STRATIFICATIONS OF REPRESENTATIONS AND CYCLIC QUIVERS

BY

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DISSERTATION

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Abstract

Given an algebraic variety X with an action of a reductive group G, geometric invariant theory splits X as the disjoint union $X = X^{ss} \sqcup X^{un}$ of the semistable and unstable locus. The Kirwan-Ness stratification refines X even more by describing X^{un} as a disjoint union of strata $X^{un} = \sqcup_{\beta \in \mathsf{KN}} S_\beta$ detemined by 1-parameter subgroups β. In this thesis we study the 1-parameter subgroups that determine the Kirwan-Ness stratifications of representations. We will describe an algorithm that finds the β's and we show that such algorithm can be simplified when our space is of the form T^*V where V is a vector space. We go on to investigate more deeply the 1-parameter subgroups in the case of the space of representations $\mathrm{Rep}(Q, v)$ of a cyclic quiver Q.

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Chapter 1

Introduction

1.1 Quotients of algebraic varieties

The study of a space can often be simplified by making use of symmetries. For example, if that space has symmetries we can construct a quotient space and study that instead. More precisely, given a space X equipped with an action of a group G, often we study the orbit space X/G. Unfortunately, the orbit space does not always keep the geometric properties that X had. For example, it often occurs that the space X is Hausdorff but X/G is not. When X is an algebraic variety a very similar thing occurs. Most of the times the orbit space X/G will not be separated (a notion similar to Hausdorff in algebraic geometry), which will mean that X/G is not an algebraic variety.

In the next chapter we will study different quotients that have better geometric properties than the orbit space. More precisely, we relax the idea of having an orbit space and prioritize having a quotient space that will be an algebraic variety. In particular, it will often happen that the new quotient identifies some orbits. Given an affine algebraic variety X equipped with an action of a reductive group G, the categorical quotient $X /\!\!/ G = \mathrm{Spec}(\mathbb{C}[X]^G)$ will again be an affine variety. Unfortunately, in most of the cases we are interested in, the categorical quotients are not "good enough". For example, let $G = \mathbb{G}_m$ act on $X = \mathbb{A}^n$ by $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$ and let k be an algebraically closed field of characteristic zero. Then $A(X)^G = k$ and $\mathrm{Spec}[A(X)^G] = \mathrm{Spec}\, k$ is a point, which doesn't reflect much about the orbit structure of the original space. In order to improve this, we will introduce a quotient developed by Mumford [1]. Geometric Invariant Theory (GIT) was developed to construct quotients $X /\!\!/_L G$ of projective varieties and understand them geometrically. This construction can be adapted to the case when X is an affine variety, obtaining this way a "better" quotient than the categorical quotient. The downside in this construction is that $X /\!\!/_L G$ depends on a choice of a line bundle L. The area that studies how the GIT quotient depends on the choice of linearization of the action is known as variation of GIT. One of the main results in the area shows that there are only finitely many different GIT quotients corresponding to the same G-action. More precisely, there is a wall and chamber structure on the space of linearizations of a reductive group action such that the quotient only changes as one crosses a wall. In this thesis we will not talk much about variation of GIT, but more about the topic can be found in [2] and [3], for example.

1.2 Kirwan-Ness stratifications

As with the categorical quotient, $X /\!\!/_L G$ may not be an orbit space anymore. However, it parametrizes the closed G-orbits in an open subset X^{ss} of X, which we call the semistable locus of X. Thus, the GIT quotient splits the space as a disjoint union $X = X^{ss} \cup X^{un}$, where we call X^{un} the unstable locus of X. Finding the semistable points can become a tedious task, but Mumford introduced a numerical criterion, known as the Hilbert-Mumford criterion, which is incredibly useful for finding X^{ss}. The essence of the criterion is that stability can be studied by only looking at the 1-parameter subgroups of G (i.e. group homomorphism $\beta : \mathbb{G}_m \to G$, where $\mathbb{G}_m$ denotes the multiplicative group). In particular, Mumford's proof indicates the existence of a preferred 1-parameter subgroup which will be witness to the instability of an unstable point. Kempf [4], Kirwan [5] and Ness [6] used that idea to stratify the unstable locus as a union of stratum labeled by 1-parameter subgroups.

Given an action of a reductive group G on a variety X, the Kirwan-Ness stratification associates a canonical stratification where the open stratum coincides with the semistable locus X^{ss}, and the unstable locus X^{un} is a disjoint union of G-invariant locally-closed subvarieties S_β of X,

$$X = X^{ss} \sqcup \bigsqcup_{\beta \in \mathsf{KN}} S_\beta.$$

Furthermore, when X is nonsingular, the subvarieties S_β will also be nonsingular. In chapter 3 we will describe the stratification for the case when X is a projective variety. We will do it by following the work by Kirwan [5], which is based on work by Kempf [4] (which is exposed in a paper by Hesselink [7]) and has close links to work by Ness[6]. However, most of the work in this thesis is focused on studying the set KN; that is, the 1-parameter subgroups that determine the stratification. Knowing these is sometimes enough; for example, McGerty and Nevins [8] used them to find information about the representation of certain algebras.

1.3 Representations of the form T^*V

In chapter 4, following the work by McGerty and Nevins [8], we will adjust the theory used to define the Kirwan-Ness stratification when X was a projective variety, to be able to define a Kirwan-Ness stratification when X is a representation. In their paper, they give an algorithm that finds the 1-parameter subgroups β that determine the strata. The main theorem in the thesis gives a more efficient algorithm when X is the cotangent bundle T^*V of a representation V. This space is of great interest since it carries a symplectic form; more precisely, when V is a finite-dimensional vector space, the cotangent bundle is of the form $T^*V = V \oplus V^*$ and it has a symplectic form. This will allow as to take the Hamiltonian reduction, which is closely related to the GIT quotient [9].

Precise definitions and statements of the main result and consequences appear in section 4.1. However, the main idea is the following:

Let V be a linear representation of G and fix a maximal torus T. Choose a character $\lambda : \mathfrak{g} \to \mathbb{G}_m$ and assume that ϵ is a small negative rational number. Let $\{e_1, e_2, \dots, e_n\}$ be a basis of V consisting of T-weight vectors, and let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be the corresponding set of weights. Consider the basis of $\mathbb{C} \oplus V$ given by $\{e_0, e_1, \dots e_n\}$ where e_0 is the standard basis vector of $\mathbb{C} \times \{0\} \subset \mathbb{C} \oplus V$. Setting $\alpha_0 = 0$, $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$ gives a list of weights of $\mathbb{C} \oplus V$.

Given a nonempty subset $I \subset \{0, 1, \dots n\}$, we denote by W_I the subspace spanned by the weights $\{\alpha_i \mid i \in I\}$ and define

$$p_I(\epsilon) := \mathrm{proj}_{W_I^{\perp}}(\epsilon\lambda).$$

Theorem 1.1. (Theorem 4.1). *Let V be a finite dimensional representation of a reductive group*

G over k. Let T^*V be equipped with the G action induced from the action on V. Then the set of KN 1-parameter subgroups that defines the stratification for the cotangent bundle T^*V is given by

$$\mathsf{KN} = \{p_I(\epsilon) \mid I \subset \{0, 1, \ldots, n\}\}.$$

Corollary 1.1. *Under the same assumptions as in the theorem and given a character λ, the semistable locus $(T^*V)^{ss}$ is nonempty if and only if there is a subset I such that $\lambda \in \operatorname{span}\{\alpha_i \mid i \in I\}$.*

Sage code that implements the algorithm can be found in chapter 5, followed by several examples.

1.4 Representation spaces of quivers

The rest of the thesis studies the case where $V = \operatorname{Rep}(Q, v)$ is the representation space of a cyclic quiver Q. Chapter 6 focuses on finding the 1-parameter subgroups that define the Kirwan-Ness stratifications of the space $T^*(\operatorname{Rep}(Q, v))$. The GIT quotient $T^*(\operatorname{Rep}(Q, v)) /\!\!/_\chi GL(v)$ is closely related to *Nakajima quiver varieties*, objects that are of great interest in geometric representation theory. Below we give a brief introduction to the topic following the work by Nakajima [10], [11].

A *quiver* $Q = (I, E, h, t)$ is a directed graph where I is the set of vertices, E is the set of edges and $h, t : E \to I$ are maps that assign its head and tail to every edge. A *representation* (V, f) of a quiver Q consists of the following data:

- To every vertex $i \in I$ corresponds a vector space V_i over k.
- To every edge $x \in E$ corresponds a linear operator $f_x : V_{t(x)} \to V_{h(t)}$.

The dimension vector of the representation is given by $v = (\dim V_i)_{i \in I}$. and once this is fixed, the representation space of a quiver Q is given by

$$\operatorname{Rep}(Q, v) = \bigoplus_{x \in E} \operatorname{Hom}_k(k^{v_{t(x)}}, k^{v_{h(x)}}).$$

The reductive group

$$GL(v) = \prod_{i \in I} GL(v_i, k)$$

acts on $\text{Rep}(Q, v)$ by conjugation, i.e. $(g \cdot f)_x = g_{h(x)} f_x g_{t(x)}^{-1}$.

The characters of $GL(v)$ are of the form

$$\chi_\theta(g) = \prod_{i \in I} \det(g_i)^{\theta_i}$$

where $\theta \in \mathbb{Z}^I$. Throwout the thesis, we will identify χ_θ and θ.

As we mentioned before, $T^*\text{Rep}(Q, v)$ has a symplectic structure and we call the associated moment map μ. The *Nakajima quiver variety* is defined to be the GIT quotient

$$\mathcal{M}_\theta = \mu(0)^{-1} /\!\!/_{\chi_\theta} GL(v).$$

Even if this is not exactly the space we will be studying, knowing the Kirwan-Ness strata S_β for $T^*\text{Rep}(Q, v)$ is enough because $\mu^{-1}(0)$ is an affine closed subvariety of $T^*\text{Rep}(Q, v)$. In particular, the Kirwan-Ness strata of the quiver variety is given by $S_\beta \cap \mu^{-1}(0)$.

The cyclic quiver can be seen as the generalization to the Jordan quiver (a quiver with one vertex and a single loop) which is known for having very nice geometric properties. For example; for certain dimension vector and $\theta = -1 \in \mathbb{Z}$, the projective morphism

$$\pi : \mathcal{M}_\theta = \text{Hilb}^n(\mathbb{C}^2) \to \mathcal{M}_0 = (\mathbb{C}^n \times \mathbb{C}^n)/\mathbb{S}_n$$

is a resolution of singularities [12]. We refer the reader to Section 6.1 for more details.

1.5 Why study the KN strata for (cyclic) quiver varieties?

The main conclusion of my result is that it makes it now conceivable that we can work out the GIT chamber structure—before, one did not know what to do about the refinement procedure, whereas now we can produce exactly the KN 1-PS and thus can hopefully deduce the GIT walls. I intend to pursue that in future work. For example, this might allow the classification of symplectic resolutions of affine quiver varieties, which is not yet known.

As I mentioned before, the cyclic quiver can be seen as the generalization of a basic but very important case: the Jordan quiver. The KN strata for the Jordan quiver were worked out in McGerty-Nevins [8] in about a page. In their paper they deduce applications to the representation theory of the type A spherical Cherednik algebra. The next important class of examples for geometric representation theory is the class of more general cyclic quivers. There the corresponding problems in representation theory, related to representations of the "spherical cyclotomic Cherednik algebra", have not been solved, though there is some related literature ([13], for example).

Another very important outcome of my result is the improvement on the computational complexity. Although growth of set of KN 1-PS appears exponential in the dimension of the representation, in natural families of representations the number of W-orbits (Weyl group orbits), hence strata, seems to grow linearly (or at least polynomially) in the natural index of the family. For example, $\mathfrak{gl}_n \times \mathbb{C}^n$ acted on by GL_n, where there are $n+1$ KN 1-parameter subgroups (that's the Jordan quiver case in McGerty-Nevins [8]). It's natural to expect a similar growth for example in the cyclic quiver situation. Thus in important situations, the computational complexity is not expected to grow so quickly as to make algorithmic computation unfeasible.

Chapter 2

Geometric invariant theory

2.1 Quotients

We have mentioned that the quotient $\text{Rep}(Q, v)/GL(v)$ describes the orbits of the $GL(v)$ action on $\text{Rep}(Q, v)$. However, the orbit space is not an algebraic variety in most of the cases. The purpose of Geometric Invariant Theory (GIT) is to construct quotients that are algebraic varieties. We will discuss two approaches: the categorical quotients and the GIT quotients. Before that let's remind some notions regarding algebraic groups.

Throwout the thesis k will denote an algebraically closed field of characteristic zero. In this section we will assume $k = \mathbb{C}$.

Definition 2.1. An *(affine) algebraic group* is a group that is also an (affine) algebraic variety, such that the product and inverse maps are morphisms of algebraic varieties.

For example, the group $GL_n(k)$ is an algebraic group. Furthermore, every affine algebraic group over k is isomorphic to a linear algebraic group over k; that is, a Zariski closed subset of a general linear group $GL_n(k)$ which is also a subgroup of $GL_n(k)$.

Definition 2.2. We say that an algebraic group G is *linearly reductive* if any of its rational representations (i.e. homomorphisms $G \to GL(V)$ of algebraic groups) are completely reducible.

A linear algebraic group over a field k is said to be *reductive* if its unipotent radical (i.e. the largest normal subgroup consisting of elements satisfying the equation $(x-1)^n = 0$ for some n) is the trivial subgroup.

Facts:

a) When $\text{Char}(k) = 0$, the definitions of linearly reductive and reductive are equivalent.

b) If G is a complex algebraic group, G is reductive if and only if it is the complexification of a compact Lie group.

Example 2.1. Any general linear group and any product of those are reductive.

In particular, notice that the group $GL(v)$ that acts on $\text{Rep}(Q, v)$ is reductive.

2.1.1 Categorical quotient

Let $X \subset \mathbb{A}^n$ be an affine variety over k defined by polynomials $f_1, \dots f_r \in \mathbb{C}[x_1, \dots, x_n]$, and let G be an algebraic group acting on X. The coordinate ring of X is the finitely generated k-algebra

$$\mathbb{C}[X] \cong \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_r).$$

The G action on X induces an action on $\mathbb{C}[X]$ given by $g \cdot f(x) = f(g^{-1} \cdot x)$. We write $\mathbb{C}[X]^G$ for the G-invariant functions, that is

$$\mathbb{C}[X]^G = \{f \in \mathbb{C}[X] \mid g \cdot f(x) = f(x)\ \forall g \in G\}.$$

Let Y be an affine variety where G acts trivially. Given any G-invariant morphism (i.e. $\varphi(g \cdot x) = g \cdot \varphi(x)$) $\varphi : X \to Y$ of affine varieties, the image of the induced morphism $\varphi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ is contained in $\mathbb{C}[X]^G$. If $\mathbb{C}[X]^G$ is a finitely generated algebra we can construct an affine variety whose coordinate ring is $\mathbb{C}[X]^G$. That affine variety is given by $\text{Spec}[\mathbb{C}[X]^G]$.

Definition 2.3. We define the categorical quotient to be

$$X /\!\!/ G := \text{Spec}[\mathbb{C}[X]^G].$$

However, there is a question that we need to answer: Is $\mathbb{C}[X]^G$ finitely generated? Determining when the ring of invariant functions $k[X]^G$ is finitely generated is known as Hilbert's 14th problem. For $G = GL_n$ over the complex numbers, Hilbert showed that the invariant ring is always finitely generated. However, for a group G built using copies of the additive group $\mathbb{G}_a$, Nagata gave a counterexample where $k[X]^G$ is not finitely generated. Furthermore, Nagata proved that for any reductive group G, the ring of invariants functions is finitely generated.

2.1.2 GIT quotients

- Projective quotient: Let $X \subset \mathbb{P}^n$ be a projective variety with homogeneous coordinate ring

$$R(X) = k[x_0, x_1, \dots, x_n]/I(X).$$

Let $\tilde{X}$ be the affine cone over X; that is,

$$\tilde{X} = \{0\} \cup \{(x_0, x_1, \dots, x_n) \in \mathbb{A}^{n+1} - \{0\} \mid [x_0 : \cdots : x_n] \in X\}.$$

Notice that $R(\mathbb{P}^n) = \mathbb{C}[\mathbb{A}^{n+1}]$, and therefore

$$R(X) = R(\mathbb{P}^n)/I(X) = \mathbb{C}[\mathbb{A}^{n+1}]/I(\tilde{X}) = \mathbb{C}[\tilde{X}].$$

Let G be a reductive group that acts linearly on $X \subset \mathbb{P}^n$; in particular, G acts on X and $\mathbb{P}^n$ with $G \cdot X \subset X$. This lifts to an action on the affine cones $\mathbb{A}^{n+1}$ and $\tilde{X}$ over $\mathbb{P}^n$ and X respectively. More precisely, G acts on $R(X) = \mathbb{C}[\tilde{X}]$ and preserves the graded parts such that $R(X)^G = \mathbb{C}[\tilde{X}]^G$ is a homogeneous graded subalgebra of $R(X)$. $R(X)^G$ is then finitely generated by Nagata's theorem, so we can define $\mathrm{Proj}(R(X)^G)$.

Definition 2.4. Let $X \subset \mathbb{P}^n$ be a projective variety. We define the projective GIT quotient to be the projective variety

$$X /\!\!/ G := \mathrm{Proj}[R^G].$$

The inclusion $R^G \hookrightarrow R$ induces a rational map

$$X \dashrightarrow X /\!\!/ G$$

which is not defined on the set $\{x \in X \mid f(x) = 0 \text{ for all } f \in R^G_+ = \otimes_{i>0} R^G_i\}$. We define the *semistable locus* to be

$$X^{ss} = \{x \in X \mid f(x) \neq 0 \text{ for some } f \in R_+^G\}.$$

Then the map $X^{ss} \to X /\!\!/ G$ is a morphism.

Example 2.2. Let $G = \mathbb{G}_m$ act on $X = \mathbb{P}^n$ by

$$t \cdot [x_0 : x_1 : \cdots : x_n] = [t^{-1}x_0 : tx_1 : \cdots : tx_n].$$

The homogeneous coordinate ring is

$$R(X) = k[x_0, \ldots, x_n]$$

and the G-invariant functions are given by

$$R(X)^G = k[x_0x_1, \ldots, x_0x_n] \cong k[y_0, \ldots, y_{n-1}].$$

Then $X /\!\!/ G := \text{Proj}[R^G] = \mathbb{P}^{n-1}$ and

$$\varphi : X = \mathbb{P}^n \dashrightarrow X /\!\!/ G = \mathbb{P}^{n-1}$$
$$[x_0 : x_1 : \cdots : x_n] \mapsto [x_0x_1 : \cdots : x_0x_n]$$

is a rational morphism. However, if we restrict the map to

$$X^{ss} = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_0 \neq 0, (x_1, \ldots, x_n) \neq 0\} \cong \mathbb{A}^n - \{0\}$$

then

$$\varphi : X^{ss} \cong \mathbb{A}^n - \{0\} \to X /\!\!/ G$$

is a morphism.

Remark: So far we have considered projective varieties $X \subset \mathbb{P}^n$. In particular, they were equipped with an embedding in the projective space. Given an abstract projective variety X acted on by a reductive algebraic group, we first need to choose a G-invariant embedding of X in a projective space. We do that by choosing an ample G-linearized line bundle on X, i.e. an ample line bundle together with a lift of the G-action.

Definition 2.5. A *character* χ on a group G is a group homomorphism from G to the multiplicative group $\mathbb{G}_m$.

Example 2.3. Let $L = X \times \mathbb{A}^1$ be the trivial line bundle on a variety X over k. A G-linearized line bundle on X is defined by a group character $\chi : G \to \mathbb{G}_m$. The character χ defines a lift of the action to L by

$$g \cdot (x, z) = (g \cdot x, \chi(g)z)$$

where $(x, z) \in X \times \mathbb{A}^1$.

Suppose that a reductive group G acts on a projective variety X with respect to a G-equivariant ample line bundle L. We define

$$R := \bigoplus_{n \geq 0} \Gamma(X, L^{\otimes n})$$

to be the associated graded algebra of sections of powers of L; then $X \cong \text{Proj}(R)$. There is an induced action of G on the space of sections $\Gamma(X, L^{\otimes n})$ and we denote the graded subalgebra of G-invariant sections by

$$R^G = \bigoplus_{n \geq 0} \Gamma(X, L^{\otimes n})^G.$$

Definition 2.6. We define the projective GIT quotient to be the projective variety

$$X /\!\!/_L G := \text{Proj}[R^G].$$

As before, the inclusion $R^G \hookrightarrow R$ induces a rational map

$$X \dashrightarrow X /\!\!/_L G$$

and this is only well defined on the semistable locus

$$X^{ss}(L) = \{x \in X : \exists \sigma \in \Gamma(X, L^{\otimes n})^G \text{ such that } \sigma(x) \neq 0\}.$$

Definition 2.7. We call $x \in X$ *semistable* if there exits $\sigma \in \Gamma(X, L^{\otimes n})^G$ such that $\sigma(x) \neq 0$. We say $x \in X$ is unstable if it is not semistable.

Definition 2.8. We call $x \in X$ *stable* if all of the following conditions are satisfied:

- x is semistable
- $G \cdot x$ is closed in the semistable locus
- the stabilizer G_x is finite.

Remarks:

1. The semistable locus, and therefore $X /\!\!/_L G$, depend on the choice of line bundle.
2. When we have a linear action of G on $X \subset \mathbb{P}^n$, the G-linearized line bundle is assumed to be $\mathcal{O}_{\mathbb{P}^n}(1)$ with the lift given by the natural action of GL_{n+1} on $\mathbb{A}^{n+1}$.

- Affine quotient: Let $X \subset \mathbb{A}^n$ be an affine variety over k with an action of a reductive algebraic group G. In most of the cases we are interested in, the categorical quotients are not "good enough". For example, let $G = \mathbb{G}_m$ act on $X = \mathbb{A}^n$ by $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$. Then $A(X)^G = k$ and $\mathrm{Spec}[A(X)^G] = \mathrm{Spec}\, k$ is a point, which doesn't have very interesting geometric properties. To improve this, we can apply the GIT quotient with the trivial G-linearized line bundle twisted by a character χ of G.

  Let $L = X \times \mathbb{A}^1$ be the trivial line bundle on X over k. Given a character $\chi : G \to \mathbb{G}_m$, the lift of the G action to L is given by

$$g \cdot (v, z) = (g \cdot v, \chi^{-1}(g) z)$$

where $(x, z) \in X \times \mathbb{A}^1$ and $g \in G$.

Definition 2.9. A function $f : X \to k$ is called *semi-invariant* or *relative-invariant* of weight χ if

$$f(g \cdot x) = \chi(g) f(x)$$

for all $g \in G$ and $x \in V$. We write $k[X]^{G,\chi}$ for the space of semi-invariant functions.

Notice that a section $\sigma : X \to L$ of the form $\sigma(x) = (x, f(x))$ is G-equivariant if and only if f is χ-semi-invariant. The GIT quotient of X by G with respect to χ is described as

$$X /\!\!/_\chi G = \mathrm{Proj}(\bigoplus_{n \geq 0} k[X]^{G,\chi^n}).$$

$X /\!\!/_\chi G$ is projective over $X /\!\!/ G$, and if $k[X]^G = k$ then $X /\!\!/_\chi G$ is a projective variety.

2.2 Semistability Criterion

In this section we provide different criterion that are used to determine the semistable locus.

2.2.1 A Topological Criterion

Let G be a reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$. This induces an action of G on the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$. The following theorem shows how analyzing the action upstairs gives a nice topological characterization of semistability.

Theorem 2.1. ([1], Proposition 2.2). *Let $\tilde{x} \in \tilde{X}$ be a point lying over x. Then,*

(a) x is semistable if and only if $0 \notin \overline{G \cdot \tilde{x}}$.

(b) x is stable if and only if $\dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$.

Note that the in [1] the term for *stable* was *properly stable.*

2.2.2 Hilbert-Mumford Criterion

Let G be a reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$. Finding the semistable points by looking at the invariant functions can be quite tedious. Mumford introduced a numerical criterion, known as the Hilbert-Mumford criterion, which is incredibly useful for finding X^{ss}. The essence of the criterion is that stability can be studied by only looking at the 1-parameter subgroups of G.

Convention: All homomorphisms of algebraic groups are assumed to be morphisms.

Definition 2.10. A *1-parameter subgroup* (1-PS) of G is a nontrivial group homomorphism $\lambda : \mathbb{G}_m \to G$.

A 1-PS $\lambda : \mathbb{G}_m \to G$ induces an action of G on $\mathbb{A}^{n+1}$ which can be diagonalized. Explicitly, there is a basis $\{e_0, \dots e_n\}$ of $\mathbb{A}^{n+1}$ such that

$$\lambda(t) \cdot e_i = t^{r_i} e_i \quad \text{for some } r_i \in \mathbb{Z}$$

The r_i's are called the *weights* of this action. We choose a point $\tilde{x} = \sum_{i=0}^{n} \tilde{x}_i e_i$ lying above $x \in X$. Then

$$\lambda(t) \cdot \tilde{x} = \sum_{i=0}^{n} t^{r_i} \tilde{x}_i e_i.$$

Definition 2.11. The *Hilbert-Mumford function* of x at λ is defined by

$$\mu(x, \lambda) := -\min\{r_i : \tilde{x}_i \neq 0\}.$$

Theorem 2.2. ([1], Theorem 2.1; *Hilbert-Mumford Criterion)*

Let G be a reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$. Then

$$x \in X^{ss} \iff \mu(x, \lambda) \geq 0 \textit{ for all 1-PSs } \lambda \textit{ of } G$$

$$x \in X^{s} \iff \mu(x, \lambda) > 0 \textit{ for all 1-PSs } \lambda \textit{ of } G.$$

We will discuss the following implications:

$$x \in X^{ss} \implies \mu(x, \lambda) \geq 0 \text{ for all 1-PSs } \lambda \text{ of } G$$

$$x \in X^{s} \implies \mu(x, \lambda) > 0 \text{ for all 1-PSs } \lambda \text{ of } G.$$

The converse is more complicated and can be found in [1]. For that, Mumford's proof uses an algebraic theorem of Iwahori and indicates the existence of a preferred 1-parameter subgroup which will be witness to the instability of an unstable point.

There is an embedding $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$ given by $z \mapsto [1 : z]$ and we call the points $[1 : 0]$ and $[0 : 1]$ in $\mathbb{P}^1$ zero and infinity respectively. Given a 1-PS λ of G and $x \in X$, the action of λ induces a morphism

$$\lambda(-) \cdot x : \mathbb{G}_m \to X.$$

Because X is complete, this morphism extends to a unique morphism $\mathbb{P}^1 \to X$ by letting $\lim_{t \to 0} \lambda(t) \cdot x$ and $\lim_{t \to \infty} \lambda(t) \cdot x$ be the images of zero an infinity respectively. Similarly, we can consider the morphism

$$\lambda(-) \cdot \tilde{x} : \mathbb{G}_m \to \tilde{X}$$

where $\tilde{x}$ is a lift of the point x. However, this morphism cannot be extended to $\mathbb{P}^1$. To be able to use the Topological Criterion, we need to study the closure of the orbit $\lambda(\mathbb{G}_m) \cdot \tilde{x}$. Notice that if the boundary is nonempty then any point in the boundary is equal to either $\lim_{t \to 0} \lambda(t) \cdot x$ or $\lim_{t \to \infty} \lambda(t) \cdot x$.

Let $x \in X$ be semistable with respect to the action of G, then it is semistable with respect to the action of any subgroup $H \subset G$; in particular when $H = \lambda(\mathbb{G}_m) \subset G$. By the Topological Criterion (theorem 2.1.) we know that x is semistable if and only if $0 \notin \overline{G \cdot \tilde{x}}$, therefore

$$0 \notin \overline{\lambda(\mathbb{G}_m) \cdot \tilde{x}} \quad \text{for any 1-PS } \lambda \text{ of } \ G.$$

From the definition of $\lambda(t) \cdot \tilde{x}$ above, $\lim_{t \to 0} \lambda(t) \cdot \tilde{x}$ exists and is equal to zero if and only if $r_i > 0$ for all i; equivalently, if and only if $\mu(x, \lambda) < 0$. We conclude that we must have $\mu(x, \lambda) \geq 0$ for all 1-PSs of G.

Let $x \in X$ be stable. By the Topological Criterion, $\dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$. In particular, $\lim_{t \to 0} \lambda(t) \cdot x$ and $\lim_{t \to \infty} \lambda(t) \cdot x$ do not exist. This will only happen if all r_i are negative; that is, if $\mu(x, \lambda) > 0$ for all 1-PSs of G.

Chapter 3

Kirwan-Ness stratifications of projective varieties

Given an action of a reductive group G on a variety X, the Kirwan-Ness stratification associates a canonical stratification where the open stratum coincides with the semistable locus X^{ss} and the unstable locus X^{un} is a disjoint union of G-invariant locally-closed subvarieties S_β of X. Furthermore, when X is nonsingular, the subvarieties S_β will also be nonsingular.
In this chapter, we will describe the stratification for the case when X is a projective variety. We will do it by following the work by Kirwan [5], which is based on work by Kempf [4] and Hesselink [7], and has close links to work by Ness [6].

3.1 Preliminaries

Let k be an algebraically closed field and let G be a reductive group that acts linearly on $\mathbb{P}^n$ so that $G \cdot X \subset X$ for a closed subvariety $X \subseteq \mathbb{P}^n$. Given any $x \in X^{un}$, Kempf associates a conjugacy class of one-parameter subgroups in a parabolic subgroup of G. This class will determine the stratum to which x belongs and we can think of those one-parameter subgroups as the ones that are most responsible for the instability of the point x.
For any $x \in X$ and one parameter subgroup $\lambda : \mathbb{G}_m \to G$, Hesselink defines a 'measure of instability' $m(x, \lambda)$ which is determined uniquely by the following two facts:

a) If $\lambda : \mathbb{G}_m \to$ GL(n+1) is given by

$$t \to \text{diag}(t^{r_0}, \dots, t^{r_n}) \quad \text{with} \quad r_0 \dots r_n \in \mathbb{Z}$$

then

$$m(x;\lambda) = \min\{r_j : x_j \neq 0\}$$

if this is nonegative and

$$m(x;\lambda) = 0$$

otherwise.

b) $m(x; g\lambda g^{-1}) = m(gx;\lambda)$ for any $g \in G$

Definition 3.1. We say that x is *unstable* if $m(x;\lambda) > 0$ for some 1-parameter subgroup λ of G.

Remark: Notice that $m(x;\lambda) = -\mu(x;\lambda)$, where $\mu(x;\lambda)$ is the Hilbert-Mumford function defined before. Thus,

$$x \in X^{ss} \iff x \notin X^{un} \iff m(x;\lambda) \leq 0$$

for all 1-parameter subgroups λ.

Definition 3.2. Let $Y(G)$ be the set of 1-parameter subgroups of X, and let $M(G)$ denote the set of rational 1-parameter subgroups, i.e.

$$M(G) = {}^{(Y(G) \times \mathbb{N})}\diagup_{\sim}$$

where $(\lambda, l) \sim (\mu, m)$ if $\lambda(t^m) = \mu(t^l)$.

In particular, when T is a torus, $Y(T)$ is a free $\mathbb{Z}$ module of finite rank, and $M(T)$ is a $\mathbb{Q}$-vector space. Moreover, when $k = \mathbb{C}$, there is a natural correspondence:

$$Y(T) \longleftrightarrow \text{lattice points in } \mathfrak{t}$$

where $\mathfrak{t}$ is the Lie algebra of the maximal compact subgroup of G. Thus, we can identify $M(T)$ with the rational points of $\mathfrak{t}$.

Remark: Notice that $m(x;\lambda)$ is defined for $\lambda \in Y(G)$. However, the definition of m can be extended uniquely over all λ in $M(G)$ such that $m(x; r\lambda) = rm(x,\lambda)$ for all $r \in \mathbb{Q}$.

We fix a maximal torus T and a Weyl group invariant, integral, positive definite quadratic form

$$q : M(T) \to \mathbb{Q}.$$

Then q extends to a squared norm on all of $M(G)$, and we will write $||\beta||$ for the norm $\sqrt{q(\beta)}$ of $\beta \in M(G)$. We define a natural partial order on $M(G)$ by

$$\beta < \beta' \text{ if } ||\beta|| < ||\beta'|| \quad \text{for } \beta, \beta' \in M(G).$$

At the same time, a norm on $M(G)$ is the square of an inner product on $M(T)$. We denote the inner product by $\bullet$, then

$$q(\beta) = \beta \bullet \beta.$$

The following definitions will be used to determine the 1-parameter subgroups that define the stratum to which a point belongs.

Definition 3.3. For any $x \in X$ let

$$q^{-1}(x)_G = \inf\{q(\lambda) : \lambda \in M(G), m(x;\lambda) \geq 1\}$$

and

$$\Lambda_G(x) = \{\lambda \in M(G) : q(\lambda) = q^{-1}(x)_G, m(x;\lambda) \geq 1\}.$$

Remarks:

- $q^{-1}(x)_G$ is the smallest norm among all rational 1-parameter subgroups that make x unstable. Then $\Lambda_G(x)$ is the set of the 1-parameter subgroups that make x unstable and have smallest norm.
- $x \in X^{un} \iff q^{-1}(x)_G < \infty \iff \Lambda_G(x) \neq \emptyset$.

The representation of T on k^{n+1} splits as the sum of scalar representations given by characters, let's denote them by $\alpha_0, \alpha_1 \ldots \alpha_n \in M(T)^*$. We identify $M(T)$ with $M(T)^*$ by using the inner product q we fixed before. This way we can think of the characters as elements in $M(T)$.

Fix $x = (x_0, \ldots, x_n) \in X$ and let β be the closest point to the origin for the norm q of the convex hull $C(x)$ of the set $\{\alpha_i : x_i \neq 0\}$ in the $\mathbb{Q}$-vector space $M(T)$.

Lemma 3.1. ([5], 12.6). *If $\beta \neq 0$ then $\Lambda_T(x) = \{\beta/q(\beta)\}$.*

Lemma 3.2. ([5], 12.7). *If $\beta = 0$ then $\Lambda_T(x) = \emptyset$.*

From these two lemmas we conclude that $\Lambda_T(x)$ determines and is determined by β.

Definition 3.4. Call the closest point to the origin of the convex hull in $M(T)$ of any nonempty subset of $\{\alpha_0, \ldots, \alpha_n\}$ a *minimal combination of weights.*

Let KN be the set of all minimal combinations of weights lying in some positive Weyl chamber. KN will be the indexing set for the stratification.

Definition 3.5. We say that a subgroup $H \subseteq G$ is *optimal for $x \in X$* if $q_H^{-1}(x) = q_G^{-1}(x)$.

Remarks:

- If H is optimal for $x \in X$ then $\Lambda_H(x) = M(H) \cap \Lambda_G(x)$. In particular,

$$\Lambda_H(x) \neq \emptyset \iff \Lambda_G(x) \neq \emptyset.$$

- By Hesselink [7], given $x \in X$ there is always some maximal torus T' of G which is optimal for x and $T' = g^{-1}Tg$ for some $g \in G$.

Lemma 3.3. *For every $x \in X$, there exists $g \in G$ such that T is optimal for gx.*

Proof. Let $x \in X$, T' a maximal torus of G which is optimal for x and $h \in G$ such that $T' = hTh^{-1}$. Them, by definition and because q is $G-$equivariant, we have the following equalities:

$$\begin{aligned}
q_G^{-1}(gx) &= \inf\{q(\lambda) : \lambda \in M(G), m(gx;\lambda) \geq 1\} \\
&= \inf\{q(\lambda) : \lambda \in M(G), m(x; g\lambda g^{-1}) \geq 1\} \\
&= q_G^{-1}(x) = q_{T'}^{-1}(x) = inf\{q(\lambda) : \lambda \in M(T'), m(x;\lambda) \geq 1\} \\
&= \inf\{q(h\lambda h^{-1}) : h\lambda h^{-1} \in M(T), m(gx; h\lambda h^{-1}) \geq 1\} = q_T^{-1}(gx).
\end{aligned}$$

□

Definition 3.6. For each nonzero $\beta \in M(T)$ let

$$S_\beta = G\{x \in X : \beta/q(\beta) \in \Lambda_G(x)\}$$

and

$$S_0 = G\{x \in X : \Lambda_G(x) = \emptyset\}.$$

Lemma 3.4. ([5], 12.15.). *X is a disjoint union of the subsets $\{S_\beta : \beta \in \mathsf{KN}\}$. In particular, $S_0 = X^{ss}$.*

Definition 3.7. A finite collection $\{S_\beta : \beta \in \mathsf{KN}\}$ of subsets form a *stratification* of X if X is the disjoint union of the strata S_β and there is a strict partial order $>$ on the indexing set KN such that

$$\bar{S_\beta} \subseteq_{\gamma \geq \beta} S_\gamma \quad \forall \beta \in \mathsf{KN}.$$

Remark: This is a weak notion of a stratification, but it is all that holds in our case. In the next sections we will see that the subsets $\{S_\beta : \beta \in \mathsf{KN}\}$ form a stratification of X and we will give a more specific definition of the strata.

3.2 Kirwan-Ness stratifications

Let G be a connected reductive group, and let T be a maximal torus of G, with W_G the corresponding Weyl group. Following the notation of the paper by Nevins and McGerty [8] we denote by

$$X(T) = Hom(T, \mathbb{G}_m)$$

the group of characters. Let $Y_\mathbb{Q} = Y(T)_\mathbb{Q} = Y(T) \otimes_\mathbb{Z} \mathbb{Q}$ and $X_\mathbb{Q} = X(T)_\mathbb{Q} = X(T) \otimes_\mathbb{Z} \mathbb{Q}$ be the $\mathbb{Q}$ vector spaces. As before, let $M(G)$ denote the set of rational 1-parameter subgroups of G and fix

$$q : Y_\mathbb{Q} \to \mathbb{Q},$$

a Weyl group invariant, integral, positive definite quadratic form.

Given a 1-parameter subgroup $\beta : \mathbb{G}_m \to G$, let P_β the parabolic subgroup of G whose Lie algebra is spanned by the nonnegative weight spaces for the $\beta(\mathbb{G}_m)$-action on $\mathfrak{g}$. Let L_β be the centralizer of $\beta(\mathbb{G}_m)$, which is a Levi subgroup for P_β, and let U_β be the unipotent radical of P_β. There is a canonical isomorphism $L_\beta \cong {}^{P_\beta}\!/_{U_\beta}$.

Example 3.1. Let $G = GL_3$ and let

$$\beta(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then,

$$L_\beta = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \text{ in } GL_3 \right\}, P_\beta = \left\{ \begin{bmatrix} x & \alpha & \gamma \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \text{ in } GL_3 \right\} \quad \text{and} \quad U_\beta = \left\{ \begin{bmatrix} 1 & \alpha & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ in } GL_3 \right\}.$$

Let X be a G-variety and $\beta \in M(G)$. Let $Z_\beta = X^{\beta(\mathbb{G}_m)}$ be the set of fixed points and define

$$Y_\beta = \{x \in X : \lim_{t \to 0} \beta(t) \cdot x \in Z_\beta\}.$$

The map $\mathrm{pr}_\beta : Y_\beta \to Z_\beta$ given by $\mathrm{pr}_\beta(x) = \lim_{t\to 0} \beta(t) \cdot x$ is P_β-equivariant, where P_β acts on Y_β in the natural way, and on Z_β via $L_\beta = {}^{P_\beta}\!/_{U_\beta}$.

Example 3.2. Let $X = \mathfrak{gl}_3$ with $G = GL_3$ acting by conjugation, and let

$$\beta(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then,

$$Z_\beta = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \text{ in } \mathfrak{gl}_3 \right\} \quad \text{and} \quad Y_\beta = \left\{ \begin{bmatrix} x & \alpha & \gamma \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \text{ in } \mathfrak{gl}_3 \right\}.$$

Note: In principle, we could write $Z_\beta = \bigsqcup_{i \in I_\beta} Z_{\beta,i}$ for the decomposition of Z_β into connected components and define $Y_{\beta,i} = \{x \in X | \lim_{t\to 0} \beta(t) \cdot x \in Z_{\beta,i}\}$. However, in the case of most interest in this thesis (and that we will study in the next sections) the subsets Z_β are connected.

Fix a finite subset $\mathsf{KN} \subset \{\beta | \beta \in Y_{\mathbb{Q}}\}$ where no two distinct βs are conjugate under the action of the Weyl group W_G of G. Then KN is partially ordered by $\beta' < \beta$ if $||\beta'|| < ||\beta||$.

Definition 3.8. A Kirwan-Ness (KN) stratification of a smooth projective variety X indexed by KN is a finite partition of X

$$X = X^{ss} \sqcup \bigsqcup_{\beta \in \mathsf{KN}} S_\beta,$$

into locally closed smooth pieces, S_β, satisfying the following:

1. X^{ss} is open in X.

2. For each $\beta \in \mathsf{KN}$ there is an open L_β-invariant subset Z_β^{ss} of Z_β such that:

   (a) setting $Y_\beta^{ss} = \mathrm{pr}_\beta^{-1}(Z_\beta^{ss})$, the map pr_β defines an affine fibration $\mathrm{pr}_\beta : Y_\beta^{ss} \to Z_\beta^{ss}$

   (b) the stratum S_β is then given by $G \cdot Y_\beta^{ss}$, and we require $S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}$ via the obvious map.

3. There is a refinement of $<$ (which we also denote by $<$) such that for each $\beta \in \mathsf{KN}$, the union $S_{\geq\beta} = S_\beta \sqcup (\bigsqcup_{\beta' > \beta} S_{\beta'})$ is closed in X.

In our case, and in typical examples of interest, X^{ss} will be the semistable locus of X with respect to a G-equivariant line bundle.

Remarks:

1. Every 1-parameter subgroup is conjugate under G to a 1-parameter subgroup in a maximal torus T.

2. The W_G-orbit of $\beta \in \mathsf{KN}$ uniquely determines S_β.

Therefore, once we fix a maximal torus T, the KN strata are labeled by the KN 1-parameter subgroups associated to the minimal combinations of weights of the T-action which lie in a single Weyl chamber.

To understand the stratum S_β we need to define Z_β^{ss} and Y_β^{ss}. For that we mention a lemma and definitions from [5].

Definition 3.9. Let

$$Z_\beta = \{(x_0 : \cdots : x_n) \in X \mid x_j = 0 \text{ if } \alpha_j \bullet \beta \neq q(\beta)\}$$

and let

$$Y_\beta = \{(x_0 : \cdots : x_n) \in X \mid x_j = 0 \text{ if } \alpha_j \bullet \beta < q(\beta), x_j \neq 0 \text{ for some } i \text{ with } \alpha_j \bullet \beta = q(\beta)\}.$$

Lemma 3.5. ([5], 12.19). *If $x \in Z_\beta$ then L_β is optimal for x.*

Notice that we can rewrite the definition of $m(x;\beta)$ as

$$m(x;\beta) = \min\{\alpha_i \bullet \beta \mid x_i \neq 0\}.$$

Definition 3.10. Let Z_β^{ss} be the subset of Z_β consisting of those $x \in Z_\beta$ such that $\beta/q(\beta) \in \Lambda_G(x)$ and let Y_β^{ss} be the inverse image of Z_β^{ss} under the map $p_\beta : Y_\beta \to Z_\beta$.

Because L_β is optimal for x, the condition that $\beta/q(\beta) \in \Lambda_G(x)$ is equivalent to

$$m(x;\lambda) \leq \lambda \bullet \beta \quad \forall \lambda \in M(L_\beta).$$

The following remark will be very useful when finding the stratum of a specific example in section 5.2.

Remark: It can be seen that there is a unique connected reductive subgroup G_β of L_β such that

$$M(G_\beta) = \{\lambda \in M(L_\beta) \mid \lambda \bullet \beta = 0\}.$$

Then Z_β^{ss} consists of $x \in Z_\beta$ which are semistable under the action of G_β on Z_β.

Chapter 4

Kirwan-Ness stratifications of a representation

In this chapter we will study the case when X is a linear representation V of G. So far, we have defined the Kirwan-Ness stratification for projective varieties. We need to adjust the theory to be able to apply geometric invariant theory to a representation and then be able to define a KN stratification in V.

We consider the vector space V as an open subset of $\mathbb{P}(\mathbb{C} \oplus V)$, where $\mathbb{C}$ is the trivial representation of G. Let $\mathrm{Sym}((\mathbb{C} \oplus V)^*)$ be the polynomials on the vector space $\mathbb{C} \oplus V$. Then G acts on

$$\mathbb{P} = \mathbb{P}(\mathbb{C} \times V) = \mathrm{ProjSym}((\mathbb{C} \oplus V)^*)$$

by $g \cdot (c, v) = (c, gv)$. Then $V \cong \{1\} \times V \subset \mathbb{P}$ is a G-invariant open subset of $\mathbb{P}$ and consists of semistable points, i.e. for any $v \in V$ there is a non-constant G-invariant homogeneous polynomial $f \in \mathrm{Sym}((\mathbb{C} \oplus V)^*)$ such that $f(1, v) \neq 0$.

Let's fix a group character $\chi : G \to \mathbb{G}_m$, and consider the projective morphism $\pi : \mathbb{P} \to \mathbb{P}$ given by the identity map. We equip π with the relatively ample line bundle $M = \mathcal{O}$, which we make into a G-equivariant line bundle via the action of G on the total space $\mathbb{P} \times \mathbb{A}^1$ given by

$$g \cdot ((c, v), w) = ((c, gv), \chi(g)w) \quad \text{where} \quad g \in G.$$

For each $\epsilon \in \mathbb{Q}_{>0}$, we can define a Q-line bundle $L = M^{\otimes \epsilon} \otimes \mathcal{O}(1)$ with the G-linearization given by

$$g \cdot ((c, v), w) = ((c, gv), \chi^{\epsilon}(g)w) \quad \text{where} \quad g \in G.$$

Restricting L^{-1} to the open set V of $\mathbb{P}$,we write the action on L^{-1} as

$$g \cdot (v, w) = (gv, \chi^{-\epsilon}(g)w).$$

This action agrees with the definition of affine GIT quotient given in chapter 2, and allows us to use Kirwan's setup to describe the KN stratification of V for the rational character χ^{ϵ}.

Lemma 4.1. ([8], Lemma 4.8). *The KN stratification of $\mathbb{P}$ induced by L is independent of ϵ for small $\epsilon \in \mathbb{Q}_{>0}$. The induced stratification of $V \subseteq \mathbb{P}$ is a KN stratification of V.*

4.1 KN 1-parameter subgroups for a representation

We define $\lambda = d\chi$ and we assume that ϵ is a small negative rational number. Therefore, $L = M^{\otimes(-\epsilon)} \otimes \mathcal{O}(1)$.

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of V consisting of T-weight vectors, and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the corresponding set of weights. Consider the basis of $\mathbb{C} \oplus V$ given by $\{e_0, e_1, \ldots e_n\}$ where e_0 is the standard basis vector of $\mathbb{C} \times \{0\} \subset \mathbb{C} \oplus V$. Setting $\alpha_0 = 0$, $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ gives a list of weights of $\mathbb{C} \oplus V$.

Given a nonempty subset $I \subset \{0, 1, \ldots n\}$, we denote by W_I the subspace spanned by the weights $\{\alpha_i \mid i \in I\}$, and by C_I the convex hull of the set $\{\alpha_i \mid i \in I\}$. Let $\alpha_i^{\epsilon} = \alpha_i + \epsilon\lambda$ and define C_I^{ϵ} to be the convex hull of the set $\{\alpha_i^{\epsilon} \mid i \in I\}$.

Recall from the previous chapter that KN, the set of all minimal combinations of weights lying in some positive Weyl chamber, will be the indexing set for the stratification of $\mathbb{P}(\mathbb{C} \oplus V)$. For $I \subset \{0, 1, \ldots, n\}$, we denote by β_I the closest point to zero in C_I^{ϵ}, i.e. $\beta_I(\epsilon)$ minimizes the distance $d(0, C_I^{\epsilon})$.

Remarks:

1. Notice that $\beta_I = \beta_I(\epsilon)$ depends on the choice of $\epsilon\lambda$.

2. If $\beta_I = 0$, the strata corresponding to β_I coincides with the semistable locus X^{ss}. Otherwise, the corresponding 1-parameter subgroup is given by $\beta_I / q(\beta_I)$.

Since we are interested in the stratification of V, we only care about the KN 1-parameter subgroups that label strata which intersects V. We denote the set of such KN 1-parameters subgroups by Λ_V. In particular, we are only looking for β_I's where $0 \in I$.

Note: Notice that for each subset $I \subseteq \{0, 1, \dots, n\}$, the distance $d(0, C_I^\epsilon)$ is a piecewise linear function on ϵ. Since there are finitely many subsets I, we can fix $C < 0$ and $\epsilon \in [C, 0]$ such that all distance functions $d(0, C_I^\epsilon)$ and minimal vectors $\beta_I(\epsilon)$ are linear. Given I, fix a *minimal* subset $J \subset I$ such that $d(0, C_I^\epsilon) = d(0, C_I^\epsilon)$ for $\epsilon \in [C, 0]$. Because q is strictly convex, the closest point is unique and $\beta_I(\epsilon) = \beta_J(\epsilon)$ for $\epsilon \in [C, 0]$.

Finding the closest point to the origin in a convex hull is not always easy, and even less in high dimensions. To facilitate this calculations we will see that the β_i's can be seen as orthogonal projections of $\epsilon\lambda$ to certain subspaces. For each subset $I \subset \{0, 1, \dots, n\}$, let

$$p_I(\epsilon) := \mathrm{proj}_{W_I^\perp}(\epsilon\lambda)$$

be the orthogonal projection to the subspace of the weights perpendicular to W_I.

Proposition 4.1. *Let $x \in V \subset \mathbb{P}(\mathbb{C} \oplus V)$ have homogeneous coordinates $x = [t_0 : \cdots : t_n]$ and let $I = I_x = \{i : 0 \leq i \leq n, t_i \neq 0\}$. If $J \subseteq I$ is minimal such that $d(0, C_I^\epsilon) = d(0, C_J^\epsilon)$, then $\beta_I(\epsilon) = p_J(\epsilon)$.*

Proof. Fix $I \subset I$ minimal such that $\beta_I(\epsilon) = \beta_J(\epsilon)$ for $\epsilon \in [C, 0]$ and the let A_J^ϵ be the affine hull $\mathrm{aff}\{\alpha_i^\epsilon : i \in I\}$, i.e.

$$A_J^\epsilon = \{\sum_{i=1}^{m} t_i \alpha_i^\epsilon : m \in \mathbb{N}, t_i \in \mathbb{Q}, \sum_{i=1}^{m} t_i = 1\}.$$

C_J^ϵ is a closed subset of A_J^ϵ and its relative interior $\mathrm{relint}(C_J^\epsilon)$ is nonempty.

Claim 1. $\beta_I(\epsilon) \in \mathrm{relint}(C_J^\epsilon)$ *for* $\epsilon \in [C, 0]$.

Assume that $\beta_I(\epsilon)$ is on the boundary of C_J^ϵ in A_J^ϵ. This boundary is the union of convex hulls $C_{J'}^\epsilon$ for some $J' \subset J$, so $\beta_I(\epsilon)$ lies in one of this $C_{J'}^\epsilon$ for small ϵ. Thus $\beta_{J'}(\epsilon) = \beta_I(\epsilon)$, which contradicts the minimality of J. We conclude that $\beta_I(\epsilon) \in \mathrm{relint}(C_J^\epsilon)$.

Claim 2. $0 \in C_J$.

Let β' be the closest point to zero in C_J. Notice that $C_J^\epsilon = \epsilon\lambda + C_J$ and $\beta_J(\epsilon)$ is the closest point to 0 in C_J^ϵ. Therefore,

$$|| \beta_I(\epsilon) || \geq || \beta' || + \epsilon || \lambda || . \tag{4.1}$$

On the other hand, notice that $x \in V \cong \{1\} \times V \subset \mathbb{P}$. In particular, $t_0 \neq 1$ and $i \in I$. So $\alpha_0^\epsilon = \epsilon\lambda \in C_I^\epsilon$ and since $\beta_J(\epsilon) = \beta_J(\epsilon)$ is the closest point, by convexity

$$(\epsilon\lambda - \beta_J(\epsilon)) \bullet \beta_J(\epsilon) \geq 0, \quad \text{i.e. } \epsilon\lambda \bullet \beta_J(\epsilon) \geq \beta_J(\epsilon)\beta_J(\epsilon) = || \beta_J(\epsilon) ||^2 . \tag{4.2}$$

For small enough ϵ, (4.3) and (4.4) will contradict unless $|| \beta' || = 0$. Therefore, we must have $\beta' = 0$ and we conclude that $0 \in C_J$.

Since A_J is an affine subspace, claim 2 implies that A_J is actually a subspace. By claim 1, $\beta_J(\epsilon)$ is also the closest point to zero in $A_J^\epsilon = \epsilon + A_J$. We conclude that $\beta_I(\epsilon) = \beta_J(\epsilon) = p_J(\epsilon)$. □

Remarks:

1. It is important for the proof of the proposition to note that $0 \in I$. This is because we are interested in the stratum S_β that intersects $V \cong \{1\} \times V \subset \mathbb{P}$. In particular, $t_0 \neq 0$.

2. If $\mathbb{P}(\mathbb{C} \oplus V)$ has a KN stratum defined by a 1-parameter subgroup of the form $\beta = p_I(\epsilon)$ for some $I \subset \{0, 1, \ldots, n\}$, then S_β intersects V. In particular, β is a KN 1-parameter subgroup for the stratification of V. More precisely, if $0 \in I$ we know from the proposition that we can find $x \in V$ with associated weights giving I, and therefore $S_{p_I(\epsilon)}$ would intersect V. If $0 \notin I$, we can replace I by $I \cup \{0\}$ without changing $p_J(\epsilon)$ and we would now be in the previous case. We conclude that the KN strata of $\mathbb{P}(\mathbb{C} \oplus V)$ that intersects V are precisely the ones of the form $\beta = p_I(\epsilon)$ for some subset I. However, this does not imply that given a subset I, $p_I(\epsilon)$ will define an stratum of V, it could happen that $p_I(\epsilon)$ is not in C_I^ϵ.

3. Notice that the T-weights of an action of G on T^*V are the weights of the action on V together with their negatives; that is, they are given by the set $\{\alpha_1, \ldots, \alpha_n, -\alpha_1, \ldots, -\alpha_n\}$. In particular, the collection of projections $p_I(\epsilon)$ that we get as I runs over weights of T^*V

is the same as those arising as I runs over weights of V. However, most of the times the KN 1-parameter subgroups labeling KN strata of V and of T^*V will not be the same. More precisely, the KN 1-parameter subgroups of V will be a subset of the ones corresponding to T^*V.

We now know that the set of KN 1-parameter subgroups is contained in $\{p_I(\epsilon) \mid I \subset \{0, 1, \ldots, n\}\}$. However, we have to refine this set and throw away the projections that do not lie in the convex hull C_I^ϵ.

Lemma 4.2. ([14], Theorem 1). *Let W be a finite-dimensional $\mathbb{Q}$-vector space, and let $S \subset W$ be a finite set with $C = conv(S)$ its convex hull. Then*

$$p \in C \iff \forall p' \in C \backslash \{p\} \ \exists s \in S \ \textit{such that} \ \ d(s, p) < d(s, p'). \tag{4.3}$$

In our case, we want to see whether $p_I(\epsilon)$ is in C_I^ϵ. Furthermore, we know from the proposition above that for small enough ϵ, there is $J \subseteq I$ such that p_J^ϵ is the closest point to zero in C_I^ϵ. Therefore, we only need to check property (4.1) for $p' = p_J(\epsilon)$ where J is any subset of I. More precisely, we have the following corollary:

Corollary 4.1. *Given $I \subset \{0, 1, \ldots, n\}$, $p_I(\epsilon)$ gives a KN 1-parameter subgroup if and only if there is no $J \subset I$ such that $d(p_J(\epsilon), \alpha_k^\epsilon) < d(p_I(\epsilon), \alpha_k^\epsilon)$ for all $k \in I$.*

Example: Let $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$ with the torus $T = \mathbb{G}_m^2$ acting with weights $\alpha_1 = (1, 0)$, $\alpha_2 = (1, -1)$ and $\alpha_3 = (-1, 1)$. We choose the group character to be $\lambda = (-2, -1)$ and ϵ to be small and negative. The weights corresponding to $\mathbb{C} \times V$ are then given by $\alpha_0^\epsilon = (-2\epsilon, -\epsilon)$, $\alpha_1^\epsilon = (1 - 2\epsilon, -\epsilon)$, $\alpha_2^\epsilon = (1 - 2\epsilon, -1 - \epsilon)$ and $\alpha_3^\epsilon = (-1 - 2\epsilon, 1 - \epsilon)$.

<u>*Method 1:*</u> We will determine the KN 1-parameter subgroups by finding the closest point to zero in each convex hull C_I^ϵ for $I \subset \{0, 1, 2, 3\}$. Remember that since we are looking for the stratum corresponding to V, we require that $0 \in I$; that is, we take all possible subsets of weights associated to $x = (x_0 : x_1 : x_2) \in \mathbb{P}(\mathbb{C} \times V)$ with $x_0 \neq 0$.

1. When I is $\{0, 1\}$, $\{0, 2\}$ or $\{0, 1, 2\}$, we find that $\beta_I = (-2\epsilon, -1\epsilon)$.

2. When I is $\{0,3\}$, $\{0,1,3\}$ or $\{0,1,2,3\}$, we find that $\beta_I = (-3\epsilon/2, -3\epsilon/2)$.

Therefore, the KN 1-parameter subgroups are given by

$$\mathsf{KN} = \{(-2\epsilon, -1\epsilon), (-3\epsilon/2, -3\epsilon/2)\}.$$

<u>*Method 2:*</u> This time we will find KN by calculating all projections $p_I(\epsilon)$. Notice that having 0 in the set I does not influence the projection $p_I(\epsilon)$, but it is important for the purpose of finding $J \subset I$ such that

$$d(p_J(\epsilon), \alpha_k^\epsilon) < d(p_I(\epsilon), \alpha_k^\epsilon)$$

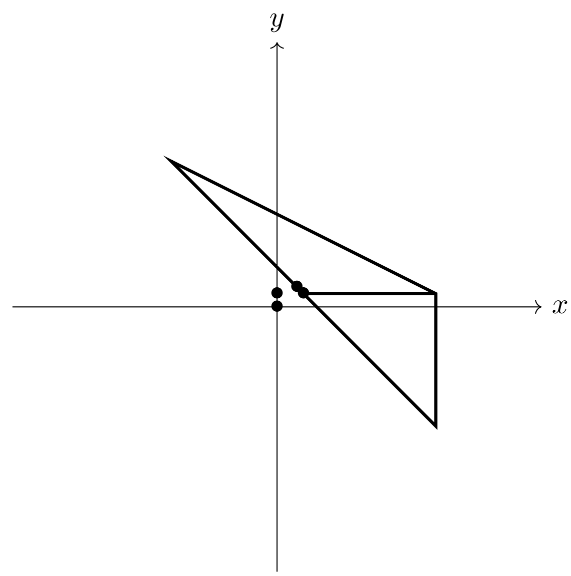
for all $k \in I$.

1. Notice that $\{\alpha_2\}$ and $\{\alpha_3\}$ span the same subspace. Therefore, they both correspond to the projection $p_I(\epsilon) = (-3\epsilon/2, -3\epsilon/2)$.

2. If $I = \{0,1\}$, we get $p_I(\epsilon) = (0, -\epsilon)$.

3. If $I = \{0\}$, $W_I^\perp$ will the the whole space. So $p_I(\epsilon) = \epsilon\lambda = (-2\epsilon, -\epsilon)$.

4. For any other possibilities of I we get that $p_I(\epsilon) = (0,0)$.



We conclude that

$$\mathsf{KN} \subseteq \{(-3\epsilon/2, -3\epsilon/2), (0, -\epsilon), (-2\epsilon, -\epsilon), (0,0)\}.$$

Notice that when $I = \{0,1\}$, the subspace $\{0\} = J \subset I$ corresponds to the projection $p_J(\epsilon) = \epsilon\lambda = (-2\epsilon, -\epsilon)$ and

$$d(p_J(\epsilon), \alpha_1^\epsilon) = ||(1,0)|| < ||(1-2\epsilon, 0)|| = d(p_I(\epsilon), \alpha_1^\epsilon).$$

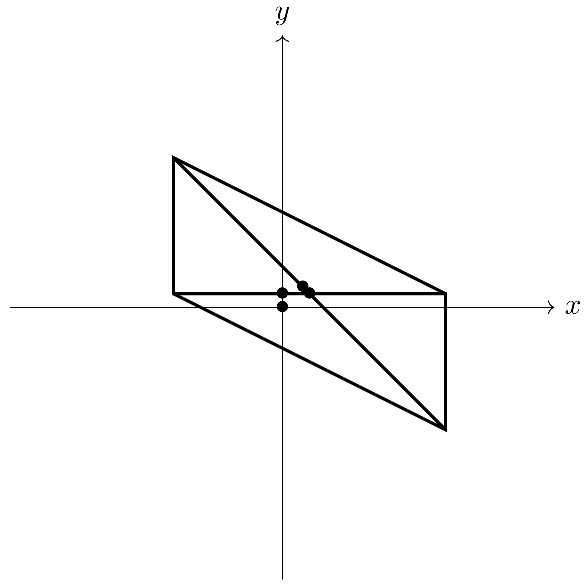
By the corollary, we conclude that $(0, -\epsilon)$ is not a KN 1-parameter subgroup. Similarly, it can be seen that $(0,0)$ is not in KN. Therefore, $\mathsf{KN} = \{(-3\epsilon/2, -3\epsilon/2), (-2\epsilon, -\epsilon)\}$.

In this case, because the dimension of our space is low, it is easy to see that $(0, -\epsilon)$ and $(0,0)$ are not contained in the convex hull C_I^ϵ.

Example: Let V, ϵ, λ and weights be as in the previous example. We now want to find the KN 1-parameter subgroups corresponding to T^*V. The torus weights on T^*V are given by $\alpha_1 = (1,0)$, $\alpha_2 = (1,-1)$, $\alpha_3 = (-1,1)$ and $\alpha_4 = (-1,0)$. As we have mentioned before, the collection of projections $p_I(\epsilon)$ that we get as I runs over weights of T^*V is the same as those arising as I runs over weights of V. Therefore, $\mathsf{KN} \subseteq \{(-3\epsilon/2, -3\epsilon/2), (0, -\epsilon), (-2\epsilon, -\epsilon), (0,0)\}$.

However, in this case we do not throw any of those projections away. From the previous example, we know that

$$\{(-3\epsilon/2, -3\epsilon/2), (-2\epsilon, -\epsilon)\} \subset \mathsf{KN}.$$



We have to check that in this case $(0, -\epsilon)$ and $(0,0)$ are not discarded.

Even if $p_{\{0,1\}}(\epsilon) = (0, -\epsilon)$ is discarded by taking $J = \{0\}$, notice that $p_{\{0,1,4\}} = (0, -\epsilon)$ and in this case there is no $J \subset \{0,1,4\}$ such that

$$d(p_J(\epsilon), \alpha_k^\epsilon) < d(p_I(\epsilon), \alpha_k^\epsilon) \text{ for all } k \in I.$$

This occurs because $\alpha_4 = -\alpha_1$. In particular, $J = \{0\}$ does not work anymore because

$$d(p_J(\epsilon), \alpha_4^\epsilon) = ||(-1,0)|| > ||(-1-2\epsilon, 0)|| = d(p_I(\epsilon), \alpha_4^\epsilon).$$

Similarly, it can be seen that $\{0,0\} \in \mathsf{KN}$. So

$$\mathsf{KN} = \{(-3\epsilon/2, -3\epsilon/2), (0, -\epsilon), (-2\epsilon, -\epsilon), (0,0)\}.$$

Equivalently, we did not throw any of the projections away because in this case all of them lie inside the convex hull.

4.2 Main result and conclusions

Theorem 4.1. *Let V be a finite dimensional representation of a reductive group G over k. Let T^*V be equipped with the G action induced from the action on V. Then the set of KN 1-parameter subgroups that defines the stratification for the cotangent bundle T^*V is given by*

$$\mathsf{KN} = \{p_I(\epsilon) \mid I \subset \{0, 1, \ldots, n\}\}.$$

Proof. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of torus weights of the action on V and define $\alpha_{-i} := -\alpha_i$ (notice that these are all of the weights of the torus action on T^*V). Let $I \subset \{1, 2, \ldots, n\}$ be a nonempty subset. Assume that p_I is not on the final list of KN 1-PS, i.e. there is $J \subset I$ such that

$$d(p_J(\epsilon), \alpha_i^\epsilon) < d(p_I(\epsilon), \alpha_i^\epsilon) \qquad \text{for all } i \in I. \tag{4.4}$$

Without loss of generality, we can assume that $\{\alpha_i\}_{i \in I}$ is linearly independent; if it is not we could take I to be the indices of the maximal linearly independent subset (it is clear that $p_I(\epsilon)$ would not change). For simplicity, we will assume that $I = \{1, 2, \ldots, k\}$ and $J = \{1, 2, \ldots, l\}$ where $l < k$.
Let's fix a basis $\mathcal{B}'$ of $\mathfrak{t}^*$ such that $\{\alpha_i\}_{i \in I} \subset \mathcal{B}'$ are the first k elements in the basis. Let $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ be the orthogonal basis we get after applying the Gram–Schmidt algorithm to $\mathcal{B}'$.
Notice that if $\alpha_i = (a_1, a_2, \ldots, a_m)_{\mathcal{B}}$ then $a_i \neq 0$.
Let $\epsilon\lambda = \epsilon(\lambda_1, \lambda_2, \ldots, \lambda_m)_{\mathcal{B}}$. Then

$$\begin{aligned} p_I(\epsilon) =& \epsilon(0, \ldots\ldots, 0, \lambda_{k+1}, \ldots, \lambda_m)_{\mathcal{B}} \\ p_J(\epsilon) =& \epsilon(0, \ldots, 0, \lambda_{l+1}, \ldots\ldots, \lambda_m)_{\mathcal{B}} \\ \alpha_i^\epsilon =& \alpha_i + \epsilon\lambda = (a_1 + \epsilon\lambda_1, \ldots, a_n + \epsilon\lambda_m)_{\mathcal{B}} \end{aligned}$$

Therefore,

$$d(p_I(\epsilon), \alpha_i^\epsilon) = ||(a_1 + \epsilon\lambda_1, \ldots\ldots, a_k + \epsilon\lambda_k, a_{k+1}, \ldots, a_m)||$$
$$d(p_J(\epsilon), \alpha_i^\epsilon) = ||(a_1 + \epsilon\lambda_1, \ldots, a_l + \epsilon\lambda_l, a_{l+1}, \ldots\ldots, a_m)||$$

and property (4.4) implies

$$a_{l+1}^2 + \cdots + a_k^2 < (a_{l+1} + \epsilon\lambda_{l+1})^2 + \cdots + (a_k + \epsilon\lambda_k)^2.$$

Since $-1 << \epsilon < 0$, we have

$$a_{l+1}^2 + \cdots + a_k^2 > (-a_{l+1} + \epsilon\lambda_{l+1})^2 + \cdots + (-a_k + \epsilon\lambda_k)^2 \tag{4.5}$$

unless $a_{l+1} = \cdots = a_k = 0$, but (because $a_i \neq 0$) that won't happen if $i \geq l+1$, i.e. if $i \in I \backslash J$. Define $\bar{I} = I \cup \{-\alpha_i : i \in I\}$.

Claim 3. *$p_{\bar{I}}(\epsilon)$ is a KN 1-PS.*

We will show that $p_{\bar{I}}(\epsilon)$ is a KN 1-PS by contradiction. Let's assume that there is a subset $K \subset \bar{I}$ that satisfies (4.4). Because of how $\bar{I}$ is defined, there is a subset $J' \subset I$ such that $\text{span}(J') = \text{span}(K)$ and therefore $p_{J'}(\epsilon) = p_K(\epsilon)$. In particular, that would mean that

$$d(p_{J'}(\epsilon), \alpha_i^\epsilon) < d(p_I(\epsilon), \alpha_i^\epsilon) \qquad \text{for all } i \in I.$$

and

$$d(p_{J'}(\epsilon), \alpha_{-i}^\epsilon) < d(p_I(\epsilon), \alpha_{-i}^\epsilon) \qquad \text{for } i \in I$$

but that contradicts (4.5).

Claim 4. $p_I(\epsilon) = p_{\bar{I}}(\epsilon)$.

It is clear that $\text{span}(\bar{I}) = \text{span}(I)$, in particular, $p_I = p_{\bar{I}}$. $\square$

Corollary 4.2. *Under the same assumptions as in the theorem, the semistable locus $(T^*V)^{ss}$ is nonempty (independently of λ) if the weights under the torus action span the space of weights.*

Proof. We only need to check that $0 \in \mathsf{KN}$ since $\beta = 0$ is the 1-parameter subgroup that corresponds to the semistable locus. Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ be a set of T-weights that span the space of weights. Then when $I = \{i \mid \alpha_i \in \mathcal{B}\}$ we get that $p_I(\epsilon) = 0$, and by the theorem $0 \in \mathsf{KN}$. $\square$

Corollary 4.3. *Under the same assumptions as in the theorem and given a character λ, the semistable locus $(T^*V)^{ss}$ is nonempty if and only if there is a subset I such that $\lambda \in \text{span}\{\alpha_i \mid i \in I\}$.*

Proof. $(T^*V)^{ss}$ is nonempty if and only if $\beta = 0$ is in KN, and this happens if and only if $p_I(\epsilon) = 0$ for some subset I; that is, when $\lambda \in \text{span}\{\alpha_i \mid i \in I\}$ for some subset I. $\square$

Remark: We want to understand the KN strata S_β better, which requires us to understand Z_β and Y_β deeper. At the end of chapter 3 we gave a description of such sets for $\beta \in \mathsf{KN}$. In this case, we know that the KN 1-parameter subgroups are of the form $\beta = p_I(\epsilon)$. From the discussion in [8] (section 4.6) we see that in our case

$$(x_0 : \cdots : x_n) \in Z_\beta \iff \text{for each } i \text{ such that } x_i \neq 0, \text{ we have } \alpha_i \bullet \beta = 0;$$

$$(x_0 : \cdots : x_n) \in Y_\beta \iff \text{for each } i \text{ such that } x_i \neq 0, \text{ we have } \alpha_i \bullet \beta \geq 0.$$

These equivalences will be used when calculating the stratum of different examples in chapter 6.

Chapter 5

Code

On this chapter you will find the code that I implemented to find the Kirwan-Ness one parameter subgroups of a representation. The code takes as input the weights under the torus action, a choice of group character and a very small negative number, and outputs the KN 1-parameter subgroups. Step 5) in the algorithm refines the set of projections, throwing away those that do not define a stratum. We have seen that when the space we are working with is of the form T^*V, where V is a G-representation, there is no refinement. Therefore, step 5) can be skipped in that case, making the code a lot more efficient.

5.1 For spaces that are not of the form T^*V

```
#1) Input data:
B=[] #Define the list of 1-PSs (before refining).
FinalB=[] #Final list of KN 1-PS (after refining).
n #Number of weights.
epsilon #Choice of small negative number.
v=matrix(1,1) #Choice of character.
W = matrix(n,1)  #Matrix which has weights as rows. Do not add the zero
weight.
W= matrix(QQbar, W)

#2) We calculate all possible subsets of n:
X=Set(range(n)) #Set {0,1,...,n-1}.
```

```
S=list( X.subsets())    #Subsets of {0,1,...,n-1}.
s=len(S) # s=number of subsets.

#3) We add the projection corresponding to the zero weight. It will
correspond to the empty subset on the calculations:
B.append(transpose(epsilon*v))

#4) We will make a list where the elements are all possible subsets
#of weights and calculate the projections:
matrix_list = []    #Empty list for the possible subsets of weights.
matrix_list.append(zero) #Weight corresponding to the empty set.
for i in [1..s-1]: #We start with i=1 because the subset corresponding
#to i=0 is the empty subset.
   A=[] #empty matrix for the weights corresponding to the numerical
#subset.
   for j in S[i]:
       A.append(W[j])
   A = matrix(A)
   if A.rank()<l:
       M,G = A.gram_schmidt(orthonormal = True) # find orthonormal
       #Matrix using gram-schmidt.
       M=matrix(M)
       b=transpose((matrix.identity(l) - transpose(M)*M)*epsilon*v)
       #Orthogonal projection.
   else:
       b=matrix(1,l)
   B.append(b)
   matrix_list.append(A)
```

```
# 5) Refine the projections:
d=dict((S[i],B[i]) for i in xrange(len(matrix_list)))  #We create a
dictionary with the subsets and projections.
choose=[] #Empty list for the KN 1-PS.
for i in xrange(len(matrix_list)):
    I=S[i]
    choose.append(i) #We include the index corresponding to the subset.
    for j in xrange(len(list( I.subsets()))-1): #we only go until
    #len(list( I.subsets()))-1 because we do not count the subset J=I.
        b=d[list( I.subsets())[j]] #Projection of J.
        count=0 # It counts how many times distJ<distI.
        K=0 #It counts the weights in I.
        for k in I:
                distJ=(b-(matrix(W[k])+epsilon*transpose(v)))*
                (b-(matrix(W[k])+epsilon*transpose(v))).transpose()
                # distance square between p(J) and shifted weights.
                distI=(B[i]-(matrix(W[k])+epsilon*transpose(v)))*
                (B[i]-(matrix(W[k])+ epsilon*transpose(v))).transpose()
                #distance square between p(I) and shifted weights.
                K=K+1
                if distJ<distI:
                        count=count+1
        if count==K: # We compare count and K. If they are
        #the same we give the value -1 to the place
        #corresponding to I.
                choose[i]=-1
```

```
for i in xrange(len(B)): #We discard the projections corresponding to -1.
    B[i].set_immutable()
    if choose[i]!=-1:
            FinalB.append(B[i])


FinalB=Set(FinalB)
B=Set(B)
```

5.2 Case where the space is of the form T^*V

The following code is specific to the case of V = Rep(Q,v) where Q is a cyclic quiver. Any other case of V can be adapted easily by changing the matrix W accordingly. Notice that in this code it is not necessary to enter the group character λ since the code works with λ as a variable (which is not possible in the code above).

```
#1) Input data
var("t,u,w,x") # We are assuming we only need 4 variables. More can be
#defined.
KN=[] #Set of KN 1-PS's
n  #Dimension of vector spaces
l  #Number of vertices in the cycle of the quiver
v=matrix(n*l,1,[t,u,w,x]) #lambda as a column vector.
epsilon=-1/100
#2) Construct the matrix of weights as rows
W = matrix(n*n*l+n,l*n)  #Empty matrix for the weights as rows
k=0
for r in [0..l-2]:
        for i in [n*r..(n-1)+n*r]:
                for j in [0..n-1]:
                        W[k,i]=1
```

```
                    W[k,j+n*(r+1)]=-1
                    k=k+1
for i in [n*(l-1)..(n-1)+n*(l-1)]:
        for j in [0..n-1]:
                W[k,i]=1
                W[k,j]=-1
                k=k+1
for  i in [0..n-1]:
        W[k,i]=1
        k=k+1

W= matrix(QQbar, W) #Matrix of weights as rows

#3) We will make a list where the elements are all possible subsets
#of weights
X = Set(range(n*n*l+n))
print X
S=list( X.subsets())
s=len(S) # s=number of subsets
B.append(transpose(epsilon*v))

for i in [1..s-2]:
        A=[]
        for j in S[i]:
                A.append(W[j])
        A = matrix(A)
        M,G = A.gram_schmidt(orthonormal = True)
        M=matrix(M)
```

```
        M= matrix(QQbar, M)
        N=(matrix.identity(n*l) - transpose(M)*M)
        N= matrix(QQ, N)
        b=transpose(N*epsilon*v)
        b.set_immutable()
        if b not in B:
            B.append(b)

print 'B=', B
```

Chapter 6

Kirwan-Ness stratification of the cyclic quiver

6.1 Introduction to quiver varieties

A *quiver* $Q = (I, E, h, t)$ is a directed graph where I is the set of vertices, E is the set of edges and $h, t : E \to I$ are maps that assign its head and tail to every edge. A *representation* (V, f) of a quiver Q consists of the following data:

- To every vertex $i \in I$ corresponds a vector space V_i over k.
- To every edge $x \in E$ corresponds a linear operator $f_x : V_{t(x)} \to V_{h(t)}$.

The dimension vector of the representation is given by $v = (\text{dim}V_i)_{i \in I}$.

Given (V, f) and (W, g) representations of a quiver Q, a *morphism* $\phi : V \to W$ is a collection of linear operators $\phi_i : V_i \to W_i$ such that $\phi_{t(x)} g_x = f_x \phi_{h(x)}$ for all edges x. Such a map is an isomorphism if and only if ϕ_i is for every vertex i. We denote by $\text{Hom}(V, W)$ the vector space consisting of morphisms $V \to W$. Notice that when V is a finite-dimensional representation of a quiver $Q = (I, E, h, t)$ with dimension vector $v = (v_i)_{i \in I}$, we can identify V_i with k^{v_i} by making a choice of basis for V_i. Then the representation V is described by a collection of matrices in $\text{Hom}_k(k^{v_{t(x)}}, k^{v_{h(x)}})$ for each edge $x \in E$. That is, once we fix a dimension vector v, the representation space of a quiver Q is given by

$$\text{Rep}(Q, v) = \bigoplus_{x \in E} \text{Hom}_k(k^{v_{t(x)}}, k^{v_{h(x)}}).$$

The reductive group

$$GL(v) = \prod_{i \in I} GL(v_i, k)$$

acts on $\text{Rep}(Q, v)$ by conjugation, i.e. $(g \cdot f)_x = g_{h(x)} f_x g_{t(x)}^{-1}$. The orbits of this action are in one to one correspondence with the isomorphism classes of representations of Q with dimension vector v. As we mentioned before, the orbit space may not preserve all of the geometric properties we want. Instead, we want to study the quotient $\mathrm{Rep}(Q, v) /\!\!/_\chi GL(v)$, where χ defines a linearisation of the trivial line bundle. The characters of $GL(v)$ are of the form

$$\chi_\theta(g) = \prod_{i \in I} \det(g_i)^{\theta_i}$$

where $\theta \in \mathbb{Z}^I$. From now on, we will identify χ_θ and θ.

For numerous interesting cases of quivers Q, the quotient $\mathrm{Rep}(Q, v) /\!\!/_\chi GL(v)$ may be empty. A way to try to avoid this is to define a framing. A *framing* of a quiver $Q = (I, E, h, t)$ is a quiver $Q^f = (I \sqcup I', E \sqcup E', h, t)$ where I' is a copy of I (with a bijection $I \to I'$, by $i \mapsto i'$) and E' are arrows from i to i'. Then a representation of Q^f is given by a representation V of Q together with

- a collection of vector spaces W_i corresponding to the vertices in I' and
- linear maps $V_i \to W_i$ corresponding to the arrows in E'.

In particular, if we fix the dimension vectors of V and W to be v and w respectively, we have

$$\mathrm{Rep}(Q^f, v, w) = \bigoplus_{x \in E} \mathrm{Hom}_k(k^{v_{t(x)}}, k^{v_{h(x)}}) \oplus \bigoplus_{i \in I} \mathrm{Hom}_k(k^{v_i}, k^{w_i}).$$

We define an action of $GL(v)$ on $\mathrm{Rep}(Q^f, v, w)$ by

$$g \cdot (f, j) = (gfg^{-1}, j \circ g^{-1})$$

where $g = (g_i)_{i \in I}$, $f = (f_x)_{x \in E}$ and

$$gfg^{-1} = (g_{h(x)} f_x g_{t(x)}^{-1})_{x \in E}$$

$$j \circ g^{-1} = (j_i \cdot g_i^{-1})_{i \in I}.$$

Our next goal will be to find a quiver whose representation space is a cotangent bundle $T^*V = V \times V^*$, where V is a vector space. This is convenient since the total space of the cotangent bundle comes equipped with a symplectic structure, i.e. a nondegenerated closed 2-form ω on T^*V.

Given a quiver $Q = (I, E, h, t)$ we define a *double quiver* $\bar{Q} = (I, E \sqcup E^{op}, h, t)$, where E^{op} has the same arrows as E but with opposite direction. Equivalently, we can write $\bar{Q} = Q \sqcup Q^{op}$, where Q^{op} is a quiver with the same vertex set as Q but with arrows in opposite direction. Given a dimension vector v, we have

$$\text{Rep}(\bar{Q}, v) \cong \text{Rep}(Q, v) \times \text{Rep}(Q^{op}, v).$$

Using that given two finite dimensional vector spaces V and W there is a pairing

$$\text{Hom}(V, W) \times \text{Hom}(W, V) \to \mathbb{C}$$

defined by $(h_1, h_2) \mapsto \text{Tr}(h_1 \circ h_2)$, we can identify $\text{Rep}(Q^{op}, v)$ with $\text{Rep}(Q, v)^*$. Thus, we have

$$\text{Rep}(\bar{Q}, v) \cong \text{Rep}(Q, v) \times \text{Rep}(Q^{op}, v) \cong \text{Rep}(Q, v) \times \text{Rep}(Q, v)^* \cong T^*(\text{Rep}(Q, v)).$$

Finally, given a quiver $Q = (I, E, h, t)$ we construct the quiver $\bar{Q^f}$. After fixing dimension vectors $v, w \in \mathbb{Z}^I$, we can choose vector space $V = \bigoplus_{i \in I} V_i$ and $W = \bigoplus_{i \in I} W_i$ such that $\text{dim}V = v$ and $\text{dim}W = w$. Then

$$\begin{aligned} \text{Rep}(\bar{Q^f} v, w) &= T^*(\text{Rep}(Q^f), v, w) \\ &= \text{Rep}(Q, v) \times \text{Rep}(Q^{op}, v) \times \text{Hom}(V, W) \times \text{Hom}(W, V). \end{aligned}$$

The group $GL(v)$ acts on $\text{Rep}(\bar{Q^f}, v, w)$ by

$$g \cdot (X, Y, j, i) = (gXg^{-1}, gYg^{-1}, g \circ j, i \circ g^{-1}).$$

Furthermore, $\text{Rep}(\bar{Q^f}, v, w)$ has a symplectic structure and the associated moment map is given by

$$\mu : \text{Rep}(\bar{Q^f}, v, w) \to \mathfrak{gl}(v)^* = \mathfrak{gl}(v)$$
$$(X, Y, j, i) \mapsto [X, Y] + j \circ i.$$

The *Nakajima quiver variety* is defined to be the GIT quotient

$$\mathcal{M}_\theta = \mu(0)^{-1} /\!\!/_{\chi_\theta} GL(v).$$

Notice that for the trivial character (we denote it by $\theta = 0$) the GIT and categorical quotient coincide; that is, $\mu(0)^{-1} /\!\!/_{\chi_\theta} GL(v) = \mu(0)^{-1} /\!\!/ GL(v)$. Then, by construction of the quiver varieties, we have a projective morphism $\pi : \mathcal{M}_\theta \to \mathcal{M}_0$.

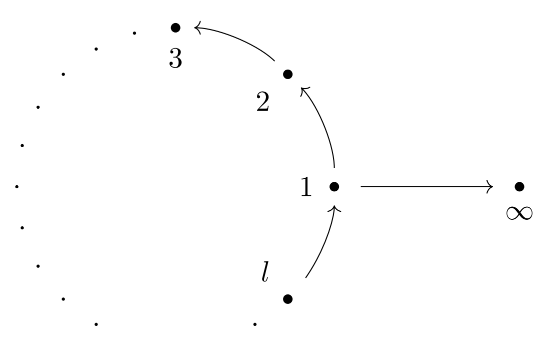
The cyclic quiver can be seen as the generalization to the Jordan quiver (a quiver with one vertex and a single loop) which is known for having very nice geometric properties. For example; when $v = n$, $w = 1$ and $\theta = -1 \in \mathbb{Z}$, the projective morphism

$$\pi : \mathcal{M}_\theta = \text{Hilb}^n(\mathbb{C}^2) \to \mathcal{M}_0 = (\mathbb{C}^v \times \mathbb{C}^v)/\mathbb{S}_v$$

is a resolution of singularities [12].

6.2 Cyclic quivers

In this chapter we study the KN 1-parameter subgroups and stratifications of the representations of the cyclic quiver Q.



Let l be the number of vertices and $v = (v_i)_i$ the dimension vector with $v_i = n$ for $i \in \{1, 2, \ldots, l\}$. The vertex ∞ corresponds to the framing and has dimension vector $w = 1$. In this case we have

$$\mathrm{Rep}(Q, v) = \underbrace{\mathfrak{gl}_n \times \cdots \times \mathfrak{gl}_n}_{l \ \ times} \times \mathbb{C}^n$$

with an action of the reductive group

$$G = \underbrace{GL_n \times \cdots \times GL_n}_{l \ \ times}$$

given by

$$(g_1, g_2, \ldots, g_l) \cdot (X_1, X_2, \ldots, X_l, i) = (g_1 X_1 g_2^{-1}, g_2 X_2 g_3^{-1}, \ldots, g_l X_l g_1^{-1}, g_1 i)$$

where $X_k = (x_{ij}^k)_{1 \leq i,j \leq n}$ for $k = 1, 2, \ldots, l$. We start by finding the weights under the torus action. For that, we fix the maximal torus of G to be $T = \underbrace{(\mathbb{C}^\times)^n \times \cdots \times (\mathbb{C}^\times)^n}_{l \ \ times}$ and we choose $\{e_i \mid i \in \{1, 2, \ldots, n \cdot l\}\}$ to be the basis of T. Here e_i denotes the standard basis vector.

Let $g = (g_i)_{i=1}^l \in T$ where $g_i = (g_i^1, \ldots, g_i^n) \in (\mathbb{C}^\times)^n$. Notice that g_i can be viewed as a diagonal $n \times n$ matrix. Then the T-weights are given by

$$e_{i+jn} - e_{k+(j+1)n},$$

$$e_{i+(l-1)n} - e_s,$$

$$e_s$$

where $1 \leq i, k \leq n$, $0 \leq j \leq l-2$, and $1 \leq s \leq n$. In particular, there are $n^2 \cdot l + n$ weights.

The induced action of G on the cotangent space $T^*(\mathrm{Rep}(Q, v))$ is given by

$$(g_1, g_2, \ldots, g_l) \cdot (X_1, X_2, \ldots, X_l, Y_1, Y_2, \ldots, Y_l, i, j) =$$

$$(g_1 X_1 g_2^{-1}, g_2 X_2 g_3^{-1}, \ldots, g_l X_l g_1^{-1}, g_1^{-1} Y_1 g_2, g_2^{-1} Y_2 g_3, \ldots, g_l^{-1} Y_l g_1, g_1 i, g_1^{-1} j).$$

Thus, the weights of the torus action are given by all of the weights α_i together with $-\alpha_i$ for $1 \leq i \leq n^2 \cdot l + n$. However, remember that we only need the T-weights on $\mathrm{Rep}(Q, v)$ to find the

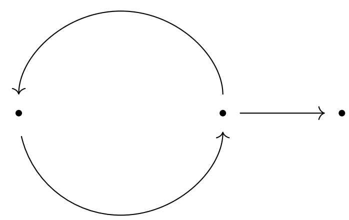
KN 1-parameter subgroups defining the Kirwan-Ness stratification of $T^*(\mathrm{Rep}(Q, v))$.

Example 6.1. When $n = 1$, the weights are given by

$$e_i - e_{i+1},$$
$$e_l - e_1,$$
$$e_1$$

where $1 \leq i \leq l - 1$.

Example 6.2. Let $n = l = 2$.



In this case we have $\mathrm{Rep}(Q, v) = \mathfrak{gl}_2 \times \mathfrak{gl}_2 \times \mathbb{C}^2$ with an action of $G = GL_2 \times GL_2$ give by

$$(g_1, g_2) \cdot (X, Y, i) = (g_1 X g_2^{-1}, g_2 Y g_1^{-1}, g_1 i).$$

The torus action is given by

$$\left(\begin{bmatrix} \lambda_1 x_{11} \gamma_1^{-1} & \lambda_1 x_{12} \gamma_2^{-1} \\ \lambda_2 x_{21} \gamma_1^{-1} & \lambda_2 x_{22} \gamma_2^{-1} \end{bmatrix}, \begin{bmatrix} \gamma_1 y_{11} \lambda_1^{-1} & \gamma_2 y_{12} \lambda_2^{-1} \\ \gamma_2 y_{21} \lambda_1^{-1} & \gamma_2 y_{22} \lambda_2^{-1} \end{bmatrix}, \begin{bmatrix} \lambda_1 i_1 \\ \lambda_2 i_2 \end{bmatrix} \right)$$

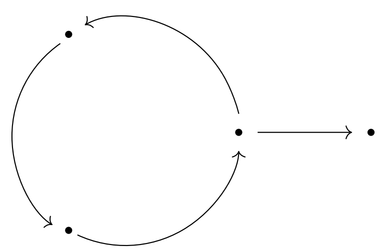
where

$$g_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ and } g_2 = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}.$$

The weights under the torus action are the rows of the following matrix:

$$\alpha = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Example 6.3. Let $l = 3$ and $n = 2$.



$\mathrm{Rep}(Q, v) = \mathfrak{gl}_2 \times \mathfrak{gl}_2 \times \mathfrak{gl}_2 \times \mathbb{C}^2$ with an action of $G = GL_2 \times GL_2 \times GL_2$ given by

$$(g_1, g_2, g_2) \cdot (X, Y, Z, i) = (g_1 X g_2^{-1}, g_2 Y g_3^{-1}, g_3 Z g_1^{-1}, g_1 i).$$

Then the torus $T = (\mathbb{C}^\times)^2 \times (\mathbb{C}^\times)^2 \times (\mathbb{C}^\times)^2$acts by

$$\left(\begin{bmatrix} \lambda_1 x_{11} \gamma_1^{-1} & \lambda_1 x_{12} \gamma_2^{-1} \\ \lambda_2 x_{21} \gamma_1^{-1} & \lambda_2 x_{22} \gamma_2^{-1} \end{bmatrix}, \begin{bmatrix} \gamma_1 y_{11} \beta_1^{-1} & \gamma_2 y_{12} \beta_2^{-1} \\ \gamma_2 y_{21} \beta_1^{-1} & \gamma_2 y_{22} \beta_2^{-1} \end{bmatrix}, \begin{bmatrix} \beta_1 z_{11} \lambda_1^{-1} & \beta_2 z_{12} \lambda_2^{-1} \\ \beta_2 z_{21} \lambda_1^{-1} & \beta_2 z_{22} \lambda_2^{-1} \end{bmatrix}, \begin{bmatrix} \lambda_1 i_1 \\ \lambda_2 i_2 \end{bmatrix} \right)$$

where $g_1 = (\lambda_1, \lambda_2)$, $g_2 = (\gamma_1, \gamma_2)$ and $g_3 = (\beta_1, \beta_2)$. The weights under the torus action are given by the rows of the following matrix:

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6.3 KN 1-parameter subgroups for fixed characters λ

As we can see from the examples, the number of weights of the action grows exponentially as the dimension n increases. This makes the calculation of the KN 1-parameter subgroups quite tedious. However, there are some choices of the character λ that can make the calculation easier.

Lemma 6.1. *Let ϵ be a small negative number and $\lambda = (-1, -1, \ldots, -1)$. Then the set of KN 1-parameters subgroups for Rep(Q, v) is given by $\mathsf{KN} = \{\epsilon\lambda\}$ for arbitraty n and l.*

Proof. Let $I = \{0\}$, then $p_I(\epsilon) = \epsilon\lambda$ and $\epsilon\lambda \in \mathsf{KN}$. If I is a subset where $p_I(\epsilon) \neq \epsilon\lambda$, then $\{0\} = J \subset I$ and

$$d(p_J(\epsilon), \alpha_i^\epsilon) < d(p_I(\epsilon), \alpha_i^\epsilon) \qquad \text{for all } i \in I.$$

The reason is that the T-weights are given by

$$e_{i+jn} - e_{k+(j+1)n},$$

$$e_{i+(l-1)n} - e_s,$$

$$e_s$$

where $1 \le i,k \le n$, $0 \le j \le l-2$, and $1 \le s \le n$. In particular, all $p_I(\epsilon)$ are of the form

$$-\epsilon \sum_i e_i$$

for some $1 \le i \le n \cdot l$, or $p_I(\epsilon) = 0$. $\square$

Example 6.4. Let $n = 1$, $l = 3$ and $\lambda = (-1,-1,-1)$. The weights are $\alpha_0 = (0,0,0)$, $\alpha_1 = (1,-1,0)$, $\alpha_2 = (0,-1,1)$, $\alpha_3 = (-1,0,1)$ and $\alpha_4 = (1,0,0)$. All possible $p_I = p_I(\epsilon)$ as I runs over the subsets of $\{0,1,2,3,4\}$ (with 0 always in I) are given by

$$\begin{aligned}
p_0 = p_{\{0,1\}} = p_{\{0,2\}} = p_{\{0,3\}} &= (\epsilon\lambda) = (\epsilon, -\epsilon, -\epsilon) \\
p_{\{0,1,4\}} &= (0,0,-\epsilon) \\
p_{\{0,2,4\}} &= (0,-\epsilon,-\epsilon) \\
p_{\{0,3,4\}} &= (0,-\epsilon,0) \\
p_{\{0,1,2,3,4\}} = p_{\{0,1,2,3\}} = p_{\{0,1,2,4\}} = p_{\{0,2,3,4\}} &= (0,0,0)
\end{aligned}$$

Let $I = \{0,1,4\}$ and $J = \{0\}$. Then,

$$\begin{aligned}
d(p_J(\epsilon), \alpha_0^\epsilon) &= 0 < 2\epsilon^2 = d(p_I, \alpha_0^\epsilon) \\
d(p_J(\epsilon), \alpha_1^\epsilon) &= 2 < 2 + 2\epsilon^2 = d(p_I, \alpha_1^\epsilon) \\
d(p_J(\epsilon), \alpha_4^\epsilon) &= 1 < (1-\epsilon)^2 + \epsilon^2 = d(p_I, \alpha_4^\epsilon)
\end{aligned}$$

Therefore $(0,0,-\epsilon) \notin \mathsf{KN}$. Similarly, all of the projections are discarded by $J = \{0\} \subset I$ except for $\epsilon\lambda$.

Remark: Notice that $0 \notin \mathsf{KN}$, so the choice of character $\lambda = (-1,-1,\ldots,-1)$ is not convenient since $(\mathrm{Rep}(Q,v))^{ss}$ would be empty.

6.4 Arbitrary choice of character λ

The set KN of KN 1-parameter subgroups depends very much on the choice of group character λ. In this section we will study how the set KN corresponding to the stratifications of $\text{Rep}(Q, v)$ and $T^*(\text{Rep}(Q, v))$ changes as we vary λ.

6.4.1 Case for X=Rep(Q,v)

We first assume that $n = 1$ and $l = 2$; that is, the dimension vector is $v = (1, 1)$. In this case we have $X = \mathfrak{gl}_1 \times \mathfrak{gl}_1 \times \mathbb{C}$ with an action of $G = GL_1 \times GL_1$ given by

$$(g_1, g_2) \cdot (X, Y, i) = (g_1 X g_2^{-1}, g_2 Y g_1^{-1}, g_1 i).$$

The weights under the torus action are given by the columns of the following matrix

$$\alpha = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix},$$

where the i^{th} column will be denoted by α_i. Taking all possible combinations of subsets of weights, we get that $\mathsf{KN} \subset \{\epsilon(\frac{x+y}{2}, \frac{x+y}{2}), \epsilon(0, y), \epsilon\lambda, (0, 0)\}$, for $\lambda = (x, y) \in \mathbb{C}^2$.

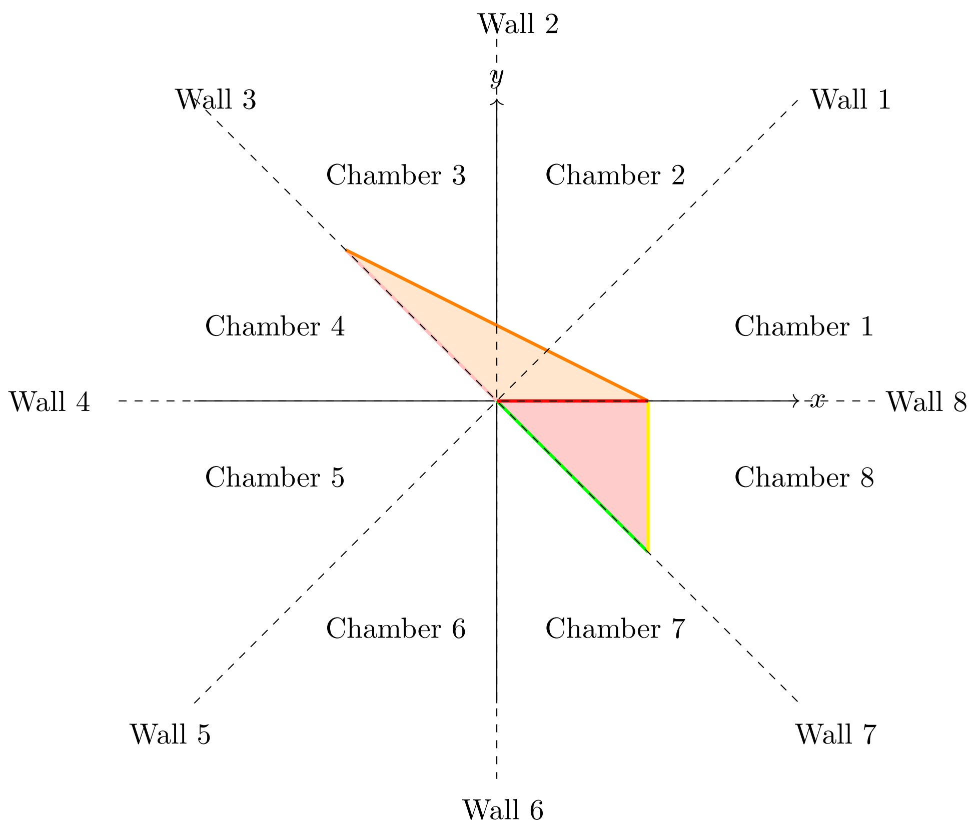


Figure 6.1: An illustration of all possible convex hulls C_I.

Depending on $\epsilon\lambda$, the closest point to the origin in C_I^ϵ will be different. In the next charts we can see how the set KN and the stratum S_β change as $\epsilon\lambda$ varies.

Note: The walls and chambers correspond to $\epsilon\lambda$ (not λ) and they are defined as follows:

- Wall 1= $\{(x,y) : x = y, x, y > 0\}$.
- Wall 2= $\{(0,y) : y > 0\}$.
- Wall 3= $\{(x,y) : x = -y, y > 0\}$.
- Wall 4= $\{(x,0) : x < 0\}$.
- Wall 5= $\{(x,y) : x = y, x, y < 0\}$.
- Wall 6= $\{(0,y) : y < 0\}$.
- Wall 7= $\{(x,y) : x = -y, y < 0\}$.
- Wall 8= $\{(x,0) : x > 0\}$.

$\epsilon\lambda$	KN 1-parameter subgroups
Chamber 1	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1,\lambda_2)$
Wall 1	$\beta_1 = \epsilon\lambda = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$
Chamber 2	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1,\lambda_2)$
Wall 2	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1,\lambda_2)$
Chamber 3	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1,\lambda_2)$ $\beta_3 = \epsilon(0,\lambda_2)$
Wall 3	$\beta_1 = \epsilon(\lambda_1,\lambda_2)$ $\beta_2 = \epsilon(0,\lambda_2)$ $\beta_3 = (0,0)$
Chamber 4	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1,\lambda_2)$ $\beta_3 = \epsilon(0,\lambda_2)$ $\beta_4 = (0,0)$
Wall 4	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1,\lambda_2)$ $\beta_3 = (0,0)$

$\epsilon\lambda$	KN 1-parameter subgroups
Chamber 5	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1,\lambda_2)$ $\beta_3 = \epsilon(0,\lambda_2)$ $\beta_4 = (0,0)$
Wall 5	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1) = \epsilon(\lambda_1,\lambda_2)$ $\beta_3 = \epsilon(0,\lambda_2)$ $\beta_4 = (0,0)$
Chamber 6	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1,\lambda_2)$ $\beta_3 = \epsilon(0,\lambda_2)$ $\beta_4 = (0,0)$
Wall 6	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1,\lambda_2) = \epsilon(0,\lambda_2)$ $\beta_3 = (0,0)$
Chamber 7	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1,\lambda_2)$ $\beta_3 = (0,0)$
Wall 7	$\beta_1 = \epsilon(\lambda_1,\lambda_2)$ $\beta_2 = (0,0)$
Chamber 8	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1,\lambda_2)$
Wall 8	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1,\lambda_2)$

Table 6.1

We can already see when X^{ss} will be empty, since this happens when $(0,0) \notin \mathsf{KN}$. Next we study how the strata changes depending on the choice of the character. Remember that S_β is given by $G \cdot Y_\beta^{ss}$ and we can calculate Y_β^{ss} using the remarks at the end of chapter 3 and 4.

Example: Let $\lambda = (-2,-1)$. Then $\epsilon\lambda$ lies in chamber 1 and $\mathsf{KN} = \{\epsilon(-3/2,-3/2), \epsilon(-2,-1)\}$. Y_β^{ss} does not change if we rescale the KN 1-parameter subgroups by a positive number, so we can assume $\mathsf{KN} = \{(1,1),(2,1)\}$.

1. Let $\beta_1 = (1,1)$. Then $\beta \bullet \alpha = (0,0,1)$ which implies that

$$Z_\beta = \{(x,y,0) \mid x,y \in \mathbb{C}\}$$

and

$$Y_\beta = \{(x,y,z) \mid x,y,z \in \mathbb{C}\}.$$

Using the remark at the end of chapter 3, we take $v = (-1,1)$ such that $v \bullet \beta = 0$. Then $v \bullet \alpha = (-2,2,-1)$ and we conclude

$$Z_\beta^{ss} = \{(x,y,0) \mid x,y \in \mathbb{C} \text{ and } y \neq 0\}.$$

Thus $Y_\beta^{ss} = \mathrm{pr}_\beta^{-1}(Z_\beta^{ss}) = \{(x,y,z) \mid y \neq 0\}$.

2. Let $\beta_2 = (2,1)$. Then $\beta \bullet \alpha = (1,-1,2)$ which implies that

$$Z_\beta = \{(0,0,0)\}$$

and

$$Y_\beta = \{(x,0,z) \mid x,z \in \mathbb{C}\}.$$

Using the remark at the end of chapter 3, we take $v = (-1,2)$ such that $v \bullet \beta = 0$. Then $v \bullet \alpha = (-3,3,-1)$ and we conclude

$$Z_\beta^{ss} = \{(0,0,0)\}.$$

Thus $Y_\beta^{ss} = \text{pr}_\beta^{-1}(Z_\beta^{ss}) = \{(x, 0, z) \mid x, z \in \mathbb{C}\}$.

Remark: Notice that if we chose any other λ in chamber 1 the calculations don't change, so we will get the same Y_β^{ss}. More explicit calculations can be found in the appendix.

$\epsilon\lambda$	β	Z_β	Z_β^{ss}	Y_β	Y_β^{ss}
Chamber 1	(1,1)	$\{(x,y,0)\}$	$\{(x,y,0)\|y \neq 0\}$	$\{(x,y,z)\}$	$\{(x,y,z)\|y \neq 0\}$
	(2,1)	$\{(0,0,0)\}$	$\{(0,0,0)\}$	$\{(x,0,z)\}$	$\{(x,0,z)\}$
Wall 1	(1,1)	$\{(x,y,0)\}$	$\{(x,y,0)\}$	$\{(x,y,z)\}$	$\{(x,y,z)\}$
Chamber 2	(1,1)	$\{(x,y,0)\}$	$\{(x,y,0)\|x \neq 0\}$	$\{(x,y,z)\}$	$\{(x,y,z)\|x \neq 0\}$
	(2,3)	$\{(0,0,0)\}$	$\{(0,0,0)\}$	$\{(0,y,z)\}$	$\{(0,y,z)\}$
Wall 2	(1,1)	$\{(x,y,0)\}$	$\{(x,y,0)\|x \neq 0\}$	$\{(x,y,z)\}$	$\{(x,y,z)\|x \neq 0\}$
	(0,1)	$\{(0,0,z)\}$	$\{(0,0,z)\}$	$\{(0,y,z)\}$	$\{(0,y,z)\}$
Chamber 3	(1,1)	$\{(x,y,0)\}$	$\{(x,y,0)\|x \neq 0\}$	$\{(x,y,z)\}$	$\{(x,y,z)\|x \neq 0\}$
	(-1,2)	$\{(0,0,0)\}$	$\{(0,0,0)\}$	$\{(0,y,0)\}$	$\{(0,y,0)\}$
	(0,1)	$\{(0,0,z)\}$	$\{(0,0,z)\|z \neq 0\}$	$\{(0,y,z)\}$	$\{(0,y,z)\|z \neq 0\}$

Table 6.2

$\epsilon\lambda$	β	Z_β	Z_β^{ss}	Y_β	Y_β^{ss}
Wall 3	(0,0)	$\{(x,y,z)\}$	X^{ss}	$\{(x,y,z)\}$	X^{ss}
	(-1,1)	$\{(0,0,0)\}$	$\{(0,0,0)\}$	$\{(0,y,0)\}$	$\{(0,y,0)\}$
	(0,1)	$\{(0,0,z)\}$	$\{(0,0,z)\|z\neq 0\}$	$\{(0,y,z)\}$	$\{(0,y,z)\|z\neq 0\}$
Chamber 4	(-1,-1)	$\{(x,y,0)\}$	$\{(x,y,0)\|x\neq 0\}$	$\{(x,y,0)\}$	$\{(x,y,0)\|x\neq 0\}$
	(-2,1)	$\{(0,0,0)\}$	$\{(0,0,0)\}$	$\{(0,y,0)\}$	$\{(0,y,0)\}$
	(0,1)	$\{(0,0,z)\}$	$\{(0,0,z)\|z\neq 0\}$	$\{(0,y,z)\}$	$\{(0,y,z)\|z\neq 0\}$
	(0,0)	$\{(x,y,z)\}$	X^{ss}	$\{(x,y,z)\}$	X^{ss}
Wall 4	(-1,-1)	$\{(x,y,0)\}$	$\{(x,y,0)\|x\neq 0\}$	$\{(x,y,0)\}$	$\{(x,y,0)\|x\neq 0\}$
	(-1,0)	$\{(0,0,0)\}$	$\{(0,0,0)\}$	$\{(0,y,0)\}$	$\{(0,y,0)\}$
	(0,0)	$\{(x,y,z)\}$	X^{ss}	$\{(x,y,z)\}$	X^{ss}
Chamber 5	(-1,-1)	$\{(x,y,0)\}$	$\{(x,y,0)\|x\neq 0\}$	$\{(x,y,0)\}$	$\{(x,y,0)\|x\neq 0\}$
	(-2,-1)	$\{(0,0,0)\}$	$\{(0,0,0)\}$	$\{(0,y,0)\}$	$\{(0,y,0)\}$
	(0,-1)	$\{(0,0,z)\}$	$\{(0,0,z)\|z\neq 0\}$	$\{(x,0,z)\}$	$\{(x,0,z)\|z\neq 0\}$
	(0,0)	$\{(x,y,z)\}$	X^{ss}	$\{(x,y,z)\}$	X^{ss}

Table 6.2 (cont)

$\epsilon\lambda$	β	Z_β	Z_β^{ss}	Y_β	Y_β^{ss}
Wall 5	(-1,-1)	$\{(x,y,0)\}$	$\{(x,y,0)\}$	$\{(x,y,0)\}$	$\{(x,y,0)0\}$
	(0,-1)	$\{(0,0,z)\}$	$\{(0,0,z)\|z\neq 0\}$	$\{(x,0,z)\}$	$\{(x,0,z)\|z\neq 0\}$
	(0,0)	$\{(x,y,z)\}$	X^{ss}	$\{(x,y,z)\}$	X^{ss}
Chamber 6	(-1,-1)	$\{(x,y,0)\}$	$\{(x,y,0)\|y\neq 0\}$	$\{(x,y,0)\}$	$\{(x,y,0)\|y\neq 0\}$
	(-2,-3)	$\{(0,0,0)\}$	$\{(0,0,0)\}$	$\{(x,0,0)\}$	$\{(x,0,0)\}$
	(0,-1)	$\{(0,0,z)\}$	$\{(0,0,z)\|z\neq 0\}$	$\{(x,0,z)\}$	$\{(x,0,z)\|z\neq 0\}$
	(0,0)	$\{(x,y,z)\}$	X^{ss}	$\{(x,y,z)\}$	X^{ss}
Wall 6	(-1,-1)	$\{(x,y,0)\}$	$\{(x,y,0)\|y\neq 0\}$	$\{(x,y,0)\}$	$\{(x,y,0)\|y\neq 0\}$
	(0,-1)	$\{(0,0,z)\}$	$\{(0,0,z)\}$	$\{(x,0,z)\}$	$\{(x,0,z)\}$
	(0,0)	$\{(x,y,z)\}$	X^{ss}	$\{(x,y,z)\}$	X^{ss}
Chamber 7	(-1,-1)	$\{(x,y,0)\}$	$\{(x,y,0)\|y\neq 0\}$	$\{(x,y,0)\}$	$\{(x,y,0)\|y\neq 0\}$
	(1,-2)	$\{(0,0,0)\}$	$\{(0,0,0)\}$	$\{(x,0,z)\}$	$\{(x,0,z)\}$
	(0,0)	$\{(x,y,z)\}$	X^{ss}	$\{(x,y,z)\}$	X^{ss}

Table 6.2 (cont)

$\epsilon\lambda$	β	Z_β	Z_β^{ss}	Y_β	Y_β^{ss}
Wall 7	(0,0)	$\{(x,y,z)\}$	X^{ss}	$\{(x,y,z)\}$	X^{ss}
	(1,-1)	$\{(0,0,0)\}$	$\{(0,0,0)\}$	$\{(x,0,z)\}$	$\{(x,0,z)\}$
Chamber 8	(1,1)	$\{(x,y,0)\}$	$\{(x,y,0)\|y\neq 0\}$	$\{(x,y,z)\}$	$\{(x,y,z)\|y\neq 0\}$
	(2,-1)	$\{(0,0,0)\}$	$\{(0,0,0)\}$	$\{(x,0,z)\}$	$\{(x,0,z)\}$
Wall 8	(1,1)	$\{(x,y,0)\}$	$\{(x,y,0)\|y\neq 0\}$	$\{(x,y,z)\}$	$\{(x,y,z)\|y\neq 0\}$
	(1,0)	$\{(0,0,0)\}$	$\{(0,0,0)\}$	$\{(x,0,z)\}$	$\{(x,0,z)\}$

Table 6.2 (cont)

6.4.2 Case for $T^*(\mathbf{Rep}(Q,v))$

In this section we will study the KN 1-parameter subgroups associated to $T^*(\text{Rep}(Q,v))$ for arbitrary group character λ. We will do that by first analyzing the cases where $n=1$, and later we will give a general pattern for n and l arbitrary.

Lemma 6.2. *Independently of the group character λ, $T^*(Rep(Q,v))^{ss}$ will always be nonempty; that is, $(0,0)$ is always a KN 1-parameter subgroup.*

Proof. This follows by corollary 4.2. □

Note: In the following discussion we will only study the nonzero KN 1-parameter subgroups.
Case 1: n=1, l=2:

The action of $G = GL_1 \times GL_1$ in $T^*(\text{Rep}(Q,v))$ is given by

$$(g_1,g_2)\cdot(X,Y,Z,W,i,j) = (g_1Xg_2^{-1}, g_2Yg_1^{-1}, g_1^{-1}Zg_2, g_2^{-1}Wg_1, g_1i, jg_1^{-1}).$$

The weights under the torus action are given by the columns of the following matrix:

$$\alpha = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 0 & 0 \end{bmatrix}$$

In this case, $\mathsf{KN} = \{\epsilon(\frac{x+y}{2}, \frac{x+y}{2}), \epsilon(0,y), \epsilon\lambda, (0,0)\}$ for any $\lambda = (x,y) \in \mathbb{C}^2$. However, the stratum will still change as λ varies, since different KN 1-parameter subgroups may coincide depending on the group characters λ. In particular, $\epsilon(\frac{x+y}{2}, \frac{x+y}{2}) = \epsilon(x,y)$ if $x = y$; $\epsilon(\frac{x+y}{2}, \frac{x+y}{2}) = (0,0)$ if $x = -y$ (semistable locus will change); $\epsilon(x,y) = \epsilon(0,y)$ if $x = 0$ and $\epsilon(0,y) = (0,0)$ is $y = 0$ (semistable locus will change).

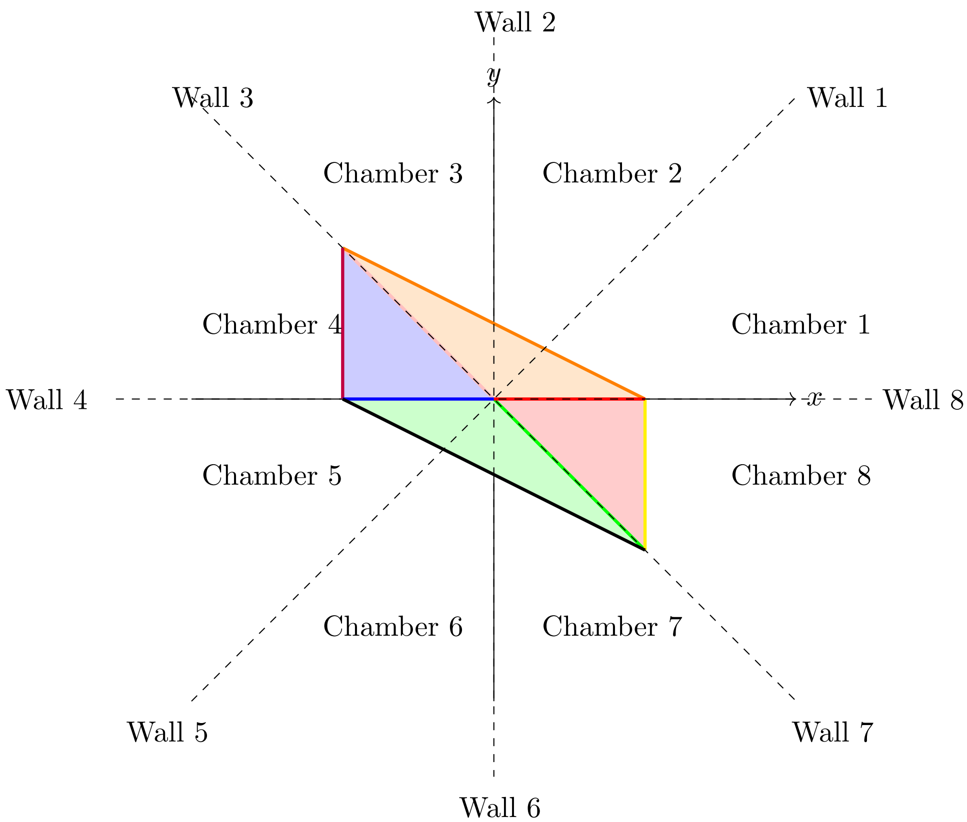


Figure 6.2: An illustration of all possible convex hulls C_I.

The walls are the same as in the previous case, but the KN 1-parameter subgroups change. The following charts show how the stratum changes as the character λ varies. More explicit calculations can be found on the appendix.

$\epsilon\lambda$	KN 1-parameter subgroups
Chamber 1	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1, \lambda_2)$ $\beta_3 = \epsilon(0, \lambda_2)$
Wall 1	$\beta_1 = \epsilon\lambda = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(0, \lambda_2)$
Chamber 2	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1, \lambda_2)$ $\beta_3 = \epsilon(0, \lambda_2)$
Wall 2	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1, \lambda_2) = \epsilon(0, \lambda_2)$
Chamber 3	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1, \lambda_2)$ $\beta_3 = \epsilon(0, \lambda_2)$
Wall 3	$\beta_1 = \epsilon(\lambda_1, \lambda_2)$ $\beta_2 = \epsilon(0, \lambda_2)$
Chamber 4	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1, \lambda_2)$ $\beta_3 = \epsilon(0, \lambda_2)$
Wall 4	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1, \lambda_2)$

$\epsilon\lambda$	KN 1-parameter subgroups
Chamber 5	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1, \lambda_2)$ $\beta_3 = \epsilon(0, \lambda_2)$
Wall 5	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1) = \epsilon(\lambda_1, \lambda_2)$ $\beta_3 = \epsilon(0, \lambda_2)$
Chamber 6	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1, \lambda_2)$ $\beta_3 = \epsilon(0, \lambda_2)$
Wall 6	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1, \lambda_2) = \epsilon(0, \lambda_2)$
Chamber 7	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1, \lambda_2)$ $\beta_3 = \epsilon(0, \lambda_2)$
Wall 7	$\beta_1 = \epsilon(\lambda_1, \lambda_2)$ $\beta_2 = \epsilon(0, \lambda_2)$
Chamber 8	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1, \lambda_2)$ $\beta_3 = \epsilon(0, \lambda_2)$
Wall 8	$\beta_1 = \frac{\epsilon(\lambda_1+\lambda_2)}{2}(1,1)$ $\beta_2 = \epsilon(\lambda_1, \lambda_2)$

Table 6.3

$\epsilon\lambda$	β	Z^{ss}_{β}	Y^{ss}_{β}
Chamber 1	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\vert y\neq 0\neq z\}$	$\{(x,y,z,w,t,0)\vert y\neq 0\neq z\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(x,0,0,w,t,0)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\vert s\neq 0\}$	$\{(0,y,z,0,t,s)\vert s\neq 0)\}$
Wall 1	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,t,0)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\vert s\neq 0\}$	$\{(0,y,z,0,t,s)\vert s\neq 0)\}$
Chamber 2	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\vert x\neq 0\neq w\}$	$\{(x,y,z,w,t,0)\vert x\neq 0\neq w\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,y,z,0,t,0)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\vert s\neq 0\}$	$\{(0,y,z,0,t,s)\vert s\neq 0)\}$
Wall 2	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\vert x\neq 0\neq w\}$	$\{(x,y,z,w,t,0)\vert x\neq 0\neq w\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,y,z,0,t,s)\}$
Chamber 3	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\vert x\neq 0\neq w\}$	$\{(x,y,z,w,t,0)\vert x\neq 0\neq w\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,y,z,0,0,s)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,0)\vert t\neq 0\}$	$\{(0,y,z,0,t,s)\vert t\neq 0\}$

Table 6.4

$\epsilon\lambda$	β	Z_β^{ss}	Y_β^{ss}
Wall 3	(0,0)	$(T^*X)^{ss}$	$(T^*X)^{ss}$
	$\epsilon(\lambda_1, \lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,y,z,0,0,s)\}$
	$\epsilon(0, \lambda_2)$	$\{(0,0,0,0,t,s) \| t \neq 0\}$	$\{(0,y,z,0,t,s) \| t \neq 0\}$
Chamber 4	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0) \| x \neq 0 \neq w\}$	$\{(x,y,z,w,0,s) \| x \neq 0 \neq w\}$
	$\epsilon(\lambda_1, \lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,y,z,0,0,s)\}$
	$\epsilon(0, \lambda_2)$	$\{(0,0,0,0,t,s) \| t \neq 0\}$	$\{(0,y,z,0,t,s) \| t \neq 0\}$
	(0,0)	$(T^*X)^{ss}$	$(T^*X)^{ss}$
Wall 4	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0) \| x \neq 0 \neq w\}$	$\{(x,y,z,w,0,s) \| x \neq 0 \neq w\}$
	$\epsilon(\lambda_1, \lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,y,z,0,0,s)\}$
	(0,0)	$(T^*X)^{ss}$	$(T^*X)^{ss}$
Chamber 5	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0) \| x \neq 0 \neq w\}$	$\{(x,y,z,w,0,s) \| x \neq 0 \neq w\}$
	$\epsilon(\lambda_1, \lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,y,z,0,0,s)\}$
	$\epsilon(0, \lambda_2)$	$\{(0,0,0,0,t,s) \| t \neq 0\}$	$\{(x,0,0,w,t,s) \| t \neq 0\}$
	(0,0)	$(T^*X)^{ss}$	$(T^*X)^{ss}$

Table 6.4 (cont)

$\epsilon\lambda$	β	Z_β^{ss}	Y_β^{ss}
Wall 5	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,s)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\|t\neq 0\}$	$\{(x,0,0,w,t,s)\|t\neq 0\}$
	(0,0)	$(T^*X)^{ss}$	$(T^*X)^{ss}$
Chamber 6	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\|y\neq 0\neq z\}$	$\{(x,y,z,w,0,s)\|y\neq 0\neq z\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(x,0,0,w,0,s)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\|t\neq 0\}$	$\{(x,0,0,w,t,s)\|t\neq 0\}$
	(0,0)	$(T^*X)^{ss}$	$(T^*X)^{ss}$
Wall 6	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\|y\neq 0\neq z\}$	$\{(x,y,z,w,0,s)\|y\neq 0\neq z\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(x,0,0,w,t,s)\}$
	(0,0)	$(T^*X)^{ss}$	$(T^*X)^{ss}$
Chamber 7	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\|y\neq 0\neq z\}$	$\{(x,y,z,w,0,s)\|y\neq 0\neq z\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(x,0,0,w,t,0)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\|s\neq 0\}$	$\{(x,0,0,w,t,s)\|s\neq 0\}$
	(0,0)	$(T^*X)^{ss}$	$(T^*X)^{ss}$

Table 6.4 (cont)

$\epsilon\lambda$	β	Z_β^{ss}	Y_β^{ss}
Wall 7	(0,0)	$(T^*X)^{ss}$	$(T^*X)^{ss}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(x,0,0,w,t,0)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\|s\neq 0\}$	$\{(x,0,0,w,t,s)\|s\neq 0\}$
Chamber 8	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\|y\neq 0\neq z\}$	$\{(x,y,z,w,t,0)\|y\neq 0\neq z\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(x,0,0,w,t,0)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\|s\neq 0\}$	$\{(x,0,0,w,t,s)\|s\neq 0\}$
Wall 8	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\|y\neq 0\neq z\}$	$\{(x,y,z,w,t,0)\|y\neq 0\neq z\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(x,0,0,w,t,0)\}$

Table 6.4 (cont)

Case 2: n=1, l=3:

In this case, the weights of $\text{Rep}(Q,v)$ under the torus action are given by the rows of the following matrix:

$$\alpha = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Let α_i denote the i^{th} row and $\alpha_0 = (0,0,0)$. Notice that $\{\alpha_1,\alpha_2,\alpha_3\}$ and $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ are not linearly independent, so we don't need to calculate the projection of these subsets (since there will be a subset J such that $p_I = p_J$). Furthermore, $\{\alpha_1,\alpha_2,\alpha_4\}$, $\{\alpha_1,\alpha_3,\alpha_4\}$ and $\{\alpha_2,\alpha_3,\alpha_4\}$ span the whole space, so we know that the projection will just be $\epsilon\lambda$. Finally, notice that the weights span the whole space, so $(0,0,0)\in \mathsf{KN}$. It only remains to find the projections when the subsets of weights contain one or two weights.

Let ϵ be a small negative number, and $\lambda = (x, y, z)$. Then the KN 1-parameters subgroups are the following:

$(0,0,0)$, $\epsilon(\frac{x+z}{2}, y, \frac{x+z}{2})$, $\epsilon(0,0,z)$,
$\epsilon(\frac{x+y}{2}, \frac{x+y}{2}, z)$, $\epsilon(0,y,z)$, $\epsilon(0, \frac{y+z}{2}, \frac{y+z}{2})$,
$\epsilon(x, \frac{y+z}{2}, \frac{y+z}{2})$, $\epsilon(\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3})$, $\epsilon(0,y,0)$.

Depending on λ some projections may coincide. In particular, the semistable locus will change whenever we cross a "wall" making one of the KN 1-parameter subgroups become $(0,0,0)$.

Case 3: n=1, l=4:

The weights under the torus action are given by the rows of the following matrix:

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Given $\lambda = (x, y, z, w)$, all possible KN 1-parameter subgroups are given by the following list:

$\epsilon(x, y, z)$
$\epsilon(\frac{x+y}{2}, \frac{x+y}{2}, z, w)$
$\epsilon(x, \frac{y+z}{2}, \frac{y+z}{2}, w)$
$\epsilon(x, y, \frac{z+w}{2}, \frac{z+w}{2})$
$\epsilon(\frac{x+w}{2}, y, z, \frac{x+w}{2})$
$\epsilon(0, y, z, w)$
$\epsilon(\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}, w)$
$\epsilon(\frac{x+y}{2}, \frac{x+y}{2}, \frac{z+w}{2}, \frac{z+w}{2})$
$\epsilon(\frac{x+y+w}{3}, \frac{x+y+w}{3}, z, \frac{x+y+w}{3})$
$\epsilon(0, 0, z, w)$
$\epsilon(z, \frac{y+z+w}{3}, \frac{y+z+w}{3}, \frac{y+z+w}{3})$
$\epsilon(\frac{x+w}{2}, \frac{y+z}{2}, \frac{y+z}{2}, \frac{x+w}{2})$
$\epsilon(0, \frac{y+z}{2}, \frac{y+z}{2}, w)$
$\epsilon(\frac{x+z+w}{3}, y, \frac{x+z+w}{3}, \frac{x+z+w}{3})$
$\epsilon(0, y, \frac{z+w}{2}, \frac{x+w}{2})$
$\epsilon(0, y, z, 0)$
$\epsilon(\frac{x+y+z+w}{4}, \frac{x+y+z+w}{4}, \frac{x+y+z+w}{4}, \frac{x+y+z+w}{4})$
$\epsilon(0, 0, 0, w)$
$\epsilon(0, 0, \frac{z+w}{2}, \frac{z+w}{2})$
$\epsilon(0, 0, y, 0)$
$\epsilon(0, \frac{y+z+w}{3}, \frac{y+z+w}{3}, \frac{y+z+w}{3})$
$\epsilon(0, \frac{y+z}{2}, \frac{y+z}{2}, 0)$
$\epsilon(0, y, 0, 0)$

General case: n=1, l arbitrary:

The weights of $\mathrm{Rep}(Q,v)$ under the torus action are given by the rows of the following matrix.

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{l-1} \\ \alpha_l \\ \alpha_{l+1} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots\cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots\cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 \end{bmatrix}$$

Given $\lambda = (x_1, x_2, \cdots, x_l)$ we calculate some of the KN 1-parameter subgroups below. We only calculate the projections for the cases when $\mid I \mid < 5$. We are aware that this is not a complete list, but the purpose in this case is only to find a pattern in the KN 1-PSs.

- When the set I only contains one number other than zero we get:

$$p_0 = \epsilon\lambda$$

$$p_{\{0,i\}} = \epsilon(x_1, \cdots, x_{i-1}, \frac{x_i + x_{i+1}}{2}, \frac{x_i + x_{i+1}}{2}, x_{i+2}, \cdots, x_l) \qquad \text{for } i \in \{1, 2, \ldots, l-1\}$$

$$p_{\{0,l-1\}} = \epsilon(x_1, x_2, \cdots, x_{l-2}, \frac{x_{l-1} + x_l}{2}, \frac{x_{l-1} + x_l}{2})$$

$$p_{\{0,l\}} = \epsilon(\frac{x_1 + x_l}{2}, x_2, \ldots, x_{l-1}, \frac{x_1 + x_l}{2})$$

$$p_{\{0,l+1\}} = \epsilon(0, x_2, x_3, \cdots, x_{l-1}, x_l)$$

- When I contains 2 numbers other than zero:

$$p_{\{0,i,i+1\}} = \epsilon(x_1, x_2, \ldots, x_{i-1}, \frac{x_i + x_{i+1} + x_{i+2}}{3}, \frac{x_i + x_{i+1} + x_{i+2}}{3}, \frac{x_i + x_{i+1} + x_{i+2}}{3}, x_{i+3}, \ldots, x_l)$$

for $i \in \{1, 2, \ldots, l-2\}$

$$p_{\{0,l-1,l\}} = \epsilon(\frac{x_1 + x_{l-1} + x_l}{3}, x_2, x_3, \ldots, x_{l-2}, \frac{x_1 + x_{l-1} + x_l}{3}, \frac{x_1 + x_{l-1} + x_l}{3})$$

$$p_{\{0,1,l\}} = \epsilon(\frac{x_1 + x_2 + x_l}{3}, \frac{x_1 + x_2 + x_l}{3}, x_3, \ldots, x_{l-1}, \frac{x_1 + x_2 + x_l}{3})$$

$$p_{\{0,i,j\}} = \epsilon(x_1, \ldots, x_{i-1}, \frac{x_i + x_{i+1}}{2}, \frac{x_i + x_{i+1}}{2}, x_{i+2}, \ldots, x_{j-1}, \frac{x_j + x_{j+1}}{2}, \frac{x_j + x_{j+1}}{2}, x_{j+2}, \ldots, x_l)$$

for $j \in \{i+2, \ldots, l-1\}$

$$p_{\{0,i,l\}} = \epsilon(\frac{x_1+x_l}{2}, x_2, x_3, \ldots, x_{i-1}, \frac{x_i+x_{i+1}}{2}, \frac{x_i+x_{l+1}}{2}, x_{i+2}, \ldots, x_{l-1}, \frac{x_1+x_l}{2})$$

for $i \in \{2, \ldots, l-2\}$

$$p_{\{0,i,l+1\}} = \epsilon(0, x_2, \ldots, x_{i-1}, \frac{x_i+x_{i+1}}{2}, \frac{x_i+x_{i+1}}{2}, x_{i+1}, \ldots, x_l) \qquad \text{for } i \in \{2,3,\ldots,l-1\}$$

$$p_{\{0,l,l+1\}} = \epsilon(0, x_2, x_3, \ldots, x_{l_1}, 0)$$

$$p_{\{0,1,l+1\}} = \epsilon(0, 0, x_3, x_4, \ldots, x_n)$$

- When I contains 3 numbers other than zero:

$$p_{\{0,i,i+1,i+2\}} = (x_1, \ldots, x_{i-1}, \frac{x_i+x_{i+1}+x_{i+2}+x_{i+3}}{4}, \frac{x_i+x_{i+1}+x_{i+2}+x_{i+3}}{4}, \frac{x_i+x_{i+1}+x_{i+2}+x_{i+3}}{4}, \frac{x_i+x_{i+1}+x_{i+2}+x_{i+3}}{4}, x_{i+4}, \ldots, x_l) \qquad \text{for } i \in \{1, \ldots, l-3\}$$

$$p_{\{0,1,l-1,l\}} = (\frac{x_1+x_2+x_{l-1}+x_l}{4}, \frac{x_1+x_2+x_{l-1}+x_l}{4}, x_2, \ldots, x_{l-2}, \frac{x_1+x_2+x_{l-1}+x_l}{4}, \frac{x_1+x_2+x_{l-1}+x_l}{4})$$

$$p_{\{0,1,2,l\}} = (\frac{x_1+x_2+x_3+x_l}{4}, \frac{x_1+x_2+x_3+x_l}{4}, \frac{x_1+x_2+x_3+x_l}{4}, x_4, \ldots, x_{l-1}, \frac{x_1+x_2+x_3+x_l}{4})$$

$$p_{\{0,l-2,l-1,l\}} = (\frac{x_1+x_{l-2}+x_{l-1}+x_l}{4}, x_2, \ldots, x_{l-3}, \frac{x_1+x_{l-2}+x_{l-1},+x_l}{4}, \frac{x_1+x_{l-2}+x_{l-1}+x_l}{4}, \frac{x_1+x_{l-2}+x_{l-1},+x_l}{4})$$

$$p_{\{0,i,i+1,j\}} = (x_1, \ldots, x_{i-1}, \frac{x_i+x_{i+1}+x_{i+2}}{3}, \frac{x_i+x_{i+1}+x_{i+2}}{3}, \frac{x_i+x_{i+1}+x_{i+2}}{3}, x_{i+3}, \ldots, x_{j-1}, \frac{x_j+x_{j+1}}{2}, \frac{x_j+x_{j+1}}{2}, x_{j+2}, \ldots, x_l) \qquad \text{for } j > i+2$$

$$p_{\{0,j,k,t\}} = (x_1, \ldots, x_{i-1}, \frac{x_i+x_{i+1}}{2}, \frac{x_i+x_{i+1}}{2}, x_{i+2}, \ldots, x_{j-1}, \frac{x_j+x_{j+1}}{2}, \frac{x_j+x_{j+1}}{2}, x_{j+1}, \ldots, x_{k-1}, \frac{x_k+x_{k+1}}{2}, \frac{x_k+x_{k+1}}{2}, x_{k+2}, \ldots, x_l)$$ where neither of $j, k, t < l+1$ are consecutive to each other.

$$p_{\{0,i,i+1,l+1\}} = (0, x_2, \ldots, x_{i-1}, \frac{x_i+x_{i+1}+x_{i+2}}{3}, \frac{x_i+x_{i+1}+x_{i+2}}{3}, \frac{x_i+x_{i+1}+x_{i+2}}{3}, x_{i+3}, \ldots, x_l)$$

where $1 < i < l$.

$$p_{\{0,j,k,l+1\}} = (0, x_2, \ldots, x_{i-1}, \frac{x_i + x_{i+1}}{2}, \frac{x_i + x_{i+1}}{2}, x_{i+2}, \ldots, x_{j-1}, \frac{x_j + x_{j+1}}{2}, \frac{x_j + x_{j+1}}{2}, x_{j+1}, \ldots, x_l)$$

where j, k are not consecutive and $1 < j, k < l$.

$$p_{\{0,i,l,l+1\}} = (0, x_2, \ldots, x_{i-1}, \frac{x_i + x_{i+1}}{2}, \frac{x_i + x_{i+1}}{2}, x_{i+2}, \ldots, x_{l-1}, 0) \qquad \text{where } 1 < i < l-1.$$

$$p_{\{0,1,2,l+1\}} = (0, 0, 0, x_4, \ldots, x_l)$$

$$p_{\{0,1,i,l+1\}} = (0, 0, x_3, \ldots, x_{i-1}, \frac{x_i + x_{i+1}}{2}, \frac{x_i + x_{i+1}}{2}, x_{i+2}, \ldots, x_l) \qquad \text{for } 2 < i < l$$

$$p_{\{0,1,l,l+1\}} = (0, 0, x_3, \ldots, x_{l-1}, 0)$$

$$p_{\{0,l-1,l,l+1\}} = (0, x_2, x_3, \ldots, x_{l-2}, 0, 0)$$

6.4.3 n and l arbitrary: Case for $T^*(\mathbf{Rep}(Q, v))$

Looking at the examples above, we realize that the projections follow a pattern. In this section we give the general formulas for the KN 1-parameter subgroups associated to the stratification of $T^*(\text{Rep}(Q, v))$. Remember that the torus weights are of the form

$$e_{i+jn} - e_{k+(j+1)n},$$
$$e_{i+(l-1)n} - e_s,$$
$$e_s$$

where $1 \leq i, k \leq n$, $0 \leq j \leq l-2$, and $1 \leq s \leq n$. Therefore, there are $n^2 \cdot l + n$ weights.

Let $\lambda = (x_1, \ldots, x_{n \cdot l})$.

1. Case $|I| = 1$: For those weights of the form $e_t - e_q$, the projection is

$$p_I = \frac{x_t + x_q}{2} e_t + \frac{x_t + x_q}{2} e_q + \sum_{i \neq t,q} x_i e_i.$$

   If the weight is of the form e_s, the projection is

$$p_I = \sum_{i \neq s} x_i e_i.$$

2. Case $|I| = 2$: If the weights are of the form $e_{t_1} - e_{q_1}$ and $e_{t_2} - e_{q_2}$ where non of the indices

coincide, then

$$p_I = \sum_{j=t_1,q_1} \frac{x_{t_1}+x_{q_1}}{2} e_j + \sum_{k=t_2,q_2} \frac{x_{t_2}+x_{q_2}}{2} e_k + \sum_{i\neq t_j,q_j} x_i e_i.$$

If one of the indices coincide (let's assume $t_1 = t_2$) then

$$p_I = \sum_{j=t_1,q_1,q_2} \frac{x_{t_1}+x_{q_1}+x_{q_2}}{3} e_j + \sum_{i\neq t_j,q_j} x_i e_i.$$

If the weights are of the form $e_t - e_q$ and e_s where the indices are different, then

$$p_I = \frac{x_t+x_q}{2} e_t + \frac{x_t+x_q}{2} e_q + \sum_{i\neq t,q,s} x_i e_i.$$

If one if the indices coincides (say $t = s$) then

$$p_I = \sum_{i\neq q,s} x_i e_i.$$

Finally, if the weights are e_{s_1} and e_{s_2} then

$$p_I = \sum_{i\neq s_1,s_2} x_i e_i.$$

3. Case $|I| = 3$: If the weights are of the form $e_{t_1} - e_{q_1}$, $e_{t_2} - e_{q_2}$ and $e_{t_3} - e_{q_3}$ where non of the indices coincide, then

$$p_I = \sum_{j=t_1,q_1} \frac{x_{t_1}+x_{q_1}}{2} e_j + \sum_{k=t_2,q_2} \frac{x_{t_2}+x_{q_2}}{2} e_k + \sum_{w=t_3,q_3} \frac{x_{t_3}+x_{q_3}}{2} e_w + \sum_{i\neq t_j,q_j} x_i e_i.$$

If less than 3 indices coincide, for example $q_1 = t_2$ then

$$p_I = \sum_{j=t_1,q_1,q_2} \frac{x_{t_1}+x_{q_1}+t_{q_2}}{3} e_j + \sum_{w=t_3,q_3} \frac{x_{t_3}+x_{q_3}}{2} e_w + \sum_{i\neq t_j,q_j} x_i e_i.$$

If 3 indices coincide, say $q_1 = t_2 = t_3$, then

$$p_I = \sum_{j=t_1,q_1,q_2,q_3} \frac{x_{t_1} + x_{q_1} + t_{q_2} + t_{q_3}}{4} e_j + \sum_{i \neq t_j, q_j} x_i e_i.$$

If the weights are $e_{t_i} - e_{q_i}$ and e_{s_j}, and non of the indices coindice, then

$$p_I = \sum_{\substack{j=t_i q_i \\ i}} \frac{x_{t_i} + x_{q_i}}{2} e_j + \sum_{i \neq t_j, q_j, s_k} x_i e_i.$$

If the weights are of the form e_{s_i} for $1 \leq i \leq 3$, then

$$p_I = \sum_{i \neq s_1, s_2, s_3} x_i e_i.$$

4. <u>Case $|I| = m$</u>: If the weights are of the form $e_{t_1} - e_{q_1}$ for $1 \leq i \leq m$ where non of the indices coincide, then

$$p_I = \sum_{\substack{j=t_i,q_i \\ 1 \leq i \leq m}} \frac{x_{t_i} + x_{q_i}}{2} e_i + \sum_{i \neq t_j, q_j} x_i e_i.$$

If $z < m$ indices coincide, let's assume $t_1 = ... = t_z$. Define $A_z = \{t_1, q_1, \ldots, q_z\}$

$$p_I = \sum_{j \in A_z} \frac{x_{t_1} + x_{q_1} + \cdots + t_{q_z}}{z+1} e_j + \sum_{w=t_i, q_i \notin A_z} \frac{x_{t_i} + x_{q_i}}{2} e_w + \sum_{\substack{i \neq t_j, q_j \\ 1 \leq j \leq m}} x_i e_i.$$

If m indices coincide, define $B = \{t_j : 1 \leq j \leq m, \text{there are no repetitions}\} \cup \{q_j : 1 \leq j \leq m, \text{there are no repetitions}\}$. Then

$$p_I = \sum_{j \in B} \frac{\sum_{k \in B} x_k}{m+1} e_j + \sum_{\substack{i \neq t_j, q_j \\ 1 \leq j \leq m}} x_i e_i.$$

$$p_I = \sum_{j \in B} \frac{\sum_{k \in B} x_k}{m+1} e_j + \sum_{\substack{i \neq t_j, q_j \\ 1 \leq j \leq m}} x_i e_i.$$

If the weights are $e_{t_1} - e_{q_1}$ for $1 \leq i \leq v$ and $e_{s_1} \dots e_{s_{m-v}}$, and non of the indices coindice, then

$$p_I = \sum_{j=t_1,q_1} \frac{x_{t_1} + x_{q_1}}{2} e_j + \sum_{k=t_2,q_2} \frac{x_{t_2} + x_{q_2}}{2} e_k + \cdots + \sum_{w=t_v,q_v} \frac{x_{t_v} + x_{q_v}}{2} e_w + \sum_{\substack{i \neq t_j, q_j, s_k \\ 1 \leq j \leq v}} x_i e_i.$$

If $z < m$ indices coincide, let's assume $t_1 = ... = t_z$. Define A_z as before, then

$$p_I = \sum_{j \in A_z} \frac{x_{t_1} + x_{q_1} + \cdots + t_{q_z}}{z+1} e_j + \sum_{w=t_i,q_i \notin A_z} \frac{x_{t_i} + x_{q_i}}{2} e_w + \sum_{\substack{i \neq t_j, q_j, s_k \\ 1 \leq j \leq m \\ 1 \leq k \leq m-v}} x_i e_i.$$

If $t_1 = \cdots = t_s$ and $t_{s+1} = \cdots = t_{z-1} = s_1$ then

$$p_I = \sum_{j \in A_s} \frac{x_{t_1} + x_{q_1} + \cdots + t_{q_s}}{s+1} e_j + \sum_{\substack{w=t_i,q_i \notin A_s \\ \text{and} \\ i \notin \{s+1,\dots,z-1\}}} \frac{x_{t_i} + x_{q_i}}{2} e_w + \sum_{\substack{i \neq t_j, q_j, s_k \\ 1 \leq j \leq m \\ 1 \leq k \leq m-v}} x_i e_i.$$

If the weights are of the form e_{s_i} for $1 \leq i \leq m$, then

$$p_I = \sum_{i \neq s_1, \dots, s_m} x_i e_i.$$

Example 6.5. Let n and l be arbitrary and $\lambda = \sum_{i=1}^{n \cdot l} e_i$. Then the KN 1-parameter subgroups are given by

$$\sum_{i \in I} e_i$$

where $I \subseteq \{1, 2, \dots, n \cdot l\}$ and $\{1, 2, \dots, n\} \nsubseteq I$.

References

[1] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, Third, ser. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]. Springer-Verlag, Berlin, 1994, vol. 34.

[2] M. Thaddeus, "Geometric invariant theory and flips", *J. Amer. Math. Soc.*, vol. 9, no. 3, pp. 691–723, 1996.

[3] I. V. Dolgachev and Y. Hu, "Variation of geometric invariant theory quotients", *Inst. Hautes Études Sci. Publ. Math.*, no. 87, pp. 5–56, 1998, With an appendix by Nicolas Ressayre.

[4] G. R. Kempf, "Instability in invariant theory", *Annals of Mathematics*, vol. 108, no. 2, pp. 299–316, 1978.

[5] F. C. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry.* Princeton University Press, 1984, vol. 31.

[6] L. Ness, "A stratification of the null cone via the moment map", *Amer. J. Math.*, vol. 106, no. 6, pp. 1281–1329, 1984, With an appendix by David Mumford.

[7] W. H. Hesselink, "Uniform instability in reductive groups", *J. Reine Angew. Math.*, vol. 303/304, pp. 74–96, 1978.

[8] K. McGerty and T. Nevins, "Compatibility of t-structures for quantum symplectic resolutions", *Duke Mathematical Journal*, vol. 165, no. 13, pp. 2529–2585, 2016.

[9] G. Kempf and L. Ness, "The length of vectors in representation spaces", in *Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978)*, ser. Lecture Notes in Math. Vol. 732, Springer, Berlin, 1979, pp. 233–243.

[10] H. Nakajima, "Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras", *Duke Mathematical Journal*, vol. 76, no. 2, pp. 365–416, 1994.

[11] ——, "Quiver varieties and Kac-Moody algebras", *Duke Mathematical Journal*, vol. 91, no. 3, pp. 515–560, 1998.

[12] ——, *Lectures on Hilbert schemes of points on surfaces*, ser. University Lecture Series. American Mathematical Society, Providence, RI, 1999, vol. 18.

[13] R. C. J. Jenkins, "Representations of rational cherednik algebras: Koszulness and localisation", PhD thesis, 2013. [Online]. Available: `http://www.maths.ed.ac.uk/sites/default/files/atoms/files/jenkins.pdf`.

[14] B. Kalantari, "A characterization theorem and an algorithm for a convex hull problem", *Annals of Operations Research*, vol. 226, no. 1, pp. 301–349, 2015.

[15] P. E. Newstead, "Introduction to moduli problems and orbit spaces, volume 51 of tata institute of fundamental research lectures on mathematics and physics", *Tata Institute of Fundamental Research, Bombay*, 1978.

[16] T. J. Hodges, "Noncommutative deformations of type-A Kleinian singularities", *Journal of Algebra*, vol. 161, no. 2, pp. 271–290, 1993.

[17] C. Teleman, "The quantization conjecture revisited", *Annals of Mathematics*, vol. 152, no. 1, pp. 1–43, 2000.

[18] C. T. Woodward, "Moment maps and geometric invariant theory", *arXiv preprint arXiv:0912.1132*, 2009.

[19] A. D. King, "Moduli of representations of finite dimensional algebras", *The Quarterly Journal of Mathematics*, vol. 45, no. 4, pp. 515–530, 1994.

[20] M. F. Atiyah and R. Bott, "The yang-mills equations over riemann surfaces", *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, vol. 308, no. 1505, pp. 523–615, 1983.

Appendix A

Calculations for Chapter 6

Case for $X = Rep(Q, v)$:

λ	β	$\beta \cdot \alpha$	Z_β	v	$v \cdot \alpha$	$< \lambda, v >$	Z_β^{ss}	Y_β	Y_β^{ss}
(-1, -1/2)	(1,1)	(0,0,1)	$\{(x, y, 0)\}$	(-1,1)	(-2,2,-1)	1/2	$\{(x, y, 0) \mid y \neq 0\}$	$\{(x, y, z)\}$	$\{(x, y, z) \mid y \neq 0\}$
	(2,1)	(1,-1,2)	$\{(0, 0, 0)\}$	(-1,2)	(-3,3,-1)	2	$\{(0, 0, 0)\}$	$\{(x, 0, z)\}$	$\{(x, 0, z)\}$
(-1, -1)	(1,1)	(0,0,1)	$\{(x, y, 0)\}$	(-1,1)	(-2,2,-1)	0	$\{(x, y, 0)\}$	$\{(x, y, z)\}$	$\{(x, y, z)\}$
(-1, -3/2)	(1,1)	(0,0,1)	$\{(x, y, 0)\}$	(-1,1)	(-2,2,-1)	-1/2	$\{(x, y, 0) \mid x \neq 0\}$	$\{(x, y, z)\}$	$\{(x, y, z) \mid x \neq 0\}$
	(2,3)	(-1,1,2)	$\{(0, 0, 0)\}$	(-3,2)	(-5,5,-3)	0	$\{(0, 0, 0)\}$	$\{(0, y, z)\}$	$\{(0, y, z)\}$
(0, -1)	(1,1)	(0,0,1)	$\{(x, y, 0)\}$	(-1,1)	(-2,2,-1)	-1	$\{(x, y, 0) \mid x \neq 0\}$	$\{(x, y, z)\}$	$\{(x, y, z) \mid x \neq 0\}$
	(0,1)	(-1,1,0)	$\{(0, 0, z)\}$	(-1,0)	(-1,1,-1)	0	$\{(0, 0, z)\}$	$\{(0, y, z)\}$	$\{(0, y, z)\}$
(1, -2)	(1,1)	(0,0,1)	$\{(x, y, 0)\}$	(-1,1)	(-2,2,-1)	-3	$\{(x, y, 0) \mid x \neq 0\}$	$\{(x, y, z)\}$	$\{(x, y, z) \mid x \neq 0\}$
	(-1,2)	(-3,3,-1)	$\{(0, 0, 0)\}$	(2,1)	(1,-1,2)	0	$\{(0, 0, 0)\}$	$\{(0, y, 0)\}$	$\{(0, y, 0)\}$
	(0,1)	(-1,1,0)	$\{(0, 0, z)\}$	(1,0)	(1,-1,1)	1	$\{(0, 0, z) \mid z \neq 0\}$	$\{(0, y, z)\}$	$\{(0, y, z) \mid z \neq 0\}$

λ	β	$\beta \cdot \alpha$	Z_β	v	$v \cdot \alpha$	$< \lambda, v >$	Z_β^{ss}	Y_β	Y_β^{ss}
(1, -1)	(0,0)	(0,0,0)	$\{(x,y,z)\}$	(0,0)	(0,0,0)	0	X^{ss}	$\{(x,y,z)\}$	X^{ss}
	(-1,1)	(-2,2,-1)	$\{(0,0,0)\}$	(1,1)	(0,0,1)	0	$\{(0,0,0)\}$	$\{(0,y,0)\}$	$\{(0,y,0)\}$
	(0,1)	(-1,1,0)	$\{(0,0,z)\}$	(1,0)	(1,-1,1)	1	$\{(0,0,z) \mid z \neq 0\}$	$\{(0,y,z)\}$	$\{(0,y,z) \mid z \neq 0\}$
(1, -1/2)	(-1,-1)	(0,0,-1)	$\{(x,y,0)\}$	(1,-1)	(2,-2,1)	3/2	$\{(x,y,0) \mid x \neq 0\}$	$\{(x,y,0)\}$	$\{(x,y,0) \mid x \neq 0\}$
	(-2,1)	(-3,3,-2)	$\{(0,0,0)\}$	(1,2)	(-1,1,1)	0	$\{(0,0,0)\}$	$\{(0,y,0)\}$	$\{(0,y,0)\}$
	(0,1)	(-1,1,0)	$\{(0,0,z)\}$	(1,0)	(1,-1,1)	1	$\{(0,0,z) \mid z \neq 0\}$	$\{(0,y,z)\}$	$\{(0,y,z) \mid z \neq 0\}$
	(0,0)	(0,0,0)	$\{(x,y,z)\}$	(0,0)	(0,0,0)	0	X^{ss}	$\{(x,y,z)\}$	X^{ss}
(1, 0)	(-1,-1)	(0,0,-1)	$\{(x,y,0)\}$	(1,-1)	(2,-2,1)	1	$\{(x,y,0) \mid x \neq 0\}$	$\{(x,y,0)\}$	$\{(x,y,0) \mid x \neq 0\}$
	(-1,0)	(-1,1,-1)	$\{(0,0,0)\}$	(0,1)	(-1,1,0)	0	$\{(0,0,0)\}$	$\{(0,y,0)\}$	$\{(0,y,0)\}$
	(0,0)	(0,0,0)	$\{(x,y,z)\}$	(0,0)	(0,0,0)	0	X^{ss}	$\{(x,y,z)\}$	X^{ss}
(1, 1/2)	(-1,-1)	(0,0,-1)	$\{(x,y,0)\}$	(1,-1)	(2,-2,1)	1/2	$\{(x,y,0) \mid x \neq 0\}$	$\{(x,y,0)\}$	$\{(x,y,0) \mid x \neq 0\}$
	(-2,-1)	(-1,1,-2)	$\{(0,0,0)\}$	(1,-2)	(3,-3,1)	0	$\{(0,0,0)\}$	$\{(0,y,0)\}$	$\{(0,y,0)\}$
	(0,-1)	(1,-1,0)	$\{(0,0,z)\}$	(1,0)	(1,-1,1)	1	$\{(0,0,z) \mid z \neq 0\}$	$\{(x,0,z)\}$	$\{(x,0,z) \mid z \neq 0\}$
	(0,0)	(0,0,0)	$\{(x,y,z)\}$	(0,0)	(0,0,0)	0	X^{ss}	$\{(x,y,z)\}$	X^{ss}
(1, 1)	(-1,-1)	(0,0,-1)	$\{(x,y,0)\}$	(1,-1)	(2,-2,1)	0	$\{(x,y,0)\}$	$\{(x,y,0)\}$	$\{(x,y,0)0\}$
	(0,-1)	(1,-1,0)	$\{(0,0,z)\}$	(1,0)	(1,-1,1)	1	$\{(0,0,z) \mid z \neq 0\}$	$\{(x,0,z)\}$	$\{(x,0,z) \mid z \neq 0\}$
	(0,0)	(0,0,0)	$\{(x,y,z)\}$	(0,0)	(0,0,0)	0	X^{ss}	$\{(x,y,z)\}$	X^{ss}

λ	β	$\beta \cdot \alpha$	Z_β	v	$v \cdot \alpha$	$< \lambda, v >$	Z_β^{ss}	Y_β	Y_β^{ss}
(1, 3/2)	(-1,-1)	(0,0,-1)	$\{(x,y,0)\}$	(1,-1)	(2,-2,1)	-1/2	$\{(x,y,0) \mid y \neq 0\}$	$\{(x,y,0)\}$	$\{(x,y,0) \mid y \neq 0\}$
	(-2,-3)	(1,-1,-2)	$\{(0,0,0)\}$	(3,-2)	(5,-5,3)	0	$\{(0,0,0)\}$	$\{(x,0,0)\}$	$\{(x,0,0)\}$
	(0,-1)	(1,-1,0)	$\{(0,0,z)\}$	(1,0)	(1,-1,1)	1	$\{(0,0,z) \mid z \neq 0\}$	$\{(x,0,z)\}$	$\{(x,0,z) \mid z \neq 0\}$
	(0,0)	(0,0,0)	$\{(x,y,z)\}$	(0,0)	(0,0,0)	0	X^{ss}	$\{(x,y,z)\}$	X^{ss}
(0, 1)	(-1,-1)	(0,0,-1)	$\{(x,y,0)\}$	(1,-1)	(2,-2,1)	-1	$\{(x,y,0) \mid y \neq 0\}$	$\{(x,y,0)\}$	$\{(x,y,0) \mid y \neq 0\}$
	(0,-1)	(1,-1,0)	$\{(0,0,z)\}$	(1,0)	(1,-1,1)	0	$\{(0,0,z)\}$	$\{(x,0,z)\}$	$\{(x,0,z)\}$
	(0,0)	(0,0,0)	$\{(x,y,z)\}$	(0,0)	(0,0,0)	0	X^{ss}	$\{(x,y,z)\}$	X^{ss}
(-1, 2)	(-1,-1)	(0,0,-1)	$\{(x,y,0)\}$	(1,-1)	(2,-2,1)	-3	$\{(x,y,0) \mid y \neq 0\}$	$\{(x,y,0)\}$	$\{(x,y,0) \mid y \neq 0\}$
	(1,-2)	(3,-3,1)	$\{(0,0,0)\}$	(2,1)	(1,-1,2)	0	$\{(0,0,0)\}$	$\{(x,0,z)\}$	$\{(x,0,z)\}$
	(0,0)	(0,0,0)	$\{(x,y,z)\}$	(0,0)	(0,0,0)	0	X^{ss}	$\{(x,y,z)\}$	X^{ss}
(-1, 1)	(0,0)	(0,0,0)	$\{(x,y,z)\}$	(0,0)	(0,0,0)	0	X^{ss}	$\{(x,y,z)\}$	X^{ss}
	1,-1)	(2,-2,-1)	$\{(0,0,0)\}$	(1,1)	(0,0,1)	0	$\{(0,0,0)\}$	$\{(x,0,z)\}$	$\{(x,0,z)\}$
(-1, 1/2)	(1,1)	(0,0,1)	$\{(x,y,0)\}$	(-1,1)	(-2,2,-1)	3/2	$\{(x,y,0) \mid y \neq 0\}$	$\{(x,y,z)\}$	$\{(x,y,z) \mid y \neq 0\}$
	(2,-1)	(3,-3,2)	$\{(0,0,0)\}$	(1,2)	(-1,1,1)	0	$\{(0,0,0)\}$	$\{(x,0,z)\}$	$\{(x,0,z)\}$
(-1, 0)	(1,1)	(0,0,1)	$\{(x,y,0)\}$	(-1,1)	(-2,2,-1)	1	$\{(x,y,0) \mid y \neq 0\}$	$\{(x,y,z)\}$	$\{(x,y,z) \mid y \neq 0\}$
	(1,0)	(1,-1,1)	$\{(0,0,0)\}$	(0,1)	(-1,1,0)	0	$\{(0,0,0)\}$	$\{(x,0,z)\}$	$\{(x,0,z)\}$

Case for $T^*Rep(Q,v)$:

λ	β	Z_β	Z_β^{ss}	Y_β	Y_β^{ss}
Chamber 1	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0)\|y\neq 0\neq z\}$	$\{(x,y,z,w,t,0)\}$	$\{(x,y,z,w,t,0)\|y\neq 0\neq z\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,0,0,0,0,0)\}$	$\{(x,0,0,w,t,0)\}$	$\{(x,0,0,w,t,0)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s)\|s\neq 0\}$	$\{(0,y,z,0,t,s)\}$	$\{(0,y,z,0,t,s)\|s\neq 0)\}$
Wall 1	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,t,0)\}$	$\{(x,y,z,w,t,0)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s)\|s\neq 0\}$	$\{(0,y,z,0,t,s)\}$	$\{(0,y,z,0,t,s)\|s\neq 0)\}$
Chamber 2	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0)\|x\neq 0\neq w\}$	$\{(x,y,z,w,t,0)\}$	$\{(x,y,z,w,t,0)\|x\neq 0\neq w\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,0,0,0,0,0)\}$	$\{(0,y,z,0,t,0)\}$	$\{(0,y,z,0,t,0)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s)\|s\neq 0\}$	$\{(0,y,z,0,t,s)\}$	$\{(0,y,z,0,t,s)\|s\neq 0)\}$
Wall 2	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0)\|x\neq 0\neq w\}$	$\{(x,y,z,w,t,0)\}$	$\{(x,y,z,w,t,0)\|x\neq 0\neq w\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s)\}$	$\{(0,y,z,0,t,s)\}$	$\{(0,y,z,0,t,s)\}$
Chamber 3	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0)\|x\neq 0\neq w\}$	$\{(x,y,z,w,t,0)\}$	$\{(x,y,z,w,t,0)\|x\neq 0\neq w\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,0,0,0,0,0)\}$	$\{(0,y,z,0,0,s)\}$	$\{(0,y,z,0,0,s)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,0)\}$	$\{(0,0,0,0,t,0)\|t\neq 0\}$	$\{(0,y,z,0,t,0)\}$	$\{(0,y,z,0,t,s)\|t\neq 0\}$
Wall 3	(0,0)	$\{(x,y,z,w,t,s)\}$	$(T^*X)^{ss}$	T^*X	$(T^*X)^{ss}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,0,0,0,0,0)\}$	$\{(0,y,z,0,0,s)\}$	$\{(0,y,z,0,0,s)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s)\|t\neq 0\}$	$\{(0,y,z,0,t,s)\}$	$\{(0,y,z,0,t,s)\|t\neq 0\}$

λ	β	Z_β	Z_β^{ss}	Y_β	Y_β^{ss}
Chamber 4	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0) \vert x\neq 0\neq w\}$	$\{(x,y,z,w,0,s)\}$	$\{(x,y,z,w,0,s) \vert x\neq 0\neq w\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,0,0,0,0,0)\}$	$\{(0,y,z,0,0,s)\}$	$\{(0,y,z,0,0,s)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s) \vert t\neq 0\}$	$\{(0,y,z,0,t,s)\}$	$\{(0,y,z,0,t,s) \vert t\neq 0\}$
	(0,0)	$\{(x,y,z,w,t,s)\}$	$(T^*X)^{ss}$	T^*X	$(T^*X)^{ss}$
Wall 4	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0) \vert x\neq 0\neq w\}$	$\{(x,y,z,w,0,s)\}$	$\{(x,y,z,w,0,s) \vert x\neq 0\neq w\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,0,0,0,0,0)\}$	$\{(0,y,z,0,0,s)\}$	$\{(0,y,z,0,0,s)\}$
	(0,0)	$\{(x,y,z,w,t,s)\}$	$(T^*X)^{ss}$	T^*X	$(T^*X)^{ss}$
Chamber 5	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0) \vert x\neq 0\neq w\}$	$\{(x,y,z,w,0,s)\}$	$\{(x,y,z,w,0,s) \vert x\neq 0\neq w\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,0,0,0,0,0)\}$	$\{(0,y,z,0,0,s)\}$	$\{(0,y,z,0,0,s)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s) \vert t\neq 0\}$	$\{(x,0,0,w,t,s)\}$	$\{(x,0,0,w,t,s) \vert t\neq 0\}$
	(0,0)	$\{(x,y,z,w,t,s)\}$	$(T^*X)^{ss}$	T^*X	$(T^*X)^{ss}$
Wall 5	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,s)\}$	$\{(x,y,z,w,0,s)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s) \vert t\neq 0\}$	$\{(x,0,0,w,t,s)\}$	$\{(x,0,0,w,t,s) \vert t\neq 0\}$
	(0,0)	$\{(x,y,z,w,t,s)\}$	$(T^*X)^{ss}$	T^*X	$(T^*X)^{ss}$
Chamber 6	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0) \vert y\neq 0\neq z\}$	$\{(x,y,z,w,0,s)\}$	$\{(x,y,z,w,0,s) \vert y\neq 0\neq z\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,0,0,0,0,0)\}$	$\{(x,0,0,w,0,s)\}$	$\{(x,0,0,w,0,s)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s) \vert t\neq 0\}$	$\{(x,0,0,w,t,s)\}$	$\{(x,0,0,w,t,s) \vert t\neq 0\}$
	(0,0)	$\{(x,y,z,w,t,s)\}$	$(T^*X)^{ss}$	T^*X	$(T^*X)^{ss}$
Wall 6	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0) \vert y\neq 0\neq z\}$	$\{(x,y,z,w,0,s)\}$	$\{(x,y,z,w,0,s) \vert y\neq 0\neq z\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s)\}$	$\{(x,0,0,w,t,s)\}$	$\{(x,0,0,w,t,s)\}$
	(0,0)	$\{(x,y,z,w,t,s)\}$	$(T^*X)^{ss}$	T^*X	$(T^*X)^{ss}$

λ	β	Z_β	Z_β^{ss}	Y_β	Y_β^{ss}
Chamber 7	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0)\|y\neq 0\neq z\}$	$\{(x,y,z,w,0,s)\}$	$\{(x,y,z,w,0,s)\|y\neq 0\neq z\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,0,0,0,0,0)\}$	$\{(x,0,0,w,t,0)\}$	$\{(x,0,0,w,t,0)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s)\|s\neq 0\}$	$\{(x,0,0,w,t,s)\}$	$\{(x,0,0,w,t,s)\|s\neq 0\}$
	(0,0)	$\{(x,y,z,w,t,s)\}$	$(T^*X)^{ss}$	T^*X	$(T^*X)^{ss}$
Wall 7	(0,0)	$\{(x,y,z,w,t,s)\}$	$(T^*X)^{ss}$	T^*X	$(T^*X)^{ss}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,0,0,0,0,0)\}$	$\{(x,0,0,w,t,0)\}$	$\{(x,0,0,w,t,0)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s)\|s\neq 0\}$	$\{(x,0,0,w,t,s)\}$	$\{(x,0,0,w,t,s)\|s\neq 0\}$
Chamber 8	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0)\|y\neq 0\neq z\}$	$\{(x,y,z,w,t,0)\}$	$\{(x,y,z,w,t,0)\|y\neq 0\neq z\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,0,0,0,0,0)\}$	$\{(x,0,0,w,t,0)\}$	$\{(x,0,0,w,t,0)\}$
	$\epsilon(0,\lambda_2)$	$\{(0,0,0,0,t,s)\}$	$\{(0,0,0,0,t,s)\|s\neq 0\}$	$\{(x,0,0,w,t,s)\}$	$\{(x,0,0,w,t,s)\|s\neq 0\}$
Wall 8	$\epsilon\frac{\lambda_1+\lambda_2}{2}(1,1)$	$\{(x,y,z,w,0,0)\}$	$\{(x,y,z,w,0,0)\|y\neq 0\neq z\}$	$\{(x,y,z,w,t,0)\}$	$\{(x,y,z,w,t,0)\|y\neq 0\neq z\}$
	$\epsilon(\lambda_1,\lambda_2)$	$\{(0,0,0,0,0,0)\}$	$\{(0,0,0,0,0,0)\}$	$\{(x,0,0,w,t,0)\}$	$\{(x,0,0,w,t,0)\}$