda \mathbb{E} \left[ \frac{k}{1 + \theta^p k} \right] \]  

(1.60)

\[ 0 = \frac{1}{c^p} - \frac{1}{\rho} \]  

(1.61)
To complete the characterization of the equilibrium, we need market clearing conditions.

\[ c^o W^o + c^p W^p = \varepsilon \]  
\[ \theta^o W^o + \theta^p W^p = S \]  
\[ (1 - \theta^o)W^o + (1 - \theta^p)W^p = 0 \]  

With (1.63) and (1.64),

\[ W^o + W^p = S \]  

With (1.62), (1.59) and (1.61),

\[ W^o + W^p = \frac{\varepsilon}{\rho} \]  

With (1.65) and (1.66),

\[ \frac{\varepsilon}{S} = \rho \]  
\[ \frac{dS}{S} = (\mu - \lambda \mathbb{E}[k]) dt + \sigma dz_t + k dN_t \]  

This confirms the Conjecture 2.

From (1.64),

\[ \theta^p = 1 + \omega (1 - \theta^o) \]  

And substitute (1.69) in (1.60):

\[ 0 = \mu^p - \lambda \mathbb{E}[k] + \rho - r^f - \sigma^2 (1 + \omega (1 - \theta^o)) + \lambda \mathbb{E} \left[ \frac{k}{1 + (1 + \omega (1 - \theta^o)) k} \right] \]
\[0 = (\mu^p - \mu^o) - \sigma^2 (1 - \theta^o) (1 + \omega) + \lambda \mathbb{E}\left[ \frac{k}{1 + (1 + \omega (1 - \theta^o)) k} \right] - \lambda \mathbb{E}\left[ \frac{k}{1 + \theta^o k} \right]\] (1.71)

**Step 2**

We now work with \(k\) by approximating it with a sequence of discrete random variables \(k^{(n)}\), defined as

\[
\{k^{(n)} = k_{l+1}^{(n)}\} = k_l^{(n)} < k \leq k_{l+1}^{(n)}
\] (1.72)

where \(-1 = k_0^{(n)} < k_1^{(n)} < \ldots < k_n^{(n)} = \infty\) is a partition of interval \((-1, \infty)\) associated with \(k^{(n)}\). As \(n\) gets greater, the partition also becomes finer. We also define

\[
\begin{align*}
\frac{k_n^{(n)}}{1 + \theta^o k_n^{(n)}} & \bigg|_{k_n^{(n)} = \infty} = \frac{1}{\theta^o} \\
\frac{k_n^{(n)}}{1 + (1 + \omega (1 - \theta^o)) k_n^{(n)}} & \bigg|_{k_n^{(n)} = \infty} = \frac{1}{1 + \omega (1 - \theta^o)}
\end{align*}
\] (1.73, 1.74)

Note by definition of expectation of continuous random variable,

\[
\mathbb{E}\left[ \frac{k}{(1 + \theta^o k)} \right] = \lim_{n \to \infty} \mathbb{E}\left[ \frac{k^{(n)}}{1 + \theta^o k^{(n)}} \right] = \lim_{n \to \infty} \sum_{l=0}^{n-1} \frac{k_l^{(n)}}{1 + \theta^o k_{l+1}^{(n)}} \mathbb{P}[k_l^{(n)} < k \leq k_{l+1}^{(n)}]
\] (1.75)

Similarly,

\[
\mathbb{E}\left[ \frac{k}{1 + (1 + \omega (1 - \theta^o)) k} \right] = \lim_{n \to \infty} \mathbb{E}\left[ \frac{k^{(n)}}{1 + (1 + \omega (1 - \theta^o)) k^{(n)}} \right]
\] (1.76)

So we can start by approximating \(\mathbb{E}\left[ \frac{k}{(1 - \theta^o k)} \right]\) and \(\mathbb{E}\left[ \frac{k}{1-(1+\omega(1-\theta^o))k} \right]\) and let \(n \to \infty\).
eventually.

As $n$ gets greater, $k_1^{(n)} < 0$. For the aforementioned reason, wealth has to stay positive always and thus $\theta^o < -\frac{1}{k_1^{(n)}}$. All we need is to prove $\forall \omega \in (0, \infty), \exists \theta^o \in \left(1, -\frac{1}{k_1^{(n)}}\right)$ solves

$$0 = (\mu^p - \mu^o) - \sigma^2 (1 - \theta^o) (1 + \omega) + \lambda \mathbb{E} \left[ \frac{k^{(n)}}{1 + (1 + \omega (1 - \theta^o)) k^{(n)}} \right] - \lambda \mathbb{E} \left[ \frac{k^{(n)}}{1 + \theta^o k^{(n)}} \right]$$

(1.77)

Denote

$$\mathcal{H}(\theta^o) = (\mu^p - \mu^o) - \sigma^2 (1 - \theta^o) (1 + \omega) + \lambda \mathbb{E} \left[ \frac{k^{(n)}}{1 + (1 + \omega (1 - \theta^o)) k^{(n)}} \right] - \lambda \mathbb{E} \left[ \frac{k^{(n)}}{1 + \theta^o k^{(n)}} \right]$$

(1.78)

It is easy to see that

$$\mathcal{H}(\theta^o) < 0, \text{ as } \theta^o \to 1$$

(1.79)

$$\mathcal{H}(\theta^o) \to \infty, \text{ as } \theta^o \to -\frac{1}{k_1^{(n)}}$$

(1.80)

Hence, according intermediate value theorem, there exists $\theta^o \in \left(1, -\frac{1}{k_1^{(n)}}\right)$ such that $\mathcal{H}(\theta^o) = 0$. Also, Conjecture 2 can be easily verified. It shows the equilibrium exists under the approximating discrete distributions $k^{(n)}$ for jump amplitude.

Step 3

Finally, by letting $n \to \infty$ in (1.72), $\mathbb{E} \left[ \frac{k^{(n)}}{1 + \theta^o k^{(n)}} \right] \to \mathbb{E} \left[ \frac{k}{1 + \theta^o k} \right]$ and $\theta^o \to 1$ as $-\frac{1}{k_1^{(n)}} \to 1$. There is no trade and the equilibrium breaks down.
1.5.3 Proof of Proposition 3

In the proof, we assume that $\lambda$, $\mu^o$ and $\mu^p$ are constants. This can be regarded as a special case of the model in Section 1.3. Moreover, the assumption is innocuous: agents with logarithmic utility functions are “myopic” in the sense that their decisions today just depend on today’s state variables without looking forward in the future. The advantage of treating $\lambda$, $\mu^o$ and $\mu^p$ constants is to reduce state variables for Bellman Equation and save the space.

Proof Steps Roadmap

1. Conjecture stock price $\frac{dS}{S}$ and value function $J$ and derive first order conditions.

2. Solve for consumption choice and verify the conjectures in step (1); also solve for credit spread.

3. Solve for portfolio choice on stock and provide conditions for the equilibrium to exist.

To be clear, we re-state the agent’s problem and equilibrium definition here. They are similar to what is stated in Section 1.3.2.

Problem 4.

$$\max_{\theta^o, \theta^{d,o}, C^o} \mathbb{E}^0 \left\{ \int_0^\infty e^{-\rho t} \log(C^o) dt \right\}$$

subject to

$$\frac{dW^o}{W^o} = \theta^o \left( \frac{dS}{S} + \frac{\xi}{S} \right) + \theta^{d,o} \frac{dB^d}{B^d} + (1 - \theta^o - \theta^{d,o}) \frac{dB^f}{B^f} - c^o$$

where $\frac{dS}{S}$, $\frac{dB^d}{B^d}$ and $\frac{dB^f}{B^f}$ are given by (1.23), (1.26) and (1.25). $\theta^o$ and $\theta^{d,o}$ are optimist’s positions on stock and risky debt.
Definition 3. A competitive equilibrium of the “defaultable debt” market is composed of \( \{ \theta^o, \theta^{d,o}, \theta^p, \theta^{d,p}, c^o, c^p \} \) and prices \( \{ r, r^f, S \} \) and debt contract variable \( \bar{k} \) such that

1. Given prices \( \{ r^d, r^f, S \} \) and debt contract variable \( \bar{k} \), \( \theta^o, \theta^{d,o}, c^o \) solve the optimist’s problem 4.

2. Given prices \( \{ r^d, r^f, S \} \) and debt contract variable \( \bar{k} \), \( \theta^p, \theta^{d,p}, c^p \) solve the pessimist’s problem similar to 4.

3. Default occurs when the optimist’s net worth drops no less than \( 100|\gamma| \% \), i.e. \( \theta^o k^s \leq \gamma \);
   If default occurs, the writedown \( k^d = k \), otherwise \( k^d = 0 \).

4. Given debt contract variable \( \bar{k} \), all market Clears, i.e.
   
   \[(a)\] \( c^o W^o + c^p W^p = \mathcal{E} \)
   
   \[(b)\] \( \theta^o W^o + \theta^p W^p = S \)
   
   \[(c)\] \( \theta^{d,o} W^o + \theta^{d,p} W^p = 0 \)

5. The optimists and pessimists are engaged in the non-cooperative, fully decentralized bargaining with no cost and determine \( \bar{k} \) in the equilibrium.

Step 1

Conjecture 4.

\[
\frac{dS}{S} = (\mu - \lambda \mathbb{E}[k]) \, dt + \sigma \, dz_t + k \, dN_t \quad (1.83)
\]

\[
\frac{\mathcal{E}}{S} = \rho \quad (1.84)
\]
\[0 = \sup_{\theta^o,\theta^d,\alpha,\sigma^o} \{ \mathcal{J}_t + \mathcal{J}_{W^\alpha} (\theta^o (\mu^o + \rho - \lambda \mathbb{E}[k]) + \theta^d \omega d + (1 - \theta^o - \theta^d) r_f - c^o) + \]
\[\frac{1}{2} \mathcal{J}_{W^\alpha} (W^\alpha)^2 (\theta^o)^2 \sigma^2 + \mathcal{J}_{\omega \omega} \left( f_1 (\omega) + \frac{f_2 (\omega)}{\sigma} (\mu^o - \mu) \right) + \frac{1}{2} \mathcal{J}_{\omega \omega} f_2^2 (\omega) \omega^2 + \mathcal{J}_{W^\alpha W^\alpha} \omega^\alpha \sigma f_2 (\omega) + \lambda \mathbb{E} \left[ \mathcal{J} (t, (1 + \theta^o k + \theta^d \omega d) W, (1 + f_{3\omega}) - \mathcal{J} (t, W, \omega)) + e^{-\rho t} \log (c^o W^\alpha) \right] \}
\] (1.85)

**Conjecture 5.**

\[\mathcal{J} = e^{-\rho t} \left[ \frac{\log (W^\alpha)}{\rho} + \mathcal{G} (\omega) \right] \] (1.86)

\[0 = \sup_{\theta^o,\theta^d,\alpha,\sigma^o} \left\{ - \log (W^\alpha) - \rho \mathcal{G} (\omega) + \frac{\theta^o (\mu^o + \rho - \lambda \mathbb{E}[k]) + \theta^d \omega d + (1 - \theta^o - \theta^d) r_f - c^o - (\theta^o \sigma)^2}{2 \rho} \right. \]
\[+ \mathcal{G} (\omega) \left( f_1 (\omega) + \frac{f_2 (\omega)}{\sigma} (\mu^o - \mu) \right) + \frac{1}{2} \mathcal{G}_{\omega \omega} f_2^2 (\omega) \omega^2 + \lambda \mathbb{E} \left[ \frac{1 + \theta^o k + \theta^d \omega d}{\rho} + \mathcal{G} (1 + f_{3\omega}) \right] - \mathcal{G} (\omega) + \log (c^o W^\alpha) \right\} \] (1.87)

**First Order conditions**

\[\mu^o + \rho - \lambda \mathbb{E}[k] - r_f - \theta^o \sigma^2 + \lambda \mathbb{E} \left[ \frac{k}{1 + \theta^o k + \theta^d \omega d} \right] = 0 \] (1.88)

\[r^d - r_f + \lambda \mathbb{E} \left[ \frac{k^d}{1 + \theta^o k + \theta^d \omega d} \right] = 0 \] (1.89)

\[c^o - \rho = 0 \] (1.90)
Similarly, we can write down the Bellman equation and first order conditions for the pessimists.

**Step 2** (1.90) and the counterpart for pessimists together with goods market clearing condition,

\[
\frac{S}{\delta S} = \rho \quad (1.91)
\]

\[
\frac{dS}{S} = \frac{d\delta}{\delta} = (\mu - \lambda \mathbb{E}[k]) \, dt + \sigma \, dz_t + k \, dN_t \quad (1.92)
\]

Given we look for a threshold equilibrium for \( k^d \), i.e.

\[
k^d = \begin{cases} 
0, & \text{if } k > \bar{k} \\
k, & \text{if } k \leq \bar{k}
\end{cases} \quad (1.93)
\]

and also

\[
\mathbb{E}[k] = \int_{-\infty}^{\infty} (e^y - 1) \, p_Y(y) \, dy
\]

\[
= \int_{-\infty}^{\infty} (e^y - 1) \frac{1}{\mathcal{B}(2, 2)} \frac{e^{-2y}}{(1+e^{-y})^3} \, dy
\]

\[
= 1 \quad (1.94)
\]

\[
\mathbb{E} \left[ \frac{k}{1 + \theta^o k + \theta^d o k^d} \right] = \int_{-\infty}^{\bar{\gamma}} \frac{e^y - 1}{1 + (\theta^o + \theta^d o)(e^y - 1)} \, p_Y(y) \, dy + \int_{\bar{\gamma}}^{\infty} \frac{e^y - 1}{1 + \theta^o (e^y - 1)} \, p_Y(y) \, dy
\]

(1.95)
\[
E \left[ \frac{k^d}{1 + \theta^o k + \theta^{d,o} k^d} \right] = \int_{-\infty}^{y} \frac{e^y - 1}{1 + (\theta^o + \theta^{d,o}) (e^y - 1)} p_Y(y) \, dy \tag{1.96}
\]

Hence, from (1.88), (1.95) and (1.96), we get

\[
r^d = \mu^o + \rho - \lambda - \theta^o \sigma^2 + \lambda \int_{y}^{\infty} \frac{e^y - 1}{1 + \theta^o (e^y - 1)} p_Y(y) \, dy
\]

\[
= \mu^o + \rho - \lambda - \theta^o \sigma^2 + \lambda E \left[ \frac{k}{1 + \theta^o k} \middle| k > \bar{k} \right] \mathbb{P} \left( k > \bar{k} \right)
\]

\[
= \mu^o + \rho - \lambda - \theta^o \sigma^2 - \lambda \left( \frac{6(\theta^o - 1)\theta^o \log(\theta^o)}{(1 - 2\theta^o)^4} + \frac{4(2\theta^o - 1)^3}{(k+2)^4} - \frac{6(\theta^o - 1)(2\theta^o - 1)}{k+2} + \frac{3(1 - 2\theta^o)^2}{(k+2)^2} \right)
\]

\[
\left( \frac{6(\theta^o - 1)\theta^o \log(\bar{k} + 2) - 6(\theta^o - 1)\theta^o \log(\theta^o \bar{k} + 1)}{(1 - 2\theta^o)^4} \right)
\tag{1.97}
\]

The credit spread, \( r^d - r^f \)

\[
r^d - r^f = -\lambda E \left[ \frac{k}{1 + (\theta^o + \theta^{d.o}) k} \middle| k \leq \bar{k} \right] \mathbb{P} \left( k \leq \bar{k} \right)
\]

\[
= -\lambda \left( \frac{\theta^o + \theta^{d.o} - 1)(\theta^o + \theta^{d.o})(\log(1 - (\theta^o + \theta^{d.o}) \bar{k} + 1) - \log((\theta^o + \theta^{d.o}) \bar{k} + 1) + \log(\bar{k} + 2))}{1 - 2(\theta^o + \theta^{d.o})^4} \right)
\tag{1.98}
\]

Note (1.98) does not involve any beliefs heterogeneity, unlike (1.97) ((1.97) contains a term \( \mu^o \)). Thus we conjecture in equilibrium,

\[
\theta^o + \theta^{d.o} = 1 \tag{1.99}
\]

\[
\theta^p + \theta^{d,p} = 1 \tag{1.100}
\]
This conjecture satisfies the market clearing condition of risk-free bonds. In addition, it shows the risk-free interest rate is a truly “shadow” rate: both agents invest zero proportion of their wealth in the risk-free bonds.

\[ r^d - r^f = -\lambda \left( \frac{4}{(\bar{k} + 2)^3} - \frac{3}{(\bar{k} + 2)^2} - 1 \right) \]  

(1.101)

**Step 3**

The last step is to solve \( \theta^o \) and \( \theta^p \). With first order conditions (1.97), its counterpart of pessimist’s, market clearing condition:

\[ \theta^o \omega + \theta^p = 1 + \omega \]  

(1.102)

Also note that in equilibrium the threshold \( \bar{k} \) must satisfy

\[ \theta^o \bar{k} = \gamma \]  

(1.103)

With these conditions, \( \theta^o \) is the solution for the following equation

\[ \mu^o - \mu^p = \mathcal{D}(\theta^o; \omega, \gamma) \]  

(1.104)
where

\[ Q (\theta^o; \omega, \gamma) = \theta^o \sigma^2 - \sigma^2 (-\theta^o \omega + \omega + 1) - \lambda \left( - \frac{6(\theta^o - 1) \theta^o \log(\theta^o)}{(1 - 2\theta^o)^4} + \frac{4(2\theta^o - 1)^3}{(\frac{\gamma}{\omega} + 2)^3} - \frac{6(\theta^o - 1)(2\theta^o - 1)}{\frac{\gamma}{\omega} + 2} + \frac{3(1 - 2\theta^o)^2(3 - 4\theta^o)}{(\frac{\gamma}{\omega} + 2)^2} - \frac{6(\theta^o - 1) \theta^o \log \left( \frac{\gamma}{\theta^o} + 2 \right) - 6(\theta^o - 1) \theta \log(\gamma + 1)}{(1 - 2\theta^o)^4} \right) \]

\[ + \lambda \left( - \frac{6(\omega - \theta^o \omega)(-\theta^o \omega + \omega + 1) \log(-\theta^o \omega + \omega + 1)}{(1 - 2(-\theta^o \omega + \omega + 1))^4} - \frac{4(2(-\theta^o \omega + \omega + 1))}{(\frac{\gamma}{\omega} + 2)^3} - \frac{6(\omega - \theta^o \omega)(2(-\theta^o \omega + \omega + 1) - 1)}{\frac{\gamma}{\omega} + 2} + \frac{3(1 - 2(-\theta^o \omega + \omega + 1))^2(3 - 4(-\theta^o \omega + \omega + 1))}{(\frac{\gamma}{\omega} + 2)^2} \right) \]

\[ - \frac{6(\omega - \theta^o \omega)(-\theta^o \omega + \omega + 1) \log (\frac{\gamma}{\theta^o} + 2) - 6(\omega - \theta^o \omega)(-\theta^o \omega + \omega + 1) \log \left( \frac{\gamma(-\theta^o \omega + \omega + 1)}{\theta^o} + 1 \right)}{(1 - 2(-\theta^o \omega + \omega + 1))^4} \]

(1.105)

\[ Q (\theta^o; \omega, \gamma) \text{ is a continuous function defined on } [0, 1 + \frac{1}{\omega}] \text{ with limit} \]

\[ \lim_{\theta^o \to 1} Q (\theta^o; \omega, \gamma) = (\omega + 1) \left( \frac{\lambda \left( (6\gamma^2 + 15\gamma + 10) \omega^2 + 5(3\gamma + 4) \omega + 10 \right)}{(\gamma + 2) \omega + 2} + \frac{\sigma^2}{\omega} + \frac{x^2}{(\omega + 2)^3} - \frac{3(\omega + 1)(\omega + 4) \omega^2 + 6(\omega + 1)}{(\gamma + 2) \omega + 2} - 6\omega \log(\gamma + 1) + 6\omega \log(\frac{\gamma}{\omega} + 2) + 6 \log \left( \frac{\gamma}{\omega} + 1 \right) \right) \]

(1.106)

\[ \lim_{\theta^o \to 0} Q (\theta^o; \omega, \gamma) = \frac{1}{8} \left( \frac{3\lambda(2\gamma + 1)^2}{(\gamma + 1)^4} - 4\sigma^2 \omega - 4\sigma^2 + \frac{2\lambda}{(\omega + 2)^4} \left( -6\omega(\omega + 2) \log(2(\gamma + 1)) - 6\omega(\omega + 2) \log \left( \frac{\omega + 2}{2} \right) \right) + \frac{(\omega + 1)(6\gamma^2 + 3\gamma (2\omega^2 + 7\omega + 1) + 4\omega^2 + 11\omega + 1) + 6(\gamma + 1)^3 \omega(\omega + 2) \log(\gamma(\omega + 2) + 1)}{(\gamma + 1)^4} \right) \]

(1.107)
\[
\lim_{\theta_0 \to 1} \mathcal{Q}(\theta_0; \omega, \gamma) = \mathcal{Q}(1, \omega, \gamma) = 0 \tag{1.108}
\]

\[
\lim_{\theta_0 \to 0} \mathcal{Q}(\theta_0; \omega, \gamma) = -\frac{6\lambda \omega(\omega + 1)(-\log(\gamma(\omega + 1)) + \log(\gamma) + \log(\omega + 1))}{(2\omega + 1)^4} + \sigma^2(-\omega) - \sigma^2 \tag{1.109}
\]

And therefore, by extreme value theorem, the \( \max_{[0, 1 + \frac{1}{2\omega}]} \mathcal{Q}(\theta_0; \omega, \gamma) \) is well defined.
1.5.4 Proof of Theorem 1

In the proof, we assume that $\lambda$, $\mu^o$ and $\mu^p$ are constants. This can be regarded as a special case of the model in Section 1.3. Moreover, the assumption is innocuous: agents with logarithmic utility functions are “myopic” in the sense that their decisions today just depend on today’s state variables without looking forward in the future. The advantage of treating $\lambda$, $\mu^o$ and $\mu^p$ constants is to reduce state variables for Bellman Equation and save the space.

The steps for proving theorem 1 is similar to those in Proposition 3. We skip over the conjecture step and state the first order conditions directly.

From (1.32):

$$0 = \mu^o + \rho - \lambda E[k] - r^f - \theta^o \sigma^2 + \lambda \left( \int_{-\infty}^{\bar{y}} \frac{e^y - 1}{1 + (\theta^o + \theta^d.o)(e^y - 1)} p_Y(y) dy + \int_{\bar{y}}^{\tilde{y}} \frac{e^y - 1}{1 + \theta^o(e^y - 1)} p_Y(y) dy - \int_{\tilde{y}}^{\infty} \frac{e^y - 1}{1 + (\theta^o + \theta^d,o)(e^y - 1)} p_Y(y) dy \right)$$

(1.110)

From (1.33):

$$r^d - r^f = -\lambda \left( \int_{-\infty}^{\bar{y}} \frac{e^y - 1}{1 + (\theta^o + \theta^d,o)(e^y - 1)} p_Y(y) dy \right)$$

(1.111)

From (1.34)

$$r^d - r^f = \lambda \left( \int_{\tilde{y}}^{\infty} \frac{e^y - 1}{1 + (\theta^o + \theta^d,o)(e^y - 1)} p_Y(y) dy \right)$$

(1.112)
Hence, (1.110) becomes

\[ r^d + r^d - r^f = \mu^o + \rho - \lambda \mathbb{E}[k] - \theta^o \sigma^2 + \lambda \mathbb{E}\left[ \frac{k}{1 + \theta^o k} | \bar{k} < k < \tilde{k} \right] \mathbb{P}\left( \bar{k} < k < \tilde{k} \right) \]  
(1.113)

\[ = \mu^o + \rho - \lambda \mathbb{E}[k] - \theta^o \sigma^2 + \lambda \int_{y}^{\tilde{y}} \frac{e^y - 1}{1 + \theta^o (e^y - 1)} p_y(y) dy \]  
(1.114)

Note \( \frac{1}{1 + \theta^o} = \frac{\log'(1 + \theta^o)W}{\log'(\rho W)} = \frac{\log'(C_\ell^+)}{\log'(C_\ell^-)} \) is the marginal rate of substitution given the jump happens at \( t \).

Similarly, we can derive the first order conditions for the pessimists. In equilibrium, \( \theta^o \) is the solution of the following equation:

\[ \mu^o - \mu^p = Q(\theta^o; \omega, \gamma, \zeta) \]  
(1.115)

where \( Q(\theta^o; \omega, \gamma, \zeta) := 1_{\theta^o \in (1, 1+1]} \mathcal{Q}(\theta^o; \omega, \gamma) + 1_{\theta^o \in [1+1, \infty)} \tilde{\mathcal{Q}}(\theta^o; \omega, \gamma, \zeta) \). \( \mathcal{Q}(\theta^o; \omega, \gamma, \zeta) \) consists of two parts: \( \theta^o \in (1, \frac{1}{\omega} + 1] \), \( \mathcal{Q}(\theta^o; \omega, \gamma) \), when belief dispersion is mild and risky bond for positive jump is not needed for trade to happen between agents; and \( \theta^o \in [\frac{1}{\omega} + 1, \infty) \), \( \tilde{\mathcal{Q}}(\theta^o; \omega, \gamma, \zeta) \), when belief dispersion is large, risky bond for positive jump is issued by short sellers to cover the position on stock. We have the following observation

**Lemma 1.** \( Q(\theta^o; \omega, \gamma, \zeta) \) is a continuous function on \((1, \infty)\)

**Proof.** All we need to do is to prove \( Q(\theta^o; \omega, \gamma, \zeta) \) is continuous on \( \theta^o = \frac{1}{\omega} + 1 \)

\[ \lim_{\theta^o \to \frac{1}{\omega} + 1} \mathcal{Q}(\theta^o; \omega, \gamma) = \lim_{\theta^o \to \frac{1}{\omega} + 1} \tilde{\mathcal{Q}}(\theta^o; \omega, \gamma, \zeta) \]  
(1.116)
Lemma 2. There exists a solution $\theta^o$ in $(1, \infty)$ to equation (1.115).

Proof. Note

\[ Q(1, \omega, \gamma, \zeta) = 0 \]  \hspace{1cm} (1.117)

\[ \lim_{\theta^o \to \infty} Q(\theta^o, \omega, \gamma, \zeta) \approx \sigma^2 \theta^o (1 + \omega) \to \infty \]  \hspace{1cm} (1.118)

Therefore, by intermediate value theorem, there exists $\theta^o$ in $(1, \infty)$ to equation (1.115).

Still, as in the proof of Proposition 3, $\theta^o + \theta^{d,o} = \theta^o + \theta^{d,o} = 1$, hence equilibrium credit spread is

\[ r^d - r^f = -\lambda \left( \frac{4}{(k + 2)^3} - \frac{3}{(k + 2)^2} \right) - 1 \]

\[ r^{d,c} - r^f = \lambda \left( \frac{2 + 3k}{(2 + k)^3} \right) \]  \hspace{1cm} (1.119)
\[ B(\theta^n, \omega, \gamma, \zeta) = \theta^n \sigma^2 - \sigma^2 (-\theta^n \omega + \omega + 1) \]

\[ -\lambda \left( -\frac{4(2\theta^n - 1)^3}{(\frac{1}{n} + 2)} + \frac{6(\theta^n - 1)(2\theta^n - 1)}{\frac{1}{n^2} + 2} - \frac{6(\theta^n - 1)\theta \log \left( \frac{\theta}{n} + 2 \right)}{(1 - 2\theta^n)^4} \right) \]

\[ + \lambda \left( \frac{4(2\theta^n - 1)^3}{(\frac{1}{n} + 2)} - \frac{6(\theta^n - 1)(2\theta^n - 1)}{\frac{1}{n^2} + 2} + \frac{6(\theta^n - 1)\theta \log \left( \frac{\theta}{n} + 2 \right)}{(1 - 2\theta^n)^4} \right) \]

\[ -6 \log(\zeta + 1)(-\theta^n \omega + \omega + 1 - \theta^n \omega)(-\theta^n \omega + \omega + 1) \log \left( \frac{\zeta}{\theta^n - \omega + \omega + 1} + 2 \right) \]

\[ + \frac{4(2(-\theta^n + \omega + 1) - 1)^3}{(\frac{1}{n} + 2)^2} - \frac{6(\theta^n - 1)(2(-\theta^n + \omega + 1) - 1)}{(1 - 2(-\theta^n + \omega + 1))^4} \]

\[ + \frac{3(1 - 2(-\theta^n + \omega + 1))^2(3 - 4(-\theta^n + \omega + 1))}{(\frac{1}{n} + 2)^2} \]

\[ - \frac{6(\theta^n - 1)(-\theta^n \omega + \omega + 1) \log \left( \frac{\zeta}{\theta^n - \omega + \omega + 1} + 2 \right)}{(1 - 2(-\theta^n + \omega + 1))^4} \]

\[ - \frac{6(\theta^n - 1)(-\theta^n \omega + \omega + 1) \log \left( \frac{\zeta}{\theta^n - \omega + \omega + 1} + 2 \right)}{(1 - 2(-\theta^n + \omega + 1))^4} \]
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Auto Loan</td>
<td>0.092</td>
<td>-0.257</td>
<td>0.487</td>
<td>0.175</td>
<td>1.000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mortgage</td>
<td>-0.139</td>
<td>-0.135</td>
<td>0.765</td>
<td>0.496</td>
<td>0.502</td>
<td>1.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>AAA Spread</td>
<td>-0.763</td>
<td>0.452</td>
<td>0.682</td>
<td>0.758</td>
<td>-0.210</td>
<td>0.310</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>Baa Spread</td>
<td>-0.606</td>
<td>0.362</td>
<td>0.562</td>
<td>0.610</td>
<td>-0.048</td>
<td>0.524</td>
<td>0.890</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Notes: Data Source: Federal Reserve Bank of St. Louis, 1981-2016

**Table 1.1**

**Correlation between different measures of Consumer and Corporate Debt Credit Spread**
<table>
<thead>
<tr>
<th>FICO Score</th>
<th>Delinquency Rate</th>
<th>Lending Risk Premium (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>750-799</td>
<td>2%</td>
<td>249.53</td>
</tr>
<tr>
<td>800-850</td>
<td>1%</td>
<td></td>
</tr>
</tbody>
</table>

Notes: Data Source: Federal Deposit Insurance Corporation, Board of Governors of the Federal Reserve System, Federal Reserve Bank of St. Louis.
<table>
<thead>
<tr>
<th>Basic Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility of aggregate endowment growth $\sigma$</td>
<td>3.44%</td>
</tr>
<tr>
<td>Long-run average growth of the aggregate endowment $\bar{\mu}$</td>
<td>1.55%</td>
</tr>
<tr>
<td>Volatility of the expected endowment growth $\sigma_{\mu}$</td>
<td>1.1%</td>
</tr>
<tr>
<td>Mean reversion of the expected endowment growth $\alpha_{\mu}$</td>
<td>0.05</td>
</tr>
<tr>
<td>Time preference $\rho$</td>
<td>0.03</td>
</tr>
<tr>
<td>Default Trigger $\gamma = \zeta$</td>
<td>-70%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Beliefs Formation Parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Behavior Bias $\phi$</td>
<td>9.7</td>
</tr>
<tr>
<td>Volatility of the signal(s) $\sigma_s$</td>
<td>0.55%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Disaster Risk Parameters</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Long-run annual probability of disaster $\bar{\lambda}$</td>
<td>3.63%</td>
</tr>
<tr>
<td>Volatility of disaster risk $\sigma_{\lambda}$</td>
<td>0.081</td>
</tr>
<tr>
<td>Mean reversion of disaster risk $\alpha_{\lambda}$</td>
<td>0.11</td>
</tr>
</tbody>
</table>
### Table 1.4
Annual Belief Dispersion, Risk-free Rate and Credit Spread.

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beliefs Dispersion</td>
<td>1.23%</td>
<td>1.29%</td>
</tr>
<tr>
<td></td>
<td>(0.71%)</td>
<td>(0.98%)</td>
</tr>
<tr>
<td>Risk-free Rate</td>
<td>1.07%</td>
<td>0.57%</td>
</tr>
<tr>
<td></td>
<td>(2.89%)</td>
<td>(1.01%)</td>
</tr>
<tr>
<td>Credit Spread</td>
<td>249.53</td>
<td>224.52</td>
</tr>
<tr>
<td></td>
<td>(102.63)</td>
<td>(139.62)</td>
</tr>
<tr>
<td>Probability of Default</td>
<td>≤ 1%</td>
<td>0.59%</td>
</tr>
</tbody>
</table>

*Notes.* Model parameters are listed in Table 1.3. Standard Deviation are in parentheses. *Data Source:* Philadelphia Fed Survey of Professional Forecasters, Federal Reserve Bank of St. Louis.
Table 1.5  
**Correlation Coefficients between beliefs dispersion, credit spread and risk-free Rate**

<table>
<thead>
<tr>
<th>Panel I: Correlation Coefficients <strong>[Data]</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Belief Dispersion</strong></td>
</tr>
<tr>
<td>Belief Dispersion</td>
</tr>
<tr>
<td>Credit Spread</td>
</tr>
<tr>
<td>Risk-free Rate</td>
</tr>
</tbody>
</table>

(A)

<table>
<thead>
<tr>
<th>Panel II: Correlation Coefficients <strong>[Model]</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Belief Dispersion</strong></td>
</tr>
<tr>
<td>Belief Dispersion</td>
</tr>
<tr>
<td>Credit Spread</td>
</tr>
<tr>
<td>Risk-free Rate</td>
</tr>
</tbody>
</table>

(B)

Notes. $\gamma$ is calibrated to match the average correlation coefficients between credit spread, risk-free rate and belief dispersion. All data in Panel I span from 1981 to 2016 on a quarterly basis. Hodrick-Prescott filter is applied to the belief dispersion to obtain the cyclical variation.
Table 1.6
Regression of Credit Spread

<table>
<thead>
<tr>
<th></th>
<th>Regression of Credit Spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.0004</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
</tr>
<tr>
<td>Beliefs Dispersion</td>
<td>0.1090</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>1.1132</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
</tr>
<tr>
<td>Relative Wealth Distribution</td>
<td>-0.0006</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.988</td>
</tr>
</tbody>
</table>

Notes. P-values are in parentheses.
<table>
<thead>
<tr>
<th>Regressor</th>
<th>Coefficient</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.02</td>
<td>(0.000)</td>
</tr>
<tr>
<td>Belief Dispersion</td>
<td>-0.2105</td>
<td>(0.000)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>-0.6071</td>
<td>(0.000)</td>
</tr>
<tr>
<td>Relative Wealth Distribution</td>
<td>0.0001</td>
<td>(0.675)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.005</td>
<td></td>
</tr>
</tbody>
</table>

*Notes. P-values are in parentheses.*
Table 1.8
Correlations between equilibrium $\bar{k}$ and $\mu^o - \mu^p$, $\lambda$ in the model.

<table>
<thead>
<tr>
<th>Beliefs Dispersion $\mu^o - \mu^p$</th>
<th>0.69</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rare Event Risk $\lambda$</td>
<td>-0.39</td>
</tr>
</tbody>
</table>
Table 1.9
Correlation between Consumer Debt-to-GDP ratio and risk-free rate

<table>
<thead>
<tr>
<th></th>
<th>Consumer Debt</th>
<th>Risk-Free Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumer Debt GDP</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>Risk-free Rate</td>
<td>-0.075</td>
<td>1.000</td>
</tr>
</tbody>
</table>

(A)

<table>
<thead>
<tr>
<th></th>
<th>Consumer Debt</th>
<th>Risk-Free Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumer Debt GDP</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>Risk-free Rate</td>
<td>0.001</td>
<td>1.000</td>
</tr>
<tr>
<td>Risk-free Rate</td>
<td>0.333</td>
<td>-</td>
</tr>
<tr>
<td>Risk-free Rate (Rare Event Risk)</td>
<td>-0.755</td>
<td>-</td>
</tr>
</tbody>
</table>

(B)

Notes. Risk-free Rate decomposition is based on regression result from Table 1.7.
Table 1.10
Correlation Coefficient between household leverage (FODSP) and risk-free rate.

<table>
<thead>
<tr>
<th>Correlation Coefficients [Data]</th>
<th>Household Leverage</th>
<th>Risk-Free Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Household Leverage</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>Risk-free Rate</td>
<td>0.033</td>
<td>1.000</td>
</tr>
</tbody>
</table>

(A)

<table>
<thead>
<tr>
<th>Correlation Coefficients [Model]</th>
<th>Household Leverage ($\theta^{d,o}$)</th>
<th>Risk-Free Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Household Leverage ($\theta^{d,o}$)</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>Risk-free Rate</td>
<td>0.000</td>
<td>1.000</td>
</tr>
<tr>
<td>Risk-free Rate (Rare Event Risk)</td>
<td>0.302</td>
<td>-</td>
</tr>
<tr>
<td>Risk-free Rate (Beliefs Dispersion)</td>
<td>-0.698</td>
<td>-</td>
</tr>
</tbody>
</table>

(B)

Notes. Risk-free Rate decomposition is based on regression result from Table 1.7.
Table 1.11  
Comparative Statics of $\gamma$

<table>
<thead>
<tr>
<th></th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-40%</td>
</tr>
<tr>
<td>Credit Spread</td>
<td>239.54</td>
</tr>
<tr>
<td>Volatility of Credit Spread</td>
<td>152.31</td>
</tr>
<tr>
<td>Probability of Default</td>
<td>0.78%</td>
</tr>
<tr>
<td>Default boundary $\bar{k}$</td>
<td>-19.90%</td>
</tr>
<tr>
<td>Risky Asset Holdings $\theta^o = \gamma \bar{k}$</td>
<td>2.010</td>
</tr>
<tr>
<td>Risk-free Rate $r^f$</td>
<td>0.575%</td>
</tr>
</tbody>
</table>
Figure 1.1

Household Debt and Corporate Debt

Household (solid line) and Corporate debt (dashed line) outstanding (in billions).

Data Source: Board of Governors of the Federal Reserve System [https://www.federalreserve.gov/releases/z1/current/accessibile/d3.htm]
Figure 1.2 Comparison of different distributions for modeling jump size parameter $Y$

The solid blue line is p.d.f of standard normal distribution; the green dot-dashed line is p.d.f of the generalized logistic distribution used in the current paper; the dashed red line is p.d.f of standard logistic distribution.
Figure 1.3

$Q(\theta^0; \omega, \gamma, \zeta)$ as a function of $\omega$ and $\theta^0$

Figure 1.3 (a) plots $Q(\theta^0; \omega, \gamma)$, the part of $Q(\theta^0; \omega, \gamma, \zeta)$ when short selling does not occur, as a function of $\theta^0$ and $\omega$.

Figure 1.3 (b) plots $\tilde{Q}(\theta^0; \omega, \gamma, \zeta)$ as a function of $\theta^0$ and $\omega$ when short selling occurs.

Note that the upper bounds of $Q(\theta^0; \omega, \gamma)$ are the starting points for $\tilde{Q}(\theta^0; \omega, \gamma, \zeta)$, i.e. the two functions can be “glued” together seamlessly.
Figure 1.4 (a) plots the position on risky assets $\theta^o$ (solid line, left axis) and $\bar{k} = \frac{\gamma}{\theta^o}$ (dashed line, right axis) as functions of belief dispersion $\mu^o - \mu^p$.

Figure 1.4 (b) plots the position on risk-free bonds $1 - \theta^o - \theta^{d,o}$ as a function of belief dispersion $\mu^o - \mu^p$. The rest parameters used in the example of comparative statics are: $r^f = 1.07\%$, $r^d = 3.57\%$, $\lambda = 3.63\%$, $\gamma = -70\%$, $\omega = 1.0$. 

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Figure 1.5(a) plots the consumer debt over GDP ratio (dashed line, left axis) and risk-free rate (solid line, right-axis, in percentage) from 1978 to 2016. Shaded areas indicate NBER recession dates.

Figure 1.5(b) plots the household leverage (dotted line, left axis) and risk-free rate (solid line, right-axis, in percentage) from 1980 to 2016. Shaded areas indicate NBER recession dates.

*Data Source:* Federal Reserve Bank of St. Louis.
Figure 1.6
Figure 1.6 (a) plots the credit spread $r^d - r^f$ as a function of relative wealth share $\omega = \frac{W_o}{W_p}$.
Figure 1.6 (b) plots the risk-free rate $r^f$ as a function of relative wealth share $\omega = \frac{W_o}{W_p}$. The rest parameters used in the example are: $\mu^o - \mu^p = 1.23\%, \rho = 3\%, \sigma = 3.44\%, \lambda = 3.63\%, \gamma = -70\%$. 
Chapter 2

Scooping Up Own Debt On the Cheap: The Effect Of Corporate Bonds Buyback on Firm’s Credit Condition

2.1 Introduction

It is well known that firms have trouble issuing new bonds to raise funding during economic recessions. However, an intriguing fact is that many firms are engaged in bonds buyback at the same time. As noted by the Wall Street Journal during the burst of dot-com bubble (Newswires (2000)):

“It’s not just investors who are bargain-hunting amid the beaten-down sectors of the corporate-bond market. Companies themselves are beginning to buy back their own debt at discount prices”

... 

“Stater Brothers Holdings Inc., a Southern California supermarket chain with strong single-B ratings, saw its bonds fall 20 points to about 80 cents on the dollar after reporting
a net loss of $9.1 million in May. Confident that it would be able to engineer a turnaround, it retired about $11 million of its debt, realizing an extraordinary gain of $1.1 million.”

The same also has occurred during the Great Recession. An article in the Wall Street Journal (Ng (2009)) has noted that

“A number of corporations are quietly buying back bonds on the cheap in the open market as the financial system works its way out of crisis mode. They are taking advantage of depressed prices to save millions of dollars in interest and debt-repayment costs.”

Not only bonds buyback is an important corporate finance strategy at micro level, it is also an economically important macroeconomic factor. Using debt repurchase data cover the 1996-2011 period, Julio (2013) documented that total debt repurchase activity has been increasing over time, from $11.7 billion in 1996 to $65.3 billion in 2011. In addition, it also shows cyclical pattern at an aggregate level. Begenau and Salomao (2014), Covas and Haan (2011) and Jermann and Quadrini (2012) found that debt repurchase is countercyclical. The countercyclicality of debt repurchase might also contribute to the procyclicality of debt maturity. Chen, Xu, and Yang (2012) found that the average debt maturity is longer in economic expansions than in recessions. This is not surprising given the fact that firms tend to buy more long-term bonds than short-term ones when buying back bonds: Julio (2013) showed that the average maturity of debt drops from 10.84 years to 6.90 years after repurchase.

Despite the significant role played by debt buyback in the financial markets and in the economy, few academic literature has studied it. In this paper, I provide a dynamic structural model for corporate debt buyback. The paper focuses on two questions. The first question is how debt buyback affects on firm’s default decision and credit rating. The second question is how liquidity, which is usually scarce during recession, affects firm’s buyback decision.
Indeed, the two questions are related. As previous literature including He and Xiong (2012), Ericsson and Renault (2006) and Chen, Lesmond, and Wei (2007) pointed out, credit risk and liquidity risk are intricately interconnected. However, to better understand their connection qualitatively and quantitatively, one also needs to consider firm’s strategies thoroughly such as rollover and debt buyback when they face those risks.

To study these questions, I employ and augment Leland and Toft (1996) model and provide a much more general framework accounting for both debt rollover and repurchase. As in Leland’s model, the firm’s assets are exogenous and follow a geometric Brownian motion. However, the firm commits itself to a stationary debt maturity structure not only by issuing new debt but also buying back outstanding debt. Depending on the firm’s current assets, issuing new debt, paying off maturing debt and buying back outstanding debt can result in capital gain or loss which equity holders have to assume. Any gain would be paid out to equity holders right away and any loss would be paid off by a new contribution from equity holders. The equity holders decide to default when assets drop to an endogenous threshold chosen by the equity holders; i.e. when the equity value reaches zero and the firm stops servicing the debt.

As for the first question, I find that the firm strategically chooses how much debt to buy back and the buyback program reduces the firm’s probability of default and consequently improves its credit ratings, relative to the case where the firm does not buy back any outstanding bonds. Moreover, the effect of corporate debt buyback on the firm’s credit risk changes with leverage. The model shows that higher leveraged firms are more actively engaged in bonds buyback and the firm’s credit condition improves more as a result of the buyback program. The model also shows that debt buyback strategy allows firm to employ more debt and the optimal leverage ratio is higher than what early models predicted.
With regard to the second question, I discuss how market liquidity condition affects equity holder’s optimal buyback strategy and, as a feedback, how the strategy dampens the adverse effect of liquidity drought. As market liquidity is well known to be pro-cyclical, the connection between market liquidity and buyback also shed light on the countercyclicality of debt buyback. The model shows that as market liquidity dries up, the firm tends to buy back more bonds from secondary market. As a consequence, the firm opportunistically exploiting the market liquidity condition reduces the unfavorable impact of liquidity deterioration on the firm’s credit condition. Following He and Xiong (2012), we quantify the effect of bonds buyback on the firm’s credit risk. Depending on the size of liquidity shock, the buyback can reduce the credit spread in an amount of around 10 to 15 basis points for Investment-Grade A firm; and reduce the credit spread in an amount of around 20 to 60 basis points for Speculative Grade BB firms. This also echoes the empirical discovery in Julio (2013) that firm with lower credit ratings are more likely to repurchase debt from secondary market.

I discuss the underlying mechanism and economic intuition behind the results. In the model, equity holders would stop servicing the debt and declare bankruptcy when the assets value is too low. Buying back outstanding bonds when their market price is low can transfer value from bondholders to equity holders. The increased equity value therefore incentivizes equity holders willing to bail out the firm until a much lower assets value and reduces the credit risk of the firm overall. We characterize the value transfer to equity holders and consistent with the results aforementioned, as liquidity cost rises, equity holders tend to buy back more bonds and total value transfer is more substantial. In addition to the value transfer, reduced debt overhang acts as an amplification mechanism. We shows that reduced overhang improves return of equity when assets value becomes higher in the future and thus amplifies the initial value transfer effect. However, as the firm buys back more outstanding bonds, the premium for investors to sell or tender their bonds also goes up. The benefit and cost of bonds buyback eventually determines the optimal buyback strategy.
Our model assumes that the equity holders make the decision on how much debt to buy back. Like most decision variables in corporate finance e.g. investment and leverage, the amount (or proportion) of debt to repurchase also subjects to agency cost. An interesting yet understudied question is to gauge the agency cost on debt buyback. To do so, we compare the equity maximizing debt buyback with the firm-value maximizing debt buyback for firms of different leverages. The model suggests that equity holders tend to under-buy-back the debt for low leverage firms while over-buy-back the debt if the leverage is high. Equity holders choose how much debt to buy back and endogenous default threshold jointly to maximize the equity value. When leverage is low, market value of the debt is less discounted relative to the principal and value transfer is limited, and as a result equity holders would like to choose a smaller proportion of the debt to repurchase and higher default boundary. Although a slight more repurchase of debt increases overall debt value, it will hurt equity holders. When the leverage is high, debt value is greatly discounted and value transfer is considerable. Equity holders thus have incentive to buy back much of the outstanding debt from the secondary market. However, a significant amount of the value becomes deadweight loss during the buyback transaction and it is not efficient to the firm overall.

Given the importance of debt financing in US financial market and economy relative to equity as well as the huge literature on share repurchase, the early research on debt buyback is really scarce. Kruse, Nohel, and Todd (2014) analyzes the impact of debt tender offers on stock market. They found that debt tender offers increase return of equity in general. Mao and Tserluevich (2014) builds up a static model and found that debt is cheaper when the outstanding bonds are held by many dispersed creditors. They also argued that debt repurchase increase firm value ex-ante as it makes capital structure more flexible. Another strand of literature focused on debt restructuring when the firm is financial distress. Gertner and Scharfstein (1991) analyzes the condition under which debt-for-equity is exchange is profitable. Cornett and Travlos (1989) analyzes wealth transfer between different security
holders during debt restructuring. The present paper is the first dynamic model to study debt repurchase and quantify the effect of debt repurchase on the firm’s credit risk. The model also identifies the lower default boundary chosen by equity holders as a new mechanism for firm value to increase ex-ante.

Our model also contributes to a large literature on corporate debt maturity structure, rollover risk and liquidity risk. Among early studies, Almeida et al. (2009) shows firms with large amount of bonds that matured during the 2008 crisis reduced more investments than the others. Harford, Klasa, and Maxwell (2014) analyzes that firm can use cash holdings to mitigate rollover risk when the debt has a short maturity. Acharya, Gale, and Yorulmazer (2011) explains frequent rollover is one important factor that leads to a sudden freeze in the availability of short-term, secured borrowing. These papers mostly focus on the unfavorable impact of liquidity risk on the firm via debt rollover. Yet, as mentioned before, in reality the firm use sophisticate corporate finance strategies, such as issuing bonds of different maturities Choi, Hackbarth, and Zechner (2014), Norden, Roosenboom, and Wang (2016) and buying back cheap bonds, to lessen the adverse effect or even take advantage of the depressed market situation. Accordingly, a more precise assessment of the effect of liquidity risk as well as firm’s debt maturity structure calls for a model featuring the strategies.

The remainder of the article proceeds as follows. In section 2.2, I introduce several stylized facts about debt buyback to motivate the model; Section 2.3 provides a general framework to study bonds rollover and buyback; Section 2.4 parameterizes the general framework and studies the effect of debt buyback on the firm’s credit risk and liquidity risk; Section 2.5 discusses other important variables firms have to take into consideration when repurchasing bonds from market and concludes.
2.2 Stylized Facts

Our model aims at replicating several stylized facts on debt buyback. To motivate our model, we draw existing empirical literature on macroeconomics, finance, accounting and law, and summarize several stylized facts on debt buyback in this section.

1. Debt buyback is countercyclical

Literature on macroeconomics and business cycles have widely documented that debt buyback is countercyclical. Using Flow of Funds Accounts of the Federal Reserve Board, [Jermann and Quadrini (2012)] shows that debt repurchase is strongly countercyclical. In Figure 2.1, we plot both debt repurchase as well as credit spread for BofA Merrill Lynch US corporate AA and B firms. Credit spread is also known for moving counter-cyclically over business cycle and it is a critical variable representing cost of debt finance that firms have to take into account. It clearly shows that strong counter-cyclicality of debt repurchase activity by firms. [Begenau and Salomao (2014)] uses CRSP/Compustat Merged Fundamentals Quarterly reached the similar conclusion for both small and large firms.

The macroeconomic literature usually focus on firm’s trade-off between equity and debt over business cycle and emphasize the effect of borrowing constraint on the trade-off. Moreover, they also usually define debt repurchase as reduction in outstanding debt. A reduction in outstanding debt does not necessarily mean that firms are “re-purchasing” debt. It could simply means that firm temporarily suspend issuing new debt after outstanding debt matures. To examine the actual debt repurchase activity in details, Figure 2.2 plots the yearly data of open market repurchase and tender offers firm engage in bonds market from [Julio (2013)], along with credit spread of BofA Merrill Lynch AA and B firms. Table 2.5 also calculate the correlation coefficients between
open market repurchase, tender offers and credit spread. The correlation coefficients show that the debt buyback activity is still countercyclical.

2. Debt buyback can be achieved in various ways

When firms try to buy back debt from secondary market, there are numerous ways to do so. The typical methods include open market repurchase, tender offer, debt-for-equity exchange.

Open market repurchase means the issuer firm directly buys back bonds from secondary market. The firm can remain anonymous and take advantage of the distressed debt pricing. However, open market repurchase can only buy back a small proportion of outstanding bonds in a short time and subject to a serious of legal restrictions. For the firm to buy back a larger proportion of bonds in a short time, tender offer is often employed. Moreover, if ever there are covenants restricting debt repurchase, a consent solicitation approved by certain number of bond holders can slack the covenant. To tender a large amount of bonds, the firm has to offer a compelling premium to the bondholders and this makes tender offer very costly. Debt-for-equity exchange (or debt-to-equity swap) is often used by firms in financial distress or short of cash holdings (Butler (2010)). Debt-for-equity exchange is similar to tender offer in many aspects. However, by exchanging debt for equity, the firm avoids using cash. All these methods have their own advantages and disadvantages and are substitutes for each other. For example, Table 2.5 shows the correlation between open market repurchase and tender offers are negatively correlated, implying the firm substitutes one with the other during the sample period.
The debt buyback approach in our model can be interpreted as either open market repurchase or tender offer or debt-for-equity swap. As in most Leland-type models, our model does not feature cash holdings. We discuss the role of cash hoardings in debt buyback in Section 2.3.

3. **Debt buyback is correlated with firm and debt characteristics**

Julio (2013) found that debt buyback improves the firm’s investment distortions. Firms with higher leverage are more likely to repurchase debt, as the improvement is more salient for higher leverage firms. He also discovered that average credit ratings for repurchased bonds prior to the repurchase are declining and while the credit ratings stabilize and even increases following the buyback. Xu (2014) found similar pattern, although she mainly focused on callable bonds.

Debt characteristics also affect firm’s buyback activities. Julio (2013) found that firms bought back more long-term bonds than short term bonds, with average maturity being 10.84 years for repurchased debt prior to repurchase whilst the average maturity shortens to 6.9 years after repurchase. Moreover, firms are more likely to buy back convertible bonds through open market repurchase instead of tender offer. Open market repurchase consists of 40% convertible bonds while tender offer is only composed of 6.7%.

2.3 **The Model**

In this section, I provide a general framework to study bonds rollover and buyback based on Leland and Toft (1996). Another important way to buy back bonds is to write the call provision in the initial contract at date-0, i.e. issuing callable bonds. This feature makes callable bonds different from other buyback methods we mentioned here. For details of pricing callable bonds and their effect on the firm’s default risk, see Acharya and Carpenter (2002), Jarrow et al. (2010) and Leland (1998).
2.3.1 Firm and Assets

Unlevered value of firm’s assets $V$ follows a geometric Brownian motion given by

$$\frac{dV}{V} = (r - \delta)dt + \sigma dz \quad (2.1)$$

where \( \{z_t\} \) is a standard Brownian motion. \( r \) is the risk-free interest rate; \( \delta \) is the payout rate; \( \sigma \) is the volatility of asset value; \( r, \sigma, \delta \) are assumed to be constants.

2.3.2 Debt Structure

Suppose firm has 1 unit of outstanding debt in total and time-to-maturities, \( s \), are distributed on a finite interval \([0, T]\). To isolate the effect of maturity structure, all of the outstanding bonds are assumed to be of equal seniority. 1 unit of outstanding debt allows us to use a probability density function \( \kappa_t(s) \) to denote the amount (or, fraction) of debt maturing in \( s \) periods from date \( t \). Specifically, there are bonds in the amount of \( \kappa_t(s)ds \) with time-to-maturity \( s \) at date \( t \), and \( \kappa_t(s) \) satisfies

$$\int_0^T \kappa_t(s)ds = 1 \quad (2.2)$$

Also let \( \kappa_t(s) \) denote the corresponding cumulative distribution function. We make the following assumptions.

Assumption 5. Equity holders control the firm and commit to a stationary structure through continuously repurchasing debt as well as rolling over maturing debt. See Figure 2.3.

Assumption 6. The debt structure \( \kappa_t(s) \) takes U-shape. Formally, \( \kappa_t(s) \leq 0, \forall s \in [0, T^*] \) and \( \kappa_t(s) \geq 0, \forall s \in [T^*, T] \).

Consistent with empirical evidences, a U-shape \( \kappa_t(s) \) implies that the firm issues more short-term debt and buys back more long-term debt. The second assumption is not critical, as the model can be modified easily to accommodate any debt structure \( \kappa_t(s) \).
A stationary debt structure means that
\[ \kappa_t(s) = \kappa_{t+\Delta t}(s), \forall \Delta t \] (2.3)

During \([t, t + \Delta t]\), debt repurchasing and issuing does not change the total debt outstanding. Hence for the debt maturity structure considered here \(\kappa_t(s)\), we have
\[
\int_0^{\Delta t} \kappa_t(s) ds + \left( \int_{T^*}^T \kappa_t(s) ds - \int_{T^*-\Delta t}^{T^*} \kappa_{t+\Delta t}(s) ds \right) = \left( \int_{T^*-\Delta t}^{T^*} \kappa_{t+\Delta t}(s) ds - \int_T^{T^*} \kappa_t(s) ds \right) + \int_{T^*-\Delta t}^T \kappa_t(s) ds
\] (2.4)

At date \(t\), the debt structure is given by \(\kappa_t(s)\). The first term accounts for the debt that matures during \([t, t + \Delta t]\); The second term accounts for debt buyback from the secondary market. Note the debt with time-to-maturities ranging over \([T^*, T]\) at date \(t\) will have time-to-maturities ranging over \([T^*-\Delta t, T-\Delta t]\) at date \(t + \Delta t\). This explains the shift of lower and upper bounds of integrals in term (2); The third term accounts for the debt rollover: debt that just matured recently is refinanced by issuing new debt with time-to-maturities ranging over \([\Delta t, T^*]\) at date \(t\); The fourth term accounts for newly issued debt with maturity \(T\). The rollover and buyback also shows in Figure 2.3.

By changing the bounds of integrals, (2.4) is equivalent to
\[
\int_0^{\Delta t} \kappa_t(s) ds + \int_{T^*}^T \left( \kappa_t(s) - \kappa_{t+\Delta t}(s-\Delta t) \right) ds = \int_{T^*-\Delta t}^{T^*} \left( \kappa_{t+\Delta t}(s-\Delta t) - \kappa_t(s) \right) ds + \int_{T^*-\Delta t}^T \kappa_{t+\Delta t}(s) ds
\] (2.5)

Substitute (2.5) with (2.3), it yields:
\[
\int_0^{\Delta t} \kappa_t(s) ds + \int_{T^*}^T \left( \kappa_t(s) - \kappa_t(s-\Delta t) \right) ds = \int_{T^*-\Delta t}^{T^*} \left( \kappa_t(s-\Delta t) - \kappa_t(s) \right) ds + \int_{T^*-\Delta t}^T \kappa_t(s) ds
\] (2.6)

Differentiate with respect to \(\Delta t\) on both sides of (2.6) and let \(\Delta t \to 0\), we have
\[
\kappa_t(0) + \int_{T^*}^T \kappa_t'(s) ds = \int_0^{T^*} (-\kappa_t'(s)) ds + \kappa_t(T)
\] (2.7)
The left side of Equation (2.7) is the total reduced bonds, including bonds that just matured and bought back. The right side of Equation (2.7) is the total bonds newly issued.

Equation (2.7) implies a particular way the firm manages the debt maturity structure by debt buyback and rollover. At each instant $dt$, $\kappa(0)dt$ amount of debt matures; for the debt with time-to-maturities $s \in [T^*, T]$, the firm buys an amount of $\kappa'(s)ds dt$ back from the open market; The firm also issues new bonds with time-to-maturities $s \in (0, T^*]$ in an amount of $-\kappa'_t(s)ds dt$, and new bonds with maturity $T$ in an amount of $\kappa(T)dt$. Eventually, the firm manages to maintain a stationary debt structure represented by $\kappa_t(s)$. Since the debt maturity structure $\kappa_t(s)$ is time-homogeneous and does not depend on $t$, we will drop subscript $t$ and use $\kappa(s)$ to denote it below.

2.3.3 Secondary Market

We follow Amihud and Mendelson (1986) and He and Xiong (2012), assuming an illiquid secondary bond market. Each bond investor subjects to an idiosyncratic Poisson liquidity shock with intensity $\lambda$. Upon the arrival of the liquidity shock, the bond investor has to sell his bond holdings at a fractional cost of $k$, and exit the market. The presence of liquidity shock, on one hand, causes higher discount and reduces the market value of new bonds; On the other hand, from the perspective of the firm, it is an great opportunity to buy back the bonds on fire sale. Intuitively, buying back bonds on cheap has many benefits, such as decreasing the leverage ratio, reducing the repayment burden in the future and alleviating debt overhang effect against new investment. Henceforth, we assume that the firm can buy back a proportion of the bonds sold by bond investors who got struck by liquidity shocks.

We implicitly assume that bond investors do not care to whom they sell the bonds, upon the arrival of the liquidity shock. This is consistent with the real bonds market. The secondary bond market is highly illiquid and fragmented than the stock market (Acharya, Amihud, and Bharath (2013), Bao, Pan, and Wang (2011), Bushman, Le, and Vasvari (2013))
As documented by Levy and Shalev (2013), the transaction of corporate bonds in the secondary market usually takes place between two deals over the phone. The dealer who sells the bonds is not aware of who the end counterparty to the transaction is, whether the other dealer is buying corporate bonds on behalf of himself or as an agent for a different party.

Although firms can buy back bonds quietly by open market repurchase, it is subject to negotiation and the amount of bonds is limited. By paying a premium, firms can buy back a larger amount of bonds in a shorter time via tender offer. In addition, the information about the stealthy repurchase is usually disclosed in the following statements and sophisticate bondholders will take the information into consideration in the future. Either way, firms are likely to pay a fractional cost \( \phi \), which is higher than \( 1 - k \) received by bond investors, for each share of bonds bought back.

Therefore, as can be seen below, given the firm’s default boundary, the debt buyback in the secondary market does not change the way investors value the bonds. However, the debt buyback has an significant effect on the firm’s endogenous default boundary and as a result, affects the market value of the bonds through bondholder’s rational expectation.

### 2.3.4 Debt Valuation

In this subsection, we characterize the value of bonds. Let \( V_B \) be the assets value when equity holders choose to default. Taking \( V_B \) as given, the current market value of one unit of debt, \( d(V, s; V_B) \), with a time-to-maturity of \( s \), coupon payment of \( c \) and a principal value of \( p \) when current assets is \( V \) satisfies the following partial differential equation (P.D.E):

\[
\begin{align*}
  r \cdot d(V, s; V_B) &= c - \lambda \cdot k \cdot d(V, s; V_B) - \frac{\partial d(V, s; V_B)}{\partial s} + \frac{\partial d(V, s; V_B)}{\partial V} (r - \delta) V + \frac{1}{2} \frac{\partial^2 d(V, s; V_B)}{\partial V^2} \sigma^2 V^2 \\
  &= (c - \lambda \cdot k \cdot d(V, s; V_B) + \frac{1}{2} \frac{\partial^2 d(V, s; V_B)}{\partial V^2} \sigma^2 V^2)
\end{align*}
\]

(2.8)
To pin down the bond price, two boundary conditions are needed. When time-to-maturity \( s = 0 \), the bond investors can claim the principal value \( p \) if the assets value \( V \) is greater than the default threshold \( V_B \), i.e.

\[
d(V, 0; V_B) = p, \forall V \geq V_B \tag{2.9}
\]

The other boundary condition describes the payoff to bondholders when equity holders choose to default. Since all bonds are of equal seniority, the assets value that goes to bonds with time-to-maturity \( s \) upon default is \( \kappa(s)V_B \). Noting the total amount of bonds with time-to-maturity \( s \) is \( \kappa(s) \), each unit of bonds will receive \( V_B \) as a consequence, i.e.

\[
d(V_B, s; V_B) = V_B, \forall s \in [0, T] \tag{2.10}
\]

The solution to (2.8) with (2.9) and (2.10) is given by

\[
d(V, s; V_B) = \frac{c}{r + \lambda k} + e^{-(r+\lambda k)s} \left( p - \frac{c}{r + \lambda k} \right) (1 - F(s)) + \left( \alpha V_B - \frac{c}{r + \lambda k} \right) G(s) \tag{2.11}
\]

where

\[
F(s) = N(h_1(s)) + \left( \frac{V}{V_B} \right)^{-2a} N(h_2(s))
\]

\[
G(s) = \left( \frac{V}{V_B} \right)^{-a+z} N(q_1(s)) + \left( \frac{V}{V_B} \right)^{-a-z} N(q_2(s))
\]

\[
q_1(s) = \frac{-b - z\sigma^2 s}{\sigma \sqrt{s}}; q_2(s) = \frac{-b + z\sigma^2 s}{\sigma \sqrt{s}}
\]

\[
h_1(s) = \frac{-b - a\sigma^2 s}{\sigma \sqrt{s}}; h_2(s) = \frac{-b + a\sigma^2 s}{\sigma \sqrt{s}}
\]

\[
a = \frac{r - \delta - \sigma^2}{\sigma^2}; b = \ln\left( \frac{V}{V_B} \right); z = \left[ \frac{(a\sigma^2)^2 + 2r\sigma^2}{\sigma^2} \right]^{\frac{1}{2}}
\]

The result is similar to the bond price derived in Leland and Toft (1996) and He and
Xiong (2012). Yet, the total market value of outstanding debt, \( D(V; s; V_B) \) depends on the debt maturity structure \( \kappa(s) \) the firm chooses to maintain.

\[
D(V; V_B) = \int_0^T \kappa(s)d(V, s; V_B) ds \tag{2.17}
\]

### 2.3.5 Equity Valuation

In this subsection, we will derive the equity value \( E \) and endogenous default threshold \( V_B \). As there are transaction costs in trading bonds, part of the firm value accrues to neither bondholders nor equity holders. To derive equity value \( E \), note that \( E \) satisfies the following differential equation

\[
rE = \left( r - \delta \right)V E_V + \frac{\sigma^2}{2}V^2 E_{VV} + \delta V - (1 - \pi)c +
\]

\[
\kappa(T)d(V, T; V_B) + \int_0^{T^*} (-\kappa'(s))d(V, s; V_B)ds - \kappa(0)p - \phi \int_{T^*}^{T} \kappa'(s)d(V, s; V_B)ds \tag{2.18}
\]

The left hand side of (2.18) is the required return of holding equity; Term (1) on right hand side of (2.18) is equity change caused by underlying assets fluctuation. Term (2) is the payout plus tax benefits of debt minus coupon payment; Term (3) is the market value of newly issued bonds\(^2\); Term (4) is equity holders’ payment on principal due; Term (5) is equity holder’s expense on bonds buyback \((\phi > 0)\). Note that (2.18) reduces to equation (11) in He and Xiong (2012) by letting \( \kappa = \frac{1}{T} \).

#### 2.3.5.1 A measure of maturity risk

Before solving (2.18) and deriving default boundary \( V_B \), we first examine the terms (3),(4) and (5), as they are the terms of rollover and buyback and affect equity holders’ decision of

---

\(^2\) The firm might also incur cost when issuing bonds, such as underwriter compensation. I leave out the cost of issuing bonds as it is not the focus of the current paper.
default. Assuming firms pay competitive price when buying back the bonds, i.e. $\phi = 1$, the second line of equation (2.18) becomes

$$\kappa(T)d(V, T; V_B) - \kappa(0)p - \int_0^T \kappa'(s)d(V, s; V_B)ds$$  \hspace{1cm} (2.19)

Note that $d(V, 0; V_B) = p$. Integrating by parts, (3.5) can be rewritten as

$$\int_0^T \kappa(s) \frac{\partial d(V, s; V_B)}{\partial s} ds$$  \hspace{1cm} (2.20)

Note (2.20) does not rely on Assumption (2) and can be interpreted as the maturity risk of debt equity holders face. It is the weighted average of sensitivity of debt market value with respect to time-to-maturity. However, as we assume that firms pay a premium when purchasing bonds in the secondary market, $\phi > 1 - k$ in (2.18). Therefore, term (3), (4) and (5) in (2.18) do not necessarily have a simple form like (2.20) any more.

### 2.3.5.2 Equity Valuation

We can solve the equity value, $E$, in the closed form by guess and verify. The expression of $E$ is provided in the appendix. The endogenous default boundary $V_B$ satisfies smooth-pasting condition

$$E_V|_{V = V_B} = 0$$  \hspace{1cm} (2.21)

and is given in Theorem 2.

**Theorem 2.**

$$V_B = \frac{(1-\pi)\kappa(0)p - \kappa(T)Q_1(T) + \int_0^T \kappa'(s)Q_1(s)ds + \phi \int_0^T \kappa'(s)Q_1(s)ds}{\frac{1}{\pi} + \alpha \left( \kappa(T)B(u, T) + B(u, T) - \int_0^T \kappa'(s)B(-u, s + B(u, s))ds - \phi \int_0^T \kappa'(s)B(-u, s + B(u, s))ds \right)}$$  \hspace{1cm} (2.22)

where

$$Q_1(T) = \left( \frac{c}{r + \lambda k} + e^{-(r+\lambda k)T}(p - \frac{c}{r + \lambda k}) \right)$$  \hspace{1cm} (2.23)
\[ Q_2(T) = \left( p - \frac{c}{r + \lambda k} \right) (b(-a, T) + b(a, T)) + \frac{c}{r + \lambda k} (B(-u, T) + B(u, T)) \] (2.24)

\[ b(u, s) = \frac{e^{-(r+\lambda k)s}}{z + u} \left( N(\mu - \lambda k \sqrt{m}) - e^{rs} N(-\lambda k \sqrt{m}) \right) \] (2.25)

\[ B(u, s) = \frac{1}{z + u} \left( N(\mu - \lambda k \sqrt{m}) - e^{s(\mu - \lambda k \sqrt{m})} N(-\lambda k \sqrt{m}) \right) \] (2.26)

\[ \eta = z - a, a = \frac{r - \delta}{\sigma^2}, z = \sqrt{a^2 + 2r\sigma^2}, u = \sqrt{a^2 + 2(r + \lambda k)} \] (2.27)

### 2.4 Debt Buyback

We are interested in the effect of debt buyback on the default decision by equity holders and how the effect changes with market liquidity risk. To focus on the questions and put the model into work, we make two additional assumptions. The goal of the assumptions is to facilitate calibration exercise while keep intuitive interpretation.

**Assumption 7.**

\[ \kappa(s) = \frac{\beta e^{\beta s}}{e^{\beta T} - 1} \] (2.28)

See Figure 2.4. The choice of debt maturity structure echoes Poisson random maturity model of [Leland (1994a)] and [Leland (1998)]. Also, it is consistent with the empirical evidence that firms buy back more long-term bonds. More importantly, the probability density function considered above has the following property

\[ \frac{\kappa'(s)}{\kappa(s)} = \beta \] (2.29)

It comes with a simple interpretation that the firm early finances and buys back bonds in the proportion of \( \beta \), and rolls over maturing bonds by issuing new bonds with maturity \( T \). [Julio (2013)] documented that \( \beta \) is between 5% to 10%. Later we will see that the calibrated model predicts that the \( \beta \) chosen by equity holders lie within this range.

**Assumption 8.** \( \phi(\beta) = (1 - k)e^{\psi \beta} \), where \( \psi > 0 \)
\( \phi(\beta) \) also has an intuitive interpretation. Note that

\[
\frac{\partial \beta}{\partial \psi} = \frac{1}{\psi}
\]  

(2.30)

Hence, given the interpretation of \( \beta \) in (2.29), \( \frac{1}{\psi} \) measures the price elasticity of debt buyback.

The base \( 1 - k \), which is received by bondholders when forced to sell, is the lowest price the firm can get. The parameter \( \psi \) and \( k \) will be calibrated later.

### 2.4.1 Model Calibration

To compare and clearly see the effect of debt buyback, I adopt most of parameters from He and Xiong (2012). I set \( T = 6 \), meaning the firm issues bonds with time-to-maturities spanning from 0 to 6 years. We also set the principal \( P = 61.68 \). Coupon \( c \) is determined in a way such that the new debt is issued at par under the condition that the firm is not engaged in debt buyback, i.e. \( d(V,T;VB)|_{\beta=0} = p \). Powers and Mann (2005) found that bondholders respond to higher tender premiums by tendering a greater percentage of their bonds and a 1% increase in tender premium increases the tendering rate by approximately 9%. Thus, we set \( \psi = 0.11 \).

The calibrated parameters are listed in Table 3.2.

### 2.4.2 Default Boundary and Credit risk

In this subsection, I examine how debt buyback affects firm’s decision on default and credit risk.

Figure 2.5 plot the endogenous default boundary \( V_B \) and equity value \( E \) for different \( \beta \), given other parameters listed in Table 3.2. 2.5a shows that there is an optimal \( \beta^* \) that maximizes equity value \( ceteris paribus \). Specifically, the equity value first increases with \( \beta \) until \( \beta^* = 6.85\% \) and then starts to decrease. The default boundary, which shows in 2.5b,
### Baseline Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt Tax Benefit Rate</td>
<td>$\pi = 0.27$</td>
</tr>
<tr>
<td>Assets Volatility</td>
<td>$\sigma = 0.23$</td>
</tr>
<tr>
<td>Interest Rate</td>
<td>$r = 8.0%$</td>
</tr>
<tr>
<td>Payout Rate</td>
<td>$\delta = 2.0%$</td>
</tr>
<tr>
<td>Bankruptcy Recovery Rate</td>
<td>$\alpha = 0.6$</td>
</tr>
<tr>
<td>Liquidity Cost</td>
<td>$k = 0.01$</td>
</tr>
<tr>
<td>Liquidity Shock Intensity</td>
<td>$\lambda = 1.0$</td>
</tr>
<tr>
<td>Current (date-0) Assets Value</td>
<td>$V = 100.0$</td>
</tr>
<tr>
<td>Maturity</td>
<td>$T = 6.0$</td>
</tr>
<tr>
<td>Coupon</td>
<td>$c = 6.12$</td>
</tr>
<tr>
<td>Debt Principal</td>
<td>$p = 61.68$</td>
</tr>
<tr>
<td>Price elasticity</td>
<td>$\psi = 0.11$</td>
</tr>
</tbody>
</table>

Table 2.1

follows an inverse pattern: it first decreases with $\beta$ and then starts to increase and has a minimum around $\beta = 20\%$. We will talk about these two extrema in subsection 2.4.5, as the difference between them clearly shows agency cost.

Figure 2.6 examines the relationship between leverage, equity maximizing $\beta^*$ and default boundary $V_B$. I vary debt principal $p$ and search for $\beta^*$ that maximizes the equity value. Coupon $c$ is determined such that new debt is issued at par given $p$ and $\beta$. Consistent with empirical evidence, the model predicts that equity holders are more actively engaged in debt buyback as leverage increases. Figure 2.6 shows that the proportion of debt repurchased steadily increases from 4% to 8% as leverage rises. Figure 2.7 compares the default boundary $V_B$ when equity holders choose to buy back debt in the proportion of $\beta^*$ to the one when they do not buy back at all. The conclusion is that strategic buyback always lowers the default boundary and the effect is more salient as leverage increases. The relationship has important implication on the firm’s optimal leverage. Early studies on firm’s optimal leverage mostly focus on firm’s debt rollover without considering that the firm can also repurchase debt from secondary market. The flexible debt buyback strategies imply that the firm can probably employ more debt than what early models predicted.
2.4.3 Liquidity and Debt Buyback

One feature of debt buyback is its countercyclicality: firms tend to buy back more debt during recession. On the other hand, market liquidity is pro-cyclical (Eisfeldt (2004), Brunnermeier and Pedersen (2009), Næs, Skjeltorp, and Ødegaard (2011)). This implies that market liquidity might impact equity holder’s choice on debt buyback. I formally explore the relationship in this subsection.

Figure 2.8 shows how equity maximizing $\beta^*$ changes with market liquidity. 2.8a plots $\beta^*$ with respect to different liquidity shock intensity $\lambda$; 2.8b plots $\beta^*$ with respect to liquidity cost $k$. They show that $\beta^*$ increases with both $\lambda$ and $k$ yet the rates are different: $\beta^*$ increases much faster with $k$. Although higher $\lambda$ and $k$ both lower the market price of bonds, higher $k$ also lowers $1 - k$, the base of buyback price the firm has to pay and thus triggers the firm to buy back more debt from secondary market.

As mentioned before, bonds buyback strategy can potentially increase the optimal leverage of the firm. To see this, I compute the optimal leverage following Leland and Toft (1996). I look for $p^*$ that maximizes equity value plus aggregate debt value, given the coupon such that the new debt is issued at par, i.e.

$$\max_p E(p; V, V_B) + D(p; V, V_B)$$

subject to

$$d(V, T; V_B) = p$$

The market leverage is then defined as

$$\frac{D(p^*; V, V_B)}{E(p^*; V, V_B) + D(p^*; V, V_B)}$$

Figure 2.9 plots the optimal leverage with respect to liquidity cost $k$ for $\beta = 0$ and $\beta = 4%$. In either case, the optimal leverage decreases with liquidity cost $k$. However, as expected,
the debt buyback strategy allows the firm to issue more debt and increases optimal leverage as a result.

Not only market liquidity affects the equity holder’s choice on debt buyback, debt buyback also alters the effect of liquidity risk on the firm. Liquidity risk, interacting with default risk, determines the credit spread of a firm together with default risk. To better illustrate the impact of debt buyback, I follow [He and Xiong (2012)] and compare responses of firms with investment-grade A and speculative-grade BB and different buyback strategies to liquidity shock represented by an increase in $k$. Specifically, A-rated firms have $\sigma = .21$ and $k = 0.5\%$; BB-rated firms have $\sigma = .23$ and $\kappa = 1\%$. Other parameters are adopted from Table 3.2. For each type credit rating of firms, I consider two maturities: $T = 6$ and $T = 10$. Principal $p$ and coupon $c$ are determined such that new bonds are issued at par with a credit spread of 100 bps for A-rated firms and with a credit spread of 330 bps for BB-rated firms, given that the firm is not engaged in debt buyback for each maturity $T$, i.e. $p$ and $c$ are the solutions to

\[
\begin{align*}
\frac{d(V, T; V_B)}{\beta = 0} &= p \\
&= \frac{\sigma}{\gamma} (1 - e^{-\gamma T}) + pe^{-\gamma T} = p
\end{align*}
\]

where $y$ is the bonds yield. I then let equity holders choose $\beta^*$ and see how credit spread changes with $\beta^*$.

Table 2.2 shows the result. Debt buyback reduces the adverse effect of liquidity cost increase on firm’s credit spread, compared to the case where firm does not buy back any debt at all. The effect is much stronger for speculative-grade BB bonds than investment-grade A bonds. However, the effect across different maturities ($T=6$ vs. $T=10$) is mixed and does not show a clear pattern.
Panel A: Investment-Grade A

<table>
<thead>
<tr>
<th>Maturity (Years)</th>
<th>k=0.5%</th>
<th>k=1%</th>
<th>k=2%</th>
</tr>
</thead>
<tbody>
<tr>
<td>T=6</td>
<td>β = 0</td>
<td>β = β*</td>
<td>β = 0</td>
</tr>
<tr>
<td>Spread (bps)</td>
<td>100</td>
<td>93.06</td>
<td>155.35</td>
</tr>
<tr>
<td>Δ Spread (bps)</td>
<td>-6.94</td>
<td>-6.94%</td>
<td>-9.19</td>
</tr>
<tr>
<td>Δ Spread (fraction)</td>
<td>-6.94%</td>
<td>155.35</td>
<td>146.17</td>
</tr>
</tbody>
</table>

Panel B: Speculative-Grade BB

<table>
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<tr>
<th>Maturity (Years)</th>
<th>k=1%</th>
<th>k=2%</th>
<th>k=4%</th>
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<tbody>
<tr>
<td>T=6</td>
<td>β = 0</td>
<td>β = β*</td>
<td>β = 0</td>
</tr>
<tr>
<td>Spread (bps)</td>
<td>330</td>
<td>305.13</td>
<td>458.94</td>
</tr>
<tr>
<td>Δ Spread (bps)</td>
<td>-24.87</td>
<td>-7.54%</td>
<td>-36.47</td>
</tr>
<tr>
<td>Δ Spread (fraction)</td>
<td>-7.54%</td>
<td>458.94</td>
<td>422.47</td>
</tr>
</tbody>
</table>

Table 2.2

2.4.4 Mechanism

So far, we have seen that the results of the calibrated model are in line with the empirical evidences. One might wonder the channel through which debt buyback strategy affects the firm’s credit risk and values. In this subsection, we focus on two mechanisms: value transfer from debt holders to equity holders and reduced debt overhang.

2.4.4.1 Value Transfer

In the model, equity holders decide to stop servicing the debt and liquidate the firm when the assets value hits a boundary \( V_B \) and the equity value becomes zero. However, when liquidity cost is high, the market value of the debt is also very low. Buying back outstanding bonds on the cheap therefore can transfer value from bondholders to equity holders and increase equity value, compared to the case where the equity holders do not buy back any bonds at all. The transferred value from debt holders thus incentivize the equity holders to bail out the firm to a lower assets value. To see this, the value transferred from debt holders to equity holders at date 0 is

\[
\beta p - \phi(\beta) \int_0^T \beta \kappa(s) d(V, s; V_B) \, ds
\]  
(2.33)
(2.33) uses the fact that $\kappa'(s) = \beta \kappa(s)$. By buying back outstanding debt in the proportion of $\beta$, equity holders have to pay $\phi(\beta) \int_0^T \beta \kappa(s) \, d(V, s; V_B) \, ds$ but avoid the principal payment in the amount of $\beta p$. Thus (2.33) represents the value transferred from debt holders to equity holders.

Figure 2.10 shows the value transfer with respect to different buyback strategies $\beta$ for $k = 0.01$ and $k = 0.012$. It confirms the idea that buying back debt on the cheap transfers value from debt holders to equity holders. Also, as liquidity cost $k$ gets higher, an increase in $\beta$ will transfer more value in favor of equity holders, resulting in lower default boundary $V_B$. However, the buyback strategy and more value transfer does not overturn the adverse effect of higher liquidity cost $k$ on the firm’s default boundary and credit risk. Figure 2.11 plots default boundary $V_B$ with respect to liquidity cost $k$, with equity holders choosing $\beta^\ast$. It shows that $V_B$ still increases with liquidity cost $k$. Together, it explains the pattern we have seen in Table 2.2: credit spread decreases when equity holders choose $\beta = \beta^\ast$ from $\beta = 0$, given $k$; but increases with $k$.

2.4.4.2 Amplification: Reduced Overhang

Figure 2.5a shows the $\beta^\ast$ the equity holders would choose when $k = 0.01$, which is much greater than the $\beta$ maximizing value transferred from debt holders to equity holders. This implies that there must be other amplification mechanism of the initial value transfer effect. I will argue that the mechanism is reduced debt overhang effect.

Debt-overhang, stated formally in Myers (1977), refers to the fact that part of earnings generated by potential new projects is appropriated by existing debt holders and reduces equity holders’ incentive to invest on the projects. The effect is more salient when the firm is under financial distress. Diamond and He (2014) also showed that a higher default threshold is another form of debt overhang in the model with endogenous default boundary. Formal modeling debt overhang requires to specify the firm’s production technology. Here I follow
Diamond and He (2014) and use the sensitivity of market value of the new debt with respect to current assets value to measure debt-overhang effect, i.e.

\[
\frac{\partial d(V,T;V_B)}{\partial V} \tag{2.34}
\]

it measures how much assets value change accrues to debt holders.

Figure 2.12 plots debt overhang effect \( \frac{\partial d(V,T;V_B)}{\partial V} \) as well as endogenous default boundary \( V_B \) with respect to debt buyback proportion \( \beta \). Both variables synchronize to decrease at first and then increase with \( \beta \). The synchronization reflects the fact that \( \beta \) only affects \( d(V,T;V_B) \) via \( V_B \). From (3.13):

\[
\frac{\partial \frac{\partial d(V,T;V_B)}{\partial V}}{\partial \beta} = \frac{\partial \frac{\partial d(V,T;V_B)}{\partial V}}{\partial V_B} \cdot \frac{\partial V_B}{\partial \beta} \tag{2.35}
\]

**Proposition 1.** If \( p > \frac{c}{r+\lambda k} > (1 + \frac{1}{2a}) \alpha V_B \) and \( a = \frac{r-\delta-\sigma^2}{\sigma^2} > 0, \frac{\partial V_B}{\partial \beta} > 0 \)

Proposition 1 indicates that given debt principal and bankruptcy cost are sufficiently high, a lower default boundary reduces debt overhang. Suppose equity holders start from \( \beta = 0 \), an slight increase in \( \beta \) transfers value from debt holders to equity holders and lowers \( V_B \) (\( \frac{\partial V_B}{\partial \beta} < 0 \)). As a consequence, the lowered \( V_B \) reduces the debt overhang effect (\( \frac{\partial \frac{\partial d(V,T;V_B)}{\partial V}}{\partial V_B} > 0 \)). The reduced overhang improves return of equity when assets value becomes high in the future, which incentives equity holders to incur more cost and buy back more debt. Reduced debt overhang amplifies the initial effect of value transfer of debt buyback. In the end, \( \beta^* \) is the optimal point where marginal cost of debt buyback equals its marginal benefit from the perspective of equity holders.

### 2.4.5 Agency Cost on Debt Buyback

Hitherto, we have retained the assumption 1 that equity holders choose the debt buyback strategy \( \beta \). When equity holders make decisions, they do not take into account the external-
ities of their decisions on debt holders, resulting conflict of interest between equity and debt holders and agency cost. The two famous and well-studied problems on conflict of interest are excessive risk taking (Jensen and Meckling (1976)) and debt overhang (Myers (1977)). In this subsection, we show that agency cost also reflects on the deb buyback strategy and equity maximizing $\beta^*$ deviates from what is optimal for the entire firm, i.e. equity value plus aggregate debt value.

To gauge the agency cost, I consider equity maximizing $\beta^*$ and firm value maximizing $\beta^{**}$ for different principal $p$ outstanding.

$$\beta^* = \arg \max \ E(\beta; p, V)$$

$$\beta^{**} = \arg \max \ \{E(\beta; p, V) + D(\beta; p, V)\}$$

Figure 2.13 plots $\beta^*$ and $\beta^{**}$ as a function $p$. Interestingly, although equity holders maximizing $\beta^*$ deviates from $\beta^{**}$, the sign of the deviation depends on the leverage: when the leverage is low, $\beta^{**} > \beta^*$, meaning equity holders tend to under-buy-back the bonds compared to what is optimal to the firm; when the leverage is high, $\beta^{**} < \beta^*$ and they tend to over-buy-back the bonds.

To further understand the economic reasons, I consider two specific cases where $p = 80$ (high leverage) and $p = 45$ (low leverage). Table 2.3 lists the buyback proportion $\beta$, endogenous default boundary $V_B$, equity value $E$, debt value $D$ when equity holders choose $\beta^*$ or firm chooses $\beta^{**}$ for each $p$, respectively. If there were no transaction cost, the firm value would have equaled the asset value plus the value of tax benefits minus the value of bankruptcy costs (Leland (1994b)). Or in other words,

$$DWL = V + \frac{\tau_c}{r} \left[1 - \left(\frac{V}{V_B}\right)^{-(a+z)}\right] - \alpha V_B \left(\frac{V}{V_B}\right)^{-(a+z)} - (E + D)$$

(2.37)
represents the deadweight loss that occurs during buyback and sales of bonds. This reflects in the last row of Table 2.3.

When maximizing their security value, equity holders decide the endogenous default boundary \( V_B \) based on smooth-pasting condition \( (2.21) \) given \( \beta \). Then equity holders choose the pair \( (\beta, V_B(\beta)) \) that yield the highest equity value. When leverage is low, the market value of the debt and bonds buyback cost is high relative to the principal outstanding, and thus the value transferred is limited. Therefore, equity holders only would like to buy back a smaller proportion of bonds and choose a higher default boundary, compared to what is optimal to the entire firm. Optimal firm buyback strategy \( \beta^{**} \) requires equity holders to buy more, as it reduces \( V_B \) and increases debt value and the increased value exceeds the buyback cost.

When leverage is high, the market value of the debt is low ceteris paribus and value transferred to equity holders from buyback is high. Under such circumstance, equity holders choose a higher \( \beta \) and lower default boundary \( V_B \), compared to what is optimal to the firm. Nevertheless, most of the value eventually does not go to equity holders but is lost in the transaction. A decrease in \( \beta \) therefore cuts the transaction cost and increase the firm value overall.

2.4.6 Empirical Evidence

One implication of the model is that the firm should buy back more bonds when bonds market liquidity dries up. In this section, we provide some empirical evidences. The evidences serve to peep “the tip of the iceberg” and are no way in place of a rigorous empirical study.

Figure 2.14 plots the debt repurchase data from Jermann and Quadrini (2012) and liquidity measure from Corwin and Schultz (2012) spanning from 2004 to 2010. Corwin and Schultz (2012) developed a bid-ask spread estimator from bonds daily high and low prices and is one
Table 2.3

<table>
<thead>
<tr>
<th></th>
<th>Agency Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$V_B$</td>
</tr>
<tr>
<td>$p=80$</td>
<td></td>
</tr>
<tr>
<td>$\beta^*$</td>
<td>0.118</td>
</tr>
<tr>
<td>$\beta^{**}$</td>
<td>0.111</td>
</tr>
<tr>
<td>$\Delta(\beta^* \rightarrow \beta^{**})$</td>
<td>0.13</td>
</tr>
<tr>
<td>$p=45$</td>
<td></td>
</tr>
<tr>
<td>$\beta^*$</td>
<td>0.022</td>
</tr>
<tr>
<td>$\beta^{**}$</td>
<td>0.055</td>
</tr>
<tr>
<td>$\Delta(\beta^* \rightarrow \beta^{**})$</td>
<td>-0.316</td>
</tr>
</tbody>
</table>

of the best performed bonds market liquidity proxies along with Roll (1984) and Hasbrouck (2009). It clearly shows that debt repurchase positively co-moves with illiquidity in the bonds market, with correlation being 0.55. Debt repurchases reaches the peak around 2008-2009 when the corporate bonds are highly discounted.

Nevertheless, in Jermann and Quadrini (2012) debt repurchase is defined as “the reduction in outstanding debt (or increase if negative)” and measured by the negative of “net increase in credit markets instruments of nonfinancial business” in the Flow of Funds accounts of the Federal Reserve Board. This also includes the instances that firms halt new bonds issuance when existing bonds mature. To better match the liquidity environment described in the model to the reality, we focus on bonds tender offer within a window period from 2004 to 2005. The choice is based on two considerations. First, by considering bonds tender offer, we focus on “clean” bonds buyback and rule out the cases where firms suspend issuing new bonds when existing bonds mature. Second, In 2005 May 5th, bonds issued by GM and Ford was downgraded by junk status by S&P. While the downgrade was expected by investors, the timing still came as a shock to the bonds market (Acharya et al. (2014)). As a result, many

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3 The three measures are highly correlated (Schestag, Schuster, and Uhrig-Homburg, 2016), so it is not critical which measure to use.
insurance companies, pension funds etc. were forced to liquidate the bonds holdings issued by GM and Ford as regulations prevent them from holding junk-rated securities (Acharya, Schaefer, and Zhang (2015)). This bonds market liquidity shock exactly captures what is described in the model. And moreover, it occurs solely within the bonds market and rules out confounding factors in other large scale economic or financial crisis (e.g. the Great Recession) that could also possibly cause debt buyback.

We employ data of total tender amount and the number of bond issues tendered from Mergent Fixed Income Securities Database (FISD). Figure 2.15a shows that the total amount of bonds tender offers strongly and positively co-moves with illiquidity measures, both peaking around May, 2005 when the bonds market liquidity shock occurs. The correlation is 0.07. Figure 2.15b shows the number of bonds tender offers in 2004-2005, with the most tender offers occurring in June, 2005.

A deeper analysis on bonds market liquidity, firm’s tender offer decision and tender offer premium calls for data on market price of bonds. However, the data on market price of bonds were sparse back in 2005. The National Association of Securities Dealers (NASD) did not report and publicize information on market transaction of bonds until July 1, 2002, and at the beginning the reporting merely covered investment-grade bonds with initial issuance size greater than 1 billion. The project started to cover 99% of the public transactions beginning from February 7, 2005. (Bao, Pan, and Wang (2011)). Furthermore, linking to Compustat for firm’s characteristics results in even less observations. The data deficiency makes it challenging to analyze the causal relationship between bonds market liquidity and firm’s tender offer decision. We call for attention of future empirical studies for a more rigorous analysis.
2.5 Conclusion

In this paper, we present a model to study bonds buyback, an important yet somehow overlooked corporate finance strategy. Moreover, debt buyback is also a major macroeconomic variable at an aggregate level. We focus on its link to the firm’s default risk and market liquidity. Firms strategically choose how much debt to buy back and the decision increases with market liquidity cost. The model shows that bonds buyback can help to reduce the firm’s default risk and lessen the adverse effect of liquidity risk on the firm. The reason lies in the fact that debt buyback transfers value from debt holders to equity holders and incentivize equity holders to bail out the firm to a much lower assets level. The higher liquidity cost is, the more the market price of debt is discounted and therefore more value transferred to equity holders. In addition, the lower default boundary also reduces the debt overhang effect and increase the return of equity.

There are two issues our model does not cover. First, the model does not leave room for cash. When the firm buys back debt from secondary market, it is more likely that the firm will use cash hoard. Imperfect capital market makes it costly to issue more equity. Especially, as noted in Myers and Majluf (1984), if firm is short of cash and has to issue more equity to finance, the firm will pass profitable opportunity with asymmetric information. Cash hoard lessens firm’s reliance on equity issuance to raise capital. Second, as an assumption to derive closed-form endogenous default boundary $V_B$, the firm issues new debt and buys back old debt such that the total outstanding principal remains the same. However, the firm usually buys back debt as a way to deleverage. The deleverage has two countervailing effects. On one hand, it reduces total debt outstanding, mitigates debt overhang to a much larger extent, resulting a higher return of equity and lower default boundary; On the other hand, the value transfer from bond holders to equity holders also decrease with leverage and it makes equity holders to buy back less bonds. To analyze the roles of cash and leverage, a more delicate and comprehensive model is needed. We leave these questions to future research.
2.6 Proofs

2.6.1 Proof of Theorem \( \boxed{2} \)

I take a guess-verify approach to solve the equity value \( E \). Note that in He and Xiong (2012), equity value \( E \) satisfies

\[
rE = (r - \delta)V_{EV} + \frac{\sigma^2}{2}V^2E_{VV} + \delta V - (1 - \pi)c + \kappa(T)d(V, T; V_B) - \kappa(0)p \tag{2.38}
\]

In (2.18), the underlined part is

\[
\kappa(T)d(V, T; V_B) + \int_0^T (-\kappa'(s))d(V, s; V_B)ds - \kappa(0)p - \phi \int_{T*}^T \kappa'(s)d(V, s : V_B)ds
\]

Every part including the integral is a linear operator of \( d(V, \cdot ; V_B) \). Therefore, we conjecture that equity value \( E \) satisfying (2.18) is given by

\[
E = V - \frac{\delta V_B}{\sigma^2} \left( \frac{V}{V_B} \right)^{-\gamma} \frac{1}{\gamma + 1} - \frac{1}{\sigma^2} \left( \frac{1}{\gamma} + 1 - \frac{V}{\gamma} \right) \left( (1 - \pi)c + \kappa(0)p - \kappa(T) \left( \frac{c}{r + \lambda k} + e^{-(r + \lambda k)T} (p - \frac{c}{r + \lambda k}) \right) + \int_0^T \left( \frac{c}{r + \lambda k} + e^{-(r + \lambda k)s} (p - \frac{c}{r + \lambda k}) \right) \kappa'(s)ds + \phi \int_{T*}^T \left( \frac{c}{r + \lambda k} + e^{-(r + \lambda k)s} (p - \frac{c}{r + \lambda k}) \right) \kappa'(s)ds \right) + \int_0^T \kappa'(s) \left( e^{-(r + \lambda k)s} (p - \frac{c}{r + \lambda k})A(s) - (\alpha V_B - \frac{c}{r + \lambda k})A(s) \right) ds - \phi \int_{T*}^T \kappa'(s) \left( e^{-(r + \lambda k)s} (p - \frac{c}{r + \lambda k})A(s) - (\alpha V_B - \frac{c}{r + \lambda k})A(s) \right) ds
\]

\[
\int_0^T \kappa'(s) \left( e^{-(r + \lambda k)s} (p - \frac{c}{r + \lambda k})A(s) - (\alpha V_B - \frac{c}{r + \lambda k})A(s) \right) ds
\]

It is easy to verify the conjecture by plugging it into (2.18).
2.7 Figures and Tables

Figure 2.1 Time series of debt repurchase and credit spread of BofA Merrill Lynch US corporate AA and B firms. The debt repurchase data is from Jermann and Quadrini (2012) and credit spread data are from Federal Reserve Bank of St. Louis.

<table>
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<tr>
<th></th>
<th>Debt Repurchase</th>
<th>B</th>
<th>AA</th>
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<tbody>
<tr>
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<td></td>
<td></td>
</tr>
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<td>1</td>
<td></td>
</tr>
<tr>
<td>AA</td>
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<td>0.8637</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2.4
Figure 2.2 Time series of open market repurchase, tender offers and credit spread of BofA Merrill Lynch US corporate AA and B firms. Yearly data of open market repurchase and tender offers are from [Julio (2013)]. Credit spread data are from Federal Reserve Bank of St. Louis.

Correlation Coefficients

<table>
<thead>
<tr>
<th></th>
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<th>TENDER</th>
<th>AA</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>OPEN</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TENDER</td>
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<td></td>
<td></td>
</tr>
<tr>
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<tr>
<td>B</td>
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<td>0.8365</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table 2.5
Figure 2.3 Debt Rollover and Buyback
Figure 2.4 Debt Buyback
Figure 2.5 Equity Value $E$ and Endogenous Default boundary $V_B$ when the firm buys back different proportion ($\beta$) of outstanding bonds. The parameters are listed in 3.2.
Figure 2.6 Equity maximizing $\beta^*$ for different principal outstanding $p$. The coupon $c$ is determined such that new debt is issued at par given $p$ and $\beta$. The rest of the parameters are listed in 3.2.
Figure 2.7 Default Boundary $V_B$ when equity holders optimally buy back debt (solid line) compared to the one when equity holders do not buy back at all (dashed line).
Figure 2.8 Equity maximizing $\beta^*$ for different liquidity shock frequency $\lambda$ and bonds transaction cost $k$. The rest of the parameters are listed in 3.2.
Figure 2.9 Optimal Leverage Ratio for different liquidity cost $k$ when firms do not buy back bonds at all (solid line) or buy back 4% annually (dashed line).
Figure 2.10 Value transfer from debt holders to equity holders for different buyback strategy $\beta$ when bonds transaction cost $k = 0.01$ (solid line) and $k = 0.012$ (dashed line). The value transfer is defined as $\beta p - \phi(\beta) \int_0^T \beta \kappa(s)d(V, s; V_B)$. 
Figure 2.11 Endogenous Default boundary $V_B$ as a function of liquidity shock cost $k$ when equity holders optimally buy back outstanding bonds.
Figure 2.12 Endogenous Default boundary $V_B$ (dashed line, right axis) and debt overhang $\frac{\partial d(V,T;V_B)}{\partial V}$ (solid line, left axis) for different buyback strategy $\beta$. 
Figure 2.13 Equity maximizing $\beta^*$ (dashed line) and firm value maximizing $\beta^{**}$ (solid line) as a function of debt principal outstanding $p$. 
Figure 2.14 Debt Repurchase from Jermann and Quadrini (2012) (dashed line, left axis) and bonds market liquidity measure from Corwin and Schultz (2012) (solid line, right axis)
Figure 2.15 Figure 2.15a shows liquidity measure from Corwin and Schultz (2012) (dashed line, left axis) and the total amount of bonds tendered in 2004-2005 (solid line, right axis). Figure 2.15b shows the number of bonds tender offers in 2004-2005.
Chapter 3

How to Roll Over Debt? The Effect of Risky Bond Yield Curve on Optimal Rollover Strategy

3.1 Introduction

The 2008 financial crisis highlights the rollover risk in corporate debt management. Firms commonly employ rollover policy: maturing bonds are paid off and replaced by issuing new bonds. Nagler (2015) found that 13% of the outstanding bonds of a given S&P 500 firm would be rolled over. He also found that the average amount of maturing bonds is $370 million and the average amount of new issuance is about $410 million, contingent upon a firm is active in debt market. However, debt rollover subjects to risk, as it is difficult for firms to issue new bonds when the firm’s assets value worsens or debt market freezes and lacks liquidity. As a matter of fact, credit risk and market liquidity are two important factors that determine corporate bonds credit spread. As He and Xiong (2012) points out, credit risk and liquidity risk are intertwined together through debt rollover.
Nevertheless, debt rollover itself still remains a black box and how to roll over debt is largely unknown. Previous research usually relies on Leland and Toft (1996) to study debt rollover and its impact on the yield curve of risky bonds (default-free yield curve plus term structure of credit spread). In Leland and Toft (1996), firms follow what we dub “naive rollover strategy”: firm issue new bonds of the same amount and maturity to replace the maturing ones. Figure 3.1a shows the induced debt maturity profile when the firm naively rolls over the bonds. The time-to-maturities of bonds are uniformly distributed on $[0, T]$. In other words, the proportion of bonds with time-to-maturity $s$ over the entire outstanding ones is $\frac{1}{T}$, $\forall s \in [0, T]$. Figure 3.1b shows the naive rollover strategy. It is represented by a Dirac delta function, since the firm only issues new bonds with maturity $T$ and the maturity bin has point mass across the entire maturity profile of new debt issuance. A natural question would be if the naive rollover strategy is optimal then. If it is not, what factors should the firm take into account when seeking an optimal rollover strategy, which assuages the adverse effect of assets and liquidity shocks on the credit conditions and increases the firm value overall? This is both a normative and positive question. In actual, the survey done by Tufano and Servaes (2006) has listed several factors that CEOs deem important in determining maturity structure of debt (See Table 3.1). They are also the important factors that affect a firm’s rollover strategy, as debt rollover largely shapes a firm’s maturity structure of debt. Among all the factors, risky bonds yield curve (including market depth) is a major one, featuring the critical feedback channel of risky bonds yield curve on rollover strategy decision.

Scant finance theory has studied these issues. In this paper, we use a dynamic model to study how the firm exploits risky bonds yield curve to come close to an optimal rollover strategy, highlighting the endogenous rollover strategy and risky bonds yield curve as well.

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The risky bond yield curve can be decomposed into treasury yield curve plus the term structure of credit spread. As can be seen in Section 3.2, the model fixes risk-free rate $r$ constant and thus assumes a flat default-free yield curve. Therefore, the curvature, level and sensitivity of the risky bond yield curve are entirely driven by the term structure of the credit spread.
as their interaction. Our model is similar to Leland and Toft (1996) and both generate risky bonds yield curve endogenously. In the model, the firm’s assets are exogenous and follow a geometric Brownian motion. Each instant the firm issues new bonds to pay off the maturing bonds and maintains a stationary debt structure. Depending on the firm’s current assets and debt market liquidity, debt rollover may result in capital gain or loss which equity holders have to assume. Any gain would be paid out to equity holders right away and any loss would be paid off by equity holders’ new contribution. The firm will service the debt until assets value reaches a threshold upon which equity value becomes zero and equity holders choose to default. Notwithstanding, unlike early literature on debt maturity accentuating rollover frequency, our model shuts down rollover frequency channel and focuses on the feedback channel of risky bond yield curve on rollover strategy.

Our analysis consists of two parts and focuses on the risky bonds yield level, sensitivity and its curvature. First, we argue that the firm can disperse maturity dates of newly issued debt when the risky bonds yield curve is concave. When the risky bonds yield curve is concave, the market value of the bonds is convex to the time-to-maturities. Dispersing the maturity dates of new issuance therefore bring in more proceeds than otherwise, incentivizing equity holders to lower the default boundary and mitigating the adverse effects of assets value and market liquidity shocks on firm’s credit conditions. The mitigation effect depends on the curvature of the current risky bond yield curve, and is strong when the curve is normal, i.e. concave upward sloping. We provide a general sufficient condition for the curve to be (locally) concave. Choi, Hackbarth, and Zechner (2014) empirically documented that corporate bonds maturity profile displays dispersion. A more relevant fact to our model, found by Norden, Roosenboom, and Wang (2016), is that firm’s incremental bond maturity choice also displays dispersion. In other words, firms issue bonds of various maturities to replace the maturing ones. Both empirical papers argue the reason is to mitigate maturity concentrations. While this is true, our paper implies another undocumented and testable
mechanism: firms are more likely to disperse the maturity dates of new issuances when the current risky bonds yield curve is concave.

Second, we argue that a better rollover strategy weighs the level against the sensitivity of the risky bonds yield curve. When debt is rolled over, the newly issued debt would cause the current market yield to change and the magnitude of the change echoes “market depth” in market micro-structure literature. For example, the risky bond yield of some maturity is low but might be highly sensitive to any new bonds issued in the maturity bin, and thus it is not clear whether newly issued bonds in that maturity bin would raise more money than others. Our model shows that when the firm switches from naive rollover to issuing new bonds based on trade-off between yield and market depth, equity holders would like to choose a lower default boundary. This is not surprising, as the better rollover strategy exploits the risky bonds yield curve and brings in more proceeds than naive rollover, reducing the amount of new contribution needed from equity holders. Therefore, a rollover strategy based on trade-off between risky bonds yield and market depth would also lessen the adverse effect of assets and market liquidity shocks on firm’s credit conditions.

Our calibration exercise is composed of “local” and “global” versions. The “local” calibration exercise takes the naive rollover and engendered debt profile as benchmark and numerically examines the effect of slight (local) perturbation from naive rollover and issuing new bonds based upon exploiting the risky bonds yield curve on the default boundary and firm value; Based on the result of local calibration, the “global” calibration proposes a new rollover strategy to take advantage of the risky bonds yield curve and compare it to the naive rollover. The calibration exercises try to reconcile model tractability and sophisticated rollover strategies employed by firms. Admittedly, a comprehensive characterization of rollover strategy would entail firm’s dynamic adjustment of debt maturity profile and leverage as well as taking account of the debt maturity profile inertia caused by frictions. However, such a model would be highly intractable and lose intuitive appeals.
It is helpful to compare our model to Modigliani and Miller (1958) and Miller and Modigliani (1961) and recognize the economic forces that are working. Modigliani-Miller theorem states that the capital structure and financial policies, such as rollover strategy and debt maturity arrangement, do not affect the value of a firm in an efficient market in the absence of taxes, bankruptcy and agency cost. Our model is in alignment with tradeoff theory and features debt tax benefits, bankruptcy cost as well as market illiquidity cost. Among the three of them, bankruptcy cost plays the essential role in our model. A better rollover strategy would help to enhance the firm value because it reduces the amount of new contribution from equity holders, incentivizing them to choose a lower endogenous default boundary and leading to a smaller bankruptcy cost and higher firm value overall.

Our model fills the gap between practice and finance theory and contributes to a large strand literature on corporate finance and debt maturity. As aforementioned, practitioners take many factors into consideration when rolling over debt, especially the risky bonds yield curve. Yet, finance theory has mainly focused the factor of rollover frequency and the unidirectional effect of rollover strategy on the risky bonds yield curve. So far as we know, our paper is the first one to discuss the feedback effect of risky bonds yield curve, including its level, curvature and sensitivity, on the rollover strategy. Furthermore, there have been extensive literature on the firm’s decision on debt maturity. Among those, He and Xiong (2012) discussed the credit condition exacerbation caused by frequent rollover of short-term bonds when market liquidity dries up. Acharya, Gale, and Yorulmazer (2011) shows that frequent rollover causes the sudden freeze in the accessibility of secured short-term debt. He and Milbradt (2016) analyzed the dynamic debt maturity choice made by firms based on the framework Leland (1994a) and Leland (1998). These works focused on the dichotomy trade-off between short-term and long-term debt but did not account for the whole debt maturity profile. Our paper, bridging Leland and Toft (1996) and Leland (1994a), studies rollover strategy and ensued debt maturity profile.
The remainder of the article proceeds as follows. In Section 3.2, we provide a general model to study bonds rollover and bridge Leland and Toft (1996) and Leland (1994a). We show that the major distinction is that our model shuts down the rollover frequency channel. In other words, it fixes the amount of debt maturing each instant. In Section 3.3, we apply the model to study rollover strategy, debt maturities dispersion and curvature of the risky bonds yield curve; In Section 3.4, we apply the model to study the effect of yield level and sensitivity in shaping a better rollover strategy; Section 3.5 concludes. All supplementary proofs are in the appendix.

3.2 A General Model of Debt Rollover

3.2.1 Firm and Asset

The unlevered value of a firm’s assets, \( V \), follows a geometric Brownian motion given by

\[
\frac{dV}{V} = (r - \delta) dt + \sigma dz
\]

where \( \{z_t\} \) is a standard Brownian motion. \( r \) is the risk-free interest rate; \( \delta \) is the payout rate to equity holders; \( \sigma \) is the volatility of assets value; \( r, \delta, \sigma \) are constants.

3.2.2 Debt Structure

The stationary debt structures from Leland and Toft (1996) and Leland (1998) are the most often cited structural frameworks for studying credit risk, corporate debt issues and debt maturity. These two frameworks assume that the firm is committed to maintaining a stationary debt structure. When an issue of debt matures, the firm “naively replaces it by issuing another debt with the same maturity.” Leland and Toft (1996) assumes a flat, uniform debt distribution, implying that a constant amount of outstanding debt is retired and replaced each instant. The time-to-maturity of the bonds is uniformly distributed between today,
time 0, and time $T$. In other words, $\forall s \in [0, T]$, an amount of $\frac{1}{t}$ of bonds will mature in $s$ periods. At each instant, the firm issues new bonds maturing in $T$ periods with the same amount of principal as the retired ones. (See Figure (3.1a)). Leland (1998) assumes that debt matures in an I.I.D random Poisson shock fashion, implying that a constant proportion of outstanding debt is retired and replaced; all bonds mature stochastically following an exponential distribution with mean $\frac{1}{\theta}$. At each instant, a proportion $\theta dt$ of bonds matures, and to replace them the firm issues new bonds of the same amount of principal with the same stochastic maturity.

We provide a more general model of debt rollover and argue that these two frameworks can be integrated, depending on how the firm rolls over its maturing debt. When some bonds mature, firms do not necessarily issue new bonds with the same amount of principal and maturity to replace them. On the contrary, to replace maturing bonds firms can issue several new bonds with different amounts of principal and periods of maturity (See Figure (3.2)). In fact, Norden, Roosenboom, and Wang (2016) empirically corroborates that many firms frequently issued bonds of multiple maturities at the same time and maintain a dispersed debt maturity structure consequently.

Suppose the firm has 1 unit of outstanding debt in total and that its maturities are distributed on $[0, T]$. All units of outstanding bonds are assumed to be of equal seniority. Since the total outstanding debt is 1, I use the probability density function $\kappa_t(s)$ to represent the amount of debt maturing in $s$ periods from date $t$. This satisfies

$$\int_0^T \kappa_t(s)ds = 1$$

Let $\mathcal{K}_t(s)$ be the corresponding cumulative distribution function. I assume $\kappa'_t(s) < 0, \forall s$. The firm commits to a stationary debt structure. Starting from date $t$, after $\Delta t$ periods, a
stationary debt structure implies
\[ \kappa_{t+\Delta t}(s) = \kappa_t(s), \forall \ 0 \leq s \leq T, \quad (3.1) \]

That is, the firm always has the same amount of bonds with the same time-to-maturity, regardless of the date that corresponds with “today”. The bonds must be rolled over in such a way that
\[ \mathcal{K}_t(\Delta t) = \int_0^{T-\Delta t} \kappa_{t+\Delta t}(s) ds - \int_{\Delta t}^{T} \kappa_t(s) ds + \int_{T-\Delta t}^{T} \kappa_{t+\Delta t}(s) \]

That is, the bonds that have matured before date \( \Delta t \) must be spread over among the bonds maturing in the future.

By changing the bounds of integral of part (1) in (3.2):
\[ \mathcal{K}_t(\Delta t) = \int_0^{T-\Delta t} (\kappa_{t+\Delta t}(s) - \kappa_t(s + \Delta t)) \ ds + \int_{T-\Delta t}^{T} \kappa_{t+\Delta t}(s) \]

From (3.1) and (3.3),
\[ \mathcal{K}_t(\Delta t) = \int_0^{T-\Delta t} (\kappa_t(s) - \kappa_t(s + \Delta t)) \ ds + \int_{T-\Delta t}^{T} \kappa_t(s) \]

Multiply \( \frac{1}{\Delta t} \) on both sides and let \( \Delta t \to 0 \),
\[ \kappa_t(0) = \int_0^{T} -\kappa'_t(s) ds + \kappa_t(T) \quad (3.5) \]
or equivalently,
\[ \int_0^{T} -\kappa'_t(s) ds = \kappa_t(T) - \kappa_t(0) \quad (3.6) \]

where \( \kappa'_t(s) \) is the derivative of the density function.
Equation (3.5) (or (3.6)) is in essence the first fundamental theorem of calculus, and implies that to keep debt structure stationary, maturing bonds are rolled over following a particular fashion. Specifically, for any time-to-maturity $s \in (0, T)$, the amount of bonds has to increase by $-\kappa'_0(s)$. Recall $\kappa'_0(s) \leq 0$ by assumption, so the firm does not need to buy back any outstanding bonds. The greater $-\kappa'_0(s)$ is, the more the firm issues new bonds with time-to-maturity $s$ to replace bonds that have just matured. Also, the firm has to issue new bonds with maturity $T$ in the amount of $\kappa_t(T)$ to maintain a stationary debt structure. Since $\kappa_t(s)$ does not explicitly depend on $t$, I drop the subscript $t$ and refer to $\kappa(s)$ henceforward.

A closer look at (3.5) reveals that

$$\int_0^T \frac{-\kappa'(s)}{\kappa(0)} ds + \frac{\kappa(T)}{\kappa(0)} = 1$$

(3.7)

This shows that $\Phi(s) := \left\{ -\frac{\kappa'(s)}{\kappa(0)} \mathbb{1}_{\{0 \leq s < T\}} \vee \frac{\kappa(T)}{\kappa(0)} \mathbb{1}_{\{s = T\}} \right\}$ is another probability density function and continuous almost everywhere.\footnote{The density function is not continuous at $s = T$. There is a point mass of $\frac{\kappa(T)}{\kappa(0)}$ at $s = T$} Note that $\Phi(s)$ is independent of $\kappa(0)$, i.e. the amount of maturing debt each instant, allowing us to fix $\kappa(0)$ and focus on the way the firm issues new debt. It effectively shuts down the rollover frequency channel of different maturities of debt and distinguishes us from previous literature on rollover risk and debt maturity. In those literature, higher rollover frequency is equivalent to larger amount of maturing debt each instant. Therefore, I interpret $\Phi(s)$ as the way the firm rolls over a single unit of maturing bonds and use $\mathbb{E}_{\Phi(s)}[\cdot]$ to denote the expectation with respect to probability density $\Phi(s)$. The first two moments can be calculated easily.

Lemma 1. Let $\bar{T} = \mathbb{E}_{\kappa(s)}[s] = \int_0^T s \kappa(s) ds$, i.e. $\bar{T}$ is the average maturity of the current
outstanding bonds. Then,

\[ \mathbb{E}_{\Phi(s)}[s] = \int_0^T s\Phi(s) \, ds = \frac{1}{\kappa(0)} \tag{3.8} \]

\[ \text{Var}_{\Phi(s)}[s] = \mathbb{E}_{\Phi(s)}[s^2] - \mathbb{E}_{\Phi(s)}[s]^2 = \frac{2T}{\kappa(0)} - \frac{1}{\kappa^2(0)} \tag{3.9} \]

Below I demonstrate how I establish a bridge between Leland (1998) and Leland and Toft (1996) by letting \( \kappa(s) = \theta e^{-\theta s} \). The framework more accurately reflects actual firm behaviors in debt maturity management and so provides a valuable basis for reasoning about optimal rollover strategy and debt maturity structure.

### 3.2.3 Debt Valuation

We follow He and Xiong (2012) and assume that the bondholders subject to a Poisson liquidity shock arriving with intensity \( \lambda \). Upon arrival of the liquidity shock, the bondholders have to liquidate all his portfolio right away at a fractional cost of \( k \). \( k \) models the availability of liquidity in the bonds market.

Consider a single bond maturing in \( t \) periods from the present that continuously pays a constant coupon flow \( c \) and has principal \( p \). Let \( \alpha \) be the fraction of asset value \( V_B \) the debt receives in the event of bankruptcy. Let \( d(V; V_B, t) \) be the market value of the debt and it satisfies

\[ r d(V; V_B, s) = \left( \frac{c}{2} - \lambda k \frac{d(V; V_B, s)}{\kappa} \right) - \frac{\partial d(V; V_B, s)}{\partial s} (r - \delta) V + \frac{1}{2} \frac{\partial^2 d(V; V_B, s)}{\partial V^2} \sigma^2 V^2 \tag{3.10} \]

The right side of (3.10) is the required return of holding bonds by investors; Part 1 on the left is the coupon received by the bondholders; Part 2 is the fractional cost incurred upon liquidation of the bond portfolios; Part 3 accounts for the bond value change caused by shortened time-to-maturity; The last part is the bond value change caused by underlying assets value fluctuation. To solve the partial differential equation (P.D.E), we need two
additional boundary conditions, corresponding to whether the firm declares bankruptcy or not before the debt matures. If the firm is able to repay the principal when the debt matures, the bond investors will claim $p$, i.e.

$$d(V; V_B, 0) = p, \forall V \geq V_B$$

(3.11)

Otherwise, if the firm defaults before the bonds mature, the bond investors can only claim $\alpha V_B$ since all units of debt are of equal seniority, i.e.

$$d(V_B; V_B, s) = \alpha V_B, \forall s \in [0, T]$$

(3.12)

The solution to (3.10) with boundary conditions (3.11) and (3.12) is given by

$$d(V; V_B) = \frac{c}{r + \lambda k} + e^{-(r + \lambda k)s} \left( p - \frac{c}{r + \lambda k} \right) (1 - F(s)) + \left( \alpha V_B - \frac{c}{r + \lambda k} \right) G(s)$$

(3.13)

where

$$F(s) = N(h_1(s)) + \left( \frac{V}{V_B} \right)^{-2a} N(h_2(s))$$

(3.14)

$$G(s) = \left( \frac{V}{V_B} \right)^{-a+z} N(q_1(s)) + \left( \frac{V}{V_B} \right)^{-a-z} N(q_2(s))$$

(3.15)

$$q_1(s) = \frac{-b - z \sigma^2 s}{\sigma \sqrt{s}}; q_2(s) = \frac{-b + z \sigma^2 s}{\sigma \sqrt{s}}$$

(3.16)

$$h_1(s) = \frac{-b - a \sigma^2 s}{\sigma \sqrt{s}}; h_2(s) = \frac{-b + a \sigma^2 s}{\sigma \sqrt{s}}$$

(3.17)

$$a = \frac{r - \delta - \frac{\sigma^2}{2}}{\sigma^2}; b = \ln \left( \frac{V}{V_B} \right); z = \frac{[(a \sigma^2)^2 + 2r \sigma^2]^\frac{1}{2}}{\sigma^2}$$

(3.18)

As a consequence, the market value of total outstanding debt is

$$D(V; V_B) = \int_0^T \kappa(s) d(V; V_B, s) \, ds$$

(3.19)
3.2.4 Equity Valuation

Analogous to debt valuation in last section, equity value $E$ satisfies the following differential equation

$$rE = (r - \delta)VE_V + \frac{\sigma^2}{2}V^2 E_{VV} + \delta V - \left(1 - \pi\right)c + \kappa(T)d(V; V_B, T) + \int_0^T -\kappa'(s)d(V; V_B, s)ds - \kappa(0)p$$  

(3.20)

The right side of (3.20) is the required return of equity holders; (1) on the left is the equity value change due to underlying assets value fluctuation; (2) is the assets payout to equity holders; (3) is the coupon payment plus tax benefits; (*) denotes the difference between the market value of newly issued bonds and the principal due. Clearly, it is closely related to how the firm rolls over the maturing debt. The (*) part which we dub “rollover term” can also be rewritten as

$$\mathcal{R}(\Phi(s)) = \kappa(0) \left[ \left( \frac{\kappa(T)}{\kappa(0)}d(V; V_B, T) + \int_0^T \frac{-\kappa'(s)}{\kappa(0)}d(V; V_B, s)ds \right) - p \right]$$  

(3.21)

$$= \kappa(0) \left( \mathbb{E}_{\Phi(s)}[d(V; V_B, s)] - p \right)$$  

(3.22)

(3.22) is the rollover gain/loss for one unit of debt outstanding times the total amount of principal due.

The equity value $E$ is solved in closed-form in Xu (2016). The endogenous default boundary $V_B$ can be solved from smooth-pasting condition

$$E_V|_{V=V_B} = 0$$  

(3.23)

and is given in Theorem 3.
Theorem 3. The endogenous default boundary $V_B$ is given by

$$V_B = \frac{(1-\pi)e^{-\kappa(0)(p+E\Phi(s)[Q_1(s)])} + \kappa(0)E\Phi(s)[Q_2(s)]}{\eta} + \frac{\delta}{\eta-1 + \kappa(0)E\Phi(s)[B(-u,s) + B(u,s)]}$$

(3.24)

where

$$Q_1(s) = \left(\frac{c}{r + \lambda k} + e^{-(r+\lambda k)s}(p - \frac{c}{r + \lambda k})\right)$$

(3.25)

$$Q_2(s) = \left(p - \frac{c}{r + \lambda k}\right)(b(-a,s) + b(a,s)) + \frac{c}{r + \lambda k}(B(-u,s) + B(u,s))$$

(3.26)

$$b(x,s) = \frac{e^{-(r+\lambda k)s}}{z+x}\left(N(x\sigma\sqrt{m}) - e^{rx}N(-z\sigma\sqrt{m})\right)$$

(3.27)

$$B(u,s) = \frac{1}{z+u}\left(N(u\sigma\sqrt{s}) - e^{\frac{1}{2}(\sigma^2-u^2)\sigma^2}sN(-z\sigma\sqrt{s})\right)$$

(3.28)

$$\eta = z - a, a = \frac{r - \delta - \frac{\sigma^2}{2}}{\sigma^2}, z = \frac{\sqrt{a^2\sigma^4 + 2r\sigma^2}}{\sigma^2}, u = \frac{\sqrt{a^2\sigma^4 + 2(r + \lambda k)\sigma^2}}{\sigma^2}$$

(3.29)

3.2.5 Connection to Leland (1994b, 1998)

The exponential distribution, the p.d.f for which is expressed as $\theta e^{-\theta s}$, is one of the distribution satisfying the assumption that its first order derivative is non-positive. By letting $\kappa(s) = \theta e^{-\theta s}$, I formally show (3.24) in theorem 3 can be reduce to Leland (1994b, 1998).

Corollary 1. By letting $T \to \infty$, $\lambda \to 0$ and $\kappa(s) = \theta e^{-\theta s}$, the endogenous default threshold and firm value reduce to Leland (1994b, 1998)

3.3 Yield Curve and Maturity Dispersion

In this section, we apply the general model of rollover strategy and analyze a new factor that drives the firm to disperse maturity dates when issuing new debt. Firms disperse maturity dates to exploit the curvature of the current term structure of the credit spread. When the term structure of credit spread is concave, the firm is able to raise more money from debt.
market by dispersing maturity dates, lessening the repayment pressure on equity holders and incentivizing them to choose a lower default boundary. To study the question, we need to be able to characterize $\kappa(t)$ in general, instead of assuming specific function form. To get around the technical difficulty, we employ Taylor expansion to analyze the expectation $E_{\Phi(s)}[\cdot]$ under the probability density function $\Phi(s)$.

To see this, we reexamine the equation (3.22)

$$\mathcal{R}(\Phi(s)) = \kappa(0) \left( E_{\Phi(s)}[d(V; V_B, s)] - p \right)$$

Under Taylor Expansion,

$$\mathcal{R}(\Phi(s)) \approx \kappa(0) \left( d(V; V_B, E_{\Phi(s)}[s] + s - E_{\Phi(s)}[s]) - p \right)$$

$$= \kappa(0) \left( d(V; V_B, \frac{1}{\kappa(0)}) \left( 1 + \frac{1}{2} \frac{\partial^2 d(V; V_B, \frac{1}{\kappa(0)})}{\partial s^2} Var_{\Phi(s)} - p \right) \right)$$

(3.31)

The last line of (3.31) uses the result of Lemma 1 that $E_{\Phi(s)} = \frac{1}{\kappa(0)}$. (3.31) reduces to Leland and Toft (1996) and He and Xiong (2012) when $\kappa(0) = \frac{1}{T}$, $\bar{T} = \frac{1}{2\kappa(0)} = \frac{T}{2}$, i.e. when the firm “naively” rolls over the maturing debt by issuing new debt with the same maturity $T$. When $\kappa(s) = \frac{1}{T}, \forall s \in [0, T]$, the firm only issues new debt with maturity $T$ to refinance the maturing ones and the dispersion part is missing as $Var_{\Phi(s)} = 0$.

Before we proceed to derive the endogenous default boundary $V_B$ under the approximation, we would like to first look at the economic intuition behind the “dispersion” term. Let $y$ denote bond’s yield. Note that
Part (1) is bond’s convexity; Part (3) is bond’s duration; Part (4) represents the curvature of the bond’s yield curve; Part (2) is the square of the slope of bond’s yield curve. We have

\[(1) > 0, (2) > 0, (3) < 0 \quad (3.30)\]

Only (4)’s sign is uncertain. For a normal yield curve, the curve is usually concave upward sloping and (4) < 0; For a inverted yield curve, the curve is usually convex downward sloping.

Due to the nature of approximation, we shall interpret (3.31) in a local fashion. Given the firm rolls over debt in a naive way and issues new debt with maturity $T$, if the current yield curve is concave at $T$, dispersing maturities of new bonds around $T$ therefore exploits the curvature of the yield curve and brings in more proceeds than otherwise (See Figure 3.3). Furthermore, although Lemma 1 characterizes $\text{Var}_{\Phi(s)}$ and associates it with the average maturity of the current outstanding bonds $\bar{T}$, we shall restrict $\text{Var}_{\Phi(s)}$ to small magnitude in the calibration for a more accurate approximation.

A natural question is when the yield curve is concave around $T$. Proposition 1 provides a sufficient condition for bond value to be convex with respect to time-to-maturity. It says that given debt principal outstanding and bankruptcy cost is sufficiently high, bond value $d(V, s; V_B)$ is convex with respect to time-to-maturity $s$ as long as $s$ is sufficiently long. Based on the aforementioned analysis, it also provides a sufficient condition for term structure of credit spread to be locally concave.

**Proposition 1.** Suppose $a = \frac{r-\delta - \frac{\sigma^2}{2}}{\sigma^2} > 0$, $p \geq \frac{c}{(r+\xi)} \geq \alpha V_B$. If $s \geq \frac{\ln(V_B)}{a\sigma^2}$, then $\frac{\partial^2 d(V, s; V_B)}{\partial s^2} > 0$.
Corollary 2 gives the default boundary $V_B$ under the approximation.

**Corollary 2.** Under the Taylor Expansion approximation, the endogenous default boundary $V_B$ becomes

$$ V_B = \frac{(1 - \pi - \kappa(0)) \left( -p + Q_1 \left( \frac{1}{\kappa(0)} \right) + \frac{Q''_1}{2} \frac{1}{\kappa(0)^2} \right) V \varphi(s)}{\eta} + \kappa(0) \left( Q_2 \left( \frac{1}{\kappa(0)} \right) + \frac{Q''_2}{2} \frac{1}{\kappa(0)^2} V \varphi(s) \right) $$

where

$$ \frac{\delta}{\eta - 1} + \alpha \kappa(0) \left( B(-u, \frac{1}{\kappa(0)}) + B(u, \frac{1}{\kappa(0)}) + \frac{\partial^2}{\partial s^2} \left( B(-u, \frac{1}{\kappa(0)}) + B(u, \frac{1}{\kappa(0)}) \right) V \varphi(s) \right) $$

$$ (3.34) $$

### 3.3.1 Model Calibration

To put the model into work, we adopt the parameters from existing literature. Table 3.2 shows the parameters values used in calibration. The annual proportion of outstanding bonds maturing, $\kappa(0)$, is a new parameter in our model. We set $\kappa(0) = \frac{1}{6} = 0.17$, corresponding to $T = 6$ in [Leland and Toft (1996)](1996). It is also close to the estimate 13% found by Nagler (2015).

### 3.3.2 Maturities Dispersion

In this subsection, we first consider “local” comparative statics. In other words, instead of assuming any specific rollover strategy, we will rely on equation (3.34) and consider the effect of “locally” dispersing maturities of newly issued debt around $T$ on endogenous default boundary $V_B$, restricting $V \varphi(s)$ to small magnitude in (3.34). Under such circumstance, $T = \frac{1}{\kappa(0)}$ and as a consequence, the proportion of debt maturing each instant decreases with $T$. Next, we will consider a “global” variation and propose a new specific rollover strategy to exploit the curvature of the yield curve. In this way, we are able to fix $\kappa(0)$.

Figure 3.4a shows the endogenous default boundary $V_B$ and $\bar{V}_B$ for different $T$ (or $\frac{1}{\kappa(0)}$) when $VAR_{\varphi(s)} = 0$ and $VAR_{\varphi(s)} = 1$ in (3.34), respectively. As we can see, a local debt
maturities dispersion does help to decrease the default boundary and the effect is more salient when the entire debt profile is short, i.e. $T$ is small. Figure 3.4(b) provides the answer. Figure 3.4(b) plots the difference between $V_B$ and $\bar{V}_B$ as well as local curvature at $T$. The concavity slowly decreases as the maturity of newly issued debt becomes longer. As a result, the marginal impact of local dispersion on the default boundary also diminishes.

The approximating nature of Taylor expansion in (3.31) destines that (3.34) is only accurate when $VAR_{\Phi(s)}$ is small. To obtain a better characterization of $V_B$ when newly issued bonds display dispersion around $T$ to a larger degree, we consider a specific class of debt maturity profiles and rollover strategies. In particular, we fixed $T = 6$ and $\kappa(0) = 0.17$ and disperse the maturities of new bonds symmetrically around $T$, ranging from $T - \Delta t$ to $T + \Delta t$. Figure 3.5 depicts the rollover strategy. The rollover strategy is consistent to the interpretation of (3.34) and more importantly, it shuts down the rollover frequency channel by fixing $\kappa(0)$ and highlights the dispersion effect.

Figure 3.6(a) and 3.6(b) show term structure of credit spread and market value of bonds with respect to time-to-maturities when the default boundary $V_B$ is determined by naive rollover strategy. Consistent with Proposition 1, the bonds value (credit spread) is concave (convex) with respect to time-to-maturity $s$ for $s > 2$. We are interested in how dispersion measured by $\Delta t$ affects default boundary $V_B$. Figure 3.7 plots the relationship between $\Delta t$ and $V_B$ and shows that $V_B$ decreases with $\Delta t$. The result not only depends on the local concavity of $d(V; V_B, s)$ at $T$ but also depends on the its global concavity on $[T - \Delta t, T + \Delta t]$.

As mentioned earlier and attested by (3.34), dispersing the maturities of newly issued bonds can take advantage of the curvature of the term structure of credit spread and increase the proceeds from new debt issuance. Fixing $\kappa(0)$, maturities dispersion increases (3.30) and lessens the repayment burden on equity holders, incentivizing them to choose a lower $V_B$. This explains the relationship between $V_B$ and maturity dispersion measure the calibrated
model displays so far.

For the same reason, dispersing maturities and taking advantage of the curvature of term structure of credit spread can also dampen the adverse effect of liquidity shock. Fig 3.8 shows $V_B$ for different liquidity cost $k$ when the firm disperses the maturities of new issuance of bonds and does not. It confirms that dispersing the maturity dates can decrease the default boundary chosen by equity holders and reduces the firm’s credit risk. The numerical example also shows that the attenuating effect of maturity dates dispersion on default boundary increases with liquidity cost.

3.4 Rollover, Yield Curve and Market Depth

When issuing new bonds to refinance the maturing ones, bonds price of different maturities have disparate sensitivities to the new issuance. For example, a low yield bond might be highly sensitive to any new bonds to the maturity bin. The sensitivity or bonds market depth is “Wall Street’s way of talking about a market’s ability to handle large trades without big moves in prices”[3]. Therefore, when firm issues new bonds, it has to trade off the yield and market depth of bonds for different maturities.

To see this more clearly, recall the rollover term (3.22)

$$\mathcal{R}(\Phi(s)) = \kappa(0) \left( \mathbb{E}_{\Phi(s)} [d(V; V_B, s) - p] \right)$$

$$= \kappa(0) \left[ \left( \Phi(T)d(V; V_B, T) + \int_0^T \Phi(s)d(V; V_B, s) ds \right) - p \right]$$

(3.35)

where

$$\Phi(s) := \left\{ -\frac{\kappa'(s)}{\kappa(0)} 1_{\{0 \leq s < T\}} \sqrt{\frac{\kappa(T)}{\kappa(0)}} 1_{\{s = T\}} \right\}$$

(3.36)

$$\Phi(T) + \int_0^T \Phi(s) ds = 1$$

(3.37)

[Bloomberg, “The Treasury Market’s Legendary Liquidity Has Been Drying Up”]

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Therefore, taking the derivative with respect to $\Phi(s)$, $\forall s \in (0, T)$:

\[
\frac{1}{\kappa(0)} \frac{\partial \mathcal{R}(\Phi(s))}{\partial \Phi(s)} = \frac{d(V; V_B, s) - d(V; V_B, T)}{1} + \Phi(s) \frac{\partial d(V; V_B, s)}{\partial \Phi(s)} + \Phi(T) \frac{\partial d(V; V_B, s)}{\partial \Phi(s)}
\]

(3.38)

\[
= \frac{d(V; V_B, s) - d(V; V_B, T)}{1} + \Phi(s) \frac{\partial d(V; V_B, s)}{\partial V_B} \frac{\partial V_B}{\partial \Phi(s)} + \Phi(T) \frac{d(V; V_B, T)}{\partial V_B} \frac{\partial V_B}{\partial \Phi(s)}
\]

(3.39)

(3.38) shows the effect of slightly shifting the amount of new debt issuance with maturity $T$ to the one with maturity $s < T$ on rollover term $\mathcal{R}(\Phi(s))$. It consists of two parts. Term (1) is the direct effect and shows the difference in proceeds raised between bonds of maturity $s$ and $T$. In other words, it represents the difference in the level of the yields associated with the two maturities; Term (2) accounts for the indirect effect and is the impact of the shift on the proceeds raised from the rest of new debt issuance with maturity $s$ and $T$. For example, suppose the yield of bonds with time-to-maturity $s$ is slightly lower than the ones with time-to-maturity $T$, it seems that issuing new bonds on maturity $s$ instead of on maturity $T$ would boost up the rollover term $\mathcal{R}(\Phi(s))$; however, if the market of bonds $d(V; V_B, s)$ is not deep in the sense that a new issuance on maturity $s$ would increase the yield to a larger extent, the indirect effect will dominate the direct effect and the firm would be better off not using new issuance of bonds with maturity $s$ to replace the ones with maturity $T$. The term (2) tallies with the definition of market depth in market micro-structure literature and therefore we dub (2) “market depth effect”. It essentially measures the sensitivity of current yield to new debt issuance for a particular maturity. (3.39) reveals the nature that drives the market depth effect: the heterogeneous response of endogenous default boundary $V_B$ to new debt issuance for different maturities.

(3.39) implies two factors that affect the depth of bonds market: the sensitivity of bonds value with respect to the default boundary $V_B$ and the sensitivity of default boundary $V_B$ with respect to $\Phi(s)$, the proportion of bonds with time-to-maturity $s$ among all newly issued bonds. We first study the sensitivity of debt market value $d(V; V_B, s)$ to default boundary
Proposition 2 characterizes $\frac{\partial d(V, V_B, s)}{\partial V_B}$ under general conditions. It says that given debt principal and bankruptcy cost sufficiently high, a higher default boundary will lower the market value of the debt as long as the time-to-maturity is sufficiently long.

**Proposition 2.** Suppose $p \geq c \frac{r}{r+\xi_k} \geq \alpha V_B$. If $s > \frac{(u+\alpha)s^2 + \log\left(\frac{V_B}{V}\right)}{u\sigma^2}$, then $\frac{\partial d(V; V_B, s)}{\partial V_B} < 0$

The current debt maturity profile inevitably affects the marginal effect of $\Phi(s)$ on default boundary $V_B$. Therefore, to characterize $\frac{\partial V_B}{\partial \Phi(s)}$, we have to set the benchmark debt maturity profile. As we are interested in whether the naive debt rollover is optimal, we set the debt maturity profile stemming from the naive rollover strategy as the benchmark, i.e. the one in Figure 3.1a. Proposition 3 characterizes $\frac{\partial V_B}{\partial \Phi(s)}$ under such circumstance.

**Proposition 3.** Suppose $p \geq c \frac{r}{r+\xi_k} \geq \alpha c(1-\pi) + \kappa(0) u (1 - \frac{1}{u-a})$ and the firm currently rolls over the debt naively. Then

$$\frac{\partial V_B}{\partial \Phi(s)} < 0, \quad \frac{\partial}{\partial s} \left( \frac{\partial V_B}{\partial \Phi(s)} \right) > 0 \quad (3.40)$$

Proposition 3 says that, given debt principal and bankruptcy cost sufficiently high, naive debt rollover strategy is not optimal. Fixing the amount of debt maturing each instant constant, increase the use of short-term bonds in debt rollover can decrease the default boundary $V_B$. Moreover, the effect of decreasing the default boundary and improving credit condition is stronger when the maturity of newly issued bonds is shorter. As can be seen in the calibrated model below, when the firm naively rolls over the debt, the market of short-term bonds is “deep” and yield is low.

**Model Calibration**

In this subsection, we calibrate the model using the parameters from Table 3.2. Our calibration exercise again has “local” and “global” two versions. In the local version, we will focus on the marginal effect of shifting the weight $\Phi(s)$ on $V_B$. We will numerically compute
\[
\frac{1}{\kappa(0)} \frac{\partial \mathcal{R}(\Phi(s))}{\partial \Phi(s)} \quad \text{and} \quad \frac{\partial V_B}{\partial \Phi(s)}
\]
and explore their relationships. In the global version, based on our aforementioned analysis, we will come up with a specific rollover strategy and debt maturity profile and see how targeting on yield curve and market depth can help firm to manage credit risk and liquidity risk.

Figure 3.9 shows the sensitivity of \(V_B\) to \(\Phi(s)\), the proportion of newly issued debt with time-to-maturity \(s\), given the amount of debt maturing each instant \(\kappa(0)\), when the firm currently rolls over the debt naively. Clearly, issuing short-term bonds slightly more reduces default boundary \(V_B\) and the effect becomes stronger as the time-to-maturity of debt is shorter. When the firm rolls over the debt naively, the firm does not fully exploit the term structure of credit spread and overlooks many favorable factors, including the difference in level and sensitivity of credit spread among various maturities. Figure 3.9 also plots \(\frac{1}{\kappa(0)} \frac{\partial \mathcal{R}(\Phi(s))}{\partial \Phi(s)}\), the sensitivity of rollover term \(\mathcal{R}(\Phi(s))\) with respect to the proportion function \(\Phi(s)\). Consistent with the evidence on \(\frac{\partial V_B}{\partial \Phi(s)}\), issuing short-term bonds slight more increases the rollover term and the effect is stronger as the time-to-maturity of debt considered, \(s\), is shorter. The rise on the rollover term lessens the amount of new contribution from equity holders, if any, and therefore let them willing to bail out the firm to a lower assets level.

The local calibration implies that issuing more short-term bonds lowers the default boundary, given the amount of debt maturing each instant. Therefore, we consider a specific rollover strategy featuring new issuance of short-term bonds. We consider a rollover strategy and debt profile illustrated in Figure 3.10. We fix \(\kappa(0) = \frac{1}{6} = 0.17\). Each instant the firm has to repay debt principal in the amount of \(\kappa(0)p\) and issues new bonds with maturities between 0 and 2 years as well as \(T\) years. To facilitate our discussion, we assume that the firm issues the same amount of bonds with maturity between 0 and 2 years, i.e. \(-\kappa'(s) = \epsilon, \forall s \in [0, 2]\), where \(\epsilon\) is the amount of bonds issued with maturity \(s\). We aim at studying the relationship between \(V_B\) and \(\epsilon\). To keep \(\kappa(0)\) still, we have to innocuously lengthen \(T\) so that \(\Phi(s)\) is integrated to 1 as a probability density function. Figure 3.11 shows the default boundary \(V_B\)
with respect to $\epsilon$. As the firm issues more short-term bonds, equity holders are more willing to bail out the firm to a lower assets value.

Therefore, for the same reason, we conjecture that the same rollover strategy can assuage the adverse effect of liquidity shock on the firm. Figure 3.12 shows $V_B$ for different liquidation cost $k$ and the amount of new short-term bonds issuance $\epsilon$. As the firm issues more short-term bonds to refinance the maturing debt, the default boundary $V_B$ becomes lower. Moreover, the effect is more conspicuous when the liquidation cost is higher. Higher liquidation cost would raise the bonds yield and coerce equity holders to contribute more capital to pay off the maturing debt, pushing up the default boundary quickly if the firm simply rolls over the debt naively; as a consequence, taking advantage of the yield curve and strategical rollover greatly helps to manage credit risk and liquidity risk.

3.5 Conclusion

In this paper, we study if the naive debt rollover in [Leland and Toft (1996)] is optimal and explore the strategies that the firm could take to exploit the yield curve, including its level, sensitivity as well as curvature, when rolling over debt, accentuating the endogenous rollover strategies and yield curve as well as their interaction. Debt rollover plays a very important role in corporate finance and has substantial impact on firm’s credit risk and liquidity risk. To study the question, we propose a model of debt rollover in general, bridging [Leland and Toft (1996)] and [Leland (1998), Leland (1994a)]. The model is able to fix the amount of bonds maturing each instant and thus shut down the rollover frequency channel, which distinguishes us from early literature on debt maturity. In addition, our model accounts for the corporate debt maturity profile, instead of focusing on dichotomy of short-term and long-term bonds.
Our model shows that the firm can disperse the maturity dates of new debt issuance when the yield curve is concave with respect to time-to-maturity, taking advantage of the curvature of the yield curve; the firm should also weigh the yield level against its sensitivity when issuing new debt, ideally targeting maturity bins with low yield and deep market. This shows that the firm can do better than simply rolling over debt naively. Taking advantage of the yield curve would lower the amount of new contribution from equity holders, if any, and incentivize them to choose a lower default boundary, leading to a lower bankruptcy cost and higher firm value. For the same reason, a better rollover strategy exploiting yield curve is able to assuage the adverse effect of assets and liquidity shocks on the firm’s credit conditions.

Like most Leland-type models, our model also assumes firm’s commitment on leverage and debt maturity structure to keep the model tractable and a constant endogenous default boundary $V_B$. In practice, firms do change leverage (Adrian, Moench, and Shin (2013), Goldstein, Ju, and Leland (2001), Collin-Dufresne, Goldstein, and Martin (2001), Titman and Tsyplakov (2007), Hennessy and Whited (2005)) and debt maturity structure (Chen, Xu, and Yang (2012), Xu (2014)). Recently, some progress are made to relax the assumptions. For example, DeMarzo and He (2014) allows the firm to dynamically adjust the leverage while He and Milbradt (2016) allows the firm to dynamically adjust the debt maturity. How dynamic adjustment on leverage and debt maturity profile affects rollover strategy and yield curve is unknown and we leave it to future research.
3.6 Proofs

3.6.1 Proof of Lemma 1

Direct calculation yields that

\[
\mathbb{E}_{\Phi(s)}[s] = \frac{\kappa(T)}{\kappa(0)} T - \int_0^T \frac{\kappa'(s)}{\kappa(0)} s \, ds \tag{3.41}
\]

\[
= \frac{\kappa(T)}{\kappa(0)} T - \frac{s \kappa(s) T}{\kappa(0)} - \int_0^T \frac{\kappa(s)}{\kappa(0)} s \, ds \tag{3.42}
\]

\[
= \frac{1}{\kappa(0)} \tag{3.43}
\]

where (3.42) is from integration by parts and (3.43) is by definition of probability density function.

Similarly, we have

\[
\text{Var}_{\Phi(s)}[s] = \mathbb{E}_{\Phi(s)}[s^2] - \mathbb{E}_{\Phi(s)}[s]^2 \tag{3.44}
\]

\[
= \frac{\kappa(T)}{\kappa(0)} T^2 - \int_0^T \frac{\kappa'(s)}{\kappa(0)} s^2 \, ds - \frac{1}{\kappa^2(0)} \tag{3.45}
\]

\[
= \frac{\kappa(T)}{\kappa(0)} T^2 - \frac{s^2 \kappa(s) T}{\kappa(0)} - 2 \int_0^T s \kappa(s) \, ds - \frac{1}{\kappa^2(0)} \tag{3.46}
\]

\[
= \frac{2T}{\kappa(0)} - \frac{1}{\kappa^2(0)} \tag{3.47}
\]

where (3.46) is from integration by parts and (3.47) is by definition of \( \bar{T} := \int_0^T s \kappa(s) \, ds \), i.e. the average maturity of existing debt structure.

3.6.2 Proof of Theorem 3

Xu (2016) provides a general model of rollover and early buyback and \( V_B \) (letting \( \phi = 1 \)) is given by
\[ V_B = \frac{(1-\pi)e^{c+\kappa(T)p - \kappa(T)Q_1(T)} + \int_0^T \kappa'(s)Q_1(s) ds}{\eta} + \frac{\kappa(T)Q_2(T) - \int_0^T \kappa'(s)Q_2(s) ds}{\delta} + \alpha \left( \kappa(T)(B(-u,T) + B(u,T)) - \int_0^T \kappa'(s)(B(-u,s) + B(u,s)) ds \right) \] 

(3.48)

where

\[ Q_1(T) = \left( \frac{c}{r + \lambda k} + e^{-(r+\lambda k)T} \left( p - \frac{c}{r + \lambda k} \right) \right) \] 

(3.49)

\[ Q_2(T) = \left( p - \frac{c}{r + \lambda k} \right) \left( b(-a,T) + b(a,T) + \frac{c}{r + \lambda k} (B(-u,T) + B(u,T)) \right) \] 

(3.50)

\[ b(u,s) = \frac{e^{-(r+\lambda k)s}}{z + u} \left( N(u\sigma \sqrt{s}) - \frac{e^{\sigma^2/2}}{\sqrt{2\pi}} \right) N(-z\sigma \sqrt{s}) \] 

(3.51)

\[ B(u,s) = \frac{1}{z + u} \left( N(u\sigma \sqrt{s}) - e^{\frac{1}{2}(\sigma^2-u^2)\sigma^2} N(-z\sigma \sqrt{s}) \right) \] 

(3.52)

\[ \eta = z - a, a = \frac{r - \delta - \sigma^2/2}{\sigma^2}, z = \frac{\sqrt{a^2\sigma^4 + 2r\sigma^2}}{\sigma^2}, u = \frac{\sqrt{a^2\sigma^4 + 2(r + \lambda k)\sigma^2}}{\sigma^2} \] 

(3.53)

Rearranging the terms in (3.48) and given the definition of \( \Phi(s) \), we have

\[ V_B = \frac{(1-\pi)e^{c-\kappa(T)(-p+\mathbb{E}_\Phi(s))}Q_1(s)}{\eta} + \frac{\kappa(0)\mathbb{E}_\Phi(s)Q_2(s)}{\delta} + \frac{\alpha \kappa(0)\mathbb{E}_\Phi(s)(B(-u,s) + B(u,s))}{\eta-1} \] 

(3.54)

3.6.3 Proof of Corollary 1

We will mainly discuss endogenous default threshold \( V_B \). Other variables including firm value, equity value and debt value can be derived in a similar way.

Let \( \kappa(t) = \lambda e^{-\lambda t} \) in (3.24) and the valuation involves two integrals to be evaluated:

\[ \int_0^\infty \lambda e^{-\lambda t} N\left( \frac{b\sigma^2 t}{\sigma \sqrt{t}} \right) dt = -N(b\sigma \sqrt{t})e^{-\lambda t}\big|_0^\infty + \int_0^\infty e^{-\lambda t} n(b\sigma \sqrt{t}) \frac{b\sigma}{2\sqrt{t}} dt \]

\[ = \frac{1}{2} + \frac{b\sigma}{2\sqrt{2\pi}} \Gamma\left( \frac{1}{2} \right) \]

\[ = \frac{1}{2} + \frac{b\sigma}{2\sqrt{2\pi}} \left( \lambda + \frac{b^2\sigma^2}{2} \right)^{1/2} \] 

(3.55)
\[
\int_0^\infty \lambda e^{-\lambda t} n\left( \frac{b \sigma^2 t}{\sigma \sqrt{t}} \right)(-1) \, dt = \frac{-2}{\sigma^2 b^2} \int_0^\infty \lambda e^{-\lambda t} dN(b \sigma \sqrt{t})
\] (3.56)

From (3.55),
\[
\int_0^\infty \lambda e^{-\lambda t} n\left( \frac{b \sigma^2 t}{\sigma \sqrt{t}} \right)(-1) \, dt = \frac{-\lambda}{\sqrt{2 \sigma (\lambda + \frac{b^2 \sigma^2}{2})}}
\] (3.57)

Hence,
\[
V_B|_{\kappa(t)=\lambda e^{-\lambda t}} = \frac{C + P \bar{y} - \tau C \bar{y}}{1 + \alpha \bar{y} + (1 - \alpha) \bar{y}}, \quad \text{where} \quad \bar{y} = \frac{r - \delta - \frac{\sigma^2}{2} + \sqrt{(r - \delta - \frac{\sigma^2}{2})^2 + 2(\lambda + r)\sigma^2}}{\sigma^2}
\] (3.58)
### 3.7 Figures and Tables

<table>
<thead>
<tr>
<th>Factors Determining Maturity Structure of Debt</th>
<th>% 4 or 5</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mitigate maturity concentrations</td>
<td>48%</td>
<td>217</td>
</tr>
<tr>
<td>Assets and liabilities matching</td>
<td>33%</td>
<td>212</td>
</tr>
<tr>
<td>Market Depth</td>
<td>33%</td>
<td>209</td>
</tr>
<tr>
<td>Expected slope of the yield curve</td>
<td>30%</td>
<td>208</td>
</tr>
<tr>
<td>Current slope of the yield curve</td>
<td>29%</td>
<td>210</td>
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<tr>
<td>Absolute credit spreads</td>
<td>24%</td>
<td>209</td>
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<tr>
<td>Current versus expected credit risk</td>
<td>19%</td>
<td>207</td>
</tr>
<tr>
<td>Credit Spreads relative to history</td>
<td>16%</td>
<td>208</td>
</tr>
<tr>
<td>Evaluated on the total interest paid</td>
<td>14%</td>
<td>216</td>
</tr>
<tr>
<td>Long-term debt riskier projects</td>
<td>12%</td>
<td>203</td>
</tr>
<tr>
<td>Mispricing of debt</td>
<td>12%</td>
<td>208</td>
</tr>
<tr>
<td>Evaluated on the interest volatility</td>
<td>8%</td>
<td>215</td>
</tr>
<tr>
<td>Other companies in industry</td>
<td>7%</td>
<td>203</td>
</tr>
</tbody>
</table>

**Table 3.1**

---

[1] The Table is from Tufano and Servaes (2006).

[2] The survey asked CEO “How important are the following factors in deciding on the maturity structure of your debt?” The answer can range from “Not important (0)” to “Very important (5)”.

[3] N: The number of correspondents

[4] The yield curve refers to the risky bonds yield curve, i.e. default-free yield curve plus term structure of credit spread.
<table>
<thead>
<tr>
<th>Baseline Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Debt Tax Benefit Rate</td>
<td>$\pi = 0.27$</td>
</tr>
<tr>
<td>Assets Volatility</td>
<td>$\sigma = 0.23$</td>
</tr>
<tr>
<td>Interest Rate</td>
<td>$r = 8.0%$</td>
</tr>
<tr>
<td>Payout Rate</td>
<td>$\delta = 2.0%$</td>
</tr>
<tr>
<td>Bankruptcy Recovery Rate</td>
<td>$\alpha = 0.6$</td>
</tr>
<tr>
<td>Liquidity Cost</td>
<td>$k = 0.01$</td>
</tr>
<tr>
<td>Liquidity Shock Intensity</td>
<td>$\lambda = 1.0$</td>
</tr>
<tr>
<td>Current (date-0) Assets Value</td>
<td>$V = 100.0$</td>
</tr>
<tr>
<td>Coupon</td>
<td>$c = 6.22$</td>
</tr>
<tr>
<td>Debt Principal</td>
<td>$p = 61.68$</td>
</tr>
<tr>
<td>Principal Proportion Due</td>
<td>$\kappa(0) = 0.17$</td>
</tr>
</tbody>
</table>

**Table 3.2**
(A) Debt Profile in Leland and Toft (1996). The horizontal axis indicates time-to-maturity of outstanding bonds; The vertical axis indicates the proportion of bonds of each time-to-maturity over the entire outstanding bonds.

(B) Debt Rollover in Leland and Toft (1996). The horizontal axis indicates new debt maturity; The vertical axis indicates the proportion of new bonds of each maturity over the entire new issuance. In Leland and Toft (1996), it is a Dirac Delta function.

Figure 3.1 Debt profile and rollover strategy in Leland and Toft (1996).
Time-to-maturity of outstanding bonds
Proportion $\kappa(0)$
maturing debt $dt$
Newly Issued Debt

Figure 3.2 Debt Rollover in general
Figure 3.3 Local dispersion around $T$ when debt is rolled over
Figure 3.4 Figure 3.4a shows the default boundary $V_B$ when $VAR_{\Phi(s)} = 0$, i.e. no dispersion (solid line) and the default boundary $\bar{V}_B$ when $VAR_{\Phi(s)} = 1$ (dashed line), as functions of the amount of principal maturing $\kappa(0)$ each instant. Figure 3.4b plots $V_B - \bar{V}_B$, the difference between $V_B$ and $\bar{V}_B$ (solid line and left axis) and $\frac{\partial^2 d(V,s;V_B)}{\partial s^2}|_{s=T}$, i.e. the local concavity of bonds value with respect to time-to-maturity at $s = T$ (dashed line and right axis).
Figure 3.5 Debt maturity profile and rollover strategy when the firm exploits the curvature of yield curve for maturities in \([T - \Delta t, T + \Delta t]\). When debt is rolled over, the firm would issue the same amount of new bonds with maturities in \([T - \Delta t, T + \Delta t]\).
(A) Bonds Yield Curve (Term structure of credit spread)

(B) Bonds Market Value with respect to time-to-maturity

Figure 3.6 Bonds Yield Curve and Market Value. All parameters follow Table 3.2.
Figure 3.7 Endogenous default boundary $V_B$ when the firm naively rolls over the debt (no dispersion); Endogenous default boundary $V_B$ as a function of $\Delta t$ (dashed line, left axis), when the firm follows the debt maturity profile and rollover strategy illustrated in Figure 3.5; Firm value, $D + E$, as a function of $\Delta t$ (dot-dashed line, right axis). All other parameters follow Table 3.2.
Figure 3.8 The endogenous default boundary as a function of liquidity cost $k$ when the firm naively rolls over the debt, i.e. no dispersion (solid line) and disperses the maturity dates of new bonds (dashed line).
Figure 3.9 The sensitivity of endogenous default boundary $V_B$ with respect to $\Phi(s)$, the proportion of bonds with time-to-maturity $s$ among all newly issued bonds (solid line, left axis); The sensitivity of rollover term $\mathcal{R}(\Phi(s))$ with respect to $\Phi(s)$ (dashed line, right axis).
Figure 3.10 Debt maturity profile and rollover strategy when the firm issues bonds with maturities between 0 and 2 year as well as \( T \) years to replace the maturing ones, holding the amount of debt maturing each instant constant. Note that the slope between 0 and 2 years is \(-\epsilon\).
Figure 3.11 Endogenous Default Boundary $V_B$ as a function of $\epsilon$, the proportion of short-term bonds issued.
Figure 3.12 Endogenous Default Boundary $V_B$ as a function of liquidity cost $k$ and the proportion of short-term bonds issued $\epsilon$. 
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