ON SEQUENCES RELATED TO BINARY PARTITION FUNCTION AND THE THUE-MORSE SEQUENCE

BY

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DISSERTATION

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Abstract

In this dissertation, we discuss properties of the family of sequences \( u_d = \{u_d(n)\}_{n \geq 0} \) for positive integer \( d \). We define them by letting \( u_d(n) \) be the coefficient of \( X^n \) in \( \prod_{j=0}^{\infty} \left( 1 - X^{2^j} \right) \left( 1 - X^{d \cdot 2^j} \right)^{-1} \). First, we discuss the binary partition function and its relationship with the sequence \( u_d \). We then give several intermediate results and identities. Afterward, we generalize the sequence with different initial values. We also look at the corresponding generating function. After this, we focus on its asymptotic behavior by illustrating the cases when \( d = 3, 5, 9 \). Finally, we explain asymptotic behavior for general cases and establish conjectures based on numerical data.

Then, we investigate another family of sequences, \( x_k = \{x_k(n)\}_{n \geq 0} \), defined by \( x_k(n) = |t_{n+k} - t_n| \) where \( t = \{t_n\}_{n \geq 0} \) is the Thue-Morse sequence. We give the frequency of 1’s and 0’s of each sequence \( x_k \) and express them in terms of recurrence relations. We note the similarity with the Stern sequence, denoted by \( s = \{s(n)\}_{n \geq 0} \). Further, we investigate the frequency of appearances of 00, 01, 10, and 11 of each sequence.

Finally, we define the correlation function related to the sequence \( x_k \), denoted by \( f(d) \), and the associated density function \( \tilde{f}(d) \). We present both recurrence relations, and closed formulas for values of \( d \) near powers of 2.
I dedicate this dissertation to you, for your kind and wholehearted encouragement.
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Chapter 1

Introduction to the Sequence $u_d$

In this chapter, we first mention the historical background of the binary partition functions, as well as discuss some more modern results. Then, we summarize some important properties, and introduce a family of sequences $u_d$ that is the main focus of this dissertation. Finally, we conclude with an overview of the following chapters.

1.1 The Binary Partition Function $b(d; n)$

Definition 1.1.1. For $d \geq 2$, the $d$-th binary partition function, denoted by $b(d; n)$, is the number of representations

\[ n = \sum_{i=1}^{\infty} \epsilon_i 2^i, \epsilon_i \in \{0, 1, \ldots, d - 1\}. \quad (1.1) \]

For instance, $b(3, 6) = 3$ because we can express 6 as a sum of powers of 2 where coefficients are taken from the set $\{0, 1, 2\}$ in 3 different ways, namely

\[
\begin{align*}
6 &= (0 \times 2^0) + (1 \times 2^1) + (1 \times 2^2) \\
&= (2 \times 2^0) + (0 \times 2^1) + (1 \times 2^2) \\
&= (2 \times 2^0) + (2 \times 2^1) + (0 \times 2^2).
\end{align*}
\]

The Euler binary partition function is defined by $b(\infty; n) = \lim_{d \to \infty} b(d; n)$. For some values of $d$, the $d$-th binary partition function has been well studied. Reznick [6] provided the following formula.
Theorem 1.1.2. For all non-negative integer \( n \), we have

\[
\begin{align*}
  b(2; n) &= 1, \\
  b(3; n) &= s(n + 1), \\
  b(4; n) &= \lfloor n/2 \rfloor + 1,
\end{align*}
\]

(1.2)

(1.3)

(1.4)

where \( s(n) \) is the Stern sequence, defined recursively by

\[
\begin{align*}
  s(0) &= 0, & s(1) &= 1, & s(2n) &= s(n), & s(2n + 1) &= s(n + 1) + s(n),
\end{align*}
\]

for all \( n \).

In addition, Reznick [6] gave a representation of \( b(2^r; n) \).

Theorem 1.1.3. For all \( r \geq 1 \), \( 0 \leq t \leq 2^{r-1} - 1 \), and \( s \geq 0 \),

\[
b(2^r; 2^r s + t) = \sum_{j=0}^{r-2} a_j(r, t) \left( \frac{s + r - 1 - j}{r - 1} \right),
\]

where \( a_j(r, t) \)'s are sequences defined recursively by

\[
\begin{align*}
  a_0(1, 0) &= a_0(2, 1) = 1, \\
  a_j(r + 1, 2k) &= a_j(r + 1, 2k + 1) = \sum_{t=0}^{k} a_j(r, t) + \sum_{t=k+1}^{2^{r-1} - 1} a_{j-1}(r, t),
\end{align*}
\]

(1.5)

(1.6)

with \( a_{-1}(r, t) = a_{r-1}(r, t) = 0 \) for all \( r, t \).

Unfortunately, no other values of \( d \) result in a well-behaved binary partition function. Reznick [6] also proved that, for each \( k \geq 1 \), there exist positive constants \( C_1, C_2 \) such that \( C_1 < C_2 \) and

\[
C_1 n^{\log_2 k} \leq b(2k; n) \leq C_2 n^{\log_2 k}.
\]

(1.7)

Protasov [5] generalized this by showing that for any integer \( d > 1 \), there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 n^{\lambda_1} \leq b(d; n) \leq C_2 n^{\lambda_2},
\]

(1.8)
where
\[ \lambda_1 = \liminf_{n \to \infty} \frac{\log b(d; n)}{\log n}, \quad \lambda_2 = \limsup_{n \to \infty} \frac{\log b(d; n)}{\log n}. \] (1.9)

### 1.2 The Generating Function of \( b(d; n) \)

The generating function of \( b(d; n) \) is easily seen to be

\[ F_d(X) := \sum_{n=0}^{\infty} b(d; n) X^n \] (1.10)

\[ = \prod_{j=0}^{\infty} \frac{1}{1 - X^{2^j} + X^{2^{j+1}} + \ldots + X^{(d-1)2^j}} \] (1.11)

\[ = \prod_{j=0}^{\infty} \frac{1 - X^{d-2^j}}{1 - X^{2^j}}. \] (1.12)

\[ F_\infty(X) := \prod_{j=0}^{\infty} \frac{1}{1 - X^{2^j}}. \] (1.13)

Reznick [6] provided elementary properties for these generating functions, which can be obtained directly from the infinite products.

**Theorem 1.2.1.** For positive integers \( c, d \), we have

(i) \( F_{cd}(X) = F_c(X^d)F_d(X) \),

(ii) \( F_\infty(X) = F_\infty(X^d)F_d(X) \),

(iii) \( F_{2d}(X) = (1 - X)^{-1}F_d(X^2) \),

(iv) \((1 - X)F_d(X) = (1 - X^d)F_d(X^2)\),

(v) \( F_d(X) = (1 - X^d)F_{2d}(X) \).

**Proof.** We simply expand the left-hand side of each equation to get the results.

(i) \( F_{cd}(X) = \prod_{j=0}^{\infty} \frac{1 - X^{cd-2^j}}{1 - X^{2^j}} = \prod_{j=0}^{\infty} \frac{1 - X^{d-2^j}}{1 - X^{2^j}} \prod_{j=0}^{\infty} \frac{1 - X^{d-2^j}}{1 - X^{2^j}} = F_c(X^d)F_d(X) \).

(ii) \( F_d(X)F_\infty(X^d) = \prod_{j=0}^{\infty} \frac{1 - X^{d-2^j}}{1 - X^{2^j}} \prod_{j=0}^{\infty} \frac{1}{1 - (X^2)^{2^j}} = \prod_{j=0}^{\infty} \frac{1}{1 - X^{2^j}} = F_\infty(X) \).

(iii) \( F_{2d}(X) = \prod_{j=0}^{\infty} \frac{1 - X^{2d-2^j}}{1 - X^{2^j}} = \prod_{j=0}^{\infty} \frac{1 - (X^2)^{d-2^j}}{1 - (X^2)^{2^j}} = \frac{1}{1 - X} \prod_{j=0}^{\infty} \frac{1}{1 - (X^2)^{2^j}} = (1 - X)^{-1}F_d(X^2) \).
1.3 Definition of the Sequence $u_d$

In this dissertation, we will investigate the reciprocal of the generating function for the binary partition function $F_d(X)$.

**Definition 1.3.1.** For each positive integer $d$, let

\[
U_d(X) := F_d(X)^{-1} = \prod_{j=0}^{\infty} \frac{1 - X^{2^j}}{1 - X^{2^{j+1}}},
\]

and define the corresponding sequence $u_d = \{u_d(n)\}_{n \geq 0}$ to be the sequence of coefficients in $U_d(X)$. In other words,

\[
U_d(X) = \sum_{n=0}^{\infty} u_d(n) X^n.
\]

In a similar fashion, let $U_\infty(X) := F_\infty(X)^{-1} = \prod_{j=0}^{\infty} \left(1 - X^{2^j}\right)$.

We note that the sequence $u_d$ can also be defined directly from $b(d; n)$, as $u_d$ is the unique sequence that satisfies

\[
u_d(0)b(d; 0) = 1,
\]

\[
\sum_{j=0}^{k} u_d(j)b(d; k - j) = 0,
\]

for all positive integer $k$. Now, we also need to introduce the Thue-Morse sequence, which we will use repeatedly throughout this dissertation.

**Definition 1.3.2.** The Thue-Morse sequence, denoted by $t = (t_n)_{n \geq 0}$, is defined recursively as follows.

\[
t_0 = 0, \quad t_{2n} = t_n, \quad t_{2n+1} = 1 - t_n.
\]
for all $n \geq 0$. The first few entries of such sequence $t$ are given below.

\[ t = 0110100110010110 \ldots \]

**Theorem 1.3.3.** For all $n \geq 0$, we have $u_\infty(n) = (-1)^t_n$.

**Proof.** We have

\[ U_\infty(X) = \mathcal{F}_\infty(X)^{-1} = \prod_{j=0}^{\infty} (1 - X^2^j) = \sum_{i=0}^{\infty} u_\infty(i)X^i. \]

If we write $i = \sum_{k \in A} 2^k$ where $A \subseteq \mathbb{N}$, then $u_\infty(i) = (-1)^{|A|} = (-1)^t_n$. \hfill \Box

Directly applying Theorem 1.2.1, we have the following properties for the generating functions.

**Theorem 1.3.4.** For positive integers $c, d$ and non-negative integer $n$, we have

(i) $U_{cd}(X) = U_c(X^d)U_d(X)$,

(ii) $U_\infty(X) = U_\infty(X^d)U_d(X)$,

(iii) $U_{2d}(X) = (1 - X)U_d(X^2)$,

(iv) $(1 - X^d)U_d(X) = (1 - X)U_d(X^2)$,

(v) $U_d(X) = (1 - X^d)^{-1}U_{2d}(X)$.

Consequently, we obtain recurrences for the sequence $u_d$.

**Theorem 1.3.5.** For positive integers $c, d$, we have

(i) $u_{cd}(n) = \sum_{i=0}^{\lfloor n/d \rfloor} u_c(i)u_d(n - di)$,

(ii) $(-1)^t_n = \sum_{i=0}^{\lfloor n/d \rfloor} (-1)^t_i u_d(n - di)$,

(iii) $u_{2d}(2n) = u_d(n)$,

(iv) $u_{2d}(2n + 1) = -u_d(n)$,

(v) $\sum_{i=0}^{d-1} u_d(2n - i) = u_d(n)$,

(vi) $\sum_{i=0}^{d-1} u_d(2n - 1 - i) = 0$. 

5
Proof.

(i) Expanding Theorem 1.3.4 (i), we have

\[ U_c(X^d)U_d(X) = \sum_{i=0}^{\infty} u_c(i)X^{di} \sum_{j=0}^{\infty} u_d(j)X^j = \sum_{i,j} u_c(i)u_d(j)X^{di+j}. \]

Then, we set \( n = di + j \) for \( i = 0, \ldots, \lfloor \frac{n}{d} \rfloor \). Comparing the coefficients of \( x^n \) on both sides of Theorem 1.3.4 (i), we get \( u_{cd}(n) = \sum_{i=0}^{\lfloor \frac{n}{d} \rfloor} u_c(i)u_d(n-di). \)

(ii) Similarly, we apply Theorem 1.3.3 to obtain the left-hand side and Theorem 1.3.4 (ii) to get

\[ U_\infty(X^d)U_d(X) = \sum_{i=0}^{\infty} u_\infty(i)X^{di} \sum_{j=0}^{\infty} u_d(j)X^j = \sum_{i,j} (-1)^i u_d(j)X^{di+j}. \]

Thus, we have \( (-1)^n = \sum_{i=0}^{\lfloor \frac{n}{d} \rfloor} (-1)^i u_d(n-di). \)

(iii) The right-hand side in Theorem 1.3.4 (iii) is

\[ (1 - X)U_d(X^2) = (1 - X) \sum_{i=0}^{\infty} u_d(i)X^{2i} = \sum_{i=0}^{\infty} u_d(i)X^{2i} - \sum_{i=0}^{\infty} u_d(i)X^{2i+1} = \sum_{i=0}^{\infty} (-1)^i u_d([i/2])X^i. \]

When \( i = 2n \), the coefficient of \( x^{2n} \) in \( (1 - X)U_d(X^2) \) equals \( u_d(n) \).

(iv) The proof is similar to (iii), but with \( i = 2n + 1 \) instead.

(v) We can rearrange the terms in Theorem 1.3.4 (iv) as

\[ U_d(X^2) = U_d(X) \frac{(1 - X^d)}{1 - X} = U_d(X)(1 + X + \ldots + X^{d-1}). \]

Expanding both sides and comparing coefficients of \( X^{2n} \) yields the result.

(vi) This follows from the equation above by comparing coefficients of \( X^{2n+1} \) instead.

\[ \Box \]

Theorem 1.3.5 (iii) suggests that it suffices to study \( u_d \) when \( d \) is odd. Moreover, Theorem 1.3.5 (iv) and (v) give

\[ u_d(2n) + u_d(2n-1) + \ldots + u_d(2n-d+1) = u_d(n), \quad (1.16) \]

\[ u_d(2n+1) + u_d(2n) + \ldots + u_d(2n-d) = 0, \quad (1.17) \]
for \( n \geq d/2 \). Upon simplifying, we obtain this corollary.

**Corollary 1.3.6.** Let \( d \geq 1 \). Then, we have

\[
\begin{align*}
    u_d(2n) &= u_d(2n - d) + u_d(n), \\
    u_d(2n + 1) &= u_d(2n - d + 1) - u_d(n),
\end{align*}
\]

(1.18) \hspace{1cm} (1.19)

for \( n \geq d/2 \).

Since these equations are only applicable when the arguments on the left-hand side are \( 2n \geq d \), we need to also provide initial conditions.

**Corollary 1.3.7.** Let \( d \geq 1 \). Then, \( u_d(n) = (-1)^t_n \) for \( n = 0, \ldots, d - 1 \).

**Proof.** For \( d \geq 1 \), we have

\[
U_d(X) = \prod_{j=0}^{\infty} \frac{1 - X^{2^j}}{1 - X^{d2^j}}
\]

\[
= \prod_{j=0}^{\infty} (1 - X^{d2^j}) \prod_{j=0}^{\infty} (1 - X^{d2^j})^{-1}
\]

\[
= \left( \sum_{k=0}^{\infty} (-1)^{t_k} X^k \right) (1 + X^d + 2X^{2d} + 3X^{3d} \ldots)
\]

\[
= \left( \sum_{k=0}^{d-1} (-1)^{t_k} X^k \right) \mod X^d.
\]

Thus, \( u_d(n) = (-1)^t_n \) for \( n = 0, 1, \ldots, d - 1 \) as desired.

Alternately, since we have \( n < d \), in any representation \( n = \sum_{i=1}^{\infty} \epsilon_i 2^i \), we must have \( \epsilon_i \leq n < d \). Thus, \( u_d(n) = u_\infty(n) = (-1)^t_n \).

\[\square\]

### 1.4 A Characterization of \( u_d \)

Summarizing the results from the previous section, we obtained a new definition for the sequence \( u_d \) which does not rely on \( b(d; n) \). This definition will be used repeatedly in this dissertation.

**Definition 1.4.1.** Let \( d \geq 1 \) and \( t \) be the Thue-Morse sequence. The sequence \( u_d = \{u_d(n)\}_{n \geq 0} \) is defined by the following recurrences

\[
\begin{align*}
    u_d(2n) &= u_d(2n - d) + u_d(n), \\
    u_d(2n + 1) &= u_d(2n - d + 1) - u_d(n),
\end{align*}
\]

(1.20) \hspace{1cm} (1.21)
for \( n \geq d/2 \), with initial conditions \( u_d(n) = (-1)^n \) for \( n = 0, \ldots, d - 1 \).

Firstly, these two recurrences can be combined to a single equation

\[
u_d(n) = u_d(n - d) + (-1)^n u_d \left( \left\lfloor \frac{n}{2} \right\rfloor \right),
\]

(1.22)

for \( n \geq d \). Despite the brevity of (1.22), we will see that the recurrences (1.20) and (1.21) are easier to work with.

| \( n \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) | \( 9 \) | \( 10 \) | \( 11 \) | \( 12 \) | \( 13 \) | \( 14 \) | \( 15 \) | \( 16 \) | \( 17 \) | \( 18 \) | \( 19 \) | \( 20 \) |
| \( u_3(n) \) | 1 | -1 | -1 | 2 | -2 | 0 | 4 | -4 | -2 | 6 | -4 | -2 | 10 | -8 | -6 | 14 | -10 | -4 | 20 | -16 | -8 |
| \( u_5(n) \) | 1 | -1 | -1 | 1 | -1 | 2 | 0 | -2 | 0 | 4 | -2 | -2 | 6 | -2 | -2 | 0 | -2 | 0 | -2 | 10 |
| \( u_7(n) \) | 1 | -1 | -1 | 1 | -1 | 1 | 1 | -1 | 0 | -2 | 0 | 2 | -2 | 2 | 0 | 0 | -2 | -2 | 4 | -2 | 2 |
| \( u_9(n) \) | 1 | -1 | -1 | 1 | -1 | 1 | 1 | -1 | -1 | 2 | 0 | -2 | 2 | -2 | 0 | 2 | -2 | 0 | 4 | -2 | -2 |
| \( u_{11}(n) \) | 1 | -1 | -1 | 1 | -1 | 1 | 1 | -1 | -1 | 1 | 1 | 1 | 0 | 0 | -2 | 0 | 0 | 2 | 0 | -2 | 2 |

Table 1.1: Values for \( u_d(n) \) for small \( d \) and \( n \)

We can get another representation by applying the recurrences repeatedly. If \( d \) is even, then we have

\[
u_d(2n) = u_d(2n - d) + u_d(n) = u_d(2n - 2d) + u_d(n) + u_d \left( n - \frac{d}{2} \right)
\]

\[
\vdots
\]

\[
= \sum_{i \geq 0} u_d \left( n - \frac{id}{2} \right),
\]

(1.23)

\[
u_d(2n + 1) = - \sum_{i \geq 0} u_d \left( n - \frac{id}{2} \right)
\]

(1.24)

where we let \( u_d(k) = 0 \) for all \( k < 0 \).

If \( d \) is odd, then we have

\[
u_d(2n) = \sum_{i \geq 0} u_d(n - id) - u_d \left( n - \frac{d+1}{2} - id \right),
\]

(1.25)

\[
u_d(2n + 1) = \sum_{i \geq 0} -u_d(n - id) + u_d \left( n - \frac{d+1}{2} - id \right).
\]

(1.26)

As values of \( u_d(n) \) might be negative, a standard combinatorial interpretation cannot be given. Never-
theless, we have

\[ U_d(X) = \left( \sum_{k=0}^{\infty} (-1)^k X^k \right) \prod_{j=0}^{\infty} (1 - X^{d2j})^{-1} \]
= \left( \sum_{k=0}^{\infty} (-1)^k X^k \right) \sum_{m=0}^{\infty} b(\infty; m) X^{dm} \]
= \sum_{n=0}^{\infty} \binom{n/d}{l} \sum_{l=0}^{b(\infty; l)(-1)^{n-l}} X^n.

Corollary 1.4.2. For all \( d \geq 1, n \geq 0 \), \( u_d(n) = \sum_{l=0}^{\lfloor n/d \rfloor} b(\infty; l)(-1)^{l(n-ld)} \).

This corollary implies that \( u_d(n) \) is equal to the sum of \( b(\infty; l) \) where \( t(n-ld) = 0 \), reduced by the sum of \( b(\infty; l^*) \) where \( t(n-l^*d) = 1 \), for \( 0 \leq l, l^* < \frac{n}{d} \).

Finally, we look at the summatory function \( S_d(n) = \sum_{i=0}^{n-1} u_d(i) \). Applying equations (1.16) and (1.17), we immediately have the following.

Theorem 1.4.3. For all \( k, d \geq 0 \), we have

\[ S_d(kd) = \sum_{i=0}^{kd-1} u_d(i) = \begin{cases} 0, & \text{if } d \text{ is even,} \\ \sum_{i=0}^{k/2} u_d \left( (2i + 1) \left( \frac{d-1}{2} \right) \right), & \text{if } d \text{ is odd}. \end{cases} \quad (1.27) \]

Proof. Let \( S_d(a, b) = \sum_{i=a}^{b-1} u_d(i) \). When \( d \) is even, we have \( S_d(ld, (l+1)d) = 0 \) for all \( l \) by the equation (1.17), and thus \( S_d(kd) = \sum_{i=0}^{k-1} S_d(id, (i+1)d) = 0 \). Likewise, when \( d \) is odd, we have \( S_d(ld, (l+1)d) = 0 \) if \( l \) is odd and \( S_d(ld, (l+1)d) = u_d \left( \frac{l(l+1)d-1}{2} \right) \) if \( l \) is even. Thus, \( S_d(kd) = \sum_{i=0}^{k-1} S_d(id, (i+1)d) = \sum_{i=0}^{k/2} u_d \left( (2i + 1) \left( \frac{d-1}{2} \right) \right) \).

\[ \square \]

1.5 Overview of this Dissertation

In Chapter 2, we introduce the sequence \( u_d(a) \) where we use the recurrence relations obtained in Definition 1.4.1, but with general initial conditions. We then express the sequence as a linear combination of the initial conditions and establish relationships among them. We conclude the chapter by proving the necessary and sufficient conditions so that \( u_d(a; n) = 0 \) for all \( n \geq d \).

In Chapter 3, we investigate the generating function \( U_d(X) \). We obtain several interesting identities and
derive an explicit generating function for the generalized sequence $u_d(a)$ introduced in Chapter 2.

In Chapter 4, we focus on the sequence $u_3$. We use an existing result that the asymptotic growth of the sequence $u_d$ is $n^{\log n}$, and prove inequalities for $u_3(3k+i)$ where $i = 0, 1, 2$. In a similar fashion, we take a look at the sequence $u_5$, where inequalities are proven and interesting behaviors are discovered. Finally, we complete this chapter by comparing the sequences $u_3$ and $u_9$.

In Chapter 5, we study $u_d$ for odd $d \geq 7$. We attempt to explain our findings from scrutinizing numerical data, point out some mysterious behavior, and make some conjectures.

In Chapter 6, we focus on another related topic of the Thue-Morse sequence $t = \{t(n)\}_{n \geq 0}$. We give the definition and list several properties, and then introduce a family of difference sequences named $x_d$.

In Chapter 7, we introduce a new function $f(d)$ related to the difference sequences $x_d$, and also another function $\tilde{f}(d) = \frac{f(d)}{d}$. We derive formulas for $f(d)$ and $\tilde{f}(d)$ for $d$ near powers of 2. We show that the average of $\tilde{f}(d)$ over all integer $d$ is $\frac{1}{2}$. 
Chapter 2

A Generalization of the Sequence $u_d$

2.1 Definition of the Sequence $u_d(a)$

We recall the recurrence relations obtained in Corollary 1.3.5, but we use general initial conditions.

**Definition 2.1.1.** Let $d \geq 1$ and $a = (a_0, a_1, \ldots, a_{d-1}) \in \mathbb{R}^d$ be a $d$-dimensional vector. Define the sequence $u_d(a) = \{u_d(a;n)\}_{n \geq 0}$ as follows: $u_d(a;k) = a_k$ when $k = 0, 1, \ldots, d - 1$, and

\[
\begin{align*}
    u_d(a;2n) &= u_d(a;2n-d) + u_d(a;n), \\
    u_d(a;2n+1) &= u_d(a;2n-d+1) - u_d(a;n),
\end{align*}
\]

when $n \geq d/2$.

We know that $u_d(a)$ is well defined since the arguments on the right-hand side of both equations are non-negative and those on the left-hand side are at least $d - 1$ when $n \geq d/2$. We think of $a$ as a $d$-dimensional vector, and $u_d(\cdot)$ as a function from $\mathbb{R}^d$ to $\mathbb{R}^\infty$.

By linearity of the recurrences, we have several immediate results.

**Proposition 2.1.2.** Let $a, b \in \mathbb{R}^d$ and $k \in \mathbb{R}$.

(i) $u_d(a;n) + u_d(b;n) = u_d(a+b;n)$,

(ii) $u_d(ka;n) = k \cdot u_d(a;n)$,

(iii) If $\alpha_d = ((-1)^{t_0},(-1)^{t_1},\ldots,(-1)^{t_{d-1}})$, then $u_d(\alpha_d) = u_d$ where $u_d$ is the original sequence from Section 1.4.

Moreover, $u_d(a;n)$ can be expressed as a linear combination of $a_0, \ldots, a_{d-1}$.

**Definition 2.1.3.** Let $e_d(i) = (a_0, \ldots, a_{d-1})$ where $a_i = 1$ and $a_j = 0$ for $j \neq i$. For each positive integer $k$, define the sequence $\lambda_d(k) = \{\lambda_d(k;n)\}_{n \geq 0}$ by

\[
\lambda_d(k;n) = u_d(e_d(k);n).
\]
Theorem 2.1.4. For all $a \in \mathbb{R}^d$ and $n \geq 0$, we have

$$u_d(a; n) = \sum_{k=0}^{d-1} \lambda_d(k; n)a_k. \quad (2.3)$$

Proof. We notice that $a = (a_0, a_1, \ldots, a_{d-1}) = a_0e_d(0) + a_1e_d(1) + \ldots + a_{d-1}e_d(d-1)$ and applying Proposition 2.1.2, we obtain

$$u_d(a; n) = \sum_{k=0}^{d-1} u_d(a_k e_d(k); n) = \sum_{k=0}^{d-1} u_d(e_d(k); n) \cdot a_k = \sum_{k=0}^{d-1} \lambda_d(k; n)a_k.$$

\[\square\]

### 2.2 Properties of the Sequence $u_d(a)$

Let $\theta_d = ((-1)^{d-1}, (-1)^{d-2}, \ldots, (-1)^{d})$. We will show that setting $a = \theta_d$ yields a sequence which is eventually zero. More precisely, $u_d(\theta_d; n) = 0$ for all $n \geq d$. This will imply Corollary 2.2.6.

Lemma 2.2.1. $u_d(\theta_d; n) = 0$ for $n = d, d + 1, \ldots, 2d - 1$.

Proof. When $n = 2k$, we have

$$u_d(\theta_d; 2k) = u_d(\theta_d; 2k - d) + u_d(\theta_d; k)$$

$$= (-1)^{d-1-2k} + (-1)^{d-1-k}$$

$$= (-1)^{d-1-k-1} + (-1)^{d-1-k}$$

$$= (-1)^{d-1-k} + (-1)^{d-1-k}$$

$$= 0.$$

Similarly, when $n = 2k + 1$, we have

$$u_d(\theta_d; 2k + 1) = u_d(\theta_d; 2k + 1 - d) - u_d(\theta_d; k)$$

$$= (-1)^{d-1-(2k+1-d)} - (-1)^{d-1-k}$$

$$= (-1)^{2k-2} - (-1)^{d-1-k}$$

$$= (-1)^{d-1-k} - (-1)^{d-1-k}$$

$$= 0.$$
Hence, \( u_d(\theta_d; n) = 0 \) for \( n = d, d + 1, \ldots, 2d - 1 \).

\[ \Box \]

**Theorem 2.2.2.** We have \( u_d(\theta_d; n) = 0 \) for all \( n \geq d \).

**Proof.** We prove this statement by induction. The base cases are when \( n = d, d + 1, \ldots, 2d - 1 \), which are verified by Lemma 2.2.1. Suppose that \( u_d(\theta_d; n) = 0 \) for \( n = d, d + 1, \ldots, k - 1 \) with \( k \geq 2d \). If \( k \) is even, then \( u_d(\theta_d; k) = u_d(\theta_d; k - d) + u_d(\theta_d; k/2) \). Since \( k - d \) and \( k/2 \) are in the range \([d, k)\), by the induction hypothesis we have \( u_d(\theta_d; k - d) = u_d(\theta_d; k/2) = 0 \), and so \( u_d(\theta_d; k) = 0 \). Likewise, if \( k \) is odd, then \( u_d(\theta_d; k) = u_d(\theta_d; k - d) - u_d(\theta_d; (k - 1)/2) = 0 \). \( \Box \)

**Lemma 2.2.3.** If \( u_d(a; n) = 0 \) for all \( n \geq k \) for some \( k > d > 0 \), then we have \( u_d(a; n) = 0 \) for all \( n \geq d \).

**Proof.** If \( k \geq d + 2 \), then we have \( u_d(a; 2(k - 1)) = u_d(a; 2(k - 1) - d) = 0 \) because \( 2(k - 1) > 2(k - 1) - d \geq k \). Thus, applying the equation \( u_d(a; 2(k - 1)) = u_d(a; 2(k - 1) - d) + u_d(a; k - 1) \), we obtain \( u_d(a; k - 1) = 0 \).

On the other hand, if \( k = d + 1 \), then we have \( u_d(a; 2d) = u_d(a; d) + u_d(a; d) = 2 \cdot u_d(a; d) \). Since \( 2d \geq k \), we have \( u_d(a; 2d) = 0 \) and thus \( u_d(a; d) = 0 \). In any case, we can always trace back the zeroes from \( n = k \) down to \( n = d \). Hence, \( u_d(a; n) = 0 \) for all \( n \geq d \). \( \Box \)

**Theorem 2.2.4.** Suppose that \( d \) is odd and \( u_d(a; n) = 0 \) for all \( n \geq k \) for some \( k \). Then, \( a = c \cdot \theta_d \) where \( c \in \mathbb{R} \).

**Proof.** By Lemma 2.2.3, \( u_d(a; n) = 0 \) for all \( n \geq d \). Also, we have \( u_d(a; 2d - 1) = u_d(d - 1) - u_d(d - 1) = 0 \). Therefore, we need to solve the following system of linear equations

\[
0 = u_d(a; d) = u_d(a; 0) + u_d(a; (d - 1)/2),
0 = u_d(a; d + 1) = u_d(a; 1) - u_d(a; (d + 1)/2),
\vdots
0 = u_d(a; 2d - 2) = u_d(a; d - 2) + u_d(a; d - 1).
\]

The arguments on the right-hand side range from \( 0 \) to \( d - 1 \), so this system consists of \( d - 1 \) equations and \( d \) variables. In other words, we want to find \( u = (u_d(a; 0), \ldots, u_d(a; d - 1)) \) such that \( M \cdot u^T = 0 \) where \( M \) is a \( d - 1 \times d \) matrix and \( 0 = (0, \ldots, 0) \in \mathbb{R}^{d - 1} \). Since \( M \) is an upper triangular matrix, it has the full column rank \( d - 1 \), and thus the null space has rank 1. Theorem 2.2.2 provides one such solution, namely \( a = \theta_d \), and since it lies in the null space, \( c \cdot \theta_d \) is also a solution for any \( c \in \mathbb{R} \). \( \Box \)

**Corollary 2.2.5.** We have \( \sum_{k=0}^{d-1} (-1)^{d-1-k} \lambda_d(k, n) = 0 \) for all \( n \geq d \).
Proof. Simply applying Theorem 2.2.2 in (2.3) with \(a = \alpha_d\) yields the result.

This establishes a relationship among the \(\lambda_d(k; n)\)'s. For instance, take \(d = 3\). We get

\[-\lambda_3(0; n) - \lambda_3(1; n) + \lambda_3(2; n) = 0,\]

for \(n \geq 3\). Moreover, with \(a = (1, -1, -1)\), we have \(u_3(a; n) = u_3(n)\), where \(u_3(n)\) is the main sequence defined in Section 1.4, so

\[\lambda_3(0; n) - \lambda_3(1; n) - \lambda_3(2; n) = u_3(n).\]

Thus, we have the following corollary.

**Corollary 2.2.6.** \(\lambda_3(1; n) = -\frac{1}{2} u_3(n)\) for \(n \geq 3\).

**Corollary 2.2.7.** \(\lambda_5(2; n) = -\frac{1}{2} u_5(n)\) for \(n \geq 5\).

*Proof.* We have \(u_5 = u_5(\alpha_5)\) with \(\alpha_5 = (1, -1, -1, 1, -1)\), so by equation (2.3)

\[\lambda_5(0, n) - \lambda_5(1, n) - \lambda_5(2, n) + \lambda_5(3, n) - \lambda_5(4, n) = u_5(n).\]

By Corollary 2.2.5, \(-\lambda_5(0, n) + \lambda_5(1, n) - \lambda_5(2, n) - \lambda_5(3, n) + \lambda_5(4, n) = 0\). Adding these equations together, we get \(-2\lambda_5(2, n) = u_5(n)\). □

These corollaries only emerge because \(d\) is small. For odd \(d \geq 7\), there is no such cancellation. For instance, when \(d = 7\), we can derive \(\lambda_7(0, n) - \lambda_7(2, n) + \lambda_7(3, n) - \lambda_7(4, n) + \lambda_7(6, n) = \frac{1}{2} u_7(n)\). Note that when \(d = 2^i\), we have \(t_{d-1-k} = -t_k\) for \(k = 0, \ldots, 2^i - 1\), which gives us \(u = u_{2^i}(n)\) for \(n \geq 2^i\). This result is obvious from the product

\[U_{2^i}(X) = \prod_{j=0}^{2^i-1} \frac{1 - X^{2^i}}{1 - X^{2^i + 2}}, \]

which is finite and has largest degree \(2^{2i} - 1\).

### 2.3 The Case \(d = 3\)

When \(d = 3\), we can say more about the sequences \(\lambda_3(k)\) where \(k = 0, 1, 2\). Throughout this section, we write \(\lambda(k) = \lambda_3(k)\) and \(\lambda(k; n) = \lambda_3(k; n)\).
Theorem 2.3.1. For all $n \geq 1$,
\[
\begin{align*}
\lambda(0, 3n + 3) &\geq \lambda(0, 3n) \geq -\lambda(0, 3n + 1) \geq -\lambda(0, 3n + 2) \geq 0, \\
\lambda(1, 3n + 3) &\leq \lambda(1, 3n) \leq -\lambda(1, 3n + 1) \leq -\lambda(1, 3n + 2) \leq 0, \\
\lambda(2, 3n + 3) &\leq \lambda(2, 3n) \leq -\lambda(2, 3n + 1) \leq -\lambda(2, 3n + 2) \leq 0.
\end{align*}
\]

Proof. We only give an induction proof for $\lambda(0; n)$. Then, $\lambda(1; n)$ and $\lambda(2; n)$ will follow the same argument. It is easy to verify the equation for $n = 0, 1$. Suppose that it holds for $n = 0, 1, \ldots, 2k - 1$. We need to show that the inequalities above hold for $n = 2k, 2k + 1$. Consider
\[
\begin{align*}
\lambda(0; 6k + 1) &= \lambda(0; 6k - 2) - \lambda(0; 3k), \\
\lambda(0; 6k + 2) &= \lambda(0; 6k - 1) + \lambda(0; 3k + 1), \\
\lambda(0; 6k + 3) &= \lambda(0; 6k) - \lambda(0; 3k + 1), \\
\lambda(0; 6k + 4) &= \lambda(0; 6k + 1) + \lambda(0; 3k + 2), \\
\lambda(0; 6k + 5) &= \lambda(0; 6k + 2) - \lambda(0; 3k + 2), \\
\lambda(0; 6k + 6) &= \lambda(0; 6k + 3) + \lambda(0; 3k + 3).
\end{align*}
\]

By the induction hypothesis, $\lambda(0; 6k - 1) \leq 0$ and $\lambda(0; 3k + 1) \leq 0$. Using the facts that
\[
\begin{align*}
\lambda(0; 6k) + \lambda(0; 6k + 1) + \lambda(0; 6k + 2) &= \lambda(0; 3k) \\
\lambda(0; 6k - 1) + \lambda(0; 6k) + \lambda(0; 6k + 1) &= 0,
\end{align*}
\]
we now have
\[
\begin{align*}
\lambda(0; 6k + 3) &= \lambda(0; 6k) - \lambda(0; 3k + 1) \geq \lambda(0; 6k), \\
\lambda(0; 6k) &= -\lambda(0; 6k + 1) - \lambda(0; 6k - 1) \geq -\lambda(0; 6k + 1), \\
-\lambda(0; 6k + 1) &= -\lambda(0; 6k - 2) + \lambda(0; 3k) \geq -\lambda(0; 6k - 1) - \lambda(0; 3k + 1) = -\lambda(0; 6k + 2), \\
-\lambda(0; 6k + 2) &= -\lambda(0; 6k - 1) - \lambda(0; 3k + 1) \geq 0 + 0 = 0.
\end{align*}
\]
Therefore, $\lambda(0, 6k + 3) \geq \lambda(0, 6k) \geq -\lambda(0, 6k + 1) \geq -\lambda(0, 6k + 2) \geq 0$. Similar computations verify the other inequalities. \qed
2.4 Doubly Infinite Sequence $\tilde{u}_d(a)$

In this section we extend the sequence $u_d(a)$ to negative infinity.

**Definition 2.4.1.** Let $a = (a_0, a_1, \ldots, a_{d-1}) \in \mathbb{Z}^d$. Define $\tilde{u}_d(a) = \{\tilde{u}_d(a;n)\}_{n \in \mathbb{Z}}$ to be a doubly infinite sequence such that

\[
\tilde{u}_d(a;2n) = \tilde{u}_d(a;2n-d) + \tilde{u}_d(a;n),
\]

\[
\tilde{u}_d(a;2n+1) = \tilde{u}_d(a;2n+1-d) - \tilde{u}_d(a;n),
\]

for all $n \in \mathbb{Z}$, and $\tilde{u}_d(a;k) = (-1)^{t_k}$ for $0 \leq n \leq d-1$.

It is not obvious that this sequence is well-defined when the argument is negative. For instance, let $d = 2k + 1$ be an odd number. Then, the equation $\tilde{u}_{2k+1}(a;2k) = \tilde{u}_{2k+1}(a;-1) + \tilde{u}_{2k+1}(a;k)$ is used to determine $\tilde{u}_{2k+1}(a;-1)$. We see that $\tilde{u}_{2k+1}(a;-1)$ is on the right-hand side instead, but since $\tilde{u}_{2k+1}(a;2k)$ and $\tilde{u}_{2k+1}(a;k)$ are given as initial conditions, the value of $\tilde{u}_{2k+1}(a;-1)$ is uniquely determined.

**Proposition 2.4.2.** Let $a \in \mathbb{R}^d$ and $\tilde{u}_d(a)$ be defined as above. If $d = 2j + 1$, then for $n \geq 1$,

\[
\tilde{u}_{2j+1}(a;-2n) = \tilde{u}_{2j+1}(a;-2n+2j+1) + \tilde{u}_{2j+1}(a;j+1-n),
\]

\[
\tilde{u}_{2j+1}(a;-2n+1) = \tilde{u}_{2j+1}(a;-2n+2j+2) - \tilde{u}_{2j+1}(a;j+1-n).
\]

On the other hand, if $d = 2j$, then for $n \geq 1$,

\[
\tilde{u}_{2j}(a;-2n) = \tilde{u}_{2j}(a;-2n+2j) - \tilde{u}_{2j}(a;j-n),
\]

\[
\tilde{u}_{2j}(a;-2n+1) = \tilde{u}_{2j}(a;-2n+1+2j) + \tilde{u}_{2j}(a;j-n).
\]

**Proof.** When $d = 2j + 1$, we rewrite equation (2.15) as $\tilde{u}_{2j+1}(a;2n-2j-1) = \tilde{u}_{2j+1}(a;2n) - \tilde{u}_{2j+1}(a;n)$ and replace $n$ by $j+1-n$ to get equation (2.17). Similarly, equation (2.14) gives $\tilde{u}_{2j+1}(a;2n-2j) = \tilde{u}_{2j+1}(a;2n+1) + \tilde{u}_{2j+1}(a;n)$ and replacing $n$ by $j+1-n$ yields equation (2.16). When $d = 2j$, we use $j-n$ instead of $j+1-n$. 

These new equations tell us that the sequence $\tilde{u}_d(a)$ is well defined when $n \leq 0$. Because we have $-2n + 1 < -2n < d/2 - n < (d+1)/2 - n \leq -2n + d$, the arguments on the right-hand side are greater than those on the left-hand side. Therefore, $\tilde{u}_d(a;-2n)$ and $\tilde{u}_d(a;-2n+1)$ can be computed from previously determined values.
Next, we will show that the negative part is in fact part of a sequence \( u_d(a') \) for some \( a' \in \mathbb{R}^d \). For any \( a = (a_0, a_1, \ldots, a_d) \in \mathbb{R}^d \), let \( \hat{a} = (a_d, a_{d-1}, \ldots, a_0) \).

**Theorem 2.4.3.** Let \( a \in \mathbb{R}^d \). Then, \( \tilde{u}_d(a; -n + d - 1) = u_d(\hat{a}; n) \) for \( n \geq 0 \).

**Proof.** Suppose that \( d \) is odd. When \( 0 \leq n \leq d - 1 \), we have \( \tilde{u}_d(a; -n + d - 1) = a_{-n+d-1} = u_d(\hat{a}; n) \) by the definition of \( \hat{a} \). When \( n \geq d \), we use induction and the recurrences in (2.1), (2.2) and (2.16) through (2.19) to show that \( \tilde{u}_d(a; -n + d - 1) = u_d(\hat{a}; n) \).

The base cases are already shown. Suppose that \( \tilde{u}_d(a; -n + d - 1) = u_d(\hat{a}; n) \) for \( n \leq k - 1 \) with \( k - 1 \geq d - 1 \). If \( k \) is odd, then so is \( -k + d - 1 \). So, when we replace \( n \) by \( \frac{k-d-1}{2} + 1 \) in (2.19), we have

\[
\tilde{u}_d(a; -k + d - 1) = \tilde{u}_d(a; -2 \left( \frac{k-d-1}{2} \right) + 1) \\
= \tilde{u}_d(a; -2 \left( \frac{k-d-1}{2} \right) + 1 + d) - \tilde{u}_d(a; \frac{d+1}{2} - \left( \frac{k-d-1}{2} \right)) \\
= \tilde{u}_d(a; -k + 2d - 1) - \tilde{u}_d(a; -\frac{k-1}{2} + d) \\
= \tilde{u}_d(a; -(k-d) + (d-1)) - \tilde{u}_d(a; -\frac{k-1}{2} + (d-1)) \\
= u_d(\hat{a}; k + d) - u_d(\hat{a}; \frac{k-1}{2}) \\
= u_d(\hat{a}; k).
\]

If \( k \) is even, we replace \( n \) by \( \frac{k-d-1}{2} \) in (2.18) to get

\[
\tilde{u}_d(a; -k + d - 1) = \tilde{u}_d(a; -2 \left( \frac{k-d-1}{2} \right)) \\
= \tilde{u}_d(a; -2 \left( \frac{k-d-1}{2} \right) + d) + \tilde{u}_d(a; \frac{d+1}{2} - \left( \frac{k-d-1}{2} \right)) \\
= \tilde{u}_d(a; -k + 2d - 1) + \tilde{u}_d(a; -\frac{k}{2} + d - 1) \\
= \tilde{u}_d(a; -(k-d) + (d-1)) - \tilde{u}_d(a; -\frac{k}{2} + (d-1)) \\
= u_d(\hat{a}; k + d) - u_d(\hat{a}; \frac{k}{2}) \\
= u_d(\hat{a}; k).
\]

If \( d \) is even, we redo the computations with the equations (2.18) and (2.19) instead of (2.16) and (2.17). □

By simply shifting the indices from \( n \) to \( n - \frac{d-1}{2} \), we obtain the following corollary.

**Corollary 2.4.4.** Let \( d \) be odd, \( a \in \mathbb{R}^d \), and \( v_d(a; n) = \tilde{u}(a; n + \frac{d-1}{2}) \). Then, \( v_d(a; n) = v_d(\hat{a}; -n) \) for all \( n \).

Theorem 2.4.3 suggests that extending the sequence to negative infinity does not offer anything new, as the extension along with initial conditions is the aforementioned sequence \( u_d(\hat{a}) \). Nevertheless, we can apply the theorem to show that the negative part of the sequence \( \tilde{u}_d(\alpha, d) \) extended from \( u_d \) is identically zero.
Corollary 2.4.5. \( \tilde{u}_d(\alpha_d; -n) = 0 \) for \( n \geq 1 \).

Proof. By definition, we have \( \hat{\alpha}_d = \theta_d \). Theorem 2.2.3 says that \( u_d(\theta_d; n) = 0 \) for all \( n \geq d \). Combining the results with Theorem 2.4.3, we have \( \tilde{u}_d(\alpha_d; -n) = u_d(\hat{\alpha}_d; n + d - 1) = u_d(\theta_d; n + d - 1) = 0 \) for \( n \geq 1 \). \( \square \)
Chapter 3

The Generating Function of \( u_d \)

3.1 The Generating Function \( U_d(a; X) \)

It is natural to study the generating function of a given sequence. Recall from Section 1.3 that

\[
U_d(X) = \sum_{n=0}^{\infty} u_d(n)X^n. \tag{3.1}
\]

We have shown that \( U_d(X) = F_d(X)^{-1} = \prod_{j=0}^{\infty} \frac{1 - X^{2j}}{1 - X^{d2j}} \). Several properties of the products are proved in Section 1.3. We have shown in Theorem 1.3.3 (iv) that

\[
U_d(X) = \frac{1 - X}{1 - X^d} U_d(X^2). \tag{3.2}
\]

**Definition 3.1.1.** For \( a = (a_0, a_1, \ldots, a_{d-1}) \in \mathbb{R}^d \), let \( u_d(a) \) be the sequence defined in Definition 2.1.1 by the recurrences

\[
u_d(a; 2n) = u_d(a; 2n - d) + u_d(a; n),
\]

\[
u_d(a; 2n + 1) = u_d(a; 2n - d + 1) - u_d(a; n),
\]

when \( n \geq d/2 \), with \( u_d(a, k) = a_k \) for \( k \leq d - 1 \). Define

\[
U_d(a; X) = \sum_{n=0}^{\infty} u_d(a; n)X^n. \tag{3.5}
\]

The goal is to obtain a functional equation and to compare \( U_d(a; X) \) with \( U_d(X) \).

**Theorem 3.1.2.** Let \( d = 2k + 1 \) be a positive odd integer. Define

\[
A_d(a; X) = \sum_{n=1}^{d-1} \left( a_n + (-1)^{n+1} a_{[n/2]} \right) X^n.
\]
Then, $U_d(a; X)$ satisfies the functional equation,

$$U_d(a; X) = \frac{1 - X}{1 - X^d} \cdot U_d(a; X^2) + \frac{A_d(a; X)}{1 - X^d}. \quad (3.6)$$

Moreover,

$$U_d(a; X) = U_d(X) + \sum_{n=0}^{\infty} \left( A_d(a; X^{2n}) \cdot \prod_{j=0}^{n-1} 1 - X^{2^{j+1}} \right). \quad (3.7)$$

**Proof.** First of all, we multiply the generating function by $(1 - X^d)$

$$(1 - X^d)U_d(a; X) = (1 - X^d) \sum_{n=0}^{\infty} u_d(a; n)X^n$$

$$= \sum_{n=0}^{d-1} a_nX^n + \sum_{n=0}^{\infty} (u_d(a; n + d) - u_d(a; n))X^{n+d}. \quad (3.8)$$

Recall that $d = 2k+1$. Then, we split the latter summation into odd and even parts and apply the recurrences (3.3) and (3.4)

$$\sum_{n=0}^{\infty} (u_d(a; n + 2k + 1) - u_d(a; n))X^{n+2k+1}$$

$$= \sum_{n=0}^{\infty} ((u_d(a; 2n + 2k + 1) - u_d(a; 2n))X^{2n+2k+1} + (u_d(a; 2n + 2k + 2) - u_d(a; 2n + 1))X^{2n+2k+2})$$

$$= \sum_{n=0}^{\infty} -u_d(a; n + k)X^{2n+2k+1} + u_d(a; n + k + 1)X^{2n+2k+2}$$

$$= \left( \sum_{n=0}^{k-1} u_d(a; n)X^{2n+1} - \sum_{n=0}^{\infty} u_d(a; n)X^{2n+1} \right) + \left( \sum_{n=0}^{k-1} -u_d(a; n)X^{2n} + \sum_{n=0}^{\infty} u_d(a; n)X^{2n} \right)$$

$$= \left( \sum_{n=0}^{\infty} u_d(a; n)X^{2n} - \sum_{n=0}^{\infty} u_d(a; n)X^{2n+1} \right) + \left( \sum_{n=0}^{k-1} u_d(a; n)X^{2n+1} - \sum_{n=0}^{k} u_d(a; n)X^{2n} \right)$$

$$= (1 - X)U_d(a; X^2) + \sum_{n=0}^{2k} (-1)^{n+1}a_{[n/2]}X^n.$$

Substituting this in (3.8), we obtain

$$(1 - X^d)U_d(a; X) = \sum_{n=0}^{d-1} a_nX^n + \left( (1 - X)U_d(a; X^2) + \sum_{n=0}^{2k} (-1)^{n+1}a_{[n/2]}X^n \right)$$

$$= (1 - X)U_d(a; X^2) + \sum_{n=0}^{d-1} (a_n + (-1)^{n+1}a_{[n/2]})X^n,$$

$$= (1 - X)U_d(a; X^2) + A_d(a; X).$$
Because $a_0 + (-1)^1 a_0 = 0$, then we have

$$U_d(a; X) = \frac{1 - X}{1 - X^d} \cdot U_d(a; X^2) + \frac{A_d(a; X)}{1 - X^d}. \quad (3.9)$$

We now iterate on $U_d(a; X^2)$. For instance, in the first iteration of replacing $U_d(a; X^2)$ with (3.9), we have

$$U_d(a; X) = \frac{1 - X}{1 - X^d} \left( \frac{1 - X^2}{1 - X^{2d}} \cdot U_d(a; X^4) + \frac{A_d(a; X^2)}{1 - X^{2d}} \right) + \frac{A_d(a; X)}{1 - X^d}$$

$$= \frac{1 - X}{1 - X^d} \cdot U_d(a; X^4) + \frac{1 - X}{1 - X^d} \cdot A_d(a; X^2).$$

Thus, by repeatedly applying (3.9) with $U_d(a; X^2)$ and by induction, we get the desired result. Since the lowest power of $X$ in $A_d(a; X^2)$ is $2^i$, the sum converges to a power series.

**Corollary 3.1.3.** Let $a = (a_0, a_1, \ldots, a_{d-1}) \in \mathbb{R}^d$. Define $\sigma(a) = -a_{\lfloor (d-1)/2 \rfloor} + \sum_{n=0}^{d-1} a_n$. Then,

$$u_d(a; 2n) + u_d(a; 2n - 1) + \ldots + u_d(a; 2n - d) = u_d(a; n) + \sigma(a), \quad (3.10)$$

$$u_d(a; 2n + 1) + u_d(a; 2n - 1) + \ldots + u_d(a; 2n - d + 1) = \sigma(a), \quad (3.11)$$

for $n \geq d/2$.

**Proof.** Using (3.9), we get

$$\frac{1 - X^d}{1 - X} U_d(a; X) = U_d(a; X^2) + \frac{1}{1 - X} A_d(a; X)$$

$$(1 + X + \ldots + X^{d-1}) U_d(a; X) = U_d(a; X^2) + \sum_{n=0}^{d-1} \left( a_n + (-1)^{n+1} a_{\lfloor (n+1)/2 \rfloor} \right) \frac{X^n}{1 - X}$$

$$= U_d(a; X^2) + \sum_{n=0}^{d-1} \left( a_n + (-1)^{n+1} a_{\lfloor (n+1)/2 \rfloor} \right) \sum_{i=0}^{\infty} X^{i+n}$$

$$= U_d(a; X^2) + \sum_{i=0}^{\infty} \left( \sum_{n=0}^{\min(i, d-1)} \left( a_n + (-1)^{n+1} a_{\lfloor (n+1)/2 \rfloor} \right) \right) X^i.$$

When $i \geq d - 1$, $\min(i, d-1) = d - 1$, so

$$\sum_{n=0}^{d-1} \left( a_n + (-1)^{n+1} a_{\lfloor (n+1)/2 \rfloor} \right) = -a_{\lfloor (d-1)/2 \rfloor} + \sum_{n=0}^{d-1} a_n = \sigma(a).$$

Finally, we can compare coefficients of $X^n$ on both sides to get the equations stated in the corollary.

We will explore several interesting generating functions $U_d(a; X)$ for some choices of $a$.  

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First of all, note that when \( a = \alpha_d = ((-1)^{t_0}, (-1)^{t_1}, \ldots, (-1)^{t_{d-1}}) \), we have

\[
A_d(\alpha_d; X) = \sum_{n=1}^{d-1} ((-1)^{t_n} + (-1)^{n+1+\lfloor n/2 \rfloor}) X^n = 0.
\]

This agrees with the formula in Theorem 3.1.2 as \( U_d(\alpha_d, X) = U_d(X) \). On the other hand, when \( d = 2k + 1 \) and \( a = e_d(k) = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^d \) where \( a_k = 1 \) and \( a_i = 0 \) for \( i \neq k \), we have

\[
A_d(e_d(k); X) = (a_k + (-1)^{k+1}a_{(k/2)})X^k + (a_{2k} + (-1)^{2k+1}a_k)X^{2k} = X^k - X^{2k}.
\]

Recall that \( \theta_d = ((-1)^{t_{d-1}}, (-1)^{t_{d-2}}, \ldots, (-1)^{t_0}) \). When \( d = 3 \), we have

\[
A_3(\theta_3; X) = -2X + 2X^2, \quad A_3(-e_3(1); X) = -X + X^2, \quad U_3(\theta_3; X) = -1 - X + X^2.
\]

Hence, by equation (3.7), we have

\[
U_3(\theta_3; X) = U_3(X) + \sum_{n=0}^{\infty} \left( \frac{-2X^{2^n} + 2X^{2^{n+1}}}{1 - X^{3 \cdot 2^n}} \cdot \prod_{j=0}^{n-1} \frac{1 - X^{2^j}}{1 - X^{3 \cdot 2^j}} \right),
\]

\[
U_3(-e_3(1); X) = U_3(X) + \sum_{n=0}^{\infty} \left( \frac{-X^{2^n} + X^{2^{n+1}}}{1 - X^{3 \cdot 2^n}} \cdot \prod_{j=0}^{n-1} \frac{1 - X^{2^j}}{1 - X^{3 \cdot 2^j}} \right).
\]

These two equations imply

\[
2U_3(-e_3(1); X) - U_3(\theta_3; X) = U_3(X),
\]

\[
U_3(-e_3(1); X) = \frac{1}{2}U_3(\theta_3; X) + \frac{1}{2}U_3(X) = \frac{1}{2}(-1 - X + X^2) + \frac{1}{2}U_3(X).
\]

This is an alternate proof to Corollary 2.2.7 that \(-\lambda_3(1; n) = \frac{1}{2}u_3(n)\). This also works for the case \( d = 5 \) since we have

\[
A_5(\theta_5; X) = -2X^2 + 2X^4, \quad A_5(-e_5(2); X) = -2X^2 + 2X^4.
\]
Then, the same cancellation can be applied. In general, when $d = 2k+1$, we have $A_d(\theta_d; X) = -2X^k + 2X^{2k}$, and we can always find $a \in \mathbb{R}^d$ such that $A_d(a; X) = -X^k + X^{2k}$ by solving a system of linear equations. Nevertheless, when $d > 5$, we observe that such $a$ does not yield a simple formula.
Chapter 4

The First Cases of \( u_d \)

We first discuss the asymptotic growth of the sequence \( \{|u_d(n)|\}_{n \geq 0} \). Then, we investigate in detail the cases when \( d = 3 \) and \( d = 5 \) and explain their behaviors.

4.1 The Asymptotic growth of \( u_d \)

Recall that \( U_d(X) = \sum_{k=0}^{\infty} u_d(n)X^n = \prod_{j=0}^{\infty} \frac{1}{1 - X^{d2^j}} \). We use a known result to find the asymptotic growth of the sequence \( \{|u_d(n)|\}_{n \geq 0} \).

Definition 4.1.1. Let \( m \) be a positive integer and \( \Phi_m(z) \) be the \( m \)-th cyclotomic polynomial. It is given by

\[
\Phi_m(z) = \prod_{1 \leq k \leq m \atop \gcd(k,m)=1} \left( x - e^{\frac{2\pi i k}{m}} \right).
\]

Dumas and Flajolet [2] applied the Mellin Transformation and the circle method to prove the following theorem.

Theorem 4.1.2. Let \( a \) and \( B \) be positive integers that are relatively prime and let \( a \) be square-free. Let \( \Phi_a(z) \) be the \( a \)-th cyclotomic polynomial. Then, the coefficient of \( z^n \) in the function

\[
f_{a,B}(z) = \prod_{k=0}^{\infty} \Phi_a(z^{B^k})^{-1}
\]

(4.1)

has asymptotic growth as \( z \to \infty \)

\[
[z^n]f_{a,B}(z) \approx \exp \left( \frac{\log^2 \rho}{2 \log B} + (1 + \kappa) \log \rho + n \rho + \frac{1}{2} \log(n\rho) \right) \times \left( \sum_{k=0}^{\infty} (n\rho)^{-\frac{1}{2}} \omega_k \left( \frac{\log \rho}{\phi(a) \log B} \right) \right),
\]

(4.2)

where \( \rho \) is given by

\[
\log \rho = -\log n + \log \log n - \log \log 2 + o(1),
\]

(4.3)

for sufficiently large \( n \), \( \omega_k(v) \) is a period 1 function and \( \kappa = -\frac{1}{2} + \frac{\log a}{\log B} \).
When \( p \) is a prime number, we have \( \Phi_p(z) = 1 + z + z^2 + \ldots + z^{p-1} \). Therefore,

\[
f_{p, 2}(z) = \prod_{k=0}^{\infty} \Phi_p^{-1}(z^{2^k}) = \prod_{k=0}^{\infty} (1 + z^{2^k} + z^{2^{2k}} + \ldots + z^{(p-1)2^k})^{-1} = U_p(z).
\]

**Corollary 4.1.3.** Let \( p \) be an odd prime. Then, for sufficiently large \( n \), \( \log|u_p(n)| = \frac{\log^2 n}{2 \log 2} + \mathcal{O}(\log n) \).

Moreover, we have \( 1 + z + z^2 + \ldots + z^{d-1} = \prod_{k|d} \Phi_k(z) \), and so \( U_d(z) = \prod_{k|d} f_{k, 2}(z) \).

**Theorem 4.1.4.** For any \( d > 0 \) and for sufficiently large \( n \), \( \log|u_d(n)| = c_d \cdot \frac{\log^2 n}{2 \log 2} + \mathcal{O}(\log n) \) for some constant \( c_d \).

Note that this is similar to the asymptotic formula for \( b(\infty; n) \) proven by Mahler [3] that \( \log b(\infty; n) \sim \frac{\log^2 n}{2 \log 2} \). We compare the sequences \( u_d \) with the function \( \log^2 n \). Let \( g_d(n) = \frac{\log((|u_d(n)| + 1)}{\log^2 n} \). Note that we add 1 to avoid cases when \( u_d(n) = 0 \), which may occur even for large \( n \). The plot shows \( g_3(n) \) for \( n \) up to \( 10^5 \), and the table shows the maximum of \( g_d(n) \) for small odd \( d \) and for \( 100 < n < 10^5 \). Note that small values of \( n \), where slight irregularity occurs, are excluded.

![Figure 4.1: A plot of \( g_d(n) \) for \( n \) up to \( 10^4 \) for small \( d \).](image)

<table>
<thead>
<tr>
<th>( d )</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \max g_d(n) )</td>
<td>0.5303</td>
<td>0.4271</td>
<td>0.3640</td>
<td>0.3549</td>
<td>0.2858</td>
<td>0.2676</td>
<td>0.2965</td>
<td>0.2800</td>
<td>0.2156</td>
</tr>
</tbody>
</table>

Table 4.1: The maximum of \( g_d(n) \), rounded to 4 decimal places, for odd \( d < 20 \) and for \( 100 < n < 10^4 \)

**Theorem 4.1.5.** \( \lim_{n \to \infty} g_d(n) \) exists for all \( d > 0 \).
Proof. This follows from Theorem 4.1.4.

**Definition 4.1.6.** Given $d \in \mathbb{N}$ and $i = 0, 1, \ldots, d - 1$, define a sequence $v_{d,i} = \{v_{d,i}(n)\}_{n \geq 0}$ by

$$v_{d,i}(n) = u_d(dn + i). \quad (4.4)$$

For the rest of the chapter, we will call each sequence $v_{d,i}$ a branch. The reason for this name will be evident when we see plots of $u_d(n)$.

**Theorem 4.1.7.** For all $n, d \geq 0$, we have

$$v_{d,i}(2n + 1) = v_{d,i}(2n) + (-1)^i d v_{d,i(d+i)/2}(n), \quad (4.5)$$

$$v_{d,i}(2n + 2) = v_{d,i}(2n + 1) + (-1)^i d v_{d,i(i)/2}(n + 1). \quad (4.6)$$

Proof. We apply the recurrences in Definition 1.4.1 and convert $u_d(n)$ to $v_{d,i}(n)$.

### 4.2 Properties of $u_3$

Even though the asymptotic growth is settled, interesting behavior can still be discovered. The plots below show points $(n, u_3(n))$ for $n$ up to 100 and 100,000 respectively. Even though we use different colors to highlight each branch, they are in fact all generated by the sequence $u_3$. To be precise, the red branch, the green branch, and the blue branch represent the points of the form $(3k, u_3(3k)), (3k + 1, u_3(3k + 1))$, and $(3k + 2, u_3(3k + 2))$ respectively.

![Figure 4.2: Plots of $u_3(n)$ for $n \leq 100$ and $n \leq 10^5$.](image)

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The next plots show points \((n, f(n))\) where 
\[
f(n) = \text{sign}(u_3(n)) \cdot \log(|u_3(n)| + 1)
\]
for \(n\) up to 100,000. We use \(|u_3(n)| + 1\) instead of \(|u_3(n)|\) to avoid singularity of logarithm when \(u_3(n) = 0\), and we also preserve the sign of \(u_3(n)\).

Figure 4.3: Plots of \(\text{sign}(u_3(n)) \cdot \log(|u_3(n)| + 1)\) for \(n \leq 100\) and \(n \leq 10^5\).

**Theorem 4.2.1.** For all \(n \geq 0\) and \(i = 0, 1, 2\), we have

\[
v_{3,0}(n) \geq -v_{3,1}(n) \geq -v_{3,2}(n) \geq 0 
\]

(4.7)

\[
|v_{3,i}(2n + 2)| \geq |v_{3,i}(2n)|
\]

(4.8)

\[
|v_{3,i}(2n + 3)| \geq |v_{3,i}(2n + 1)|.
\]

(4.9)

**Proof.** Use induction on \(n\). The base cases with \(n = 0, 1\) can be directly verified. Suppose all inequalities hold for \(n\) up to \(2k - 1\). When \(n = 2k, 2k + 1\), we have

\[
v_{3,0}(2k) + v_{3,1}(2k) = v_{3,0}(2k - 1) + v_{3,1}(2k - 1) + (v_{3,0}(k) - v_{3,0}(k)) = v_{3,0}(2k - 1) + v_{3,1}(2k - 1) \geq 0
\]

\[
v_{3,2}(2k) - v_{3,1}(2k) = v_{3,2}(2k - 1) - v_{3,1}(2k - 1) + (v_{3,1}(k) + v_{3,0}(k)) \geq v_{3,2}(2k - 1) - v_{3,1}(2k - 1) \geq 0
\]

\[
v_{3,0}(2k + 1) + v_{3,1}(2k + 1) = v_{3,0}(2k) + v_{3,1}(2k) + (-v_{3,1}(k) + v_{3,2}(k)) \geq v_{3,0}(2k) + v_{3,1}(2k) \geq 0
\]

\[
v_{3,2}(2k + 1) + v_{3,1}(2k + 1) = v_{3,2}(2k) - v_{3,1}(2k) + (-2v_{3,2}(k)) \geq v_{3,1}(2k) - v_{3,2}(2k) \geq 0.
\]
Note that all expressions in parentheses are non-negative by induction. Similarly, we have

\[ v_{3,0}(2k) = v_{3,0}(2k - 1) + v_{3,0}(k) = v_{3,0}(2k - 2) + (v_{3,0}(k) - v_{3,1}(k - 1)) \geq v_{3,0}(2k - 2) \geq 0, \]
\[ v_{3,1}(2k) = v_{3,1}(2k - 1) - v_{3,0}(k) = v_{3,1}(2k - 2) - (v_{3,0}(k) - v_{3,2}(k - 1)) \leq v_{3,1}(2k - 2) \leq 0, \]
\[ v_{3,2}(2k) = v_{3,2}(2k - 1) + v_{3,1}(k) = v_{3,2}(2k - 2) - (-v_{3,1}(k) + v_{3,2}(k - 1)) \leq v_{3,2}(2k - 2) \leq 0 \]
\[ v_{3,0}(2k + 1) = v_{3,0}(2k) - v_{3,1}(k) = v_{3,0}(2k - 1) + (-v_{3,1}(k) + v_{3,0}(k)) \geq v_{3,0}(2k - 1) \geq 0, \]
\[ v_{3,1}(2k + 1) = v_{3,1}(2k) + v_{3,2}(k) = v_{3,1}(2k - 1) - (-v_{3,2}(k) + v_{3,0}(k)) \leq v_{3,1}(2k - 1) \leq 0, \]
\[ v_{3,2}(2k + 1) = v_{3,2}(2k) - v_{3,2}(k) = v_{3,2}(2k - 1) - (v_{3,2}(k) - v_{3,1}(k)) \leq v_{3,2}(2k - 1) \leq 0. \]

Again, the expressions in parentheses are non-negative by induction. This proves inequalities in (4.8) and (4.9).

The proof provides several insights about the behavior of \( v_{3,1} \). First, we need to look at branches mod 6 instead of mod 3 because they are based on different recurrences, as stated in Theorem 4.1.7. In particular, \( v_{3,1}(2n + 2) \geq v_{3,1}(2n + 1) \) but \( v_{3,1}(2n + 1) \leq v_{3,1}(2n) \) for all \( n \), which explains zigzags in the plots.

Second, monotonicity in (4.8) and (4.9) requires inequalities in (4.7). For example, we have \( v_{3,1}(2k) = v_{3,1}(2k - 2) - (v_{3,0}(k) - v_{3,2}(k - 1)) \) so the behavior of the branch \( v_{3,1} \) is influenced by the difference of the branches \( v_{3,0} \) and \( v_{3,2} \). Nevertheless, as we will see later in this and the next chapter, this pattern does not hold for all \( d \).

### 4.3 Properties of \( u_5 \)

When \( d = 5 \), the sequence \( u_5 \) gets more interesting. We expect similar behavior as in the case when \( d = 3 \), like inequalities between different branches \( v_{5,i}(n) \), but they are not as simple as before. Below are the plots of \( (n, u_5(n)) \) and \( \log |u_5(n)| \) for \( n \) up to 100 and 100,000 respectively.
Figure 4.4: Plots of $u_5(n)$ for $n \leq 100$ and $n \leq 10^5$.

Figure 4.5: Plots of $\text{sign}(u_5(n)) \cdot \log(|u_5(n)| + 1)$ for $n \leq 100$ and $n \leq 10^5$.

**Theorem 4.3.1.** Let $v_{5,i}(n) = u_5(5n + i)$ for $i = 0, 1, 2, 3, 4$. Then, for $n \geq 5$, we have

\begin{align*}
v_{5,0}(n) &> -v_{5,1}(n) > v_{5,3}(n) > 0, \\
v_{5,0}(n) &> -v_{5,2}(n) > -v_{5,4}(n) > 0, \\
v_{5,4}(n) &> v_{5,1}(n), \\
-v_{5,2}(n) &> v_{5,3}(n) \\
|v_{5,i}(2n + 2)| &\geq |v_{5,i}(2n)| \quad (4.14) \\
|v_{5,i}(2n + 3)| &\geq |v_{5,i}(2n + 1)|. \quad (4.15)
\end{align*}

**Proof.** Use the recurrences in Theorem 4.1.7 and an induction on $n$ similar to the proof of Theorem 4.2.1.
We describe the behavior in plain language. The theorem implies that the branches \( v_{5,0} \) and \( v_{5,3} \) are positive whereas \( v_{5,1}, v_{5,2} \) and \( v_{5,4} \) are negative. When comparing \( |v_{5,i}(n)| \) for each \( i \), we see that for all \( n \), \( |v_{5,0}(n)| > |v_{5,i}(n)| \). In addition, we know that \( |v_{5,1}(n)| \) and \( |v_{5,2}(n)| \) are greater than \( |v_{5,3}(n)| \) and \( |v_{5,4}(n)| \) for all \( n \).

Nevertheless, magnitudes of \( v_{5,1} \) and \( v_{5,2} \) are not comparable as the quantity \( |v_{5,2}(n)| - |v_{5,1}(n)| = v_{5,1}(n) - v_{5,2}(n) \) seems to change sign infinitely often. This is also true for \( v_{5,3} \) and \( v_{5,4} \).

Conjecture 4.3.2. The functions \( \delta_1(n) = v_{5,1}(n) - v_{5,2}(n) \) and \( \delta_2(n) = v_{5,3}(n) + v_{5,4}(n) \) change sign infinitely often. Formally, for all \( n \) there exist \( k_-, k_+ > n \) such that \( \delta_i(k_-) < 0 < \delta_i(k_+) \) for \( i = 1, 2 \).

4.4 Observations about \( u_9 \)

We intentionally skip analysis of the sequence \( u_7 \) because it does not have strict inequalities. This will be discussed in detail in Chapter 5. For now, we look at the sequence \( u_9 \).
By Theorem 1.3.4 in Chapter 1, we have

\[ u_9(n) = \left\lfloor \frac{n}{3} \right\rfloor \sum_{i=0}^{\left\lfloor \frac{n}{3} \right\rfloor} u_3(i)u_3(n - 3i) \]

\[ = u_3(n)u_3(0) + u_3(n - 3)u_3(1) + \ldots + u_3(n - 3\left\lfloor \frac{n}{3} \right\rfloor)u_3(\left\lfloor \frac{n}{3} \right\rfloor) \]

\[ = u_3(n) - u_3(n - 3) - u_3(n - 6) + \ldots + u_3(n - 3\left\lfloor \frac{n}{3} \right\rfloor)u_3(\left\lfloor \frac{n}{3} \right\rfloor). \]

This suggests that \( u_9 \) exhibits a similar behavior to \( u_3 \). In addition, for large \( n \), \( u_9(n) \) seems to be positive if and only if \( n \) is divisible by 3.
Chapter 5

General Cases of $u_d$

In this chapter, we examine the behavior of $u_d$ for odd $d \geq 7$. Contrary to the cases when $d$ is 3, 5 and 9 in the previous chapter, the behavior of $u_d$ seem to be less predictable. We attempt to analyze the sequences for small $d$, and give several conjectures.

5.1 Numerical Data

We begin by providing a definition that will describe precisely the behavior we observe.

**Definition 5.1.1.** Let $u_d$ be the sequence as defined in Definition 1.4.1. The sequence $u_d$ is **stable** on branch $k$ if there exists an integer $K$ such that either $u_d(nd + k) > 0$ for all $n \geq K$, or $u_d(nd + k) < 0$ for all $n \geq K$. The sequence $u_d$ is **completely stable** if it is stable on all the branches $k = 0, 1, \ldots, d - 1$.

We have shown that $u_3$ and $u_5$ are completely stable. Consequently, we have that $u_{3 \cdot 2^i}$ and $u_{5 \cdot 2^i}$ are completely stable for all $i \geq 0$. Nevertheless, we will see shortly that this is always not the case for $d > 5$.

Recall Theorem 4.1.3 that $\log(|u_d(n)|+1)$ is asymptotic to $\log^2 n$. Let $s_d(n) = \text{sign}(u_d(n)) \cdot \log(|u_d(n)|+1)$. Below are the plots of $(n, s(n))$ for odd $d$ where $7 \leq d \leq 17$. When $d \geq 11$, we discard legend labels and use duplicated colors. Plots for larger values of $d$ can be found in the Appendix.

We briefly point out interesting behavior.

- When $d = 7$, some branches switch side, meaning they drastically increase or decrease and change sign. This behavior can also be seen when $d = 11, 13$ and 17.

- Some branches do not totally change sign. For example, when $d = 13$, the orange branch (when $n \equiv 4 \mod 13$), changes sign at around $n = 3 \times 10^4$, and then turns back at around $n = 4 \times 10^4$.

- When $d = 15$, all branches seem smooth except the sudden drops when $n = 45590$ and 46175, both on the branch $n \equiv 5 \mod 15$. The sequence $u_{15}$ seems to be stable.

- When $d = 17$, sign changes occur frequently when $n < 10^4$, but seem to disappear afterwards. We do not have enough numerical data to conjecture whether the sequence $u_{17}$ is completely stable or not.
Figure 5.1: Plots of \((n, s_d(n))\) when \(n \leq 10^5\).

The following sections analyze each plot in more details.
5.2 Observations about $u_7, u_{11},$ and $u_{15}$

For $u_7$, we note all the occurrences when a branch changes sign. The next table lists all $n < 10^5$ such that $u_7(n)$ and $u_7(n + 14)$ are non-zero and have different signs. The values of $u_7(n)$ and $u_7(n + 14)$ with magnitude greater than $10^6$ are rounded to 3 significant digits. We note that $u_7(100) = 0$ and $u_7(n) \neq 0$ for $100 < n < 10^5$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n \mod 7$</th>
<th>$u_7(n)$</th>
<th>$u_7(n + 14)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>120</td>
<td>1</td>
<td>-6</td>
<td>2</td>
</tr>
<tr>
<td>521</td>
<td>3</td>
<td>30</td>
<td>-252</td>
</tr>
<tr>
<td>542</td>
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<td>42</td>
<td>-146</td>
</tr>
<tr>
<td>634</td>
<td>4</td>
<td>-32</td>
<td>222</td>
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<tr>
<td>635</td>
<td>5</td>
<td>160</td>
<td>-166</td>
</tr>
<tr>
<td>655</td>
<td>4</td>
<td>-244</td>
<td>76</td>
</tr>
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<td>5</td>
<td>26</td>
<td>-402</td>
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<tr>
<td>895</td>
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<td>284</td>
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</tr>
<tr>
<td>4851</td>
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<td>$-4.17 \times 10^6$</td>
<td>$5.6 \times 10^5$</td>
</tr>
<tr>
<td>4900</td>
<td>0</td>
<td>$-5.08 \times 10^6$</td>
<td>$2.16 \times 10^5$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n \mod 7$</th>
<th>$u_7(n)$</th>
<th>$u_7(n + 14)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17243</td>
<td>2</td>
<td>2.08 $\times 10^8$</td>
<td>-4.47 $\times 10^9$</td>
</tr>
<tr>
<td>17432</td>
<td>2</td>
<td>2.68 $\times 10^9$</td>
<td>-2.51 $\times 10^9$</td>
</tr>
<tr>
<td>22891</td>
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<td>-1.14 $\times 10^{10}$</td>
</tr>
<tr>
<td>22982</td>
<td>1</td>
<td>1.45 $\times 10^{10}$</td>
<td>-8.58 $\times 10^{10}$</td>
</tr>
<tr>
<td>81651</td>
<td>3</td>
<td>-3.32 $\times 10^{15}$</td>
<td>6.43 $\times 10^{13}$</td>
</tr>
<tr>
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<td>-6.42 $\times 10^{14}$</td>
<td>2.76 $\times 10^{15}$</td>
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<tr>
<td>94938</td>
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<td>1.70 $\times 10^{15}$</td>
<td>-8.07 $\times 10^{15}$</td>
</tr>
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<td>95001</td>
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<td>1.28 $\times 10^{13}$</td>
<td>-9.89 $\times 10^{15}$</td>
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<td>96255</td>
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<td>4.19 $\times 10^{15}$</td>
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<tr>
<td>96332</td>
<td>5</td>
<td>-6.61 $\times 10^{15}$</td>
<td>2.43 $\times 10^{15}$</td>
</tr>
</tbody>
</table>

Table 5.1: All occurrences such that $u_7(n) \times u_7(n + 14) < 0$ for $n \leq 10^5$.

The pattern, along with numerical data of $u_7(n)$ for $n$ up to $2 \times 10^6$, leads us to make the following conjecture.

**Conjecture 5.2.1.** There are infinitely many $n$ such that $u_7(n)u_7(n + 14) < 0$. In other words, $u_7$ is not stable on branch $k$ for any $k$. Moreover, let $s_7 = \{s_7(n)\}_{n \geq 1}$ be the increasing sequence such that $s_7(n)$ is the $n$-th occurrence where $u_7(s_7(n))u_7(s_7(n) + 14) < 0$. Then, the subsequence $\{s_7(n) \mod 7\}_{n \geq 8}$ is the sequence 6, 6, 0, 0, 2, 2, 1, 3, 3, 4, 4, 5, repeated.

When $d = 11$, sign changes are less predictable. Nevertheless, they seem to occur only on the branches where $n$ is congruent to 0, 1, 2, 3, 5, 7, and 10 mod 11, with no particular order.

**Conjecture 5.2.2.** The sequence $u_{11}$ is stable on branch $k$ for $k = 4, 6, 8, 9$ and not stable on branch $k$ for $k = 0, 1, 2, 3, 5, 7, 10, 11$.

When $d = 15$, after the sudden drops when $n = 45590$ and 46175, the branches seem to stabilize and stay away from the line $y = 0$. 

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Conjecture 5.2.3. For all \( n \geq 46205 \), we have \( u_{15}(n) > 0 \) if \( n \equiv 0 \mod 3 \) and \( u_{15}(n) < 0 \) if \( n \equiv 1, 2 \mod 3 \).

5.3 A Sufficient Condition for Stable Sequences

We conclude this chapter by stating a conjecture.

Conjecture 5.3.1. The sequence \( u_d \) is completely stable if \( d \equiv 0 \mod 3 \). Moreover, for such \( d \), there exists \( N_d \) such that \( u_d(3n) > 0, u_d(3n + 1) < 0, u_d(3n + 2) < 0 \) for all \( n \geq N_d \).

Recall the proof on inequalities of \( u_3(n) \). We begin with a series of recurrences:

\[
\begin{align*}
u_d(2dn) &= u_d(2dn - d) + u_d(dn), \\
u_d(2dn + 1) &= u_d(2dn - d + 1) - u_d(dn), \\
\vdots \\
u_d(2dn + 2d - 1) &= u_d(2dn + d - 1) - u_d(dn + d - 1).
\end{align*}
\]

Then we use induction to show that the quantities on the right-hand side are either always positive or always negative. For instance, if \( u_d(2dn - d) > 0 \) and \( u_d(dn) > 0 \) by induction hypothesis, then \( u_d(2dn) > 0 \). Based on this observation, we want to know if there is a way to assign signs to each branch \( u_d(nd + k) \) so that as many recurrences as possible have the same sign on both sides. We notice that if we assume that \( u_d(3n) > 0, u_d(3n + 1) < 0, u_d(3n + 2) < 0 \), then most recurrences will have the property we want.

Nevertheless, we are unable to find an appropriate method to prove the claim. One possible approach is by looking at the formula of \( u_d(n) \) in Corollary 1.4.2, which is

\[
u_{3d}(3n) = \sum_{l=0}^{\lfloor n/d \rfloor} b(\infty; l)(-1)^{l(3n-3ld)}.
\]

Newman [4] proved that the function

\[
\lambda(n) = n^{-\alpha} \sum_{j=0}^{n} (-1)^{3n-j},
\]

where \( \alpha = \frac{\log 3}{\log 4} \), lies in the range \((\frac{1}{20}, 5)\) for all \( n > 0 \). Even though he stated that the limit of \( \lambda(n) \) as \( n \) approaches \( \infty \) does not exist, this theorem implies that \( t_{3n} \) equals 0 most of the time. Thus, it seems reasonable that \( u_{3d}(3n) \) is positive.
Chapter 6

The Difference Sequence $x_k$

We study a family of difference sequences $x_k$, which is related to the famous Thue-Morse sequence. We draw a connection between the difference sequences and the $u_d$ sequences we have discussed earlier. We prove several unexpected theorems based on frequency of values and pairs of consecutive terms, and conclude the chapter by defining another sequence based on the difference sequences.

6.1 Introduction of the Thue-Morse Sequence

In 1906, Axel Thue [7] introduced a binary sequence, which is now called the Thue-Morse sequence, or the Prouhet-Thue-Morse sequence, in order to answer the following questions. Is there an infinite binary sequence that contains no cube (no three consecutive identical blocks $www$ where $w$ is a binary block of any positive length)? Also, is there an infinite binary sequence that has no overlap (no block of the form $awawa$ where $a \in \{0, 1\}$ and $w$ is a binary block)? The answer to both questions is positive, and the first few entries of such a sequence $t$ are given below.

$$t = 0110100110010110 \cdots$$

In this chapter, we will first provide a formal definition of the Thue-Morse sequence, along with several interesting properties. Then, we will introduce a family of $k$-difference sequences, related to the Thue-Morse sequence, and a corresponding discrepancy function. These will be used to establish the connection with our original $u_d$ sequence in previous chapters. Finally, we conclude with an investigation of pair correlations resulted from the sequences.

6.2 Definition of the Thue-Morse Sequence

We first repeat a formal definition of the Thue-Morse sequence given in Chapter 1.
Definition 6.2.1. The Thue-Morse sequence, denoted by $t = (t_n)_{n \geq 0}$, is defined recursively as follows.

\begin{align*}
  t_0 &= 0 \\
  t_{2n} &= t_n \\
  t_{2n+1} &= \overline{t_n}
\end{align*}

for all $n \geq 0$, where $\overline{a} = 1 - a$ for $a \in \{0, 1\}$.

By Theorem 1.3.3 and Corollary 1.3.7, we have $u_d(n) = (-1)^{c(n)} = (-1)^{t_n}$ for $n < d$ and $u_\infty(n) = (-1)^{t_n}$ for all $n$. This establishes a relationship between the sequence $u_d$ and the Thue-Morse sequence.

6.3 Definition of the Difference Sequence $x_k$

We are interested in the following family of sequences.

Definition 6.3.1. For $k \geq 0$, the $k$-difference sequence $x_k = \{x_k(n)\}_{n \geq 0}$ is defined by

$$x_k(n) = |t_{n+k} - t_n|.$$ 

(6.4)

Generally speaking, the $k$-difference sequence $x_k$ tells us whether the $n$-th term and the $(n+k)$-term are the same, i.e. both zeroes or both ones. In Table 6.1 we list the first few entries of the sequences $x_k$ for small $k$.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1(n)$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_2(n)$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$x_3(n)$</td>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_4(n)$</td>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_5(n)$</td>
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<td>0</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.1: Values for $x_k(n)$ for small $n$

6.4 A Characterization of $x_k$

Now we interpret the family of sequences in different ways.
Definition 6.4.1. Define the matrices \( M_k = [a_{i,j}^{(k)}], 1 \leq i, j \leq 2^k \) recursively by \( M_0 = [0] \) and for any \( k \geq 1 \) and \( 1 \leq i, j \leq 2^{k-1} \), we have
\[
a_{2i-1,2j-1}^{(k)} = a_{2i,2j}^{(k)} = a_{i,j}^{(k-1)}, \quad a_{2i-1,2j}^{(k)} = a_{2i,2j-1}^{(k)} = 1 - a_{i,j}^{(k-1)}. \tag{6.5}
\]

Here are the first few matrices of the sequence \( M \).

\[
M_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad \ldots
\]

Alternately, we can generate \( M_k \) by replacing 0’s and 1’s in \( M_{k-1} \) with \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) respectively.

Theorem 6.4.2. For any \( k \geq 0 \) and \( 1 \leq i, j \leq 2^{k-1} \), we have
\[
a_{2i-1,2j-1}^{(k)} = a_{2i,2j}^{(k)} = a_{i,j}^{(k-1)}, \quad a_{2i-1,2j}^{(k)} = a_{2i,2j-1}^{(k)} = 1 - a_{i,j}^{(k-1)}. \tag{6.6}
\]

Proof. Use induction on \( k \). \( \Box \)

Corollary 6.4.3. Let \( k \geq 0 \). Then, for any \( n > k \) and \( 1 \leq i, j \leq 2^k \), we have \( a_{i,j}^{(k)} = a_{i,j}^{(n)} \).

Proof. Use Theorem 6.4.2 and induction on \( n \). \( \Box \)

This corollary allows us to define the infinite matrix \( M_\infty = [a_{i,j}^{\infty}], i, j \in \mathbb{Z} \) by letting \( a_{i,j}^{\infty} = a_{i,j}^{k} \) for any \( k \) such that \( 2^k \geq \max\{i, j\} \).

Lemma 6.4.4. For all \( i, j \in \mathbb{Z} \), we have \( a_{i,j}^{\infty} = |t_{i-1} - t_{j-1}| \equiv t_{i-1} + t_{j-1} \mod 2 \)

Theorem 6.4.5. For all \( k, n \geq 0 \), we have \( x_k(n) = a_{n+1,n+k+1}^{\infty} \).

Proof. By Lemma 6.4.4, we have \( a_{n+1,n+k+1}^{\infty} = |t_n - t_{n+k}| = x_k(n) \). \( \Box \)
Theorem 6.4.5 tells us that the sequence $x_k$ appears as a diagonal of $\mathcal{M}$ starting from the entry $a_{k+1,1}^\infty$ and continuing in the down-right direction. We will see later in this chapter that this insight provides a quick proof of some properties of each sequence $x_k$.

Allouche et al. [1] studied a sequence called $c$ and described it as an encoder of the lengths of blocks in the Thue-Morse sequence. Specifically, if we express $t = 01101001 \ldots = 0^{d_1}1^{d_2}0^{d_3}1^{d_4} \ldots$ where $d_i$ is the size of the $i$-th block in the Thue-Morse sequence, then we have $c_i = \sum_{j=1}^{i} d_i$, i.e. the total size of the first $i$ blocks. They show that if $\chi(n)$ is the characteristic function of $c$, i.e. $\chi(n) = 1$ if $n$ is in the sequence $c$ and 0 otherwise, then $\chi(n) = x_1(n-1)$. Additionally, we have this corollary.

**Corollary 6.4.6.** Let $\nu_2(n)$ denote the largest power of 2 dividing $n$. Then,

(i) $x_1(n) = 1$ if and only if $\nu_2(n+1)$ is even.

(ii) $x_3(n) = 1$ if and only if $\nu_2(\alpha(n)+1)$ is odd where $\alpha(n) = \lfloor \frac{n}{2} \rfloor$ if $n$ is even and $\alpha(n) = \lfloor \frac{n}{4} \rfloor$ if $n$ is odd.

**Proof.** (i) is proved in Proposition 1 in Allouche et al. [1] using induction. We prove (ii) by noticing that

$$x_3(2n) = x_1(n) = 1 \iff \chi(n+1) = 0 \iff \nu_2(n+1) \equiv 1 \mod 2,$$

Similarly, we have

$$x_3(2n+1) = x_1(\lfloor \frac{n}{2} \rfloor) = 1 \iff \chi(\lfloor \frac{n}{2} \rfloor + 1) = 0 \iff \nu_2(\lfloor \frac{n}{2} \rfloor + 1) \equiv 1 \mod 2.$$

Nevertheless, for $x_k(n)$ where $k$ is odd and greater than 3, the patterns are more complicated and we cannot yet describe them in simple terms.

### 6.5 The Density of 0’s and 1’s in $x_k$

It is natural to ask how often 0’s and 1’s appear in each sequence $x_k$. For $i = 0, 1$ and $k \geq 0$, define

$$C_i(k; n) = \# \{ 0 \leq m \leq n - 1 \mid x_k(m) = i \},$$

$$A_i(k; n) = \frac{C_i(k; n)}{n}.$$

We begin with two useful and simple lemmas which will be applied repeatedly.

**Lemma 6.5.1.** If $i, j \in \{0, 1\}$, then we have
(i) $|i - j| + |i - \overline{j}| = 1,$

(ii) $|\overline{i} - j| = |i - \overline{j}|.$

**Proof.** (i) If $i = j$, then $|i - j| + |i - \overline{j}| = 0 + 1 = 1.$ Otherwise, $i = \overline{j}$ and so $|i - j| + |i - \overline{j}| = 1 + 0 = 1.$

(ii) By (i), we have $|\overline{i} - j| = 1 - |i - \overline{j}|.$

**Proposition 6.5.2.** For all $k, n \geq 0$, we have

(i) $C_0(k; n) = \sum_{j=0}^{n-1} x_k(j),$

(ii) $C_1(k; n) = \sum_{j=0}^{n-1} x_k(j),$

(iii) $C_0(k; n) + C_1(k; n) = n,$ $A_0(k; n) + A_1(k; n) = 1,$

(iv) $\lim_{n \to \infty} A_0(1; n) = \frac{1}{3},$ $\lim_{n \to \infty} A_1(1; n) = \frac{2}{3}.$

**Proof.** (i), (ii) and (iii) follow from the definition. (iv) can be computed directly. For all $j \geq 0$, we know that $x_1(2j) = |t_{2j+1} - t_{2j}| = |t_{\overline{j}} - t_j| = 1$ and $x_1(2j+1) = |t_{2j+2} - t_{2j+1}| = |t_{j+1} - t_{\overline{j}}| = x_1(j)$. Thus, we have

$$C_1(1; 2n) = \sum_{j=0}^{2n-1} x_1(j) = \sum_{j=0}^{n-1} (1 + x_1(j)) = n + C_0(1; n) = 2n - C_1(1; n),$$

$$C_1(1; 2n + 1) = C_1(1; 2n) + x_1(2n) = 2n + 1 - C_1(1; n).$$

Now, we show that $\{A_1(1; n)\}_{n \geq 0}$ is a Cauchy sequence. We observe that

$$A_1(1, 2n + 1) - A_1(1; 2n) = \left(1 - \frac{C_1(1; n)}{2n + 1}\right) - \left(1 - \frac{C_1(1; n)}{2n}\right) = \frac{C_1(1; n)}{2n(2n + 1)} < \frac{1}{2(2n + 1)}.$$ 

So, for any odd $j$, we have $A_1(1; j) < A_1(1; 2\lfloor \frac{j}{2} \rfloor) + \frac{1}{2j}$, and thus for any $j > l$, we have

$$|A_1(1; j) - A_1(1; l)| \leq |A_1(1; 2\lfloor \frac{j}{2} \rfloor) - A_1(1; 2\lfloor \frac{l}{2} \rfloor)| + \frac{1}{2j} + \frac{1}{2l}$$

$$\leq \left| \left(1 - \frac{1}{2}A_1(1; \lfloor \frac{j}{2} \rfloor)\right) - \left(1 - \frac{1}{2}A_1(1; \lfloor \frac{l}{2} \rfloor)\right) \right| + \frac{1}{l}$$

$$\leq \frac{1}{2} |A_1(1; \lfloor \frac{j}{2} \rfloor) - A_1(1; \lfloor \frac{l}{2} \rfloor)| + \frac{1}{2\lfloor \frac{j}{2} \rfloor}.$$
If we choose \( j > l > 2^k \) for some \( k \geq 0 \) and apply the inequality above twice, we obtain

\[
|A_1(1; j) - A_1(1; l)| \leq \frac{1}{2} |A_1(1; \lfloor \frac{j}{2^k} \rfloor) - A_1(1; \lfloor \frac{l}{2^k} \rfloor)| + \frac{1}{2} |A_1(1; \lfloor \frac{j}{2^k} \rfloor) - A_1(1; \lfloor \frac{l}{2^k} \rfloor)|
\]

By using induction, we may conclude that

\[
|A_1(1; j) - A_1(1; l)| \leq \frac{1}{2} |A_1(1; \lfloor \frac{j}{2^{k+1}} \rfloor) - A_1(1; \lfloor \frac{l}{2^{k+1}} \rfloor)| + \frac{k}{2^{k+1}} |A_1(1; \lfloor \frac{j}{2^{k+1}} \rfloor) - A_1(1; \lfloor \frac{l}{2^{k+1}} \rfloor)|.
\]

For any \( \epsilon > 0 \), pick \( k \) large enough so that \( \frac{k+1}{2^{k+1}} < \epsilon \). Then, for any \( j > l \geq 2^k \), we have

\[
|A_1(1; j) - A_1(1; l)| \leq \frac{1}{2^{k+1}} |A_1(1; \lfloor \frac{j}{2^{k+1}} \rfloor) - A_1(1; \lfloor \frac{l}{2^{k+1}} \rfloor)| + \frac{k}{2^{k+1}} \leq \frac{1}{2^{k+1}} + \frac{k}{2^{k+1}} \leq \epsilon.
\]

Because \( \{A_1(1; n)\}_{n \geq 0} \) is a Cauchy sequence, \( \lim_{n \to \infty} A_1(1; n) = c \) exists. Taking a limit on both sides of the equation \( A_1(1; 2n) = 1 - \frac{1}{2} A_1(1; n) \) gives us \( c = 1 - \frac{c}{2} \) and thus \( c = \frac{2}{3} \). Consequently, by (iii), we have

\[
\lim_{n \to \infty} A_0(1; n) = \lim_{n \to \infty} 1 - A_1(1; n) = \frac{1}{3}.
\]

**Theorem 6.5.3.** For \( i = 0, 1 \) and \( k \geq 1 \), \( A_i(k) = \lim_{n \to \infty} A_i(k; n) \) exists. Furthermore, we have the recurrences

\[
A_i(2k) = A_i(k), \tag{6.7}
\]

\[
A_i(2k + 1) = 1 - \frac{1}{2} (A_i(k) + A_i(k + 1)). \tag{6.8}
\]

**Proof.** We show that \( \lim_{n \to \infty} A_i(k; n) \) exists and that the recurrences hold by induction. The base case is shown in Proposition 6.5.2 (iv). Suppose that the limit exists up to \( 2k - 1 \). Since \( x_{2k}(2n) = x_{2k}(2n + 1) = x_k(n) \) for all \( n \geq 0 \), we have

\[
A_0(2k) = \lim_{n \to \infty} \frac{1}{2n} \sum_{j=0}^{2n-1} x_{2k}(j) = \lim_{n \to \infty} \frac{1}{2n} \sum_{j=0}^{n-1} 2(x_k(j)) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} x_k(j) = A_0(k).
\]

Thus, \( \lim_{n \to \infty} A_i(k; 2n) \) exists, with \( A_i(2k) = 1 - A_0(2k) = 1 - A_0(k) = A_1(k) \).
Likewise, we have

\[ x_{2k+1}(2n) = |t_{2n+2k+1} - t_{2n}| = |t_{n+k} - t_n| = x_k(n), \]
\[ x_{2k+1}(2n + 1) = |t_{2n+2k+2} - t_{2n+1}| = |t_{n+k+1} - t_n| = x_{k+1}(n). \]

These equations imply that

\[ A_0(2k + 1) = \lim_{n \to \infty} \frac{1}{2n} \sum_{j=0}^{2n-1} x_{2k+1}(j) \]
\[ = \lim_{n \to \infty} \frac{1}{2n} \sum_{j=0}^{n-1} (x_k(j) + x_{k+1}(j)) \]
\[ = \frac{1}{2}(A_1(k) + A_1(k + 1)) \]
\[ = 1 - \frac{1}{2}(A_0(k) + A_0(k + 1)). \]

Thus, \( \lim_{n \to \infty} A_i(k; 2n + 1) \) exists, with

\[ A_1(2k + 1) = 1 - \left( 1 - \frac{1}{2}(1 - A_1(k) + 1 - A_1(k + 1)) \right) = 1 - \frac{1}{2}(A_1(k) + A_1(k + 1)). \]

This is reminiscent of the Stern sequence we defined in Chapter 1, which suggests that there is no closed form. Several properties can be transferred from those of the Stern sequence, as listed in the following theorem.

| \( k \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( A_0(k) \) | \( \frac{1}{3} \) | \( \frac{1}{3} \) | \( \frac{2}{3} \) | \( \frac{1}{3} \) | \( \frac{1}{2} \) | \( \frac{1}{3} \) | \( \frac{7}{12} \) | \( \frac{1}{2} \) | \( \frac{5}{12} \) | \( \frac{2}{3} \) | \( \frac{5}{12} \) | \( \frac{1}{2} \) | \( \frac{7}{12} \) |
| \( A_1(k) \) | \( \frac{2}{3} \) | \( \frac{2}{3} \) | \( \frac{1}{3} \) | \( \frac{2}{3} \) | \( \frac{1}{2} \) | \( \frac{1}{3} \) | \( \frac{7}{12} \) | \( \frac{1}{2} \) | \( \frac{7}{12} \) | \( \frac{1}{3} \) | \( \frac{7}{12} \) | \( \frac{1}{2} \) | \( \frac{5}{12} \) |

Table 6.2: Values for \( A_0(k) \) and \( A_1(k) \) for small \( k \)

**Theorem 6.5.4.** Let \( r > 0 \) and \( 2^r \leq k \leq 2^{r+1} \). Then, for \( i = 0, 1 \), we have the following.

(i) \( A_i(k) \in \left[ \frac{1}{3}, \frac{2}{3} \right] \),

(ii) \( A_i(k) = A_i(3 \cdot 2^r - k) \),

(iii) \( A_i(4k + 1) + A_i(4k + 3) = 1 \),

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\[(a) \quad y = A_1(k) \text{ for } k \leq 100 \]

\[(b) \quad y = A_1(k) \text{ for } k \leq 10^4 \]

Figure 6.1: Plots of $A_1(k)$ for $n \leq 100$ and $n \leq 10^4$.

\[(iv) \quad \sum_{j=2^r}^{2^{r+1}-1} A_0(j) = 2^{r-1}.\]

**Proof.**

(i) We use induction on $k$. For the base case, we already have $A_1(k) = \frac{1}{3}$. If we assume that $A_i(l) \in \left[\frac{1}{3}, \frac{2}{3}\right]$ for $l < 2k$, then $A_i(2k) = A_i(k) \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and $A_i(2k+1) = 1 - \frac{1}{2} (A_i(k) + A_i(k+1)) \leq 1 - \frac{1}{2} \left(\frac{1}{3} + \frac{1}{3}\right) = \frac{2}{3}$ and $1 - \frac{1}{2} (A_i(k) + A_i(k+1)) \geq 1 - \frac{1}{2} \left(\frac{2}{3} + \frac{2}{3}\right) = \frac{1}{3}$.

(ii) We use induction on $r$. When $r = 1$, we have $A_1(2) = A_1(3 \cdot 2 - 2) = A_1(4)$ and $A_1(3) = A_1(3 \cdot 2 - 3) = A_1(3)$. Suppose that $A_i(k) = A_i(3 \cdot 2^r - k)$ for $r < s$ and $k \leq 2^s$. If $k$ is even, then we write $k = 2l$ and so $A_i(2l) = A_i(l) = A_i(3 \cdot 2^{s-1} - l) = A_i(3 \cdot 2^s - 2l)$. Similarly, if $k$ is odd, then we write $k = 2l + 1$ and so

\[A_i(2l+1) = 1 - \frac{1}{2} (A_i(l) + A_i(l+1)) = 1 - \frac{1}{2} \left(A_i(3 \cdot 2^{s-1} - l) + A_i(3 \cdot 2^{s-1} - (l+1))\right) = A_i(3 \cdot 2^s - (2l+1)).\]

(iii) By simply applying the recurrences, we get

\[A_i(4k + 1) + A_i(4k + 3) = 2 - \frac{1}{2} (A_i(2k) + 2A_i(2k + 1) + A_i(2k + 2)) = 2 - \frac{1}{2} (A_i(k) + 2(1 - \frac{1}{2} (A_i(k) + A_i(k+1))) + A_i(k+1)) = 1.\]

(iv) When $r = 1$, we have $A_0(2) + A_0(3) + A_0(4) = \frac{1}{3} + \frac{2}{3} = 1 = 2^0$. Suppose that the equation holds for
Then, we have

\[
\sum_{j=2^r}^{2^{r+1}-1} A_0(i) = \sum_{j=2^{r-1}}^{2^r-1} A_0(2j) + A_0(2j + 1)
\]

\[
= \sum_{j=2^{r-1}}^{2^r-1} A_0(j) + 1 - \frac{1}{2}(A_0(j) + A_0(j + 1))
\]

\[
= 2^{s-1} + \frac{1}{2} \sum_{j=2^{r-1}}^{2^r-1} (A_0(j) - A_0(j + 1))
\]

\[
= 2^{s-1} + A_0(2^{s-1}) - A_0(2^s)
\]

\[
= 2^{s-1}.
\]

This suggests that the average of the values of \(A_i(k)\) is \(\frac{1}{2}\).

The next natural step is to normalize \(A_i(k)\). Let \(B_i(k) = 3 - 6A_i(k)\). Then, we have

\[
B_i(2k) = B_i(k)
\]

\[
B_i(2k + 1) = -\frac{1}{2}(B_i(k) + B_i(k + 1)).
\]

It follows immediately that \(B_0(k) = -B_1(k), B_0(1) = B_0(2) = 1\) and \(B_i(k) \in [-1, 1]\). This new function \(B_i\) has several properties.

**Theorem 6.5.5.** Let \(B_i(k)\) be defined as above for \(i = 0, 1\).

(i) For \(k \geq 0\), we have \(B_i(4k + 1) + B_i(4k + 3) = 0\).

(ii) For \(r \geq 2\), we have \(\sum_{j=2^r}^{2^{r+1}-1} B_i(j) = 0\).

(iii) Let \(r \geq 3\) and \(k\) be an odd integer in \([2^r, 2^{r+1}]\). Then, we have \(B_i(k) = \frac{p}{2^r}\) for some odd integer \(p\).

**Proof.** (i) This follows immediately from Theorem 6.5.4 (iii).

(ii) This follows from Theorem 6.5.4 (iv) because we have

\[
\sum_{j=2^r}^{2^{r+1}-1} B_0(j) = \sum_{j=2^r}^{2^{r+1}-1} 3 - 6A_0(j) = 3 \cdot 2^r - 6(2^{r-1}) = 0.
\]

Consequently, \(\sum_{j=2^r}^{2^{r+1}-1} B_1(j) = -B_0(j) = 0\).
(iii) We proceed by using induction on $r$. The base case when $r = 3$ can be shown by direct computation for $B_i(k)$ with $k = 9, 11, 13, 15$. Suppose the statement holds for $3 \leq s < r$. When we have $s = r$ and an odd integer $k = 2m + 1 \in [2^r, 2^{r+1}]$, we get $B_i(2m + 1) = -\frac{1}{2}(B_i(m) + B_i(m + 1))$ where $m \in [2^{r-1}, 2^r]$.

Without loss of generality, suppose that $m$ is odd. Then, we have $B_i(m) = \frac{p}{2^{r-1}}$ for some odd integer $p$. Moreover, since $m + 1$ is even, we can write $m + 1 = b \cdot 2^a$ where $b$ is odd and $a < r$. Thus, we have $B_i(m + 1) = B_i(b) = \frac{q}{2^a}$ where $q$ is an odd integer.

We conclude the section by noting that Theorem 6.5.5 (iii) implies that for odd $k$ in $[2^r, 2^{r+1}]$, the value of $B_i(k)$ never appears for $k < 2^r$. This tells us that, for instance, $B_0(k) = 0$ if and only if $k = 5 \cdot 2^r$ or $7 \cdot 2^r$, and $B_0(k) = -1$ if and only if $k = 3 \cdot 2^r$ for some $r \geq 0$.

### 6.6 Consecutive Terms of $x_k$

We study the correlation between consecutive terms of the sequence $x_k$.

**Definition 6.6.1.** For $k, d \in \mathbb{N}$ and $i, j \in \{0, 1\}$, define

$$A(i,j)(k; d) = \# \{0 \leq n \leq d \mid (x_k(n), x_k(n + 1)) = (i, j)\},$$

$$A(i,j)(k) = \limsup_{d \to \infty} \frac{A(i,j)(k; d)}{d}. \quad (6.12)$$

Note that we define $A(i,j)(k)$ as the outer limit for now, and we will later show that the limit indeed exists.

**Theorem 6.6.2.** Let $r \in \mathbb{N}^+$, and $k < 2^r$ be an odd number. Then, we have

$$A_{2^r+k}(0, 0) + A_{2^r+1+k}(0, 0) = 2A_{2^r+2+k}(0, 0). \quad (6.13)$$

The proof of this theorem requires several lemmas, where we consider the equation for each modulus. In particular, define

$$A_k(r; j) = \lim_{n \to \infty} \frac{1}{l} \# \{0 \leq m < l \mid ((x_k(m2^r + j), x_k(m2^r + j + 1)) = (0, 0))\}. \quad (6.14)$$

Given non-negative integers $r, j$, this function tells us the density of non-negative integers $n \equiv j \mod 2^r$ such
that \( x_k(n) = x_k(n+1) = 0 \). We will show that for each \( 0 \leq j < 2^r - 1 \), we have \( A_{2^r+k}(r,j) + A_{2^r+1+k}(r,j) = 2A_{2^r+2+k}(r,j) \), and thus prove the theorem.

From this point on, let \( k_i = i2^r + k \) where \( i = 1, 2, 4 \). For any \( n \in \mathbb{N} \), write \( n = m2^r + j \) with \( 0 \leq j < 2^r \). Since \( x_k(n) = (t_{n+k} + t_n \mod 2) \), we will work in mod 2 for simplicity. Then,

\[
x_k(n) \equiv t_{k_i+n} + t_n \tag{6.15}
\]

\[
\equiv t_i(2^r + i2^{r-1} + j) + t_m2^r + j \tag{6.16}
\]

\[
\equiv t_i(2^r + (i+m)2^r + j) + t_m2^r + j \tag{6.17}
\]

We want to determine which values \( n \) satisfy \((x_k(n), x_k(n+1)) = (0, 0)\). There are 4 cases to consider.

**Lemma 6.6.3.** Let \( r, k \geq 0 \). Then, for \( 0 \leq j < 2^r - 1 - k \), we have \( A_{2^r+k}(r,j) + A_{2^r+1+k}(r,j) = 2A_{2^r+2+k}(r,j) \).

**Proof.** Since \( k + j + 1 < 2^r \), we have

\[
x_k(n) \equiv t_i(2^r + (i+m)2^r + j) + t_m2^r + j \]

\[
\equiv t_i + m + t_k + j + t_m + t_j = x_i(m) + x_k(j),
\]

\[
x_k(n+1) \equiv x_i(m) + x_k(j+1).
\]

Then, \((x_k(n), x_k(n+1)) = (0, 0)\) if and only if \( x_k(j) = x_k(j+1) = 0 \) and \( x_i(m) = 0 \), or \( x_k(j) = x_k(j+1) = 1 \) and \( x_i(m) = 1 \)

- For each \( j \) such that \( x_k(j) = x_k(j+1) = 0 \), then we need \( x_i(m) = 0 \). Table 7.2 shows that \( A_0(i) = \frac{1}{4} \)

- For each \( j \) such that \( x_k(j) = x_k(j+1) = 1 \), then we need \( x_i(m) = 1 \). Table 7.2 shows that \( A_1(i) = \frac{2}{4} \)

Hence, \( A_{k_1}(r;j) + A_{k_2}(r;j) = 2A_{k_4}(r;j) \) for \( 0 \leq j < 2^r - 1 - k \).

**Lemma 6.6.4.** Let \( r, k \geq 0 \). Then, for \( 2^r - k \leq j < 2^r - 1 \), we have \( A_{2^r+k}(r,j) + A_{2^r+1+k}(r,j) = 2A_{2^r+2+k}(r,j) \).
Proof. Write \( k' = 2^r - k \). Since \( 2^r \leq k + j < 2^{r+1} \), we have

\[
x_{k_i}(n) \equiv t_{(i+m)2^r + (k+j)} + t_{m2^r + j}
\]
\[
\equiv t_{(i+m+1)2^r + (k+j-2^r)} + t_m + t_j
\]
\[
\equiv t_{i+m+1} + t_j - k' + t_m + t_j
\]
\[
\equiv t_{i+m+1} + t_m + t_j - k' + t_j
\]
\[
\equiv x_{i+1}(m) + x_{k'}(j - k'),
\]
\[
x_{k_i}(n+1) \equiv x_{i+1}(m) + x_{k'}(j - k' + 1).
\]

This is similar to Case 1, where we have \((x_{k_i}(n), x_{k_i}(n+1)) = (0, 0)\) if and only if \( x_{k'}(j - k') = x_{k'}(j - k' + 1) = 0 \) and \( x_{i+1}(m) = 0 \), or \( x_{k'}(j - k') = x_{k'}(j - k' + 1) = 1 \) and \( x_{i+1}(m) = 1 \).

- For each \( j \) such that \( x_{k'}(j - k') = x_{k'}(j - k' + 1) = 0 \), then we need \( x_{i+1}(m) = 0 \). Table 7.2 shows that \( A_0(2) = \frac{1}{3}, A_0(3) = \frac{2}{3}, A_0(5) = \frac{1}{2} \), so we have \( A_{k_1}(r; j) + A_{k_2}(r; j) = 1 = 2A_{k_1}(r; j) \).

- Likewise, for each \( j \) such that \( x_{k'}(j - k') = x_{k'}(j - k' + 1) = 1 \), then we need \( x_{i+1}(m) = 1 \). Table 7.2 shows that \( A_1(2) = \frac{2}{3}, A_1(3) = \frac{1}{3}, A_1(5) = \frac{1}{2} \), so we have \( A_{k_1}(r; j) + A_{k_2}(r; j) = 1 = 2A_{k_1}(r; j) \).

- Finally, \( A_{k_1}(r; j) + A_{k_2}(r; j) = 2A_{k_1}(r; j) = 0 \) for other values of \( j \).

Hence, \( A_{k_1}(r; j) + A_{k_2}(r; j) = 2A_{k_1}(r; j) \) for \( 2^r - k \leq j < 2^r - 1 \). \( \square \)

We need the following lemma to simplify calculations.

**Lemma 6.6.5.** For all \( r \geq 0 \) and \( k < 2^r - 1 \), we have \( t_{2^r - 1 - k} \equiv r + t_k \mod 2 \).

**Proof.** Since \( 2^r - 1 = (111\ldots1)2 \), \( 2^r - k \) in base 2 is \( 2^r - 1 \) has 0 exactly at the position where \( k \) in base 2 has 1. Thus, \( t_{2^r - 1 - k} \equiv r - t_k \equiv r + t_k \mod 2 \). \( \square \)

**Lemma 6.6.6.** Let \( r, k \geq 0 \). Then, when \( j = 2^r - 1 - k \), we have \( A_{2^r+k}(r; j) + A_{2^r+1+k}(r; j) = 2A_{2^r+2+k}(r; j) \).

\[
A_{2^r+k}(r; j) + A_{2^r+1+k}(r; j) = 2A_{2^r+2+k}(r; j).
\]
Proof. Since $k + j + 1 = 2^r$, we have a different formula for $x_k(n + 1)$. Using the lemma, we have

$$x_k(n) \equiv t_{i+m} + t_m + t_{k+j} + t_j$$
$$\equiv x_i(m) + t_{2r-1} + t_{2r-1-k}$$
$$\equiv x_i(m) + r + r + t_k$$
$$\equiv x_i(m) + t_k,$$

$$x_k(n + 1) \equiv t_{(i+m)2r+(k+j+1)} + t_m2^r+j$$
$$\equiv t_{(i+m+1)2r} + t_m + t_{j+1}$$
$$\equiv t_{i+m+1} + t_m + t_{2r-k}$$
$$\equiv x_{i+1}(m) + t_{2r-1-(k-1)}$$
$$\equiv x_{i+1}(m) + t_{k-1}.$$

Suppose that $t_k = 0$. Then, since $k$ is odd, we have $t_{k-1} = 1 - t_k = 1$. Thus, we want $x_i(m) = 0$ and $x_{i+1}(m) = 1$ for $i = 1, 2, 4$.

- $(x_1(m), x_2(m)) = (0, 1)$ when $t_{m+1} + t_m \equiv 0, t_{m+2} + t_m = 1$. This implies that $(t_m, t_{m+1}, t_{m+2}) = (0, 0, 1)$ or $(1, 1, 0)$, with density $\frac{1}{3}$.

- $(x_2(m), x_3(m)) = (0, 1)$ when $(t_m, t_{m+2}, t_{m+3}) = (0, 0, 1), (1, 1, 0)$. Since the Thue-Morse sequence is cube-free, we want $(t_m, t_{m+2}, t_{m+3}) = (0, 1, 0, 0, 1, 1, 0).$ Using the generator definition of Thue-Morse sequence, we see that the word 0101 can be generated by either 11 → 0101 or 111 → 101010. Since the Thue-Morse sequence is cube-free, we know it has to be 11 for 0101 and 00 for 1010. Since 00 and 11 appear one-third of the time each $(x_2(m), x_3(m))$ is equal to (0, 1) with density $\frac{1}{3}$. Applying properties of the Thue-Morse sequence, we find that the only possible words $t_m t_{m+1} t_{m+2} t_{m+3}$ are 001101, 011001, 100110 and 110010.

001101 can only be generated from 10 → 0101 → 1001011 → 1001011001101001 and 110010 from 01. Since the sequence becomes 8 times the length, and 01 and 10 appear in the Thue-Morse sequence with density 1/6, the have $(x_2(m), x_3(m)) = (0, 1)$ for $\frac{1}{5} (\frac{1}{3} + \frac{1}{3}) = \frac{1}{12}$.

On the other hand, 011001 and 100110 are generated by 010 and 101 respectively. This yields $\frac{1}{2} (\frac{1}{6} + \frac{1}{6}) = \frac{1}{6}$. Thus, the total is $\frac{1}{12} + \frac{1}{6} = \frac{1}{4}$.

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Following this line of reasoning, we can replace the pairs by the ratios.

<table>
<thead>
<tr>
<th>$t_k$</th>
<th>0</th>
<th>1</th>
</tr>
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<tbody>
<tr>
<td>$A_{k_1}(r; j)$</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>$A_{k_2}(r; j)$</td>
<td>1/6</td>
<td>1/2</td>
</tr>
<tr>
<td>$A_{k_4}(r; j)$</td>
<td>1/4</td>
<td>5/12</td>
</tr>
<tr>
<td>$A_{k_1}(r; j) + A_{k_2}(r; j) - 2A_{k_4}(r; j)$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6.3: Values for $A_{k_i}(r, j)$ for $j = 2^r - 1 - k$.

Hence, $A_{k_1}(r; j) + A_{k_2}(r; j) = 2A_{k_4}(r; j)$ when $j = 2^r - 1 - k$.

**Lemma 6.6.7.** Let $r, k \geq 0$. Then, when $j = 2^r - 1$, we have $A_{2^r+k}(r, j) + A_{2^r+k+1}(r, j) = 2A_{2^r+2k}(r, j)$.

**Proof.** Similar to the case when $j = 2^r - 1 - k$, $x_{k_i}(n)$ and $x_{k_i}(n+1)$ have different formulas.

\[
x_{k_i}(n) \equiv t_{i+m+1}2^r+(k-1) + t_{m2^r+2r-1} \\
    \equiv t_{i+m+1} + t_{k-1} + t_m + t_{2^r-1} \\
    \equiv x_{i+1}(m) + t_{k-1} + r,
\]

\[
x_{k_i}(n+1) \equiv t_{i+m+1}2^r+k + t_{m2^r+2r} \\
    \equiv t_{i+m+1} + t_k + t_{m+1} \\
    \equiv x_{i}(m+1) + t_k.
\]

Notice that since $k$ is odd, $t_{k-1} = 1 - t_k$. This allows us to determine the set of solutions for each $k_i$ based on $r$ and $t_k$.

<table>
<thead>
<tr>
<th>$t_k$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_{k_i+1}(m), x_{k_i}(m+1)$</td>
<td>(1,0)</td>
<td>(0,0)</td>
</tr>
</tbody>
</table>

Table 6.4: Values for $x_{k_i+1}(m)$ and $x_{k_i}(m+1)$ for $j = 2^r - 1$ and $i = 1, 2, 4$.

Again, by analyzing all possible combinations, we derive Table 7.5.

<table>
<thead>
<tr>
<th>$t_k$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{k_1}(r; j)$</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>$A_{k_2}(r; j)$</td>
<td>1/6</td>
<td>1/6</td>
</tr>
<tr>
<td>$A_{k_4}(r; j)$</td>
<td>1/4</td>
<td>1/12</td>
</tr>
<tr>
<td>$A_{k_1}(r; j) + A_{k_2}(r; j) - 2A_{k_4}(r; j)$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6.5: Values for $A_{k_i}(r; j)$ for $j = 2^r - 1$. 

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Proof of Theorem 6.6.2. We’ve shown in Lemmas 6.6.4 - 6.6.7 that 

\[ A_{k_1}(r; j) + A_{k_2}(r; j) - 2A_{k_4}(r; j) = 0 \]

for all \( j \). Therefore, \( A_{k_1}(0, 0) + A_{k_2}(0, 0) = 2A_{k_4}(0, 0) \). \( \square \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{(0,0)}(k) )</td>
<td>0</td>
<td>1/6</td>
<td>1/3</td>
<td>1/4</td>
<td>1/6</td>
<td>1/2</td>
<td>1/6</td>
<td>7/24</td>
<td>1/4</td>
<td>1/4</td>
<td>1/12</td>
<td>7/12</td>
<td>1/12</td>
<td>1/3</td>
</tr>
<tr>
<td>( A_{(0,1)}(k) )</td>
<td>1/3</td>
<td>1/6</td>
<td>1/3</td>
<td>1/12</td>
<td>1/3</td>
<td>1/6</td>
<td>1/3</td>
<td>1/24</td>
<td>1/3</td>
<td>1/6</td>
<td>1/3</td>
<td>1/12</td>
<td>1/3</td>
<td>1/6</td>
</tr>
<tr>
<td>( A_{(1,0)}(k) )</td>
<td>1/3</td>
<td>1/6</td>
<td>1/3</td>
<td>1/12</td>
<td>1/3</td>
<td>1/6</td>
<td>1/3</td>
<td>1/24</td>
<td>1/3</td>
<td>1/6</td>
<td>1/3</td>
<td>1/12</td>
<td>1/3</td>
<td>1/6</td>
</tr>
<tr>
<td>( A_{(1,1)}(k) )</td>
<td>1/3</td>
<td>1/2</td>
<td>1/3</td>
<td>7/12</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>15/24</td>
<td>1/12</td>
<td>1/3</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/3</td>
</tr>
</tbody>
</table>

Table 6.6: Values for \( A_{(i,j)}(k) \) with \( i,j \in \{0,1\} \)

Corollary 6.6.8. Let \( r \in \mathbb{N}^+ \), and \( 2^{r-1} < k < 2^r \) be an odd number. Let \( f(s) = \frac{2}{3} \left(1 - \left(-\frac{1}{2}\right)^s\right) \). Then, for any \( s \geq 0 \),

\[ A_{2^r+r+k} = (1 - f(s))A_{2^r+k} + f(s)A_{2^r+1+k} \]

Proof. We prove the corollary by induction. The equation holds for \( s = 0, 1 \) since \( f(0) = 0 \) and \( f(1) = 1 \). Suppose that it holds for \( s < n \). Using the observation that \( f(n-1) + f(n) = 2f(n+1) \), we have

\[
A_{2^r+n+1} = \frac{1}{2} (A_{2^r+n} + A_{2^r+n-1}) \\
= \frac{1}{2} (1 - f(n))A_{2^r+k} + f(n)A_{2^r+1+k} + (1 - f(n-1))A_{2^r+k} + f(n-1)A_{2^r+1+k}) \\
= \frac{1}{2} ((2 - f(n) - f(n-1))A_{2^r+k} + (f(n) + f(n-1))A_{2^r+1+k}) \\
= (1 - f(n+1))A_{2^r+k} + f(n+1)A_{2^r+1+k}. 
\]
Chapter 7

The Correlation Function $f(k)$

We study a function based on the difference sequence $x_k$ from Chapter 6.

### 7.1 Definition of the Functions $f(k)$ and $\tilde{f}(k)$

We first recall the definition of the difference sequences $x_k$, and then use those to define the correlation function. For $k \geq 0$, the $k$-difference sequence $x_k = \{x_k(n)\}_{n \geq 0}$ is defined by

$$x_k(n) = |t_{n+k} - t_n|.$$  

(7.1)

**Definition 7.1.1.** The correlation function, denoted $f(k)$ for $k \geq 0$, and the density function, denoted $\tilde{f}(k)$, are defined by

$$f(k) = \sum_{n=0}^{k-1} x_k(n),$$  

(7.2)

$$\tilde{f}(k) = \frac{f(k)}{k}. $$  

(7.3)

Combinatorically speaking, $f(k)$ counts how many times the $n$-th and $(n + k)$-th term differ, where $n$ ranges from 0 to $k - 1$. Likewise $\tilde{f}(k)$ gives the ratio of such occurrences.

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(k)$</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>8</td>
<td>3</td>
<td>6</td>
<td>7</td>
<td>4</td>
<td>9</td>
<td>6</td>
<td>5</td>
<td>16</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>$\tilde{f}(k)$</td>
<td>1</td>
<td>1/3</td>
<td>1</td>
<td>3/5</td>
<td>1/3</td>
<td>3/7</td>
<td>1</td>
<td>1/3</td>
<td>3/5</td>
<td>7/11</td>
<td>1/3</td>
<td>9/13</td>
<td>3/7</td>
<td>1/3</td>
<td>1</td>
<td>7/17</td>
<td>1/3</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.1: Values for $f(k)$ and $\tilde{f}(k)$

The left plot seems to cover the line $y = k/2$, which we will prove that this is the case in the next section. The right plot has two interesting features. First, all values are either 1 or in between 0.3 and 0.7, and we will show that they are indeed in $[\frac{1}{3}, \frac{2}{3}] \cup \{1\}$. Second, the ranges $[2^m, 2^{m+1}]$ have a pattern similar to each other.
7.2 Properties of the Functions \( f(k) \) and \( \tilde{f}(k) \)

Recall Lemma 6.5.1 that for \( i, j \in \{0, 1\} \), we have \( |i - j| + |i - \tilde{j}| = 1 \) and \( |\tilde{i} - j| = \tilde{i} - j \).

**Theorem 7.2.1.** For all \( k \geq 0 \), we have

\[
\begin{align*}
    f(2k) &= 2f(k), \\
    f(2k + 1) &= 2k + 2 - f(k) - f(k + 1).
\end{align*}
\]
Proof. We use Lemma 6.5.1 to simplify \( f(2k) \) and \( f(2k+1) \).

\[
f(2k) = \sum_{i=0}^{2k-1} |t_{i+2k} - t_i|
= \sum_{i=0}^{k-1} |t_{2i+2k} - t_{2i}| + |t_{2i+2k+1} - t_{2i+1}|
= \sum_{i=0}^{k-1} (|t_{i+k} - t_i| + |t_{i+k} - t_i|)
= \sum_{i=0}^{k-1} 2(|t_{i+k} - t_i|)
= 2f(k).
\]

\[
f(2k+1) = \sum_{i=0}^{2k} |t_{i+2k+1} - t_i|
= \sum_{i=0}^{k} |t_{2i+2k+1} - t_{2i}| + \sum_{i=0}^{k-1} |t_{2i+2k+2} - t_{2i+1}|
= \sum_{i=0}^{k} |t_{i+k} - t_i| + \sum_{i=0}^{k-1} |t_{i+k+1} - t_i|
= k + 1 - \sum_{i=0}^{k} |t_{i+k} - t_i| + k - \sum_{i=0}^{k-1} |t_{i+k+1} - t_i|
= 2k + 1 - (f(k) + |t_{2k} - t_k|) - (f(k+1) - |t_{2k+1} - t_k|)
= 2k + 2 - f(k) - f(k+1).
\]

The last equation utilizes the fact that \( t_{2k} = t_k \) and \( t_{2k+1} = \overline{t_k} \) by the definition of the Thue-Morse sequence.

\[\square\]

**Corollary 7.2.2.** For all \( k \geq 0 \), we have

\[
\tilde{f}(2k) = \tilde{f}(k),
\]

\[
\tilde{f}(2k+1) = 1 - \frac{k\tilde{f}(k) + (k+1)\tilde{f}(k+1) - 1}{2k+1}.
\]

Although the recurrence relation for odd parameters seems complicated, we can still obtain a simple formula for parameters near powers of two. We list several formulas as a corollary below, and will include more in the appendix.

**Corollary 7.2.3.** For \( d \geq 1 \),
(i) If \( k = 2^{2d+2} \), then we have
\[
\tilde{f}(k-5) = \frac{1}{2} + \frac{3}{2(k-5)}, \quad \tilde{f}(k-3) = \frac{2}{3} - \frac{1}{3(k-3)}, \quad \tilde{f}(k-1) = \frac{1}{3},
\]
\[
\tilde{f}(k+5) = \frac{1}{2} - \frac{3}{2(k+5)}, \quad \tilde{f}(k+3) = \frac{2}{3} - \frac{5}{3(k+3)}, \quad \tilde{f}(k+1) = \frac{1}{3} + \frac{4}{3(k+1)}.
\]

(ii) If \( k = 2^{2d+1} \), then we have
\[
\tilde{f}(k-5) = \frac{1}{2} - \frac{1}{2(k-5)}, \quad \tilde{f}(k-3) = \frac{2}{3} + \frac{1}{3(k-3)}, \quad \tilde{f}(k-1) = \frac{1}{3} + \frac{2}{2(k-1)},
\]
\[
\tilde{f}(k+5) = \frac{1}{2} + \frac{5}{2(k+5)}, \quad \tilde{f}(k+3) = \frac{2}{3} - \frac{1}{3(k+3)}, \quad \tilde{f}(k+1) = \frac{1}{3}.
\]

Proof. We apply Theorem 7.2.2 and use induction on \( d \). For instance, consider \( \tilde{f}(2^d + 1) \).

- If \( d = 2m \) and we assume that \( \tilde{f}(2^{2m-1} + 1) = \frac{4}{3} \), then we have \( f(2^{2m-1} + 1) = \frac{2^{2m-1} + 1}{3} \), and so
\[
\begin{aligned}
f(2^{2m} + 1) &= 2^{2m} + 2 - f(2^{2m-1}) - f(2^{2m-1} + 1) \\
&= 2^{2m} + 2 - \frac{2^{2m} + 1}{3} \\
&= \frac{2^{2m} + 5}{3}, \\
\tilde{f}(2^{2m} + 1) &= \frac{1}{3} - \frac{4}{3(2^{2m} + 1)}. 
\end{aligned}
\]

- If \( d = 2m + 1 \) and we assume that \( \tilde{f}(2^{2m} + 1) = \frac{1}{3} + \frac{4}{3(2^{2m} + 1)} \), then we have \( f(2^{2m} + 1) = \frac{2^{2m} + 5}{3} \), and so
\[
\begin{aligned}
f(2^{2m+1} + 1) &= 2^{2m+1} + 2 - f(2^{2m}) - f(2^{2m} + 1) \\
&= 2^{2m+1} + 2 - 2^{2m} - \frac{2^{2m} + 5}{3} \\
&= \frac{2^{2m+1} + 1}{3}, \\
\tilde{f}(2^{2m} + 1) &= \frac{1}{3}.
\end{aligned}
\]

The other equations can be verified in a similar fashion.

Theorem 7.2.4. For any \( k \geq 0 \), we have \( f(4k + 1) + f(4k + 3) = 4k + 2 \).
Proof. We apply Theorem 7.2.2 several times.

\[
f(4k + 1) + f(4k + 3) = (4k + 2 - f(2k) - f(k + 1)) + (4k + 4 - f(2k + 1) - f(2k + 2))
\]
\[
= 8k + 6 - (f(2k) + 2f(2k + 1) + f(2k + 2))
\]
\[
= 8k + 6 - (2f(k) + 2(2k + 2 - f(k) - f(k + 1)) + 2f(k + 1))
\]
\[
= 4k + 2.
\]

\[ \square \]

To provide an insight on how the terms in the summation add up nicely, we can prove Theorem 7.2.4 by properly grouping the summations.

Lemma 7.2.5. Let \( S_i = |t_{i+4k+1} - t_i| + |t_{i+4k+3} - t_i| \). For all \( i \in \mathbb{Z} \), we have

(i) \( S_{4i} = 1 \).

(ii) \( S_{4i+3} = 1 \).

(iii) \( S_{4i+1} + S_{4i+2} = 2 \).

Proof. We simply calculate each term.

\[ S_{4i} = |t_{4i+4k+1} - t_{4i}| + |t_{4i+4k+3} - t_{4i}| = |t_{2i+2k} - t_{2i}| + |t_{2i+2k+1} - t_{2i}| = |t_{i+k} - t_{i}| + |t_{i+k} - t_{i}| = 1. \]
\[ S_{4i+3} = |t_{4i+4k+4} - t_{4i+3}| + |t_{4i+4k+6} - t_{4i+3}| = |t_{i+k+1} - t_{i}| + |t_{i+k+1} - t_{i}| = 1. \]
\[ S_{4i+1} = |t_{4i+4k+2} - t_{4i+1}| + |t_{4i+4k+4} - t_{4i+1}| = |t_{i+k} - t_{i}| + |t_{i+k} - t_{i}|. \]
\[ S_{4i+2} = |t_{4i+4k+3} - t_{4i+2}| + |t_{4i+4k+5} - t_{4i+2}| = |t_{i+k} - t_{i}| + |t_{i+k} - t_{i}|. \]
\[ S_{4i+1} + S_{4i+2} = (|t_{i+k} - t_{i}| + |t_{i+k} - t_{i}|) + (|t_{i+k+1} - t_{i}| + |t_{i+k+1} - t_{i}|) = 2. \]

Applying the lemma, we find that

\[
f(4k + 1) + f(4k + 3) = \sum_{i=0}^{4k} |t_{i+4k+1} - t_i| + \sum_{i=0}^{4k+2} |t_{i+4k+3} - t_i|
\]
\[
= |t_{8k+1} - t_{4k}| + |t_{8k+3} - t_{4k}| + |t_{8k+4} - t_{4k+1}| + |t_{8k+5} - t_{4k+2}| + \sum_{i=0}^{4k-1} S_i
\]
\[
= |t_{4k} - t_{4k}| + |t_{2k} - t_{2k}| + |t_{4k+2} - t_{4k}| + |t_{4k+2} - t_{4k+2}|
\]
\[
+ \sum_{i=0}^{k-1} (S_{4i} + S_{4i+1} + S_{4i+2} + S_{4i+3})
\]
\[
= 2 + 4k.
\]
Corollary 7.2.6. The sequence \( a_k := \frac{1}{k} \sum_{i=1}^{k} \tilde{f}(i) \) is convergent, with \( \lim_{k \to \infty} a_k = \frac{1}{2} \).

Proof. It is clear from the definition that \( \tilde{f}(k) = O(1) \). We apply Corollary 7.2.3 to get

\[
\sum_{i=1}^{2k} \tilde{f}(i) = \sum_{i=1}^{k} (\tilde{f}(2i - 1) + \tilde{f}(2i))
\]

\[
= \tilde{f}(1) + \tilde{f}(2) + \sum_{i=2}^{k} \left( 1 - \frac{(i-1)\tilde{f}(i-1) + i\tilde{f}(i) - 1}{2i-1} + \tilde{f}(i) \right)
\]

\[
= 2 + (k - 1) + \sum_{i=2}^{k} \frac{1}{2i-1} + \sum_{i=2}^{k} \left( \frac{(i-1)\tilde{f}(i-1) + i\tilde{f}(i)}{2i-1} + \tilde{f}(i) \right)
\]

\[
= k + O(\ln k) + \left( \frac{-\tilde{f}(1)}{3} - \frac{k\tilde{f}(k)}{2k-1} + \sum_{i=2}^{k-1} \tilde{f}(i) \left( 1 - \frac{i}{2i-1} - \frac{i}{2i+1} \right) \right)
\]

\[
= k + O(\ln k) + O(1) + \sum_{i=2}^{k-1} \tilde{f}(i) \frac{i}{4i-1}
\]

\[
a_{2k} = \frac{1}{2k} (k + O(\ln k)) = \frac{1}{2} + O \left( \frac{\ln k}{k} \right)
\]

\[
a_{2k+1} = \frac{1}{2k+1} (k + \tilde{f}(2k+1) + O(\ln k)) = \frac{1}{2} + O \left( \frac{\ln k}{k} \right).
\]

Therefore, \( \lim_{k \to \infty} a_k = \frac{1}{2} \). \( \square \)
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<tr>
<td>$\tilde{f}(k)$</td>
<td>The density function</td>
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</table>
References


Appendix A

More plots for $u_d(n)$

This appendix contains plots of $(n, s_d(n))$ where $s_d(n) = \text{sign}(u_d(n)) \cdot \log(|u_d(n)| + 1)$, for odd $19 \leq d \leq 49$ and for $n$ up to $10^5$.

(a) $d = 19$

(b) $d = 21$

(c) $d = 23$

(d) $d = 25$
Appendix B

A Table for $x_k$

Below is an extension of Table 6.2.

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Table B.1: Values for $A_0(k)$ for $k = 1$ to 100
Appendix C

A Table for $f(k)$

Below is an extension of Table 7.1.

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Table C.1: Values for $f(k)$ for $k = 1$ to 100