PRISM TABLEAUX AND ALTERNATING SIGN MATRICES

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2018

Urbana, Illinois

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ABSTRACT

A. Lascoux and M.-P. Schützenberger introduced Schubert polynomials to study the cohomology ring of the complete flag variety $\mathcal{F}l(\mathbb{C}^n)$. Each Schubert polynomial corresponds to the class defined by a Schubert variety $\mathfrak{X}_w \subseteq \mathcal{F}l(\mathbb{C}^n)$. Schubert polynomials are indexed by elements of the symmetric group and form a basis of the ring $\mathbb{Z}[x_1, x_2, \ldots]$. The expansion of the product of two Schubert polynomials in the Schubert basis has been of particular interest. The structure coefficients are known to be nonnegative integers. As of yet, there are only combinatorial formulas for these coefficients in special cases, such as the Littlewood-Richardson rule for multiplying Schur polynomials.

Schur polynomials form a basis of the ring of symmetric polynomials. They have a combinatorial formula as a weighted sum over semistandard tableaux. In joint work with A. Yong, the author introduced prism tableaux. A prism tableau consists of a tuple of tableaux, positioned within an ambient grid. With A. Yong, the author gave a formula for Schubert polynomials using prism tableaux. We continue the study of prism tableaux, detailing their connection to the poset of alternating sign matrices (ASMs).

Schubert polynomials can be interpreted as multidegrees of the matrix Schubert varieties of Fulton. We study a more general class of determinantal varieties, indexed by ASMs. More generally, one can consider subvarieties of the space of $n \times n$ matrices cut out by imposing rank conditions on maximal northwest submatrices. We show that, up to an affine factor, such a variety is isomorphic to an ASM variety. The multidegrees of ASM varieties can be expressed as a sum over prism tableaux.

In joint work with A. Yong and R. Rimányi, the author studies representations of quivers and their connection to the dilogarithm identities of M. Reineke. We give a bijective proof to establish an identity of generating series. This bijection uses a generalization of Durfee squares. From this identity, we give a new proof of M. Reineke's identities in type A.

To my parents.

ACKNOWLEDGMENTS

I am extremely indebted to my advisor, Alexander Yong. I appreciate the time and energy he invested into my education. His support, advice, and encouragement have been invaluable. I thank Philippe Di Francesco, Rinat Kedem, and Bruce Reznick for serving on my thesis committee. I am grateful to have had the opportunity to learn from each of them during my time at Illinois. Parts of this thesis are based on collaborations with Richárd Rimányi and Alexander Yong. I thank them both for illuminating conversations and exchanges.

I thank the Mathematics Department at UIUC for supporting my studies and for providing generous travel support. This work was partially supported by the Ruth V. Shaff and Genevie I. Andrews Fellowship, UIUC Campus Research Board RB15060, and NSF DMS Grant 1500691. Part of this thesis was completed during the trimester "Combinatorics and Interactions" hosted at the Institut Henri Poincaré. Support for travel to IHP and lodging was provided by NSF Conference Grant 1643027.

To my fellow mathematics graduate students, thanks for the company, coffee runs, and cookouts. In particular, I thank Melinda, Claire, Matej, Sarah, and Nick for their friendship. I am grateful for my mathematical siblings, Dominic, Oliver, Cara, Colleen, and Harshit, for many helpful conversations. I owe an extra debt of gratitude to my mathematical twin, Cara, for the countless hours she spent driving me to conferences.

I thank Leonardo for being my muse. Finally, I am immensely grateful for my family and their constant help and encouragement. Without them, this work would not have been possible.

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CHAPTER 1

INTRODUCTION

1.1 The Flag Variety

Fix an n-dimensional vector space V over $\mathbb{C}$. A **complete flag** is a nested sequence of subspaces of V

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V \text{ so that } \dim(V_i) = i \text{ for all } i = 1, \ldots, n. \tag{1}$$

The **complete flag variety** $\mathcal{F}l(V)$ is the set of complete flags in V.

We follow [KM2005] for conventions on flag varieties. Fix an ordered basis and identify elements of V with row vectors in $\mathbb{C}^n$. Let $\mathsf{GL}(n)$ be the **general linear group** of $n \times n$ invertible matrices over $\mathbb{C}$. An element $M \in \mathsf{GL}(n)$ defines a complete flag as follows. Let v_i be the ith row of M and let $V_i = \langle v_1, \ldots, v_i \rangle$ be the span of the first i rows of M. Since M is invertible,

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n$$

is a complete flag. Replacing v_i with cv_i (with $c \neq 0$) does not change the resulting flag. Nor does replacing v_j with $v_j + cv_i$ whenever $i < j$. In other words, two matrices define the same flag whenever they are related by a sequence of downwards sweeping row operations.

We may interpret this in the language of homogeneous spaces. Let $\mathsf{B}_-, \mathsf{B}_+ \subseteq \mathsf{GL}(n)$ be the subgroups of lower and upper triangular matrices respectively. Lower triangular matrices act on $\mathsf{GL}(n)$ by left multiplication. This action corresponds to downward sweeping row operations. As such, we may identify $\mathcal{F}l(\mathbb{C}^n)$ with the quotient $\mathsf{B}_- \backslash \mathsf{GL}(n)$. The group B_+ acts on $\mathsf{GL}(n)$ by right multiplication. This action descends to the quotient $\mathcal{F}l(\mathbb{C}) = \mathsf{B}_- \backslash \mathsf{GL}(n)$. The B_+ orbits on $\mathcal{F}l(\mathbb{C})$ are called **Schubert cells**.

The **symmetric group** $\mathcal{S}_n$ is the set of bijections from $\{1, 2, \ldots, n\}$ to itself. Each permutation $w \in \mathcal{S}_n$ defines a flag as follows. Write e_i for the ith standard basis vector. Then

$$E_i^{(w)} := \langle e_{w(1)}, e_{w(2)}, \ldots, e_{w(i)} \rangle$$

and we define $E^{(w)}$ to be the flag

$$E_1^{(w)} \subseteq E_2^{(w)} \subseteq \cdots \subseteq E_n^{(w)}.$$

For each Schubert cell Ω, there is a unique $w \in \mathcal{S}_n$ so that $E^{(w)} \in \Omega$. Write Ω_w for this cell.

The flag variety decomposes as a disjoint union of Schubert cells

$$\mathcal{F}l(\mathbb{C}^n) = \bigcup_{w \in \mathcal{S}_n} \Omega_w. \tag{2}$$

This is called the **Bruhat decomposition** of the flag variety. The closure $\mathfrak{X}_w := \overline{\Omega_w}$ is called a **Schubert variety**. Each Schubert variety defines a class $\sigma_w = [\mathfrak{X}_w]$ in the cohomology ring $\mathsf{H}^*(\mathcal{F}l(\mathbb{C}^n), \mathbb{Z})$. Furthermore, $\{\sigma_w : w \in \mathcal{S}_n\}$ is a $\mathbb{Z}$-linear basis for $\mathsf{H}^*(\mathcal{F}l(\mathbb{C}^n), \mathbb{Z})$. As such,

$$\sigma_u \cup \sigma_v = \sum_w C_{u,v}^w \sigma_w \text{ with } C_{u,v}^w \in \mathbb{Z}. \tag{3}$$

The structure coefficient $C_{u,v}^w$ counts the number of points in the intersection of three generically translated Schubert varieties. Therefore, $C_{u,v}^w$ is a nonnegative integer. A central open problem in Schubert calculus is to give a combinatorial rule for $C_{u,v}^w$. This has been achieved in special cases, such as the Littlewood-Richardson rule [LR1934].

The expansion in (3) can be reformulated in an algebraic setting via the Borel isomorphism:

$$\mathsf{H}^*(\mathcal{F}l(\mathbb{C}^n), \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, \ldots, x_n]/I^{\mathcal{S}_n} \tag{4}$$

where $I^{\mathcal{S}_n}$ is the ideal generated by the symmetric polynomials with no constant term. A. Lascoux and M.-P. Schützenberger introduced *Schubert polynomials* to study this isomorphism [LS1982]. Schubert polynomials $\{\mathfrak{S}_w : w \in \mathcal{S}_n\}$ have a recursive definition using *divided difference operators*. See Section 2.4 for details. The explicit isomorphism in (4) is given by mapping σ_w to the coset in $\mathbb{Z}[x_1, x_2, \ldots, x_n]/I^{\mathcal{S}_n}$ which contains $\mathfrak{S}_w$.

Schubert polynomials have nonnegative, integer coefficients. Numerous combinatorial explanations for this positivity have been studied. For instance, there are *Kohnert diagrams* (see [Koh1990] and [Ass2017]), *compatible sequences* [BJS1993], and *balanced tableaux* [FGRS1997], among many others. We will recall the *pipe dream formula*, which has its origin in the *pseudo-line configurations* of [FK1996] and was further studied in [BB1993] and [KM2005]. In Chapter 3, we use pipe dreams to resolve a conjecture of R. Stanley regarding two term Schubert polynomials. We show that $\mathfrak{S}_w(1, 1, \ldots) = 2$ if and only if w contains the pattern 132 exactly once.

If $v \in \mathcal{S}_n$ is a *Grassmannian* permutation, then $\mathfrak{S}_v$ is a *Schur polynomial*. Schur polynomials form a basis for the ring of symmetric polynomials. They have been studied through the development of a theory of *tableau combinatorics* (see [Man2001] and [Ful1997]). Each Schur polynomial is a generating series over a set of *semistandard tableaux*. See Section 2.3 for this definition.

In joint work with A. Yong, the author defined *prism tableaux*, a generalization of semistandard tableaux [WY2018]. The authors used prism tableaux to provide a new combinatorial formula for $\mathfrak{S}_w$. In [Wei2017a], the author studied a more general class of prism tableaux and detailed their connection to *alternating sign matrices*. The main results from this direction of research will be summarized in the next section.

1.2 Prism Tableaux and Alternating Sign Matrices

An **alternating sign matrix**[1] (ASM) is a square matrix with entries in $\{-1, 0, 1\}$ so that

(A1) the nonzero entries in each row and column alternate in sign and

(A2) each row and column sums to 1.

Let $\mathsf{ASM}(n)$ be the set of all $n \times n$ ASMs. The enumeration of ASMs has drawn much interest, the sequence for $n \geq 1$ being

$$1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, \ldots . \tag{5}$$

There is a closed form expression for (5). The celebrated *alternating sign matrix conjecture* of W. H. Mills–D. P. Robbins–H. Rumsey [MRR1983] asserts that

$$\#\mathsf{ASM}(n) = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

The original proof was given by D. Zeilberger [Zei1996]. A second proof was given by G. Kuperberg [Kup1996] using the six-vertex model of statistical mechanics. See *Proofs and Confirmations: The Story of the Alternating-Sign Matrix Conjecture*, by D. Bressoud, for the link between ASMs and hypergeometric series, plane partitions, and lattice paths [Bre1999].

Each $A = (a_{ij})_{i,j=1}^{n} \in \mathsf{ASM}(n)$ has an associated **corner sum function**

$$r_A(i, j) = \sum_{k=1}^{i} \sum_{\ell=1}^{j} a_{k\ell}. \tag{6}$$

[1] The text in this section first appeared in work of the author [Wei2017a, Section 1].

Corner sum functions define a lattice structure on $\mathsf{ASM}(n)$; say

$$A \leq B \text{ if and only if } r_A(i,j) \geq r_B(i,j) \text{ for all } 1 \leq i,j \leq n. \tag{7}$$

Restricted to permutation matrices, (7) is the **Bruhat order** on the symmetric group $\mathcal{S}_n$. A. Lascoux and M.-P. Schützenberger showed that $\mathsf{ASM}(n)$ is the smallest lattice which contains $\mathcal{S}_n$ as an order embedding [LS1996, Lemme 5.4].

Fix tuples of partitions and positive integers

$$\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(k)}) \text{ and } \mathbf{d} = (d_1, \ldots, d_k) \text{ so that } d_i \geq \ell(\lambda^{(i)}) \text{ for all } i. \tag{8}$$

Here $\ell(\lambda)$ denotes the *length* of λ (see Section 2.2). We associate to each $(\boldsymbol{\lambda}, \mathbf{d})$ an ASM, denoted $A_{\boldsymbol{\lambda},\mathbf{d}}$, which is the least upper bound of a list of *Grassmannian* permutations. For any ASM, there exists some $(\boldsymbol{\lambda}, \mathbf{d})$ so that $A = A_{\boldsymbol{\lambda},\mathbf{d}}$. In Section 6.2 we describe two combinatorial procedures to take a pair $(\boldsymbol{\lambda}, \mathbf{d})$ and produce a specified ASM.

Following [Wei2017a], a *prism tableau* for $(\boldsymbol{\lambda}, \mathbf{d})$ is a k-tuple of *reverse semistandard tableaux*, with shapes and labels determined by the pair $(\boldsymbol{\lambda}, \mathbf{d})$. We write $\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$ for the set of *minimal prism tableaux* for $(\boldsymbol{\lambda}, \mathbf{d})$ which have no *unstable triples*. These terms are defined in Section 6.1. Each prism tableau has an associated weight monomial $\texttt{wt}(\mathcal{T})$. Let

$$\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}} = \sum_{\mathcal{T} \in \texttt{Prism}(\boldsymbol{\lambda},\mathbf{d})} \texttt{wt}(\mathcal{T}). \tag{9}$$

Call $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}}$ an **ASM polynomial**.

If $\boldsymbol{\lambda} = (\lambda)$ and $\mathbf{d} = (d)$, the polynomial $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}}$ is the Schur polynomial $s_\lambda(x_1, \ldots, x_d)$. This follows immediately from the definition of s_λ as a weighted sum over *semistandard tableaux*. The purpose of [WY2018] was to provide a prism formula for Schubert polynomials. We prove the following generalization.

Theorem 1.1. $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}} = \displaystyle\sum_{w \in \texttt{MinPerm}(A_{\boldsymbol{\lambda},\mathbf{d}})} \mathfrak{S}_w.$

Here, $\texttt{MinPerm}(A)$ denotes the set permutations above A in $\mathsf{ASM}(n)$ which have the minimum possible length. Our proof of Theorem 1.1 is purely combinatorial; we give a bijection between $\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$ and the set of facets of a union of the *subword complexes* of [KM2004]. Each Schubert polynomial is a weighted sum over the facets of its corresponding subword complex [FK1996, BB1993, KM2005]. In Section 6.4, we define a map from the set of all prism tableaux for $(\boldsymbol{\lambda}, \mathbf{d})$ to a simplicial complex $\Delta(\mathcal{Q}_{n\times n}, A_{\boldsymbol{\lambda},\mathbf{d}})$, which is itself a union subword complexes. Restricted to $\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$, this map is a bijection onto the set of maximal dimensional facets in $\Delta(\mathcal{Q}_{n\times n}, A_{\boldsymbol{\lambda},\mathbf{d}})$ (see Theorem 6.3).

The polynomial $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}}$ also has a geometric interpretation; it is the *multidegree* of an *alternating sign matrix variety*. Write $\mathsf{Mat}(n)$ for the space of $n \times n$ matrices over an algebraically closed field $\Bbbk$. Given $M \in \mathsf{Mat}(n)$, let $M_{[i],[j]}$ be the submatrix of M which consists of the first i rows and j columns of M. We define the **alternating sign matrix variety**

$$X_A := \{M \in \mathsf{Mat}(n) : \mathrm{rank}(M_{[i],[j]}) \le r_A(i,j) \text{ for all } 1 \le i,j \le n\}. \tag{10}$$

If $w \in \mathcal{S}_n$, then X_w is a **matrix Schubert variety** as defined in [Ful1992].

ASM varieties are stable under multiplication by the group of invertible, diagonal matrices $\mathsf{T} \subset \mathsf{GL}(n)$. There is a corresponding $\mathbb{Z}^n$ grading and multidegree

$$\mathcal{C}(X_A; \mathbf{x}) \in \mathbb{Z}[x_1, \ldots, x_n].$$

Whenever $w \in \mathcal{S}_n$, we have $\mathfrak{S}_w = \mathcal{C}(X_w; \mathbf{x})$. This was shown in [KM2005] and is equivalent to earlier statements in the language of equivariant cohomology [FR2003] and degeneracy loci [Ful1992]. We show $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}}$ is the multidegree of the ASM variety $X_{A_{\boldsymbol{\lambda},\mathbf{d}}}$.

Theorem 1.2. *Fix* $\boldsymbol{\lambda}$ *and* $\mathbf{d}$ *as in (8). Then*

$$\mathcal{C}(X_{A_{\boldsymbol{\lambda},\mathbf{d}}}; \mathbf{x}) = \mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}}.$$

The irreducible components of X_A are always matrix Schubert varieties. Theorem 1.2 follows from Theorem 1.1 and the additivity of multidegrees.

We also discuss the explicit connection of prism tableaux to the Gröbner geometry of X_A. Let $\mathbf{z} = (z_{ij})_{i,j=1}^n$ be the generic $n \times n$ matrix. Define the **ASM ideal** by

$$I_A := \langle \text{minors of size } r_A(i,j) + 1 \text{ in } \mathbf{z}_{[i],[j]} \rangle. \tag{11}$$

It is immediate that I_A provides set-theoretic equations for X_A. For any $A \in \mathsf{ASM}(n)$, I_A is radical. This follows from the Frobenius splitting argument given in [Knu2009, Section 7.2]. We make the connection to ASM varieties explicit.

Proposition 1.1 ([Knu2009])**.** *Fix any antidiagonal term order* $\prec$ *on* $\Bbbk[\mathbf{z}]$.

(I) The essential (and hence defining) generators of I_A *form a Gröbner basis under* $\prec$.

(II) I_A *is radical and its initial ideal is a square-free monomial ideal.*

(III) The Stanley-Reisner complex of $\texttt{init}(X_A)$ *is* $\Delta(\mathcal{Q}_{n\times n}, A)$.

Since $\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$ is in weight preserving bijection with the facets of maximum dimension in $\Delta(\mathcal{Q}_{n\times n}, A_{\boldsymbol{\lambda},\mathbf{d}})$, this yields a second proof of Theorem 1.2.

1.3 Partition Identities and Quiver Representations

We now describe joint work with R. Rimányi and A. Yong found in [RWY2018]. Define the **quantum dilogarithm series**

$$\mathbb{E}(z) = \sum_{k=0}^{\infty} \frac{(-z)^k q^{k^2/2}}{(1-q)(1-q^2)\cdots(1-q^k)} \in \mathbb{Q}(q^{1/2})[[z]]. \tag{12}$$

In each term of (12), the denominator may be written more compactly using the q-**shifted factorial**,

$$(q)_k = (1-q)(1-q^2)\cdots(1-q^k). \tag{13}$$

This has an interpretation in terms of partitions; the reciprocal of $(q)_k$ is the generating series for partitions with at most k parts [And1984, Theorem 1.1].

There are many interesting identities among quantum dilogarithms. We highlight the following, which specializes to the *pentagon identity* of Rogers' dilogarithm.

Theorem 1.3 ([Sch1953] [FV1993], [FK1994])**.** *Suppose x and y are formal q commuting variables, with $yx = qxy$. Then*

$$\mathbb{E}(x)\mathbb{E}(y) = \mathbb{E}(y)\mathbb{E}(-q^{1/2}xy)\mathbb{E}(x). \tag{14}$$

M. Reineke extended (14) to give a family of identities, one for each *Dynkin quiver* (see [Rei2010] and [Kel2011]). The quantum pentagon identity corresponds to the quiver which has two vertices connected by a single edge.

In Chapter 9, we show that Theorem 1.3 can be proven using the combinatorial tool of *Durfee rectangles*. We give a proof of M. Reineke's identity in type A by proving related identities using iterated Durfee rectangles on *multipartitions*.

1.4 Organization

The main body of this thesis is as follows. We review the relevant background material regarding Schur polynomials and Schubert polynomials in Chapter 2. In particular, we describe combinatorial rules for these polynomials, the semistandard tableau model for Schur polynomials and the pipe dream model for Schubert polynomials. In Chapter 3 we prove a conjecture of R. Stanley regarding the number of monomials in a Schubert polynomial. The content of this chapter is found in [Wei2017b].

Our focus in Chapter 4 is on the poset structure of the set of alternating sign matrices. We present a generalization of the Rothe diagram to ASMs. This in turn, gives a new interpretation of the bijection between ASMs and antichains of bigrassmannian permutations. In Chapter 5, we review generalities relating to Coxeter groups, the Bruhat order, and subword complexes. We prove if w^J is a minimal length coset representative for w in a parabolic subgroup W/W_J, then any reduced word for w contains w^J as a subword exactly once (see Proposition 5.3).

Chapter 6 is work which appears in the preprint [Wei2017a]. Prism tableaux were introduced in joint work with A. Yong [WY2018]. We consider generating series over sets of prism tableaux. These polynomials expand as a multiplicity free sum of Schubert polynomials. In Chapter 7, we use prism tableaux to give a formula for the multidegrees of ASM varieties, which generalize matrix Schubert varieties.

In Chapter 8 we review background on the representation theory of quivers. Chapter 9 is joint work with R. Rimányi and A. Yong [RWY2018]. We discuss a specific connection between type A quiver representations and the quantum dilogarithms identities of M. Reineke [Rei2010]. We give a bijective proof of a related identity and use this to provide an elementary proof of M. Reineke's dilogarithm identities in type A.

CHAPTER 2

SCHUR POLYNOMIALS AND SCHUBERT POLYNOMIALS

2.1 Permutations and the Rothe Diagram

We follow [Man2001] as a reference. Let $\mathcal{S}_n$ be the symmetric group of permutations on $\{1, 2, \ldots, n\}$. We will most commonly represent permutations in one-line notation, i.e. $w = 4721635$ is the permutation which maps 1 to 4, 2 to 7, 3 to 2, and so on. We will sometimes use cycle notation. For instance, we can write 4721635 as a product of disjoint cycles $(27563)(14) = (14)(27563)$. In particular, we are interested in the factorization of a permutation as the product of **transpositions** $t_{ij} := (i\ \ j)$ or **simple transpositions** $s_i := (i\ \ i+1)$.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

A **permutation matrix** is a matrix with entries in $\{0, 1\}$ so that each row and column has exactly one nonzero entry. Encode $w \in \mathcal{S}_n$ as the unique $n \times n$ permutation matrix which has a 1 in the $(i, w(i))$ entry for each $1 \leq i \leq n$. The permutation matrix for $w = 4721635$ is pictured to the right. Observe that

$$(w(i), i) = (w(i), w^{-1}(w(i))).$$

Since this holds for all i, taking the transpose of the permutation matrix for w produces the permutation matrix for w^{-1}.

An **inversion** of w is a pair $i < j$ so that $w(j) > w(i)$. Call (i, j) an **inversion pair** for w. There is a **descent** at position i of w if $w(i) > w(i+1)$. This is equivalent to saying $(i, i+1)$ is an inversion pair. The **length** of a permutation records the number of inversions,

$$\ell(w) = \#\{(i, j) : 1 \leq i < j \leq n \ \text{ and } \ w(j) > w(i)\}. \tag{15}$$

The **sign** of w is

$$\mathrm{sgn}(w) = (-1)^{\ell(w)}. \tag{16}$$

Write

$$w_0 = n\ n-1\ n-2\ \ldots\ 1 \tag{17}$$

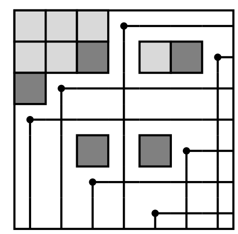
to denote the **longest** permutation in $\mathcal{S}_n$.

In 1800, H. Rothe developed a visual way to study the inversions of a permutation [Rot1800]. The **Rothe diagram** of a permutation is the set

$$D(w) := \{(i,j) : 1 \leq i,j \leq n,\ \ w(i) > j,\ \text{ and }\ w^{-1}(j) > i\}. \tag{18}$$

From (18), it follows that

$$D(w^{-1}) = D(w)^t. \tag{19}$$



We may visualize $D(w)$ as the complement of hooks in an $n \times n$ grid. Label cells using matrix conventions, i.e. (i,j) is the cell which is i rows from the top and j columns from the left. For each $i = 1, \ldots, n$, place a black dot in position $(i, w(i))$. Then, strike out all boxes to the right and below each of the plotted points. The boxes which remain form $D(w)$. For example, $D(4721635)$ is pictured to the right.

The **essential set** of a permutation w consists of the southeast most boxes in each connected component of the diagram

$$\text{Ess}(w) := \{(i,j) \in D(w) : (i,j+1), (i+1,j) \notin D(w)\}. \tag{20}$$

In the example above, the essential boxes of $D(4721635)$ are shaded in dark gray. Each permutation has an associated rank function r_w, where

$$r_w(i,j) := \#\{k : 1 \leq k \leq i\ \text{ and }\ w(k) \leq j\}. \tag{21}$$

Equivalently, $r_w(i,j)$ counts the number of 1's in the permutation matrix of w which sit weakly northwest of the (i,j) entry. Fulton showed that each permutation is uniquely determined by the restriction its rank function to the essential set [Ful1992, Lemma 3.10].

Lemma 2.1. *(I) Elements in $D(w)$ are in bijection with inversions of w. Explicitly, if $(i,j) \in D(w)$ then $(i, w^{-1}(j))$ is an inversion pair.*

(II) There is a descent at position i of w if and only if $(i,j) \in \text{Ess}(w)$ for some $1 \leq j \leq n$.

Proof. (I) By (18), if $(i, j) \in D(w)$ then

$$w(i) > j = w(w^{-1}(j)) \text{ and } w^{-1}(j) > i.$$

As such, $(i, w^{-1}(j))$ is an inversion pair. Conversely, if (i, j) is an inversion pair, then by definition

$$i < j = w^{-1}(w(j)) \text{ and } w(i) > w(j).$$

Therefore, $(i, w(j)) \in D(w)$.

(II) If w has a descent at position i, then $(i, i+1)$ is an inversion pair. By the argument above, $(i, w(i+1)) \in D(w)$. Whenever $j \geq w(i+1)$, we have that $(i, j) \notin D(w)$. In particular, any box in the ith row of $D(w)$ which is weakly to the right of $(i, w(i+1))$ does not have any box directly below it in $D(w)$. For instance, the right most box in the ith row of $D(w)$ is an essential box.

Conversely, suppose $(i, j) \in \text{Ess}(w)$. Then $(i, j) \in D(w)$ but $(i+1, j) \notin D(w)$. Since $(i+1, j) \notin D(w)$, we have

$$w(i+1) \leq j \text{ or } w^{-1}(j) \leq i+1.$$

Case 1: Suppose $w(i+1) \leq j$.

Since $(i, j) \in D(w)$, we have $w(i) > j$. Therefore,

$$w(i+1) \leq j < w(i)$$

and there is a descent in position i.

Case 2: Suppose $w^{-1}(j) \leq i+1$.

Since $(i, j) \in D(w)$ we have $w^{-1}(j) > i$. As these are integers,

$$w^{-1}(j) \geq i+1. \tag{22}$$

Since $w^{-1}(j) \leq i+1$, using (22) yields

$$w^{-1}(j) \leq i+1 \leq w^{-1}(j).$$

Therefore $i+1 = w^{-1}(j)$ and so $w(i+1) = j$. Since $w(i) > j = w(i+1)$, there is a descent at position i. $\square$

As an immediate consequence of Lemma 2.1, we have

$$\ell(w) = \#D(w). \tag{23}$$

Permutations are uniquely determined by the number of boxes in each row of $D(w)$. Let

$$c_w(i) := \#\{j : (i,j) \in D(w)\}. \tag{24}$$

The **code**[2] of w is the sequence

$$\mathbf{c}_w = (c_w(1), c_w(2), \ldots, c_w(n)). \tag{25}$$

For instance $\mathbf{c}_{4721635} = (3,5,1,0,2,0,0)$.

Write $\mathbb{N} = \{0,1,2,\ldots\}$ for the set of nonnegative integers and $\mathbb{Z}^+ = \{1,2,\ldots\}$ for the set of positive integers. A **weak composition** is a sequence $\alpha = (a_1, a_2, \ldots, a_n)$ with $a_i \in \mathbb{N}$. If weak compositions of different lengths agree on all nonzero entries, we consider them to be the same. Let $\delta = (n-1, n-2, \ldots, 0)$. Say $\alpha \subseteq \delta$ if $a_i \leq \delta_i$ for $1 \leq i \leq n$.

Lemma 2.2. *(I) The map $w \mapsto \mathbf{c}_w$ defines a bijection from $\mathcal{S}_n$ to weak compositions α such that $\alpha \subseteq \delta$.*

(II) There is a descent at position i of w if and only if $c_w(i) > c_w(i+1)$.

Proof. See [Man2001, Proposition 2.1.2]. □

There is a natural inclusion $\iota : \mathcal{S}_n \to \mathcal{S}_{n+1}$ given by

$$\iota(w(i)) = \begin{cases} w(i) & \text{for all } 1 \leq i \leq n \text{ and} \\ i & \text{when } i = n+1. \end{cases} \tag{26}$$

It will sometimes be convenient to consider permutations as elements of the **infinite symmetric group**

$$\mathcal{S}_\infty = \left(\bigcup_{n=1}^{\infty} \mathcal{S}_n \right) / \sim, \tag{27}$$

where $\sim$ is the equivalence relation generated by $w \sim \iota(w)$. More simply, $\mathcal{S}_\infty$ is the set of permutations of $\mathbb{Z}^+$ which fix all but finitely many numbers. We sometimes write $w \in \mathcal{S}_n$ to emphasize that w has a representative in $\mathcal{S}_n$, i.e. $w(i) = i$ whenever $i > n$.

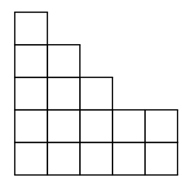
[2]Often, $\mathbf{c}_w$ is called the Lehmer code of w, after D. H. Lehmer [Leh1960]. However, $\mathbf{c}_w$ made an earlier appearance in 1888 in work of C.-A. Laisant [Lai1888] who encoded permutations as elements of the *factorial number system*. The correspondence between permutations and factorial numbers is bijective.

The permutations w and $\iota(w)$ have the same set of inversions. As such, the length, diagram, essential set, and descents of a permutation are all stable under ι and are well defined on classes in $\mathcal{S}_\infty$. Furthermore, $\mathbf{c}_{\iota(w)}$ is obtained from $\mathbf{c}_w$ by appending a zero. Therefore, they are the same as weak compositions. As such, the bijection in Lemma 2.2 extends to a bijection from $\mathcal{S}_\infty$ to the set of all weak compositions.

2.2 Grassmannian and Bigrassmannian Permutations

In this section we explain the connection between *Grassmannian* permutations and *partitions*. Parts of this section first appeared in [Wei2017a, Section 2.2]. A **partition** is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Each λ_i is called a **part** of λ. The **length** of a partition $\ell(\lambda)$ is the number of nonzero parts. We consider partitions to be the same whenever they agree on all nonzero parts.

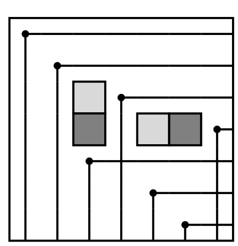
We represent a partition visually as a **Young diagram**, a collection of left justified rows of boxes so that the bottom row has λ_1 boxes, the next has λ_2, and so on. For example, the Young diagram of $\lambda = (5, 5, 3, 2, 1)$ is pictured to the right. We use the French convention; the longest row is situated at the bottom of the Young diagram. Write $a \times b$ for the partition whose Young diagram has a rows of length b.



If a permutation has a unique descent, it is called **Grassmannian**. Let $\mathcal{G}_n$ denote the set of Grassmannian permutations in $\mathcal{S}_n$. If $u \in \mathcal{G}_n$, write $\mathtt{des}(u)$ for the position of its descent. A permutation is **bigrassmannian** if both it, and its inverse, are Grassmannian. Write $\mathcal{B}_n$ for the set of bigrassmannian permutations in $\mathcal{S}_n$.

If $u \in \mathcal{G}_n$ with $k = \mathtt{des}(u)$, then let $\lambda^{(u)} = (c_u(k), c_u(k-1), \ldots, c_u(1))$. We will show shortly that $\lambda^{(u)}$ is a partition. Visually, the Young diagram of u can be obtained by left justifying the boxes of $D(u)$.

Example 2.1. *Let $u = 1247356$. The diagram $D(u)$ is pictured below.*



From this picture, we see that $\mathbf{c}_u = (0, 0, 1, 3, 0, 0, \ldots)$. As such, $\lambda^{(u)} = (3, 1)$. □

Lemma 2.3. *(I) If $u \in \mathcal{G}_n$, then $\lambda^{(u)}$ is a partition and $\lambda^{(u)} \subseteq \mathtt{des}(u) \times (n - \mathtt{des}(u))$.*

(II) The map $u \mapsto (\lambda^{(u)}, \mathtt{des}(u))$ *defines a bijection between* $\mathcal{G}_n$ *and pairs* (λ, d) *with*

$$\lambda \subseteq d \times (n-d) \ \textit{ and } \ \lambda \neq \emptyset$$

i.e. nonempty partitions with $\lambda_1 \leq n - d$ *and* $\ell(\lambda) \leq d$.

Proof. (I) Fix $u \in \mathcal{G}_n$ and let $k = \mathtt{des}(u)$. By Lemma 2.2, since u has a unique descent, its code is of the form

$$c_u(1) \leq c_u(2) \leq \ldots \leq c_u(k) \ \text{ and } \ c_u(i) = 0 \ \text{ whenever } \ i > k.$$

Since $\lambda^{(u)} \subseteq \delta$, we have $\lambda_1 = c_u(k) \leq n-k$. Furthermore, $\lambda^{(u)}$ has at most k parts. Therefore, $\lambda^{(u)} \subseteq k \times (n-k)$.
(II) Since the map $w \mapsto \mathbf{c}_w$ is injective, the map $u \mapsto (\lambda^{(u)}, \mathtt{des}(u))$ is also injective. There are $\binom{n}{k} - 1$ Grassmannian permutations with a descent at position k. To see this, notice that if $u \in \mathcal{G}_n$ then u is uniquely determined by picking $\mathtt{des}(u)$ numbers from the set $\{1, 2, \ldots, n\}$. Sorting this list from least to greatest gives the first $\mathtt{des}(u)$ entries of u in one-line notation. We exclude the choice $\{1, 2, \ldots, k\}$ which corresponds to the identity since the identity has no descents. There are also $\binom{n}{k} - 1$ nonempty partitions constrained to a $k \times (n-k)$ box. Since the source and the target are equinumerous, the injection $u \mapsto (\lambda^{(u)}, \mathtt{des}(u))$ is a bijection. □

Write $[\lambda, d]_g$ for the Grassmannian with $\lambda^{([\lambda,d]_g)} = \lambda$ and $\mathtt{des}([\lambda, d]_g) = d$. If $\lambda = (0)$ is the empty partition, then for any d, we let $[\lambda, d]_g = \mathrm{id}$.

If $u \in \mathcal{B}_n$ then $D(u)$ has a unique essential box. In particular, this implies $D(u)$ is a rectangle. Elements of $\mathcal{B}_n$ are naturally labeled by triples of integers (i, j, r) which satisfy the following conditions:

(B1) $1 \leq i, j$,

(B2) $0 \leq r < \min(i, j)$, and

(B3) $i + j - r \leq n$.

Let I_r denote the $r \times r$ identity matrix. Then write

$$[i, j, r]_b := \left(\begin{array}{c|c|c|c} I_r & & & \\ \hline & & I_{i-r} & \\ \hline & I_{j-r} & & \\ \hline & & & I_{n-i-j+r} \end{array} \right) \tag{28}$$

for the (unique) bigrassmannian encoded by this triple. In the case $r = \min(i, j)$, let $[i, j, r]_b$ be the identity permutation.

There are multiple labeling conventions for bigrassmannians in the literature (see e.g. [LS1996], [Rea2002], [Kob2013]). We have chosen ours so the following properties hold.

Lemma 2.4. *Let $u = [i, j, r]_b \in \mathcal{B}_n$. Then the following hold.*

(I) $\mathrm{Ess}(u) = \{(i, j)\}$.

(II) $r_u(i, j) = r$.

(III) $\mathtt{des}(u) = i$.

(IV) $\lambda^{(u)} = (i - r_u(i, j)) \times (j - r_u(i, j))$.

Proof. Lemma 2.4 is immediate from (28). □

In Section 4.3, we will review the role of $\mathcal{B}_n$ in terms of the *Bruhat* order on the symmetric group. Bigrassmannian permutations form the set of *basic* elements of this poset [LS1996].

2.3 Schur Polynomials

We follow [Man2001] as a reference on symmetric polynomials. There is a natural action of $\mathcal{S}_n$ on $\mathbb{Z}[x_1, \ldots, x_n]$ by permutation of indices. Given $w \in \mathcal{S}_n$ and $f \in \mathbb{Z}[x_1, \ldots, x_n]$, define

$$w \cdot f(x_1, \ldots, x_n) := f(x_{w(1)}, \ldots, x_{w(n)}). \tag{29}$$

A polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ is **symmetric** if it is invariant under the action of $\mathcal{S}_n$. Write

$$\Lambda_n := \{f : w \cdot f = f \text{ for all } w \in \mathcal{S}_n\} \tag{30}$$

for the ring of symmetric polynomials.

A polynomial is **alternating** if $t_{ij} \cdot f = -f$ for all $1 \leq i < j \leq n$. Equivalently, an alternating polynomial has the property that $w \cdot f = \mathrm{sgn}(w) f$. Recall, that $\delta = (n-1, n-2, \ldots, 1, 0)$. Define

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j + n - j})_{i,j=1}^n. \tag{31}$$

Immediately, from definition, we see that $a_{\lambda+\delta}$ is alternating; exchanging x_i and x_{i+1} has the effect of swapping columns in the determinant. The special case a_δ is called the **Vander-**

monde determinant. The **Schur polynomial** is the ratio

$$s_\lambda := \frac{a_{\lambda+\delta}}{a_\delta}. \tag{32}$$

Any alternating polynomial is divisible by the Vandermonde determinant. As such, s_λ is actually a polynomial. Furthermore, as a ratio of alternating polynomials, s_λ is symmetric. An additional property of Schur polynomials is more mysterious from the perspective of (32); the coefficients of the monomials in s_λ are nonnegative integers.

Schur polynomials play an essential role in the theory of symmetric functions; they form a $\mathbb{Z}$-linear basis of Λ_n.

Proposition 2.1. *The set $\{s_\lambda : \lambda \text{ has at most } n \text{ parts}\}$ is a basis for Λ_n.*

Proof. See [Ful1997, Section 6.1]. □

As such, we may expand the product of two Schur polynomials in the Schur basis

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu. \tag{33}$$

The structure coefficients $c^\nu_{\lambda,\mu}$ in (33) are called **Littlewood-Richardson coefficients**. Littlewood-Richardson coefficients are always non-negative integers. A significant achievement in algebraic combinatorics was the development of Littlewood-Richardson rules for computing $c^\nu_{\lambda,\mu}$ [LR1934].

A **semistandard tableau of shape** λ is a filling of the Young diagram of λ with non-negative integers so that

(I) along rows (from left to right) the labels are weakly increasing and

(II) along columns (from bottom to top) the labels strictly increase.

More often, we will consider **reverse semistandard tableaux**. These are fillings of a Young diagram so that

(I) along rows (from left to right) the labels are weakly decreasing and

(II) along columns (from bottom to top) the labels strictly decrease.

Write $\texttt{SSYT}(\lambda, n)$ (or $\texttt{RSSYT}(\lambda, n)$) for the set of semistandard (or reverse semistandard) tableaux of shape λ, filled with labels from the set $[n]$.

Example 2.2. *Below are two fillings of the Young diagram of $\lambda = (4, 4, 2, 1)$.*

$$T = \begin{array}{|c|c|c|c|} \hline 7 \\ \cline{1-1} 5 & 7 \\ \hline 4 & 4 & 6 & 6 \\ \hline 2 & 2 & 5 & 5 \\ \hline \end{array} \qquad\qquad T' = \begin{array}{|c|c|c|c|} \hline 1 \\ \cline{1-1} 3 & 1 \\ \hline 4 & 4 & 2 & 2 \\ \hline 6 & 6 & 4 & 3 \\ \hline \end{array}$$

Notice that $T \in \texttt{SSYT}(\lambda, 7)$ and $T' \in \texttt{RSSYT}(\lambda, 7)$. □

Each tableau has a **weight monomial**

$$\texttt{wt}(T) := \prod_{i=1}^{\infty} x_i^{m_i} \tag{34}$$

where m_i is the number of times that the label i appears in T.

The Schur polynomial $s_\lambda(x_1, x_2, \ldots, x_n)$ is a weighted sum over elements of $\texttt{SSYT}(\lambda, n)$.

Theorem 2.1 ([Lit1938, Theorem VI])**.**

$$s_\lambda(x_1, x_2, \ldots, x_n) = \sum_{T \in \texttt{SSYT}(\lambda, n)} \texttt{wt}(T). \tag{35}$$

For the discussion in Chapter 6, it is more convenient to use reverse semistandard tableaux. Lemma 2.5 shows that this is a harmless change of convention.

Let $\Phi_n(T)$ be the tableau obtained by replacing the labels in T of value i with $n - i + 1$ for all $1 \leq i \leq n$. For instance, returning to Example 2.2, notice that $\Phi_n(T) = T'$.

Lemma 2.5. *The map*

$$\Phi_n : \texttt{SSYT}(\lambda, n) \to \texttt{RSSYT}(\lambda, n) \tag{36}$$

is a bijection. Furthermore, if w_0 is the longest permutation in $\mathcal{S}_n$, then

$$\texttt{wt}(\Phi_n(T)) = w_0 \cdot \texttt{wt}(T). \tag{37}$$

Since s_λ is symmetric, combining Theorem 2.1 and Lemma 2.5 yields

$$s_\lambda = \sum_{T \in \texttt{RSSYT}(\lambda, n)} \texttt{wt}(T). \tag{38}$$

Example 2.3. *If $\lambda = (3, 1)$ then $s_\lambda(x_1, x_2) = x_1 x_2^3 + x_1^2 x_2^2 + x_1^3 x_2$. Conflating tableaux with their weights, we could have expressed this as*

$$s_\lambda(x_1, x_2) = \begin{array}{|c|c|c|} \cline{1-1} 2 \\ \hline 1 & 2 & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \cline{1-1} 2 \\ \hline 1 & 1 & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \cline{1-1} 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array}.$$

Alternatively, using reverse semistandard tableaux, we see

$$s_\lambda(x_1, x_2) = \begin{array}{|c|cc}\hline 1 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} + \begin{array}{|c|cc}\hline 1 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{1} \\ \hline \end{array} + \begin{array}{|c|cc}\hline 1 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array}.$$

Both models give an expression for s_λ. □

2.4 Schubert Polynomials

We now recall facts about the *Schubert polynomials* of A. Lascoux and M.-P. Schützenberger. Parts of this section first appeared in [Wei2017b]. The **divided difference** operator ∂_i acts on $f \in \mathbb{Z}[x_1, x_2, \ldots]$ by $\partial_i(f) := \frac{f - s_i \cdot f}{x_i - x_{i+1}}$. Divided difference operators satisfy the following relations.

Lemma 2.6. *(I) $\partial_i \circ \partial_{i+1} \circ \partial_i = \partial_{i+1} \circ \partial_i \circ \partial_{i+1}$*

(II) $\partial_i \circ \partial_j = \partial_j \circ \partial_i$ if $|i - j| > 1$.

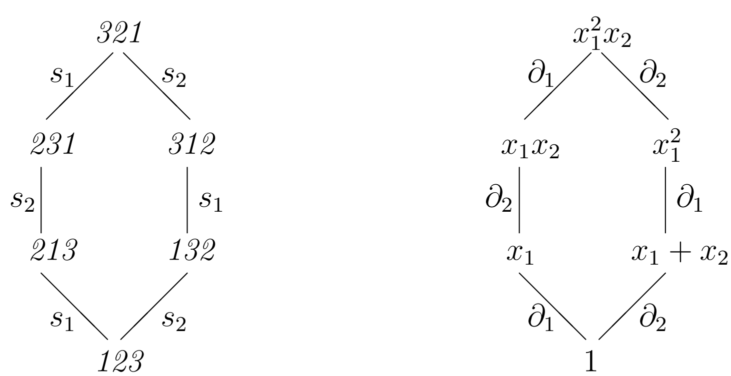
(III) $\partial_i \circ \partial_i = 0$.

Schubert polynomials are defined recursively. For the longest permutation, $w_0 \in \mathcal{S}_n$, we set $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$. If $w(i) < w(i+1)$, then ws_i covers w in the **weak Bruhat order**. In this case, set $\mathfrak{S}_w := \partial_i(\mathfrak{S}_{ws_i})$. The polynomials $\{\mathfrak{S}_w : w \in \mathcal{S}_n\}$ are called **Schubert polynomials**. If $s_{i_1} \cdots s_{i_k}$ is any *reduced expression* for $w^{-1}w_0$, then

$$\mathfrak{S}_w = (\partial_{i_1} \circ \cdots \circ \partial_{i_k})(\mathfrak{S}_{w_0}). \tag{39}$$

Thus, as a consequence of Lemma 2.6, Schubert polynomials are well defined.

Example 2.4. *Consider the diagrams below.*



On the left, are the covering relations in the weak Bruhat order on $\mathcal{S}_3$. On the right, the corresponding Schubert polynomials are shown. □

Schubert polynomials are stable under the inclusion $\iota : \mathcal{S}_n \hookrightarrow \mathcal{S}_{n+1}$, i.e. $\mathfrak{S}_w = \mathfrak{S}_{\iota(w)}$. As such, we index Schubert polynomials by elements of $\mathcal{S}_\infty$.

Theorem 2.2. $\{\mathfrak{S}_w : w \in \mathcal{S}_\infty\}$ *is a basis for* $\mathbb{Z}[x_1, x_2, \ldots]$.

Proof. See [Man2001, Proposition 2.5.4]. □

Schubert polynomials directly generalize Schur polynomials. In particular, $\mathfrak{S}_w$ is a Schur polynomial if and only if w is Grassmannian (or the identity). In this way, the Schubert basis is a lift of the Schur basis for the inclusion $\Lambda_d \hookrightarrow \mathbb{Z}[x_1, x_2, \ldots]$.

Theorem 2.3.

$$s_\lambda(x_1, \ldots, x_d) = \mathfrak{S}_{[\lambda,d]_g}(x_1, x_2, \ldots). \tag{40}$$

Proof. See [Man2001, Proposition 2.6.8]. □

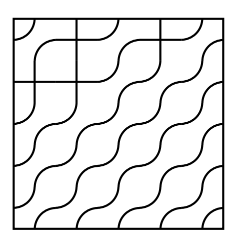
Schubert polynomials have nonnegative integer coefficients and can be written as a weighted sum over *pipe dreams*. Pipe dreams appear in the literature under various names; they are the *pseudo-line configurations* of S. Fomin and A. N. Kirillov [FK1996] and the *RC-graphs* of N. Bergeron and S. C. Billey [BB1993]. They were studied from a geometric perspective by A. Knuston and E. Miller [KM2005].

Identify $\mathbb{Z}^+ \times \mathbb{Z}^+$ with the semi-infinite grid. We use matrix notation, i.e. (i, j) indicates the ith row from the top and the jth column from the left. A **pipe dream** is a tiling of this grid with a finite number of $+$'s (pluses). The rest of the cells are filled with ╯╭'s (elbows). For simplicity, we will sometimes draw the elbows as dots.

We identify each pipe dream with a subset of $\mathbb{Z}^+ \times \mathbb{Z}^+$ by recording the coordinates of the pluses. Associate a weight monomial to $\mathcal{P}$

$$\mathtt{wt}(\mathcal{P}) = \prod_{(i,j)\in\mathcal{P}} x_i.$$

Equivalently, the exponent of x_i counts the number of pluses which appear in row i of $\mathcal{P}$.



We may interpret $\mathcal{P}$ as a collection of overlapping strands, using the rule that a strand never bends at a right angle. The $+$'s indicate the positions where two strands cross. Each row on the left edge of $\mathbb{Z}^+ \times \mathbb{Z}^+$ is connected by some strand to a unique column along the top, and vice versa. If the ith row is connected to the jth column, let $w_\mathcal{P}(i) := j$. There exists some n so that $w_\mathcal{P}(i) = i$ for all $i > n$, so $w_\mathcal{P} \in S_\infty$. In practice, we identify $w_\mathcal{P}$ with its representative in some finite symmetric group. For example, if $\mathcal{P}$ is the pipe dream pictured to the right, then we write $w_{\mathcal{P}} = 15324$.

If $\#\mathcal{P} = \ell(w_{\mathcal{P}})$ then $\mathcal{P}$ is **reduced**. Let

$$\texttt{Pipes}(w) := \{\mathcal{P} : w_{\mathcal{P}} = w \text{ and } \mathcal{P} \text{ is reduced}\} \tag{41}$$

denote the set of reduced pipe dreams for w.

Theorem 2.4 ([BB1993, FK1996]).

$$\mathfrak{S}_w = \sum_{\mathcal{P} \in \texttt{Pipes}(w)} \texttt{wt}(\mathcal{P}). \tag{42}$$

There are two pipe dreams which have an explicit description in terms of w. Recall

$$c_w(i) = \#\{j : j > i \text{ and } w(j) < w(i)\}. \tag{43}$$

Then the **bottom pipe dream** is

$$\mathcal{B}_w = \{(i,j) : j \leq c_w(i)\}. \tag{44}$$

Graphically, $\mathcal{B}_w$ is obtained from $D(w)$ by replacing each box with a plus and then left justifying within each row. We define the **top pipe dream** as the transpose of the bottom pipe dream of w^{-1}:

$$\mathcal{T}_w := \mathcal{B}^t_{w^{-1}}.$$

By (19), $\mathcal{T}_w$ is obtained from $D(w)$ by top justifying pluses within columns.

In [BB1993], N. Bergeron and S. C. Billey gave a procedure to obtain any pipe dream in $\texttt{Pipes}(w)$ algorithmically, starting from $\mathcal{B}_w$. A **ladder move** is an operation on pipe dreams which produces a new pipe dream by a replacement of the following type:

$$\begin{matrix} \cdot & \cdot \\ + & + \\ + & + \\ \vdots & \vdots \\ + & + \\ + & \cdot \end{matrix} \quad \mapsto \quad \begin{matrix} \cdot & + \\ + & + \\ + & + \\ \vdots & \vdots \\ + & + \\ \cdot & \cdot \end{matrix}$$

In the above picture, the columns and rows are consecutive. If $\mathcal{P} \mapsto \mathcal{P}'$ is a ladder move, then $\mathcal{P} \in \texttt{Pipes}(w)$ if and only if $\mathcal{P}' \in \texttt{Pipes}(w)$. In other words, $\texttt{Pipes}(w)$ is closed under ladder moves [BB1993]. Furthermore, $\texttt{Pipes}(w)$ is connected by ladder moves.

Theorem 2.5 ([BB1993, Theorem 3.7]). *If $\mathcal{P} \in \texttt{Pipes}(w)$, then $\mathcal{P}$ can be obtained by a sequence of ladder moves from $\mathcal{B}_w$.*

We will sometimes restrict focus to a special type of ladder move. A **simple ladder move** is a replacement of the following form:

$$\begin{matrix} \cdot & \cdot \\ + & \cdot \end{matrix} \quad \mapsto \quad \begin{matrix} \cdot & + \\ \cdot & \cdot \end{matrix}$$

For nice classes of permutations, such as Grassmannian permutations, $\texttt{Pipes}(v)$ is connected by simple ladder moves. This observation can be used to give an explicit bijection between $\texttt{RSSYT}(\lambda, d)$ and $\texttt{Pipes}([\lambda, d]_g)$; see for instance [Kog2000] and [KMY2009]. We use this bijection in Section 6.4.

CHAPTER 3

SCHUBERT POLYNOMIALS AND 132-PATTERNS

This chapter is motivated by a conjecture of R. P. Stanley [Sta2017, Conj. 4.1] concerning Schubert polynomials. It is based on work which first appeared in [Wei2017b].

3.1 Specializations of Schubert Polynomials

We are interested in the following specialization of a Schubert polynomial:

$$\nu_w := \mathfrak{S}_w(1, 1, \ldots, 1). \tag{45}$$

Immediately from (42), $\nu_w = \#\texttt{Pipes}(w)$. Here, we show that the permutations for which $\nu_w = 2$ are characterized by permutation pattern containment. Let

$$P_{132}(w) := \{(i, j, k) : i < j < k \text{ and } w(i) < w(k) < w(j)\}. \tag{46}$$

Write $\eta_w := \#P_{132}(w)$. If $\eta_w \geq 1$ then w **contains** the pattern 132. We prove that η_w provides a lower bound for ν_w.

Theorem 3.1 (The 132-bound)**.** *For any $w \in \mathcal{S}_n$, $\nu_w \geq \eta_w + 1$.*

As a corollary, we obtain the following conjecture of R. P. Stanley [Sta2017, Conj. 4.1].

Corollary 3.1. *$\nu_w = 2$ if and only if $\eta_w = 1$.*

Proof. Let $w \in \mathcal{S}_n$. If $\eta_w = 0$ then $\nu_w = 1$ [Mac1991, Chapter 4]. If $\eta_w = 1$ then $\nu_w = 2$ [Sta2017, Section 4]. Otherwise, $\eta_w \geq 2$. Then we apply Theorem 3.1 and obtain

$$\nu_w \geq \eta_w + 1 \geq 3.$$

As such, $\nu_w = 2$ if and only if $\eta_w = 1$. □

The outline of the proof is as follows. In Lemma 3.4, we show that any sequence of ladder moves connecting $\mathcal{B}_w$ to $\mathcal{T}_w$ must contain only simple ladder moves. We use this special structure to count the exact number of pipe dreams in any such sequence, providing a lower bound for $\#\texttt{Pipes}(w)$.

3.2 Proof of Theorem 3.1

We start by interpreting η_w as a weighted sum over $D(w)$.

Lemma 3.1.

$$\eta_w = \sum_{(i,j)\in D(w)} r_w(i,j).$$

Proof. Suppose $(i,j,k) \in P_{132}(w)$. Then $w(j) > w(k)$ and $w^{-1}(w(k)) = k > j$. By (18), we have $(j, w(k)) \in D(w)$. Furthermore, $i \leq j$ and $w(i) \leq w(k)$. Then by (21),

$$\#\{\ell : (\ell, j, k) \in P_{132}(w)\} \leq \#\{\ell : \ell \leq j \text{ and } w(\ell) \leq w(k)\} = r_w(j, w(k)).$$

Then

$$\eta_w \leq \sum_{(i,j)\in D(w)} r_w(i,j). \tag{47}$$

On the other hand, suppose $(i,j) \in D(w)$. Then

$$w(i) > j = w(w^{-1}(j)) \text{ and } w^{-1}(j) > i.$$

Take

$$k \in \{k : k \leq i \text{ and } w(k) \leq j\}.$$

Since $(i,j) \in D(w)$, we must have $k < i$ and $w(k) < j$. Then

$$k < i < w^{-1}(j) \text{ and } w(k) < w(w^{-1}(j)) < w(i)$$

and so

$$(k, i, w^{-1}(j)) \in P_{132}(w).$$

As such, if $(i,j) \in D(w)$,

$$\#\{\ell : (\ell, i, w^{-1}(j)) \in P_{132}(w)\} \geq r_w(i,j).$$

Therefore,

$$\eta_w \geq \sum_{(i,j)\in D(w)} r_w(i,j). \tag{48}$$

Then combining (47) and (48) gives

$$\eta_w = \sum_{(i,j)\in D(w)} r_w(i,j). \qquad \square$$

If $\mathcal{P} \in \texttt{Pipes}(w)$, let $\mathbf{a}_\mathcal{P} := (a_\mathcal{P}(1), \ldots, a_\mathcal{P}(n))$ where

$$a_\mathcal{P}(k) = \#\{(i,j) \in \mathcal{P} : i + j - 1 = k\}. \tag{49}$$

Equivalently, $a_\mathcal{P}(k)$ is the number of pluses that occur in the kth antidiagonal of $\mathcal{P}$.

Lemma 3.2. *Suppose there is a path of ladder moves from $\mathcal{P}$ to $\mathcal{Q}$:*

$$\mathcal{P} = \mathcal{P}_0 \mapsto \mathcal{P}_1 \mapsto \cdots \mapsto \mathcal{P}_N = \mathcal{Q}. \tag{50}$$

Each ladder move in (50) is simple if and only if $\mathbf{a}_\mathcal{P} = \mathbf{a}_\mathcal{Q}$.

Proof. ($\Rightarrow$) Assume each $\mathcal{P}_i \mapsto \mathcal{P}_{i+1}$ is a simple ladder move. Then $\mathcal{P}_{i+1}$ is obtained from $\mathcal{P}_i$ by moving a single plus to a new position in the same antidiagonal. As such, $\mathbf{a}_{\mathcal{P}_i} = \mathbf{a}_{\mathcal{P}_{i+1}}$ for each i. Therefore $\mathbf{a}_\mathcal{P} = \mathbf{a}_\mathcal{Q}$.
($\Leftarrow$) We prove the contrapositive. Suppose there is a nonsimple ladder move in the sequence (50). It acts by removing a plus from the ith antidiagonal and replacing it in the jth antidiagonal with $i < j$. In particular, we may pick j to be the maximum such label. By the maximality, no plus moves into the jth antidiagonal from a different antidiagonal. Then $a_\mathcal{P}(j) > a_\mathcal{Q}(j)$ and so $\mathbf{a}_\mathcal{P} \neq \mathbf{a}_\mathcal{Q}$. $\square$

Fix an indexing set I. A labeling of a pipe dream is an injective map $\mathcal{L}_\mathcal{P} : \mathcal{P} \to I$. Suppose $\mathcal{P} \mapsto \mathcal{P}'$ is a simple ladder move. Then $\mathcal{P}'$ inherits a labeling from $\mathcal{P}$ as follows:

$$\mathcal{L}_{\mathcal{P}'}(i,j) = \begin{cases} \mathcal{L}_\mathcal{P}(i,j) & \text{if } (i,j) \in \mathcal{P} \\ \mathcal{L}_\mathcal{P}(i+1,j-1) & \text{otherwise.} \end{cases}$$

Since $\mathcal{P} \mapsto \mathcal{P}'$ is a simple ladder move, $\mathcal{P}'$ is obtained from $\mathcal{P}$ by adding some (i,j) to $\mathcal{P}$ and removing $(i+1, j-1)$. Therefore, $\mathcal{L}_{\mathcal{P}'}$ is well defined. If there is a path of simple ladder moves from $\mathcal{P}$ to $\mathcal{Q}$, then $\mathcal{Q}$ inherits the labeling $\mathcal{L}_\mathcal{Q}$ from $\mathcal{L}_\mathcal{P}$ inductively.

Lemma 3.3. *Let $\mathcal{L}_\mathcal{P}$ be a labeling. Suppose $\mathcal{Q}$ can be reached from $\mathcal{P}$ by simple ladder moves. Then $\mathcal{Q}$ inherits the same labeling from $\mathcal{P}$ regardless of the choice of sequence.*

Proof. Suppose $\mathcal{P} \mapsto \mathcal{P}'$ is a simple ladder move. Then within any antidiagonal, both pipe dreams have the same set of labels in the same relative order. Iterate this argument along a path of simple ladder moves from $\mathcal{P}$ to $\mathcal{Q}$. Then, in each antidiagonal, $\mathcal{P}$ and $\mathcal{Q}$ have the same set of labels, still in the same relative order. As such, the labeling is uniquely determined and independent of the choice of path. $\square$

Lemma 3.4. *(I) The map*

$$(i, j) \mapsto (i, j - r_w(i, j))$$

is a bijection between $D(w)$ and $\mathcal{B}_w$.

(II) The map

$$(i, j) \mapsto (i - r_w(i, j), j)$$

is a bijection between $D(w)$ and $\mathcal{T}_w$.

(III) $\mathcal{B}_w$ and $\mathcal{T}_w$ are connected by simple ladder moves.

Proof. (I) Suppose $\ell > i$ and $w(\ell) < w(i)$. Then since $w^{-1}(w(\ell)) = \ell > i$ and $w(i) > w(\ell)$, by (18), we have $(i, w(\ell)) \in D(w)$. Therefore,

$$w(\ell) \in \{j : (i, w(j)) \in D(w)\}.$$

If $(i, \ell) \in D(w)$, then $w(i) > \ell = w(w^{-1}(\ell))$ and $w^{-1}(\ell) > i$. Then

$$w^{-1}(\ell) \in \{j : j > i \text{ and } w(j) < w(i)\}.$$

Therefore, the two sets are in bijection and

$$\#\{j : (i, j) \in D(w)\} = \#\{j : j > i \text{ and } w(j) < w(i)\} = c_w(i).$$

Then the ith row of $D(w)$ has as many boxes as there are pluses in the ith row of $\mathcal{B}_w$.

Let $j_1 < j_2 < \cdots < j_{c_w(i)}$ be the sequence obtained by sorting the set $\{j : (i, j) \in D(w)\}$.

$$\begin{aligned} j_\ell - r_w(i, j_\ell) &= j_\ell - \#\{k : k \leq i \text{ and } w(k) \leq j_\ell\} \\ &= \#\{k : k > i \text{ and } w(k) \leq j_\ell\} \\ &= \#\{j : (i, j) \in D(w) \text{ and } j \leq j_\ell\} \\ &= \ell. \end{aligned}$$

Therefore $(i, j_\ell) \mapsto (i, \ell)$. Since $1 \leq \ell \leq c_w(i)$ the map is well defined. This holds for any $\ell \in \{1, \ldots, c_w(i)\}$ so the map is surjective. By definition, $j_\ell = j_{\ell'}$ if and only if $\ell = \ell'$, giving injectivity. Therefore, this is a bijection.

(II) Let ϕ be the map defined by $(i,j) \mapsto (j,i)$. Restricted to $D(w)$, ϕ is a bijection between $D(w)$ and $D(w^{-1})$. By the definition of $\mathcal{T}_w$, the restriction

$$\phi : \mathcal{B}_{w^{-1}} \to \mathcal{T}_w$$

is also a bijection.

Let $\psi : \mathcal{P}(w^{-1}) \to \mathcal{B}_w$ be the map in (I). Then the composition

$$D(w) \xrightarrow{\phi} D(w^{-1}) \xrightarrow{\psi} \mathcal{B}_{w^{-1}} \xrightarrow{\phi} \mathcal{T}_w$$

is a bijection. Computing directly,

$$\begin{aligned}\phi(\psi(\phi(i,j))) &= \phi(\psi(j,i))\\ &= \phi(j, i - r_{w^{-1}}(j,i))\\ &= (i - r_{w^{-1}}(j,i), j).\end{aligned}$$

Applying (21),

$$\begin{aligned}r_{w^{-1}}(j,i) &= \#\{k : k \leq j \text{ and } w^{-1}(k) \leq i\}\\ &= \#\{\ell : w(\ell) \leq j \text{ and } w^{-1}(w(\ell)) \leq i\}\\ &= \#\{\ell : \ell \leq i \text{ and } w(\ell) \leq j\}\\ &= r_w(i,j).\end{aligned}$$

Therefore, $\phi(\psi(\phi(i,j))) = (i - r_w(i,j), j)$.

(III) By Theorem 2.5, there is a path of ladder moves from $\mathcal{B}_w$ to $\mathcal{T}_w$. Applying (49) and the bijections in parts (I) and (II),

$$\begin{aligned}\mathbf{a}_{\mathcal{B}_w}(k) &= \#\{(i,j) \in D(w) : i + (j - r_w(i,j)) - 1 = k\}\\ &= \#\{(i,j) \in D(w) : (i - r_w(i,j)) + j - 1 = k\}\\ &= \mathbf{a}_{\mathcal{T}_w}(k).\end{aligned}$$

By Lemma 3.2, the path uses only simple ladder moves. □

In light of the previous lemma, we may label the pluses of $\mathcal{B}_w$ using the map $(i,j) \mapsto (i, j - r_w(i,j))$, i.e. we refer to the plus which is the image of (i,j) as $+_{(i,j)}$. Likewise we label $\mathcal{T}_w$ using the map $(i,j) \mapsto (i - r_w(i,j), j)$.

Lemma 3.5. *The above labeling of $\mathcal{T}_w$ is the same as the labeling it inherits from $\mathcal{B}_w$.*

Proof. It is enough to show that within any given antidiagonal the labels in $\mathcal{B}_w$ and $\mathcal{T}_w$ are the same and have the same relative order. If $(i,j) \in D(w)$, then $+_{(i,j)}$ is in position $(i, j - r_w(i,j))$ in $\mathcal{B}_w$ and in position $(i - r_w(i,j), j)$ in $\mathcal{T}_w$. Since $i + j - r_w(i,j) = i - r_w(i,j) + j$, they are in the same antidiagonal.

Now consider the rth antidiagonal in $\mathcal{B}_w$. Suppose the sorted list of pluses from top to bottom is

$$+_{(i_1,j_1)}, +_{(i_2,j_2)}, \cdots, +_{(i_k,j_k)}.$$

Since the map from $D(w)$ is by left justification, we must have $i_1 < i_2 < \cdots < i_k$. As $i_\ell + j_\ell - 1 = r$ for all ℓ, it follows that $j_1 > j_2 > \cdots > j_k$. Since the map from $D(w)$ to $\mathcal{T}_w$ is by top justification, the sorted list of pluses from top to bottom must also be

$$+_{(i_1,j_1)}, +_{(i_2,j_2)}, \cdots, +_{(i_k,j_k)}.$$

Therefore, the labeling which $\mathcal{T}_w$ inherits from $\mathcal{B}_w$ coincides with the labeling determined by the map $(i,j) \mapsto (i - r_w(i,j), j)$. □

We conclude with the proof of the 132-bound.

Proof of Theorem 3.1. By Lemma 3.4, there is a path of simple ladder moves connecting $\mathcal{B}_w$ to $\mathcal{T}_w$, say

$$\mathcal{B}_w = \mathcal{P}_0 \mapsto \mathcal{P}_1 \mapsto \cdots \mapsto \mathcal{P}_N = \mathcal{T}_w. \tag{51}$$

Let $n_{i,j} = \#\{k : \mathcal{P}_k \mapsto \mathcal{P}_{k+1} \text{ moves } +_{(i,j)}\}$. By definition, $\mathcal{P}_k \mapsto \mathcal{P}_{k+1}$ moves exactly one plus, labeled by an element of $D(w)$. Therefore,

$$N = \sum_{(i,j)\in D(w)} n_{i,j}. \tag{52}$$

Claim 3.1. *If $(i,j) \in D(w)$ then $n_{i,j} = r_w(i,j)$.*

Proof. By Lemma 3.5, $+_{(i,j)}$ must move from position $(i, j - r_w(i,j))$ in $\mathcal{B}_w$ to position $(i - r_w(i,j), j)$ in $\mathcal{T}_w$. At each step $+_{(i,j)}$ remains stationary or it moves up a row and one column to the right. Therefore, $+_{(i,j)}$ must move exactly $i - (i - r_w(i,j)) = r_w(i,j)$ times to go from row i to row $i - r_w(i,j)$. □

Then

$$\begin{aligned}
\eta_w &= \sum_{(i,j)\in D(w)} r_w(i,j) && \text{(by Lemma 3.1)} \\
&= \sum_{(i,j)\in D(w)} n_{i,j} && \text{(by Claim 3.1)} \\
&= N && \text{(by (52)).}
\end{aligned}$$

Each $\mathcal{P}_i$ in the sequence (51) is distinct. Therefore,

$$\#\texttt{Pipes}(w) \geq N + 1.$$

Therefore

$$\nu_w = \#\texttt{Pipes}(w) \geq N + 1 = \eta_w + 1. \qquad \square$$

CHAPTER 4

ALTERNATING SIGN MATRICES

In this chapter, we introduce alternating sign matrices and study them from a poset theoretic perspective. Parts of this chapter first appeared in [Wei2017a].

4.1 Rothe Diagrams for ASMs

Recall, an **alternating sign matrix** (ASM) is a square matrix with entries in $\{-1, 0, 1\}$ so that the nonzero entries in each row and column alternate in sign and sum to 1. We start by presenting a generalization of Rothe diagrams to ASMs. Following [MRR1983], say $A = (a_{ij})_{i,j=1}^n \in \mathsf{ASM}(n)$ has an **inversion** in position (i, j) if

$$\sum_{(k,l):i<k \text{ and } j<l} a_{il}a_{kj} = 1. \tag{53}$$

Let

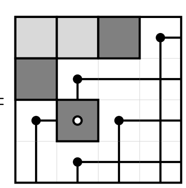
$$D(A) := \{(i, j) : (i, j) \text{ is an inversion of } A\} \subset [n] \times [n] \tag{54}$$

be the **Rothe diagram** of A. We represent $D(A)$ graphically. Our convention is to visually indicate the ASM by placing a black dot for each 1 in A and a white dot for each -1. The **essential set** $\mathrm{Ess}(A)$ consists of the southeast most corners of each connected component of $D(A)$,

$$\mathrm{Ess}(A) := \{(i, j) \in D(A) : (i + 1, j), (i, j + 1) \notin D(A)\}.$$

Example 4.1.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad D(A) =$$



The boxes of the diagram of A are shaded gray. The essential boxes are dark gray. □

The sum in (53) factorizes

$$\sum_{(k,l):i<k \text{ and } j<l} a_{il}a_{kj} = \left(\sum_{k=i+1}^{n} a_{kj}\right)\left(\sum_{l=j+1}^{n} a_{il}\right) = \left(1-\sum_{k=1}^{i} a_{kj}\right)\left(1-\sum_{l=1}^{j} a_{il}\right). \tag{55}$$

See [BMH1995]. By conditions (A1) and (A2) the factors in the RHS of (55) product are always 0 or 1. In order for (i,j) to be an inversion, both must be 1. Visually, this amounts to striking out hooks to the right and below each black dot which stop just before they encounter a box which contains a white dot. The boxes which remain are the elements of $D(A)$. If w is a permutation matrix, $D(w)$ and $\mathrm{Ess}(w)$ coincide with the usual Rothe diagram and essential set, as defined in Section 2.1. Notice that $D(A)$ is similar to the ASM diagram defined by A. Lascoux [Las2008]. However, our conventions on inversions differ; we include the set of *negative inversions*.

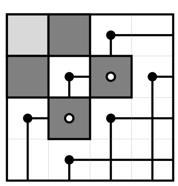
Any ASM is uniquely determined by the restriction of the corner sum function to its essential set. This generalizes the statement for permutations from [Ful1992, Lemma 3.10]. Our proof follows by showing that the essential set encodes an antichain of bigrassmannian permutations, from which the ASM in question can be recovered. See Proposition 4.2.

Recall that the length of $w \in \mathcal{S}_n$ is the number of inversions. Equivalently $\ell(w) = \#D(w)$. Say

$$\mathtt{deg}(A) = \min\{\ell(w) : w \in \mathcal{S}_n \text{ and } w \geq A\}. \tag{56}$$

In general $\mathtt{deg}(A) \neq \#D(A)$. See the following example.

Example 4.2. *Suppose A is the ASM whose diagram is pictured below.*

$D(A) =$ 

$$r_A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \qquad r_{3412} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Since $r_{3412} < r_A$ we have $3412 \geq A$. Therefore

$$\mathtt{deg}(A) \leq \ell(3412) = 4 < \#D(A).$$

By checking all $w \geq A$, the reader may verify that $\mathtt{deg}(A) = 4$. □

4.2 Preliminaries on Posets and Lattices

We follow [LS1996] and [Rea2002] as references. A **partially ordered set** (or poset) is a set $\mathcal{P}$ equipped with a binary relation $\leq$ which satisfies the axioms of *reflexivity*, *antisymmetry*, and *transitivity*. If $a \leq b$ and $a \neq b$ we write $a < b$. Given $a, b \in \mathcal{P}$ we say b **covers** a if $a < b$ and whenever $a \leq c \leq b$, we have $c = a$ or $c = b$.

An element $a \in \mathcal{P}$ is **minimal** in $\mathcal{P}$ if whenever $b \in \mathcal{P}$ so that $b \leq a$ we have $a = b$. Similarly, $a \in \mathcal{P}$ is **maximal** in $\mathcal{P}$ if whenever $b \in \mathcal{P}$ so that $b \geq a$ we have $a = b$. Write $\texttt{MIN}(\mathcal{P})$ for the set of minimal elements in $\mathcal{P}$ and $\texttt{MAX}(\mathcal{P})$ for the maximal elements.

The **join** of $\mathcal{S} \subseteq \mathcal{P}$ (when it exists) is the least upper bound of $\mathcal{S}$. Similarly, the **meet** is the greatest lower bound. The join and meet are denoted $\vee$ and $\wedge$ respectively.

An element $a \in \mathcal{P}$ is **basic** if $a \neq \vee\mathcal{S}$ whenever $a \notin \mathcal{S}$. The set of basic elements in $\mathcal{P}$ is called the **base** of $\mathcal{P}$. Let $\mathbb{P}(\mathcal{S})$ denote the **power set** of $\mathcal{S}$, that is the set of all subsets of $\mathcal{S}$. There is a natural poset structure on $\mathbb{P}(\mathcal{S})$ by inclusion of sets. Given any subset $\mathcal{C} \subseteq \mathcal{P}$, define $\pi_{\mathcal{C}} : \mathcal{P} \to \mathbb{P}(\mathcal{C})$ by $\pi_{\mathcal{C}}(a) = \{c \in \mathcal{C} : c \leq a\}$. The base is characterized by the following property.

Proposition 4.1 ([LS1996, Proposition 2.4])**.** *Let $\mathcal{B}$ be the base of a finite poset $\mathcal{P}$. The projection $\pi_{\mathcal{B}}$ is an order isomorphism onto its image. Furthermore, if any $\mathcal{C} \subseteq \mathcal{P}$ has this property, then $\mathcal{B} \subseteq \mathcal{C}$.*

As a consequence, any element $a \in \mathcal{P}$ is uniquely encoded by the set $\pi_{\mathcal{B}}(a)$. Furthermore, $a = \vee\pi_{\mathcal{B}}(a)$ (see [Rea2002, Proposition 9]). In particular, $a = \vee\texttt{MAX}(\pi_{\mathcal{B}}(a))$.

A **lattice** $\mathcal{L}$ is a poset in which every pair of elements has a join and a meet. Basic elements in a lattice are also known as **join-irreducibles** and have the characterization that they cover a unique element. A **sublattice** of $\mathcal{L}$ is a subset $\mathcal{L}' \subseteq \mathcal{L}$ which is itself a lattice and has the *same* operations of join and meet as $\mathcal{L}$.

Assume $\mathcal{S}$ is a totally ordered set and I some indexing set. Let

$$\mathcal{S}^I := \{(a_i)_{i \in I} : a_i \in \mathcal{S} \text{ for all } i \in I\}.$$

There is a natural partial order on $\mathcal{S}^I$ by entrywise comparison. Explicitly, if $\mathbf{a} = (a_i)_{i \in I}$ and $\mathbf{b} = (b_i)_{i \in I}$ in $\mathcal{S}^I$, then

$$\mathbf{a} \leq \mathbf{b} \text{ if and only if } a_i \leq b_i \text{ for all } i \in I.$$

Write

$$\mathbf{max}(\mathbf{a}, \mathbf{b}) := (\max(a_i, b_i))_{i \in I} \text{ and } \mathbf{min}(\mathbf{a}, \mathbf{b}) := (\min(a_i, b_i))_{i \in I}.$$

Lemma 4.1. *(I) $\mathcal{S}^I$ is a lattice with $\mathbf{a} \vee \mathbf{b} = \mathbf{max}(\mathbf{a}, \mathbf{b})$ and $\mathbf{a} \wedge \mathbf{b} = \mathbf{min}(\mathbf{a}, \mathbf{b})$.*

(II) If a subset of $\mathcal{S}^I$ is closed under joins and meets, then it is a sublattice of $\mathcal{S}^I$ (and hence is itself a lattice).

Proof. Since $\mathcal{S}^I$ is a Cartesian product of lattices, (I) is immediate. Likewise, (II) follows from the definition of a sublattice. □

A lattice is **complete** if every subset has a join and meet. Any finite lattice is automatically complete. The **Dedekind-MacNeille** completion of $\mathcal{P}$ is the smallest complete lattice which contains $\mathcal{P}$ as an order embedding. Any finite poset has the same base as its Dedekind-MacNeille completion [Rea2002, Proposition 28].

4.3 The Dedekind-MacNeille Completion of the Symmetric Group

Recall $r_A(i,j) = \sum_{k=1}^{i} \sum_{\ell=1}^{j} a_{k\ell}$. Write

$$\mathsf{R}(n) := \{r_A : A \in \mathsf{ASM}(n)\}.$$

For convenience, define $r_A(i,j) = 0$ whenever $i = 0$ or $j = 0$. Then

$$a_{ij} = r_A(i,j) - r_A(i,j-1) - r_A(i-1,j) + r_A(i-1,j-1) \tag{57}$$

recovers the (i,j) entry of A [RR1986]. As such, the map $A \mapsto r_A$ defines a bijection between $\mathsf{ASM}(n)$ and $\mathsf{R}(n)$. The following lemma characterizes corner sums of ASMs.

Lemma 4.2 ([RR1986, Lemma 1])**.** *Let A be an $n \times n$ matrix. Then $A \in \mathsf{ASM}(n)$ if and only if the following conditions hold:*

(R1) $r_A(i,n) = r_A(n,i) = i$ for all $i = 1, \dots, n$ and

(R2) $r_A(i,j) - r_A(i-1,j)$ and $r_A(i,j) - r_A(i,j-1) \in \{0,1\}$ for all $1 \le i,j \le n$.

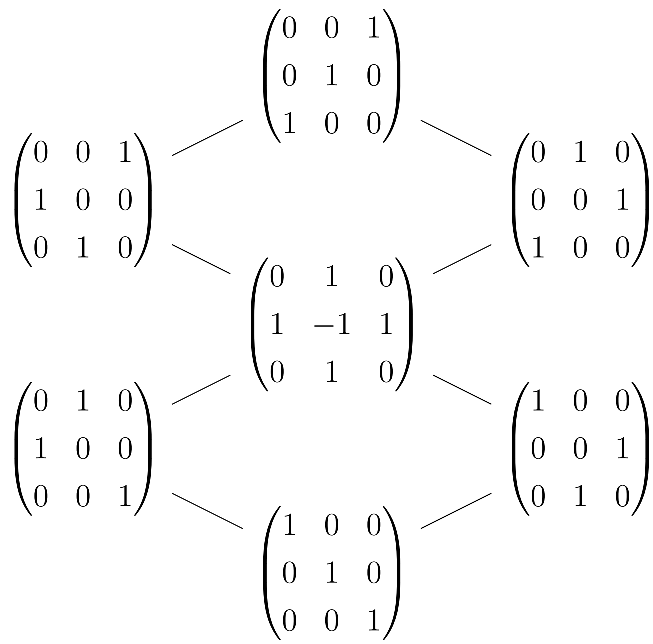
Lemma 4.3. *$\mathsf{R}(n)$ is a distributive lattice with join and meet given by $r_A \vee r_B = \mathbf{max}(r_A, r_B)$ and $r_A \wedge r_B = \mathbf{min}(r_A, r_B)$, respectively.*

Lemma 4.3 follows from Lemma 4.1 by verifying that (R1) and (R2) are preserved under taking minima and maxima.[3] Consequentially, $\mathsf{ASM}(n)$ inherits the structure of a lattice

[3]The lattice of ASMs was initially studied by N. Elkies, G. Kuperberg, M. Larsen, and J. Propp [EKLP1992]. The definition in *ibid* is in terms of height functions, which are in obvious order reversing bijection with corner sum matrices. The order on $\mathsf{ASM}(n)$ can also be defined using monotone triangles; this perspective was used in [LS1996].

from $\mathsf{R}(n)$.

Example 4.3. *The lattice of 3×3 ASMs is pictured below.*



Notice that in this case, there is only one ASM which is not a permutation matrix. Restricting the order to permutation matrices produces the Bruhat order on $\mathcal{S}_3$. □

Lemma 4.4 ([LS1996, Lemma 5.4])**.** *The Dedekind-MacNeille completion of $\mathcal{S}_n$ is isomorphic to $\mathsf{ASM}(n)$. The base of the $\mathcal{S}_n$, and hence $\mathsf{ASM}(n)$, is $\mathcal{B}_n$.*

In [LS1996], A. Lascoux and M.-P. Schützenberger also found the base for type B Coxeter groups. M. Geck and S. Kim determined the base for all finite Coxeter groups [GK1997].

Let

$$\mathtt{bigr}(A) = \mathtt{MAX}(\pi_{\mathcal{B}_n}(A)) \tag{58}$$

be the maximal bigrassmannians in $\pi_{\mathcal{B}_n}(A)$. Then as a consequence of Lemma 4.4 and Proposition 4.1

$$A = \vee \pi_{\mathcal{B}_n}(A) = \vee \mathtt{bigr}(A). \tag{59}$$

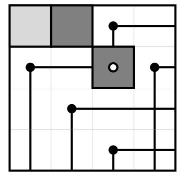
We also define

$$\mathtt{Perm}(A) := \mathtt{MIN}(\{w \in \mathcal{S}_n : w \geq A\}) \tag{60}$$

and

$$\mathtt{MinPerm}(A) := \{w \in \mathtt{Perm}(A) : \ell(w) = \mathtt{deg}(A)\}. \tag{61}$$

Example 4.4. *Let A be the ASM whose diagram is pictured below.*



By direct verification, we may compute $\texttt{Perm}(A) = \{3412, 4123\}$*. Since* $\ell(3412) = 4$ *and* $\ell(4123) = 3$*, we have* $\texttt{MinPerm}(A) = \{4123\}$*.* □

4.4 Corner Sums and Bigrassmannians

In this section, we discuss the specific connection of $\texttt{bigr}(A)$ to $\mathrm{Ess}(A)$. Furthermore, we review known facts about bigrassmannian permutations and the Bruhat order.

The definition of $\mathrm{Ess}(A)$ generalizes Fulton's definition of the essential set of a permutation matrix. However, there is another characterization in terms of corner sum matrices. This is taken as the definition elsewhere in the literature, for example see [For2008] or [Kob2013]. We prove these definitions are equivalent.

Lemma 4.5. $\mathrm{Ess}(A) = \{(i,j) : r_A(i,j) = r_A(i-1,j) = r_A(i,j-1)$ *and* $r_A(i,j)+1 = r_A(i+1,j) = r_A(i,j+1)\}$*.*

Proof. By (55), if $(i,j) \in D(A)$ if and only if $\sum_{k=1}^{i} a_{kj} = 0$ and $\sum_{l=1}^{j} a_{il} = 0$. Since

$$r_A(i,j) - r_A(i-1,j) = \sum_{l=1}^{j} a_{il} \text{ and } r_A(i,j) - r_A(i,j-1) = \sum_{k=1}^{i} a_{kj}$$

we have

$$(i,j) \in D(A) \text{ if and only if } r_A(i,j) = r_A(i-1,j) = r_A(i,j-1). \tag{62}$$

($\subseteq$) Assume $(i,j) \in \mathrm{Ess}(A)$. By definition, $(i+1,j),(i,j+1) \notin D(A)$. Since $(i,j) \in D(A)$, applying (62) and (R2), we have

$$r_A(i,j) = r_A(i,j-1) \leq r_A(i+1,j-1) \leq r_A(i+1,j).$$

If $r_A(i+1,j) = r_A(i,j)$ then $r_A(i+1,j) = r_A(i+1,j-1)$. Then by (62), we have that $(i+1,j-1) \in D(A)$, contradicting $(i,j) \in \mathrm{Ess}(A)$. Therefore, $r_A(i,j)+1 = r_A(i+1,j)$. The argument for $r_A(i,j)+1 = r_A(i,j+1)$ is entirely analogous.

($\supseteq$) By assumption, $r_A(i,j) = r_A(i-1,j) = r_A(i,j-1)$. Then applying (62), $(i,j) \in D(A)$. Since $r_A(i,j) \neq r_A(i+1,j)$ and $r_A(i,j) \neq r_A(i,j+1)$, we conclude

$$(i+1,j),(i,j+1) \notin D(A).$$

Therefore, $(i,j) \in \mathrm{Ess}(A)$. □

Lemma 4.6. $[i,j,r]_b = \wedge\{A \in \mathsf{ASM}(n) : r_A(i,j) \leq r\}$.

See [BS2017, Theorem 30] for a proof. An analogous statement in terms of monotone triangles appears in [LS1996]. Note in particular,

$$\text{if } r_A(i,j) \leq r \text{ then } [i,j,r]_b \leq A. \tag{63}$$

This is a special case of the generalized essential criterion given in [Kob2013].

Lemma 4.7. *Fix* $A \in \mathsf{ASM}(n)$*.*

(I) For all $1 \leq i,j \leq n$*,* $[i,j,r_A(i,j)]_b \in \mathcal{B}_n$ *or* $[i,j,r_A(i,j)]_b = \mathrm{id}$*.*

(II) $A = \vee\{[i,j,r_A(i,j)]_b : 1 \leq i,j \leq n\}$*.*

Proof. (I) From (R2) we must have $r_A(i,j) \leq \min\{i,j\}$. If this is an equality, we have $[i,j,r_A(i,j)]_b = \mathrm{id}$ and we are done. Then assume not. By (R1), $r_A(i,n) = i$. As a consequence of (R2), $n-j \geq i - r_A(i,j)$. Then $i+j-r_A(i,j) \leq n$. As such, the conditions (B1)-(B3) are satisfied.
(II) Let $A' = \vee\{[i,j,r_A(i,j)]_b : 1 \leq i,j \leq n\}$. By Lemma 4.6, A is an upper bound to each $[i,j,r_A(i,j)]_b$. Therefore, $A \geq A'$ and $r_A \leq r_{A'}$. Since $r_{A'}$ is entrywise the minimum of the corner sum matrices of the $[i,j,r_A(i,j)]_b$'s, in particular,

$$r_{A'}(i,j) \leq r_{[i,j,r_A(i,j)]_b}(i,j) = r_A(i,j).$$

Then $r_{A'} \leq r_A$. As such, $r_{A'} = r_A$ and so $A' = A$. □

Lemma 4.8. *Assume* $A \neq I_n$*. If* $(i,j) \notin \mathrm{Ess}(A)$*, then there is some* (i',j') *so that*

$$[i,j,r_A(i,j)]_b < [i',j',r_A(i',j')]_b.$$

Proof. Let $u = [i,j,r_A(i,j)]_b$. If $r_A(i,j) = \min\{i,j\}$ then u is the identity and hence smaller than any bigrassmannian. Then assume $r_A(i,j) < \min\{i,j\}$. Since $A \neq I_n$, there is some (i',j') exists for which $[i',j',r_A(i',j')]_b \in \mathcal{B}_n$ (e.g. some $(i',j') \in \mathrm{Ess}(A)$).

Applying Lemma 4.5 and (R2), there are four potential ways for (i,j) to fail to be in $\mathrm{Ess}(A)$.

Case 1: $r_A(i,j) = r_A(i-1,j)+1$.

Since we have assumed $r_A(i,j) < \min\{i,j\}$ and $r_A(0,j) = 0$, we must have $i > 1$. As such, let $u' = [i-1, j, r_A(i-1,j)]_b$. Then

$$r_{u'}(i,j) = r_{u'}(i-1,j)+1 = r_A(i-1,j)+1 = r_A(i,j) = r_u(i,j)$$

so by Lemma 4.6, $u \le u'$.

Case 2: $r_A(i,j) = r_A(i,j-1)+1$.

The argument is entirely analogous to Case 1.

Case 3: $r_A(i,j) = r_A(i+1,j)$.

Now let $u' = [i+1, j, r_A(i+1,j)]_b$. Then

$$r_{u'}(i,j) = r_{u'}(i+1,j) = r_A(i+1,j) = r_A(i,j) = r_u(i,j).$$

Applying Lemma 4.6, we have $u < u'$.

Case 4: $r_A(i,j) = r_A(i,j+1)$.

This is essentially the same as Case 3. □

The following proposition shows how to recover $\texttt{bigr}(A)$ from $\mathrm{Ess}(A)$.

Proposition 4.2. $\texttt{bigr}(A) = \{[i,j,r_A(i,j)]_b : (i,j) \in \mathrm{Ess}(A)\}$.

Proposition 4.2 is discussed in [LS1996, Section 5], using essential points of monotone triangles. It can be found in a slightly more general context in [For2008, Theorem 5.1]. As an immediate consequence, A is determined by the restriction of r_A to $\mathrm{Ess}(A)$. This generalizes [Ful1992, Lemma 3.10].

Proof of Proposition 4.2. First note that

$$\texttt{bigr}(A) \subseteq \{[i,j,r_A(i,j)]_b : (i,j) \in \mathrm{Ess}(A)\}. \tag{64}$$

If $A = I_n$ then $\texttt{bigr}(A) = \{\} = \mathrm{Ess}(A)$. As such, assume not.

By Lemma 4.8, whenever $(i,j) \notin \mathrm{Ess}(A)$, there is some (i',j') so that

$$[i,j,r_A(i,j)]_b < [i',j',r_A(i',j')]_b.$$

We may iteratively apply the Lemma 4.8 to construct a chain of inequalities

$$[i, j, r_A(i, j)]_b < [i', j', r_A(i', j')]_b < \cdots < [i'', j'', r_A(i'', j'')]_b$$

with $(i'', j'') \in \mathrm{Ess}(A)$. Therefore

$$\begin{aligned} A &= \vee\{[i, j, r_A(i, j)]_b : 1 \le i, j \le n\} && \text{(by Lemma 4.7)} \\ &= \vee\{[i, j, r_A(i, j)]_b : (i, j) \in \mathrm{Ess}(A)\}. \end{aligned}$$

In particular, by (64) any bigrassmannian below A has an upper bound in

$$\{[i, j, r_A(i, j)]_b : (i, j) \in \mathrm{Ess}(A)\}.$$

Claim 4.1. $\{[i, j, r_A(i, j)]_b : (i, j) \in \mathrm{Ess}(A)\}$ *is an antichain, i.e. its elements are all incomparable.*

Proof. Take $(i, j), (i', j') \in \mathrm{Ess}(A)$. Write $u = [i, j, r_A(i, j)]_b$ and $u' = [i', j', r_A(i', j')]_b$.
Case 1: $r_{u'}(i, j) \le r$.

Since $u' \le A$, we have $r_{u'} \ge r_A$. In particular, $r_{u'}(i, j) \ge r$. Then $r_{u'}(i, j) = r$. By condition (R2), $r_{u'}(i-1, j), r_{u'}(i, j-1) \in \{r-1, r\}$ and $r_{u'}(i+1, j), r_{u'}(i, j+1) \in \{r, r+1\}$. But since $(i, j) \in \mathrm{Ess}(A)$ and $r_{u'} \ge r_A$, applying Lemma 4.5 we are forced to have

$$r_{u'}(i-1, j) = r_{u'}(i, j-1) = r = r_A(i-1, j) = r_A(i, j-1)$$

and

$$r_{u'}(i+1, j) = r_{u'}(i, j+1) = r+1 = r_A(i+1, j) = r_A(i, j+1).$$

Then $(i, j) \in \mathrm{Ess}(u')\{(i', j')\}$. As such, $u' = u$.
Case 2: $r_{u'}(i, j) > r$.

As such $r_{u'}(i, j) > r_u(i, j)$. Then we conclude $u \not\ge u'$.

We may reverse the roles of u and u' in the above argument. As such, either u and u' are incomparable or $u = u'$. □

As a consequence of Claim 4.1, we have shown that $\{[i, j, r_A(i, j)]_b : (i, j) \in \mathrm{Ess}(A)\}$ is an antichain of bigrassmannian permutations whose least upper bound is A. Therefore, $\mathtt{bigr}(A) = \{[i, j, r_A(i, j)]_b : (i, j) \in \mathrm{Ess}(A)\}$. □

Lemma 4.9. *Suppose $A \in \mathsf{ASM}(n)$ and $u \in \mathcal{G}_n$. If $r_A(\mathtt{des}(u), j) \le r_u(\mathtt{des}(u), j)$ for all $j = 1, \ldots, n$, then $u \le A$.*

Proof. Let $i = \texttt{des}(u)$. Since $r_A(i,j) \leq r_u(i,j)$, we have $A \geq [i,j,r_u(i,j)]_b$. Since u is Grassmannian, all of its essential boxes occur in row i. Then by Lemma 4.6 we have $u' \leq A$ for all $u' \in \mathrm{Ess}(u)$. Therefore A is an upper bound to $\mathrm{Ess}(u)$. Then $A \geq u = \vee \mathrm{Ess}(u)$. □

4.5 Inclusions of ASMs

There is a natural inclusion $\iota : \mathsf{ASM}(n) \to \mathsf{ASM}(n+1)$ defined by

$$A \mapsto \left(\begin{array}{c|c} A & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right).$$

We write

$$\mathsf{ASM}(\infty) := \left(\bigcup_{n=1}^{\infty} \mathsf{ASM}(n)\right) / \sim$$

where $\sim$ is the equivalence relation generated by $A \sim \iota(A)$. Let

$$\mathcal{S}_\infty = \left(\bigcup_{n=1}^{\infty} \mathcal{S}_n\right) / \sim .$$

When context is clear, we will freely identify an equivalence class its representatives. We write $A \in \mathsf{ASM}(n)$ to indicate that A has a representative which is an element of $\mathsf{ASM}(n)$.

Observe that

$$A \leq B \text{ if and only if } \iota(A) \leq \iota(B). \tag{65}$$

To see this, notice that $r_{\iota(A)}(i, n+1) = r_{\iota(A)}(n+1, i) = i$ for any $A \in \mathsf{ASM}(n)$. Thus $\mathsf{ASM}(\infty)$ inherits the structure of a poset from the finite case. In particular, for any n, there is an order embedding

$$\mathsf{ASM}(n) \hookrightarrow \mathsf{ASM}(\infty).$$

To compare two classes in $\mathsf{ASM}(\infty)$, we may take N large enough so that there are representatives in $A, B \in \mathsf{ASM}(N)$. Due to (65) the resulting order does not depend on the choice of N. Pairwise, joins and meets still exist so $\mathsf{ASM}(\infty)$ is a lattice. However it is *not* complete; in particular, the entire lattice $\mathsf{ASM}(\infty)$ has no upper bound.

Note that if $u \in \mathcal{G}_n$, we have

$$(\lambda^{(u)}, \texttt{des}(u)) = (\lambda^{(\iota(u))}, \texttt{des}(\iota(u))).$$

Therefore, the bijection in Lemma 2.3 is stable under inclusion. Write $\mathcal{G}_\infty$ and $\mathcal{B}_\infty$ for the sets of Grassmannian and bigrassmannian permutations in $\mathcal{S}_\infty$. Diagrams are also stable under inclusion, i.e. $D(A) = D(\iota(A))$. Therefore

$$\texttt{bigr}(\iota(A)) = \{\iota(u) : u \in \texttt{bigr}(A)\}. \tag{66}$$

As such, elements of $\mathsf{ASM}(\infty)$ are encoded by (finite) antichains in $\mathcal{B}_\infty$.

4.6 Partial ASMs

We now discuss another poset which is closely related to $\mathsf{ASM}(n)$. A **partial alternating sign matrix** is a matrix with entries in $\{-1, 0, 1\}$ so that

(I) the nonzero entries in each row and column alternate in sign,

(II) each row and column sums to 0 or 1, and

(III) the first nonzero entry of any row or column is 1.

A **partial permutation** is a partial ASM with entries in $\{0, 1\}$. Write $\mathsf{PA}(n)$ for the set of $n \times n$ partial ASMs and $\mathsf{P}(n)$ for the set of $n \times n$ partial permutation matrices. We sometimes say A (or w) is an **honest** ASM (or **honest** permutation) to emphasize that $A \in \mathsf{ASM}(n)$ (or $w \in \mathcal{S}_n$).

As in the case of ASMs, we may endow $\mathsf{PA}(n)$ with the structure of a poset by comparison of corner sum functions. M. Fortin studied $\mathsf{PA}(n)$, showing that it is the Dedekind-MacNeille completion of $\mathsf{P}(n)$ [For2008, Section 6]. Here, partial permutation matrices are identified with *partial injective functions*. The poset structure defined by corner sum matrices agrees with the extended Bruhat order defined by L. E. Renner in [Ren2006].

Lemma 4.10. *Every $A \in \mathsf{PA}(n)$ has a canonical completion to an honest ASM $\widetilde{A} \in \mathsf{ASM}(N)$, with $n \leq N \leq 2n$.*

Proof. The construction is similar to the one in for partial permutations found in [MS2004, Proposition 15.8]. Starting from the top row of A, if sum of row i is zero, append a new column to A with a 1 in the ith row. Continue in this way from top to bottom. Then starting from the leftmost column, if column j sums to zero, add a new row with a 1 in position j. Let $\widetilde{A}$ be the matrix obtained by this procedure.

By construction, $\widetilde{A}$ satisfies (A1); nonzero entries alternate in sign along rows and columns. Also, the entries of within each row and column of $\widetilde{A}$ sum to 1, so (A2) holds. As such, the sum of all entries in A counts the total number of rows, as well as the number of columns.

Then $\widetilde{A}$ is square. At most n columns and rows were added. Therefore, $\tilde{A} \in \mathsf{ASM}(N)$ for some $n \leq N \leq 2n$. $\square$

Example 4.5. *If* $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$ *then* $\widetilde{A} = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$. *Since the sum of the entries of* A *is* 1, $N = 2n - 1 = 5$. $\square$

For $w \in \mathsf{P}(n)$ we define the **length** of w to be $\ell(w) := \ell(\widetilde{w})$. Similarly, we define the diagram $D(A) := D(\widetilde{A})$. By construction, $D(A)$ is contained in the $n \times n$ grid.

Lemma 4.11. $r_A \geq r_B$ *if and only if* $r_{\widetilde{A}} \geq r_{\widetilde{B}}$.

Proof. If $r_{\widetilde{A}} \geq r_{\widetilde{B}}$ it is immediate that $r_A \geq r_B$.

Now assume that $r_A \geq r_B$. By construction, the essential set of both $\widetilde{A}$ and $\widetilde{B}$ is contained in the first n rows and columns. As such, for any $(i, j) \in \text{Ess}(\widetilde{A})$, we have $r_{\widetilde{B}}(i, j) \leq r_{\widetilde{A}}(i, j)$ and therefore $[i, j, r_{\widetilde{A}}(i, j)]_b \leq B$. Then $\widetilde{B}$ is an upper bound to $\texttt{bigr}(A)$ and so $A \leq B$ which implies $r_{\widetilde{A}} \geq r_{\widetilde{B}}$. $\square$

Taking the inclusion of $\widetilde{A}$ into $\mathsf{ASM}(2n)$ is an order embedding $\mathsf{PA}(n) \hookrightarrow \mathsf{ASM}(2n)$. As such, we may study the order on $\mathsf{PA}(n)$ by identifying each partial ASM with its image under the above inclusion.

A **partial bigrassmannian** is an element $b \in \mathsf{P}(n)$ so that $\widetilde{b} \in \mathcal{S}_{2n}$ is bigrassmannian. Again, these are indexed by triples (i, j, r) but we omit condition (B3). Write $[i, j, r_{ij}]_b$ for the partial bigrassmannian in $\mathsf{P}(n)$. By [For2008], these are the basic elements of $\mathsf{PA}(n)$.

Notice, that restrictions of honest ASMs to northwest submatrices produce partial ASMs. Take $A \in \mathsf{ASM}(N)$. Then if $n \leq N$, we have $A_{[n],[n]} \in \mathsf{PA}(n)$. Notice $A \leq B$ implies $A_{[n],[n]} \leq B_{[n],[n]}$. The converse certainly does not hold. However, in the case $A = \widetilde{A}_{[n],[n]}$, we do have $A \leq B$ whenever $A_{[n],[n]} = B_{[n],[n]}$. This follows since $\text{Ess}(A) \subseteq n \times n$ and so $u \leq B$ for all $u \in \texttt{bigr}(A)$.

CHAPTER 5

COXETER GROUPS AND SUBWORD COMPLEXES

5.1 Coxeter Groups and the Bruhat Order

We follow [BB2006] as a reference. Fix a set S. A **Coxeter matrix** is a map from

$$S \times S \to \{1, 2, \ldots, \infty\}$$

so that $m(s, s') = m(s', s)$ for all $s, s' \in S$ and $m(s, s') = 1$ if and only if $s = s'$. A Coxeter matrix defines a group W with the presentation

$$W = \langle S \mid (ss')^{m(s,s')} : m(s, s') < \infty \rangle. \tag{67}$$

W is called a **Coxeter group** and the pair (W, S) a **Coxeter system**.

We can encode the data of a Coxeter matrix as a (partially labeled) graph. The vertices of this graph correspond to the elements of S. There is an edge between s and s' whenever $m(s, s') \geq 3$. If $m(s, s') \geq 4$, we indicate this by labeling the edge with $m(s, s')$.

Example 5.1. *Consider the symmetric group $\mathcal{S}_n$. Let $s_i = (i \;\; i+1)$ denote a simple transposition and consider the set $S = \{s_i : i = 1, \ldots, n-1\}$. Then the pair $(\mathcal{S}_n, S)$ forms a Coxeter system. Each simple transposition squares to the identity:*

$$s_i^2 = e.$$

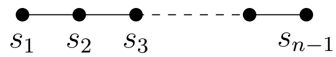
The set of simple reflections respect two additional types of relations, the commutations

$$s_i s_j = s_j s_i \;\; \textit{if} \; |i - j| > 1$$

and braid relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

The corresponding Coxeter graph appears below.



This Coxeter system is known as type A_n. □

The **Coxeter length** is

$$\ell(w) = \min(\{k : w = s_1 s_2 \dots s_k \text{ with } s_i \in S \text{ for all } i\}). \tag{68}$$

For $\mathcal{S}_n$, the Coxeter length agrees with the definition (15). A minimal length expression for $w \in S$ is called a **reduced expression** for w.

A **word** is an ordered list $\mathbf{s} = (s_1, \dots, s_m)$ of simple reflections in S. A **subword** of $\mathbf{s}$ is an ordered subsequence $\mathbf{t} = (s_{i_i}, \dots, s_{i_k})$. A word $\mathbf{s} = (s_1, \dots, s_m)$ **represents** $w \in W$ if $w = s_1 \cdots s_m$ and $\ell(w) = m$, i.e. the ordered product is a reduced expression for w. We say $\mathbf{s}$ **contains** w if $\mathbf{s}$ has a subword which represents w. Write $\mathtt{RSW}(\mathbf{s}, w)$ for the set of subwords of $\mathbf{s}$ which represent w.

Example 5.2. *Suppose $W = \mathcal{S}_3$ and $S = \{s_1, s_2\}$. Let $\mathbf{s} = (s_1, s_2, s_1, s_2)$. The word $\mathbf{s}$ has 3 subwords which represent $231 = s_1 s_2$. We list them below as tuples, writing a dash to indicate that the transposition in that position is not included in the subword.*

$$(s_1, s_2, -, -) \quad (s_1, -, -s_2) \quad (-, -, s_1, s_2).$$

The above subwords are distinct as subwords of $\mathbf{s}$. However, as words, they are all equivalent to (s_1, s_2). For another example, $\mathbf{s}$ contains $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ twice:

$$(s_1, s_2, s_1, -) \quad (-, s_2, s_1, s_2).$$

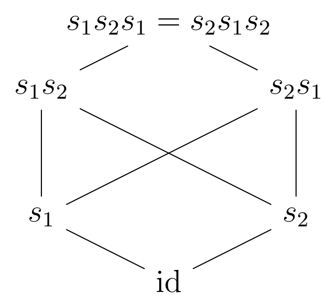
In this case, the above subwords are distinct as words, but they both represent w_0. □

We use subwords to define a partial order on W called the **Bruhat order**:

$$w \geq v \text{ if and only if some (and hence every) reduced word for } w \text{ contains } v. \tag{69}$$

There are several equivalent characterizations of the Bruhat order. See [Hum1992, Section 5.10]. For the symmetric group, (69) is equivalent to the order on $\mathcal{S}_n$ as defined in (7). See [BB2006, Theorem 2.1.5] for a proof.

Example 5.3. *Let $W = \mathcal{S}_3$.*



Pictured above is the Hasse diagram for the Bruhat order on $\mathcal{S}_3$. □

5.2 Subword Complexes

Recall that $\mathbb{P}(S)$ denotes the power set of S. A **simplicial complex** Δ is a subset of $\mathbb{P}([N])$ so that whenever $f \in \Delta$ and $f' \subseteq f$, we have $f' \in \Delta$. An element $f \in \Delta$ is called a **face**. The **dimension** of f is $\dim(f) = |f| - 1$. Write

$$\dim(\Delta) = \max\{\dim(f) : f \in \Delta\}.$$

If $f \in \Delta$, the **codimension** of f is $\operatorname{codim}(f) = \dim(\Delta) - \dim(f)$. The set of faces of Δ ordered by inclusion form a poset. Let

$$F(\Delta) = \mathtt{MAX}(\Delta) \tag{70}$$

denote the set of **facets** of Δ, i.e. the maximal faces. Then define

$$F_{\mathtt{max}}(\Delta) = \{f \in \Delta : \operatorname{codim}(f) = 0\}. \tag{71}$$

Necessarily, $F_{\mathtt{max}}(\Delta) \subseteq F(\Delta)$. When this containment is an equality, Δ is called **pure**.

Given two simplicial complexes $\Delta_1, \Delta_2 \subseteq \mathbb{P}([N])$, we may refer without ambiguity to the intersection (or union) of Δ_1 and Δ_2; it is precisely their intersection (or union) as sets. A straightforward verification shows that $\Delta_1 \cap \Delta_2$ and $\Delta_1 \cup \Delta_2$ are themselves simplicial complexes.

Lemma 5.1. *Fix simplicial complexes $\Delta_1, \ldots, \Delta_k \subseteq \mathbb{P}([N])$. Let $\Delta = \Delta_1 \cap \cdots \cap \Delta_k$. Then*

$$F(\Delta) \subseteq \{f_1 \cap \cdots \cap f_k : f_i \in F(\Delta_i)\}.$$

Proof. Fix $f \in F(\Delta) \subseteq \Delta$. Then $f \in \Delta_i$ for all i. For each i, there exists some $f_i \in F(\Delta_i)$

such that $f \subseteq f_i$. Therefore,

$$f \subseteq f_1 \cap \cdots \cap f_k \subseteq f_i \text{ for all } i = 1, \ldots, k. \tag{72}$$

Then $f_1 \cap \cdots \cap f_k \in \Delta_i$ for all i. As such,

$$f \subseteq f_1 \cap \cdots \cap f_k \in \Delta_1 \cap \ldots \cap \Delta_k = \Delta. \tag{73}$$

Since $f \in F(\Delta)$, the containment in (73) is actually an equality. □

For the remainder of this section, we follow [KM2004] as a reference. Given a fixed word $\mathbf{s}$ with n letters, there is a natural identification of subwords of $\mathbf{s}$ with subsets of $[n]$. Then we write $\mathbf{s} - \mathbf{t}$ to be the set difference as subsets of $[n]$. Define the **subword complex**

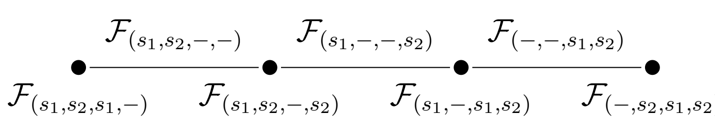
$$\Delta(\mathbf{s}, w) = \{\mathbf{s} - \mathbf{t} : \mathbf{t} \text{ contains } w\}.$$

We will abbreviate $\mathcal{F}_{\mathbf{t}} := \mathbf{s} - \mathbf{t}$. Immediately by definition,

$$\mathcal{F}_{\mathbf{t}} \subseteq \mathcal{F}_{\mathbf{t}'} \text{ if and only if } \mathbf{t} \supseteq \mathbf{t}'. \tag{74}$$

Then if $\mathbf{t}'$ contains w, $\mathbf{t}$ does as well. Therefore, $\Delta(\mathbf{s}, w)$ is a simplicial complex.

Example 5.4. *As in Example 5.2, let* $W = \mathcal{S}_3$ *and* $\mathbf{s} = (s_1, s_2, s_1, s_2)$. *Then* $\Delta(\mathbf{s}, s_1 s_2)$ *is pictured below.*



We can verify, for instance, that since $(s_1, s_2, -, -) \cup (s_1, -, -, s_2) = (s_1, s_2, -, s_2)$, *we have* $\mathcal{F}_{(s_1,s_2,-,-)} \cap \mathcal{F}_{(s_1,-,-,s_2)} = \mathcal{F}_{(s_1,s_2,-,s_2)}$. □

The **Demazure algebra** of (W, S) over a ring R is freely generated by $\{e_w : w \in W\}$ with multiplication given by

$$e_w e_s = \begin{cases} e_{ws} & \text{if } \ell(ws) > \ell(w) \\ e_w & \text{if } \ell(ws) < \ell(w). \end{cases}$$

If $\mathbf{s} = (s_1, \ldots, s_k)$, the **Demazure product** $\delta(\mathbf{s})$ is defined by the product in the Demazure algebra $e_{s_1} \cdots e_{s_m} = e_{\delta(\mathbf{s})}$. The Demazure product is well behaved with respect to the Bruhat order.

Lemma 5.2 ([KM2004, Lemma 3.4])**.** *$\delta(\mathbf{s}) \geq w$ if an only if $\mathbf{s}$ contains w.*

In particular, $\delta(\mathbf{s}) = \sup\{w : \mathbf{s} \text{ contains } w\}$.

Example 5.5. *Suppose $W = \mathcal{S}_3$ and $S = \{s_1, s_2\}$. Let $\mathbf{s} = (s_1, s_2, s_1, s_2)$. By applying a braid relation, we see the ordered product of the transpositions in S is*

$$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_2 = s_2 s_1.$$

As such, $\mathbf{s}$ is not a reduced word. We have

$$\ell(s_1) < \ell(s_1 s_2) < \ell(s_1 s_2 s_1).$$

However, $\ell(s_1 s_2 s_1) > \ell(s_1 s_2 s_1 s_2)$. Then $\delta(\mathbf{s}) = s_1 s_2 s_1$.

Alternatively, observe that $\mathbf{s}$ contains $w_0 = s_1 s_2 s_1$. Since w_0 is greater than all other elements of $\mathcal{S}_3$, we apply Lemma 5.2 and confirm that $\delta(\mathbf{s}) = w_0$. □

The faces of $\Delta(\mathbf{s}, w)$ have a natural description in terms of the Demazure product. Then

$$\Delta(\mathbf{s}, w) = \{\mathcal{F}_{\mathbf{t}} : \mathbf{t} \subseteq \mathbf{s} \text{ and } \delta(\mathbf{t}) \geq w\}. \tag{75}$$

The facets of $\Delta(\mathbf{s}, w)$ are indexed by the subwords of $\mathbf{s}$ which represent w. Explicitly, the map $\mathbf{t} \mapsto \mathcal{F}_{\mathbf{t}}$ defines a bijection from $\mathtt{RSW}(\mathbf{s}, w)$ to $F(\Delta(\mathbf{s}, w))$.

In Example 5.4, the pictured subword complex is homeomorphic to a 1-dimensional ball. In general, $\Delta(\mathbf{s}, w)$ is always homeomorphic to a ball or a sphere.

Theorem 5.1 ([KM2004, Corollary 3.8])**.** *$\Delta(\mathbf{s}, w)$ is homeomorphic to a sphere if $\Delta(\mathbf{s}) = w$. Otherwise, it is homeomorphic to a ball.*

5.3 Type A Subword Complexes and Alternating Sign Matrices

Throughout this section, let $W = \mathcal{S}_\infty$ and $S = \{(i, i+1) : i = 1, 2, \ldots\}$ be the set of simple transpositions. Given $A \in \mathsf{ASM}(n)$, define

$$\Delta(\mathbf{s}, A) = \{\mathcal{F}_{\mathbf{t}} : \delta(\mathbf{t}) \geq A\}. \tag{76}$$

This is itself a simplicial complex, but need not be a subword complex. Immediately from the definition,

$$\text{if } A \geq B \text{ then } \Delta(\mathbf{s}, A) \subseteq \Delta(\mathbf{s}, B). \tag{77}$$

We will show that $\Delta(\mathbf{s}, A)$ is a union of subword complexes. In particular, if $A \in \mathsf{ASM}(n)$, each of these subword complexes are labeled by permutations with representatives in $\mathcal{S}_n$.

Lemma 5.3. *Suppose $w \in \mathcal{S}_\infty$ is an upper bound to $\{w_1, \ldots, w_k\} \subseteq \mathcal{S}_m$. Then there exists $w' \in \mathcal{S}_m$ so that $\vee\{w_1, \ldots, w_k\} \leq w' \leq w$.*

Proof. Let $\mathbf{s}$ be a reduced word for w. By (69), $\mathbf{s}$ contains a subword, $\mathbf{s}_i$ which represents w_i for each i. Let $\mathbf{s}' = \bigcup_{i=1}^{k} \mathbf{s}_i \subseteq \mathbf{s}$. By Lemma 5.2, since $\mathbf{s}'$ contains each of the w_i's, we have $\delta(\mathbf{s}') \geq w_i$ for all i. Therefore, $\delta(\mathbf{s}')$ is an upper bound to $\{w_1, \ldots, w_k\}$. Again, by Lemma 5.2, $\mathbf{s}'$ contains $\delta(\mathbf{s}')$ and hence $\mathbf{s}$ contains $\delta(\mathbf{s}')$. As such,

$$w = \delta(\mathbf{s}) \geq \delta(\mathbf{s}').$$

Finally, the word s' uses only simple transpositions from $\mathcal{S}_m$, so $\delta(\mathbf{s}') \in \mathcal{S}_m$. □

As a corollary, we obtain the following.

Corollary 5.1. *(I)* $\texttt{Perm}(A) = \texttt{MIN}(\{w \in \mathcal{S}_\infty : w \geq A\})$.

(II) $\texttt{Perm}(A) = \texttt{MIN}(\{w \in \mathsf{P}(n) : w \geq A\})$.

Proof. (I) This is immediate from Lemma 5.3.
(II) Fix $w \in \mathsf{P}(n)$. Consider the inclusions $\widetilde{A}, \widetilde{w} \in \mathsf{ASM}(2n)$. Then $\widetilde{w} \geq A$ is an upper bound to $\texttt{bigr}(\widetilde{A}) = \texttt{bigr}(A)$. Applying Lemma 5.3, we obtain $w' \in \mathcal{S}_n$ with $\widetilde{A} \leq w' \leq \widetilde{w}$. Since $w' \in \mathcal{S}_n$, we may take its representative $\widetilde{w'} \in \mathsf{ASM}(2n)$. Then $\widetilde{A} \leq \widetilde{w'} \leq \widetilde{w}$. Applying Lemma 4.11, we see that $A \leq w' \leq w$. As such, the statement follows. □

Proposition 5.1. *Fix a word $\mathbf{s}$ and $A \in \mathsf{ASM}(n)$.*

(I) $\Delta(\mathbf{s}, A) = \bigcup_{w \in \texttt{Perm}(A)} \Delta(\mathbf{s}, w)$.

(II) If $A = \vee\{A_1, \ldots, A_k\}$ then

$$\Delta(\mathbf{s}, A) = \bigcap_{i=1}^{k} \Delta(\mathbf{s}, A_i).$$

(III) $F(\Delta(\mathbf{s}, A)) = \{\mathcal{F}_{\mathbf{t}} : \mathbf{t} \subseteq \mathbf{s}$ *is a reduced expression for some* $w \in \texttt{Perm}(A)\}$.

Proof. (I) Since $w \geq A$, applying (77) we have

$$\Delta(\mathbf{s}, A) \supseteq \bigcup_{w \in \texttt{Perm}(A)} \Delta(\mathbf{s}, w).$$

If $\mathcal{F}_{\mathbf{t}} \in \Delta(\mathbf{s}, A)$ then $\delta(\mathbf{t}) \geq A$. By (60) and Corollary 5.1, there exists $w \in \texttt{Perm}(A)$ so that $\delta(\mathbf{t}) \geq w \geq A$. Then $\mathcal{F}_{\mathbf{t}} \in \Delta(\mathbf{s}, w)$. Therefore, $\Delta(\mathbf{s}, A) \subseteq \bigcup_{w \in \texttt{Perm}(A)} \Delta(\mathbf{s}, w)$.

(II) Since $A \geq A_i$, applying (77), we have that $\Delta(\mathbf{s}, A) \subseteq \Delta(\mathbf{s}, A_i)$ for each $i = 1, \ldots, k$. Therefore,

$$\Delta(\mathbf{s}, A) \subseteq \bigcap_{i=1}^{k} \Delta(\mathbf{s}, A_i).$$

If $\mathcal{F}_{\mathbf{t}} \in \Delta(\mathbf{s}, A_i)$ for all i, then $\delta(\mathbf{t}) \geq A_i$ for all i. Since $A = \vee\{A_1, \ldots, A_k\}$ we must have $\delta(\mathbf{t}) \geq A$. As such, $\mathcal{F}_{\mathbf{t}} \in \Delta(\mathbf{s}, A)$.

(III) Suppose $\mathbf{t}$ is a reduced expression for some $w \in \texttt{Perm}(A)$. If $\mathbf{t}$ contains the subword $\mathbf{t}'$ and $\mathcal{F}_{\mathbf{t}'} \in \Delta(\mathbf{s}, A)$ then

$$w = \delta(\mathbf{t}) \geq \delta(\mathbf{t}') \geq A.$$

By (60), $\delta(\mathbf{t}') = w$. Since $\mathbf{t}$ is a reduced expression for w, we have $\mathbf{t} = \mathbf{t}'$. Therefore $\mathcal{F}_{\mathbf{t}} \in \Delta(\mathbf{s}, w)$. □

5.4 Parabolic Subgroups

Let $J \subseteq S$. The group $W_J = \langle s : s \in J \rangle$ is called a **parabolic subgroup** of W. Let

$$W^J = \{w \in W : ws > w \text{ for all } s \in J\}.$$

Equivalently, $w \in W^J$ if and only if no reduced expression for w ends with a letter from J [BB2006, Lemma 2.4.3]. The elements of W^J are the minimal length coset representatives of W/W_J. Indeed, each coset has a unique minimal length representative in W^J.

Any $w \in W$ has a unique factorization

$$w = w^J w_J \text{ with } w^J \in W^J \text{ and } w_J \in W_J \tag{78}$$

[BB2006, Proposition 2.4.4]. Furthermore, this factorization is length additive

$$\ell(w) = \ell(w^J) + \ell(w_J).$$

There is a natural projection map $P^J : W \to W^J$ defined by $P^J(w) = w^J$ which preserves the Bruhat order.

Proposition 5.2 ([BB2006, Proposition 2.5.1])**.** *If $v \leq w$ then $v^J \leq w^J$.*

The **right descent set** of w is

$$D_R(w) = \{s \in S : \ell(ws) < \ell(w)\}. \tag{79}$$

Let $J_s = S - \{s\}$.

Lemma 5.4 ([BB2006, Corollary 2.6.2])**.** *We have $u \leq w$ if and only if $P^{J_s}(u) \leq P^{J_s}(w)$ for all $s \in D_R(u)$.*

Lemma 5.5. *(I) If $v \in W^J$ so that $v \leq w$, then $v \leq w^J$.*

(II) $w = \sup\{w^{J_s} : s \in S\}$.

Proof. (I) If $v \in W^J$ then $v = v^J$. Furthermore, by Proposition 5.2, if $v \leq w$, we have

$$v = v^J \leq w^J.$$

(II) By (78), w is an upper bound to $\{w^{J_s} : s \in S\}$. Let v be an upper bound to $\{w^{J_s} : s \in S\}$. By Proposition 5.2, since $v \geq w^{J_s}$, we have $v^{J_s} \geq w^{J_s}$ for all $s \in S$. In particular, $v^{J_s} \geq w^{J_s}$ for all $s \in D_R(w)$. Applying Lemma 5.4, we see $w \leq v$. Therefore, w is the least upper bound of $\{w^{J_s} : s \in S\}$. □

The next proposition says that any reduced word for w contains a unique subword which represents w^J.

Proposition 5.3. *Fix $w \in W$ and a reduced word $\mathbf{s} = (s_1, \ldots, s_k)$ for w. Then for all $J \subseteq S$, we have $\#\mathtt{RSW}(\mathbf{s}, w^J) = 1$.*

Proof. The statement is trivially true if $\ell(w) = 0$. Suppose $\ell(w) = 1$, i.e. $\mathbf{s} = (s)$. Then if $s \in J$, we have $w^J = \mathrm{id}$. Otherwise $s \notin J$ and so $w^J = s$. In either case, $\mathbf{s}$ contains w^J as a subword exactly once.

Fix $k > 1$ and assume the statement holds for all $v \in W$ with $\ell(v) = k - 1$.

Take $w \in W$ so that $\ell(w) = k$ and let $\mathbf{s} = (s_1, \ldots, s_k)$ be a reduced word for w. Let $v = s_1 w$. Then $\mathbf{s}_v = (s_2, \ldots, s_k)$ is a reduced word for v and $v < w$.

Case 1: For any subword $\mathbf{s}' = (s_{i_1}, \ldots, s_{i_j})$ representing w^J, we have $i_1 \neq 1$.

Then w^J is contained in $\mathbf{s}_v$ and therefore $w^J \leq v$. Since $w^J \leq v \leq w$, applying Proposition 5.2 yields

$$w^J = (w^J)^J \leq v^J \leq w^J. \tag{80}$$

As such, $w^J = v^J$. By the induction hypothesis, $\mathbf{s}_v$ contains a unique subword representing $v^J = w^J$. Therefore, $\mathbf{s}$ also contains a unique subword representing w^J.

Case 2: There exists a subword $\mathbf{s}' = (s_{i_1}, \ldots, s_{i_j})$ representing w^J, with $i_1 = 1$.

First note that $s_1 w^J = s_{i_2} \ldots s_{i_j} \in W^J$. Therefore,

$$v = s_1 w = s_1 w^J w_J.$$

Since $s_1 w^J \in W^J$ and $w_J \in W_J$, by uniqueness of the factorization $v = v^J v_J$, we have $v^J = s_1 w^J$ and $v_J = w_J$. Furthermore,

$$v^J = s_{i_2} \cdots s_{i_j} \leq w^J.$$

Since $v^J < w^J$, we have that $w^J \not\leq v$. Otherwise, applying the same argument as in (80) would imply $w^J = v^J$.

Since $w^J \not\leq v$, there is no subword of $\boldsymbol{s}_v$ which represents w^J. Then, in fact, every subword $(s_{i_1}, \ldots, s_{i_j})$ of $\mathbf{s}$ representing w^J must have $i_1 = 1$. If $(s_{i_1}, \ldots, s_{i_j})$ is a subword of $\mathbf{s}$ representing w^J, since $i_1 = 1$, we have $(s_{i_2}, \ldots, s_{i_j})$ is a subword of $\mathbf{s}_v$ representing v^J. By the induction hypothesis there is only one such subword. Therefore, there is a unique subword of $\mathbf{s}$ representing w^J. □

CHAPTER 6

PRISM TABLEAUX AND ALTERNATING SIGN MATRICES

The content of this chapter is taken from [Wei2017a]. The original definitions regarding prism tableaux were introduced in joint work with A. Yong [WY2018].

6.1 Prism Tableaux

Recall, a reverse semistandard tableau for λ is a filling of the Young diagram of λ with positive integers so that labels

(T1) weakly decrease within rows (from left to right) and

(T2) strictly decrease (from bottom to top) within columns.

Fix $\boldsymbol{\lambda}$ and $\mathbf{d}$ as in (8). We define

$$\texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d}) = \texttt{RSSYT}(\lambda^{(1)}, d_1) \times \cdots \times \texttt{RSSYT}(\lambda^{(k)}, d_k). \tag{81}$$

An element of $\texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$ is called a **prism tableau**.

For the discussion which follows, it is not enough to merely think of a prism tableau as a tuple of reverse semistandard tableaux. Rather, we think of each of the component tableaux as having a position in the $\mathbb{Z}^+ \times \mathbb{Z}^+$ grid. As before, we use matrix coordinates to refer boxes in the grid. An **antidiagonal** of $\mathbb{Z}^+ \times \mathbb{Z}^+$ consists of the boxes

$$\{(i,1),(i-1,2),\ldots,(1,i)\}.$$

We identify the shape of each $\lambda^{(i)}$ with

$$\lambda^{(i)} = \{(a,b) : b \leq \lambda^{(i)}_{d_i-a+1}\} \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+. \tag{82}$$

The **prism shape** for $(\boldsymbol{\lambda}, \mathbf{d})$ is obtained by overlaying the $\lambda^{(i)}$'s:

$$\mathbb{S}(\boldsymbol{\lambda}, \mathbf{d}) := \bigcup_{i=1}^{k} \{(a,b) : b \leq \lambda^{(i)}_{d_i - a + 1}\}. \tag{83}$$

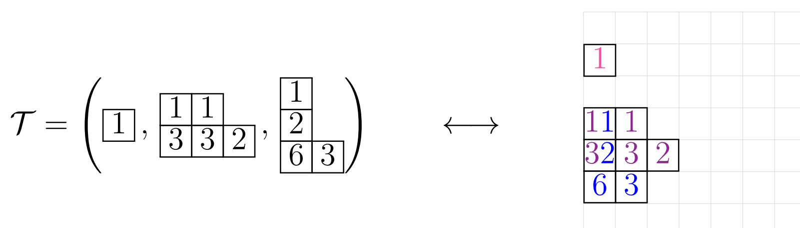
From this perspective, a prism tableau for $(\boldsymbol{\lambda}, \mathbf{d})$ is a filling of $\mathbb{S}(\boldsymbol{\lambda}, \mathbf{d})$ which assigns a label of color i from the set $\{1, 2, \ldots, d_i\}$ to each $(a,b) \in \lambda^{(i)}$ so that labels of color i weakly decrease along rows from left to right and strictly decrease along columns from bottom to top. Such fillings are in immediate bijection with $\texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$. As such, we freely identify these two representations of a prism tableau.

Weight $\mathcal{T}$ as follows:

$$\texttt{wt}(\mathcal{T}) = \prod_{i=1}^{\infty} x_i^{n_i}$$

where n_i is the number of antidiagonals which contain the label i (in any color).

Example 6.1. *Let* $\boldsymbol{\lambda} = ((1), (3,2), (2,1,1))$ *and* $\mathbf{d} = (2,5,6)$*. Below, we give an example of* $\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$.



The corresponding weight monomial is $\texttt{wt}(\mathcal{T}) = x_1^3 x_2^2 x_3^3 x_6$. □

Let

$$\texttt{deg}(\boldsymbol{\lambda}, \mathbf{d}) = \min\{\texttt{deg}(\texttt{wt}(\mathcal{T})) : \mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})\}. \tag{84}$$

$\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$ is **minimal** if $\texttt{deg}(\texttt{wt}(\mathcal{T})) = \texttt{deg}(\boldsymbol{\lambda}, \mathbf{d})$. Let ℓ_c be a label ℓ of color c. Labels $\{\ell_c, \ell_d, \ell'_e\}$ in the same antidiagonal form an **unstable triple** if $\ell < \ell'$ and replacing the ℓ_c with ℓ'_c gives a prism tableau. Write

$$\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d}) = \{\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d}) : \mathcal{T} \text{ is minimal and has no unstable triples}\}. \tag{85}$$

A tuple $(\boldsymbol{\lambda}, \mathbf{d})$ defines an ASM in the following way. We can always choose n large enough so that $\lambda^{(i)} \subseteq d_i \times (n - d_i)$ for all $i = 1, \ldots, k$. Then let

$$\mathbf{u}_{\boldsymbol{\lambda}, \mathbf{d}} := ([\lambda^{(1)}, d_1]_g, \ldots, [\lambda^{(k)}, d_k]_g) \tag{86}$$

and

$$A_{\boldsymbol{\lambda},\mathbf{d}} := \vee \mathbf{u}_{\boldsymbol{\lambda},\mathbf{d}} \in \mathsf{ASM}(n). \tag{87}$$

The main goal of this chapter is to study the polynomial

$$\sum_{\mathcal{T} \in \mathtt{Prism}(\boldsymbol{\lambda},\mathbf{d})} \mathtt{wt}(\mathcal{T}). \tag{88}$$

Theorem 1.1 states that (88) expands as a multiplicity free sum of Schubert polynomials indexed by elements of $\mathtt{MinPerm}(A_{\boldsymbol{\lambda},\mathbf{d}})$. We prove Theorem 1.1 in Section 6.4.

6.2 Combinatorial Prism Models

We now describe two ways of taking an ASM as a input and producing a pair $(\boldsymbol{\lambda}, \mathbf{d})$ so that $A = A_{\boldsymbol{\lambda},\mathbf{d}}$. Both procedures are entirely combinatorial. We start with bigrassmannian prism tableaux, which were defined in [WY2018].

Definition 6.1 (Bigrassmannian Prism Tableaux)**.** *Suppose*

$$\mathrm{Ess}(A) = \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}.$$

Let

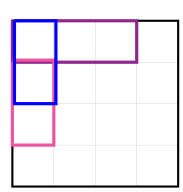
$$\beta^{(\ell)} = (i_\ell - r_A(i_\ell, j_\ell)) \times (j_\ell - r_A(i_\ell, j_\ell)). \tag{89}$$

Define $\boldsymbol{\beta}_A = (\beta^{(1)}, \ldots, \beta^{(k)})$ *and* $\mathbf{b}_A = \{i_1, \ldots, i_k\}$*. The* **bigrassmannian prism shape** *is* $\mathbb{S}_B(A) := \mathbb{S}(\boldsymbol{\beta}_A, \mathbf{b}_A)$*. Write* $\mathtt{Prism}_B(A) := \mathtt{Prism}(\boldsymbol{\beta}_A, \mathbf{b}_A)$*.*

Example 6.2. *Let A be as in Example 4.1. Then* $\mathrm{Ess}(A) = \{(1,3), (2,1), (3,2)\}$*.*

(i_ℓ, j_ℓ)	$r_A(i_\ell, j_\ell)$	$\beta^{(\ell)}$
$(1,3)$	0	1×3
$(2,1)$	0	2×1
$(3,2)$	1	2×1

Using the table above, we construct the shape $\mathbb{S}_B(A)$*.*



There are only three prism fillings of $\mathbb{S}_B(A)$.

$$\mathcal{T}_1 = \begin{array}{|c|c|c|}\hline 11 & 1 & 1 \\ \hline 22 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \qquad \mathcal{T}_2 = \begin{array}{|c|c|c|}\hline 11 & 1 & 1 \\ \hline 21 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \qquad \mathcal{T}_3 = \begin{array}{|c|c|c|}\hline 11 & 1 & 1 \\ \hline 21 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

Their weight monomials are $\mathtt{wt}(\mathcal{T}_1) = x_1^3x_2x_3$, $\mathtt{wt}(\mathcal{T}_2) = x_1^3x_2x_3$, and $\mathtt{wt}(\mathcal{T}_3) = x_1^3x_2^2$. These all have the same degree, and so each tableaux is minimal. We can obtain $\mathcal{T}_1$ from $\mathcal{T}_2$ by replacing the pink 1 with a 2. Therefore $\mathcal{T}_2$ has an unstable triple and we conclude that

$$\mathtt{Prism}_B(A) = \{T_1, T_3\}.$$

Then $\mathfrak{A}_{\boldsymbol{\beta}_A,\mathbf{b}_A} = x_1^3x_2x_3 + x_1^3x_2^2$. □

We now introduce the parabolic prism model. Our definition uses the *monotone triangles* of W. H. Mills, D. P. Robbins, and H. Rumsey [MRR1983]. Given

$$A = (a_{ij})_{i,j=1}^n \in \mathsf{ASM}(n)$$

let C_A be the matrix of partial column sums, i.e. $C_A(i,j) = \sum_{\ell=1}^i a_{\ell j}$. The ith row of m_A records (in increasing order) the positions of the 1s in the ith row of C_A. The array m_A is called a **monotone triangle**.

Example 6.3.

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad C_A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad m_A = \begin{array}{ccccccc} & & & 3 & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

Pictured above is an ASM, its column sum matrix, and its montone triangle. □

There is explicit dictionary between monotone triangles and corner sum matrices. Entry (i,j) of m_A indicates the position of the jth ascent in row i of r_A, i.e,

$$m_A(i,j) = a \text{ if and only if } r_A(i,a-1) = j-1 \text{ and } r_A(i,a) = j. \tag{90}$$

Given A and $1 \leq \ell \leq n$, we define

$$\lambda^{(A,\ell)} = (m_A(\ell,\ell) - \ell, m_A(\ell,\ell-1) - (\ell-1), \ldots, m_A(\ell,1) - 1). \tag{91}$$

Since m_A strictly increases along rows, $\lambda^{(A,\ell)}$ is a partition. By construction,

$$\lambda^{(A,\ell)} \subseteq \ell \times (n-\ell).$$

Notice if $u \in \mathcal{G}_n$, then $\lambda^{(u,\mathtt{des}(u))} = \lambda^{(u)}$.

Definition 6.2 (Parabolic Prism Tableaux)**.** *Write*

$$\{i : (i,j) \in \mathrm{Ess}(A)\} = \{i_1, \ldots, i_k\}$$

for the indices of essential rows of A. Let

$$\boldsymbol{\rho}_A = (\lambda^{(A,i_1)}, \lambda^{(A,i_2)}, \ldots, \lambda^{(A,i_k)}) \text{ and } \mathbf{p}_A = (i_1, \ldots, i_k).$$

Then define the **parabolic prism shape**

$$\mathbb{S}_P(A) = \mathbb{S}(\boldsymbol{\rho}_A, \mathbf{p}_A).$$

We abbreviate $\mathtt{Prism}_P(A) := \mathtt{Prism}(\boldsymbol{\rho}_A, \mathbf{p}_A)$.

In the following lemma, we note that each $[\lambda^{(w,\ell)}, \ell]_g$ is actually a minimal length coset representative for w with respect to a maximal parabolic subgroup of $\mathcal{S}_n$. Write $w^{(i)}$ for the minimal length coset representative of w in $W/\langle s_1, \ldots, \widehat{s_i} \ldots, s_{n-1}\rangle$.

Lemma 6.1. *If $w \in \mathcal{S}_n$, then $w^{(\ell)} = [\lambda^{(w,\ell)}, \ell]_g$.*

Proof. We obtain $w^{(\ell)}$ from w by sorting the elements in the sets $\{w(1), w(2), \ldots, w(\ell)\}$ and $\{w(\ell+1), \ldots, w(n)\}$ and then concatenating these sequences (see [BB2006, Lemma 2.4.7]). As such, row ℓ of $m_{w^{(\ell)}}$ and m_w agree. Therefore, $\lambda^{(w,\ell)} = \lambda^{(w^{(\ell)},\ell)} = \lambda^{(w^{(\ell)})}$. If they are empty partitions, then $\mathrm{id} = w^{(\ell)} = [\lambda^{(w,\ell)}, \ell]_g$. Otherwise, we apply Lemma 2.3 and conclude $w^{(\ell)} = [\lambda^{(w,\ell)}, \ell]_g$. □

In special cases, minimal parabolic prism tableaux do not have unstable triples. In particular, the following holds.

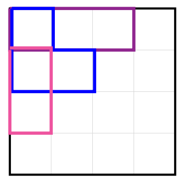
Theorem 6.1. *If $\mathcal{T}$ is a minimal parabolic prism tableau for a permutation w, then $\mathcal{T}$ does not have unstable triples.*

We postpone the proof to the next section. The conclusion of Theorem 6.1 can fail if w is not a permutation matrix.

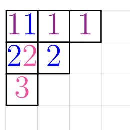
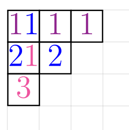
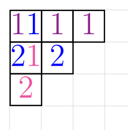
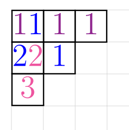
Example 6.4. *Let A be as in Example 4.1. Then*

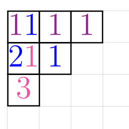
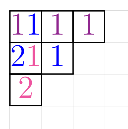
$$m_A = \begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}.$$

The essential rows are $\mathbf{p}_A = (1,2,3)$ and $\boldsymbol{\rho}_A = ((3),(2,1),(1,1))$.



Below, we list the possible prism fillings of $(\boldsymbol{\rho}_A, \mathbf{p}_A)$.

$\mathcal{T}_1 =$  $\mathcal{T}_2 =$  $\mathcal{T}_3 =$  $\mathcal{T}_4 =$ 

$\mathcal{T}_5 =$  $\mathcal{T}_6 =$ 

i	1	2	3	4	5	6
$\texttt{wt}(\mathcal{T}_i)$	$x_1^3x_2^2x_3$	$x_1^3x_2^2x_3$	$x_1^3x_2^2$	$x_1^3x_2x_3$	$x_1^3x_2x_3$	$x_1^3x_2^2$
minimal	*no*	*no*	*yes*	*yes*	*yes*	*yes*

Among the minimal tableaux, $\mathcal{T}_4$ is obtained by replacing the unstable triple in $\mathcal{T}_5$. Likewise, replacing the unstable triple $\mathcal{T}_6$ produces $\mathcal{T}_3$. As such, $\texttt{Prism}_P(A) = \{\mathcal{T}_3, \mathcal{T}_4\}$. Then

$$\mathfrak{A}_{\boldsymbol{\rho}_A, \mathbf{p}_A} = x_1^3x_2^2 + x_1^3x_2x_3.$$

Notice that $\mathfrak{A}_{\boldsymbol{\beta}_A, \mathbf{b}_A} = \mathfrak{A}_{\boldsymbol{\rho}_A, \mathbf{p}_A}$. This holds in general as a consequence of Theorem 1.1 and the next proposition. □

Proposition 6.1. *(I) $A = A_{\boldsymbol{\beta}_A, \mathbf{b}_A}$.*

(II) $A = A_{\boldsymbol{\rho}_A, \mathbf{p}_A}$.

Proof. (I) Let $\mathrm{Ess}(A) = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ and

$$\beta^{(\ell)} = (i_\ell - r_A(i_\ell, j_\ell)) \times (j_\ell - r_A(i_\ell, j_\ell))$$

as in (89). By construction,

$$[\beta^{(\ell)}, i_\ell]_g = [i_\ell, j_\ell, r_A(i_\ell, j_\ell)]_b. \tag{92}$$

Therefore,

$$\begin{aligned} A &= \vee \mathtt{bigr}(A) && \text{(by (59))} \\ &= \vee\{[i, j, r_A(i, j)]_b : (i, j) \in \mathrm{Ess}(A)\} && \text{(by Proposition 4.2)} \\ &= \vee\{[\beta^{(1)}, i_1]_g, \ldots, [\beta^{(k)}, i_k]_g\} && \text{(by (92))} \\ &= A_{\boldsymbol{\beta}_A, \mathbf{b}_A} && \text{(by (87)).} \end{aligned}$$

(II) Let $u = [\lambda^{(A,i)}, i]_g$. Since $\lambda^{(A,i)} = \lambda^{(u,i)}$, we must have

$$m_A(i, j) = m_u(i, j) \text{ for all } j = 1, \ldots, i.$$

Applying (90), we have

$$r_A(i, j) = r_u(i, j) \text{ for all } j = 1, \ldots, n. \tag{93}$$

Let $\mathbf{p}_A = (i_1, \ldots, i_k)$ be the essential rows of A and let $\boldsymbol{\rho}_A = (\lambda^{(A,i_1)}, \ldots, \lambda^{(A,i_k)})$. Then $\mathbf{u}_{\boldsymbol{\rho}_A, \mathbf{p}_A} = ([\lambda^{(A,i_1)}, i_1]_g, \ldots, [\lambda^{(A,i_k)}, i_k]_g)$.

By (93) and Lemma 4.9, $[\lambda^{(A,i_\ell)}, i_\ell]_g \leq A$ for all $\ell = 1, \ldots, k$. As such, A is an upper bound to $\mathbf{u}_{\boldsymbol{\rho}_A, \mathbf{p}_A}$ and hence

$$A_{\boldsymbol{\rho}_A, \mathbf{p}_A} = \vee \mathbf{u}_{\boldsymbol{\rho}_A, \mathbf{p}_A} \leq A. \tag{94}$$

On the other hand, by (93), $r_{[\lambda^{(A,i)}, i]_g}(i, j) = r_A(i, j)$. Then by Lemma 4.6,

$$[i, j, r_A(i, j)]_b \leq [\lambda^{(A,i)}, i]_g \text{ for all } 1 \leq i, j \leq n.$$

In particular, if $u \in \mathtt{bigr}(A)$, then there is some i_ℓ in the list $\mathbf{p}_A$ so that

$$u \leq [\lambda^{(A_{i_\ell})}, i_\ell]_g \leq A_{\boldsymbol{\rho}_A, \mathbf{p}_A}.$$

Therefore, $A_{\boldsymbol{\rho}_A,\mathbf{p}_A}$ is an upper bound to $\texttt{bigr}(A)$ and hence

$$A = \vee\texttt{bigr}(A) \leq A_{\boldsymbol{\rho}_A,\mathbf{p}_A}. \tag{95}$$

Therefore, by (94) and (95), $A = A_{\boldsymbol{\rho}_A,\mathbf{p}_A}$. □

We note that the parabolic model could also have been defined using a partition shape for *every* row of A. This has the drawback of having more redundant labels in each tableau. However, the prism shapes have a direct connection to the poset of ASMs:

$$A \leq B \text{ if and only if } \lambda^{(A,i)} \subseteq \lambda^{(B,i)} \text{ for all } i = 1, \ldots, n. \tag{96}$$

The description of Bruhat order in (96) generalizes the following description of the poset of Grassmannian permutations with a fixed descent. Take $u, v \in \mathcal{G}_n$ with $\texttt{des}(u) = \texttt{des}(v)$. Then

$$u \leq v \text{ if and only if } \lambda^{(u)} \subseteq \lambda^{(v)}. \tag{97}$$

6.3 Pipe Dreams and the Square Word

Recall s_i is the simple transposition $(i \;\; i+1) \in \mathcal{S}_{2n}$. Define the **square word**

$$\mathcal{Q}_{n\times n} = s_n\, s_{n-1} \,\ldots\, s_1 \quad s_{n+1}\, s_n \,\ldots\, s_2 \quad \ldots \quad s_{2n-1}\, s_{2n-2} \ldots s_n.$$

Order the boxes of the $n \times n$ grid by reading along rows from right to left, starting with the top row and working down to the bottom. This ordering identifies each letter of $\mathcal{Q}_{n\times n}$ with a cell in the $n \times n$ grid.

A **plus diagram** is a subset of the $n \times n$ grid. We indicate (i,j) is in the plus diagram by marking its position in the grid with a $+$. The identification of the letters in $\mathcal{Q}_{n\times n}$ with the grid defines a natural bijection between subwords of $\mathcal{Q}_{n\times n}$ and plus diagrams. As such, we freely identify each word with its plus diagram. Notice that plus diagrams may be immediately identified with pipe dreams. Furthermore, $\mathcal{P}$ as a subword of $\mathcal{Q}_{n\times n}$ represents w, if and only if $\mathcal{P}$ is a reduced pipe dream for w.

Example 6.5. *When $n = 3$, we have*

$$\mathcal{Q}_{n\times n} = s_3 s_2 s_1 s_4 s_3 s_2 s_5 s_4 s_3.$$

Below, we label the entries of the 3×3 grid with their corresponding simple transpositions.

We also give a subword of $\mathcal{Q}_{3\times 3}$ and its corresponding plus diagram.

$$\begin{matrix} s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \\ s_3 & s_4 & s_5 \end{matrix} \qquad s_3 - - - s_3\, s_2 - - - \qquad \begin{matrix} \cdot & \cdot & + \\ + & + & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

Notice that $\mathcal{P}$ is not a reduced expression, $s_3 s_3 s_2 = s_2$. Therefore, it is not a facet of $\Delta(\mathcal{Q}_{n\times n}, A)$ for any $A \in \mathsf{ASM}(n)$. □

For brevity, write $\Delta_A := \Delta(\mathcal{Q}_{n\times n}, A)$. Assign $\mathcal{F}_\mathcal{P}$ the weight

$$\texttt{wt}(\mathcal{F}_\mathcal{P}) = \prod_{i=1}^{n} x_i^{n_i} \text{ where } n_i = \#\{j : (i,j) \in \mathcal{P}\}.$$

By Theorem 5.1, if $w \in \mathcal{S}_n$, the complex Δ_w is a pure simplicial complex. Since its facets are in transparent bijection with pipe dreams, the following holds.

Theorem 6.2 ([FK1996, BB1993, KM2005])**.**

$$\mathfrak{S}_w = \sum_{\mathcal{F}_\mathcal{P} \in F(\Delta_w)} \texttt{wt}(\mathcal{F}_\mathcal{P}). \tag{98}$$

For permutations, Δ_w is the Stanley-Reisner complex of a degeneration of the Schubert determinantal ideal I_w [KM2005, Theorem B]. The same holds for I_A and Δ_A. See Section 7.6 and Section 7.7 for details.

As a consequence of Theorem 6.2, we have the following corollary.

Corollary 6.1. *(I)* $\displaystyle\sum_{w \in \texttt{Perm}(A)} \mathfrak{S}_w = \sum_{\mathcal{F}_\mathcal{P} \in F(\Delta_A)} \texttt{wt}(\mathcal{F}_\mathcal{P}).$

(II) $\displaystyle\sum_{w \in \texttt{MinPerm}(A)} \mathfrak{S}_w = \sum_{\mathcal{F}_\mathcal{P} \in F_{\texttt{max}}(\Delta_A)} \texttt{wt}(\mathcal{F}_\mathcal{P}).$

Proof. (I) By Proposition 5.1,

$$F(\Delta_A) = \bigcup_{w \in \texttt{Perm}(A)} F(\Delta_w). \tag{99}$$

$\mathcal{F}_\mathcal{P}$ is a facet of Δ_w if and only if $\mathcal{P}$ represents w. A subword can represent at most one permutation, so the union in (99) is disjoint. Therefore, applying (98), we have

$$\sum_{w \in \texttt{Perm}(A)} \mathfrak{S}_w = \sum_{w \in \texttt{Perm}(A)} \sum_{\mathcal{F}_\mathcal{P} \in F(\Delta_w)} \texttt{wt}(\mathcal{F}_\mathcal{P}) = \sum_{\mathcal{F}_\mathcal{P} \in F(\Delta_A)} \texttt{wt}(\mathcal{F}_\mathcal{P}).$$

(II) Observe that

$$F_{\mathtt{max}}(\Delta_A) = \bigcup_{w \in \mathtt{MinPerm}(A)} F(\Delta_w). \tag{100}$$

Again this union is disjoint. As such, the result follows. ◻

6.4 Proof of Theorem 1.1

Take $T \in \mathtt{RSSYT}(\lambda, d)$ and write T_{ij} for the entry of T which is in the ith row and jth column in the ambient $\mathbb{Z}^+ \times \mathbb{Z}^+$ grid (as in (82)). Define a plus diagram $\mathcal{P}_T$ by placing a plus in position $(T_{ij}, i+j-T_{ij})$ for each label in T. This in turn defines a map $T \mapsto \mathcal{F}_{\mathcal{P}_T}$. Define $\Phi_{\lambda,d}(T) = \mathcal{F}_{\mathcal{P}_T}$.

Proposition 6.2. *$\Phi_{\lambda,d} : \mathtt{RSSYT}(\lambda, d) \to F(\Delta_{[\lambda,d]_g})$ is a bijection.*

Proposition 6.2 is well known. For a proof, see e.g. [KMY2009, Proposition 5.3]. Define

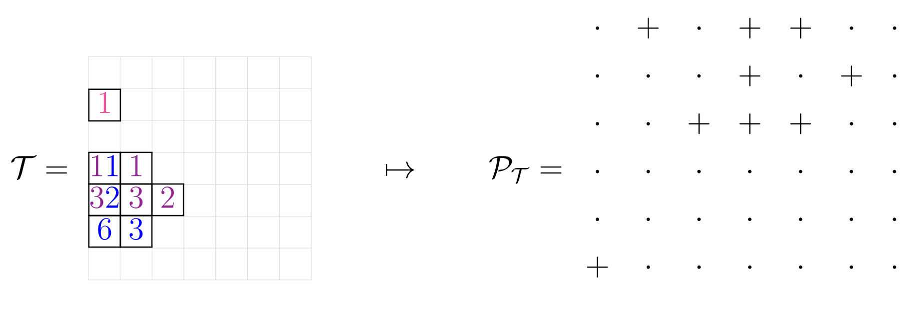
$$\Phi_{\boldsymbol{\lambda},\mathbf{d}} : \mathtt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d}) \to \Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}} \tag{101}$$

where

$$\Phi_{\boldsymbol{\lambda},\mathbf{d}}(T^{(1)}, \ldots, T^{(k)}) = \Phi_{\lambda^{(1)},d_1}(T^{(1)}) \cap \cdots \cap \Phi_{\lambda^{(k)},d_k}(T^{(k)}). \tag{102}$$

Equivalently, $\Phi_{\boldsymbol{\lambda},\mathbf{d}}(T^{(1)}, \ldots, T^{(k)}) = \mathcal{F}_{\mathcal{P}_{\mathcal{T}}}$ where $\mathcal{P}_{\mathcal{T}} = \bigcup_{i=1}^k \mathcal{P}_{T^{(i)}}$. By part (II) of Proposition 5.1, $\Phi_{\boldsymbol{\lambda},\mathbf{d}}$ is well defined.

Example 6.6. *Continuing Example 6.1, we have the following map.*



Notice that $\mathtt{wt}(\mathcal{T}) = \mathtt{wt}(\mathcal{P}_{\mathcal{T}}) = x_1^3 x_2^2 x_3^3 x_6$. ◻

Lemma 6.2. *The map $\Phi_{\boldsymbol{\lambda},\mathbf{d}}$ is weight preserving.*

Proof. The plus diagram $\mathcal{P}_T$ has a plus in position (i, j) if and only if there is some a so that $T_{a,i+j-a} = i$.

Let $\mathcal{F}_{\mathcal{P}} = \Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T})$. Then $\mathcal{P} = \mathcal{P}_{T^{(1)}} \cup \cdots \cup \mathcal{P}_{T^{(k)}}$. Therefore, $(i,j) \in \mathcal{P}$ if and only if $(i,j) \in \mathcal{P}_{T^{(\ell)}}$ for some $\ell \in [k]$. As such, $(i,j) \in P$ if and only if the label i appears in the $(i+j)$th antidiagonal of $\mathcal{T}$. Therefore,

$$\mathtt{wt}(\mathcal{F}_{\mathcal{P}}) = \prod_{(i,j)\in P} x_i = \prod_i x_i^{n_i} = \mathtt{wt}(\mathcal{T}). \qquad \square$$

Notice by Lemma 5.1 and Proposition 6.2,

$$F(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}}) \subseteq \Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathtt{AllPrism}(\boldsymbol{\lambda},\mathbf{d})). \tag{103}$$

If $\mathcal{T}$ is minimal, $\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T}) \in F_{\mathtt{max}}(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$. This implies

$$\mathtt{deg}(\boldsymbol{\lambda},\mathbf{d}) = \mathtt{deg}(A_{\boldsymbol{\lambda},\mathbf{d}}). \tag{104}$$

For permutation matrices, $\mathtt{Perm}(w) = w$ and so $\mathtt{deg}(w) = \ell(w)$. This shows the original definition for a minimal prism tableau given in [WY2018] agrees with the definition stated here.

Call $\mathcal{T}$ **facet** if $\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T}) \in F(\Delta_A)$. Let $\mathtt{Facet}(\boldsymbol{\lambda},\mathbf{d}) \subseteq \mathtt{AllPrism}(\boldsymbol{\lambda},\mathbf{d})$ denote the set of facet prism tableaux. Write $\mathtt{StableFacet}(\boldsymbol{\lambda},\mathbf{d}) \subseteq \mathtt{Facet}(\boldsymbol{\lambda},\mathbf{d})$ for the set of facet tableaux which have no unstable triples. By (103),

$$\mathtt{Perm}(A_{\boldsymbol{\lambda},\mathbf{d}}) = \{w : \Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T}) \in \Delta_w \text{ for some } \mathcal{T} \in \mathtt{Facet}(\boldsymbol{\lambda},\mathbf{d})\}. \tag{105}$$

Example 6.7. *Let A be as in Example 4.4. Set $\boldsymbol{\lambda} = ((2),(2))$ and $\mathbf{d} = (1,2)$. Notice that $\mathbb{S}(\boldsymbol{\lambda},\mathbf{d})$ is* both *the parabolic and the bigrassmannian prism shape for A. Therefore, $A_{\boldsymbol{\lambda},\mathbf{d}} = A$. There are three prism fillings of $\mathbb{S}(\boldsymbol{\lambda},\mathbf{d})$, listed below.*

$$\mathcal{T}_1 = \begin{array}{|c|c|cc}\hline 1 & 1 & & \\ \hline 1 & 1 & & \\ \hline & & & \\ & & & \end{array} \qquad \mathcal{T}_2 = \begin{array}{|c|c|cc}\hline 1 & 1 & & \\ \hline 2 & 1 & & \\ \hline & & & \\ & & & \end{array} \qquad \mathcal{T}_3 = \begin{array}{|c|c|cc}\hline 1 & 1 & & \\ \hline 2 & 2 & & \\ \hline & & & \\ & & & \end{array}$$

Only $\mathcal{T}_1$ is minimal, so $\mathtt{Prism}(\boldsymbol{\lambda},\mathbf{d}) = \{\mathcal{T}_1\}$. Therefore $\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}} = \mathtt{wt}(\mathcal{T}_1) = x_1^3$. The above prism tableaux correspond to the following plus diagrams.

$$P_1 = \begin{array}{cccc} + & + & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad P_2 = \begin{array}{cccc} + & + & + & \cdot \\ + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad P_3 = \begin{array}{cccc} + & + & \cdot & \cdot \\ + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

Since $P_2 \supsetneq P_1$, we have $\mathcal{F}_{\mathcal{P}_2} \subsetneq \mathcal{F}_{\mathcal{P}_1}$. Therefore $\mathcal{T}_2$ is not a facet prism tableau. There are no plus diagrams in the image of $\Phi_{\boldsymbol{\lambda},\mathbf{d}}$ which are strictly contained in P_1 or P_3, so by (103), $\mathcal{F}_{\mathcal{P}_1}, \mathcal{F}_{P_3} \in F(\Delta_A)$. Then $\mathcal{T}_1, \mathcal{T}_3 \in \texttt{Facet}(\boldsymbol{\lambda}, \mathbf{d})$.

The word corresponding to P_1 is $s_3 s_2 s_1 = 4123$ and the word for P_3 is $s_2 s_1 s_3 s_2 = 3412$. Therefore $\texttt{Perm}(A) = \{4123, 3124\}$ and $\texttt{MinPerm}(A) = \{4123\}$. □

Theorem 6.3. *(I) $F(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$ is in weight preserving bijection with $\texttt{StableFacet}(\boldsymbol{\lambda}, \mathbf{d})$.*

(II) The bijection in (I) restricts to a bijection between $F_{\texttt{max}}(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$ and $\texttt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$.

Theorem 1.1 follows as an immediate consequence of Theorem 6.3 and Corollary 6.1.

$$\mathfrak{A}_{\boldsymbol{\lambda},\mathbf{d}} = \sum_{T \in \texttt{Prism}(\boldsymbol{\lambda},\mathbf{d})} \texttt{wt}(T) = \sum_{w \in \texttt{MinPerm}(A_{\boldsymbol{\lambda},\mathbf{d}})} \mathfrak{S}_w. \tag{106}$$

Similarly, we have

$$\sum_{\mathcal{T} \in \texttt{StableFacet}(\boldsymbol{\lambda},\mathbf{d})} \texttt{wt}(\mathcal{T}) = \sum_{w \in \texttt{Perm}(A_{\boldsymbol{\lambda},\mathbf{d}})} \mathfrak{S}_w. \tag{107}$$

For our proof of Theorem 6.3, we analyze the fibers of $\Phi_{\boldsymbol{\lambda},\mathbf{d}}$

$$\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{\mathcal{P}}) = \{\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d}) : \Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T}) = \mathcal{F}_{\mathcal{P}}\}. \tag{108}$$

For an arbitrary face of $\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}}$, this fiber may be empty. However, by (103), facets have nonempty fibers. In Proposition 6.4 we show that the fiber of any facet has the structure of a lattice. Furthermore, the maximum element of $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{\mathcal{P}})$ is the only tableau in the fiber with no unstable triples.

Order $\texttt{RSSYT}(\lambda, d)$ and $\texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{u})$ by entrywise comparison.

Proposition 6.3. *(I) $\texttt{RSSYT}(\lambda, d)$ is a lattice.*

(II) $\texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{u})$ is a lattice.

Proof. (I) Given $T, U \in \texttt{RSSYT}(\lambda, d)$, we claim $T \wedge U = \mathbf{min}(T, U)$ and $T \vee U = \mathbf{max}(T, U)$. Note that

$$\text{if } a_1 \leq a_2 \text{ and } b_1 \leq b_2 \text{ then } \min\{a_1, b_1\} \leq \min\{a_2, b_2\}. \tag{109}$$

Similarly,

$$\text{if } a_1 < a_2 \text{ and } b_1 < b_2 \text{ then } \min\{a_1, b_1\} < \min\{a_2, b_2\}. \tag{110}$$

The same statements hold when replacing min with max. Therefore, (T1) and (T2) are preserved under taking entrywise minima and maxima. Furthermore, $\mathbf{min}(T, U)$ and $\mathbf{max}(T, U)$

use only labels from $[d]$. Then

$$\mathbf{min}(T,U), \mathbf{max}(T,U) \in \texttt{RSSYT}(\lambda, d).$$

By applying Lemma 4.1, we see that $\texttt{RSSYT}(\lambda, d)$ is a lattice.
(II) By (I), $\texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$ is a product of lattices. Then $\texttt{AllPrism}(\boldsymbol{\lambda}, \mathbf{d})$ is itself a lattice. Again, $\mathcal{T} \wedge \mathcal{U} = \mathbf{min}(\mathcal{T}, \mathcal{U})$ and $\mathcal{T} \vee \mathcal{U} = \mathbf{max}(\mathcal{T}, \mathcal{U})$. □

Write

$$\texttt{RSSYT}_{\mathcal{P}}(\lambda, d) := \{T \in \texttt{RSSYT}(\lambda, d) : \Phi_{\lambda,d}(T) \supseteq \mathcal{F}_{\mathcal{P}}\}. \tag{111}$$

Lemma 6.3. *(I) Suppose $T, U \in \texttt{RSSYT}_{\mathcal{P}}(\lambda, d)$. Then*

$$\Phi_{\lambda,d}(T \vee U) \supseteq \mathcal{F}_{\mathcal{P}} \text{ and } \Phi_{\lambda,d}(T \wedge U) \supseteq \mathcal{F}_{\mathcal{P}}.$$

As such, $\texttt{RSSYT}_{\mathcal{P}}(\lambda, d)$ is a lattice.

(II) Suppose $T, U \in \texttt{RSSYT}_{\mathcal{P}}(\lambda, d)$ with $T < U$. Then there exists $V \in \texttt{RSSYT}_{\mathcal{P}}(\lambda, d)$ so that $T < V \leq U$ and V differs from T by increasing the value of a single entry.

(III) Take $\mathcal{T}, \mathcal{U} \in \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{\mathcal{P}})$. Then

$$\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T} \vee \mathcal{U}) \supseteq \mathcal{F}_{\mathcal{P}} \text{ and } \Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T} \wedge \mathcal{U}) \supseteq \mathcal{F}_{\mathcal{P}}.$$

Proof. (I) Let $T, U \in \texttt{RSSYT}_{\mathcal{P}}(\lambda, d)$. Fix an antidiagonal D of λ. Let $\{a_1, a_2, \ldots, a_m\}$ and $\{b_1, b_2, \ldots, b_m\}$ be the ordered lists of labels which appear in antidiagonal D of T and U respectively. Then the entries in antidiagonal D of $T \vee U$ are

$$\{\max(a_1, b_1), \ldots, \max(a_m, b_m)\} \subseteq \{a_1, \ldots, a_m\} \cup \{b_1, \ldots, b_m\}.$$

Since this holds for every antidiagonal,

$$\mathcal{P}_{T \vee U} \subseteq \mathcal{P}_T \cup P_U.$$

Therefore,

$$\Phi_{\lambda,d}(T \vee U) \supseteq \Phi_{\lambda,d}(T) \cap \Phi_{\lambda,d}(U) \supseteq \mathcal{F}_{\mathcal{P}}.$$

The argument for $T \wedge U$ is the same.
(II) Suppose $T, U \in \texttt{RSSYT}_{\mathcal{P}}(\lambda, d)$ and $T < U$. Since $T < U$ all entries of T are (weakly) less than the entries of U. Let $S = \{(i, j) : T_{ij} < U_{ij}\}$. Since $T \neq U$, there is some entry of T that is strictly less than the corresponding entry in U, so $S \neq \emptyset$.

Since S is finite and nonempty, there is some $(i,j) \in S$ so that $(i+1,j),(i,j-1) \notin S$. Then replace the (i,j) entry of T with U_{ij} and call this V. Then $V \in \mathtt{RSSYT}(\lambda, d)$. Furthermore, the entries in each antidiagonal of V form a subset of the union of the antidiagonal entries of T and U so

$$\Phi_{\lambda,d}(V) \supseteq \Phi_{\lambda,d}(T) \cap \Phi_{\lambda,d}(U) \supseteq \mathcal{F}_{\mathcal{P}}.$$

Then we have produced $V \in \mathtt{RSSYT}_{\mathcal{P}}(\lambda, d)$ so that $T < V < U$.
(III) By definition, $\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T}) = \Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{U}) = \mathcal{F}_{\mathcal{P}}$. Then $\Phi_{\lambda^{(i)},d_i}(T^{(i)}) \supseteq \mathcal{F}_{\mathcal{P}}$ and

$$\Phi_{\lambda^{(i)},d_i}(U^{(i)}) \supseteq \mathcal{F}_{\mathcal{P}} \text{ for all } i = 1,\dots,k.$$

Applying (I), we have

$$\Phi_{\lambda^{(i)},d_i}(T^{(i)} \vee U^{(i)}) \supseteq \mathcal{F}_{\mathcal{P}} \text{ and } \Phi_{\lambda^{(i)},d_i}(T^{(i)} \wedge U^{(i)}) \supseteq \mathcal{F}_{\mathcal{P}}.$$

Therefore,

$$\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T} \vee \mathcal{U}) = \Phi_{\lambda^{(1)},d_1}(T^{(1)} \vee U^{(1)}) \cap \cdots \cap \Phi_{\lambda^{(k)},d_k}(T^{(k)} \vee U^{(k)}) \supseteq \mathcal{F}_{\mathcal{P}}$$

and

$$\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T} \wedge \mathcal{U}) = \Phi_{\lambda^{(1)},d_1}(T^{(1)} \wedge U^{(1)}) \cap \cdots \cap \Phi_{\lambda^{(k)},d_k}(T^{(k)} \wedge U^{(k)}) \supseteq \mathcal{F}_{\mathcal{P}}.$$ □

Proposition 6.4. *Fix* $\mathcal{F}_{\mathcal{P}} \in F(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$.

(I) $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{\mathcal{P}})$ *is a lattice.*

(II) Suppose $\mathcal{T},\mathcal{U} \in \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{\mathcal{P}})$ *with* $\mathcal{T} < \mathcal{U}$. *Then* $\mathcal{T}$ *has an unstable triple.*

(III) $\#\mathtt{StableFacet}(\boldsymbol{\lambda},\mathbf{d}) \cap \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{\mathcal{P}}) = 1$.

Proof. (I) We have that $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{\mathcal{P}})$ is a subposet of $\mathtt{AllPrism}(\boldsymbol{\lambda},\mathbf{d})$. Therefore, it is enough to show that $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{\mathcal{P}})$ is closed under taking joins and meets.

By Lemma 6.3, $\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T} \wedge \mathcal{U}) \supseteq \mathcal{F}_{\mathcal{P}}$. Since $\mathcal{F}_{\mathcal{P}} \in F(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$, this containment is actually an equality. Therefore, $\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T} \wedge \mathcal{U}) \in \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{\mathcal{P}})$, i.e., $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{\mathcal{P}})$ is closed under joins. The argument for meets is the same. Then we conclude that $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{\mathcal{P}})$ is a lattice.
(II) Suppose $\mathcal{U} > \mathcal{T}$. In particular, for some i, we have $U^{(i)} > T^{(i)}$. By the part 2 of Lemma 6.3, there is $V \in \mathtt{RSSYT}_{\mathcal{P}}(\lambda^{(i)}, d_i)$ with $U^{(i)} \geq V > T^{(i)}$ so that V differs from $T^{(i)}$ by increasing the value a single entry.

Since $\Phi_{\lambda^{(\ell)},d_\ell}(T^{(\ell)}) \supseteq \mathcal{F}_\mathcal{P}$ for all $\ell = 1,\dots,k$ and $\Phi_{\lambda^{(i)},d_i}(V) \supseteq \mathcal{F}_\mathcal{P}$, we have

$$\Phi_{\boldsymbol{\lambda},\mathbf{d}}(T^{(1)},\dots,T^{(i-1)},V,T^{(i+1)},\dots T^{(k)}) \supseteq \mathcal{F}_\mathcal{P}. \tag{112}$$

Since $\mathcal{F}_\mathcal{P}$ is a facet, (112) is an equality. Then

$$(T^{(1)},\dots,T^{(i-1)},V,T^{(i+1)},\dots T^{(k)}) \in \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_\mathcal{P}).$$

Therefore, $\mathcal{T}$ has an unstable triple.
(III) By (I), $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_\mathcal{P})$ is a lattice. In particular, it is finite and nonempty so has a unique maximum element.

By (II), if $\mathcal{T}$ is not the maximum of $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_\mathcal{P})$, then it has an unstable triple. Conversely, if $\mathcal{T}$ has an unstable triple, then by definition, there is $\mathcal{T}' \in \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_\mathcal{P})$ with $\mathcal{T} < \mathcal{T}'$. As such, $\mathcal{T}$ is not the maximum. Therefore, $\texttt{StableFacet}(\boldsymbol{\lambda},\mathbf{d}) \cap \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_\mathcal{P})$ is the maximum of $\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_\mathcal{P})$. Then

$$\#\texttt{StableFacet}(\boldsymbol{\lambda},\mathbf{d}) \cap \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_\mathcal{P}) = 1. \qquad \square$$

Proof of Theorem 6.3. (I) Define $\Psi : F(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}}) \to \texttt{StableFacet}(\boldsymbol{\lambda},\mathbf{d})$ by mapping $\mathcal{F}_\mathcal{P}$ to the unique element in $\texttt{StableFacet}(\boldsymbol{\lambda},\mathbf{d}) \cap \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_\mathcal{P})$. By Proposition 6.4 part (III), this is well defined. Injectivity follows since

$$\Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_\mathcal{P}) \cap \Phi^{-1}_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{F}_{\mathcal{P}'}) = \emptyset$$

whenever $\mathcal{P} \neq \mathcal{P}'$.

Given $\mathcal{T} \in \texttt{StableFacet}(\boldsymbol{\lambda},\mathbf{d})$, let $\mathcal{F}_\mathcal{P} = \Phi_{\boldsymbol{\lambda},\mathbf{u}}$. By the definition of $\texttt{StableFacet}(\boldsymbol{\lambda},\mathbf{d})$, we have $\mathcal{F}_\mathcal{P} \in F(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$. Then $\mathcal{T} = \Psi(\mathcal{F}_\mathcal{P})$. As such, Ψ is surjective.

Since $\Phi_{\boldsymbol{\lambda},\mathbf{u}}$ is weight preserving, $\Phi(F_{\texttt{max}}(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})) = \texttt{Prism}(\boldsymbol{\lambda},\mathbf{d})$. $\square$

We conclude by showing that minimal parabolic prism tableaux for permutations do not have unstable triples.

Lemma 6.4. *Fix a minimal prism tableaux $\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda},\mathbf{d})$. Let w be the permutation represented by $\mathcal{P}_\mathcal{T}$. If $\mathcal{T}$ has an unstable triple, then there exists i so that $[\lambda^{(i)},d_i]_g \neq w^{(i)}$.*

Proof. Fix $(\boldsymbol{\lambda},\mathbf{d})$. Take a minimal tableau $\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\lambda},\mathbf{d})$. Then

$$\Phi_{\boldsymbol{\lambda},\mathbf{d}}(\mathcal{T}) = \mathcal{F}_{\mathcal{P}_\mathcal{T}} \in F_{\texttt{max}}(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$$

and $\mathcal{P}_\mathcal{T}$ is a reduced word for some $w \in \mathcal{S}_n$.

Suppose $\mathcal{T} = (T^{(1)}, \ldots, T^{(k)})$ has an unstable triple. Then there is some index i so that $\mathcal{T}' = (T^{(1)}, \ldots, T, \ldots T^{(k)})$ for some T which differs from $T^{(i)}$ by increasing the value of a single entry.

By Proposition 6.2, $\mathcal{P}_T$ and $\mathcal{P}_{T^{(i)}}$ are distinct subwords of $\mathcal{P}_{\mathcal{T}}$ and they both represent $[\lambda^{(i)}, d_i]_g$. Applying Proposition 5.3, we see that $[\lambda^{(i)}, d_i]_g \neq w^{(i)}$. $\square$

We conclude by showing that minimal parabolic prism tableaux for permutations do not have unstable triples.

Proof of Theorem 6.1. Fix $w \in \mathcal{S}_n$ and a minimal tableau $\mathcal{T} \in \texttt{AllPrism}(\boldsymbol{\rho}_w, \mathbf{p}_w)$. From Proposition 6.1, $w = A_{\boldsymbol{\rho}_w, \mathbf{p}_w}$. Since $\mathcal{T}$ is minimal, $\mathcal{P}_{\mathcal{T}}$ represents w. By Lemma 6.1, we have $[\boldsymbol{\rho}_i, \mathbf{p}_i]_g = w^{(i)}$ for all i. Applying Lemma 6.4, we see that $\mathcal{T}$ does not have unstable triples. $\square$

CHAPTER 7

MULTIDEGREES OF ASM VARIETIES

In this chapter, we study a class of subvarieties of the space of $n \times n$ matrices. These varieties generalize the matrix Schubert varieties of Fulton. Their multidegrees have a natural interpretation in terms of prism tableaux. This work originally appeared in [Wei2017a].

7.1 Stanley-Reisner Theory

Let $\mathbb{k}[\mathbf{z}] = \mathbb{k}[z_1, \ldots, z_N]$. Given $\mathbf{v} = (v_1, \ldots, v_N) \in \mathbb{N}^N$, write $\mathbf{z}^{\mathbf{v}} := \prod_{i=1}^N z_i^{v_i}$. If $\mathbf{v} \in \{0,1\}^N$, then $\mathbf{z}^{\mathbf{v}}$ is a **square-free monomial**. An ideal is called a **square-free monomial ideal** if it has a generating set of square-free monomials. Stanley-Reisner theory describes the correspondence between square-free monomial ideals in $\mathbb{k}[\mathbf{z}]$ and simplicial complexes $\Delta \subseteq \mathbb{P}([N])$. We give a brief overview. For more background, see [MS2004, Chapter 1].

Notice square-free monomials in $\mathbb{k}[\mathbf{z}]$ correspond to faces in $\mathbb{P}([N])$. Given $f \in \mathbb{P}([N])$, write $\mathbf{z}^f = \displaystyle\prod_{i \in f} z_i$.

Definition 7.1. *The* **Stanley-Reisner ideal** *of* Δ *is*

$$I_\Delta = \langle \mathbf{z}^f : f \notin \Delta \rangle.$$

The quotient $\mathbb{k}[\mathbf{z}]/I_\Delta$ *is called the* **Stanley-Reisner ring** *of* Δ*.*

Write $\mathfrak{m}_f = \langle z_i : i \in f \rangle$ and let $\overline{f} = [N] - f$.

Theorem 7.1. *The map* $\Delta \mapsto I_\Delta$ *is a bijection between square-free monomial ideals in* $\mathbb{k}[\mathbf{z}]$ *and simplicial complexes* $\Delta \subseteq \mathbb{P}([N])$*. The ideal* I_Δ *can be expressed as an intersection of monomial prime ideals*

$$I_\Delta = \bigcap_{f \in \Delta} \mathfrak{m}_{\overline{f}}. \tag{113}$$

Proof. See [MS2004, Theorem 1.7]. □

Explicitly, the inverse map takes a square-free monomial ideal I to

$$\Delta(I) := \{f \subseteq [N] : \mathbf{z}^f \notin I\}.$$

Given a square-free monomial ideal I, we say $\Delta(I)$ is the **Stanley-Reisner complex** associated to I.

The following lemma is straightforward from Definition 7.1, but we give the details.

Lemma 7.1. *Let $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ be a set square-free monomial ideals with $I_\alpha \subseteq \Bbbk[z_1, \ldots, z_N]$. Then $\Delta(\sum_{\alpha\in\mathcal{A}} I_\alpha) = \bigcap_{\alpha\in\mathcal{A}} \Delta(I_\alpha)$.*

Proof. A generating set for $\sum_{\alpha\in\mathcal{A}} I_\alpha$ can be obtained by concatenation of the generating sets for the I_α's. Therefore, it is a square-free monomial ideal. Notice a monomial $m \in \sum_{\alpha\in\mathcal{A}} I_\alpha$ if and only if $m \in I_\alpha$ for some $\alpha \in \mathcal{A}$.

Assume $f \subseteq [N]$. Then,

$$\begin{aligned} f \in \Delta(\sum_{\alpha\in\mathcal{A}} I_\alpha) &\iff \mathbf{z}^f \notin \sum_{\alpha\in\mathcal{A}} I_\alpha \\ &\iff \mathbf{z}^f \notin I_\alpha \text{ for all } \alpha \in \mathcal{A} \\ &\iff f \in \Delta(I_\alpha) \text{ for all } \alpha \in \mathcal{A} \\ &\iff f \in \bigcap_{\alpha\in\mathcal{A}} \Delta(I_\alpha). \end{aligned}$$

□

7.2 Multidegrees

In this section, we review multidegrees. See [MS2004, Chapter 8] for an introduction. We say $\Bbbk[\mathbf{z}]$ is **multigraded** by $\mathbb{Z}^n$ if there is a semigroup homomorphism $\mathtt{d} : \mathbb{N}^N \to \mathbb{Z}^n$. We may interpret $\mathtt{d}$ as a map from monomials in $\Bbbk[\mathbf{z}]$ to elements of $\mathbb{Z}^n$. As such, we write $\mathtt{d}(\mathbf{z}^\mathbf{v}) := \mathtt{d}(\mathbf{v})$.

Write $\Bbbk[\mathbf{z}]_\mathbf{a}$ for the $\Bbbk$ vector space which has as a basis the monomials of degree $\mathbf{a}$,

$$\{\mathbf{z}^\mathbf{v} : \mathtt{d}(\mathbf{z}^\mathbf{v}) = \mathbf{a}\}.$$

As a vector space, $\Bbbk[\mathbf{z}] = \bigoplus_{\mathbf{a}\in\mathcal{A}} \Bbbk[\mathbf{z}]_\mathbf{a}$. A $\Bbbk[\mathbf{z}]$-module M is multigraded by $\Bbbk[\mathbf{z}]$ if it has a direct sum decomposition $M = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} M_\mathbf{a}$ which satisfies

$$\Bbbk[\mathbf{z}]_\mathbf{a} \cdot M_\mathbf{b} \subseteq M_{\mathbf{a}+\mathbf{b}}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$. We will assume that the multigrading is **positive**, that is each of the graded pieces of $\Bbbk[\mathbf{z}]$ are finite dimensional as $\Bbbk$ vector spaces.

Let $\mathcal{C}$ be a function from finitely generated, graded S modules to $\mathbb{Z}[x_1, \ldots, x_n]$. We say $\mathcal{C}$ is **additive** if for each M

$$\mathcal{C}(M; \mathbf{x}) = \sum_{i=1}^{k} \mathtt{mult}(M, \mathfrak{p}_i)\mathcal{C}(\Bbbk[\mathbf{z}]/\mathfrak{p}_i; \mathbf{x}). \tag{114}$$

Here, $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$ is the set of maximal dimensional associated primes of M and $\mathtt{mult}(M, \mathfrak{p})$ is the *multiplicity* of M at $\mathfrak{p}$. See [Eis1995, Section 3.6].

Fix a monomial term order on $\Bbbk[\mathbf{z}]$. If $f \in \Bbbk[\mathbf{z}]$, write $\mathtt{init}(f)$ for its lead term. The **initial ideal** of I is

$$\mathtt{init}(I) := \{\mathtt{init}(f) : f \in I\}.$$

$\mathcal{C}$ is **degenerative** if given a graded free presentation F/K, we have

$$\mathcal{C}(F/K; \mathbf{x}) = \mathcal{C}(F/\mathtt{init}(K); \mathbf{x}). \tag{115}$$

Write $\langle \mathbf{a}, \mathbf{x} \rangle := a_1x_1 + a_2x_2 + \cdots + a_nx_n$.

Theorem 7.2. *[MS2004, Theorem 8.44] There is a unique function $\mathcal{C}$ which is additive and degenerative so that*

$$\mathcal{C}(\Bbbk[\mathbf{z}]/\langle z_{i_1}, \ldots, z_{i_k} \rangle; \mathbf{x}) = \prod_{\ell=1}^{k} \langle \mathtt{d}(z_{i_\ell}), \mathbf{x} \rangle. \tag{116}$$

$\mathcal{C}(M; \mathbf{x})$ is called the **multidegree** of M.

Lemma 7.2. *Suppose I is a square-free monomial ideal in $\Bbbk[\mathbf{z}]$. Then*

$$\mathcal{C}(\Bbbk[\mathbf{z}]/I; \mathbf{x}) = \sum_{f \in F_{\mathtt{max}}(\Delta(I))} C(\Bbbk[\mathbf{z}]/\mathfrak{m}_{\overline{f}}; \mathbf{x}).$$

Proof. Since I is square-free, it has the prime decomposition

$$I = \bigcap_{f \in F(\Delta(I))} \mathfrak{m}_{\overline{f}}.$$

Squarefree monomial ideals are radical, and so

$$\mathtt{mult}(\Bbbk[\mathbf{z}]/I, \Bbbk[\mathbf{z}]/\mathfrak{m}_{\overline{f}}) = 1$$

whenever $f \in F(\Delta(I))$. The maximal dimensional associated primes of $\Bbbk[\mathbf{z}]$ are

$$\{\mathfrak{m}_{\overline{f}} : f \in F_{\mathtt{max}}(\Delta(I))\}.$$

Applying additivity,

$$\mathcal{C}(\Bbbk[\mathbf{z}]/I; \mathbf{x}) = \sum_{f \in F_{\mathtt{max}}(\Delta(I))} \mathcal{C}(\Bbbk[\mathbf{z}]/\mathfrak{m}_{\overline{f}}; \mathbf{x}). \qquad \square$$

7.3 Matrix Schubert Varieties

In this section, we follow [MS2004, Chapter 15] as a general reference. Let $\mathsf{Mat}(n)$ denote the space of $n \times n$ matrices with coefficients in an algebraically closed field $\Bbbk$. Let $\mathbf{z} = (z_{ij})_{i,j=1}^{n}$ be a matrix of generic variables and write $\Bbbk[\mathbf{z}] = \Bbbk[z_{11}, z_{12}, \ldots, z_{nn}]$ for the coordinate ring of $\mathsf{Mat}(n)$.

The **classical determinantal variety** $X_r \subseteq \mathsf{Mat}(n)$ is the set of $n \times n$ matrices of rank at most r

$$X_r = \{M \in \mathsf{Mat}(n) : \operatorname{rank}(M) \leq r\}.$$

X_r is an irreducible subvariety of $\mathsf{Mat}(n)$. Its corresponding radical ideal is generated by the size $r+1$ minors in $\mathbf{z}$

$$I_r = \langle \text{minors of size } r+1 \text{ in } \mathbf{z}\rangle.$$

This is a special case of a *matrix Schubert variety.*

Recall $M_{[i],[j]}$ is the submatrix of M obtained by taking the first i columns and j rows. Given a partial permutation matrix $w \in \mathsf{P}(n)$, we define the **matrix Schubert variety**

$$X_w = \{M \in \mathsf{Mat}(n) : \operatorname{rank}(M_{[i],[j]}) \leq r_w(i,j) \text{ for all } 1 \leq i, j \leq n\}. \qquad (117)$$

Fulton showed that X_w is irreducible [Ful1992].

Recall $\mathsf{B}_-, \mathsf{B}_+ \subset \mathsf{GL}(n)$ are the Borel subgroups of lower triangular and upper triangular matrices respectively. There is a left action of $\mathsf{B}_- \times \mathsf{B}_+$ on $\mathsf{Mat}(n)$ given by

$$(b_1, b_2) \cdot M := b_1 M b_2^{-1}. \qquad (118)$$

Write $\Omega_w = \mathsf{B}_- \times \mathsf{B}_+ \cdot w$ for the orbit which contains the partial permutation w. We recall some facts about $\mathsf{B}_- \times \mathsf{B}_+$ orbits.

Proposition 7.1. *(I) If $M \in \Omega_w$, then $\operatorname{rank}(M_{[i],[j]}) = r_w(i,j)$ for all $1 \leq i, j \leq n$.*

(II) There is a unique $w \in \mathsf{P}(n)$ in each $\mathsf{B}_- \times \mathsf{B}$ orbit.

(III) $X_w = \overline{\Omega_w}$. Furthermore, X_w is irreducible and has dimension $n^2 - \ell(w)$.

Proof. (I) The action of $\mathsf{B}_- \times \mathsf{B}_+$ on $M \in \mathsf{Mat}(n)$ is by row operations which sweep downwards and column operations which sweep to the right. Restricted to $M_{[i],[j]}$ this action is just row and column operations within $M_{[i],[j]}$. Therefore, $\mathrm{rank}(M_{[i],[j]})$ is stable under this action for all $1 \leq i, j \leq n$. In particular, $\mathrm{rank}(w_{[i],[j]}) = r_w(i,j)$, so the result follows.
(II) This is proved in [MS2004, Proposition 15.27].
(III) See [MS2004, Theorem 15.31]. □

Recall T is the torus of invertible diagonal matrices. There is an action of T on $\mathsf{Mat}(n)$ by left multiplication. Given $M \in \mathsf{Mat}(n)$, the rank of *any* submatrix is preserved under the action of T. In particular, X_w is T stable. Define a degree map $\mathtt{d}$ by $\mathtt{d}(z_{ij}) = i$. The associated multigrading corresponds to the action of T on $\mathsf{Mat}(n)$. In particular, T stable subvarieties of $\mathsf{Mat}(n)$ have coordinate rings that are $\Bbbk[\mathbf{z}]$-graded modules. When $\Bbbk[\mathbf{z}]/I$ is the coordinate ring of $X \subseteq \mathsf{Mat}(n)$, write $\mathcal{C}(X;\mathbf{x}) := \mathcal{C}(\Bbbk[\mathbf{z}]/I;\mathbf{x})$. In this situation, (114) becomes

$$\mathcal{C}(X;\mathbf{x}) = \sum_{i=1}^{k} \mathcal{C}(X_i;\mathbf{x}) \tag{119}$$

where $\{X_1, \ldots, X_k\}$ are the maximal dimensional irreducible components of X. Since I is radical, (119) is a multiplicity free sum.

By [KM2005, Theorem A], when $w \in \mathcal{S}_n$,

$$\mathcal{C}(X_w;\mathbf{x}) = \mathfrak{S}_w. \tag{120}$$

One of the major goals of [KM2005] was to exhibit a geometrically natural explanation for previously known combinatorial models for Schubert polynomials. We explain a similar interpretation for prism tableau and the Gröbner geometry of *alternating sign matrix varieties.* This is discussed in Section 7.7.

7.4 ASM Varieties

We now present a generalization of matrix Schubert varieties. Given $A \in \mathsf{ASM}(n)$, we define the **alternating sign matrix variety**

$$X_A := \{M \in M_{[i],[j]} : \mathrm{rank}(M_{[i],[j]}) \leq r_A(i,j) \text{ for all } 1 \leq i,j \leq n\}.$$

Immediately by definition,

$$\text{if } A \le B \text{ then } X_A \supseteq X_B. \tag{121}$$

X_A has the following set theoretic descriptions as unions and intersections of other ASM varieties.

Proposition 7.2. *(I)* $X_A = \bigcup_{w \in \texttt{Perm}(A)} X_w$.

(II) If $A = \vee\{A_1, \ldots, A_k\}$, *then* $X_A = \bigcap_{i=1}^{k} A_i$.

Proof. (I) ($\subseteq$) Fix $M \in X_A$. Then $M \in \Omega_w$ for some $w \in \mathsf{P}(n)$ and $r_w \le r_A$. By Corollary 5.1, there exists $w' \in \texttt{Perm}(A)$ so that $w \ge w'$. Then $M \in X_w \subseteq X_{w'}$. Hence $M \in \bigcup_{w \in \texttt{Perm}(A)} X_w$.
($\supseteq$) If $w \in \texttt{Perm}(A)$ then $w \ge A$. Then by (121), $X_A \supseteq X_w$. Therefore,

$$X_A \supseteq \bigcup_{w \in \texttt{Perm}(A)} X_w.$$

(II) ($\subseteq$) We have $A \ge A_i$ for all i. Then by (121), $X_A \subseteq \bigcap_{i=1}^{k} X_{A_i}$.
($\supseteq$) Take $M \in \bigcap_{i=1}^{k} X_{A_i}$. Then $M \in \Omega_w$ for some $w \in \mathsf{P}(n)$. Since $M \in X_{A_i}$ for all i, we have $w \ge A_i$ for all i. Then $w \ge A = \vee\{A_1, \ldots, A_k\}$. Therefore $M \in \Omega_w \subseteq X_A$. $\square$

Fulton showed that each X_w is defined by a smaller set of **essential** conditions,

$$X_w = \{M \in M_{[i],[j]} : \operatorname{rank}(M_{[i],[j]}) \le r_w(i,j) \text{ for all } (i,j) \in \operatorname{Ess}(w)\}. \tag{122}$$

By Proposition 7.2, $X_A = \bigcap_{u \in \texttt{bigr}(A)} X_u$. Therefore, ASM varieties are also defined by essential conditions.

$$X_A = \{M \in M_{[i],[j]} : \operatorname{rank}(M_{[i],[j]}) \le r_A(i,j) \text{ for all } (i,j) \in \operatorname{Ess}(A)\}. \tag{123}$$

The rank of *any* submatrix is preserved under the action of T, so X_A is T stable. As such, we may consider its multidegree.

Proposition 7.3. $\mathcal{C}(X_A; \mathbf{x}) = \sum_{w \in \texttt{MinPerm}(A)} \mathfrak{S}_w$.

Proof. As a consequence of Proposition 7.2, the top dimensional irreducible components of X_A are $\{X_w : w \in \texttt{MinPerm}(A)\}$. Then using the additivity property of multidegrees and

(120), we have

$$\mathcal{C}(X_A;\mathbf{x}) = \sum_{w\in\texttt{MinPerm}(A)} \mathcal{C}(X_w;\mathbf{x}) = \sum_{w\in\texttt{MinPerm}(A)} \mathfrak{S}_w. \qquad \square$$

Theorem 1.2 follows as an immediate consequence of Proposition 7.3 and Theorem 1.1.

7.5 Northwest Rank Conditions

It is possible to consider more general rank conditions than those defined by corner sums of ASMs. Let $\mathbf{r} = (r_{ij})_{i,j=1}^{n}$ with $r_{ij} \in \mathbb{N} \cup \{\infty\}$. The **northwest rank variety** is

$$X_{\mathbf{r}} := \{M \in M_{[i],[j]} : \operatorname{rank}(M_{[i],[j]}) \leq r_{ij} \text{ for all } 1 \leq i, j \leq n\}. \tag{124}$$

Fulton showed that $X_{\mathbf{r}}$ is irreducible if and only if $\mathbf{r} = r_w$ for some $w \in \mathsf{P}(n)$. It is stable under the $\mathsf{B}_- \times \mathsf{B}_+$ orbit, so decomposes as a union of (partial) matrix Schubert varieties. Z. Xu-an and G. Hongzhu classified northwest rank varieties and gave an algorithm to decompose them into their irreducible components [ZG2008].

We give an alternative discussion using the order theoretic properties of partial ASMs. *A priori*, $X_{\mathbf{r}}$ appears to be a more general object than an ASM variety. We will show, up to an affine factor, $X_{\mathbf{r}}$ is isomorphic to some ASM variety. Furthermore, $X_{\mathbf{r}} = X_{r_A}$ for some $A \in \mathsf{PA}(n)$. Therefore, northwest rank varieties are indexed by partial ASMs.

Lemma 7.3. *Let $A \in \mathsf{PA}(n)$ and $\widetilde{A} \in \mathsf{ASM}(N)$ its completion to an honest ASM. Then*

$$X_A \times \Bbbk^{N^2-n^2} \cong X_{\widetilde{A}}.$$

Proof. By construction, $r_{\widetilde{A}}(i,j) = r_A(i,j)$ for all $1 \leq i, j \leq n$. Therefore, if $M \in X_{\widetilde{A}}$ then $M_{[n],[n]} \in X_A$. Conversely, fix $L \in X_A$. We have $L \in \Omega_w$ for some $w \in \mathsf{P}(n)$, with $w \geq A$. Let L' be any matrix in $\mathsf{Mat}(N)$ so that $L'_{[n],[n]} = L$. Then $L' \in \Omega_v$ for some $v \in \mathsf{P}(N)$. Consider the completions $\widetilde{w}, \widetilde{v} \in \mathsf{ASM}(\infty)$. Since $A \leq w$, we have $\widetilde{A} \leq \widetilde{w} \in \mathsf{ASM}(\infty)$.

By construction, $\widetilde{w}$ is the minimum among elements of $\mathsf{ASM}(\infty)$ which restrict to w in $\mathsf{P}(n)$. Since $v_{[n],[n]} = w$, we have $\widetilde{A} \leq \widetilde{w} \leq \widetilde{v} \in \mathsf{ASM}(\infty)$. Then $\widetilde{A} \leq \widetilde{v}_{[N],[N]} = v \in \mathsf{PA}(N)$. Therefore, $L' \in \Omega_v \subseteq X_{\widetilde{A}}$. As such, $X_{\widetilde{A}} \cong \{L \times \Bbbk^{N^2-n^2} : L \in X_A\}$. $\square$

Fix a rank function $\mathbf{r} = (r_{ij})_{i,j=1}^{n}$. Let

$$A_{\mathbf{r}} = \vee\{[i,j,r_{ij}]_b : r_{ij} < n\} \in \mathsf{PA}(n). \tag{125}$$

Proposition 7.4. $X_{A_{\mathbf{r}}} = X_{\mathbf{r}}$.

Proof. If $r_{ij} \geq n$, it is a vacuous rank condition on matrices in $\mathsf{Mat}(n)$. As such, we may ignore these entries of $\mathbf{r}$. By definition, $X_{A_{\mathbf{r}}} = \bigcap X_{[i,j,r_{ij}]_b}$ with the intersection taken over (i,j) indexing nonvacuous rank conditions.

If $M \in X_{\mathbf{r}}$ we have $M \in X_{[i,j,r_{ij}]_b}$ for all $1 \leq i,j \leq n$. Therefore, $M \in X_{A_{\mathbf{r}}}$. Conversely, if $M \in X_{A_{\mathbf{r}}}$, then $\operatorname{rank}(M_{[i,j]}) \leq r_A(i,j) \leq r_{ij}$ whenever r_{ij} is a nonvacuous rank condition. As such, $M \in X_{\mathbf{r}}$. □

Notice that unions of matrix Schubert varieties need not be northwest rank varieties.

Example 7.1. *Let $X = X_{132} \cup X_{213}$. If $X = X_{\mathbf{r}}$, then $r_{132}, r_{213} \leq \mathbf{r}$. As such, $r_{132} \wedge r_{213} \leq \mathbf{r}$. But $r_{132} \wedge r_{213} = r_{123}$. Since $\mathtt{dim}(X_{123}) > \mathtt{dim}(X)$, it follows that X cannot be defined by a list of northwest rank conditions.* □

7.6 ASM Determinantal Ideals

We now turn our discussion to defining ideals for ASM varieties. Recall the ASM ideal

$$I_A := \langle \text{minors of size } r_A(i,j)+1 \text{ in } \mathbf{z}_{[i],[j]} \rangle. \tag{126}$$

A matrix has rank at most r if and only if all of its minors of size $r+1$ vanish. As such, I_A set-theoretically cuts out X_A. Furthermore, I_A has generators which are homogeneous for the $\mathbb{Z}^n$ grading on $\Bbbk[\mathbf{z}]$.

Lemma 7.4. *(I) If $r_A \leq r_B$ then $I_A \supseteq I_B$.*

(II) $I_A = \sum_{u \in \mathtt{bigr}(A)} I_u = \langle$*minors of size* $r_A(i,j)+1$ *in* $\mathbf{z}_{[i],[j]} : (i,j) \in \operatorname{Ess}(A)\rangle$.

Proof. (I) Define

$$I^r_{i,j} = \langle \text{minors of size } r+1 \text{ in } \mathbf{z}_{[i],[j]} \rangle. \tag{127}$$

We may compute each minor by iteratively doing row expansions. As such,

$$\text{if } r \leq r' \text{ then } I^r_{i,j} \supseteq I^{r'}_{i,j}. \tag{128}$$

Therefore, suppose $r_A \leq r_B$. Then

$$I_A = \sum_j \sum_i I_{i,j}^{r_A(i,j)} \supseteq \sum_j \sum_i I_{i,j}^{r_B(i,j)} = I_B.$$

(II) For each (i,j) there is $u \in \texttt{bigr}(A)$ so that $r_A(i,j) = r_u(i,j)$. As such, $I_{i,j}^{r_A(i,j)} \subseteq I_u$ for some u in $\texttt{bigr}(A)$. By part (I), $I_u \subseteq I_A$, for all $u \in \texttt{bigr}(A)$. Therefore

$$I_A = \sum_j \sum_i I_{i,j}^{r_A(i,j)} \subseteq \sum_{u \in \texttt{bigr}(A)} I_u \subseteq I_A. \qquad \square$$

To distinguish between the two generating sets of I_A, we refer to

$$\texttt{Gen}(A) = \{\text{minors of size } \ r_A(i,j)+1 \ \text{ in } \ \mathbf{z}_{[i],[j]}\}$$

as the **defining generators** of I_A. Call

$$\texttt{EssGen}(A) = \{\text{minors of size } \ r_A(i,j)+1 \ \text{ in } \ \mathbf{z}_{[i],[j]} : (i,j) \in \mathrm{Ess}(A)\}$$

the **essential generators** of I_A.

Example 7.2. *Let A be as in Example 4.4. We have* $\mathrm{Ess}(A) = \{(1,2),(2,3)\}$. *Furthermore,* $r_A(1,2) = 0$ *and* $r_A(2,3) = 1$. *Applying Lemma 7.4 yields*

$$\begin{aligned} I_A &= \left\langle z_{11}, z_{12}, \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} \right\rangle \\ &= \langle z_{11}, z_{12,}, z_{13}z_{21}, z_{13}z_{22} \rangle \\ &= \langle z_{11}, z_{12}, z_{21}, z_{22} \rangle \cap \langle z_{11}, z_{12}, z_{13} \rangle \\ &= I_{3412} \cap I_{4123}. \end{aligned}$$

This agrees with the irreducible decomposition $X_A = X_{3412} \cup X_{4123}$. *Notice by additivity,* $\mathcal{C}(\Bbbk[\mathbf{z}]/I_A; \mathbf{x}) = \mathcal{C}(\Bbbk[\mathbf{z}]/I_{4123}; \mathbf{x}) = x_1^3 = \mathfrak{S}_{4123}$. $\square$

7.7 Initial Ideals and Prism Tableaux

An **antidiagonal** term order on $\Bbbk[\mathbf{z}]$ is a term order for which the lead term of any minor in Z is the product of its antidiagonal terms. From now on, fix an antidiagonal term order $\prec$ on $\Bbbk[\mathbf{z}]$. A **Gröbner basis** for I is a set $\{g_1, \ldots, g_k : g_i \in \Bbbk[\mathbf{z}]\}$ so that

(I) $I = \langle g_1, \ldots, g_k \rangle$, and

(II) $\texttt{init}(I) = \langle \texttt{init}(g_1), \ldots, \texttt{init}(g_k) \rangle$.

Proof of Proposition 1.1. (I) If $w \in \mathcal{S}_n$, by Section 7.2 of [Knu2009], there is a Frobenius splitting for which X_w is compatibly split. Since $X_A = \bigcap_{u \in \mathtt{bigr}(A)} X_u$, it is also compatibly split.

By the argument in [Stu1990], $\mathtt{EssGen}(u)$ is a Gröbner basis for I_u. Since $\mathcal{B}_n$ is the base of $\mathsf{ASM}(n)$, we may apply part (II) of [KM2005, Theorem 6]. As

$$\mathtt{EssGen}(A) = \bigcup_{u \in \mathtt{bigr}(A)} \mathtt{EssGen}(u),$$

it is a Gröbner basis for I_A. Since $\mathtt{Gen}(A) \supseteq \mathtt{EssGen}(A)$, we have that $\mathtt{Gen}(A)$ is also a Gröbner basis for I_A.

(II) The lead terms of $\mathtt{EssGen}(A)$ are square-free, hence $\mathtt{init}(I_A)$ is radical. Since I_A degenerates to a radical ideal, it is itself radical.

(III) By [KM2005, Theorem B], if $w \in S_n$,

$$\Delta(\mathtt{init}(I_w)) = \Delta(\mathcal{Q}_{n \times n}, w). \tag{129}$$

From part (I),

$$\mathtt{init}(I_A) = \sum_{u \in \mathtt{bigr}(A)} \mathtt{init}(I_u).$$

Therefore,

$$\begin{aligned}
\Delta(\mathtt{init}(I_A)) &= \bigcap_{u \in \mathtt{bigr}(A)} \Delta(\mathtt{init}(I_u)) && \text{(by Lemma 7.1)} \\
&= \bigcap_{u \in \mathtt{bigr}(A)} \Delta(\mathcal{Q}_{n \times n}, u) && \text{(by (129))} \\
&= \Delta(\mathcal{Q}_{n \times n}, A) && \text{(by part (II) of Proposition 5.1.)}
\end{aligned}$$ □

The discussion in [Knu2009] assumes $\Bbbk = \mathbb{Q}$. However, since the defining generators of I_A have coefficients in $\{\pm 1\}$, the generators are actually Gröbner over $\mathbb{Z}$, and so the statement holds more generally. Applying Lemma 7.2, we can also compute $\mathcal{C}(X_A; \mathbf{x})$ as the weighted sum over

$$F_{\mathtt{max}}(\Delta(\mathtt{init}(I_A))) = F_{\mathtt{max}}(\Delta_A).$$

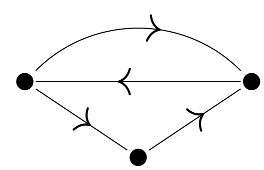
Theorem 6.3 gives a weight preserving bijection between $\mathtt{Prism}(\boldsymbol{\lambda}, \mathbf{d})$ and $F_{\mathtt{max}}(\Delta_{A_{\boldsymbol{\lambda},\mathbf{d}}})$. This produces a specific connection between the Gröbner geometry of $X_{A_{\boldsymbol{\lambda},\mathbf{d}}}$ and prism tableaux.

CHAPTER 8

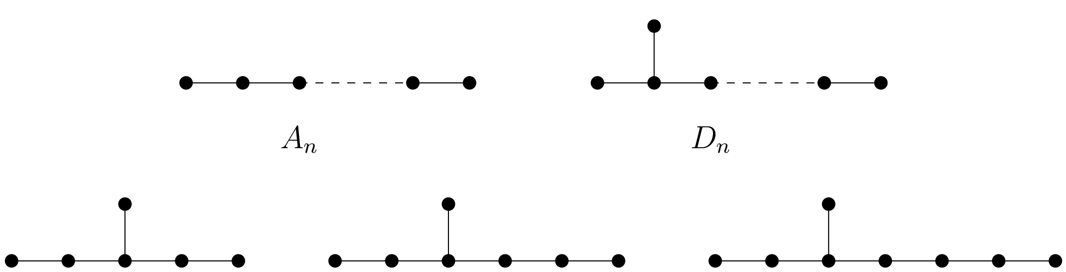
QUIVER REPRESENTATIONS AND QUIVER LOCI

In this chapter, we recall some background regarding quiver representations. The central question in the representation theory of quivers is to understand isomorphism classes of representations. We focus mostly on the nicest case, that of *Dynkin quivers*. By Gabriel's theorem, up to isomorphism, Dynkin quivers have finitely many indecomposable representations. For introductory references on quiver representations, we refer the reader to [Bri2008] and [Sch2014]. Part of the following chapter was taken from [RWY2018] which is joint work with R. Rimányi and A. Yong.

8.1 Quiver Representations



A **quiver** $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ is a directed graph with vertex set $\mathcal{Q}_0$ and arrows $\mathcal{Q}_1$. Throughout, we will assume $\mathcal{Q}$ has finitely many vertices. For $a \in \mathcal{Q}_1$, let $h(a)$ be the head of the arrow and $t(a)$ its tail. An example of a quiver is pictured to the right. A quiver is **acyclic** if it does not have any directed cycles of arrows. In particular, acyclic quivers do not have **loops**, that is arrows which start and end at the same vertex. The quiver pictured above is not acyclic. A quiver is **Dynkin** if its underlying (undirected) graph is a **Dynkin diagram** of type ADE. These graphs are pictured below.



Notice that Dynkin quivers are acylic.

Take $\mathbb{C}$ to be our ground field. A representation V of $\mathcal{Q}$ is an assignment of a finite dimensional vector space V_i to each $i \in \mathcal{Q}_0$ and a linear transformation

$$\mathsf{V}_a : \mathsf{V}_{t(a)} \to \mathsf{V}_{h(a)}$$

for each arrow $a \in \mathcal{Q}_1$. Each representation V of $\mathcal{Q}$ has an associated **dimension vector**

$$\mathbf{d}_{\mathsf{V}} = (\mathbf{d}_{\mathsf{V}}(1), \ldots, \mathbf{d}_{\mathsf{V}}(n)) \in \mathbb{N}^{\mathcal{Q}_0} \text{ where } \mathbf{d}_{\mathsf{V}}(i) = \dim \mathsf{V}_i$$

which records the dimension of the vector spaces associated to the vertices of $\mathcal{Q}$. We will sometimes write $\mathbf{dim}\mathsf{V} := \mathbf{d}_{\mathsf{V}}$. Let $\mathbb{N}^{\mathcal{Q}_0}$ denote the set of dimension vectors of $\mathcal{Q}$. A **morphism** $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ of representations of $\mathcal{Q}$ is a collection of linear transformations

$$(\mathsf{T}_i : \mathsf{V}_i \to \mathsf{W}_i)_{i \in \mathcal{Q}_0} \text{ such that } \mathsf{T}_{h(a)}\mathsf{V}_a = \mathsf{W}_a\mathsf{T}_{t(a)} \text{ for every arrow } a \in \mathcal{Q}_1.$$

Write $\mathsf{Hom}(\mathsf{V}, \mathsf{W})$ for the space of morphisms from V to W. If each of the T_i's are isomorphisms, then V and W are **isomorphic** representations. Notice, for V and W to be isomorphic, they must have the same dimension vectors.

Given representations V and W of $\mathcal{Q}$, we can build a new representation $\mathsf{V} \oplus \mathsf{W}$ by

$$(\mathsf{V} \oplus \mathsf{W})_i := \mathsf{V}_i \oplus \mathsf{W}_i \text{ for all } i \in \mathcal{Q}_0 \text{ and } (\mathsf{V} \oplus \mathsf{W})_a := \mathsf{V}_a \oplus \mathsf{W}_a \text{ for each } a \in \mathcal{Q}_1. \quad (130)$$

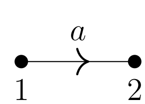
By construction, the dimension is additive over taking direct sums:

$$\mathbf{dim}\mathsf{V} \oplus \mathsf{W} = \mathbf{dim}\mathsf{V} + \mathbf{dim}\mathsf{W}.$$

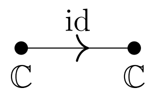
W is a **subrepresentation** of V if for each $i \in \mathcal{Q}_0$, we have that W_i is a subspace of V_i and $\mathsf{W}_a : \mathsf{W}_{t(a)} \to \mathsf{W}_{h(a)}$ is the restriction of the map $\mathsf{V}_a : \mathsf{V}_{t(a)} \to \mathsf{V}_{h(a)}$. Up to isomorphism, this is equivalent to saying there is an injective morphism from W to V.

A representation is **simple** if it has no proper subrepresentation. Up to isomorphism, the simple representations of an acyclic quiver are in bijection with its vertices. Explicitly, the simple representation $\mathsf{S}_{(i)}$ assigns $\mathbb{C}$ to vertex i and 0 to all other vertices. A representation which is a direct sum of simple representations is called **semisimple**.

Example 8.1. *Let $\mathcal{Q}$ be the quiver pictured below.*



Define a representation $\mathsf{V}_{[1,2]}$ *by assigning a copy of* $\mathbb{C}$ *to each vertex and letting the morphism between them be the identity map.*



There is a map $\mathsf{T} : \mathsf{V}_{[1,2]} \to \mathsf{S}_{(1)}$ *defined by* $\mathsf{T}_1 = [k]$ *and* $\mathsf{T}_2 = 0$*. For any* $k \in \mathbb{C}$*, the following square commutes.*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{[1]} & \mathbb{C} \\ {\scriptstyle [k]}\downarrow & & \downarrow{\scriptstyle 0} \\ \mathbb{C} & \xrightarrow{0} & 0 \end{array}$$

As such, T *is a morphism of quiver representations. In particular,* $\mathsf{Hom}(\mathsf{V}_{[1,2]}, \mathsf{S}_{(1)}) \cong \mathbb{C}$*. On the other hand, for any value of* $k \neq 0$ *the diagram*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{0} & 0 \\ {\scriptstyle [k]}\downarrow & & \downarrow{\scriptstyle 0} \\ \mathbb{C} & \xrightarrow{[1]} & \mathbb{C} \end{array}$$

does not commute. Therefore, the only morphism of quiver representations from $\mathsf{S}_{(1)}$ *to* $\mathsf{V}_{[1,2]}$ *is trivial. As such* $\mathsf{Hom}(\mathsf{S}_{(1)}, \mathsf{V}_{[1,2]}) \cong 0$*. Similarly, one may verify that* $\mathsf{Hom}(\mathsf{S}_{(2)}, \mathsf{V}_{[1,2]}) \cong \mathbb{C}$ *and* $\mathsf{Hom}(\mathsf{V}_{[1,2]}, \mathsf{S}_{(2)}) \cong 0$*.*

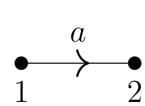
In particular, we have shown that $\mathsf{S}_{(2)}$ *is a subrepresentation of* $\mathsf{V}_{[1,2]}$*, but* $\mathsf{S}_{(1)}$ *is not.* □

A representation is **indecomposable** if it does not admit a nontrivial decomposition as a direct sum of two representations, i.e. V is indecomposable if $\mathsf{V} \cong \mathsf{U} \oplus \mathsf{W}$ implies $\mathsf{U} = 0$ or $\mathsf{W} = 0$. As a first example, simple representations are automatically indecomposable. Complete reducibility says that any finite dimensional representation V can be written as a direct sum

$$\mathsf{V} = \bigoplus_{i=1}^{N} \mathsf{V}^{(i)} \tag{131}$$

where each $\mathsf{V}^{(i)}$ is indecomposable. Furthermore, up to isomorphism and permuting factors, this decomposition is unique.

Example 8.2. *As in Example 8.1, let* $\mathcal{Q}$ *be the quiver pictured below.*



Let $\mathsf{V}_{[1,2]}$ *be as before. In Example 8.1, we showed that* $\mathsf{S}_{(1)}$ *is not a subrepresentation of*

$\mathsf{V}_{[1,2]}$, *as there is no nontrivial morphism from* $\mathsf{S}_{(1)}$ *to* $\mathsf{V}_{[1,2]}$*. Then in particular,*

$$\mathsf{V}_{[1,2]} \ncong \mathsf{S}_{(1)} \oplus \mathsf{S}_{(2)}. \tag{132}$$

If $\mathsf{V} \cong \mathsf{U} \oplus \mathsf{W}$*, either* $\mathsf{U}_1 \cong \mathbb{C}$ *or* $\mathsf{W}_1 \cong \mathbb{C}$*. Without loss of generality, assume* $\mathsf{U}_1 \cong \mathbb{C}$*. Then* $\mathsf{W}_1 = 0$*. If* $W_2 = \mathbb{C}$*, then we must have* $\mathsf{U}_2 = 0$ *and we are in the situation of the right hand side of (132). Therefore,* $\mathsf{W} = 0$ *and* $\mathsf{V}_{[1,2]}$ *is indecomposable.* □

By (131), to understand isomorphism classes of representations of $\mathcal{Q}$, it is enough to understand the isomorphism classes of indecomposable representations. A quiver is of **finite type** if it has finitely many isomorphism classes of indecomposable representations.

Theorem 8.1 ([Gab1972])**.** *A quiver is of finite type if and only if it is a disjoint union of Dynkin quivers.*

Proof. See [Bri2008, Theorem 1.1.7]. □

The **Euler form**

$$\chi_{\mathcal{Q}} : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$$

is defined by

$$\chi_{\mathcal{Q}}(\mathbf{d}_1, \mathbf{d}_2) = \sum_{i \in Q_0} \mathbf{d}_1(i)\mathbf{d}_2(i) - \sum_{a \in Q_1} \mathbf{d}_1(t(a))\mathbf{d}_2(h(a)). \tag{133}$$

We often use the abbreviation

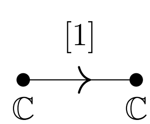
$$\chi_{\mathcal{Q}}(\mathsf{V}, \mathsf{W}) := \chi_{\mathcal{Q}}(\mathbf{dim}\mathsf{V}, \mathbf{dim}\mathsf{W}).$$

A **positive root** is a dimension vector $\mathbf{d} = (d_i)_{i \in \mathcal{Q}_0}$ so that $\chi_Q(\mathbf{d}, \mathbf{d}) = 1$. The positive roots do not depend on the orientation of the arrows of $\mathcal{Q}$, only on the underlying graph.

Theorem 8.2 ([Gab1972])**.** *Suppose* $\mathcal{Q}$ *is of finite representation type. Them the map* $[\mathsf{V}] \mapsto \mathbf{dim}\mathsf{V}$ *defines a bijection from isomorphism classes of indecomposable representations of* $\mathcal{Q}$ *to positive roots of* $\mathcal{Q}$*.*

Proof. See [Bri2008, Theorem 2.4.3]. □

Example 8.3. *For an* A_2 *quiver, the positive roots are* $(1,0)$*,* $(0,1)$*, and* $(1,1)$*. The first two positive roots correspond to the simple representations* $\mathsf{S}_{(1)}$ *and* $\mathsf{S}_{(2)}$*. The third corresponds to the indecomposable representation pictured below.*



We may confirm by direct computation that there are no other positive roots. If $\mathbf{d} = (d_1, d_2)$*, then*

$$\chi_{\mathcal{Q}}(\mathbf{d}, \mathbf{d}) = d_1^2 + d_2^2 - d_1 d_2 = (d_1 - d_2)^2 + d_1 d_2.$$

If $d_1 \geq 2$ *or* $d_2 \geq 2$ *then* $\chi_{\mathcal{Q}}(\mathbf{d}, \mathbf{d}) \geq 2$*. Since* $(0,0)$ *is not a positive root, the only positive roots are the ones listed above.* □

Given V and W an **extension** of V by W is a short exact sequence of morphisms

$$0 \to \mathsf{W} \to \mathsf{E} \to \mathsf{V} \to 0.$$

Two extensions are **equivalent** if the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{W} & \longrightarrow & \mathsf{E} & \longrightarrow & \mathsf{V} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \wr & & \downarrow & & \\ 0 & \longrightarrow & \mathsf{W} & \longrightarrow & \mathsf{E}' & \longrightarrow & \mathsf{V} & \longrightarrow & 0 \end{array}$$

Write $\mathrm{Ext}^1(\mathsf{V}, \mathsf{W})$ for the space of extensions of V by W up to equivalence.

$\mathsf{Hom}(\mathsf{V}, \mathsf{W})$ and $\mathrm{Ext}^1(\mathsf{V}, \mathsf{W})$ are finite dimensional vector spaces. The Euler form relates their dimensions as follows:

$$\chi_{\mathcal{Q}}(\mathsf{V}, \mathsf{W}) = \dim \mathsf{Hom}(\mathsf{V}, \mathsf{W}) - \dim \mathrm{Ext}^1(\mathsf{V}, \mathsf{W}), \tag{134}$$

(see [Bri2008, Corollary 1.4.3]).

8.2 The Representation Space

Let $\mathsf{Mat}(m, n)$ be the space of $m \times n$ matrices and fix $\mathbf{d} \in \mathbb{N}^{Q_0}$. The **representation space** is

$$\mathsf{Rep}_{\mathcal{Q}}(\mathbf{d}) := \bigoplus_{a \in \mathcal{Q}_1} \mathsf{Mat}(\mathbf{d}(h(a)), \mathbf{d}(t(a))).$$

A matrix in $\mathsf{Mat}(m, n)$ determines a map from $\mathbb{C}^n$ to $\mathbb{C}^m$. Explicitly, an $m \times n$ matrix acts on an $n \times 1$ column vector by matrix multiplication. As such, points of $\mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$ determine $\mathbf{d}$ dimensional representations of $\mathcal{Q}$. Conversely, by fixing a basis, we see that any $\mathbf{d}$ dimensional representation is isomorphic to some $\mathsf{V} \in \mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$.

Let

$$\mathsf{GL}_{\mathcal{Q}}(\mathbf{d}) := \prod_{x \in \mathcal{Q}_0} \mathsf{GL}(\mathbf{d}(x)).$$

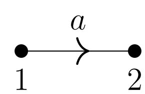
$\mathsf{GL}_{\mathcal{Q}}(\mathbf{d})$ acts on $\mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$ by base change. Explicitly, given $\mathsf{G} = (\mathsf{G}_i)_{i \in \mathcal{Q}_0}$

$$\mathsf{G} \cdot \mathsf{V} = (\mathsf{G}_{h(a)} \mathsf{V}_a \mathsf{G}_{t(a)}^{-1})_{a \in \mathcal{Q}_1}. \tag{135}$$

Lemma 8.1. *Fix* V *and* W *in* $\mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$*. Then* V *and* W *are isomorphic if and only if they lie in the same* $\mathsf{GL}_{\mathcal{Q}}(\mathbf{d})$ *orbit.*

In particular, orbits in the representation space are in bijection with isomorphism classes of $\mathbf{d}$ dimensional representations of $\mathcal{Q}$.

Example 8.4. *Let* $\mathcal{Q}$ *be as below.*



Fix a dimension vector $\mathbf{d} = (d_1, d_2)$*. Then* $\mathsf{Rep}_{\mathcal{Q}}(\mathbf{d}) = \mathsf{Mat}(d_2, d_1)$*. Given* $M \in \mathsf{Mat}(d_2, d_1)$*, there is some* $G_2 \in \mathsf{GL}(d_2)$ *so that* $G_2 M$ *is in reduced row echelon form. Furthermore, there is* $G_1 \in \mathsf{GL}(d_2)$ *so that*

$$G_2 M G_1^{-1} = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

where I_r *denotes the* $r \times r$ *identity matrix.*

Since rank is preserved under the action of the general linear group, the $\mathsf{GL}_{\mathcal{Q}}(\mathbf{d})$ *orbits on* $\mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$ *are indexed by the rank of the matrices in each orbit.* □

Write $\mathcal{O}_{\mathcal{Q}}(\mathbf{d})$ for the set of orbits in $\mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$. If $\gamma \in \mathcal{O}_{\mathcal{Q}}(\mathbf{d})$, let $\operatorname{codim}_{\mathbb{C}}(\gamma)$ denote the complex codimension of γ in $\mathsf{Rep}_{Q}(\mathbf{d})$. Fix any point $\mathsf{V} \in \eta$. The codimension of γ may be expressed in terms of extensions of V.

Lemma 8.2 (Voigt)**.** *If* $\mathsf{V} \in \gamma$ *then* $\operatorname{codim}_{\mathbb{C}}(\gamma) = \operatorname{dimExt}^1(\mathsf{V}, \mathsf{V})$.

Proof. See [Rin1980, Lemma 2.3]. □

8.3 Type A Quiver Representations

Assume $\mathcal{Q}$ is a type A quiver and label its vertices from left to right with the set $\{1, 2, \ldots, n\}$. If all arrows point in the same direction, we say that $\mathcal{Q}$ is **equioriented**. Positive roots of $\mathcal{Q}$ are in bijection with intervals of vertices in $\mathcal{Q}$. Explicitly, $\mathbf{d} = (d_1, \ldots, d_n)$ is a positive root of $\mathcal{Q}$ if and only it there exists $i \leq j$ so that

$$d_k = \begin{cases} 1 & \text{if } i \leq k \leq j \\ 0 & \text{otherwise.} \end{cases}$$

Given $1 \leq i \leq j \leq n$, define $\mathsf{V}_{[i,j]}$ to be the representation which assigns the vector space $\mathbb{C}$ to vertex k if $k \in [i,j]$ and zero otherwise. The map corresponding to an arrow is the identity whenever mapping from $\mathbb{C}$ to $\mathbb{C}$ and zero otherwise. Write $\mathbf{d}_{[i,j]} := \mathbf{dim}\mathsf{V}_{[i,j]}$.

A **lacing diagram** [ADF1980] $\mathcal{L}$ is a graph so that:

(I) the vertices are arranged in n columns labeled $1, 2, \ldots, n$ (left to right) and

(II) the edges connect adjacent columns form a partial matching.

A **strand** is a connected component of $\mathcal{L}$. A strand is of **type** $[i,j]$ if it starts in column i and ends in column j. Write

$$m_{[i,j]}(\mathcal{L}) = \#\{\text{strands of type } [i,j] \text{ in } \mathcal{L}\}. \tag{136}$$

There is an explicit dictionary between representations of $\mathcal{Q}$ and lacing diagrams. Each lacing diagram may be interpreted as a sequence of **partial permutation matrices**. This sequence defines a representation $\mathsf{V}_{\mathcal{L}} \in \mathsf{Rep}_{\mathcal{Q}}(\mathbf{d})$. We do not give the details here, as we are not concerned with the representations themselves, but merely their dimension vectors. See [KMS2006] for the equioriented case and [BR2007] for quivers of arbitrary orientation.

Let $\mathcal{L}_{[i,j]}$ be the lacing diagram which consists of a single strand of type $[i,j]$. Notice that $\mathsf{V}_{[i,j]} \cong \mathsf{V}_{\mathcal{L}_{[i,j]}}$. Strands in $\mathcal{L}$ reveal the irreducible decomposition of $\mathsf{V}_{\mathcal{L}}$

$$\mathsf{V}_{\mathcal{L}} \cong \bigoplus_{1 \leq i \leq j \leq n} \mathsf{V}_{[i,j]}^{\oplus m_{[i,j]}(\mathcal{L})}. \tag{137}$$

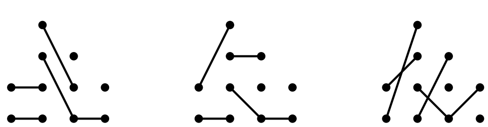
We associate a dimension vector to $\mathcal{L}$. Write

$$\mathbf{dim}(\mathcal{L}) = (\mathbf{d}_{\mathcal{L}}(1), \ldots, \mathbf{d}_{\mathcal{L}}(n))$$

where $\mathbf{d}_{\mathcal{L}}(k)$ is the number of vertices in column k of $\mathcal{L}$. Equivalently, by counting the number of strands which use a vertex of column k, we have

$$\mathbf{d}_{\mathcal{L}}(k) = \sum_{1 \leq i \leq k \leq j \leq n} m_{[i,j]}(\mathcal{L}). \tag{138}$$

Translating from lacing diagrams to representations, we have $\mathbf{dim}(\mathcal{L}) = \mathbf{dim}(V_{\mathcal{L}})$.



Two lacing diagrams are **equivalent** if they only differ by reordering of vertices within columns. For example, the lacing diagrams pictured to the left are all equivalent. Alternatively, we may say

$$[\mathcal{L}] = [\mathcal{L}'] \text{ if and only if } m_{[i,j]}(\mathcal{L}) = m_{[i,j]}(\mathcal{L}') \text{ for all } 1 \leq i \leq j \leq n.$$

Therefore, we will write $m_{[i,j]}([\mathcal{L}]) := m_{[i,j]}(\mathcal{L})$. Using (137), it follows that isomorphism classes of representations are in bijection with equivalence classes of lacing diagrams:

$$\mathsf{V}_{\mathcal{L}} \cong \mathsf{V}_{\mathcal{L}'} \text{ if and only if } [\mathcal{L}] = [\mathcal{L}'].$$

Let

$$\mathcal{C}_{\mathcal{Q}}(\mathbf{d}) = \{[\mathcal{L}] : \mathbf{dim}(\mathcal{L}) = \mathbf{d}\}$$

denote the set of equivalence classes of $\mathbf{d}$ dimensional lacing diagrams. Given $\eta = [\mathcal{L}] \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$, write $\gamma_\eta \in \mathcal{O}_{\mathcal{Q}}(\mathbf{d})$ for the orbit which contains $\mathsf{V}_{\mathcal{L}}$. The map $\eta \mapsto \gamma_\eta$ defines a bijection from $\mathcal{C}_{\mathcal{Q}}(\mathbf{d}) \to \mathcal{O}_{\mathcal{Q}}(\mathbf{d})$.

CHAPTER 9

PARTITION IDENTITIES AND QUIVER REPRESENTATIONS

The following chapter is joint work with R. Rimányi and A. Yong [RWY2018].

9.1 Introduction

The main goal of this chapter is to establish a specific connection between classical partition combinatorics and the theory of quiver representations. Our motivation is to give an elementary proof for a family of identities introduced by M. Reineke [Rei2010]. The identities are closely related to cluster algebras (see e.g., work of V. V. Fock–A. B. Goncharov [FG2009] and references therein), wall crossing phenomena (see e.g., the paper [DM2016] of B. Davison–S. Meinhardt as well as the references therein), and Donaldson-Thomas invariants and Cohomological Hall Algebras (see, e.g., the work of M. Kontsevich–Y. Soibelman [KS2011]). This work is intended to be an initial step towards understanding the rich combinatorics encoded by advanced dilogarithm identities, such as B. Keller's identities [Kel2011]. We give a new explanation for M. Reineke's identities in type A via generating series arguments.

We follow the conventions of [Rim2013]. Recall the **quantum dilogarithm series**

$$\mathbb{E}(z) = \sum_{k=0}^{\infty} \frac{(-z)^k q^{k^2/2}}{(1-q)(1-q^2)\dots(1-q^k)}. \tag{139}$$

To state M. Reineke's identities, we will first define the *quantum algebra* of $\mathcal{Q}$. We start by defining the following form:

$$\lambda_{\mathcal{Q}}(\mathbf{d}_1, \mathbf{d}_2) = \chi_{\mathcal{Q}}(\mathbf{d}_2, \mathbf{d}_1) - \chi_{\mathcal{Q}}(\mathbf{d}_1, \mathbf{d}_2). \tag{140}$$

Write $\mathbb{N}$ for the set of nonnegative integers. Following [Rim2013], the **quantum algebra** $\mathbb{A}_{\mathcal{Q}}$ is generated over $\mathbb{Q}(q^{1/2})$ by

$$\{z_{\mathbf{d}} : \mathbf{d} \in \mathbb{N}^n\}$$

with multiplication given by

$$z_{\mathbf{d}_1} z_{\mathbf{d}_2} = -q^{1/2\lambda_{\mathbf{Q}}(\mathbf{d}_1,\mathbf{d}_2)} z_{\mathbf{d}_1+\mathbf{d}_2}.$$

Theorem 9.1 ([Rei2010])**.** *If $\mathcal{Q}$ is Dynkin, there exists an ordering on the dimension vectors for the simple representations $\alpha_1, \ldots, \alpha_n$ and the indecomposable representations $\beta_1, \ldots, \beta_N$ so that*

$$\mathbb{E}(z_{\alpha_1}) \cdots \mathbb{E}(z_{\alpha_n}) = \mathbb{E}(z_{\beta_1}) \cdots \mathbb{E}(z_{\beta_N}). \tag{141}$$

Proving Theorem 9.1 is equivalent to showing that for every $\mathbf{d} \in \mathbb{N}^n$ the coefficient of $z_{\mathbf{d}}$ is equal on both sides of the expression (141). This calculation of these coefficients is carried out in [Rim2013]. Here, the identity is restated in terms of the geometry of quiver representations.

Given $\gamma \in \mathcal{O}_{\mathcal{Q}}(\mathbf{d})$, pick any representation $\mathsf{V} \in \gamma$. Then by complete reducibility,

$$\mathsf{V} \cong \bigoplus_{i=1}^{N} \mathsf{V}_{\beta_i}^{\oplus m_{\beta_i}},$$

where V_{β_i} is an indecomposable representation so that $\dim(\mathsf{V}_{\beta_i}) = \beta_i$. In fact, any $\mathsf{V}' \in \gamma$ has this same irreducible decomposition; the m_{β_i}'s are constant on orbits. Then we define $m_{\beta_i}(\gamma)$ to be the multiplicity of V_{β_i} in the irreducible decomposition of any $\mathsf{V} \in \gamma$.

Theorem 9.2 ([Rim2013])**.** *For each dimension vector $\mathbf{d} = (\mathbf{d}(1), \mathbf{d}(2), \ldots, \mathbf{d}(n))$,*

$$\prod_{i=1}^{n} \frac{1}{(q)_{\mathbf{d}(i)}} = \sum_{\gamma \in \mathcal{O}_{\mathcal{Q}}(\mathbf{d})} q^{\operatorname{codim}_{\mathbb{C}}(\gamma)} \prod_{i=1}^{N} \frac{1}{(q)_{m_{\beta_i}(\gamma)}}. \tag{142}$$

We now restrict our focus to type A quivers. Assume $\mathcal{Q}$ is a type A quiver. We label the vertices from left to right with the set $\{1, 2, \ldots, n\}$. Recall

$$\mathcal{C}_{\mathcal{Q}}(\mathbf{d}) = \{[\mathcal{L}] : \mathbf{dim}(\mathcal{L}) = \mathbf{d}\}$$

denotes the set of equivalence classes of $\mathbf{d}$ dimensional lacing diagrams. Given the class $\eta = [\mathcal{L}] \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$, write $\gamma_\eta \in \mathcal{O}_{\mathcal{Q}}(\mathbf{d})$ for the orbit which contains $\mathsf{V}_{\mathcal{L}}$. The map $\eta \mapsto \gamma_\eta$ defines a bijection from $\mathcal{C}_{\mathcal{Q}}(\mathbf{d}) \to \mathcal{O}_{\mathcal{Q}}(\mathbf{d})$.

We now associate certain statistics to η. Set parameters

$$s_i^k(\eta) = m_{[i,k-1]}(\eta) \text{ and} \tag{143}$$

$$t_j^k(\eta) = m_{[j,k]}(\eta) + m_{[j,k+1]}(\eta) + \cdots + m_{[j,n]}(\eta). \tag{144}$$

Fix a sequence of permutations

$$\mathbf{w} = (w^{(1)}, \ldots, w^{(n)}), \text{ where } w^{(i)} \in \mathcal{S}_i \text{ and } w^{(i)}(i) = i. \tag{145}$$

The partition combinatorics behind Theorem 9.3 below suggests the **Durfee statistic**:

$$r_{\mathbf{w}}(\eta) = \sum_{1 \le i < j \le k \le n} s^k_{w^{(k)}(i)}(\eta) t^k_{w^{(k)}(j)}(\eta). \tag{146}$$

With these definitions, we now state our main theorem.

Theorem 9.3 (Quiver Durfee Identity)**.** *For* $\mathbf{d} = (\mathbf{d}(1), \ldots, \mathbf{d}(n))$ *and* $\mathbf{w}$ *as in (145),*

$$\prod_{k=1}^{n} \frac{1}{(q)_{\mathbf{d}(k)}} = \sum_{\eta \in \mathcal{C}_Q(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{k=1}^{n} \frac{1}{(q)_{t^k_k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t^k_i(\eta) + s^k_i(\eta) \\ s^k_i(\eta) \end{bmatrix}_q . \tag{147}$$

Here

$$\begin{bmatrix} i+j \\ j \end{bmatrix}_q = \frac{(q)_{i+j}}{(q)_i (q)_j}$$

is the q-**binomial coefficient**, the generating series for partitions with at most i rows and j columns [And1984, Theorem 3.1]. Indeed, we will show in Lemma 9.1 that each side of (147) has an interpretation as the generating series of a set of multipartitions. By doing some algebraic cancellations, Theorem 9.3 implies the following:

Corollary 9.1.

$$\prod_{i=1}^{n} \frac{1}{(q)_{\mathbf{d}(i)}} = \sum_{\eta \in \mathcal{C}_Q(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{1 \le i \le j \le n} \frac{1}{(q)_{m_{[i,j]}(\eta)}}. \tag{148}$$

This is our link to Reineke's identity. In Definition 9.2, we assign each type A quiver a sequence of permutations $\mathbf{w}_Q$. We then show this choice satisfies

Theorem 9.4.

$$r_{\mathbf{w}_Q}(\eta) = \operatorname{codim}_{\mathbb{C}}(\gamma_\eta).$$

For type A, Theorem 9.2 follows as a consequence of Corollary 9.1 and Theorem 9.4.

The chapter is organized as follows. In Section 9.2, we recall some background on generating series. In Section 9.3, we define sets S and T so that the left hand side of (147) is a generating series for S and the right hand side is a generating series for T. We give an explicit bijection between S and T, thus proving Theorem 9.3. By simple algebraic cancellations, we prove Corollary 9.1. Finally, in Section 9.4, we prove Theorem 9.4, thus completing our proof.

9.2 Generating Series for Partitions

First we recall some background on generating series. Let A be a set equipped with a weight function

$$\mathtt{wt}_A : A \to \mathbb{N}.$$

Suppose

$$a_i := \#\{a \in A : \mathtt{wt}(a) = i\} < \infty$$

for each i. Then the **generating series** for A is

$$G(A, q) := \sum_{a \in A} q^{\mathtt{wt}_A(a)}. \tag{149}$$

Equivalently, by collecting like terms,

$$G(A, q) = \sum_{i=0}^{\infty} a_i q^i. \tag{150}$$

Generating series are well behaved under taking products and disjoint unions of sets. Define

$$\mathtt{wt}_{A \times B}(a, b) = \mathtt{wt}_A(a) + \mathtt{wt}_B(b).$$

Then

$$G(A \times B, q) = G(A, q)G(B, q). \tag{151}$$

For disjoint unions, the generating series is additive:

$$G(A \sqcup B, q) = G(A, q) + G(B, q). \tag{152}$$

Each partition has an associated weight

$$\mathtt{wt}(\lambda) = |\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i. \tag{153}$$

Equivalently, $\mathtt{wt}(\lambda)$ is the total number of boxes in the Young diagram of λ. A **multipartition** is simply a tuple of partitions $\boldsymbol{\lambda} = (\lambda^{(i)})_{i \in I}$. We weight $\boldsymbol{\lambda}$ by defining

$$\mathtt{wt}(\boldsymbol{\lambda}) = \sum_{i \in I} \mathtt{wt}(\lambda^{(i)}).$$

Let $p_k = \{\lambda : \mathtt{wt}(\lambda) = k\}$. Famously due to Euler, the generating series for the set of all partitions is

$$\sum_{k=0}^{\infty} p_k q^k = \prod_{i=1}^{\infty} \frac{1}{1-q^i}. \tag{154}$$

Throughout, we will be interested in subsets of partitions which have constraints placed on the total number of rows or columns in their Young diagram. Let

$$\mathcal{P}(i,j) = \{\lambda : \ell(\lambda) \leq i \text{ and } \lambda_1 \leq j\}.$$

Here we allow for i or j to be infinite. When i and j are finite,

$$G(\mathcal{P}(i,j), q) = \begin{bmatrix} i+j \\ i \end{bmatrix}_q. \tag{155}$$

The generating series for $\mathcal{P}(\infty, k)$, as well as $\mathcal{P}(k, \infty)$, is obtained by truncating the product in (154):

$$\frac{1}{(q)_k} = \prod_{i=1}^{k} \frac{1}{1-q^i}. \tag{156}$$

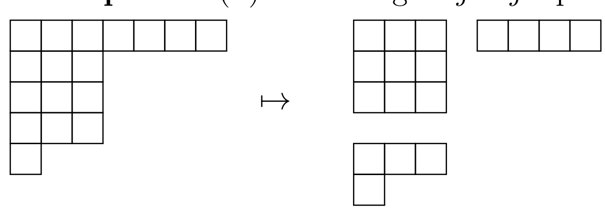
Write $i \times j$ for the rectangular partition with i parts of size j and let $\mathcal{R}(i,j) = \{i \times j\}$. Immediately from (149),

$$G(\mathcal{R}(i,j), q) = q^{ij}. \tag{157}$$

The following identity is due to Euler:

$$\frac{1}{(q)_\infty} = \sum_{j=0}^{\infty} \frac{q^{j^2}}{((q)_j)^2}. \tag{158}$$

We sketch a textbook bijective proof. The **Durfee square** $D(\lambda)$ is the largest $j \times j$ square partition that fits inside λ. Draw $D(\lambda)$ inside of λ so that it is justified against the top left corner. By cutting λ along the boundary of $D(\lambda)$, we may divide λ into three smaller partitions, as pictured to the right. This decomposition defines a bijection:

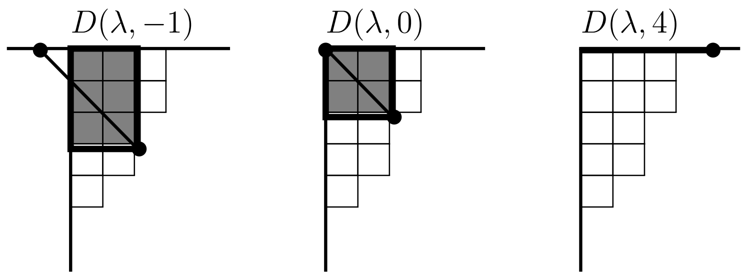


$$\mathcal{P}(\infty,\infty) \xrightarrow{\sim} \bigcup_{j=0}^{\infty} \mathcal{R}(j,j) \times \mathcal{P}(j,\infty) \times \mathcal{P}(\infty, j).$$

See [And1984, pp 27-28] for details and related identities.

The present work uses a generalization of the Durfee square. Fix $r \in \mathbb{Z}$. The **Durfee rectangle** $D(\lambda, r)$ is the largest $i \times (i+r)$ rectangular partition contained in λ. By convention, we say *any* 0-width or 0-height rectangle is contained in λ. Equivalently, $D(\lambda, r)$ is the rectangle with top left corner positioned at $(0,0)$ and bottom right corner where the line $x + y = r$ intersects the (infinite) boundary line of the partition.

Example 9.1. *Let* $\lambda = (3,3,2,2,1)$. *Pictured below are the Durfee rectangles* $D(\lambda, r)$ *for* $r = -1, 0, 4$.



Notice that $D(\lambda, 4) = 0 \times 4$ *rectangle since the line* $x + y = 4$ *intersects the boundary of* λ *at the point* $(4, 0)$. □

Decomposing λ using $D(\lambda, r)$ gives a proof of the following identity of B. Gordon and L. Houten [GH1968, pp. 91-92]:

$$\frac{1}{(q)_\infty} = \sum_{i=\max\{0,-r\}}^{\infty} \frac{q^{i(r+i)}}{(q)_i (q)_{r+i}}. \tag{159}$$

The A_2 case of Theorem 9.3 can be proved using a truncated version of (159). We sketch the explicit connection here. Fix $r \leq k$. We can split $\lambda \in \mathcal{P}(\infty, k)$ into three partitions using $D(\lambda, r)$. This defines a bijection

$$\mathcal{P}(\infty, k) \xrightarrow{\sim} \bigcup_{i=\max\{0,-r\}}^{k-r} \mathcal{R}(i, r+i) \times \mathcal{P}(\infty, r+i) \times \mathcal{P}(i, k-(r+i))$$

which corresponds to the following identity of generating series:

$$\frac{1}{(q)_k} = \sum_{i=\max\{0,-r\}}^{k-r} \frac{q^{i(r+i)}}{(q)_{r+i}} \begin{bmatrix} k-(r+i)+i \\ i \end{bmatrix}_q. \tag{160}$$

We may rephrase (160) in the language of lacing diagrams. Set $n = 2$ and fix a dimension vector $\mathbf{d} = (k-r, k)$. Choose a $\mathbf{d}$-dimensional lacing diagram $\mathcal{L}$ such that $m_{[1,1]}(\mathcal{L}) = i$. Since $m_{[1,1]}(\mathcal{L}) + m_{[1,2]}(\mathcal{L}) = k - r$, necessarily $m_{[1,2]}(\mathcal{L}) = k - r - i$. Similarly, $m_{[2,2]}(\mathcal{L}) = r + i$.

We reindex the sum in (160) and obtain

$$\frac{1}{(q)_{\mathbf{d}(2)}} = \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} \frac{q^{m_{[1,1]}(\eta)m_{[2,2]}(\eta)}}{(q)_{m_{[2,2]}(\eta)}} \begin{bmatrix} m_{[1,1]}(\eta)+m_{[1,2]}(\eta) \\ m_{[1,1]}(\eta) \end{bmatrix}_q. \tag{161}$$

For any η, we have $t^1_1(\eta) = \mathbf{d}(1)$. Dividing both sides of (161) by $(q)_{\mathbf{d}(1)}$ and using the equations (143) and (144) gives

$$\frac{1}{(q)_{\mathbf{d}(1)}(q)_{\mathbf{d}(2)}} = \frac{1}{(q)_{\mathbf{d}(1)}} \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} \frac{q^{s^2_1(\eta)t^2_2(\eta)}}{(q)_{t^2_2(\eta)}} \begin{bmatrix} s^2_1(\eta)+t^2_1(\eta) \\ s^2_1(\eta) \end{bmatrix}_q. \tag{162}$$

We have $\mathbf{d}(1) = t^1_1(\eta)$ for any $\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})$. Then we obtain

$$\frac{1}{(q)_{\mathbf{d}(1)}(q)_{\mathbf{d}(2)}} = \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} \frac{q^{s^2_1(\eta)t^2_2(\eta)}}{(q)_{t^1_1(\eta)}(q)_{t^2_2(\eta)}} \begin{bmatrix} s^2_1(\eta)+t^2_1(\eta) \\ s^2_1(\eta) \end{bmatrix}_q. \tag{163}$$

This is the $n = 2$ case of Theorem 9.3.

For $n > 2$, the proof of Theorem 9.3 uses multiple Durfee rectangles. This technique is similar to the *Durfee dissections* of A. Schilling [SW1998]. See also the work of C. Boulet on successive Durfee rectangles [Bou2010]. We also note the resemblance to the *Durfee systems* of P. Bouwknegt [Bou2002]. Also see the references to *loc. cit.* for other work on generalized Durfee square identities. Our main point of difference is that these identities do not directly concern lacing diagrams.

9.3 Proof of Theorem 9.3

Throughout this section, fix a dimension vector $\mathbf{d} = (\mathbf{d}(1), \dots \mathbf{d}(n))$ and a sequence of permutations as in (145):

$$\mathbf{w} = (w^{(1)}, \dots, w^{(n)}) \text{ with } w^{(i)} \in \mathcal{S}_i \text{ and } w^i(i) = i.$$

Define

$$S = \mathcal{P}(\infty, \mathbf{d}(1)) \times \cdots \times \mathcal{P}(\infty, \mathbf{d}(n)). \tag{164}$$

Let

$$R(\eta) = \{\boldsymbol{\mu} = (\mu^k_{i,j}) : \mu^k_{i,j} \in \mathcal{R}(s^k_{w^{(k)}(i)}(\eta), t^k_{w^{(k)}(j)}(\eta)), 1 \le i < j \le k \le n\}, \tag{165}$$

i.e. it consists of a single element, a tuple of rectangles. For ease of notation, we write $s^k_k(\eta) = \infty$ for each k. Let

$$P(\eta) = \{\boldsymbol{\nu} = (\nu^k_i) : \nu^k_i \in \mathcal{P}(s^k_{w^{(k)}(i)}(\eta), t^k_{w^{(k)}(i)}(\eta)), 1 \le i \le k \le n\}. \tag{166}$$

Define

$$T(\eta) = R(\eta) \times P(\eta). \tag{167}$$

Finally, we let

$$T = \bigcup_{\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})} T(\eta). \tag{168}$$

Weight $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)}) \in S$ by defining

$$\mathtt{wt}_S(\boldsymbol{\lambda}) = \sum_{k=1}^{n} |\lambda^{(k)}|.$$

Assign $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in T$ the weight

$$\mathtt{wt}_T(\boldsymbol{\mu}, \boldsymbol{\nu}) = \sum_{1 \le i < j < k \le n} |\mu^k_{i,j}| + \sum_{1 \le i \le k \le n} |\nu^k_i|.$$

Lemma 9.1. *(I) The generating series for S is*

$$G(S, q) = \prod_{k=1}^{n} \frac{1}{(q)_{\mathbf{d}(k)}}.$$

(II) The generating series for T is

$$G(T, q) = \sum_{\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{k=1}^{n} \frac{1}{(q)_{t^k_k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t^k_i(\eta) + s^k_i(\eta) \\ s^k_i(\eta) \end{bmatrix}_q.$$

Proof. (I) By (164),

$$S = \mathcal{P}(\infty, \mathbf{d}(1)) \times \cdots \times \mathcal{P}(\infty, \mathbf{d}(n)).$$

Then,

$$\begin{aligned}
G(S,q) &= \prod_{k=1}^{n} G(\mathcal{P}(\infty, \mathbf{d}(k)), q) && \text{(by (151))}\\
&= \prod_{i=1}^{n} \frac{1}{(q)_{\mathbf{d}(k)}} && \text{(by (156)).}
\end{aligned}$$

(II) First, observe that

$$\begin{aligned}
G(R(\eta),q) &= \prod_{1\le i<j\le k\le n} G(\mathcal{R}(s^k_{w^{(k)}(i)}(\eta), t^k_{w^{(k)}(j)}(\eta)), q) && \text{(by (151) and (165))}\\
&= \prod_{1\le i<j\le k\le n} q^{s^k_{w^{(k)}(i)}(\eta) t^k_{w^{(k)}(j)}(\eta)} && \text{(by (157))}\\
&= q^{r_{\mathbf{w}}(\eta)} && \text{(by (146)).}
\end{aligned}$$

Now,

$$\begin{aligned}
G(P(\eta),q) &= \prod_{1\le i\le k\le n} G(\mathcal{P}(s^k_{w^{(k)}(i)}(\eta), t^k_{w^{(k)}(i)}(\eta)), q) && \text{(by (166))}\\
&= \prod_{1\le i\le k\le n} G(\mathcal{P}(s^k_i(\eta), t^k_i(\eta)), q) && \text{(by permuting indices)}\\
&= \prod_{k=1}^{n} G(\mathcal{P}(s^k_k(\eta), t^k_k(\eta)), q) \prod_{i=1}^{k-1} G(\mathcal{P}(s^k_i(\eta), t^k_i(\eta)), q)\\
&= \prod_{k=1}^{n} \frac{1}{(q)_{t^k_k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t^k_i(\eta)+s^k_i(\eta)\\ s^k_i(\eta)\end{bmatrix}_q && \text{(by (156) and (155)).}
\end{aligned}$$

Therefore,

$$\begin{aligned}
G(T,q) &= \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} G(T(\eta),q) && \text{(by (152) and (168))}\\
&= \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} G(R(\eta)\times P(\eta),q) && \text{(by (167))}\\
&= \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} G(R(\eta),q)G(P(\eta),q) && \text{(by (151))}\\
&= \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{k=1}^{n} \frac{1}{(q)_{t^k_k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t^k_i(\eta)+s^k_i(\eta)\\ s^k_i(\eta)\end{bmatrix}_q . && \square
\end{aligned}$$

We now define the general "cutting" operation we use to map from S to T. Fix two weakly increasing sequences of nonnegative integers

$$\mathbf{m} = (m_0 \leq m_1 \leq \cdots \leq m_{k_m}) \text{ and } \mathbf{n} = (n_0 \leq n_1 \leq \cdots \leq n_{k_n}).$$

Given a partition λ, let $\lambda^{(i,j)}(\mathbf{m},\mathbf{n})$ be the partition formed by restricting the Young diagram of λ to rows $[m_{i-1}+1, m_i]$ and columns $[n_{j-1}+1, n_j]$. Here, we allow for infinite m_{k_m} and n_{k_n}. Immediately from the definition,

$$\lambda^{(i,j)}(\mathbf{m},\mathbf{n}) \in \mathcal{P}(m_i - m_{i-1}, n_j - n_{j-1}). \tag{169}$$

Furthermore,

$$\lambda^{(i,j)}(\mathbf{m},\mathbf{n}) \in \mathcal{R}(m_i - m_{i-1}, n_j - n_{j-1}) \tag{170}$$

if and only if the Young diagram of λ has a box in position (m_i, n_j).

The following lemma describes how the size of $D(\lambda, r)$ varies as r changes.

Lemma 9.2. *Fix λ and suppose $r' \leq r$. If $D(\lambda, r) = s \times (s+r)$ and $D(\lambda, r') = s' \times (s'+r')$ then*

(I) $s \leq s'$ and

(II) $s' + r' \leq s + r$.

Proof. (I) Suppose $s+r' < 0$. We have $0 \leq s'+r'$, since it is the width of $D(\lambda, r')$. Therefore, $s \leq s'$. Otherwise, if $s + r' \geq 0$, then

$$s \times (s+r') \subseteq s \times (s+r) \subseteq \lambda.$$

Since $D(\lambda, r') = s' \times (s'+r')$, we have $s \leq s'$.
(II) If $s = s'$ then

$$s' + r' \leq s' + r = s + r.$$

Then suppose $s < s'$. Since $s+1 \leq s'$ and

$$D(\lambda, r') = s' \times (r' + s') \subseteq \lambda,$$

we have $(s+1) \times (r'+s') \subseteq \lambda$. Since $D(\lambda, r) = s \times (s+r)$, by definition, $(s+1) \times (s+1+r) \not\subseteq \lambda$. And so

$$s' + r' \leq \lambda_{s+1} < s + 1 + r,$$

i.e. $s' + r' \leq s + r$. □

Define a map $\Psi_k : T \to \mathcal{P}(\infty, \mathbf{d}(k))$ by "gluing" the partitions of T with superscript k as indicated in Figure 9.1. Then let $\Psi = \Psi_1 \times \ldots \times \Psi_n$.

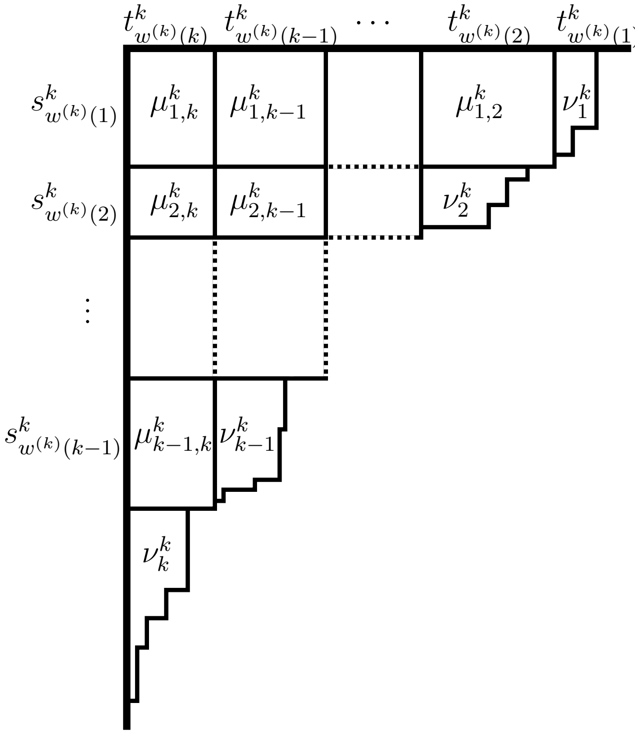


Figure 9.1: Description of the map $\Psi_k : T \to S$.

The proposed inverse $\Phi : S \to T$ is defined as follows. We will recursively define parameters

$$t^k_j(\boldsymbol{\lambda}) \text{ for } 1 \leq j \leq k \leq n$$

by induction on k. Our initial condition is that $t^1_1(\boldsymbol{\lambda}) = \mathbf{d}(1)$. Assume the sequence

$$t^{k-1}_1(\boldsymbol{\lambda}), \ldots, t^{k-1}_{k-1}(\boldsymbol{\lambda})$$

has been previously determined and that

$$t^{k-1}_j(\boldsymbol{\lambda}) \geq 0 \text{ for all } 1 \leq j \leq k-1.$$

Let

$$\delta^k_i(\boldsymbol{\lambda}) = D(\lambda^{(k)}, \mathbf{d}(k) - \sum_{\ell=1}^{i} t^{k-1}_{w^{(k)}(\ell)}(\boldsymbol{\lambda})) \text{ for } i = 0, \ldots, k-1. \tag{171}$$

Note in particular that $\delta^k_0(\boldsymbol{\lambda}) = 0 \times \mathbf{d}(k)$ for all $1 \leq k \leq n$. Suppose

$$\delta^k_i(\boldsymbol{\lambda}) = a^k_i(\boldsymbol{\lambda}) \times b^k_i(\boldsymbol{\lambda}). \tag{172}$$

For ease of indexing, write $b^k_k(\boldsymbol{\lambda}) = 0$. Let

$$t^k_{w^{(k)}(i)}(\boldsymbol{\lambda}) = b^k_{i-1}(\boldsymbol{\lambda}) - b^k_i(\boldsymbol{\lambda}) \text{ for } i = 1, \ldots, k. \tag{173}$$

We also define

$$s^k_{w^{(k)}(i)}(\boldsymbol{\lambda}) = a^k_i(\boldsymbol{\lambda}) - a^k_{i-1}(\boldsymbol{\lambda}) \text{ for } i = 1, \ldots, k-1. \tag{174}$$

By the hypothesis, $t^{k-1}_j(\boldsymbol{\lambda}) \geq 0$ for all $1 \leq j \leq k-1$. Therefore,

$$\mathbf{d}(k) - \sum_{\ell=1}^{i} t^{k-1}_{w^{(k)}(\ell)}(\boldsymbol{\lambda}) \leq \mathbf{d}(k) - \sum_{\ell=1}^{i-1} t^{k-1}_{w^{(k)}(\ell)}(\boldsymbol{\lambda}) \text{ for all } i\text{'s}.$$

Then we may apply Lemma 9.2 to the δ_i^k's, to obtain sequences

$$\mathbf{a}^k(\boldsymbol{\lambda}) = (a_0^k(\boldsymbol{\lambda}) \leq a_1^k(\boldsymbol{\lambda}) \leq \cdots \leq a_{k-1}^k(\boldsymbol{\lambda}) \leq a_k^k(\boldsymbol{\lambda})) \tag{175}$$

with $a_k^k(\boldsymbol{\lambda}) = \infty$ and

$$\mathbf{b}^k(\boldsymbol{\lambda}) = (b_k^k(\boldsymbol{\lambda}) \leq b_{k-1}^k(\boldsymbol{\lambda}) \leq \cdots \leq b_1^k(\boldsymbol{\lambda}) \leq b_0^k(\boldsymbol{\lambda})). \tag{176}$$

By (175) and (176), the $s_i^k(\boldsymbol{\lambda})$'s and $t_j^k(\boldsymbol{\lambda})$'s are all nonnegative. Continue until $k = n$.

We then map $\boldsymbol{\lambda} \mapsto (\boldsymbol{\mu}, \boldsymbol{\nu})$ where

$$\mu_{i,j}^k = \lambda^{(k)}(\mathbf{a}^k(\boldsymbol{\lambda}), \mathbf{b}^k(\boldsymbol{\lambda}))_{i,k-j+1}$$

and

$$\nu_i^k = \lambda^{(k)}(\mathbf{a}^k(\boldsymbol{\lambda}), \mathbf{b}^k(\boldsymbol{\lambda}))_{i,k-i+1}.$$

In the proof, we will justify that this map is well defined, i.e. $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in T$. This involves finding a class $\eta(\boldsymbol{\lambda}) \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$ so that $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in T(\eta(\boldsymbol{\lambda}))$. We define our candidate now.

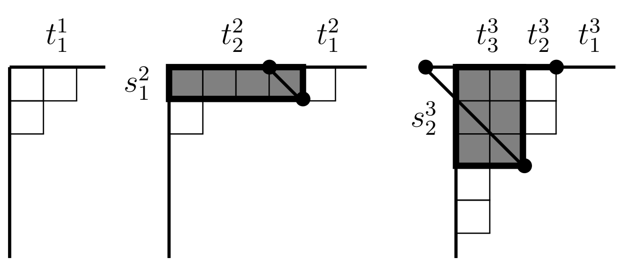
Definition 9.1. *Let $\eta(\boldsymbol{\lambda})$ be the equivalence class of a lacing diagram uniquely defined by:*

- $m_{[i,j]}(\eta(\boldsymbol{\lambda})) = s_i^{j+1}(\boldsymbol{\lambda})$ *for* $1 \leq i \leq j \leq n-1$
- $m_{[i,n]}(\eta(\boldsymbol{\lambda})) = t_i^n(\boldsymbol{\lambda})$ *for* $i = 1 \ldots n$.

Since each $m_{[i,j]}(\eta(\boldsymbol{\lambda})) \geq 0$, we have that $\eta(\boldsymbol{\lambda})$ is well defined.

Example 9.2. *Assume* $\mathbf{w} = (1, 12, 123)$*. Fix a dimension vector* $\mathbf{d} = (3, 6, 5)$ *and partitions*

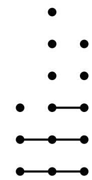
$$\lambda^{(1)} = (2,1), \lambda^{(2)} = (5,1), \text{ and } \lambda^{(3)} = (3,3,2,1,1).$$



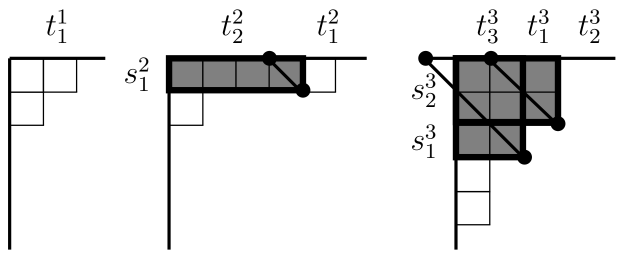
Then $\delta_1^2(\boldsymbol{\lambda}) = D(\lambda^{(2)}, 6-3) = 1 \times 4$ *and so* $t_1^2(\boldsymbol{\lambda}) = 2$*, and* $t_2^2(\boldsymbol{\lambda}) = 4$*. From this, we have*

$$\delta_1^3(\boldsymbol{\lambda}) = D(\lambda^{(3)}, 5-2) = 0 \times 3 \text{ and } \delta_2^3(\boldsymbol{\lambda}) = D(\lambda^{(3)}, 5-2-4) = 3 \times 2.$$

Therefore, $t_1^3(\boldsymbol{\lambda}) = 2$, $t_2^3(\boldsymbol{\lambda}) = 1$, and $t_3^3(\boldsymbol{\lambda}) = 2$. This corresponds to $\eta(\boldsymbol{\lambda}) = [\mathcal{L}]$ with $\mathcal{L}$ as given below.

$\mathcal{L} =$ 

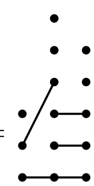
Alternatively, suppose $\mathbf{w} = (1, 12, 213)$. Keeping the same $\mathbf{d}$ and $\lambda^{(i)}$'s gives



As before, $\delta_1^2(\boldsymbol{\lambda}) = D(\lambda^{(2)}, 6-3) = 1 \times 4$. Consequently,

$$\delta_1^3(\boldsymbol{\lambda}) = D(\lambda^{(3)}, 5-4) = 2 \times 3 \text{ and } \delta_2^3(\boldsymbol{\lambda}) = D(\lambda^{(3)}, 5-4-2) = 3 \times 2.$$

This yields $\eta(\boldsymbol{\lambda}) = [\mathcal{L}']$, where $\mathcal{L}'$ is pictured below.

$\mathcal{L}' =$ 

Notice that the different choices for $\mathbf{w}$ yielded different equivalence classes. □

Immediately from the definitions (143) and (144), we have

$$t_i^k(\eta) + s_i^k(\eta) = t_i^{k-1}(\eta). \tag{177}$$

for any $\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$. We show the parameters defined in (173) and (174) satisfy the same recursion.

Lemma 9.3. $t^k_{w^{(k)}(i)}(\boldsymbol{\lambda}) + s^k_{w^{(k)}(i)}(\boldsymbol{\lambda}) = t^{k-1}_{w^{(k)}(i)}(\boldsymbol{\lambda})$ *for $1 \le i < k \le n$.*

Proof. By (172) and the definition of a Durfee rectangle,

$$b_i^k(\boldsymbol{\lambda}) - a_i^k(\boldsymbol{\lambda}) = \mathbf{d}(k) - \sum_{\ell=1}^{i} t^{k-1}_{w^{(k)}(\ell)}(\boldsymbol{\lambda}). \tag{178}$$

Applying (173) and (174),

$$\begin{aligned}
t^k_{w^{(k)}(i)}(\boldsymbol{\lambda}) + s^k_{w^{(k)}(i)}(\boldsymbol{\lambda}) &= b^k_{i-1}(\boldsymbol{\lambda}) - b^k_i(\boldsymbol{\lambda}) + a^k_i(\boldsymbol{\lambda}) - a^k_{i-1}(\boldsymbol{\lambda}) \\
&= (b^k_{i-1}(\boldsymbol{\lambda}) - a^k_{i-1}(\boldsymbol{\lambda})) - (b^k_i(\boldsymbol{\lambda}) - a^k_i(\boldsymbol{\lambda})) \\
&= \left(\mathbf{d}(k) - \sum_{\ell=1}^{i-1} t^{k-1}_{w^{(k)}(\ell)}(\boldsymbol{\lambda})\right) - \left(\mathbf{d}(k) - \sum_{\ell=1}^{i} t^{k-1}_{w^{(k)}(\ell)}(\boldsymbol{\lambda})\right) \\
&= t^{k-1}_{w^{(k)}(i)}(\boldsymbol{\lambda}). \qquad \square
\end{aligned}$$

The next lemma collects various properties of $\eta(\boldsymbol{\lambda})$. In particular, it justifies our choice in notation for $s^k_i(\boldsymbol{\lambda})$ and $t^k_j(\boldsymbol{\lambda})$.

Lemma 9.4. *(I)* $s^k_i(\eta(\boldsymbol{\lambda})) = s^k_i(\boldsymbol{\lambda})$

(II) $t^k_j(\eta(\boldsymbol{\lambda})) = t^k_j(\boldsymbol{\lambda})$

(III) $\eta(\boldsymbol{\lambda}) \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$.

Proof. (I) This is immediate from Definition 9.1.
(II) By Lemma 9.3,

$$t^k_i(\boldsymbol{\lambda}) = t^{k+1}_i(\boldsymbol{\lambda}) + s^{k+1}_i(\boldsymbol{\lambda}).$$

Iterating, we obtain

$$\begin{aligned}
t^k_i(\boldsymbol{\lambda}) &= t^{k+2}_i(\boldsymbol{\lambda}) + s^{k+2}_i(\boldsymbol{\lambda}) + s^{k+1}_i(\boldsymbol{\lambda}) \\
&= \dots \\
&= t^n_i(\boldsymbol{\lambda}) + \sum_{\ell=k+1}^{n} s^\ell_i(\boldsymbol{\lambda}) \\
&= t^n_i(\eta(\boldsymbol{\lambda})) + \sum_{\ell=k+1}^{n} s^\ell_i(\eta(\boldsymbol{\lambda})) && \text{(by Definition 9.1)} \\
&= m_{[i,n]}(\eta(\boldsymbol{\lambda})) + \sum_{\ell=k+1}^{n} m_{[i,\ell-1]}(\eta(\boldsymbol{\lambda})) && \text{(by (144) and (143))} \\
&= t^k_i(\eta(\boldsymbol{\lambda})) && \text{(by (144))}.
\end{aligned}$$

(III) For each k,

$$\mathbf{d}(k) = b^k_0(\boldsymbol{\lambda}) - b^k_k(\boldsymbol{\lambda})$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} b_{i-1}^k(\boldsymbol{\lambda}) - b_i^k(\boldsymbol{\lambda}) && \text{(by (173))} \\
&= \sum_{i=1}^{k} t_{w^{(k)}(i)}^k(\boldsymbol{\lambda}) \\
&= \sum_{i=1}^{k} t_i^k(\boldsymbol{\lambda}) && \text{(permute the terms of the sum)} \\
&= \sum_{i=1}^{k} t_i^k(\eta(\boldsymbol{\lambda})) && \text{(by part (II))} \\
&= \sum_{1 \le i \le k \le j \le n} m_{[i,j]}(\eta(\boldsymbol{\lambda})) && \text{(by (144)).}
\end{aligned}
$$

By (138), we have $\eta(\boldsymbol{\lambda}) \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$. □

Theorem 9.5. *$\Psi : T \to S$ is a weight-preserving bijection, i.e., $\mathtt{wt}_T(\boldsymbol{\mu}, \boldsymbol{\nu}) = \mathtt{wt}_S(\Psi(\boldsymbol{\mu}, \boldsymbol{\nu}))$.*

Proof. Ψ is weight-preserving: That $\mathtt{wt}_T(\boldsymbol{\mu}, \boldsymbol{\nu}) = \mathtt{wt}_S(\Psi(\boldsymbol{\mu}, \boldsymbol{\nu}))$ is clear since Ψ preserves the total number of boxes.

Ψ is well-defined: If $\mathbf{dim}(\eta) = (\mathbf{d}(1), \ldots, \mathbf{d}(n))$ then

$$
\mathbf{d}(k) = \sum_{i=1}^{k} \sum_{j=k}^{n} m_{[i,j]}(\eta) = \sum_{i=1}^{k} t_i^k(\eta) \ \text{ for } \ k = 1, \ldots, n.
$$

Therefore, $\Psi_k(\boldsymbol{\mu}, \boldsymbol{\nu})$ has parts of size at most $\mathbf{d}(k)$ for each k, i.e. $\Psi_k(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathcal{P}(\infty, \mathbf{d}(k))$ for each k. Therefore, $\Psi(\boldsymbol{\mu}, \boldsymbol{\nu}) \in S$.

Φ is well-defined:

By (169),

$$
\lambda^{(k)}(\mathbf{a}^k(\boldsymbol{\lambda}), \mathbf{b}^k(\boldsymbol{\lambda}))_{i,k-j+1} \in \mathcal{P}(a_i^k(\boldsymbol{\lambda}) - a_{i-1}^k(\boldsymbol{\lambda}), b_{j-1}^k(\boldsymbol{\lambda}) - b_j^k(\boldsymbol{\lambda})).
$$

By (174) and (173),

$$
s_{w^{(k)}(i)}(\eta(\boldsymbol{\lambda})) = a_i^k(\boldsymbol{\lambda}) - a_{i-1}^k(\boldsymbol{\lambda}) \ \text{ and } \ t_{w^{(k)}(j)}^k(\eta(\boldsymbol{\lambda})) = b_{j-1}^k(\boldsymbol{\lambda}) - b_j^k(\boldsymbol{\lambda}).
$$

Therefore,

$$
\lambda^{(k)}(\mathbf{a}^k(\boldsymbol{\lambda}), \mathbf{b}^k(\boldsymbol{\lambda}))_{i,k-j+1} \in \mathcal{P}(s_{w^{(k)}(i)}^k(\eta(\boldsymbol{\lambda})), t_{w^{(k)}(j)}^k(\eta(\boldsymbol{\lambda}))).
$$

By definition,

$$
\nu_i^k = \lambda^{(k)}(\mathbf{a}^k(\boldsymbol{\lambda}), \mathbf{b}^k(\boldsymbol{\lambda}))_{i,k-i+1}
$$

and so

$$\nu_i^k \in \mathcal{P}(s^k_{w^{(k)}(i)}(\eta(\boldsymbol{\lambda})), t^k_{w^{(k)}(i)}(\eta(\boldsymbol{\lambda})))$$

as desired.

Similarly, by (169),

$$\mu^k_{i,j} = \lambda^{(k)}(\mathbf{a}^k(\boldsymbol{\lambda}), \mathbf{b}^k(\boldsymbol{\lambda}))_{i,k-j+1}$$

and so

$$\mu^k_{i,j} \in \mathcal{P}(s_{w^{(k)}(i)}(\eta(\boldsymbol{\lambda})), t^k_{w^{(k)}(j)}(\eta(\boldsymbol{\lambda}))).$$

Since $\delta^k_i(\boldsymbol{\lambda}) \subset \lambda^{(k)}$, the box $(a^k_i(\boldsymbol{\lambda}), b^k_i(\boldsymbol{\lambda})) \in \lambda^{(k)}$. Likewise, since $\delta^k_{k-j+1}(\boldsymbol{\lambda}) \subset \lambda^{(k)}$, we have

$$(a^k_{k-j+1}(\boldsymbol{\lambda}), b^k_{k-j+1}(\boldsymbol{\lambda})) \in \lambda^{(k)}.$$

Therefore, $(a^k_i(\boldsymbol{\lambda}), b^k_{k-j+1}(\boldsymbol{\lambda})) \in \lambda^{(k)}$ As such, by (170),

$$\mu^k_{i,j} \in \mathcal{R}(s_{w^{(k)}(i)}(\eta(\boldsymbol{\lambda})), t^k_{w^{(k)}(j)}(\eta(\boldsymbol{\lambda}))).$$

Therefore, $\Phi(\boldsymbol{\lambda}) \in T(\eta(\boldsymbol{\lambda})) \subseteq T$.

$\Psi \circ \Phi = \mathrm{Id}$:

The map Φ acts by cutting the $\lambda^{(k)}$'s into various pieces and Ψ glues these shapes together into their original configurations. Then for every $\boldsymbol{\lambda} \in S$, we have $\Psi(\Phi(\boldsymbol{\lambda})) = \boldsymbol{\lambda}$.

$\Phi \circ \Psi = \mathrm{Id}$:

Fix $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in T$. Then in particular, $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in T(\eta)$ for some $\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$. Let $\boldsymbol{\lambda} := \Psi(\boldsymbol{\mu}, \boldsymbol{\nu})$. We must argue $\eta = \eta(\boldsymbol{\lambda})$. If so, $\Phi(\Psi(\boldsymbol{\mu}, \boldsymbol{\nu})) = (\boldsymbol{\mu}, \boldsymbol{\nu})$.

Since $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in T(\eta)$, each $\Psi_k(\boldsymbol{\mu}, \boldsymbol{\nu})$ contains a rectangle

$$\epsilon^k_j = \left(\sum_{i=1}^{j} s^k_{w^{(k)}(i)}(\eta)\right) \times \left(\sum_{i=j+1}^{k} t^k_{w^{(k)}(i)}(\eta)\right) \tag{179}$$

for all $1 \leq j < k$ as in Figure 9.1. By definition, $\mathbf{dim}(\eta) = \mathbf{d}$. Then it follows

$$\sum_{i=j+1}^{k} t^k_{w^{(k)}(i)}(\eta) = \mathbf{d}(k) - \left(\sum_{i=1}^{j} t^k_{w^{(k)}(i)}(\eta)\right).$$

As in (177), $t^k_i(\eta) + s^k_i(\eta) = t^{k-1}_i(\eta)$. Then by substitution we have

$$\sum_{i=j+1}^{k} t^k_{w^{(k)}(i)}(\eta) = \mathbf{d}(k) - \sum_{i=1}^{j} t^{k-1}_{w^k(i)}(\eta) + \sum_{i=1}^{j} s^k_{w^{(k)}(i)}(\eta). \tag{180}$$

Substitution of (180) into (179) yields

$$\epsilon_j^k = s \times (s + \mathbf{d}(k) - \sum_{i=1}^{j} t_{w^k(i)}^{k-1}(\eta))$$

contained in $\lambda^{(k)}$. Here, $s = \sum_{i=1}^{j} s_i^k(\eta)$. In particular, by construction, the bottom right corner of ϵ_j^k intersects the boundary of $\lambda^{(k)}$ (see Figure 9.1), i.e. s is the maximum value for which $\epsilon_j^k \subseteq \lambda^{(k)}$. Then by the definition of a Durfee rectangle,

$$\epsilon_j^k = D(\lambda^{(k)}, \mathbf{d}(k) - \sum_{i=1}^{j} t_{w^k(i)}^{k-1}(\eta)).$$

By (171) and Claim 9.4 part (II),

$$\delta_j^k(\boldsymbol{\lambda}) = D(\lambda^{(k)}, \mathbf{d}(k) - \sum_{i=1}^{j} t_{w^k(i)}^{k-1}(\eta(\boldsymbol{\lambda}))).$$

We seek to show $\delta_j^k(\boldsymbol{\lambda}) = \epsilon_j^k$ for all $1 \le j < k \le n$. Our argument is by induction on k. By definition, $t_1^1(\eta) = \mathbf{d}(1) = t_1^1(\eta(\boldsymbol{\lambda}))$. Then

$$\begin{aligned}
\delta_1^2(\boldsymbol{\lambda}) &= D(\lambda^{(2)}, \mathbf{d}(2) - t_1^1(\eta)) \\
&= D(\lambda^{(2)}, \mathbf{d}(2) - t_1^1(\eta(\boldsymbol{\lambda}))) \\
&= \epsilon_1^2,
\end{aligned}$$

so the Durfee rectangles agree. Assume $\delta_j^{k-1}(\boldsymbol{\lambda}) = \epsilon_j^{k-1}$ for all $1 \le j < k-1$. Then in particular, $t_i^{k-1}(\eta) = t_i^{k-1}(\eta(\boldsymbol{\lambda}))$ for all $1 \le i \le k-1$ and so

$$\sum_{i=1}^{j} t_{w^k(i)}^{k-1}(\eta) = \sum_{i=1}^{j} t_{w^k(i)}^{k-1}(\eta(\boldsymbol{\lambda})). \tag{181}$$

Then $\delta_j^k = \epsilon_j^k$ since both are Durfee rectangles defined by the *same* parameter. Hence, $\delta_j^k = \epsilon_j^k$. Therefore,

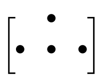
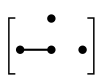
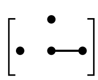
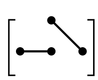
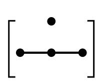
$$s_i^k(\eta) = s_i^k(\eta(\boldsymbol{\lambda})) \text{ for all } 1 \le i < k \le n$$

and

$$t_i^k(\eta) = t_i^k(\eta(\boldsymbol{\lambda})) \text{ for all } 1 \le i \le k \le n.$$

Hence $\eta = \eta(\boldsymbol{\lambda})$. □

Table 9.1: Pictured below are the equivalence classes for $\mathbf{d} = (1, 2, 1)$ and their corresponding parameters $s_j^k(\eta)$ and $t_j^k(\eta)$. The last column gives $G(T(\eta), q)$.

$\eta = [\mathcal{L}]$	$(s_j^k(\eta))$	$(t_j^k(\eta))$	$G(T(\eta), q)$
	$\begin{array}{cc\|c} 2 & 1 & j/k \\ \hline & 1 & 2 \\ 2 & 0 & 3 \end{array}$	$\begin{array}{ccc\|c} 3 & 2 & 1 & j/k \\ \hline & & 1 & 1 \\ & 2 & 0 & 2 \\ 1 & 0 & 0 & 3 \end{array}$	$q^4 \left(\frac{1}{(q)_1}\right)\left(\frac{1}{(q)_2}\right)\left(\frac{1}{(q)_1}\right) = \frac{q^4}{(1-q)^3(1-q^2)}$
	$\begin{array}{cc\|c} 2 & 1 & j/k \\ \hline & 0 & 2 \\ 1 & 1 & 3 \end{array}$	$\begin{array}{ccc\|c} 3 & 2 & 1 & j/k \\ \hline & & 1 & 1 \\ & 1 & 1 & 2 \\ 1 & 0 & 0 & 3 \end{array}$	$q^2 \left(\frac{1}{(q)_1}\right)\left(\frac{1}{(q)_1}\right)\left(\frac{1}{(q)_1}\right) = \frac{q^2}{(1-q)^3}$
	$\begin{array}{cc\|c} 2 & 1 & j/k \\ \hline & 1 & 2 \\ 1 & 0 & 3 \end{array}$	$\begin{array}{ccc\|c} 3 & 2 & 1 & j/k \\ \hline & & 1 & 1 \\ & 2 & 0 & 2 \\ 0 & 1 & 0 & 3 \end{array}$	$q^2 \left(\frac{1}{(q)_1}\right)\left(\frac{1}{(q)_2}\right)\left(\begin{bmatrix}2\\1\end{bmatrix}_q\right) = \frac{q^2}{(1-q)^3}$
	$\begin{array}{cc\|c} 2 & 1 & j/k \\ \hline & 0 & 2 \\ 0 & 1 & 3 \end{array}$	$\begin{array}{ccc\|c} 3 & 2 & 1 & j/k \\ \hline & & 1 & 1 \\ & 1 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{array}$	$q \left(\frac{1}{(q)_1}\right)\left(\frac{1}{(q)_1}\right) = \frac{q}{(1-q)^2}$
	$\begin{array}{cc\|c} 2 & 1 & j/k \\ \hline & 0 & 2 \\ 1 & 0 & 3 \end{array}$	$\begin{array}{ccc\|c} 3 & 2 & 1 & j/k \\ \hline & & 1 & 1 \\ & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}$	$\left(\frac{1}{(q)_1}\right)\left(\frac{1}{(q)_1}\right) = \frac{1}{(1-q)^2}$

Proof of Theorem 9.3. By Theorem 9.5, S and T are in weight preserving bijection. Therefore,

$$G(S, q) = G(T, q).$$

Applying Lemma 9.1 gives the result. □

Example 9.3. *Let $n = 3$ and $\mathbf{d} = (1, 2, 1)$ and $\mathbf{w} = (1, 12, 123)$. Then*

$$r_{\mathbf{w}}(\eta) = (s_1^2(\eta)t_2^2(\eta)) + (s_1^3(\eta)t_2^3(\eta) + s_1^3(\eta)t_3^3(\eta) + s_2^3(\eta)t_3^3(\eta))$$

and

$$G(P(\eta), q) = \frac{1}{(q)_{t_1^1(\eta)}} \frac{1}{(q)_{t_2^2(\eta)}} \begin{bmatrix} t_1^2(\eta) + s_1^2(\eta) \\ s_1(\eta)^2 \end{bmatrix}_q \frac{1}{(q)_{t_3^3(\eta)}} \begin{bmatrix} t_1^3(\eta) + s_1^3(\eta) \\ s_1^3(\eta) \end{bmatrix}_q \begin{bmatrix} t_2^3(\eta) + s_2^3(\eta) \\ s_2^3(\eta) \end{bmatrix}_q.$$

Table 9.1 lists the equivalence classes for $\mathbf{d} = (1, 2, 1)$ and their corresponding terms on the *right hand side of (147). We then verify,*

$$\begin{aligned}
G(T,q) &= \frac{q^4}{(1-q)^3(1-q^2)} + \frac{q^2}{(1-q)^3} + \frac{q^2}{(1-q)^3} + \frac{q}{(1-q)^2} + \frac{1}{(1-q)^2} \\
&= \frac{1}{(1-q)^3(1-q^2)}(q^4 + 2q^2(1-q^2) + q(1-q)(1-q^2) + (1-q)(1-q^2)) \\
&= \frac{1}{(q)_1(q)_2(q)_1} \\
&= G(S,q).
\end{aligned}$$

□

We now give the proof of Corollary 9.1.

Proof. By (177),

$$t_i^k(\eta) + s_i^k(\eta) = t_i^{k-1}(\eta). \tag{182}$$

Furthermore by (143) and (144),

$$s_i^k(\eta) = m_{[i,k-1]}(\eta) \text{ and } t_i^n(\eta) = m_{[i,n]}(\eta).$$

Thus,

$$\begin{aligned}
\prod_{k=1}^{n} \frac{1}{(q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t_i^k(\eta) + s_i^k(\eta) \\ s_i^k(\eta) \end{bmatrix}_q &= \prod_{k=1}^{n} \frac{1}{(q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \frac{(q)_{t_i^k(\eta)+s_i^k(\eta)}}{(q)_{t_i^k(\eta)}(q)_{s_i^k(\eta)}} \\
&= \prod_{k=1}^{n} \frac{1}{(q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \frac{(q)_{t_i^{k-1}(\eta)}}{(q)_{t_i^k(\eta)}(q)_{s_i^k(\eta)}} \\
&= \left(\prod_{k=1}^{n} \frac{1}{(q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \frac{(q)_{t_i^{k-1}(\eta)}}{(q)_{t_i^k}(\eta)} \right) \left(\prod_{k=1}^{n} \prod_{i=1}^{k-1} \frac{1}{(q)_{s_i^k(\eta)}} \right) \\
&= \left(\prod_{k=1}^{n} \prod_{i=1}^{k} \frac{1}{(q)_{t_i^k(\eta)}} \right) \left(\prod_{k=2}^{n} \prod_{i=1}^{k-1} (q)_{t_i^{k-1}(\eta)} \right) \left(\prod_{k=1}^{n} \prod_{i=1}^{k-1} \frac{1}{(q)_{s_i^k(\eta)}} \right) \\
&= \left(\prod_{k=1}^{n} \prod_{i=1}^{k} \frac{1}{(q)_{t_i^k(\eta)}} \right) \left(\prod_{k=1}^{n-1} \prod_{i=1}^{k} (q)_{t_i^k(\eta)} \right) \left(\prod_{k=1}^{n} \prod_{i=1}^{k-1} \frac{1}{(q)_{s_i^k(\eta)}} \right) \\
&= \left(\prod_{i=1}^{n} \frac{1}{(q)_{t_i^n(\eta)}} \right) \left(\prod_{k=1}^{n} \prod_{i=1}^{k-1} \frac{1}{(q)_{s_i^k(\eta)}} \right) \\
&= \left(\prod_{i=1}^{n} \frac{1}{(q)_{m_{[i,n]}(\eta)}} \right) \left(\prod_{k=1}^{n} \prod_{i=1}^{k-1} \frac{1}{(q)_{m_{[i,k-1]}(\eta)}} \right) \\
&= \prod_{1 \le i \le j \le n} \frac{1}{(q)_{m_{[i,j]}(\eta)}}.
\end{aligned}$$

□

The proof of Theorem 9.5 implies an enriched form of Theorem 9.3. Let

$$(a;q)_k = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1}).$$

For $\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})$, let $u_j(\eta)$ be the number of strands that terminate at column j in some (equivalently any) lace diagram $\mathcal{L} \in \eta$. That is,

$$u_j(\eta) = \sum_{i=1}^{j} s_i^{j+1}(\eta). \tag{183}$$

Corollary 9.2 (of Theorem 9.5)**.**

$$\prod_{k=1}^{n} \frac{1}{(qz;q)_{\mathbf{d}(k)}} = \sum_{\eta\in\mathcal{C}_{\mathcal{Q}}(\mathbf{d})} q^{r_{\mathbf{w}}(\eta)} \prod_{k=1}^{n} z^{u_{k-1}(\eta)} \frac{1}{(qz;q)_{t_k^k(\eta)}} \prod_{i=1}^{k-1} \begin{bmatrix} t_i^k(\eta)+s_i^k(\eta) \\ s_i^k(\eta) \end{bmatrix}_q . \tag{184}$$

Proof. The left hand side of (184) is the generating series for S with respect to the weight that uses q to mark the number of boxes and z to mark length of the partitions involved. Now, suppose $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)}) \in S$. Under the indicated decomposition in Figure 9.1,

$$\ell(\lambda^{(k)}) = \ell(\nu_k^k) + \sum_{i=1}^{k-1} s^k_{w^{(k)}(i)}(\eta(\boldsymbol{\lambda})) = \ell(\nu_k^k) + u_{k-1}(\eta(\boldsymbol{\lambda})),$$

where the second equality holds by (183) and reordering terms. The corollary follows immediately from this and Theorem 9.5 combined. □

Theorem 9.3 is the $z=1$ case of Corollary 9.2. By analysis as in Section 9.2, we obtain as a special case this Durfee rectangle identity:

$$\frac{1}{(qz;q)_k} = \sum_{i=\max\{0,-r\}}^{k-r} \frac{z^i q^{i(r+i)}}{(qz;q)_{r+i}} \begin{bmatrix} k-(r+i)+i \\ i \end{bmatrix}_q .$$

From Corollary 9.2 one can deduce an enriched form of Theorem 9.2.

9.4 Proof of Theorem 9.4

Assume $\mathcal{Q}$ is a type A quiver. Label its vertices from left to right with the numbers $1, 2, \ldots, n$. Write a_i for the arrow whose left endpoint is vertex i. Let $\mathcal{I}$ be the set of intervals in $\mathcal{Q}$, i.e.

$$\mathcal{I} = \{[i,j] : i \leq j \text{ and } i,j \in [n]\}.$$

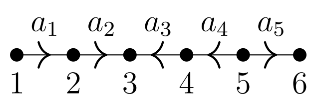
Recall ι is the natural inclusion from $\mathcal{S}_{i-1}$ to $\mathcal{S}_i$. Let $w_0^{(i-1)}$ denote the longest permutation in $\mathcal{S}_{i-1}$. We associate a sequence of permutations to $\mathcal{Q}$ as follows:

Definition 9.2. *Let $w_{\mathcal{Q}}^{(1)} = 1$ and $w_{\mathcal{Q}}^{(2)} = 12$. For $i \geq 3$ Set*

$$w_{\mathcal{Q}}^{(i)} = \begin{cases} \iota(w_{\mathcal{Q}}^{(i-1)}) & \text{if } a_{i-2} \text{ and } a_{i-1} \text{ point in the same direction} \\ \iota(w_{\mathcal{Q}}^{(i-1)} w_0^{(i-1)}) & \text{if } a_{i-2} \text{ and } a_{i-1} \text{ point in opposite directions.} \end{cases}$$

Write $\mathbf{w}_{\mathcal{Q}} := (w_{\mathcal{Q}}^{(1)}, \ldots, w_{\mathcal{Q}}^{(n)})$. By construction, $\mathbf{w}_{\mathcal{Q}}$ is of the form (145).

Example 9.4. *Let $\mathcal{Q}$ be the quiver pictured below.*



Then $\mathbf{w}_{\mathcal{Q}} = (1, 12, 123, 3214, 32145, 541236)$. □

Definition 9.2 is our link between $\operatorname{codim}_{\mathbb{C}}(\gamma_\eta)$ and the Durfee statistic. The outline of the proof of Theorem 9.4 is as follows. We start by defining two subsets of $\mathcal{I} \times \mathcal{I}$, $\texttt{BoxStrands}(\mathbf{w})$ and $\texttt{ConditionStrands}(\mathcal{Q})$. In Proposition 9.1, we show that

$$r_{\mathbf{w}}(\eta) = \sum_{(I,J) \in \texttt{BoxStrands}(\mathbf{w})} m_I(\eta) m_J(\eta).$$

Proposition 9.2 states

$$\operatorname{codim}_{\mathbb{C}}(\gamma_\eta) = \sum_{(I,J) \in \texttt{ConditionStrands}(\mathcal{Q})} m_I(\eta) m_J(\eta).$$

In Proposition 9.3, we show

$$\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) = \texttt{ConditionStrands}(\mathcal{Q}).$$

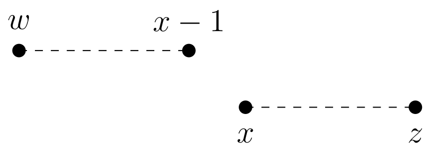
Combining these propositions completes the proof.

Given a sequence $\mathbf{w} = (w^{(1)}, \ldots, w^{(n)})$ which satisfies (145), define

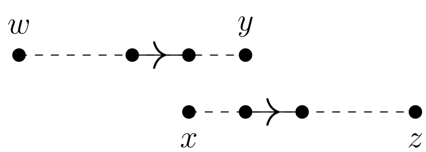
$$\texttt{BoxStrands}(\mathbf{w}) = \{([w^{(k)}(i), k-1], [w^{(k)}(j), \ell]) : 1 \leq i < j \leq k \leq \ell \leq n\} \subseteq \mathcal{I} \times \mathcal{I}. \quad (185)$$

To define $\texttt{ConditionStrands}(\mathcal{Q})$, we consider pairs of intervals $(I, J) \in \mathcal{I} \times \mathcal{I}$ of the following three types:

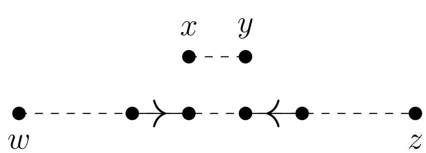
(I) $I = [w, x-1]$ and $J = [x, z]$ with $w < x \leq z$



(II) $I = [w, y]$ and $J = [x, z]$ with $w < x \leq y < z$ and the arrows a_{x-1} and a_y point in the same direction, e.g.,



(III) $I = [x, y]$ and $J = [w, z]$ with $w < x \leq y < z$ and the arrows a_{x-1} and a_y point in different directions, e.g.,



With this, we let

$$\texttt{ConditionStrands}(\mathcal{Q}) = \{(I, J) : (I, J) \text{ satisfies (I), (II), or (III)}\}. \quad (186)$$

The set $\texttt{BoxStrands}(\mathbf{w})$ has an immediate connection to the Durfee statistic $r_{\mathbf{w}}(\eta)$.

Proposition 9.1.

$$r_{\mathbf{w}}(\eta) = \sum_{(I,J) \in \texttt{BoxStrands}(\mathbf{w})} m_I(\eta) m_J(\eta).$$

Proof. By (146),

$$r_{\mathbf{w}}(\eta) = \sum_{1 \leq i < j \leq k \leq n} s^k_{w^{(k)}(i)}(\eta) t^k_{w^{(k)}(j)}(\eta).$$

Using (143) and (144), we have:

$$\begin{aligned} r_{\mathbf{w}}(\eta) &= \sum_{1\le i<j\le k\le n} m_{[w^{(k)}(i),k-1]}(\eta)\left(\sum_{\ell=k}^{n} m_{[w^{(k)}(j),\ell]}(\eta)\right) \\ &= \sum_{1\le i<j\le k\le \ell\le n} m_{[w^{(k)}(i),k-1]}(\eta) m_{[w^{(k)}(j),\ell]}(\eta) \\ &= \sum_{(I,J)\in\texttt{BoxStrands}(\mathbf{w})} m_I(\eta) m_J(\eta). \end{aligned}$$

□

Here, we give an alternate expression for $\mathrm{codim}_{\mathbb{C}}(\gamma_\eta)$ in terms of the Euler form. Define

$$U = \{(I,J) : \chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) < 0\}.$$

Lemma 9.5.

$$\mathrm{codim}_{\mathbb{C}}(\gamma_\eta) = \sum_{(I,J)\in U} m_I(\eta) m_J(\eta)(-\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J)).$$

Proof. By [Rei2001], Section 2, there exists a total order on $\mathcal{I}$ so that

$$\mathsf{Hom}(\mathsf{V}_I, \mathsf{V}_J) \text{ and } \mathrm{Ext}^1(\mathsf{V}_J, \mathsf{V}_I) = 0 \text{ whenever } I < J \text{ and } I \neq J. \tag{187}$$

Indecomposables for Dynkin quivers have no nontrivial self extensions, that is,

$$\mathrm{Ext}^1(\mathsf{V}_I, \mathsf{V}_I) = 0 \text{ for all } I \in \mathcal{I},$$

[Bri2008, Theorem 2.4.3]. Then $\mathrm{dim}\mathrm{Ext}^1(\mathsf{V}_I, \mathsf{V}_J) = 0$ whenever $I \geq J$.

Writing

$$\mathsf{V}_\eta = \bigoplus_{I\in\mathcal{I}} \mathsf{V}_I^{\oplus m_I(\eta)}$$

as a direct sum of indecomposables, we have

$$\mathrm{Ext}^1(\mathsf{V}_\eta, \mathsf{V}_\eta) \cong \bigoplus_{(I,J)\in\mathcal{I}\times\mathcal{I}} \mathrm{Ext}^1(\mathsf{V}_I, \mathsf{V}_J)^{\oplus m_I(\eta) m_J(\eta)}.$$

Then

$$\mathrm{codim}_{\mathbb{C}}(\gamma_\eta) = \mathrm{dim}\mathrm{Ext}^1(\mathsf{V}_\eta, \mathsf{V}_\eta) = \sum_{(I,J)\in\mathcal{I}\times\mathcal{I}} m_I(\eta) m_J(\eta) \mathrm{dim}\mathrm{Ext}^1(\mathsf{V}_I, \mathsf{V}_J).$$

Since $\mathrm{Ext}^1(\mathsf{V}_\eta, \mathsf{V}_\eta)$ vanishes when $I \geq J$,

$$\mathrm{codim}_{\mathbb{C}}(\gamma_\eta) = \sum_{(I,J):I<J} m_I(\eta) m_J(\eta) \mathrm{dimExt}^1(\mathsf{V}_I, \mathsf{V}_J),$$

(see [Rim2013]). Combining (134) and (187) gives

$$\mathrm{codim}_{\mathbb{C}}(\gamma_\eta) = \sum_{(I,J):I<J} m_I(\eta) m_J(\eta)(-\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J)). \tag{188}$$

Using the ordering on $\mathcal{I}$ and (134), it follows that

$$\text{if } \; I < J, \; \text{ then } \; \chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) \leq 0 \; \text{ and } \; \chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) \geq 0. \tag{189}$$

Since $\mathcal{Q}$ is a Dynkin quiver, if $I = J$, then $\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) > 0$ [Bri2008]. Thus we may reindex the sum, taking only those (I, J) for which $\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) < 0$. Therefore,

$$\mathrm{codim}_{\mathbb{C}}(\gamma_\eta) = \sum_{(I,J)\in U} m_I(\eta) m_J(\eta)(-\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J)).$$ □

Lemma 9.6. *Fix intervals I and J. If $[x, y] \subseteq I, J$ then*

$$\sum_{i=x}^{y} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=x}^{y-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) = 1. \tag{190}$$

Proof. Since $[x, y] \subseteq I, J$, $\mathbf{d}_I(i) = \mathbf{d}_J(i) = 1$ for all $i \in [x, y]$. Therefore,

$$\sum_{i=x}^{y} \mathbf{d}_I(i)\mathbf{d}_J(i) = y - x + 1. \tag{191}$$

Regardless of the orientation of a_i, if $i \in [x, y-1]$ then $t(a_i), h(a_i) \in [x, y]$. Because $[x, y] \subseteq I, J$, we have $\mathbf{d}_I(t(a_i)) = \mathbf{d}_J(h(a_i)) = 1$. Then

$$\sum_{i=x}^{y-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) = (y-1) - x + 1. \tag{192}$$

Subtracting (192) from (191) gives (190). □

Let

$$\texttt{StrandPairs} = \{(I, J) = ([x_1, x_2], [y_1, y_2]) \in \mathcal{I} \times \mathcal{I} : x_2 \leq y_2\}.$$

From (185) and the definitions (I)-(III), it follows that

$$\texttt{ConditionStrands}(\mathcal{Q}) \subset \texttt{StrandPairs}.$$

Lemma 9.7. *Let* $(I, J) \in \texttt{StrandPairs}$. *Then*

$$(I, J) \in \texttt{ConditionStrands}(\mathcal{Q}) \iff \chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) < 0 \text{ or } \chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) < 0.$$

Moreover, if $\chi_Q(\mathsf{V}_I, \mathsf{V}_J) < 0$, *then* $\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) = -1$ *and likewise* $\chi_Q(\mathsf{V}_J, \mathsf{V}_I) < 0$ *implies* $\chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) = -1$.

Proof. Since we have assumed $\mathcal{Q}$ is a type A quiver, we have

$$\chi_{\mathcal{Q}}(\mathbf{d}_1, \mathbf{d}_2) = \sum_{i=1}^{n} \mathbf{d}_1(i)\mathbf{d}_2(i) - \sum_{i=1}^{n-1} \mathbf{d}_1(t(a_i))\mathbf{d}_2(h(a_i)). \tag{193}$$

Given an interval I, write $\mathbf{d}_I$ for the dimension vector of V_I. By (193), we have

$$\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) = \chi_{\mathcal{Q}}(\mathbf{d}_I, \mathbf{d}_J) = \sum_{i=1}^{n} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=1}^{n-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)).$$

We analyze this expression repeatedly throughout our argument.
($\Rightarrow$) By direct computation, we will show if $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$ then

$$\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) = -1 \text{ or } \chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) = -1,$$

which is the last assertion of the claim.
Case 1: $(I, J) = ([w, x-1], [x, z])$ is of type (I).
Subcase i: a_{x-1} points to the right.

$$\begin{aligned}
\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) &= \sum_{i=1}^{n} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=1}^{n-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) \\
&= -\sum_{i=1}^{n-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) \quad \text{(since } I \cap J = \emptyset\text{)} \\
&= -\mathbf{d}_I(t(a_{x-1}))\mathbf{d}_J(h(a_{x-1})) \\
&= -\mathbf{d}_I(x-1)\mathbf{d}_J(x) \\
&= -1.
\end{aligned}$$

Subcase ii: a_{x-1} points to the left.

Let $\mathcal{Q}^{\text{op}}$ be the quiver obtained by reversing the direction of all arrows in $\mathcal{Q}$. Then $\chi_{\mathcal{Q}}(\mathbf{d}_J, \mathbf{d}_I) = \chi_{\mathcal{Q}^{\text{op}}}(\mathbf{d}_I, \mathbf{d}_J)$. Therefore,

$$\chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) = \chi_{\mathcal{Q}}(\mathbf{d}_J, \mathbf{d}_I) = \chi_Q^{\text{op}}(\mathbf{d}_I, \mathbf{d}_J) = -1$$

by Subcase 1.i.

Case 2: $(I, J) = ([w, y], [x, z])$ is of type (II).

Subcase i: a_{x-1} and a_y point to the right.

$$\begin{aligned}
\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) &= \sum_{i=x}^{y} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=x-1}^{y} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) \\
&= \left(\sum_{i=x}^{y} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=x}^{y-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) \right) - \mathbf{d}_I(t(a_{x-1}))\mathbf{d}_J(h(a_{x-1})) \\
&\quad - \mathbf{d}_I(t(a_y))\mathbf{d}_J(h(a_y)) \\
&= 1 - \mathbf{d}_I(t(a_{x-1}))\mathbf{d}_J(h(a_{x-1})) - \mathbf{d}_I(t(a_y))\mathbf{d}_J(h(a_y)) \quad \text{(Lemma 9.6)} \\
&= 1 - \mathbf{d}_I(x-1)\mathbf{d}_J(x) - \mathbf{d}_I(y)\mathbf{d}_J(y+1) \\
&= -1.
\end{aligned}$$

Subcase ii: a_{x-1} and a_y point to the left.

$\chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) = -1$ by the $\mathcal{Q}^{\text{op}}$ argument, as in Subcase 1.i.

Case 3: $(I, J) = ([x, y], [y, z])$ is of type (III).

Subcase i: a_{x-1} points right and a_y points left.

$$\begin{aligned}
\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) &= \sum_{i=x}^{y} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=x-1}^{y} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) \\
&= \left(\sum_{i=x}^{y} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=x}^{y-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) \right) - \mathbf{d}_I(t(a_{x-1}))\mathbf{d}_J(h(a_{x-1})) \\
&\quad - \mathbf{d}_I(t(a_y))\mathbf{d}_J(h(a_y)) \\
&= 1 - \mathbf{d}_I(t(a_{x-1}))\mathbf{d}_J(h(a_{x-1})) - \mathbf{d}_I(t(a_y))\mathbf{d}_J(h(a_y)) \quad \text{(Lemma 9.6)} \\
&= 1 - \mathbf{d}_I(x-1)\mathbf{d}_J(x) - \mathbf{d}_I(y-1)\mathbf{d}_J(y) \\
&= -1.
\end{aligned}$$

Subcase ii: a_{x-1} points left and a_y points right.

$\chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) = -1$ by the $\mathcal{Q}^{\text{op}}$ argument, as in Subcase 1.i.

Thus we have shown whenever $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$,

$$\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) = -1 \text{ or } \chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) = -1.$$

($\Leftarrow$) Let $(I, J) = ([x_1, x_2], [y_1, y_2]) \in \texttt{StrandPairs}$ and first assume $\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) < 0$.
Case 1: $I \cap J = \emptyset$. Then $\mathbf{d}_I(i) = 0$ or $\mathbf{d}_J(i) = 0$ for all $i \in [1, n]$ and so

$$\chi_{\mathcal{Q}}(\mathbf{d}_I, \mathbf{d}_J) = -\sum_{i=1}^{n-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)).$$

Since $\chi_{\mathcal{Q}}(\mathbf{d}_I, \mathbf{d}_J) < 0$ there must exist an arrow a_i with $t(a_i) \in [x_1, x_2]$ and $h(a_i) \in [y_1, y_2]$. Then $i = x_2$, a_i points to the right, and $y_1 = x_2 + 1$. This implies (I, J) is of type (I).
Case 2: Assume $I \cap J \neq \emptyset$. Since we assume $x_2 \leq y_2$

$$I \cap J = [x_1, x_2] \cap [y_1, y_2] = [z, x_2]$$

where $z \in \{x_1, y_1\}$. Then

$$\begin{aligned}
\chi_{\mathcal{Q}}(\mathbf{d}_I, \mathbf{d}_J) &= \sum_{i=1}^{n} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=1}^{n-1} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) \\
&= \sum_{i=z}^{x_2} \mathbf{d}_I(i)\mathbf{d}_J(i) - \sum_{i=z-1}^{x_2} \mathbf{d}_I(t(a_i))\mathbf{d}_J(h(a_i)) && \text{(Lemma 9.6)} \\
&= 1 - \mathbf{d}_I(t(a_{z-1}))\mathbf{d}_J(h(a_{z-1})) - \mathbf{d}_I(t(a_{x_2}))\mathbf{d}_J(h(a_{x_2})).
\end{aligned}$$

Since $\chi_{\mathcal{Q}}(\mathbf{d}_I, \mathbf{d}_J) < 0$, we must have

$$\mathbf{d}_I(t(a_{z-1})) = \mathbf{d}_J(h(a_{z-1})) = \mathbf{d}_I(t(a_{x_2})) = \mathbf{d}_J(h(a_{x_2})) = 1.$$

Therefore,

$$t(a_{z-1}), t(a_{x_2}) \in I = [x_1, x_2] \tag{194}$$

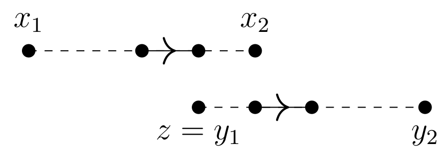
and

$$h(a_{z-1}), h(a_{x_2}) \in J = [y_1, y_2]. \tag{195}$$

If an arrow a_i points to the right, then $h(a_i) = i+1$ and $t(a_i) = i$. If a_i points left, $h(a_i) = i$ and $t(a_i) = i + 1$. We proceed by analyzing the direction of a_{x_2} and a_{z-1}. First consider a_{x_2}. If a_{x_2} points left, then $t(a_{x_2}) = x_2 + 1$ and so $x_2 + 1 \in [x_1, x_2]$, which is a contradiction. Therefore, we may assume a_{x_2} points right.

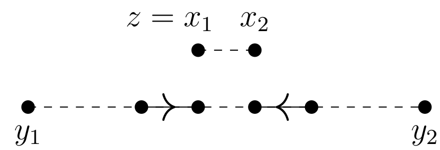
Now consider the direction of a_{z-1}. If a_{z-1} points to the right, then $t(a_{z-1}) = z-1 \in [x_1, x_2]$

by (194) and so $z > x_1$. Since $z \in \{x_1, y_1\}$, we must have $z = y_1$.



Therefore (I, J) is of type (II).

If a_{z-1} points left, now we have by (195) $h(a_{z-1}) = z - 1 \in [y_1, y_2]$. Therefore $z - 1 > y_1$ and so $z \neq y_1$ which implies $z = x_1$. Hence we have:



Therefore, (I, J) is of type (III).

By near identical arguments, $\chi_{\mathcal{Q}}(\mathbf{d}_J, \mathbf{d}_I) < 0$ when

(I) a_{z-1} and a_{x_2} both point left, $z = y_1$, and $x_2 < y_2$; i.e., (I, J) is of type (II)

(II) a_{z-1} points right, a_{x_2} points left, $z = x_1$ and $x_2 < y_2$ so (I, J) is of type (III). □

In particular, we have the following corollary.

Corollary 9.3. *If* $\chi_Q(\mathsf{V}_I, \mathsf{V}_J) < 0$ *then* $\chi_Q(\mathsf{V}_I, \mathsf{V}_J) = -1$.

Proof. If $(I, J) \in$ `StrandPairs`, this is immediate by Lemma 9.7. Otherwise, we have $(J, I) \in$ `StrandPairs`. Then by Lemma 9.7 $(J, I) \in$ `ConditionStrands`$(\mathcal{Q})$. As such,

$$\chi_Q(\mathsf{V}_I, \mathsf{V}_J) = -1.$$

□

Recall $U = \{(I, J) : \chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) < 0\}$. We let

$$U_1 = \{(I, J) = ([x_1, x_2], [y_1, y_2]) : (I, J) \in U \text{ and } x_2 \leq y_2\}, \text{ and}$$

$$U_2 = \{(I, J) = ([x_1, x_2], [y_1, y_2]) : (I, J) \in U \text{ and } x_2 > y_2\}.$$

Trivially,

$$U = U_1 \sqcup U_2. \tag{196}$$

Let

$$\widetilde{U_2} = \{(J, I) : (I, J) \in U_2\}.$$

Lemma 9.8. `ConditionStrands`$(\mathcal{Q}) = U_1 \sqcup \widetilde{U_2}$.

Proof. If $(I, J) \in \widetilde{U_2}$, then $(J, I) \in U_2 \subset U$ and so $\chi_Q(\mathsf{V}_J, \mathsf{V}_I) < 0$. Therefore $\chi_Q(\mathsf{V}_I, \mathsf{V}_J) \geq 0$ and hence $(I, J) \notin U$. As such, $(I, J) \notin U_1$. Therefore, $U_1 \cap \widetilde{U_2} = \emptyset$.
($\subseteq$) If $(I, J) \in \mathtt{ConditionStrands}(\mathcal{Q})$, by Lemma 9.7, $\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) < 0$ or $\chi_{\mathcal{Q}}(\mathsf{V}_J, \mathsf{V}_I) < 0$. In the first case, from the definition, $(I, J) \in U_1$. In the second case, again by definition, $(J, I) \in U_2$, which implies $(I, J) \in \widetilde{U}_2$.
($\supseteq$) We have $U_1, \widetilde{U_2} \subseteq \mathtt{StrandPairs}$. Thus by Lemma 9.7,

$$U_1, \widetilde{U_2} \subseteq \mathtt{ConditionStrands}(\mathcal{Q}).$$

□

Proposition 9.2.

$$\operatorname{codim}_{\mathbb{C}}(\gamma_\eta) = \sum_{(I,J)\in\mathtt{ConditionStrands}(\mathcal{Q})} m_I(\eta) m_J(\eta).$$

Proof. If $(I, J) \in U$, then $\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) < 0$. Applying Corollary 9.3, $\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J) = -1$. Then by Lemma 9.5,

$$\begin{aligned} \operatorname{codim}_{\mathbb{C}}(\gamma_\eta) &= \sum_{(I,J)\in U} m_I(\eta) m_J(\eta)(-\chi_{\mathcal{Q}}(\mathsf{V}_I, \mathsf{V}_J)) \\ &= \sum_{(I,J)\in U} m_I(\eta) m_J(\eta). \end{aligned}$$

Therefore, applying (196)

$$\begin{aligned} \operatorname{codim}_{\mathbb{C}}(\gamma_\eta) &= \sum_{(I,J)\in U_1} m_I(\eta) m_J(\eta) + \sum_{(I,J)\in U_2} m_I(\eta) m_J(\eta) \\ &= \sum_{(I,J)\in U_1} m_I(\eta) m_J(\eta) + \sum_{(I,J)\in \widetilde{U_2}} m_I(\eta) m_J(\eta) \\ &= \sum_{(I,J)\in\mathtt{ConditionStrands}(\mathcal{Q})} m_I(\eta) m_J(\eta) \qquad \text{(Lemma 9.8)} \end{aligned}$$

as claimed. □

Our final goal is to show $\mathtt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) = \mathtt{ConditionStrands}(\mathcal{Q})$. We start with a lemma.

Lemma 9.9. *All elements of* $\mathtt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$ *and* $\mathtt{ConditionStrands}(\mathcal{Q})$ *may be written in the form:*

$$(I, J) = ([x, k-1], [y, \ell]), \ \textit{with } x \neq y, \ k \leq \ell. \tag{197}$$

Proof. If

$$([w_{\mathcal{Q}}^{(k)}(i), k-1], [w_{\mathcal{Q}}^{(k)}(j), \ell]) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}),$$

then

$$w_{\mathcal{Q}}^{(k)}(i) \neq w_{\mathcal{Q}}^{(k)}(j) \text{ and } k \leq \ell.$$

Hence, by setting $x = w_{\mathcal{Q}}^{(k)}(i)$ and $y = w_{\mathcal{Q}}^{(k)}(j)$, we are done.

Now suppose

$$([x_1, x_2], [y_1, y_2]) \in \texttt{ConditionStrands}(\mathcal{Q}).$$

By (I)-(III), $x_1 \neq y_1$ and $x_2 < y_2$. Then set $x = x_1$, $y = y_1$, $k = x_2 + 1$, and $\ell = y_2$. $\square$

With Lemma 9.9 in mind, to prove $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) = \texttt{ConditionStrands}(\mathcal{Q})$, it is enough to show given (I, J) of the form in (197),

$$(I, J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \iff (I, J) \in \texttt{ConditionStrands}(\mathcal{Q}).$$

We first handle the special case when I and J are disjoint.

Lemma 9.10. *Let (I, J) be as in (197) and suppose $I \cap J = \emptyset$. Then $(I, J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$ if and only if $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$.*

Proof. If $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$, then by the disjointness hypothesis it must be of type (I), i.e.

$$(I, J) = ([x, k-1], [k, \ell]).$$

Now, since $x \leq k-1$ and $w_{\mathcal{Q}}^{(k)} \in \mathcal{S}_k$ with $w_{\mathcal{Q}}^{(k)}(k) = k$, there exists $i < k$ such that $w_{\mathcal{Q}}^{(k)}(i) = x$. Therefore,

$$([x, k-1], [k, \ell]) = ([w_{\mathcal{Q}}^{(k)}(i), k-1], [w_{\mathcal{Q}}^{(k)}(k), \ell]) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}).$$

Conversely, assume

$$(I, J) = ([w_{\mathcal{Q}}^{(k)}(i), k-1], [w_{\mathcal{Q}}^{(k)}(j), \ell]) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$$

and $I \cap J = \emptyset$. Then $w_{\mathcal{Q}}^{(k)}(j) > k-1$ which means $w_{\mathcal{Q}}^{(k)}(j) = k$ and $j = k$ by the definition of $w_{\mathcal{Q}}^{(k)}$. Furthermore, $w_{\mathcal{Q}}^{(k)}(i) \leq k-1$ since $i < j = k$. Therefore,

$$(I, J) = ([w_{\mathcal{Q}}^{(k)}(i), k-1], [k, \ell])$$

is of type (I), and hence in $\texttt{ConditionStrands}(\mathcal{Q})$. $\square$

We now prove the following.

Proposition 9.3. $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) = \texttt{ConditionStrands}(\mathcal{Q})$.

Proof. Let (I, J) be as in (197). We seek to show

$$(I, J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \iff (I, J) \in \texttt{ConditionStrands}(\mathcal{Q}).$$

We will proceed by induction on k. In the base case $k = 2$, we must have $x = 1$ and so $y \geq 2$. As such, $I \cap J = \emptyset$ and so we are done by Lemma 9.10. Fix $k > 2$ and assume the claim holds for $k - 1$. That is, given a pair of intervals $([x', k-2], [y', \ell'])$ so that x', y' and ℓ' satisfy $x' \neq y'$ and $k - 1 \leq \ell'$ we have

$$([x', k-2], [y', \ell']) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \iff ([x', k-2], [y', \ell']) \in \texttt{ConditionStrands}(\mathcal{Q}). \tag{198}$$

Now let (I, J) be as in (197), i.e.,

$$(I, J) = ([x, k-1], [y, \ell]) \text{ with } x \neq y,\ k \leq \ell.$$

Again, by Lemma 9.10, if $I \cap J = \emptyset$ we are done, so assume $I \cap J \neq \emptyset$. Then $y < k$.

Now, since $1 \leq x, y \leq k$, there exist i and j such that

$$1 \leq i, j \leq k \text{ with } x = w^{(k)}(i) \text{ and } y = w^{(k)}(j).$$

Then from (185)

$$(I, J) = ([w^{(k)}_{\mathcal{Q}}(i), k-1], [w^{(k)}_{\mathcal{Q}}(j), \ell]) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \iff i < j. \tag{199}$$

Throughout, when $x \leq k - 2$ we write $I' := [x, k-2]$. We will break the argument into two main cases.

Case 1: a_{k-2} and a_{k-1} point in the same direction.

By definition, $w^{(k)}_{\mathcal{Q}} = \iota(w^{(k-1)}_{\mathcal{Q}})$. Then if $x \leq k - 2$, it follows that

$$\begin{aligned}(I', J) &= ([x, k-2], [y, \ell]) \\ &= ([w^{k-1}_{\mathcal{Q}}(i), k-2], [w^{k-1}_{\mathcal{Q}}(j), \ell])\end{aligned}$$

and so

$$(I', J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \text{ if and only if } i < j. \tag{200}$$

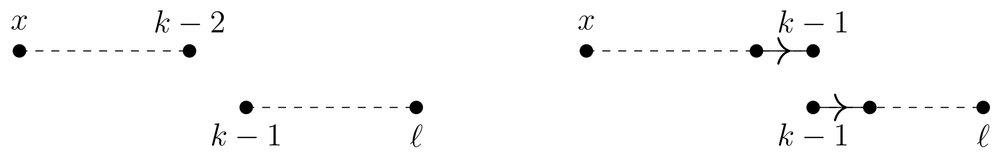
We have four possible subcases, based on the relative values of x and y.

Subcase i: $x < y = k-1$.

The pair (I, J) is of type (II), and hence $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$. Furthermore, note that

$$(I', J) = ([x, k-2], [k-1, \ell])$$

is of type (I), and so in $\texttt{ConditionStrands}(\mathcal{Q})$. The intervals for (I', J) and (I, J) look like this:



By the inductive hypothesis (198), $(I', J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$. By (200), $i < j$. Therefore, by (199), $(I, J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$.

As such, (I, J) is in both $\texttt{ConditionStrands}(\mathcal{Q})$ and $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$.

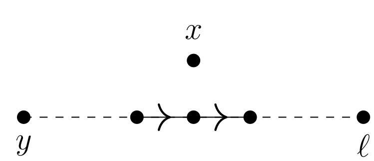
Subcase ii: $x < y < k-1$.

$$\begin{aligned}
(I, J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) &\iff i < j \text{ by (199)} \\
&\iff (I', J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \text{ by (200)} \\
&\iff (I', J) \in \texttt{ConditionStrands}(\mathcal{Q}) \text{ by (198)} \\
&\iff a_{x-1} \text{ points in the same direction as } a_{k-2} \\
&\iff a_{x-1} \text{ points in the same direction as } a_{k-1} \\
&\iff (I, J) \in \texttt{ConditionStrands}(\mathcal{Q}).
\end{aligned}$$

The following picture depicts (I', J) and (I, J) respectively when (I', J) and (I, J) are in $\texttt{ConditionStrands}(\mathcal{Q})$.

Subcase iii: $y < x = k-1$.

Pictured below are the intervals I and J.



Since $y < x$ and this case assumes a_{k-2} and a_{k-1} point in the same direction, (I, J) cannot be of type (III) and is not in $\texttt{ConditionStrands}(\mathcal{Q})$. Since

$$w_{\mathcal{Q}}^{(k)} = \iota w_{\mathcal{Q}}^{(k-1)} \text{ and } w_{\mathcal{Q}}^{(k-1)}(k-1) = k-1,$$

it follows that $i = k - 1$. Since

$$y = w_{\mathcal{Q}}^{(k)}(j) = w_{\mathcal{Q}}^{(k-1)}(j) < k - 1,$$

it follows that $i > j$, and so by (199)

$$(I, J) \notin \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}).$$

Therefore, (I, J) is in neither $\texttt{ConditionStrands}(\mathcal{Q})$ nor $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$.
Subcase iv: $y < x < k - 1$.

$$\begin{aligned}(I,J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) &\iff i < j \quad \text{by (199)} \\ &\iff (I', J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \text{ by (200)} \\ &\iff (I', J) \in \texttt{ConditionStrands}(\mathcal{Q}) \text{ by (198)} \\ &\iff a_{x-1} \text{ points in the opposite direction as } a_{k-2} \\ &\iff a_{x-1} \text{ points in the opposite direction as } a_{k-1} \\ &\iff (I, J) \in \texttt{ConditionStrands}(\mathcal{Q}).\end{aligned}$$

Below are $(I'J)$ and (I, J) respectively, in the case $(I', J), (I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$.

Case 2: a_{k-2} and a_{k-1} point in opposite directions.
By definition,

$$w_{\mathcal{Q}}^{(k)} = \iota(w_{\mathcal{Q}}^{(k-1)} w_0^{(k-1)}).$$

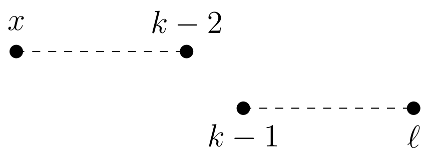
If $x \leq k - 2$, and $y \leq k - 1$ it follows that

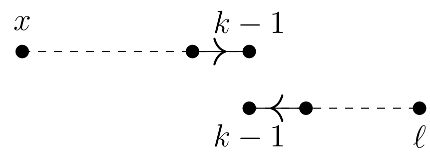
$$\begin{aligned}(I', J) &= ([x, k-2], [y, \ell]) \\ &= ([w_{\mathcal{Q}}^{(k-1)}(k-i), k-2], [w_{\mathcal{Q}}^{(k-1)}(k-j), \ell])\end{aligned}$$

and so

$$(I', J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \text{ if and only if } k-i < k-j \text{ if and only if } i > j. \tag{201}$$

Subcase i: $x < y = k-1$.





Since a_{k-2} and a_{k-1} point in opposite directions, $(I, J) \notin \texttt{ConditionStrands}(\mathcal{Q})$. The assumption $y = k-1$ implies $(I', J) \in \texttt{ConditionStrands}(\mathcal{Q})$. By (198) $(I', J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$. Since $x, y < k$, we have

$$x = w_{\mathcal{Q}}^{(k)}(i) = w_{\mathcal{Q}}^{(k-1)}(k-i) \text{ and } y = w_{\mathcal{Q}}^{(k)}(j) = w_{\mathcal{Q}}^{(k-1)}(k-j).$$

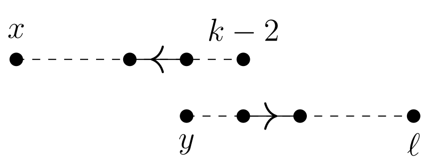
Then $k-i < k-j$, so $i > j$ and $(I, J) \notin \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$, by (199).

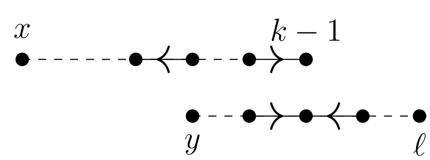
Hence (I, J) is neither in $\texttt{ConditionStrands}(\mathcal{Q})$ nor $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$.

Subcase ii: $x < y < k-1$.

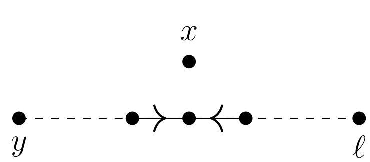
$$\begin{aligned}
(I, J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) &\iff i < j \text{ by (199)} \\
&\iff (I', J) \notin \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \text{ by (201)} \\
&\iff (I', J) \notin \texttt{ConditionStrands}(\mathcal{Q}) \text{ by (198)} \\
&\iff a_{y-1} \text{ points in the opposite direction as } a_{k-2} \\
&\iff a_{y-1} \text{ points in the same direction as } a_{k-1} \\
&\iff (I, J) \in \texttt{ConditionStrands}(\mathcal{Q}).
\end{aligned}$$

Below, we have $(I', J) \notin \texttt{ConditionStrands}(\mathcal{Q})$ and $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$.





Subcase iii: $y < x = k-1$. Here (I, J) looks like:



Since Case 2 assumes a_{k-2} and a_{k-1} point in opposite directions, (I, J) is type (II) and so $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$. Now,

$$k - 1 = x = w_{\mathcal{Q}}^{(k)}(i) = w_{\mathcal{Q}}^{(k-1)}(k - i)$$

which implies $i = 1$. Then $j > i$, so $(I, J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$. Therefore, (I, J) is both in $\texttt{ConditionStrands}(\mathcal{Q})$ and $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})$.
Subcase iv: $y < x < k - 1$.

$$\begin{aligned}
(I, J) \in \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) &\iff i < j \text{ by (199)} \\
&\iff (I', J) \notin \texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) \text{ by (201)} \\
&\iff (I', J) \notin \texttt{ConditionStrands}(\mathcal{Q}) \text{ by (198)} \\
&\iff a_{x-1} \text{ points in the same direction as } a_{k-2} \\
&\iff a_{x-1} \text{ points in the opposite direction as } a_{k-1} \\
&\iff (I, J) \in \texttt{ConditionStrands}(\mathcal{Q}).
\end{aligned}$$

Pictured below are (I', J) and (I, J), in the case that $(I', J) \notin \texttt{ConditionStrands}(\mathcal{Q})$ and $(I, J) \in \texttt{ConditionStrands}(\mathcal{Q})$.

Thus, we have $\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}}) = \texttt{ConditionStrands}(\mathcal{Q})$. □

Proof of Theorem 9.4.

$$\begin{aligned}
r_{\mathbf{w}_{\mathcal{Q}}}(\eta) &= \sum_{(I,J)\in\texttt{BoxStrands}(\mathbf{w}_{\mathcal{Q}})} m_I(\eta)m_J(\eta) && \text{(by Proposition 9.1)} \\
&= \sum_{(I,J)\in\texttt{ConditionStrands}(\mathcal{Q})} m_I(\eta)m_J(\eta) && \text{(by Proposition 9.3)} \\
&= \operatorname{codim}_{\mathbb{C}}(\gamma_\eta) && \text{(by Proposition 9.2).}
\end{aligned}$$

As such, the result follows. □

We conclude with our proof of Reineke's identity for type A quivers.

Proof of Theorem 9.2 (in type A). The map $\eta \mapsto \gamma_\eta$ defines a bijection from $\mathcal{C}_{\mathcal{Q}}(\mathbf{d})$ to $\mathcal{O}_{\mathcal{Q}}(\mathbf{d})$. Since $\mathcal{Q}$ is type A, there is a bijection between $\{\beta_1, \ldots, \beta_N\}$ and $\mathcal{I}$. Furthermore, whenever

$I \mapsto \beta_i$, we have $m_I(\eta) = m_{\beta_i}(\gamma_\eta)$. Then starting from Corollary 9.1, it follows that

$$\begin{aligned}
\prod_{i=1}^{n} \frac{1}{(q)_{\mathbf{d}(i)}} &= \sum_{\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})} q^{r_{\mathbf{w}_{\mathcal{Q}}}(\eta)} \prod_{1 \leq i \leq j \leq n} \frac{1}{(q)_{m_{[i,j]}(\eta)}} \\
&= \sum_{\eta \in \mathcal{C}_{\mathcal{Q}}(\mathbf{d})} q^{\operatorname{codim}_{\mathbb{C}}(\gamma_\eta)} \prod_{i=1}^{N} \frac{1}{(q)_{m_{\beta_i}(\gamma_\eta)}} \\
&= \sum_{\gamma \in \mathcal{O}_{\mathcal{Q}}(\mathbf{d})} q^{\operatorname{codim}_{\mathbb{C}}(\gamma)} \prod_{i=1}^{N} \frac{1}{(q)_{m_{\beta_i}(\gamma)}}.
\end{aligned}$$

□

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