Matching $\mu$-Logic

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Abstract—Matching logic is a logic for specifying and reasoning about structure by means of patterns and pattern matching. This paper makes two contributions. First, it proposes a sound and complete proof system for matching logic in its full generality. Previously, sound and complete deduction for matching logic was known only for particular theories providing equality and membership. Second, it proposes matching $\mu$-logic, an extension of matching logic with a least fixpoint $\mu$-binder. It is shown that matching $\mu$-logic captures as special instances many important logics in mathematics and computer science, including first-order logic with least fixpoints, modal $\mu$-logic as well as dynamic logic and various temporal logics such as infinite/finite-trace linear temporal logic and computation tree logic, and notably reachability logic, the underlying logic of the K framework for programming language semantics and formal analysis. Matching $\mu$-logic therefore serves as a unifying foundation for specifying and reasoning about fixpoints and induction, programming languages and program specification and verification.

I. Motivation

Matching logic [1] (shortened as ML) is a first-order logic (FOL) variant for specifying and reasoning about structure by means of patterns and pattern matching. In the practice of program verification, ML is used to specify static properties of programs in reachability logic [2] (shortened as RL), which takes an operational semantics of a programming language as axioms and yields a program verifier that can prove any reachability properties of any programs written in that language. As a successful implementation of ML and RL, the K framework (http://kframework.org) has been used to define the formal semantics of various real languages such as C [3], Java [4], JavaScript [5], and to verify complex program properties [6].

A sound and complete Hilbert-style proof system $\mathcal{P}$ of ML is given in [1], whose proof of completeness is by reduction to pure predicate logic. However, the proof system $\mathcal{P}$ is only applicable to theories where a set of special definedness symbols are given together with appropriate axioms that can be used to define both equality and membership as derived constructs. This leaves the question of whether there is any proof system of ML that gives sound and complete deduction for all theories, open. Our first contribution is to answer this question by proposing a new proof system $\mathcal{H}$ of ML that is complete without requiring definedness or any other symbols.

Our second and main contribution was stimulated by limitations of RL itself as a logic to reason about dynamic behaviors of programs. Specifically, as its name suggests, RL can only define and reason about reachability claims. In particular, it is not capable of expressing liveness or many other interesting properties that temporal or dynamic logics can naturally express. Therefore, we propose matching $\mu$-logic (shortened as MmL), which extends ML with a least fixpoint $\mu$-binder. It turns out that MmL subsumes not only RL, but also a variety of common logics/calculi that are used to reason about fixpoints and induction, especially for program verification and model checking, including first-order logic with least fixpoints (LFP) [7], modal $\mu$-logic [8] (as well as various temporal logics [9], [10] and dynamic logic (DL) [11–13]). For each of these we prove a conservative extension result, showing the faithfulness of our definitions.

We organize the rest of the paper as follows. We start with a quick but comprehensive overview of ML in Section II and then present the new proof system $\mathcal{H}$ in Section III. We present MmL in Section IV and show how to define recursive/co-recursive symbols as syntactic sugar in Section V. Then we discuss how MmL subsumes all the following: first-order logic with least fixpoints (Section VI); modal $\mu$-logic and its fragment logics (Section VII); reachability logic (Section VIII). We compare with related work and conclude the paper with a proposal of future work in Sections IX and X.

II. Matching Logic Preliminaries

Matching logic (ML) is a variant of many-sorted FOL that makes no distinction between operation and predicate symbols, allowing them to be uniformly used to build patterns. Patterns define both structural and logical constraints, and are interpreted in models as sets of elements (those that match them). We offer a compact but comprehensive review of ML below. A detailed discussion of ML can be found in [1].

A. Matching logic syntax

Definition 1. A matching logic signature or simply a signature $\Sigma = (S, \text{Var}, \Sigma)$ is a triple with a nonempty set $S$ of sorts, an $S$-indexed set $\text{Var} = \{ \text{Var}_s \}_{s \in S}$ of countably infinitely many sorted variables denoted $x, y, s, x, y, s$, etc., and an $(S \times S)$-indexed countable set $\Sigma = \{ \Sigma_{s_1, s_2, s} \}_{s_1, s_2, s \in S}$ of many-sorted symbols. When $n = 0$, we write $\sigma \in \Sigma_1$, and say $\sigma$ is a constant symbol. Matching logic $\Sigma$-patterns, or simply $(\Sigma)$-patterns, are defined inductively for all sorts $s, s', s_1, \ldots, s_n \in S$ as follows:

We use different primitives $\{\neg, \lor, \forall\}$ than [1], which uses $\{\wedge, \forall, \exists\}$. These are more appropriate for our new proof system $\mathcal{H}$ (Fig. 1 in Section III).
Definition 2. \( \varphi, \bar{\varphi} \) so we write the following derived construct as syntactic sugar:

\[
\varphi_1 \land \varphi_2 \equiv \neg \varphi_1 \rightarrow \varphi_2 \quad \exists x : s . \varphi \equiv \neg \forall x : s . \neg \varphi
\]

We use \( \text{Pattern}^{ML}(\Sigma) = \{ \text{Pattern}^{ML}(\Sigma) \}_{\Sigma \in S} \) to denote the \( S \)-indexed set of \( \Sigma \)-patterns generated by the above grammar (modulo \( \alpha \)-equivalence, see later). We feel free to drop the signature \( \Sigma \) and simply write \( \text{Pattern}^{ML} = \{ \text{Pattern}^{ML}(\Sigma) \}_{\Sigma \in S} \).

The signature \( \Sigma = (S, \text{Var}, \Sigma) \) is abbreviated as \((S, \Sigma)\) or just \( \Sigma \) when \( \text{Var} \) and \( S \) are understood or not important. When we write a pattern, we assume it is well-formed without explicitly specifying the necessary conditions. When \( \varphi \in \Sigma_{1,s} \) is a constant symbol, we write \( \sigma \) to mean the pattern \( \sigma(i) \).

We adopt the following derived construct as syntactic sugar:

\[
\varphi_1 \lor \varphi_2 \equiv \neg \varphi_1 \rightarrow \varphi_2 \quad \forall x : s . \varphi \equiv \neg \exists x : s . \neg \varphi
\]

Note that “top” \( T_s \), the pattern that matches everything (see Proposition 3) is closed. We drop sort \( s \) whenever possible, so we write \( x, y, T, \bot \) instead of \( x : s, y : s, T_s, \bot_s \). Standard precedences are adopted to avoid parentheses. The scope of “\( \forall \)” and “\( \exists \)” goes as far as possible to the right.

As in FOL, “\( \forall \)” (and “\( \exists \)” are binders, and we adopt the standard notions of free variables, \( \alpha \)-renaming, and capturing-avoiding substitution. We use \( FV(\varphi) \) to denote the set of all free variables in \( \varphi \). When \( FV(\varphi) = \emptyset \), we say \( \varphi \) is closed. We regard patterns that are \( \alpha \)-equivalent as the same, i.e., \( \varphi \equiv \bar{\varphi} \) if \( \varphi, \bar{\varphi} \) are \( \alpha \)-equivalent. We write \( \varphi[\psi/x] \) to mean the result of substituting \( \psi \) for every free occurrence of \( x \) in \( \varphi \), where \( \alpha \)-renaming happens implicitly to prevent variable capture. We abbreviate \( \varphi[\psi_1/x_1], \ldots, \psi_n/x_n] \) as \( \varphi[\psi_1/x_1, \ldots, \psi_n/x_n] \).

B. Matching logic semantics

ML symbols are interpreted as relations, and thus ML patterns evaluate to sets of elements (those “matching” them).

Definition 2. Let \( \Sigma = (S, \Sigma) \) be a signature. A matching logic \( \Sigma \)-model \( M = (\{M_s\})_{s \in S}, \ldots, M) \), or just a \( (\Sigma-) \) model, consists of:

- a nonempty carrier set \( M_s \) for each sort \( s \) in \( S \);
- an interpretation \( \sigma_M : M_s \times \cdots \times M_s \rightarrow \text{P}(M_s) \) for each \( \sigma \in \Sigma_{1,s,\ldots,s} \), where \( \text{P}(M_s) \) is the powerset of \( M_s \).

For notational simplicity, we overload the letter \( M \) and use it to also mean the \( S \)-indexed set of carrier sets \( \{M_s\}_{s \in S} \). The usual FOL models are special cases of ML models, where \( |\sigma_M(a_1, \ldots, a_n)| = 1 \) for all \( a_1 \in M_{s_1}, \ldots, a_n \in M_{s_n} \). Partial FOL models are also special cases where \( |\sigma_M(a_1, \ldots, a_n)| \leq 1 \), since we can capture the undefinedness of the partial function \( \sigma_M \) on \( a_1, \ldots, a_n \) with \( \sigma_M(a_1, \ldots, a_n) = \emptyset \).

We tacitly use the same letter \( \sigma_M \) to mean its pointwise extension, \( \sigma_M : \text{P}(M_{s_1}) \times \cdots \times \text{P}(M_{s_n}) \rightarrow \text{P}(M_s) \), defined as:

\[
\sigma_M(A_1, \ldots, A_n) = \bigcup \{ \sigma_M(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n \}
\]

for all \( A_1 \subseteq M_{s_1}, \ldots, A_n \subseteq M_{s_n} \).

Proposition 3. For all \( A_i, A'_i \subseteq M_{s_i}, 1 \leq i \leq n \), the pointwise extension \( \sigma_M \) has the following property of propagation:

\[
\sigma_M(A_1, \ldots, A_n) = \emptyset \text{ if } A_j = \emptyset \text{ for some } 1 \leq j \leq n,
\]

\[
\sigma_M(A_1 \cup A'_1, \ldots, A_n \cup A'_n) = \bigcup_{1 \leq j \leq n, B_j \in \{A_j, A'_j\}} \sigma_M(B_1, \ldots, B_n).
\]

Proposition 4. Let \( \Sigma = (S, \text{Var}, \Sigma) \) and let \( M \) be a \( \Sigma \)-model. Given a function \( \rho : \text{Var} \rightarrow M \), called an \( M \)-valuation, let its extension \( \bar{\rho} : \text{Pattern}^{ML} \rightarrow \text{P}(M) \) be inductively defined as:

- \( \bar{\rho}(\{x\}) = \{\rho(x)\} \), for all \( x \in \text{Var}_s \);
- \( \bar{\rho}(\varphi_1 \rightarrow \varphi_2) = M_\varphi \setminus (\bar{\rho}(\varphi_1) \setminus \bar{\rho}(\varphi_2)) \), for \( \varphi_1, \varphi_2 \in \text{Pattern}_s \);
- \( \bar{\rho}(\neg \varphi) = M_\varphi \setminus \bar{\rho}(\varphi) \), for all \( \varphi \in \text{Pattern}_s \);
- \( \bar{\rho}(\forall x . \varphi) = \bigcap_{a \in M_x} \bar{\rho}[a/x](\varphi) \), for all \( x \in \text{Var}_s \);
- \( \bar{\rho}(\sigma(\varphi_1, \ldots, \varphi_n)) = \sigma_M(\bar{\rho}(\varphi_1), \ldots, \bar{\rho}(\varphi_n)) \), for \( \sigma \in \Sigma_{1,s,\ldots,s} \);

where "\( \setminus \)" is set difference and \( \rho[a/x] \) denotes the \( M \)-valuation \( \rho' \) with \( \rho'(x) = a \) and \( \rho'(y) = \rho(y) \) for all \( y \neq x \).

Intuitively, a pattern is evaluated to the set of all elements that “match” it. For example, the variable \( x \) (as a pattern) is matched by exactly one element, \( \rho(x) \); the pattern \( \neg \varphi \) is matched by exactly those that do not match \( \varphi \); etc. The next proposition shows that all derived constructs have the expected semantics: “\( \land \)” means conjunction, “\( \lor \)” means disjunction, “\( \rightarrow \)” means the total set, “\( \bot \)” means the empty set, etc.

Definition 6. We say a matching logic pattern \( \varphi \) holds in \( M \), written \( M \models_{ML} \varphi \), if \( \bar{\rho}(\varphi) = M \) for all \( \rho : \text{Var} \rightarrow M \). Let \( \Gamma \) be a set of patterns, called axioms. We write \( M \models_{ML} \Gamma \) iff \( M \models_{ML} \varphi \) for all axioms \( \varphi \in \Gamma \). We write \( \Gamma \models_{ML} \varphi \) iff \( M \models_{ML} \varphi \) for all models \( M \models_{ML} \Gamma \). When \( \Gamma \) is empty, we abbreviate \( \Gamma \models_{ML} \varphi \) as \( \models_{ML} \varphi \), and say that \( \varphi \) is valid. This is, a pattern is valid iff it is matched by all elements in all models. We call the pair \( (\Sigma, \Gamma) \) a matching logic \( \Sigma \)-theory, or simply a \( (\Sigma-) \) theory. Model \( M \) is said to be a model of the theory \( (\Sigma, \Gamma) \) iff \( M \models_{ML} \Gamma \).

C. Important notations

Several mathematical instruments of practical importance, such as definedness, totality, equality, membership, set containment, functions and partial functions, and constructors, can all be defined using patterns. We give a compact summary of the definitions and notations that are needed in this paper.

Definition 7. For any (not necessarily distinct) sorts \( s, s' \), let us consider a unary symbol \( [\_]_{s'} \in \Sigma_{s,s'} \), called the definedness symbol, and the pattern/axiom \( [x:s]_{s'} \), called (Definedness).
We define \( \textit{totality} \) \( \square, \textit{equality} \) \( \equiv \), \textit{membership} \( \in \), and \textit{set containment} \( \subseteq \) as derived constructs:

\[
\begin{align*}
[\varphi]_S \equiv & \exists[\neg \varphi]_S, \varphi_1 \equiv \varphi_2 \equiv \{ \varphi \leftrightarrow \varphi \} \nu_S, \\
x \in \varphi \equiv & \{ x \wedge \varphi \} \nu_S, \varphi_1 \subseteq \varphi_2 \equiv \{ \varphi_1 \rightarrow \varphi_2 \} \nu_S
\end{align*}
\]

and feel free to drop the (not necessarily distinct) sorts \( s, s' \).

The (\textbf{Definiteness}) axiom ensures that \( (\square)^M(a) = M^n \) in all models \( M \) for all \( a \in M^n \). Therefore, for all valuations \( \rho \), we have \( \bar{\rho}((\varphi^s)_S') = M^n \) if \( \bar{\rho}(\varphi) \neq \emptyset \), and \( \bar{\rho}((\varphi^s)_S') = \emptyset \) otherwise. That is, \( (\varphi^s)_S \) says, in the sort universe \( s' \), if \( \varphi \) is defined or not in its sort universe \( s \). We can prove all constructs in Definition 2 have the expected semantics: \( \bar{\rho}((\varphi^s)_S) = M^n \) if \( \bar{\rho}(\varphi) \neq \emptyset \) and \( \bar{\rho}(\varphi) = \emptyset \) otherwise; \( \bar{\rho}(\varphi^s_1) = \bar{\rho}(\varphi^s_2) \) and \( \bar{\rho}(\varphi^s_1 \equiv \varphi^s_2) = \emptyset \) otherwise; etc.

Functions and partial functions can be defined by axioms:

- (Function) \( \exists y. \sigma(x_1, \ldots, x_n) = y \)
- (Partial Function) \( \exists y. \sigma(x_1, \ldots, x_n) \subseteq y \)

\( (\text{Function}) \) requires \( \sigma(x_1, \ldots, x_n) \) contains exactly one element and \( (\text{Partial Function}) \) requires it contains at least one element (recall \( y \) is evaluated to a singleton set). For brevity, we use the function notation \( \sigma: s_1 \times \cdots \times s_n \rightarrow s \) to mean we automatically assume the (Function) axiom of \( \sigma \). Similarly, partial functions are written as \( \sigma: s_1 \times \cdots \times s_n \rightarrow s \).

Constructors are extensively used in building programs and data, as well as semantic structures to define and reason about languages and programs. They can be defined in the “no junk, no confusion” spirit. Let \( \Sigma = (S, \Sigma) \) be a signature, let \( C = \{ c_i \in \Sigma^s_{1, \ldots, s_m} \mid 1 \leq i \leq n \} \subseteq \Sigma \) be a set of constructor symbols, and consider the following axioms:

- (No Junk) for all sorts \( s \in S \):
  \[ \bigvee_{c_i \in C \text{ with } s_i = s} \exists x_1^i, \ldots, x_{s_i}^i, c_i(x_1^i, \ldots, x_{s_i}^i) \]

- (No Confusion I) for all \( i \neq j \) and \( s_i = s_j \):
  \( \neg(c_i(x_1^i, \ldots, x_{s_i}^i) \wedge c_j(x_1^j, \ldots, x_{s_j}^j)) \)

- (No Confusion II) for all \( 1 \leq i \leq n \):
  \( c_i(x_1^i, \ldots, x_{s_i}^i) \wedge c_i(y_1^i, \ldots, y_{s_i}^i) \rightarrow c_i(x_1^i \wedge y_1^i, \ldots, x_{s_i}^i \wedge y_{s_i}^i) \)

Intuitively, (No Junk) says everything is constructed; (No Confusion I) says different constructs build different things; and (No Confusion II) says constructors are injective. We refer to the the last two axioms as (No Confusion).

D. \textbf{Defining first-order logic in matching logic}

Given a FOL signature \( (S, \Sigma, \Pi) \) with function symbols \( \Sigma \) and predicate symbols \( \Pi \), the syntax of FOL is given by:

\[
t_s \ ::= \ x \in \text{VAR}_s \mid \ f(t_{s_1}, \ldots, t_{s_n}) \mid \ f \in \Sigma_{s_1, \ldots, s_n, s}
\]

\[
\varphi \ ::= \ \pi(t_{s_1}, \ldots, t_{s_n}) \mid \ \pi \in \Sigma_{s_1, \ldots, s_n} \mid \ \varphi \rightarrow \varphi \mid \neg \varphi \mid \forall x. \varphi
\]

To subsume the syntax, we define a ML signature \( \Sigma_{\text{FOL}} = (\Sigma_{\text{FOL}}, \Sigma_{\text{FOL}}) \), where \( \Sigma_{\text{FOL}} = \Sigma \cup \{ \text{Pred} \} \) contains a distinguished sort \( \text{Pred} \), and \( \Sigma_{\text{FOL}} = \{ f: s_1 \times \cdots \times s_n \rightarrow s \mid f \in \Sigma_{s_1, \ldots, s_n, s} \} \cup \{ \pi: s_1 \times \cdots \times s_n \rightarrow \text{Pred} \mid \pi \in \Pi_{s_1, \ldots, s_n} \} \) contains FOL function symbols as ML functions and FOL predicate symbols as ML functions that return \( \text{Pred} \). Let \( \Gamma_{\text{FOL}} \) be the resulting \( \Sigma_{\text{FOL}} \)-theory. Notice that we use the function notations so \( \Gamma_{\text{FOL}} \) contains the (Function) axioms for all symbols in \( \Sigma_{\text{FOL}} \).

\textbf{Proposition 8.} For all FOL formulas \( \varphi \), we have \( \varphi \) is a \( \Sigma_{\text{FOL}} \)-pattern of sort \( \text{Pred} \) and \( \Gamma_{\text{FOL}} \varphi \) if and only if \( \Gamma_{\text{FOL}} \Gamma_{\text{ML}} \varphi \).

E. \textbf{Matching logic proof system} \( \mathcal{P} \) with defined symbols

\( \mathcal{P} \) has a conditional sound and complete Hilbert-style proof system, which we refer to as \( \mathcal{P} \) in this paper. We refer readers to [1] for details (see also Fig. [2]). Here we denote its provability relation as \( \Gamma \vdash \varphi \). The proof system \( \mathcal{P} \) can prove all patterns \( \varphi \) that are valid in \( \Gamma \) under the condition that \( \Gamma \) contains defined symbols and (Definiteness) axioms. In fact, many proof rules in \( \mathcal{P} \) use the equality “\( \equiv \)” and membership “\( \in \)” constructs, both of which are defined using the defined symbols. This means \( \mathcal{P} \) is not applicable at all to any theories that do not contain defined symbols.

We wrap up this subsection by reviewing the soundness and completeness theorem of \( \mathcal{P} \). In Section III we propose a new ML proof system \( \mathcal{H} \) that is sound and complete without requiring the theories to contain defined symbols.

\textbf{Theorem 9 (Soundness and Completeness of} \( \mathcal{P} \)). \textit{For all theories} \( \Gamma \) containing the defined symbols and axioms (Definition 2) and all patterns \( \varphi \), we have \( \Gamma \vdash_{\text{FOL}} \varphi \) if and only if \( \Gamma \vdash_{\text{ML}} \varphi \).

III. \textbf{A New Proof System of Matching Logic}

Our first main contribution in this paper is a new ML proof system \( \mathcal{H} \) that is sound and complete without requiring defined symbols and axioms, and thus extends the completeness result in [1], re-stated in Theorem 9.

We first need the following definition of context:

\textbf{Definition 10.} A context \( C \) is a pattern with a distinguished placeholder variable \( \Box \). We write \( C[\varphi] \) to mean the result of replacing \( \Box \) with \( \varphi \) from any \( \sigma \)-renaming, so free variables in \( \varphi \) may become bound in \( C[\varphi] \), different from capture-avoiding substitution. A single symbol context has the form \( C_\sigma \equiv \sigma(\varphi_1, \ldots, \varphi_{i-1}, \Box, \varphi_{i+1}, \ldots, \varphi_n) \) where \( \sigma \in \Sigma_{s_1, \ldots, s_n} \) and \( \varphi_1, \ldots, \varphi_{i-1}, \varphi_{i+1}, \ldots, \varphi_n \) are patterns of appropriate sorts. A nested symbol context is inductively defined as:

- \( \Box \) is a nested symbol context, called the identity context;
- if \( C_\sigma \) is a single symbol context, and \( C \) is a nested symbol context, then \( C_\sigma[C[\Box]] \) is a nested symbol context.

Intuitively, a context \( C \) is a nested symbol context iff the path to \( \Box \) in \( C \) contains only symbols and no logic connectives.

The proof system \( \mathcal{H} \) (Fig. [1] above the double line) has 13 proof rules that are divided into four categories. The first category consists of the Lukasiewicz complete axiomatization of propositional logic (15) (four rules). The second category completes the (complete) axiomatization of first-order logic (16) (three rules). The third category contains four rules that capture the property of propagation (Proposition 3). The
which requires definedness symbols and axioms.

In particular, when \( H \) are dual to each other, as illustrated below:

Theorem 14 (Deduction Theorem of \( \mathcal{H} \)). Let \( \Gamma \) be an axiom set containing definedness symbols and axioms (see Definition \( \mathcal{P} \)), and let \( \varphi, \psi \) be two patterns. If \( \Gamma \cup \{ \psi \} \vdash \varphi \) and the proof does not use (Universal Generalization) on free variables in \( \psi \), then \( \Gamma \vdash [\varphi] \rightarrow \varphi \). In particular, if \( \psi \) is closed, then \( \Gamma \cup \{ \psi \} \vdash \varphi \) implies \( \Gamma \vdash [\varphi] \rightarrow \varphi \). Notice that \( [\varphi] \) is an abbreviation of \( [\varphi]_{\sigma} \) if \( \varphi \) has sort \( s \) and \( \psi \) has sort \( s' \). Also, the reverse theorem holds: \( \Gamma \vdash [\varphi] \rightarrow \varphi \) implies \( \Gamma \cup \{ \psi \} \vdash \varphi \), without any additional conditions.

The verbose condition about (Universal Generalization) in Theorem 14 also appears in the deduction theorem in FOL (see, for example, [16]). Notice that we can not conclude \( \Gamma \vdash \psi \rightarrow \varphi \) in general. The theorem is proved by an induction on the length of the proof, but we here instead give an intuitive semantic explanation. Suppose \( \Gamma \cup \{ \psi \} \vdash \varphi \) for some closed pattern \( \psi \) (so we can ignore valuations). Then for all models \( M \in_{ML} \Gamma \), if \( \psi \) holds then \( \varphi \) also holds. This actually means \( M \vdash \varphi \), as \( [\psi] \) is evaluated to the empty set if \( \psi \) does not hold in \( M \). Note that \( M \vdash \varphi \rightarrow \psi \) is too strong as a conclusion, for it requires the valuation of \( \psi \) is always contained in \( \varphi \), even in models \( M \) where \( \psi \) does not hold.

The third property is that we can prove all proof rules in \( \mathcal{P} \) using the new proof system \( \mathcal{H} \), with definedness axioms as additional axioms. This immediately gives us the following completeness result of \( \mathcal{H} \) as a corollary of Theorem 9.

Theorem 15. For all axiom sets \( \Gamma \) containing (Definedness) axioms and all patterns \( \varphi \), we have \( \Gamma \vdash \varphi \) implies \( \Gamma \vdash [\varphi] \rightarrow \varphi \).

Finally, we state our main completeness result for \( \mathcal{H} \):

| (Proposition 1) | \( \varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_1) \) |
| (Proposition 2) | \( (\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_3)) \rightarrow (\varphi_1 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \varphi_3) \) |
| (Proposition 3) | \( (\neg \varphi_1 \rightarrow \neg \varphi_2) \rightarrow (\varphi_2 \rightarrow \varphi_1) \) |
| (Modus Ponens) | \( \varphi_1 \varphi_1 \rightarrow \varphi_2 \) |
| (Variable Substitution) | \( \forall x. \varphi \rightarrow \varphi[y/x] \) |
| (Universal Generalization) | \( \forall x. (\varphi_1 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \forall x. \varphi_2) \) if \( x \notin FV(\varphi_1) \) |
| (Proposition 4) | \( C_\sigma[\upsilon] \rightarrow \bot \) |
| (Proposition 5) | \( C_\sigma[\varphi_1 \lor \varphi_2] \rightarrow C_\sigma[\varphi_1] \lor C_\sigma[\varphi_2] \) |
| (Proposition 6) | \( C_\sigma[\exists x. \varphi] \rightarrow \exists x. C_\sigma[\varphi] \) if \( x \notin FV(C_\sigma[\exists x. \varphi]) \) |
| (Framing) | \( C_\sigma[\varphi_1] \rightarrow C_\sigma[\varphi_2] \) |
| (Existence) | \( \exists x, x \) |
| (Singleton Variable) | \( \neg(C_1[x \land \varphi] \land C_2[x \land \neg \varphi]) \) where \( C_1 \) and \( C_2 \) are nested symbol contexts. |
| (Set Variable Substitution) | \( \varphi[\psi/X] \) |
| (Pre-Fixpoint) | \( \varphi[\mu X. \varphi/X] \rightarrow \mu X. \varphi \) |
| (Knaster-Tarski) | \( \mu X. \varphi \rightarrow \psi \) |

Fig. 1. Sound and complete proof system \( \mathcal{H} \) of matching logic (above the double line) and the proof system \( \mathcal{H}_\mu \) of matching \( \mu \)-logic
Theorem 16 (Completeness of $\mathcal{H}$). $\Vdash_{\text{ML}} \varphi$ implies $\vdash_{\mathcal{H}} \varphi$.

The proof of Theorem 16 is rather complex (see Appendix [12]). We drew inspiration from Blackburn and Tzakova [18], who proved a completeness result for a version of hybrid modal logic with the V-binder, using a mixture of modal and first-order techniques: the idea of canonical models from modal logic and the idea of witnessed sets from first-order logic. Theorem 16 can be seen as a nontrivial generalization of the completeness result in [18]. Specifically, we extend the hybrid modal logic of [18] in two directions. First, we consider multiple sorts, each coming with its own universe of worlds and logical infrastructure; the approach in [18] has only one sort, that of “formulae”. Second, we allow arbitrarily many modal operators of arbitrary arities (see Proposition 12); the approach in [18] only considers the usual, unary “necessity” modal operator $\Box$ (and its dual $\Diamond$). Polyadic, non-hybrid (i.e., without V-binder) variants of modal logic are known (see, e.g., [19]), but at our knowledge our work in this paper is the first to combine polyadic modal operators and FOL quantifiers.

IV. From Matching Logic to Matching $\mu$-Logic

In this section, we extend ML with the least fixpoint $\mu$-binder. We call the extended logic matching $\mu$-logic (MmL), and study its syntax, semantics, and proof system. Many definitions, notations, and properties of ML that are introduced in Section I and II also work for MmL, so we only focus on parts where they differ to prevent redundancy.

A. Matching $\mu$-logic syntax

Definition 17. A matching $\mu$-logic signature $\Sigma = (S, \text{VAR}, \Sigma)$ or simply a signature is the same as a matching logic signature except that $\text{VAR} = \text{EVAR} \cup \text{SVAR}$ is now a disjoint union of two $\Sigma$-indexed sets of variables: the element variables denoted as $x_1$, $y_1$, etc. in EVAR, and the set variables denoted as $X_1$, $Y_1$, etc. in SVAR. Matching $\mu$-logic $\Sigma$-patterns, or simply $\Sigma$-patterns or just patterns, are defined inductively by the following grammar for all sorts $s, s' \in S$:

$$\varphi_s := x_1 : s \in \text{EVAR}_s \mid X_1 : s \in \text{SVAR}_s \mid \cdots \mid \mu X_1 : s \varphi_s \text{ if } \varphi_s \text{ is positive in } X, s'$$

where the “...” part is the same as in ML. We say $\varphi_s$ is positive in $X$ if every free occurrence of $X$s is under an even number of negations, where for counting negations the formula $\varphi_1 \rightarrow \varphi_2$ is interpreted as $\neg \varphi_1 \lor \varphi_2$. We let $\text{Pattern}(\Sigma) = \{ \text{Pattern}_s \}_{s \in S}$ denote the set of all matching $\mu$-logic $\Sigma$-patterns and feel free to drop the signature $\Sigma$.

From now on, we automatically assume we are talking about MmL unless we explicitly say otherwise.

Intuitively, element variables are like ML variables in that they are evaluated to elements, while set variables are evaluated to subsets. The least fixpoint pattern $\mu X : s \varphi_s$ gives the least solution (under the subset relation) of the equation $X : s = \varphi_s$ of set variable $X$ and $s$, and the condition of positive occurrence guarantees the existence of such a least solution. The notion of free variables, $\alpha$-renaming, and capture-avoiding substitution are extended to set variables and the $\mu$-binder. The dual version of the least fixpoint $\nu$-binder is the greatest fixpoint $\nu$-binder, defined as $\nu X : s \varphi_s \equiv \neg \mu X : s \neg \varphi_s \neg X : s \varphi_s$, given that $\varphi_s$ is positive in $X : s$, (which implies that $\neg \varphi_s \neg X : s \varphi_s$ is also positive in $X : s$, justifying the definition).

B. Matching $\mu$-logic semantics

We first review a variant of the Knaster-Tarski theorem [20]:

Theorem 18 (Knaster-Tarski). Let $M$ be a nonempty set and $\mathcal{F} : \mathcal{P}(M) \rightarrow \mathcal{P}(M)$ be a monotone function, i.e., $\mathcal{F}(A) \subseteq \mathcal{F}(B)$ for all subsets $A \subseteq B$. Then $\mathcal{F}$ has a unique least fixpoint $\mu \mathcal{F}$ and a unique greatest fixpoint $\nu \mathcal{F}$, where:

$$\mu \mathcal{F} = \bigcap \{ A \in \mathcal{P}(M) \mid \mathcal{F}(A) \subseteq A \}$$

$$\nu \mathcal{F} = \bigcup \{ A \in \mathcal{P}(M) \mid A \subseteq \mathcal{F}(A) \}$$

We call $\mathcal{A}$ a pre-fixpoint of $\mathcal{F}$ whenever $\mathcal{F}(A) \subseteq A$, and a post-fixpoint of $\mathcal{F}$ whenever $A \subseteq \mathcal{F}(A)$.

MmL models are exactly ML models where sorts are associated with their carrier sets and symbols are interpreted as relations. Valuations are extended such that element variables are mapped to sets and set variables are mapped to subsets. Patterns are evaluated the same way for the ML constructs, but extended with the valuation of least fixpoint patterns $\mu X : s \varphi$ as the true least fixpoints in models. Formally:

Definition 19. Let $\Sigma = (S, \text{VAR}, \Sigma)$ be a signature with $\text{VAR} = \text{EVAR} \cup \text{SVAR}$, and $M = (\{ M_s \}_{s \in S}, \mu_m)$ be a $\Sigma$-model. A valuation $\rho : \text{VAR} \rightarrow \{ \text{EVAR} \cup \text{SVAR} \}$ is a function such that $\rho(x) \in M_s$ for all $x \in \text{EVAR}_s$ and $\rho(X) \in \mathcal{P}(M_s)$ for all $X \in \text{SVAR}_s$. Its extension $\bar{\rho} : \text{Pattern} \rightarrow \mathcal{P}(M)$ is defined as in Definition 4 extended with:

- $\bar{\rho}(\{\rho(x)\})$ for all $x \in \text{EVAR}_s$;
- $\bar{\rho}(X) = \rho(X)$ for all $X \in \text{SVAR}_s$;
- $\bar{\rho}(\mu X : s \varphi) = \mu \mathcal{F}_{\varphi, X}$ for all $X : s \in \Sigma$;

where $\mathcal{F}_{\varphi, X}(A) = \mu \mathcal{F}_{\varphi, X}(A)$ for all $A \subseteq M_s$.

Here $\rho[A/X]$ denotes the valuation $\rho'$ such that $\rho'(X) = A$ and $\rho'(Y) = \rho(Y)$ for all $Y \neq X$. Notice that we need to verify that $\mathcal{F}_{\varphi, X}$ is monotone. This is done by using the fact that $\varphi$ is positive in $X$, and we omit the verification details. The notions $\rho \models \varphi$, $\Gamma \models \varphi$, and $\rho \models \varphi$ for all MmL models $M$, patterns $\varphi$, and axiom sets $\Gamma$ are defined in the expected way.

Proposition 20. For all axiom sets $\Gamma$ of matching logic patterns (without $\mu$) and all matching logic patterns $\varphi$ (without $\mu$), we have $\Gamma \models_{\text{ML}} \varphi$ if and only if $\Gamma \models \varphi$.

C. Example: capturing precisely term algebras

Many approaches to specifying formal semantics of programming languages are applications of initial algebra semantics [21]. In this subsection, we show how term algebras, a particular example of initial algebras, can be precisely captured using MmL patterns as axioms. For simplicity, we discuss only single-sorted term algebras, but the result can be extended to the many-sorted settings without any major technical difficulties using the techniques introduced in Section V.
Definition 21. Let $\Sigma = ((\text{Term}), \Sigma)$ be a signature with one sort Term and at least one constant symbol. A $\Sigma$-term or simply a term is inductively defined as follows:

$$t ::= c \in \Sigma_{\text{Term}} \mid c(t_1, \ldots, t_n) \text{ for } c \in \Sigma_{\text{Term}}$$

The $\Sigma$-term algebra $T_\Sigma = ((T_\Sigma^{\Sigma}), \tau_\Sigma)$ consisting of:

- a carrier set $T_\Sigma^{\Sigma}$ containing all $\Sigma$-terms;
- a function $c_\tau_\Sigma : T_\Sigma^{\Sigma} \times \cdots \times T_\Sigma^{\Sigma} \rightarrow T_\Sigma^{\Sigma}$ for all symbols $c \in \Sigma_{\text{Term}}$ defined as $c_\tau_\Sigma(t_1, \ldots, t_n) = c(t_1, \ldots, t_n)$.

Proposition 22. Let $\Sigma = ((\text{Term}), \Sigma)$ be a signature with one sort Term and at least one constant symbol. Define a $\Sigma$-theory $\Gamma_\Sigma^{\text{term}}$ with (Function) and (No Confusion) axioms\footnote{See Section II-C} for all symbols in $\Sigma$, plus the following axiom:

$$\text{(Inductive Domain)} \quad \mu D. \forall c(D, \ldots, D)$$

Then for all $\Sigma$-models $M \vDash \Gamma_\Sigma^{\text{term}}$, $M$ is isomorphic to $T_\Sigma$. In addition, for all extended signatures $\Sigma^+ \supseteq \Sigma$ and $\Sigma^+$-models $M \vDash \Gamma_\Sigma^{\text{term}}$, we have $M_{T_\Sigma^{\Sigma}}$ is isomorphic to $T_\Sigma$, where $M_{T_\Sigma^{\Sigma}}$ is the reduct of $M$ over the sub-signature $\Sigma$.

Intuitively, the (Inductive Domain) axiom forces that for all models $M$, the carrier set $M_{\text{term}}$ must be the the smallest set that is closed under all symbols in $\Sigma$, while the (Function) and (No Confusion) axioms forces all symbols in $\Sigma$ to be interpreted as injective functions, and different constructors construct different terms.

Proposition 22 immediately tells us that MmL cannot have a proof system that is both sound and complete, because one can capture precisely the model $(\mathbb{N}, +, \times)$ of natural numbers with addition and multiplication with MmL axioms, and the model $(\mathbb{N}, +, \times)$, by Gödel’s first incompleteness theorem [22], is not axiomatizable.

Proposition 23. Let $\Sigma = ((\text{Nat}), \{0 \in \Sigma_{\text{Nat}}, \text{succ} \in \Sigma_{\text{Nat}}, \text{Nat}\})$ and the $\Sigma$-theory $\Gamma_\Sigma^{\text{term}}$ be defined as in Proposition 22, where the (Inductive Domain) takes the following form:

$$\text{(Inductive Domain)} \quad \mu D. \forall x \in \Sigma$$

Let the signature $\Sigma^{\text{Nat}}$ extend $\Sigma$ with two functions:

$$\text{plus} : \text{Nat} \times \text{Nat} \rightarrow \text{Nat} \quad \text{mult} : \text{Nat} \times \text{Nat} \rightarrow \text{Nat}$$

and the $\Sigma^{\text{Nat}}$-theory $\Gamma_\Sigma^{\text{term}}$ extend $\Gamma_\Sigma^{\text{term}}$ with the standard axioms:

$$\text{plus}(0, y) = y \quad \text{plus}(s(x), y) = s(\text{plus}(x, y))$$
$$\text{mult}(0, y) = 0 \quad \text{mult}(s(x), y) = s(\text{mult}(x, y))$$

Then, $\Gamma_\Sigma^{\text{term}}$ captures precisely $(\mathbb{N}, +, \times)$, meaning that for all models $M \vDash \Gamma_\Sigma^{\text{term}}$, $M$ is isomorphic to $(\mathbb{N}, +, \times)$.

We finish this subsection by comparing Proposition 22 with the nontrivial result that the term algebra $T_\Sigma$ has a complete axiomatization in FOL where the only predicate symbol is equality [23]. We refer to this complete FOL axiomatization as $\Gamma_{\text{FOL}}(T_\Sigma)$. This means that for all FOL formulas $\varphi$, $\Gamma_{\text{FOL}}(T_\Sigma) \vDash_{\text{FOL}} \varphi$ if and only if $T_\Sigma \vDash \varphi$. This result is weaker than Proposition 22 because by Löwenheim-Skolem theorem [24], the FOL theory $\Gamma_{\text{FOL}}(T_\Sigma)$ has models of arbitrarily large cardinals (if $\Sigma$ contains at least one non-constant constructor), meaning that there are models $M \vDash \Gamma_{\text{FOL}}(T_\Sigma)$ with strictly more elements than $T_\Sigma$, and thus cannot be isomorphic to $T_\Sigma$. It is just the case that $M$ (and all FOL models of $\Gamma_{\text{FOL}}(T_\Sigma)$) satisfies exactly the same FOL formulas as $T_\Sigma$, known in literature as elementary equivalence. Proposition 22 on the other hand, shows that the MmL theory $\Gamma_\Sigma^{\text{term}}$ captures $T_\Sigma$ up to isomorphism.

D. Matching $\mu$-logic proof system

Proposition 23 implies that MmL cannot have a sound and complete proof system. The best we can do then is to aim for a proof system that is good enough in practice. We take the ML proof system $\mathcal{H}$ and extend it with three additional proof rules (see Fig. 1). Rules (Pre-Fixpoint) and (Knaster-Tarski) are standard proof rules about least fixpoints as in modal $\mu$-logic [8]. Rule (Set Variable Substitution) allows us to prove from $\Gamma \vdash \varphi$ any substitution $\Gamma \vdash \varphi[\psi/x]$ for $x \in \text{SVar}$. Note the condition that $X$ is a set variable is crucial. In general, we cannot prove from $\Gamma \vdash \varphi$ that $\Gamma \vdash \varphi[\psi/x]$ for $x \in \text{ENot EVar}$, because it does not hold semantically. As shown in [1], it only holds when $\psi$ is functional, that is, when $\psi$ evaluates to a singleton set. Indeed, suppose that $\psi$ is not functional, say it is the pattern $0 \lor \text{succ}(0)$ over the signature of natural numbers in Proposition 23, which evaluates to a set of two elements. Then we can pick $\varphi$ to be the tautology $\exists y. x = y$, and then $\varphi[\psi/x]$ becomes $\exists y. \psi = y$, which states that $\psi$ evaluates to a singleton set (the valuation of $y$), which is a contradiction.

We let $\mathcal{H}_p$ denote the extended 16-rule proof system in Fig. 1 and from here on we write $\Gamma \vdash \varphi$ instead of $\Gamma \vdash_{\mathcal{H}_p} \varphi$.

Theorem 24 (Soundness of $\mathcal{H}_p$). $\Gamma \vdash \varphi$ implies $\Gamma \vdash \varphi$.

E. Instance: Peano arithmetic

The purpose of this subsection is to illustrate the power of the two proof rules (Pre-Fixpoint) and (Knaster-Tarski), by showing that they derive the (Induction) axiom schema in the FOL axiomatization of Peano arithmetic [25, 26].

$$\text{(Induction)} \quad \varphi(0) \land \forall x. (\varphi(x) \rightarrow \varphi(\text{succ}(x))) \rightarrow \forall x. \varphi(x)$$

where $\varphi(x)$ is a FOL formula with a distinguished variable $x$.

We encode the FOL syntax of Peano arithmetic following the technique in Section II-D that is, we define a signature $\Sigma_{\text{Peano}} = ((\text{Nat}, \text{Pred}), \Sigma^\mathcal{H})$ where $\Sigma^\mathcal{H}$ is defined in Proposition 23 that contains the functions $0, \text{succ}, \text{plus}, \text{mult}$, and let $\Gamma_{\text{Peano}}$ contain the same equations as axioms as $\Gamma^\mathcal{H}$. Notice that the only $\Sigma_{\text{Peano}}$-patterns of sort Pred are those built from equalities between two patterns of sort Nat.

Proposition 25. Under the above notations, we have:

$$\Gamma_{\text{Peano}} \vdash \varphi(0) \land \forall x. (\varphi(x) \rightarrow \varphi(\text{succ}(x))) \rightarrow \forall x. \varphi(x).$$
V. Defining Recursive Symbols as Syntactic Sugar

Intuitively, the least fixpoint pattern \( \mu X. \varphi \) specifies a recursive set that satisfies the equation \( X = \varphi \), where \( \varphi \) may contain recursive occurrences of \( X \). For example, the pattern \( \mu X. 0 \lor \text{succ(succ}(X)) \) specifies the set of all even numbers, which conceptually defines a recursive constant symbol:

\[
even \in \Sigma_{\text{Nat, Nat}} \quad \text{even} \equiv \mu \text{ifp} \ 0 \lor \text{succ(succ(even))}.
\]

Here, “\( \equiv \text{ifp} \)” is merely a notation, meaning that we want even to be the least set that satisfies the equation (since the total set is always a trivial solution).

The challenge is how to generalize the above and define recursive non-constant symbols. For example, suppose we want to define a symbol \( \text{collatz} \in \Sigma_{\text{Nat, Nat}} \) as follows:

\[
\text{collatz}(n) \equiv \text{ifp} \\
 n \lor (\text{even}(n) \land \text{collatz}(n/2)) \lor (\text{odd}(n) \land \text{collatz}(3n + 1))
\]

with the intuition that \( \text{collatz}(n) \) gives the set of all numbers in the Collatz sequence\(^3\) starting from \( n \). However, the \( \mu \)-symbol in \( \text{MnL} \) can only be applied on set variables, not on symbols, so the following attempt is syntactically wrong:

\[
\text{collatz}(n) = \mu \sigma(n). \quad \| \mu \text{ can only bind a set variable}
\]

\[
\text{collatz}(n) \lor (\text{even}(n) \land \sigma(n/2)) \lor (\text{odd}(n) \land \sigma(3n + 1))
\]

One possible solution could be to extend \( \text{MnL} \) with the above syntax and allow the \( \mu \)-symbol to quantify set variables, not only on set variables. The semantics and proof system could be extended accordingly. This is exactly how first-order logic with least fixpoints extends FOL [1]. But do we really have to? After all, our proof rules (Prefixpoint) and (Knaster-Tarski) in Fig. 1 are nothing but a logical incarnation of the Knaster-Tarski theorem, which has been repeatedly demonstrated to serve as a solid if not the main foundation for recursion. Therefore, we conjecture that the \( \mathcal{H} \) proof system in Fig. 1 is sufficient in practice, and thus would rather resist extending \( \text{MnL} \). That is, we conjecture that it should be possible to define one’s desired approach to recursion/induction/fixed-points using ordinary \( \text{MnL} \) theories; as an analogy, in Section 1.C we showed how we can define definedness, totality, equality, membership, containment, functions, partial functions, etc. (see 1C for more) as theories, without a need to extend matching logic.

In particular, we can solve the recursive symbol challenge above by using the principle of currying-uncurrying to “mimic” the unary symbol \( \text{collatz} \in \Sigma_{\text{Nat, Nat}} \) with a constant symbol \( \text{collatz} \in \Sigma_{1, \text{Nat} \@ \text{Nat}} \), where \( \text{Nat} \@ \text{Nat} \) is the product sort (defined later; the intuition is that \( \text{Nat} \@ \text{Nat} \) has the product set \( \mathbb{N} \times \mathbb{N} \) as its carrier set), and thus reducing the challenge of defining a least relation in \( [\mathbb{N} \rightarrow \mathcal{P}(\mathbb{N})] \) to defining a least subset of \( \mathcal{P}(\mathbb{N} \times \mathbb{N}) \), without the need to extend the logic.

A. Principle of currying-uncurrying and product sorts

The principle of currying-uncurrying [27, 28] is used in various settings (e.g., simply-typed lambda calculus [29]) as a means to reduce the study of multi-argument functions to the simpler single-argument functions. We here present the principle in its adapted form that fits best with our needs.

**Proposition 26.** Let \( M_{s_1}, \ldots, M_{s_n}, M \) be nonempty sets. The principle of currying-uncurrying means the isomorphism

\[
\mathcal{P}(M_{s_1} \times \cdots \times M_{s_n} \bowtie M) \cong \mathcal{P} \left( M_{s_1} \bowtie \cdots \bowtie M_{s_n} \rightarrow \mathcal{P}(M) \right)
\]

defined for all \( a_1 \in M_{s_1}, \ldots, a_n \in M_{s_n}, b \in M \), \( \alpha \subseteq M_{s_1} \times \cdots \times M_{s_n} \times M \), and \( f : M_{s_1} \times \cdots \times M_{s_n} \rightarrow \mathcal{P}(M) \) as:

\[
\text{curry}(a_1, \ldots, a_n) = \{ b \in M_1 \mid (a_1, \ldots, a_n, b) \in \alpha \}
\]

\[
\text{uncurry}(f) = \{ (a_1, \ldots, a_n, b) \mid b \in f(a_1, \ldots, a_n) \}.
\]

In other words, we can mimic an \( n \)-ary symbol \( \sigma \in \Sigma_{1, \ldots, s_n, t} \) with a constant symbol of the product sort \( s_1 \bowtie \cdots \bowtie s_n \bowtie t \), whose (intended) carrier set is exactly the product set \( M_{s_1} \times \cdots \times M_{s_n} \times M \). This leads to the following definition.

**Definition 27.** Let \( s, t \) be two sorts, not necessarily distinct. The product sort \( s \bowtie t \) is a sort that is different from \( s \) and \( t \). The pairing symbol \( (\_ \bowtie \_)_s; s \times t \rightarrow s \bowtie t \) is a function symbol and the projection symbol \( \text{proj}_k_s; s \bowtie t \times s \rightarrow t \) is a partial function symbol. These are governed by the axioms

\[
\text{(Injectivity)} \quad ((v, k), v') \equiv (k = k') \land (v = v')
\]

\[
\text{(Key-Value)} \quad (k, v)(k') = (k = k') \land v
\]

\[
\text{(Product Domain)} \quad \exists k \exists v.(k, v)
\]

forcing the carrier of \( s \bowtie t \) to be the product of the carriers of \( s \) and \( t \), and pairing/projection are interpreted as expected.

Product of multiple sorts as well as the associated pairing/projection operations can be defined as derived constructs as follows. Let \( s_1, \ldots, s_n, t \) be sorts, not necessarily distinct, and \( \varphi_1, \ldots, \varphi_n, \psi \) be patterns of appropriate sorts. We define:

\[
\varphi_1 \bowtie \cdots \bowtie \varphi_n \bowtie \psi \equiv \{ (\varphi_1, \ldots, \varphi_n, \psi) \}
\]

\[
\psi(\varphi_1, \ldots, \varphi_n) \equiv \psi(\varphi_1) \ldots (\varphi_n).
\]

Notice that we tactically use the same syntax \( \langle \_, \ldots, \_ \rangle \) for both symbol applications and projections to blur their distinction. In particular, if \( \sigma \in \Sigma_{s_1, s_2, \ldots, s_n, \bowtie t} \) is a constant symbol of the product sort, then \( \sigma(\varphi_1, \ldots, \varphi_n) \) is a well-formed pattern if \( \varphi_1, \ldots, \varphi_n \) have appropriate sorts.

B. Defining recursive symbols in matching \( \mu \)-logic

**Definition 28.** Let \( \Sigma = (S, \Sigma) \) be a signature and \( \sigma \in \Sigma_{s_1, \ldots, s_n, s} \). We use the notation \( \sigma(x_1, \ldots, x_n) \equiv \text{ifp} \varphi \) to mean the axiom:

\[
\sigma(x_1, \ldots, x_n) = \langle \mu \varphi. s_1 \bowtie \cdots \bowtie s_n \bowtie t, \exists x_1 \ldots \exists x_n. (x_1, \ldots, x_n, \varphi) \rangle(x_1, \ldots, x_n)
\]

A symbol \( \sigma \in \Sigma_{s_1, \ldots, s_n, s} \) obeying this axiom is called recursive.
The following theorem says that if $\varphi$ “behaves like a symbol”, meaning that it has the property of propagation (Proposition 5), then we can obtain variants of (Pre-Fixpoint) and (Knaster-Tarski) for the recursive symbol $\sigma$.

**Theorem 29.** Let $\Sigma$ be a signature with a recursive symbol $\sigma \in \Sigma_{x_1,\ldots,x_n}$ defined as $\sigma(x_1,\ldots,x_n) = lfp \varphi$. Let $\Gamma$ be a $\Sigma$-theory such that for all $\varphi_1,\ldots,\varphi_n$:

$$\Gamma \vdash (\exists z_1 \ldots \exists z_n. \varphi_1 \land \cdots \land \varphi_n \land \varphi_1(z_1/1,\ldots,z_n/n) \land \cdots \land \varphi_n(z_1/1,\ldots,z_n/n))$$

$$\rightarrow \varphi_1(x_1/1,\ldots,x_n/n) \land \cdots \land \varphi_n(x_1/1,\ldots,x_n/n).$$

Then the following hold:

- **Pre-Fixpoint:** $\Gamma \vdash \varphi \rightarrow \sigma(x_1,\ldots,x_n)$;
- **Knaster-Tarski:** if $\Gamma \vdash \varphi[\psi/\sigma] \rightarrow \psi$ then $\Gamma \vdash \sigma(x_1,\ldots,x_n) \rightarrow \psi$, where $\varphi[\psi/\sigma]$ is the result of replacing all patterns of the form $\sigma(x_1,\ldots,x_n)$ in $\varphi$ with $\psi(x_1/1,\ldots,x_n/n)$.

**VI. Instance: First-Order Logic with Least Fixpoints**

First-order logic with least fixpoints (LFP) [7] extends the syntax of first-order logic formulas with:

$$\varphi := [[\text{LFP}_{R,x_1,\ldots,x_n} \varphi](t_1,\ldots,t_n)$$

where $R$ is a predicate variable and $\varphi$ is a formula that is positive in $R$. Intuitively, "[[\text{LFP}_{R,x_1,\ldots,x_n} \varphi]]" behaves as the least fixpoint predicate of the operation that maps $R$ to $\varphi$. Due to its complexity and our limited space, we skip the formal definition of the semantics and simply denote the validity relation in LFP as $\models_{\text{LFP}} \varphi$. A comprehensive study on LFP can be found in [30].

Given the notations of recursive symbols defined in Section V it is straightforward to subsume LFP by extending the theory $\Gamma^{\text{FO}}$ defined in Section II-D with:

$$[[\text{LFP}_{R,x_1,\ldots,x_n} \varphi](t_1,\ldots,t_n) \equiv (\mu R: S_1 \otimes \cdots \otimes S_n \otimes \text{Pred}.\exists x_1 \ldots \exists x_n. (\psi_1,\ldots,\psi_n)\langle t_1,\ldots,t_n \rangle)$$

for all predicate variables $R$ with argument sorts $s_1,\ldots,s_n$. What is different is that we add one additional axiom, $\forall x:\text{Pred} \forall y: \text{Pred} x = y$, to constrain the (dummy) carrier set of $\text{Pred}$ is a singleton set, so that all MmL models are also FO/LFP models. This fact is used to prove the “only if” part in the next theorem.

**Theorem 30.** If $\varphi$ is any LFP formula, then $\models_{\text{LFP}} \varphi$ iff $\models^{\text{LFP}} \not\models \varphi$.

**VII. Instances: Modal $\mu$-Calculus and Temporal Logics**

We have seen how MmL symbols and patterns can be used to specify both structure and constraints, as terms (Section IV-C) and FOL (Section II-D), as well as various induction, recursion and least-fixed point schemas (Sections IV-E and V) over these. These suffice to express and prove program assertions, including complex state abstractions (see also how separation logic falls as a fragment of matching logic in [1]), in contexts where MmL is chosen as a static state assertion formalism in program verification frameworks based on Hoare logic [31], dynamic logic [11], or reachability logic [2]. However, as explained in Section VI our ultimate goal is to support not only static state assertions, but any program properties, including ones that are usually specified using Hoare, dynamic, or reachability logics. We start the discussion in this section, by showing how MmL symbols and patterns can also be used to specify dynamic transition relations such as modal $\mu$-logic modalities and dynamic logic; in Section VIII we then discuss how MmL also subsumes reachability logic, which subsumes Hoare logic [6].

**A. Modal $\mu$-logic syntax, semantics, and proof system**

The syntax of modal $\mu$-logic [8] is parametric on a countably infinite set $\text{PVAR}$ of propositional variables. Modal $\mu$-logic formulas are given by the grammar:

$$\varphi ::= p \in \text{PVAR} \mid \varphi \land \varphi \mid \neg \varphi \mid o \varphi \mid \mu X. \varphi$$

where $p$ is any propositional variable. Due to its $\mu$-logic semantics is given using transition systems $S = (S,R)$, with $S$ a nonempty set of states and $R \subseteq S \times S$ a transition relation, and valuations $V: \text{PVAR} \rightarrow \mathcal{P}(S)$, as follows:

- $\models^S V(p) = V(p)$;
- $\models^S V(\varphi_1 \land \varphi_2) = \models^S V(\varphi_1) \cap \models^S V(\varphi_2)$;
- $\models^S V(\neg \varphi) = S \setminus \models^S V(\varphi)$;
- $\models^S V(\varphi) = \{ s \in S \mid s R t \implies t \models^S V(\varphi) \}$ for all $s \in S$;
- $\models^S V(\mu X. \varphi) = \bigcap \{ A \subseteq S \mid \models^S V(\varphi) \subseteq \{ x \mid x \models^S V(\varphi) \} \}$.

A modal $\mu$-logic formula is valid, denoted $\models_{\text{M}} \varphi$, if for all transition systems $S$ and all valuations $V$, we have $\models^S V(\varphi) = S$. A proof system of modal $\mu$-logic is firstly given in [8] and then proved to be complete in [32]. It extends the proof system of propositional logic with the following proof rules:

- $(K) \quad o(\varphi_1 \rightarrow \varphi_2) \rightarrow (o \varphi_1 \rightarrow o \varphi_2)$
- $(N) \quad \varphi \vdash o \varphi \vdash o \varphi_1 \rightarrow o \varphi_2$
- $(\mu_1) \quad \models^S V(\mu X. \varphi) \rightarrow \mu X. \varphi$
- $(\mu_2) \quad \varphi \vdash o(\varphi_1 \rightarrow \varphi_2) \rightarrow (o \varphi_1 \rightarrow o \varphi_2)$
- $(\mu_2) \quad \models^S V(\varphi) \rightarrow \models^S V(\varphi)$

We denote the corresponding provability relation as $\models_{\text{M}} \varphi$. Notice that (K) and (N) are provable in MmL (Proposition 12), and $(\mu_1)$ and $(\mu_2)$ are exactly (Pre-Fixpoint) and (Knaster-Tarski). This means that we can easily mimic all modal $\mu$-logic proofs in MmL (i.e. “(2) $\Rightarrow$ (3)”) in Theorem 31.

**B. Defining modal $\mu$-logic in matching $\mu$-logic**

To subsume the syntax, we define a signature (of transition systems) $\Sigma^{TS} = ((\text{State}), \bullet \in \Sigma^{TS}_{\text{State, State}})$ where we call the symbol "$\bullet" one-path next. We regard propositional variables in PVAR as set variables. We write $o \varphi$ instead of $o(\varphi)$, and define $o \varphi = o \varphi' = o \varphi'$. Then every modal $\mu$-logic formula $\varphi$ is an MmL $\Sigma^{TS}$-pattern of sort $\text{State}$. Finally, note that no axioms are needed; let $\Gamma'$ be the empty $\Sigma^{TS}$-theory.

An important observation is that the $\Sigma^{TS}$-models are exactly the transition systems, where $\bullet \in \Sigma^{TS}_{\text{State, State}}$ is interpreted as

3The modal $\mu$-logic literature often uses $\psi(o \varphi)$ and $\phi(o \varphi)$ instead of $\psi$ and $\phi$. We here use the latter to avoid confusion with the "always" $\psi$ and "eventually" $\phi$ in LTL and CTL.
the transition relation \( R \). Specifically, for any transition system \( S = (S,R) \), we can regard \( S \) as a \( \Sigma^{TS} \)-model where \( S \) is the carrier set of \( \text{State} \) and \( \bullet_s(t) = \{ s \in S \mid s \mathcal{R} t \} \) contains all \( R \)-predecessors of \( t \). This might seem counter-intuitive at the first glance: why “one-path next” is interpreted as the predecessor instead of the successor of \( R \)? See the following illustration:

\[
\begin{align*}
\ldots s & \xrightarrow{R} s' \xrightarrow{R} s'' \ldots & \text{ // states} \\
\bullet \varphi & \bullet \varphi & \varphi & \varphi & \text{ // patterns}
\end{align*}
\]

In other words, \( \bullet \varphi \) is matched by states which have at least one next state that satisfies \( \varphi \), conforming to the intuition. Another interesting observation is about \( \bullet \varphi \) and its dual, \( \varphi \equiv \cdots \bullet \neg \varphi \), called “all-path next”. The difference is that \( \varphi \) is matched by \( s \) if for all states \( t \) such that \( R \mathcal{R} t \), we have \( t \) matches \( \varphi \). In particular, if \( s \) has no successor, then \( s \) matches \( \varphi \) for all \( \varphi \). This is formally summarized in Proposition 32.

We now feel free to take any transition system \( S \) as an MmL \( \Sigma^{TS} \)-model. The following theorem shows that our definition of modal \( \mu \)-logic in MmL is faithful, both syntactically and semantically. What is interesting about the theorem is its proof, which can be applied to other all logics discussed in this paper, and obtain similar results for all of them.

**Theorem 31.** The following properties are equivalent for all modal \( \mu \)-logic formulas \( \varphi \): (1) \( \varphi \mu \varphi \); (2) \( \varphi \mu \mu \varphi \); (3) \( \Gamma \mu \varphi \); (4) \( \varphi \mu \varphi \); (5) \( M \models \varphi \) for all \( \Sigma^{TS} \)-models \( M \) such that \( M \models \Gamma \mu \); (6) \( S \models \varphi \mu \varphi \) for all transition systems \( S \).

**Proof sketch:** We only need to prove “(2) \( \Rightarrow \) (3)” and “(5) \( \Rightarrow \) (6)”, as the rest are already proved/known. (1) \( \Rightarrow \) (2) follows by the completeness of modal \( \mu \)-logic, which is nontrivial but known. (2) \( \Rightarrow \) (3) follows by proving all modal \( \mu \)-logic proof rules as theorems in MmL. (3) \( \Rightarrow \) (4) follows by the soundness of MmL (Theorem 24). (4) \( \Rightarrow \) (5) follows by definition. (5) \( \Rightarrow \) (6) follows by proving the contrapositive statement, “\( \varphi \mu \varphi \) implies \( \Gamma \mu \not\models \varphi \),” by taking a transition system \( S = (S,R) \) and a valuation \( V \) such that \( [\varphi]_V \not\models S \), and showing that if we regard \( S \) as a \( \Sigma^{TS} \)-model and \( V \) as an \( S \)-valuation in MmL, then \( S \models \mu \Gamma \) and \( [\varphi]_V \not\models S \), which means that \( \Gamma \mu \not\models \varphi \). Finally, (6) \( \Rightarrow \) (1) follows by definition.

Therefore, modal \( \mu \)-logic can be regarded as an empty theory in a vanilla MmL without quantifiers, over a signature containing only one sort and only one symbol, which is unary. It is worth mentioning that variants of modal \( \mu \)-logic with more modal operators have been proposed (see [33] for a survey). At our knowledge, however, all such variants consider only unary modal operators and they are only required to obey the usual (K) and (N) proof rules of modal logic. In contrast, MmL allows polyadic symbols while still obeying the desired (K) and (N) rules (see Proposition 12), allows arbitrary further constraining axioms in MmL theories, and also quantification over element variables and many-sorted universes.

**C. Studying transition systems in MmL**

The above suggests that MmL may offer a unifying playground to specify and reason about transition systems, by means of \( \Sigma^{TS} \)-theories/models. We can define various temporal/dynamic operations and modalities as derived constructs from the basic “one-path next” symbol “\( \bullet \)” and the \( \mu \)-binder, without the need to extend the syntax and semantics of the logic. We can constrain the models/transition systems of interest using additional axioms, without the need to modify/extend the proof system of the logic. In what follows, we show that by defining proper constructs as syntactic sugar and adding proper axioms, we can capture precisely LTL (both finite- and infinite-trace), CTL, dynamic logic (DL), and reachability logic (RL).

Let us add more temporal modalities as derived constructs (we have seen “all-path next” \( \varphi \) in Section VII-B):

- “eventually” \( \bigcirc \varphi \equiv \mu X. \varphi \lor \chi X \)
- “always” \( \Box \varphi \equiv \nu X. \varphi \lor \chi X \)
- “until” \( \varphi_1 U \varphi_2 \equiv \mu X. \varphi_2 \lor (\varphi_1 \land \chi X) \)
- “well-founded” \( \text{WF} \equiv \mu X. \varphi \lor X \)

**Proposition 32.** Let \( S = (S,R) \) be a transition system regarded as a \( \Sigma^{TS} \)-model, and let \( p \) be any valuation and \( s \in S \). Then:

- \( s \in \bar{p}(\varphi \bullet \varphi) \) if there exists \( t \in S \) such that \( s \mathcal{R} t \), \( t \in \bar{p}(\varphi) \);
- \( s \in \bar{p}(\varphi \mu \varphi) \) if for all \( t \in S \) such that \( s \mathcal{R} t \), \( t \in \bar{p}(\varphi) \); in particular, if \( s \in \bar{p}(\varphi \bigcirc \varphi) \) if \( s \) has no R-successor;
- \( s \in \bar{p}(\varphi \bigcirc \varphi) \) if there exists \( t \in S \) such that \( s \mathcal{R} t \), \( t \in \bar{p}(\varphi) \); in particular, if \( s \in \bar{p}(\varphi \bigcirc \varphi) \) if \( s \) has no R-successor;
- \( s \in \bar{p}(\varphi \bigcirc \varphi) \) if for all \( t \in S \) such that \( s \mathcal{R} t \), \( t \in \bar{p}(\varphi) \);
- \( s \in \bar{p}(\varphi \bigcirc \varphi) \) if there exists \( n \geq 0 \) and \( t_1, \ldots, t_n \in S \) such that \( s \mathcal{R} t_1 \cdots \mathcal{R} t_n \), \( t_n \in \bar{p}(\varphi) \), and \( s, t_1, \ldots, t_{n-1} \in \bar{p}(\varphi) \);
- \( s \in \bar{p}(\text{WF}) \) if \( s \) is R-well-founded, meaning that there is no infinite sequence \( t_1, t_2, \ldots, s \) with \( s \mathcal{R} t_1 \mathcal{R} t_2 \mathcal{R} \ldots \);

where \( R^* = \bigcup_{n \geq 0} R^n \) is the reflexive transitive closure of \( R \).

**D. Instances: temporal logics**

Since MmL can define modal \( \mu \)-logic (as an empty theory over a unary symbol), it is not surprising that it can also define various temporal logics such as LTL and CTL as theories whose axioms constrain the underlying transition relations. What is interesting, in our view, is that the resulting theories are simple, intuitive, and faithfully capture both the syntax (provability) and the semantics of these temporal logics.

1) **Instance: infinite-trace LTL.** The LTL syntax, namely

\[ \varphi ::= p \in \text{PVAR} \mid \varphi \land \varphi \mid \neg \varphi \lor \varphi \mid \varphi U \varphi \]

is already subsumed in MmL with the derived constructs we give in Section VII-C. Other common LTL modalities such as “always \( \bigcirc \varphi \)” are defined from the “until \( U \)” modality in the usual way. Infinite-trace LTL takes as models transition systems whose transition relations are linear and infinite into the future. We assume readers are familiar with the semantics and proof system of infinite-trace LTL (if not, see [10]) and skip their formal definitions. We use “\( \text{infLTL} \)” and “\( \text{infLTL} \)” to denote infinite-trace LTL validity and provability, respectively.

To capture the characteristics of both “infinite future” and “linear future”, we add the following two patterns as axioms:

\[ (\text{Inf}) \quad \bullet T \quad (\text{Lin}) \quad \varphi \lor \varphi \]

and denote the resulting \( \Sigma^{TS} \)-theory as \( \text{infLTL} \). Intuitively, (Inf) forces all states \( s \) to have at least one successor, and thus all traces are infinite, and (Lin) forces all states \( s \) to have only a linear future. The following theorem shows that our
Theorem 33. The following properties are equivalent for all infinite-trace LTL formulas $\varphi$: (1) $\models_{\text{infLTL}} \varphi$; (2) $\models_{\text{ctl}} \varphi$; (3) $\Gamma_{\text{ctl}} \vdash \varphi$; (4) $\Gamma_{\text{ctl}} \not\models \varphi$.

Therefore, infinite-trace LTL can be regarded as a theory containing two axioms, (Inf) and (Lin), over the same signature as the theory corresponding to modal $\mu$-logic.

2) Instance: finite-trace LTL: Finite execution traces play an important role in program verification and monitoring. Finite-trace LTL considers models that are linear but have only finite future. The following syntax of finite-trace LTL:

$\varphi ::= p \in \text{PVar} \mid \varphi \land \varphi \mid \neg \varphi \mid o\varphi \mid \varphi U_w \varphi$

differs from infinite-trace LTL in that the “until” modality “$U$” is weak, meaning that $\varphi_1 U_w \varphi_2$ does not necessarily imply that $\varphi_2$ eventually holds. Again, we assume readers are familiar with the semantics and proof system of finite-trace LTL (if not, see [10]) and use “$\models_{\text{finLTL}}$” and “$\Gamma_{\text{finLTL}}$” to denote its validity and provability, respectively.

To subsume the above syntax, we define in MmL:

“weak until” $\varphi_1 U_w \varphi_2 \equiv \mu X, \varphi_2 \land (\varphi_1 \land o X)$

To capture the characteristics of both finite future and linear future, we add the following two patterns as axioms:

(Fin) $WF \equiv \mu X, o X$ (Lin) $\bullet \varphi \rightarrow o\varphi$

and call the resulting $\Sigma_{\text{TS}}$-theory $\Gamma_{\text{finLTL}}$. Intuitively, (Fin) forces all states to be well-founded, meaning that there is no infinite execution trace in the underlying transition systems.

Theorem 34. The following properties are equivalent for all finite-trace LTL formula $\varphi$: (1) $\models_{\text{finLTL}} \varphi$; (2) $\models_{\text{ctl}} \varphi$; (3) $\Gamma_{\text{finLTL}} \vdash \varphi$; (4) $\Gamma_{\text{ctl}} \not\models \varphi$.

Therefore, finite-trace LTL can be regarded as a theory containing two axioms, (Fin) and (Lin), over the same signature as the theory corresponding to modal $\mu$-logic.

3) Instance: CTL: CTL’s models are transition systems which are infinite into the future and allow states to have a branching future (rather than linear). The syntax of CTL is

$\varphi ::= p \in \text{PVar} \mid \varphi \land \varphi \mid \neg \varphi \mid AX \varphi \mid EX \varphi \mid \varphi AU \varphi \mid \varphi EU \varphi$

extended with the following derived constructs:

$EF\varphi \equiv true \ \varphi \quad AG\varphi \equiv \neg EF \neg \varphi$

$AF\varphi \equiv true \ \varphi \quad EG\varphi \equiv \neg AG \neg \varphi$

The names of the CTL modalities suggest their meaning: the first letter means either “on all paths” (A) or “on one path” (E), and the second letter means “next” (X), “until” (U), “always” (G), or “eventually” (F). For example, “AX” is “all-path next”, “EU” is “one-path until”, etc. We refer readers to [34] for CTL definitions, semantics and proof system. Here we denote its validity and provability as “$\models_{\text{ctl}}$” and “$\Gamma_{\text{ctl}}$”, respectively.

To define CTL as an MmL theory, we add only the axiom (Inf) for infinite future and use the following syntactic sugar:

$AX\varphi \equiv o\varphi \quad \varphi_1 AU \varphi_2 \equiv \mu f, \varphi_2 \lor (\varphi_1 \land o f)$

$EX\varphi \equiv \varphi \quad \varphi_1 EU \varphi_2 \equiv \mu f, \varphi_2 \lor (\varphi_1 \land o f)$

The resulting $\Sigma_{\text{TS}}$-theory is denoted as $\Gamma^\text{ctl}$.

Theorem 35. For all CTL formula $\varphi$, the following are equivalent: (1) $\models_{\text{ctl}} \varphi$; (2) $\models_{\text{ctl}} \varphi$; (3) $\Gamma_{\text{ctl}} \vdash \varphi$; (4) $\Gamma_{\text{ctl}} \not\models \varphi$.

Therefore, CTL can be regarded as a theory over the same signature as the theory corresponding to modal $\mu$-logic, but containing one axiom, (Inf). It may be insightful to look at all three temporal logics discussed in this section through the lenses of MmL, as theories over a unary symbol signature: modal $\mu$-logic is the empty and thus the least constrained theory; CTL comes immediately next with only one axiom, (Inf), to enforce infinite traces; infinite-trace LTL further constrains with the linearity axiom (Lin); finally, finite-trace LTL replaces (Inf) with (Fin). We believe that MmL can serve as a convenient and uniform framework to define and study temporal logics. For example, finite-trace CTL can be trivially obtained as the theory containing only the axiom (Fin), LTL with both finite and infinite traces is the theory containing only the axiom (Lin), and CTL with unrestricted (finite or infinite branch) models is the empty theory (i.e., modal $\mu$-logic).

E. Instance: dynamic logic

Dynamic logic (DL) [11–13] is a common logic used for program reasoning. The DL syntax is parametric in a set PVar of propositional variables and a set APgm of atomic programs, each belonging to a different formula syntactic category:

$\varphi ::= p \in \text{PVar} \mid \varphi \rightarrow \varphi \mid false \mid [\alpha] \varphi$

$\alpha ::= a \in \text{APgm} \mid [\alpha] \mid [\alpha \lor \alpha] \mid [\alpha^*] \varphi$

The first line defines propositional formulas. The second line defines program formulas, which represent programs built from atomic ones with the primitive regular expression constructors. Define $[\alpha] \varphi \equiv [\alpha](\neg \varphi)$. Common program constructs such as if-then-else, while-do, etc., can be defined as derived constructs using the four primitive ones; see [11–13]. We let “$\models_{\text{DL}}$” and “$\Gamma_{\text{DL}}$” denote the validity and provability of DL.

It is known that DL can be embedded in the variant of modal $\mu$-logic with multiple modalities [33]. The idea is to define a modality $[\alpha]$ for every atomic program $a \in \text{APgm}$, and then to define the four program constructs as least/greatest fixpoints. We can easily adopt the same approach and associate an empty MmL theory to DL, over a signature containing as many unary symbols as atomic programs. However, MmL allows us to propose a better embedding, unrestricted by the limitations of modal $\mu$-logic. Indeed, the embedding in [33] suffers from at least two limitations that we can avoid with MmL. First, sometimes transitions are not just labeled with discrete programs, such as in hybrid systems [35] and cyber-physical systems [36] where the labels are continuous values such as elapsing time. We cannot introduce for every time $t \in R_{\geq}$ a modality $[t]$, as only countably many modalities are allowed. Instead, we may want to axiomatize the domains of such possibly continuous values and treat them as any other data. Second, we may want to quantify over such values, be they discrete or continuous, and we would not be able to do so (even in MmL) if they are encoded as modalities/symbols.

Let us instead define a signature (of labeled transition systems) $\Sigma = (\text{State}, \text{Pgm}, \Sigma_{\text{f}, \text{Pgm}} \cup \{ \bullet \in \Sigma_{\text{f}, \text{Pgm}} \text{State, State} \})$.
where the “one-path next” • is a binary symbol taking an additional Pgm argument, and for all atomic programs a ∈ APom we add a constant symbol a ∈ ΣTS. Just as all ΣTS-models are exactly transition systems (Section VII-B), we have that all ΣLTS-models are exactly labeled transition systems. We define compound programs as derived constructs as follows:

\[ \langle a \rangle \varphi \equiv \bullet (a, \varphi) \]

\[ [a] \varphi \equiv \neg (a) \neg \varphi \]

\[ (\text{Seq}) [a ; b] \varphi \equiv [a] \varphi \land [b] \varphi \]

\[ (\text{Choice}) [a \lor b] \varphi \equiv [a] \varphi \lor [b] \varphi \]

\[ (\text{Test}) [\psi ?] \varphi \equiv (\psi \rightarrow \varphi) \]

Like for the embedding of modal μ-logic (Section VII-B), no axioms are needed. Let ΓDL denote the empty ΣLTS-theory.

**Theorem 36.** For all DL formulas \( \varphi \), the following are equivalent: (1) \( \vdash_{DL} \varphi \); (2) \( \models_{DL} \varphi \); (3) \( \Gamma^{DL} \vdash \varphi \); (4) \( \Gamma^{DL} \models \varphi \).

We point out that the iterative operator \([a]^*\varphi\) is axiomatized with two axioms in the proof system of DL (see, e.g., [13]):

\[ (\text{DL-Iter}) \varphi \land [a]^* \varphi \leftrightarrow [a]^* \varphi \]

\[ (\text{DL-Iter}) \varphi \land [a]^* \varphi \rightarrow [a]^* \varphi \]

while we just regard it as syntactic sugar, via (Iter). One may argue that (Iter) desugars to the ν-binder, though, which obyes the proof rules (Pre-Fixpoint) and (Knsater-Tarski) that essentially have the same appearance as (DL-Iter1) and (DL-Iter2). We agree. And that is exactly why we think one should work in one uniform and fixed logic, such as MmL, where general fixpoint axioms are given to specify and reason about any fixpoint properties of any domains and to develop general-purpose automatic tools and provers, rather than designing special-purpose logics and tools that work on only certain domains and then extending existing logics or designing new logics when new domains are considered.

**VIII. Instance: reachability logic**

Reachability logic (RL) [2] is an approach to program verification using operational semantics. Different from other approaches such as Hoare-style verification, RL has a language-independent proof system that offers sound and relatively complete deduction for all languages. RL is the logic underlying the K framework [37], which has been used to define the formal semantics of various real languages such as C [3], Java [4], and JavaScript [5], yielding program verifiers for all these languages at no additional cost [6].

In spite of its generality w.r.t. languages, reachability logic is unfortunately limited to specifying and deriving only reachability properties. This limitation was one of the factors that motivated the development of MmL. Fig. 8 shows a few of RL’s proof rules; notice that unlike Hoare logic’s proof rules, RL’s proof rules are not specific to any particular programming language. The programming language is given through its operational semantics as a set of axiom rules, to be used via the (Axiom) proof rule. The characteristic feature of RL is its (Circularity) rule, which supports reasoning about circular behavior and recursive program constructs. In this subsection, we show how RL is faithfully defined in MmL and all its proof rules, including (Circularity), can be proved in MmL.

**Fig. 2.** Some selected proof rules in the proof system of reachability logic

**A. RL syntax, semantics, and proof system**

RL is parametric in a model of ML (without \( \mu \)) called the configuration model. Specifically, fix a signature (of static program configurations) \( \Sigma^{eq} \) which may have various sorts and symbols, among which there is a distinguished sort Config. Fix a \( \Sigma^{eq} \)-model \( M^{eq} \) called the configuration model, where \( M^{eq} \) is the set of all configurations. RL’s formulas are called reachability rules, or simply rules, and have the form \( \varphi_1 \Rightarrow \varphi_2 \) where \( \varphi_1, \varphi_2 \) are ML (without \( \mu \)) \( \Sigma^{eq} \)-patterns. A reachability system \( S \) is a finite set of rules, which yields a transition system \( S = (M^{eq}_{cfg}, R) \) where \( s \in S \) if there exist a rule \( \varphi_1 \Rightarrow \varphi_2 \) and a valuation \( \rho \) such that \( s \in \rho(\varphi_1) \) and \( t \in \rho(\varphi_2) \). A rule \( \varphi_1 \Rightarrow \varphi_2 \) is \( S \)-valid, denoted \( S \vdash \varphi_1 \Rightarrow \varphi_2 \), if for all \( M^{eq}_{cfg} \)-valuations \( \rho \) and configurations \( s = \rho(\varphi_1) \), either there is an infinite trace \( s \cdot R \cdot R \cdot \ldots \in S \) or there is a configuration \( t \) such that \( s \cdot R \cdot t \). Therefore, the validity in reachability logic is defined in the spirit of partial correctness.

The sound and relatively complete proof system of RL derives reachability logic sequents of the form \( C \vdash \varphi_1 \Rightarrow \varphi_2 \) where \( A \) (called axioms) and \( C \) (called circularities) are finite sets of rules. Initially we start with \( A = \emptyset \) and \( C = \emptyset \). As the proof proceeds, more rules can be added to \( C \) via (Circularity) and then moved to \( A \) via (Transitivity), which can then be used via (Axiom). We write \( S \vdash_{RL} \varphi_1 \Rightarrow \varphi_2 \) to mean that \( S \vdash_{RL} \varphi_1 \Rightarrow \varphi_2 \). Notice (Consequence) consults the configuration model \( M^{eq} \) for validity, so the completeness result is relative to \( M^{eq} \). We recall the following result [2]:

**Theorem 37.** For all reachability systems \( S \) satisfying some reasonable technical assumptions (see [2]) and all rules \( \varphi_1 \Rightarrow \varphi_2 \), we have \( S \vdash_{RL} \varphi_1 \Rightarrow \varphi_2 \) if and only if \( S \vdash_{RL} \varphi_1 \Rightarrow \varphi_2 \).

**B. Defining reachability logic in matching μ-logic**

We define the extended signature \( \Sigma^{RL} = \Sigma^{eq} \cup \{ \bullet \in \Sigma_{Cfg,Cfg} \} \) where “•” is a unary symbol (one-path next). To capture the semantics of reachability rules \( \varphi_1 \Rightarrow \varphi_2 \), we define:

“weak eventually” \( \varphi \Rightarrow \varphi \)

“reaching star” \( \varphi \Rightarrow^* \varphi \equiv \varphi \Rightarrow \varphi \)

“reaching plus” \( \varphi \Rightarrow^+ \varphi \equiv \varphi \Rightarrow \varphi \Rightarrow \varphi \)

Notice that the “weak eventually” \( \varphi \Rightarrow \varphi \) is defined similarly to “eventually” \( \varphi \Rightarrow \varphi \) but instead of using least fixpoint \( \mu \)-binder, we define it as a greatest fixpoint. One can prove that \( \varphi \Rightarrow \varphi = \neg \WF \lor \varphi \), that is, a configuration \( \gamma \)
satisfies $\diamond_w \varphi$ if either it satisfies $\varphi$, or it is not well-founded, meaning that there exists an infinite execution path from $\gamma$. Also notice that “reaching plus” $\varphi_1 \Rightarrow^+ \varphi_2$ is a stronger version of “reaching star”, requiring that $\diamond_w \varphi_2$ should hold after at least one step. This progressive condition is crucial to the soundness of RL reasoning: as shown in (Transitivity), circularities are flushed into the axiom set only after one reachability step is established. This leads us to the following translation from RL sequents to MmL patterns.

**Definition 38.** Given a rule $\varphi_1 \Rightarrow \varphi_2$, define the MmL pattern $\square (\varphi_1 \Rightarrow \varphi_2) \equiv \Box (\varphi_1 \Rightarrow^+ \varphi_2)$ and extend it to a rule set $A$ as follows: $\square A \equiv \bigwedge_{\varphi_1 \Rightarrow \varphi_2 \in A} \Box (\varphi_1 \Rightarrow \varphi_2)$. Define the translation $\text{RL2MmL}$ from RL sequents to MmL patterns as follows:

$$\text{RL2MmL}(A \vdash \varphi_1 \Rightarrow \varphi_2) = (\forall \Box A) \land (\forall \Box C) \rightarrow (\varphi_1 \Rightarrow^* \varphi_2)$$

where $* = *$ if $C$ is empty and $* = +$ if $C$ is nonempty. We use $\forall \varphi$ as a shorthand for $\forall x \varphi$ where $x = FV(\varphi)$.

Hence, the translation of $A \vdash \varphi_1 \Rightarrow \varphi_2$ depends on whether $C$ is empty or not. When $C$ is nonempty, the RL sequent is stronger in that it requires at least one step in $\varphi_1 \Rightarrow \varphi_2$. Axioms (those in $A$) are also stronger than circularities (those in $C$) in that axioms always hold, while circularities only hold after at least one step because of the leading all-path next “$\Rightarrow^*$”;

and since the “next” is a “weak” one, it does not matter which step is actually made as circularities hold on all next states.

**Theorem 39.** Let $\Gamma_{\text{RL}} = \{ \varphi \in \text{Pattern}_{\text{ML}} \mid M^{\text{eq}} \vdash \varphi \}$ be the set of all ML patterns (without $\mu$) of sort $\text{Cfg}$ that hold in $M^{\text{eq}}$. For all RL systems $S$ and rules $\varphi_1 \Rightarrow \varphi_2$ satisfying the same technical assumptions in [2], the following are equivalent: (1) $S \vdash_{\text{RL}} \varphi_1 \Rightarrow \varphi_2$; (2) $S \vdash_{\text{ML}} \varphi_1 \Rightarrow \varphi_2$; (3) $\Gamma_{\text{RL}} \vdash_{\text{RL2MmL}} (S \vdash_0 \varphi_1 \Rightarrow \varphi_2)$; (4) $\Gamma_{\text{RL}} \vdash_{\text{RL2MmL}} (S \vdash_0 \varphi_1 \Rightarrow \varphi_2)$.

Therefore, provided that an oracle for validity of ML patterns (without $\mu$) in $M^{\text{eq}}$ is available, the MmL proof system is capable of deriving any reachability property that can be derived with the RL proof system. This result makes MmL an even more fundamental logic framework for the $K$ approach and thus for programming language specification and verification than RL, because it can express significantly more properties than partial correctness reachability.

**IX. Future and Related Work**

We discuss future work, open problems, and related work.

**A. Relation to modal logics**

Due to the duality between ML symbols and modal logic modalities (Section [III], Proposition [12]), ML can be regarded as a non-trivial extension of modal logics. There are various directions to extend the basic propositional modal logic in the literature [17]. One is the hybrid extension, where first-order quantifiers “$\forall$” and “$\exists$” are added to the logic, as well as state variables/names that allow to specify one particular state. Another is the polyadic extension, where modalities can take not just one argument, but any number of arguments, and there can be multiple modalities. ML can be seen as a combination of both extensions, further extended with multiple sort universes. The completeness of $\mathcal{H}$ (Theorem [16]) also extends the completeness results of its fragment logics, including hybrid modal logic [18] and many-sorted polyadic modal logic [38].

**B. Stronger completeness results of $\mathcal{H}$**

There are various notions of completeness in modal logics. We give three of them under the context of ML and its proof system $\mathcal{H}$, from the strongest to the weakest:

- Global completeness: $\Gamma \vdash_{\text{ML}} \varphi$ implies $\Gamma \vdash_{\text{H}} \varphi$;
- Strong local completeness: $\Gamma \vdash_{\text{ML}} \varphi$ implies $\Gamma \vdash_{\text{loc}} \varphi$;
- Weak local completeness: $\Gamma \vdash_{\text{ML}} \varphi$ implies $\Gamma \vdash_{\text{loc}} \varphi$;

where $\Gamma \vdash_{\text{ML}} \varphi$, called local semantic entailment, is defined as for all models $M$, all valuations $\rho$, and all $a \in M$, if $a \in \rho(\psi)$ for all $\psi \in \Gamma$ then $a \in \rho(\varphi)$; $\Gamma \vdash_{\text{loc}} \varphi$, called local provability, is defined as there exists a finite subset $\Gamma \subseteq \Gamma$ such that $\Gamma \vdash_{\text{H}} \varphi$.

The completeness result for $\mathcal{H}$ that we present in Theorem [16] is a weak local completeness result, but the way we actually prove it is by proving the strong local completeness theorem and then let $\Gamma = \emptyset$. We did not present in this paper the strong local completeness theorem due to its complex form.

What is not known and left as future work is global completeness. Theorem [15] shows that global completeness holds when $\Gamma$ contains definedness symbols and axioms.

**C. Alternative semantics of matching $\mu$-logic**

MmL cannot have a sound and complete proof system because we can precisely define $\langle N, +, \times \rangle$ (see Proposition [23]). On the other hand, the proof system $\mathcal{H}_{\mu}$ turned out to be strong enough to prove all the proof rules of all the proof systems of all the logics discussed in this paper. Therefore, a natural question is whether we can find alternative models for MmL that make $\mathcal{H}_{\mu}$ complete. A promising direction towards such an alternative semantics is to consider the so-called Henkin semantics or general semantics, where the least fixpoint pattern $\mu X. \varphi$ is not evaluated to the true least fixpoint in the models, but to the least fixpoint that is definable in the logic.

**X. Conclusion**

We made two main contributions in this paper. Firstly, we proposed a new sound and complete proof system $\mathcal{H}$ for matching logic (ML). Secondly, we extended ML with the least fixpoint $\mu$-binder and proposed matching $\mu$-logic (MmL). We showed the expressiveness of MmL by defining a variety of common logics about induction/fixedpoints/verification in MmL. We hope that MmL may serve as a promising unifying foundation for specifying and reasoning about induction, fixpoints, as well as model checking and program verification.

**References**


Appendix A
Matching Logic Proof System $\mathcal{P}$

We show the matching logic proof system $\mathcal{P}$ proposed in Fig. 8.

Appendix B
Proof of Theorem 13

We prove the soundness theorem of $\mathcal{H}$ (Theorem 13). We only discuss ML (without $\mu$) in this section, so we drop all unnecessary annotations. Specifically, we abbreviate “$\Pi_{\text{ML}}$” as “$\Pi$” and “$\Pi_{\text{ML}}^c$” as “$\Pi_c$.”

We write $\rho_1 \not\simeq \rho_2$ to mean that $\rho_1, \rho_2$ differ only at $x$.

As in FOL, we can prove that $\bar{\rho}(\forall x. \varphi) = \bigcap_{a \in M} \bar{\rho}[a/x](\varphi) = \bigcap \{\bar{\rho}(\varphi) \mid \rho' \simeq \rho\}$ and $\bar{\rho}(\exists x. \varphi) = \bigcup_{a \in M} \bar{\rho}[a/x](\varphi) = \bigcup \{\bar{\rho}(\varphi) \mid \rho' \not\simeq \rho\}$.

**Lemma 40.** The following propositions hold:

1. $\varphi \vdash \varphi \Rightarrow \varphi (\text{Reflexivity})$
2. $\varphi \vdash \varphi \rightarrow \psi \rightarrow \varphi (\text{Modus Ponens})$
3. $\Gamma, \varphi \vdash \psi \Rightarrow \Gamma, \varphi \vdash \psi \rightarrow \gamma (\text{Simplification})$
4. $\Gamma, \varphi \vdash \psi \rightarrow \varphi \Rightarrow \Gamma, \varphi \vdash \psi \rightarrow \varphi (\text{Generalization})$
5. $\varphi \Leftrightarrow \psi \vdash \varphi \Leftrightarrow \psi (\text{Introduction})$
6. $\varphi \Leftrightarrow \psi \vdash \psi \Leftrightarrow \varphi (\text{Elimination})$

Proof: Some of the propositions are proved in [11]. To make this proof self-contained, we present the proofs of all propositions. Let $M$ be a model and $\rho$ be a valuation.

1. $\bar{\rho}(\varphi \rightarrow \psi) = \bigcap_{a \in M} \bar{\rho}[a/x](\varphi) \cap \bigcap_{a \in M} \bar{\rho}[a/x](\psi)$
2. $\bar{\rho}(\varphi \rightarrow \psi \rightarrow \gamma) = \bigcap_{a \in M} \bar{\rho}[a/x](\varphi) \cap \bigcap_{a \in M} \bar{\rho}[a/x](\psi) \cap \bigcap_{a \in M} \bar{\rho}[a/x](\gamma)$
3. $\varphi \rightarrow \psi \rightarrow \gamma = \bigcap_{a \in M} \bar{\rho}[a/x](\varphi) \cap \bigcap_{a \in M} \bar{\rho}[a/x](\psi) \cap \bigcap_{a \in M} \bar{\rho}[a/x](\gamma)$

Now we are ready to prove Theorem 13.

**Proof of Theorem 13**

Carry out induction on the length of the Hilbert-style proof $\Gamma \vdash \varphi$.

(Base Case). Suppose the length is 1. Then $\varphi$ is either an axiom in $\mathcal{H}$ or $\varphi \in \Gamma$. If $\varphi$ is an axiom, then $\varphi$ by Lemma 40.

(Induction Step). Suppose the proof $\Gamma \vdash \varphi$ has $n + 1$ steps:

...$
\varphi_{n+1} \equiv \varphi$

By induction hypothesis, $\Gamma \equiv \varphi, \ldots, \varphi_{n+1} \equiv \varphi$. If $\varphi$ is an axiom or $\varphi \in \Gamma$, then $\varphi$ by the same reason as in (Base Case). If the last step is one of (Modus Ponens), (Universal Generalization), or (Framing), then $\Gamma \equiv \varphi$ by Lemma 40, respectively.

Appendix C
Properties of Proof System $\mathcal{H}$

We discuss properties of $\mathcal{H}$. In particular, we prove Proposition 12 and Theorem 14.

Our final goal is to prove all proof rules in $\mathcal{P}$ using the proof system $\mathcal{H}$ (plus (Definedness) axioms, i.e., Theorem 15).

We only discuss ML (without $\mu$) in this section, so we drop all unnecessary annotations. Specifically, we abbreviate “$\Pi_{\text{ML}}$” as...
as “c” and “c+1” as “c”. We sometimes call a nested symbol context just a symbol context (see Definition 10).

Proposition 41 (Sound FOL Reasoning). Let $\Sigma = (S, \Sigma)$ be a matching logic signature. Let $(S, \Pi, F)$ be any first-order logic signature with $\Pi = \{\Pi_i\}_{i \in S}$ a set of constant predicate symbols and $F = \emptyset$ a set of function symbols. For any predicate logic formula $\Psi(\pi_1, \ldots, \pi_n)$ where $\pi_1, \ldots, \pi_n \in \Pi$, if $\Psi(\pi_1, \ldots, \pi_n)$ is derivable in FOL, then $\Psi(\pi_1, \ldots, \pi_n)$ is derivable in matching logic, where $\pi_i$ has the same sort as $\pi_i$ for $1 \leq i \leq n$.

Proof: Notice that $F = \emptyset$, so the only FOL terms are variables. Under that condition, the first seven rules in Fig. 1 form a complete FOL proof system as in [16].

Proposition 42 (Sound Frame Reasoning). For any $\sigma \in \Sigma_{\pi_1, \ldots, \pi_n, x_i}$ and $\pi_i, \pi_i' \in \text{pattern}_{\pi_i}$ such that $\Gamma \vdash \pi_i \rightarrow \pi_i'$ for any $1 \leq i \leq n$, then $\Gamma \vdash \sigma(\pi_1, \ldots, \pi_n) \rightarrow \sigma(\pi_1', \ldots, \pi_n')$. For any symbol context $C$ and $\phi, \phi'$ such that $\Gamma \vdash \phi_1 \rightarrow \phi_1'$, then $\Gamma \vdash C[\phi] \rightarrow C[\phi']$.

Proof: For the first case, it suffices to show that

$$\Gamma \vdash \sigma(\pi_1, \pi_2, \ldots, \pi_n) \rightarrow \sigma(\pi_1', \pi_2, \ldots, \pi_n)$$

$$\Gamma \vdash \sigma(\pi_1', \pi_2, \ldots, \pi_n) \rightarrow \sigma(\pi_1', \pi_2, \ldots, \pi_n)$$

$$\cdots$$

$$\Gamma \vdash \sigma(\pi_1', \pi_2, \ldots, \pi_n) \rightarrow \sigma(\pi_1', \pi_2, \ldots, \pi_n)$$

which directly follow by (Framing).

For the second case, the proof is by structure induction on $C$. If $C$ is the identity context, the conclusion is obvious. If $C$ has the form $C_i[C']$, the conclusion follows from induction hypothesis and (Framing).

Proposition 43 (Propagation through Symbol Contexts). For any symbol context $C$ and patterns $\phi_1, \phi_2, \phi$, the following propositions hold.

- $\Gamma \vdash C[\bot] \leftrightarrow \bot$
- $\Gamma \vdash C[\phi_1 \lor \phi_2] \leftrightarrow C[\phi_1] \lor C[\phi_2]$
- $\Gamma \vdash C[\exists x. \varphi] \leftrightarrow \exists x. C[\varphi]$ if $x \notin FV(C[\exists x. \varphi])$

The following results are often useful in practice, whose proofs can be obtained by standard propositional reasoning with the above propositions:

- $\Gamma \vdash C[\phi_1 \lor \phi_2] \iff \Gamma \vdash C[\phi_1] \lor C[\phi_2]$
- $\Gamma \vdash C[\exists x. \varphi] \iff \Gamma \vdash \exists x. C[\varphi]$ if $x \notin FV(C[\exists x. \varphi])$

Proof: The proof is by structure induction on the symbol context $C$. If $C$ is the identity context then the conclusion is obvious. Now assume $C = C_i[C']$ where $C'$ is a symbol context for which the conclusion holds.

Firstly, let us prove $\Gamma \vdash C_i[C'[\bot]] \leftrightarrow \bot$. The implication from right to left is by simple propositional reasoning. For the other direction, notice by induction hypothesis $\Gamma \vdash C'[\bot] \leftrightarrow \bot$ and by (Framing) $\Gamma \vdash C_i[C'[\bot]] \rightarrow C_i[\bot]$. In addition by (Propagation$_\bot$), $\Gamma \vdash C_i[\bot] \rightarrow \bot$, and the rest of the proof is by standard propositional reasoning.

Secondly, let us prove $\Gamma \vdash C_i[C'[\phi_1 \lor \phi_2]] \leftrightarrow C_i[C'[\phi_1]] \lor C_i[C'[\phi_2]]$. For the implication from right to left, it suffices to prove $\Gamma \vdash C_i[C'[\phi]] \rightarrow C_i[C'[\phi_1 \lor \phi_2]]$ for $i = 1, 2$. By (Framing), it suffices to prove $\Gamma \vdash C'[\phi] \rightarrow C'[\phi_1 \lor \phi_2]$, which follows from the induction hypothesis. For the implication from left to right, the proof is the same as above.
Proposition 44 (Congruence of Provably Equivalence). For any context C (not necessarily just symbol context), \( \Gamma \vdash \varphi_1 \leftrightarrow \varphi_2 \) implies \( \Gamma \vdash C[\varphi_1] \leftrightarrow C[\varphi_2] \).

Proof: The proof is by induction on the structure of C. If C is the identity context the conclusion is obvious. If C is of the form \( \neg C' \), \( \psi \rightarrow C' \), or \( C' \rightarrow \psi \) where C' is a context and \( \psi \) is a pattern (notice \( \psi \) does not have the placeholder variable \( \square \) in it), the conclusion is by standard propositional reasoning. If C has the form \( \forall x.C' \), the conclusion follows from standard FOL reasoning. If C has the form \( C_\sigma[C'] \), the conclusion follows from Proposition 42.

Proposition 44 allows us to in-place replace \( \varphi_1 \) for \( \varphi_2 \) in any context as long as \( \Gamma \vdash \varphi_1 \leftrightarrow \varphi_2 \), which is obviously a powerful and convenient result. In some of the following proofs, when we carry out structural induction on a pattern \( \psi \), we take \( I = \{ \wedge, \neg, \exists, \} \) as primitives instead of \( J = \{ \rightarrow, \neg, \forall \} \) for technical simplicity. Proposition 44 justifies this approach, as we can transform any pattern \( \varphi \) to another pattern, say \( \varphi' \), that uses only constructs in \( I \) and \( \Gamma \vdash \varphi \leftrightarrow \varphi' \). Then, \( \varphi \) and \( \varphi' \) are interchangeable in any context.

Definition 45. Define the dual of a symbol \( \sigma \) as follows:

\[
\bar{\sigma}(\varphi_1, \ldots, \varphi_n) \equiv \neg \sigma(\neg \varphi_1, \ldots, \neg \varphi_n).
\]

Lemma 46. \( \Gamma \vdash \varphi \) implies \( \Gamma \vdash \neg C[\neg \varphi] \) for symbol context C.

Proof:

\[
\begin{array}{|c|}
\hline
1 & \varphi & \text{hypostthesis} \\
2 & \neg \varphi \rightarrow \bot & \text{by 1, FOL reasoning} \\
3 & C[\neg \varphi] \rightarrow C[\bot] & \text{by 2, (FRAMING)} \\
4 & C[\bot] \rightarrow \bot & \text{by (PROPAGATION)} \\
5 & C[\neg \varphi] \rightarrow \bot & \text{by 3 and 4, FOL reasoning} \\
6 & \neg C[\neg \varphi] & \text{by 5, FOL reasoning} \\
\hline
\end{array}
\]

Now we are ready to prove Proposition 12.

Proof of Proposition 12. Let the single symbol context \( C_\sigma = \sigma(\varphi_1, \ldots, \varphi_{i-1}, \exists \varphi_i, \ldots, \varphi_n) \) for some symbol \( \sigma \in \Sigma \).

(K). Note that we just need to prove the case of one argument, i.e., to prove \( \Gamma \vdash \neg C_\sigma[\neg(\varphi \rightarrow \varphi')] \rightarrow \neg C_\sigma[\neg \varphi] \rightarrow \neg C_\sigma[\neg \varphi'] \). The case of multiple arguments can be incrementally proved by simple propositional reasoning.

To prove the “one argument” case, we apply simple propositional reasoning and obtain \( \Gamma \vdash C_\sigma[\varphi \land \varphi'] \lor C_\sigma[\neg \varphi] \lor \neg C_\sigma[\neg \varphi'] \). By Proposition 42, the goal becomes \( \Gamma \vdash C_\sigma[\varphi \land \varphi'] \lor C_\sigma[\neg \varphi] \lor \neg C_\sigma[\neg \varphi'] \lor \neg C_\sigma[\neg \varphi'] \lor \neg C_\sigma[\neg \varphi'] \). By Proposition 42 again, we obtain \( \Gamma \vdash C_\sigma[\varphi'] \lor C_\sigma[\neg \varphi] \lor \neg C_\sigma[\neg \varphi'] \lor \neg C_\sigma[\neg \varphi'] \lor \neg C_\sigma[\neg \varphi'] \lor \neg C_\sigma[\neg \varphi'] \). Done.

(N) is proved in Lemma 46, letting C to be \( C_\sigma \).

In what follows, we move towards proving Theorem 15 by showing that all proof rules of \( \mathcal{P} \) in Fig. 3 can be proved in \( \mathcal{H} \). We will need (a lot of) lemmas.

The next lemma is useful in establishing an equality.

Lemma 47. \( \Gamma \vdash \varphi_1 \leftrightarrow \varphi_2 \) implies \( \Gamma \vdash \varphi_1 = \varphi_2 \).

Proof:

\[
\begin{array}{|c|}
\hline
1 & \varphi_1 \leftrightarrow \varphi_2 & \text{hypostthesis} \\
2 & \neg[\neg(\varphi_1 \leftrightarrow \varphi_2)] & \text{by 1, Lemma 46} \\
3 & \varphi_1 = \varphi_2 & \text{by 2, definition of equality} \\
\hline
\end{array}
\]

Lemma 48. (Equality Introduction) can be proved in \( \mathcal{H} \).

Proof:

\[
\begin{array}{|c|}
\hline
1 & \varphi \leftrightarrow \varphi & \text{propositional tautology} \\
2 & \varphi = \varphi & \text{by 1, Lemma 47} \\
\hline
\end{array}
\]

Lemma 49. (Membership Introduction) can be proved in \( \mathcal{H} \).

Proof:

\[
\begin{array}{|c|}
\hline
1 & \forall x.(x \in \varphi) & \text{by 1, Lemma 47} \\
2 & \forall x.(x \rightarrow \varphi) & \text{(PROPOSITION)} \\
3 & x \rightarrow \varphi & \text{by 1 and 2, (MODUS PONENS)} \\
4 & x \rightarrow x & \text{propositional tautology} \\
5 & x \rightarrow x \land \varphi & \text{by 3 and 4, FOL reasoning} \\
6 & [x] \rightarrow [x \land \varphi] & \text{by 5, (FRAMING)} \\
7 & [x] & \text{definedness axiom} \\
8 & [x \land \varphi] & \text{by 6 and 7, (MODUS PONENS)} \\
9 & x \in \varphi & \text{by 8, definition of membership} \\
10 & \forall x. (x \in \varphi) & \text{by 9, (UNIVERSAL GENERALIZATION)} \\
\hline
\end{array}
\]

Lemma 50. (Membership Elimination) can be proved in \( \mathcal{H} \).

Proof:

\[
\begin{array}{|c|}
\hline
1 & \forall x. (x \in \varphi) & \text{hypostthesis} \\
2 & \forall y. (x \in \varphi) & \text{(VARIABLE SUBSTITUTION)} \\
3 & x = x & \text{by 1} \\
4 & x \in \varphi & \text{by 1 and 2, (MODUS PONENS)} \\
5 & [x \land \varphi] & \text{by 3, definition of membership} \\
6 & (x \land \varphi) & \text{by 4 and 5, (SINGLETON VARIABLE)} \\
7 & x \land \varphi & \text{by 6, FOL reasoning} \\
8 & x \rightarrow \varphi & \text{by 4 and 6, (MODUS PONENS)} \\
9 & \forall x. (x \rightarrow \varphi) & \text{by 7, (UNIVERSAL GENERALIZATION)} \\
10 & \exists x. x & \text{by 8, FOL reasoning} \\
11 & \varphi & \text{by 10 and 9, (MODUS PONENS)} \\
\hline
\end{array}
\]

Lemma 51. (Membership Variable) can be proved in \( \mathcal{H} \).

Proof: By Lemma 47, we just need to prove both \( \vdash (x \in y) \rightarrow (x = y) \) and \( \vdash (x = y) \rightarrow (x \in y) \). We first prove \( \vdash (x = y) \rightarrow (x \in y) \).
Lemma 52. (Membership) can be proved in $\mathcal{H}$.

Proof: We first prove $\vdash (x \in \neg \varphi) \rightarrow \neg (x \in \varphi)$.

1. $\neg((x \in \varphi) \land [x \in \neg \varphi])$ by (Singleton Variable)
2. $[x \in \neg \varphi] \rightarrow \neg (x \in \varphi)$ by 1, FOL reasoning
3. $(x \in \neg \varphi) \rightarrow (x \in \varphi)$ by 2, definition

We then prove $\vdash \neg (x \in \varphi) \rightarrow (x \in \neg \varphi)$.

1. $[x]$ definedness axiom
2. $[x] \lor [\varphi]$ by 1, FOL reasoning
3. $[x \lor \varphi]$ by 2, FOL reasoning
4. $\neg (x \lor \varphi)$ by 3, FOL reasoning
5. $[x \lor \neg \varphi] \lor [\neg x \land x]$ by 4, Proposition 43
6. $\neg (x \lor \neg \varphi) \lor \neg (x \land x)$ by 5, FOL reasoning
7. $\neg (x \lor \neg \varphi) \lor (x \land x)$ by 6, FOL reasoning
8. $[x \land x] \rightarrow (x = x)$ by definition

Lemma 53. $\vdash (x \in (\varphi_1 \lor \varphi_2)) \leftrightarrow (x \in \varphi_1) \lor (x \in \varphi_2)$.

Proof: Use (Propagation$_\lor$) and FOL reasoning.

Lemma 54. (Membership$_\land$) can be proved in $\mathcal{H}$.

Proof: Use Lemma 52 and 53 and the fact that $\vdash \varphi_1 \land \varphi_2 \leftrightarrow \neg (\neg \varphi_1 \lor \neg \varphi_2)$.

Lemma 55. (Membership$_\exists$) can be proved in $\mathcal{H}$.

Proof: Use (Propagation$_\exists$) and FOL reasoning.

Lemma 56. $\vdash C[\varphi] \rightarrow [\varphi]$ for any symbol context $C$.

Proof: Let $x$ be a fresh variable in the following proof.

- $\neg((x \in \varphi) \land [x \in \neg \varphi])$
- $[x \land \neg \varphi] \lor \neg (x \land x)$
- $\neg (x \land \neg \varphi) \lor \neg (x \land x)$
- $\neg (x \land \neg \varphi) \lor (x \land x)$
- $[x \land x] \rightarrow (x = x)$

We then prove $\vdash (x \in \varphi) \rightarrow (x = x)$.

- $\neg((x \in \varphi) \land \neg (x \in \varphi))$
- $[x \lor \varphi] \rightarrow \neg (x \in \varphi)$
- $[x \lor \neg \varphi] \lor [\neg x \land x]$ by 1, FOL reasoning
- $\neg (x \lor \neg \varphi) \lor \neg (x \land x)$ by 2, Proposition 43
- $\neg (x \lor \neg \varphi) \lor (x \land x)$ by 3, FOL reasoning
- $[x \land x] \rightarrow (x = x)$ by definition

Lemma 57. $\vdash C[\varphi] \rightarrow [\varphi]$ and $\vdash [\varphi] \rightarrow \neg C[\neg \varphi]$ for all symbols $\sigma$. In particular, $\vdash \varphi \rightarrow [\varphi]$ and $\vdash [\varphi] \rightarrow \varphi$.

We are now ready to prove the deduction theorem (Theorem 14).

Proof of Theorem 14: Carry out induction on the length of the proof $\Gamma \vdash (\psi \rightarrow \varphi)$.

(Base Case) Suppose the length is one, then either $\psi$ is an axiom in $\mathcal{H}$ or $\varphi \in \Gamma \cup \{\psi\}$. In either case, it is obvious that $\Gamma \vdash [\psi] \rightarrow \varphi$ (noticing Corollary 57 for the case $\varphi = \psi$).

(Induction Step) Suppose the proof $\Gamma \vdash \psi \rightarrow \varphi$ has $n + 1$ steps:

$$\psi_1, \ldots, \psi_n, \psi.$$ If $\varphi$ is an axiom in $\mathcal{H}$ or $\varphi \in \Gamma \cup \{\psi\}$, then $\Gamma \vdash [\psi] \rightarrow \varphi$ for the same reason as (Base Case). If the last step is (Modus Ponens) on $\psi_i$ and $\psi_j$ for some $1 \leq i, j \leq n$ such that $\psi_j$ has the form $\phi_i \rightarrow \varphi$, by induction hypothesis, $\Gamma \vdash [\psi_i] \rightarrow \phi_i$ and $\Gamma \vdash [\phi_i] \rightarrow [\varphi_i]$. By FOL reasoning, $\Gamma \vdash [\psi_i] \rightarrow [\phi_i]$. If the last step is (Universal Generalization) on $\phi_i$ for some $1 \leq i \leq n$, then $\varphi_i$ must have the form $\forall x. \psi_i$, where $x$ does not occur free in $\psi_i$. By induction hypothesis, $\Gamma \vdash [\psi_i] \rightarrow [\phi_i]$. By FOL reasoning, $\Gamma \vdash [\psi_i] \rightarrow \forall x. \psi_i$.

If the last step is (Framing) on $\phi_i$ for some $1 \leq i \leq n$, then $\psi_i$ must have the form $\phi_i \rightarrow \phi_i'$, and $\varphi_i$ must have the form $C[\phi_i'] \rightarrow C[\phi_i']$ for some symbol $\sigma$. By induction hypothesis, $\Gamma \vdash [\psi_i] \rightarrow [\phi_i']$. We now prove $\Gamma \vdash [\psi_i] \rightarrow (C[\phi_i'] \rightarrow C[\phi_i'])$.

1. $[\psi_i] \rightarrow (\phi_i' \rightarrow \phi_i')$
2. $\phi_i' \rightarrow \phi_i' \lor [\neg \psi_i]$ by 1, FOL reasoning
3. $C[\neg [\neg \psi_i]] \rightarrow [\neg \psi_i]$ by 2, (Framing)
4. $C[\phi_i'] \rightarrow [\neg \psi_i]$ by 3, FOL reasoning
5. $C[\phi_i'] \lor [\neg \psi_i]$ by 4, Proposition 43
6. $C[\phi_i'] \lor [\neg \psi_i]$ by 3, FOL reasoning
7. $C[\phi_i'] \rightarrow C[\phi_i'] \lor [\neg \psi_i]$ by 5, 6, FOL reasoning
8. $[\psi_i] \rightarrow (C[\phi_i'] \rightarrow C[\phi_i'])$ by 7, FOL reasoning
Lemma 58. **(Equality Elimination)** can be proved in $\mathcal{H}$.

**Proof:** Recall the definition of equality ($\varphi_1 = \varphi_2$) as $[\varphi_1 \leftrightarrow \varphi_2]$. Theorem [14] together with Proposition [44] give us a nice way to deal with equality premises. To prove $\vdash (\varphi_1 = \varphi_2) \rightarrow (\psi[\varphi_1/x] \rightarrow \psi[\varphi_2/x])$, we apply Theorem [14] and prove $\{\varphi_1 \leftrightarrow \varphi_2\} \vdash [\psi[\varphi_1/x] \rightarrow \psi[\varphi_2/x]]$, which is proved by Proposition [44]. Note that the (formal) proof given in Proposition [44] does not use (Universal Generalization) at all, so the conditions of Theorem [14] are satisfied.

Lemma 59. **(Functional Substitution)** can be proved in $\mathcal{H}$.

**Proof:** Let $z$ be a fresh variable that does not occur free in $\varphi$ and $\varphi'$, and is distinct from $x$. Notice the side condition that $y$ does not occur free in $\varphi'$.

We then prove $\vdash \varphi \rightarrow \exists y.(\lceil y \land \varphi \rceil \land y)$. Let $x$ be a fresh variable distinct from $y$.

| 1 | $\varphi' = z \leftrightarrow z = \varphi'$ | definition |
| 2 | $z = \varphi' \rightarrow (\varphi[z/x] \rightarrow \varphi'[x])$ | Lemma 58 |
| 3 | $(\forall x. \varphi) \rightarrow \varphi[z/x]$ | by axiom |
| 4 | $\varphi' = z \rightarrow ((\forall x. \varphi) \rightarrow \varphi'[x])$ | FOL reasoning |
| 5 | $\varphi' = z \rightarrow ((\forall x. \varphi) \rightarrow \varphi'[x])$ | FOL reasoning |
| 6 | $\varphi' = z \rightarrow ((\forall x. \varphi) \rightarrow \varphi'[x])$ | FOL reasoning |
| 7 | $\forall x.(z \rightarrow ((\forall x. \varphi) \rightarrow \varphi'[x]))$ by 6 |
| 8 | $(\exists z. \varphi' = z) \rightarrow ((\forall x. \varphi) \rightarrow \varphi'[x])$ FOL reasoning |
| 9 | $(\forall x. \varphi) \land (\exists z. \varphi' = z) \rightarrow \varphi'[x]$ FOL reasoning |
| 10 | $(\forall x. \varphi) \land (\exists z. \varphi' = y) \rightarrow \varphi'[x/y]$ FOL reasoning |

Lemma 60. $\vdash C_r[\varphi_1 \land (x \in \varphi_2)] = C_r[\varphi_1] \land (x \in \varphi_2)$.

**Proof:** We first prove $\vdash C_r[\varphi_1 \land (x \in \varphi_2)] \rightarrow C_r[\varphi_1] \land (x \in \varphi_2)$. By FOL reasoning, it suffices to show both $\vdash C_r[\varphi_1 \land (x \in \varphi_2)] \rightarrow C_r[\varphi_1]$ and $\vdash C_r[\varphi_1 \land (x \in \varphi_2)] \rightarrow (x \in \varphi_2)$. The first follows immediately by (Framing) and FOL reasoning. The second can be proved as:

| 1 | $\{x\}$ |
| 2 | $\{x \land \neg \varphi_2\} \lor (x \land \varphi_2)$ |
| 3 | $\{x \land \neg \varphi_2\} \lor \{x \land \varphi_2\}$ |
| 4 | $\neg \{x \land \neg \varphi_2\} \rightarrow \{x \land \varphi_2\}$ |
| 5 | $C_r[\{x \land \varphi_2\}] \rightarrow \neg \{x \land \neg \varphi_2\}$ |
| 6 | $C_r[\{x \land \varphi_2\}] \rightarrow \{x \land \varphi_2\}$ |
| 7 | $C_r[\varphi_1 \land (x \in \varphi_2)] \rightarrow C_r[\{x \land \varphi_2\}]$ |
| 8 | $C_r[\varphi_1 \land (x \in \varphi_2)] \rightarrow \{x \land \varphi_2\}$ |
| 9 | $C_r[\varphi_1 \land (x \in \varphi_2)] \rightarrow (x \in \varphi_2)$ |

Lemma 61. $\vdash \exists y.(\lceil x = y \rceil \land \varphi) = \varphi[x/y]$ where $x$, $y$ distinct.

**Proof:** The proof is by induction on the structural of $\varphi$ and Lemma 60.

Lemma 62. $\vdash \varphi = \exists y.(\lceil y \land \varphi \rceil \land y)$ if $y \not\in \text{FV}(\varphi)$.

**Proof:** We first prove $\vdash \neg ((\forall y. \varphi[y/x]) \rightarrow \varphi[y/x]) \rightarrow \varphi$. Carry out induction on the structure of $\varphi$. The only nontrivial case is when $\varphi \equiv \exists z. \psi$. Without loss of generality, let us assume $z$ is distinct from $x$ and $y$. If not, apply $\alpha$-renaming to make them different. Then

| 1 | $\neg ((\forall y. \varphi[y/x]) \rightarrow \varphi[y/x]) \land \varphi[y/x]$ (Singleton Variable) |
| 2 | $(\forall y. \varphi[y/x] \land y) \rightarrow \varphi$ by 1, FOL reasoning |
| 3 | $(\forall y. \varphi[y/x] \land y) \rightarrow \varphi$ by 2, axiom |
| 4 | $\exists y.(\lceil y \land \varphi \rceil \land y) \rightarrow \varphi$ by 3, FOL reasoning |
that patterns in matching logic match elements in domain. A useful intuition about consistent sets \( \varphi, \varphi \)
and semantics, (see Lemma 79). They play an important role and thus \( \text{MCS} \), adding any pattern
there exist a model \( M \) contains patterns that cannot match common elements in any
local property. Pattern set \( \Gamma \) every \( \varrho \). When \( \varrho \) or \( \varrho \), there
is a variable \( y \) such that \( (\exists x. \varphi) \rightarrow \varphi(y/x) \) \( \varphi \). By abuse
of language, we say the family set \( \varGamma = \{\Gamma_s\}_{s \in S} \) is a witnessed
MCS if every \varGamma_s is a witnessed
MCS.

In the following, we show any consistent set \( \varGamma \) can be extended to a witnessed MCS \( \varGamma^* \). The extension, however, requires an extension of the set of variables. To see why such an extension is needed, consider the following example. Let \( \Sigma = (S, \text{VAR}, \Sigma) \) be a signature, \( s \in S \) be a sort, and \( \varGamma = \{\neg x \mid x \in \text{VAR}_s\} \) be a pattern set containing all variable negations. We leave it for the readers to show that \( \varGamma \) is consistent. Here, we claim the consistent set \( \varGamma \) cannot be extended to a witnessed MCS \( \varGamma^* \) in the signature \( \varSigma \). The proof is by contradiction. Assume \( \varGamma^* \) exists. By Proposition 67 and (EXISTENCE), \( \exists x. x \in \varGamma^* \). Because \( \varGamma^* \) is a witnessed MCS, there is a variable \( y \) such that \( (\exists x. x) \rightarrow y \in \varGamma^* \), and by Proposition 67, \( y \in \varGamma^* \). On the other hand, \( \neg y \not\in \varGamma^* \). This
contradicts the consistency of \( \varGamma^* \).

**Lemma 69 (Extension Lemma).** Let \( \varSigma = (S, \text{VAR}, \Sigma) \) be a signature and \( \varGamma \) be a consistent set of sorts \( S \). Extend the variable set \( \text{VAR} \) to \( \varGamma^* \) with countably infinitely many new variables, and denoted the extended signature as \( \varSigma^+ = (\text{VAR}^+, S, \Sigma) \). There exists a pattern set \( \varGamma^* \) in the extended signature \( \varSigma \) such that \( \varGamma \subseteq \varGamma^* \) and \( \varGamma^* \) is a witnessed MCS.

**Proof:** We use \( \text{PATTERN}_s \) set pattern \( \varGamma^* \) denote the set of all patterns of sorts \( s \) in the original and extended signatures, respectively. Enumerate all patterns \( \varphi_1, \varphi_2, \cdots \in \text{PATTERN}^*_s \). For
every sort \( s \), enumerate all variables \( x_1: s, x_2: s, \ldots \in \text{VAR}^+_s \setminus \text{VAR}_s \). We will construct a non-decreasing sequence of pattern sets \( \varGamma_0 \subseteq \varGamma_1 \subseteq \varGamma_2 \cdots \subseteq \text{PATTERN}^*_s \), with \( \varGamma_0 = \varGamma \). Notice that \( \varGamma_0 \) contains variables only in \( \text{VAR} \). Eventually, we will let \( \varGamma^* = \bigcup_{n \geq 0} \varGamma_n \) and prove it has the intended properties.

For every \( n \geq 1 \), we define \( \varGamma_n \) as follows. If \( \varGamma_{n-1} \cup \{\varphi_n\} \) is inconsistent, then \( \varGamma_n = \varGamma_{n-1} \). Otherwise, if \( \varphi_n \) is not of the form \( \exists x: s'. \psi \):
\[
\varGamma_n = \varGamma_{n-1} \cup \{\varphi_n\}
\]
if \( \varphi_n \equiv \exists x: s'. \psi \) and \( x_i: s' \) is the first variable in \( \text{VAR}_s \setminus \text{VAR}_s \)
that does not occur free in \( \varGamma_{n-1} \) and \( \psi \):
\[
\Gamma_n = \varGamma_{n-1} \cup \{\varphi_n\} \cup \{\psi[x_i: s'/x: s']\}
\]
Notice that in the second case, we can always pick a variable \( x_i: s' \) that satisfies the conditions because by construction, \( \varGamma_{n-1} \cup \{\varphi_n\} \) uses at most finitely many variables in \( \text{VAR}^s \setminus \text{VAR} \).

We show that \( \varGamma_n \) is consistent for every \( n \geq 0 \) by induction. The base case is to show \( \varGamma_0 \) is consistent in the extended
signature. Assume it is not. Then there exists a finite subset
\(\Delta_0 \subseteq \text{fin} \Gamma_0\) such that \(\vdash \Delta_0 \rightarrow \bot\). The proof of \(\Delta_0 \rightarrow \bot\) is a finite sequence of patterns in \(\text{Pattern}^r\). We can replace every occurrence of the variable \(y \in \text{VAR}^r \setminus \text{VAR}\) (y can have any sort) with a variable \(y' \in \text{VAR}\) that has the same sort as \(y\) and does not occur (no matter bound or free) in the proof. By induction on the length of the proof, the resulting sequence is also a proof of \(\Delta_0 \rightarrow \bot\), and it consists of only patterns in \(\text{Pattern}\). This contradicts the consistency of \(\Gamma_0\) as a subset of \(\text{Pattern}_n\), and this contradiction finishes our proof of the base case.

Now assume \(\Gamma_{n-1}\) is consistent for \(n \geq 1\). We will show \(\Gamma_n\) is also consistent. If \(\Gamma_{n-1} \cup \{\varphi_n\}\) is inconsistent or \(\varphi_n\) does not have the form \(\exists x:s'.\psi\), \(\Gamma_n\) is consistent by construction. Assume \(\Gamma_{n-1} \cup \{\varphi_n\}\) is consistent, \(\varphi_n \equiv \exists x:s'.\psi\), but \(\Gamma_n = \Gamma_{n-1} \cup \{\varphi_n\} \cup \{\psi[x_i; s'/x; s']\}\) is not consistent. Then there exists a finite subset \(\Delta \subseteq \text{fin} \Gamma_{n-1} \cup \{\varphi_n\}\) such that \(\vdash \Delta \rightarrow \neg\psi[x_i; s'/x; s']\). By (Universal Generalization), \(\forall s_i: s'. (\vdash \Delta \rightarrow \neg\psi[x_i; s'/x; s'])\). Notice that \(s_i; s' \not\in \text{FV}(\Delta)\) by construction, so by FOL reasoning \(\vdash \Delta \rightarrow \neg\exists s_i: s'. (\psi[x_i; s'/x; s'])\). Since \(s_i; s' \not\in \text{FV}(\psi)\), by \(a\)-renaming, \(\exists x:s'. (\psi[x_i; s'/x; s']) \equiv \exists x:s'. \psi \equiv \varphi_n\), and thus \(\vdash \Delta \rightarrow \neg\varphi_n\). This contradicts the assumption that \(\Gamma_{n-1} \cup \{\varphi_n\}\) is consistent.

Since \(\Gamma_n\) is consistent for any \(n \geq 0\), \(\Gamma^+ = \bigcup_n \Gamma_n\) is also consistent. This is because the derivation that shows inconsistency would use only finitely many patterns in \(\Gamma^+\). In addition, we know \(\Gamma^+\) is maximal and witnessed by construction.

We will prove that for every witnessed MCS \(\Gamma = \{\Gamma_1\}_{i \in S}\), there exists a model \(M\) and a valuation \(\rho\) such that for every \(\varphi \in \Gamma\), \(\rho(\varphi) \neq \theta\). The next definition defines the canonical model which contains all witnessed MCSs as its elements. We will construct our intended model \(M\) as a submodel of the canonical model.

**Definition 70** (Canonical Model). Given a signature \(\Sigma = (S, \Sigma)\). The canonical model \(W = \{(W_s)_{s \in S, s \in W}\}\) consists of:

- a carrier set \(W_s = \{\Gamma \mid \Gamma\) is a witnessed MCS of sort \(s\}\) for every sort \(s \in S\);
- an interpretation \(\sigma_W: W_{s_1} \times \cdots \times W_{s_n} \rightarrow \mathcal{P}(W_s)\) for every symbol \(\sigma \in \Sigma_{s_1, \ldots, s_n}\), defined as \(\sigma_W(\Gamma_1, \ldots, \Gamma_n)\) if and only if for any \(\varphi \in \Gamma_i\), \(1 \leq i \leq n\), \(\sigma(\varphi_1, \ldots, \varphi_n) \in \Gamma\); In particular, the interpretation for a constant symbol \(\sigma \in \Sigma_{s, s}\) is \(\sigma_W(\Gamma) = (\Gamma \in W_s \mid \sigma \in \Gamma)\).

The carrier set \(W\) is not empty, thanks to Lemma 69.

The canonical model has a nontrivial property stated as the next lemma. The proof of the lemma is difficult, so we leave it to the end of the subsection.

**Theorem 71** (Existence Lemma). Let \(\Sigma = (S, \Sigma)\) be a signature and \(\Gamma\) be a witnessed MCS of sort \(s \in S\). Given a symbol \(\sigma \in \Sigma_{s_1, \ldots, s_n}\) and patterns \(\varphi_1, \ldots, \varphi_n\) of appropriate sorts. If \(\sigma(\varphi_1, \ldots, \varphi_n) \in \Gamma\), then there exist witnessed MCSs \(\Gamma_1, \ldots, \Gamma_n\) of appropriate sorts such that \(\varphi_i \in \Gamma_i\) for every \(1 \leq i \leq n\), and \(\sigma_W(\Gamma_1, \ldots, \Gamma_n)\).

**Definition 72** (Generated Models). Let \(\Sigma = (S, \Sigma)\) be a signature and \(W = \{(W_s)_{s \in S, s \in W}\}\) be the canonical model. Given a witnessed MCS \(\Gamma = \{\Gamma_s\}_{s \in S}\). Define \(Y = \{Y_s\}_{s \in S}\) be the smallest sets such that \(Y_s \subseteq W_s\) for every sort \(s\), and the following inductive properties are satisfied:

- \(\Gamma_s \in Y_s\) for every sort \(s\);
- If \(\Delta \in Y_s\) and there exist a symbol \(\sigma \in \Sigma_{s_1, \ldots, s_n}\) and witnessed MCSs \(\Delta_1, \ldots, \Delta_n\) of appropriate sorts such that \(\Delta \in \sigma_W(\Delta_1, \ldots, \Delta_n)\), then \(\Delta_1 \in Y_{s_1}, \ldots, \Delta_n \in Y_{s_n}\).

Let \(Y = (Y_{-\gamma})\) be the model generated from \(\Gamma\), where \(\sigma_W(\Delta_1, \ldots, \Delta_n) = Y_s \cap \sigma_W(\Delta_1, \ldots, \Delta_n)\) for every \(\sigma \in \Sigma_{s_1, \ldots, s_n}\) and \(\Delta_1 \in Y_{s_1}, \ldots, \Delta_n \in Y_{s_n}\).

We give some intuition about the generated model \(Y = (Y_{-\gamma})\). The interpretation \(\sigma\) is just the restriction of the interpretation \(\sigma_M\) on \(Y\). The carrier set \(Y\) is defined inductively. Firstly, \(Y\) contains \(\Gamma\). Given a set \(\Delta \in Y\). If sets \(\Delta_1, \ldots, \Delta_n\) are “generated” from \(\Delta\) by a symbol \(\sigma\), meaning that \(\Delta \in \sigma_W(\Delta_1, \ldots, \Delta_n)\), then they are also in \(Y\). Of course, a set \(\Delta\) is in \(Y\) maybe because it is generated from a set \(\Delta'\) by a symbol \(\sigma'\), while \(\Delta'\) is generated from a set \(\Delta''\) by a symbol \(\sigma''\), and so on. This generating path keeps going and eventually ends at \(\Gamma\) in finite number of steps. By definition, every member of \(Y\) has at least one such generating path, which we formally define as follows.

**Definition 73** (Generating Paths). Let \(\Gamma = \{\Gamma_s\}_{s \in S}\) be a witnessed MCS and \(Y\) be the model generated from \(\Gamma\). A generating path \(\pi\) is either the empty path \(\epsilon\), or a sequence of pairs \((\sigma_1, p_1), \ldots, (\sigma_k, p_k)\) where \(\sigma_1, \ldots, \sigma_k\) are symbols (not necessarily distinct) and \(p_1, \ldots, p_k\) are natural numbers representing positions. The generating path relation, denoted as \(GP\), is a binary relation between witnessed MCSs in \(Y\) and generating paths, defined as the smallest relation that satisfies the following conditions:

- \(GP(\Gamma_s, \epsilon)\) holds for every sort \(s\);
- If \(GP(\Delta, \pi)\) holds for a set \(\Delta \in Y_s\) and a generating path \(\pi\), and there exist a symbol \(\sigma \in \Sigma_{s_1, \ldots, s_n}\) and witnessed MCSs \(\Delta_1, \ldots, \Delta_n\) such that \(\Delta \in \sigma_W(\Delta_1, \ldots, \Delta_n)\), then \(GP(\Delta, (\pi, (\sigma, i)))\) holds for every \(1 \leq i \leq n\).

We say that \(\Delta\) has a generating path \(\pi\) in the generated model if \(GP(\Delta, \pi)\) holds. It is easy to see that every witnessed MCS in \(Y\) has at least one generating path, and if a witnessed MCS of sort \(s\) has the empty path \(\epsilon\) as its generating path, it must be \(\Gamma_s\) itself.

**Definition 74** (Symbol Contexts for Generating Paths). Given a generating path \(\pi\). Define the symbol context \(C_\pi\) inductively as follows. If \(\pi = \epsilon\), then \(C_\pi\) is the identity context \(\emptyset\). If \(\pi = (\sigma_0, (\sigma, i))\) where \(\sigma_0 \in \Sigma_{s_1, \ldots, s_n}\) and \(1 \leq i \leq n\), then \(C_\pi = C_{\sigma_0}[\sigma(T_{s_1}, \ldots, T_{s_i}, \Delta, T_{s_{i+1}}, \ldots, T_{s_n})]\).

A good intuition about Definition 74 is given as the next lemma.

**Lemma 75**. Let \(\Gamma\) be a witnessed MCS and \(Y\) be the model generated from \(\Gamma\). Let \(\Delta \in Y\). If \(\Delta\) has a generating path \(\pi\), then \(C_\pi[\sigma] \in \Gamma\) for any pattern \(\varphi \in \Delta\).

**Proof**: The proof is by induction on the length of the generating path \(\pi\). If \(\pi\) is the empty path \(\epsilon\), then \(\Delta\) must be
Lemma 76 (Singleton Variables). Let \( \Gamma \) be a witnessed MCS and \( Y \) be the model generated from \( \Gamma \). For every \( \Gamma_1, \Gamma_2 \in Y \) of the same sort and every variable \( x \), if \( x \in \Gamma_1 \cap \Gamma_2 \), then \( \Gamma_1 = \Gamma_2 \).

Proof: Let \( \pi_i \) be a generating path of \( \Gamma_i \) for \( i = 1, 2 \). Assume \( \Gamma_1 \neq \Gamma_2 \). Then there exists a pattern \( \varphi \) such that \( \varphi \in \Gamma_1 \) and \( \neg \varphi \in \Gamma_2 \). Because \( x \in \Gamma_1 \cap \Gamma_2 \), we know \( x \land \varphi \in \Gamma_1 \) and \( x \land \neg \varphi \in \Gamma_2 \). By Definition 75, \( C_{\pi_i}[x \land \varphi], C_{\pi_i}[x \land \neg \varphi] \in \Gamma_i \), and thus \( C_{\pi_1}[x \land \varphi] \land C_{\pi_2}[x \land \neg \varphi] \in \Gamma \). On the other hand, \( \neg(C_{\pi_1}[x \land \varphi] \land C_{\pi_2}[x \land \neg \varphi]) \) is an instance of (Singleton Variable) and thus is included in \( \Gamma \). This contradicts the consistency of \( \Gamma \).

We will establish an important result about generated models in Lemma 77 (the Truth Lemma), which links the semantics and syntax and is essential to the completeness result. Roughly speaking, the lemma says that for any generated model \( Y \) and any witnessed MCS \( \Delta \in Y \), a pattern \( \varphi \) is in \( \Delta \) if and only if the interpretation of \( \varphi \) in \( Y \) contains \( \Delta \). To prove the lemma, it is important to show that every variable is interpreted to a singleton. Lemma 76 ensures that every variable belongs to at most one witnessed MCS. To make sure it is interpreted to exactly one MCS, we complete our model by adding a dummy element \( * \) to the carrier set, and interpreting all variables which are interpreted to none of the MCSs to the dummy element. This motivates the next definition.

Definition 77 (Completed Models and Completed Valuations). Let \( \Gamma = \{ \Gamma_i \}_{i \in S} \) be a witnessed MCS and \( Y \) be the \( \Gamma \)-generated model. A \( \Gamma \)-completed model, denoted as \( M = \{ (M_i)_{i \in S} \} \), is inductively defined as follows for all sorts \( s \in S \):

- \( M_s = Y_s \) if every \( x:s \) belongs at least one MCS in \( Y_s \);
- \( M_s = Y_s \cup \{ * \} \) otherwise.

We assume \( * \) is an entity that is different from any MCSs, and \( *_{s_1} \neq *_{s_2} \) if \( s_1 \neq s_2 \). For every \( \sigma \in \Sigma_{s_1 \ldots s_n} \), define its interpretation

\[
\sigma_M(\Delta_1, \ldots, \Delta_n) = \begin{cases} 
\emptyset & \text{if some } \Delta_i = *_{s_i} \\
\sigma_Y(\Delta_1, \ldots, \Delta_n) \cup \{ *_{s_i} \} & \text{if all } \Delta_j \neq *_{s_j} \text{ and some } \Delta_i = \Gamma_{s_i} \\
\sigma_{\varphi_0}(\Delta_1, \ldots, \Delta_n) & \text{otherwise}
\end{cases}
\]

The completed valuation \( \rho: \text{VAR} \rightarrow M \) is defined as

\[
\rho(x:s) = \begin{cases} 
\Delta & \text{if } x:s \in \Delta \\
* & \text{otherwise}
\end{cases}
\]

The valuation \( \rho \) is a well-defined function, because by Lemma 76 if there are two witnessed MCSs \( \Delta_1 \) and \( \Delta_2 \) such that \( x \in \Delta_1 \) and \( x \in \Delta_2 \), then \( \Delta_1 = \Delta_2 \).

Now we come back to prove Lemma 71. We need the following technical lemma.

Lemma 78. Let \( \sigma \in \Sigma_{s_1 \ldots s_n} \) be a symbol, \( \Phi_1, \ldots, \Phi_n, \phi \) be patterns of appropriate sorts, and \( y_1, \ldots, y_n \) be variables of appropriate sorts such that \( y_1, \ldots, y_n \) are distinct, and \( y_1, \ldots, y_n \notin \text{fv}(\Phi) \cup \bigcup_{1 \leq i \leq n} \text{fv}(\Phi_i) \). Then

\[
\vdash \sigma(\Phi_1, \ldots, \Phi_n) \\
\implies \exists y_1, \ldots, \exists y_n.
\]

Proof: Notice that for every \( 1 \leq i \leq n \),

\[
\vdash \exists y_i, \phi(y_i/x).
\]

By easy matching logic reasoning,

\[
\vdash \sigma(\Phi_1, \ldots, \Phi_n) \\
\implies \sigma(\Phi_1 \land (\exists x, \phi \rightarrow \exists y_i, \phi(y_i/x))),
\]

\[
\ldots,
\]

\[
\Phi_n \land (\exists x, \phi \rightarrow \exists y_n, \phi(y_n/x))
\]

Then use Proposition 43 to move the quantifiers \( \exists y_1, \ldots, \exists y_n \) to the top.

Now we are ready to prove Lemma 78.

Proof of Lemma 78. Recall that \( \Gamma \in \text{w}(\Gamma_1, \ldots, \Gamma_n) \) means for every \( \phi_i \in \Gamma_i, 1 \leq i \leq n, \sigma(\phi_1, \ldots, \phi_n) \in \Gamma \).

The main technique that we will be using here is similar to Lemma 69. We start with the singleton sets \( \{ \varphi_i \} \) for each \( 1 \leq i \leq n \) and extend them to witnessed MCSs \( \Gamma_i \), while this time we also need to make sure the results \( \Gamma_1, \ldots, \Gamma_n \) satisfy the desired property \( \Gamma \in \text{w}(\Gamma_1, \ldots, \Gamma_n) \). Another difference compared to Lemma 69 is that this time we do not extend our set of variables, because our starting point, \( \{ \varphi_i \} \), contains just one pattern and uses only finitely many variables. Readers will see how these conditions play a role in the upcoming proof.

Enumerate all patterns of sorts \( s_1, \ldots, s_n \) as follows

\[
\psi_0, \psi_1, \psi_2, \ldots \in \bigcup_{1 \leq i \leq n} \text{Pattern}_{s_i}.
\]

Notice that \( s_1, \ldots, s_n \) do not need to be all distinct. To ease our notation, we define a “choice” operator, denoted as \( [\varphi_i]_{s_i} \), as follows

\[
[\varphi_i]_{s_i} = \begin{cases} 
\varphi_s & \text{if } s = s' \\
\text{nothing} & \text{otherwise}
\end{cases}
\]

For example, \( \varphi_s \land [\psi]_{s} \) means \( \varphi_s \land \psi \) if \( \psi \) also has sort \( s \). Otherwise, it means \( \varphi_s \). The choice operator propagates with all logic connectives in the natural way. For example, \( [\neg \psi]_{s} = [\neg \varphi]_{s} \).

In the following, we will define a non-decreasing sequence of pattern sets \( \Gamma^{(0)}_i \subseteq \Gamma^{(1)}_i \subseteq \Gamma^{(2)}_i \subseteq \cdots \subseteq \text{Pattern}_{s_i} \) for each \( 1 \leq i \leq n \), such that the following conditions are true for all \( 1 \leq i \leq n \) and \( k \geq 0 \):

1. If \( \psi_k \) has sort \( s_i \), then either \( \psi_k \) or \( \neg \psi_k \) belongs to \( \Gamma^{(k+1)}_{s_i} \).
2) If $\psi_k$ has the form $\exists x.\phi_k$ and it belongs to $\Gamma_i^{(k+1)}$, then there exists a variable $z$ such that $(\exists x.\phi_k) \rightarrow \phi_k[z/x]$ also belongs to $\Gamma_i^{(k+1)}$.

3) $\Gamma_i^{(k)}$ is finite.

4) Let $\pi_i^{(k)} = \bigwedge_{i=1}^n \Gamma_i^{(k)}$ for every $1 \leq i \leq n$. Then 
$$\sigma(\pi_i^{(k)}, \ldots, \pi_n^{(k)}) \in \Gamma.$$ 

5) $\Gamma_i^{(k)}$ is consistent.

Among the above five conditions, condition (2)–(5) are like “safety” properties while condition (1) is like a “liveness” properties. We will eventually let $\Gamma_i = \bigcup_{k \geq 0} \Gamma_i^{(k)}$ and prove that $\Gamma$ has the desired property. Before we present the actual construction, we give some hints on how to prove these conditions hold. Conditions (1)–(3) will be satisfied directly by construction, although we will put a notable effort in satisfying condition (2). Condition (4) will be proved hold by induction on $k$. Condition (5) is in fact a consequence of condition (4) as shown below. Assume condition (4) holds but condition (5) fails. This means that $\Gamma_i^{(k)}$ is not consistent for some $1 \leq i \leq n$, so $\pi_i^{(k)} \rightarrow \bot$. By (Framing)

$$+ \sigma(\pi_1^{(k)}, \ldots, \pi_i^{(k)}, \ldots, \pi_n^{(k)}) \rightarrow \sigma(\pi_1^{(k)}, \ldots, \bot, \ldots, \pi_n^{(k)})$$

Then by Proposition 43 and FOL reasoning,

$$+ \sigma(\pi_1^{(k)}, \ldots, \pi_i^{(k)}, \ldots, \pi_n^{(k)}) \rightarrow \bot$$

Since $\sigma(\pi_1^{(k)}, \ldots, \pi_i^{(k)}, \ldots, \pi_n^{(k)}) \in \Gamma$ by condition (4), we know $\bot \in \Gamma$ by Proposition 67. And this contradicts the fact that $\Gamma$ is consistent.

Now we are ready to construct the sequence $\Gamma_i^{(0)} \subseteq \Gamma_i^{(1)} \subseteq \Gamma_i^{(2)} \subseteq \ldots$ for $1 \leq i \leq n$. Let $\Gamma_i^{(0)} = \{\varphi_i\}$ for $1 \leq i \leq n$. Obviously, $\Gamma_i^{(0)}$ satisfies conditions (3) and (4). Condition (5) follows as a consequence of condition (4). Conditions (1) and (2) are not applicable.

Suppose we have already constructed sets $\Gamma_i^{(k)}$ for every $1 \leq i \leq n$ and $k \geq 0$, which satisfy the conditions (1)–(5). We now show how to construct $\Gamma_i^{(k+1)}$. In order to satisfy condition (1), we should add either $\psi_k$ or $\neg \psi_k$ to $\Gamma_i^{(k)}$, if $\Gamma_i^{(k)}$ has the same sort as $\psi_k$. Otherwise, we simply let $\Gamma_i^{(k+1)}$ be the same as $\Gamma_i^{(k)}$. The question here is: if $\Gamma_i^{(k)}$ has the same sort as $\psi_k$, which pattern should we add to $\Gamma_i^{(k)}$, $\psi_k$ or $\neg \psi_k$? Obviously, condition (3) will still hold no matter which one we choose to add, so we just need to make sure that we do not break conditions (2) and (4).

Let us start by satisfying condition (4). Consider pattern $\sigma(\pi_1^{(k)}, \ldots, \pi_n^{(k)})$, which, by condition (4), is in $\Gamma$. This tells us that the pattern

$$\sigma(\pi_1^{(k)} \land [\psi_k \lor \neg \psi_k], \ldots, \pi_n^{(k)} \land [\psi_k \lor \neg \psi_k])$$

is also in $\Gamma$. Recall that $[\_]$ is the choice operator, so if $\psi_k$ has sort $s_i$, then $\pi_i^{(k)} \land [\psi_k \lor \neg \psi_k]$, is $\pi_i^{(k)} \land [\psi_k \lor \neg \psi_k]$. Otherwise, it is $\pi_i^{(k)}$. Use Proposition 43 and FOL reasoning, and notice that the choice operator propagates with the disjunction $\lor$ and the negation $\neg$, we get

$$\sigma(\pi_1^{(k)} \land [\psi_k], \ldots, \pi_n^{(k)} \land [\neg \psi_k]) \in \Gamma.$$ 

Then we use Proposition 43 again and move all the disjunctions to the top, and we end up with a disjunction of $2^n$ patterns:

$$\sigma(\pi_1^{(k)} \land [-\neg \psi_k], \ldots, \pi_n^{(k)} \land [-\neg \psi_k]) \in \Gamma$$

where $[-]$ means either nothing or $\neg$. Notice that some $[\psi_k]$’s might be nothing, so some of these $2^n$ patterns may be the same.

Notice that $\Gamma$ is an MCS. By Proposition 67 among these $2^n$ patterns there must exists one pattern that is in $\Gamma$. We denote that pattern as

$$\sigma(\pi_1^{(k)} \land [-\neg \psi_k], \ldots, \pi_n^{(k)} \land [-\neg \psi_k]) \in \Gamma$$

For any $1 \leq i \leq n$, if $[-\neg \psi_k]$ does not have the form $\exists x.\phi$, we simply define $\Gamma_i^{(k+1)} = \Gamma_i^{(k)} \cup \{[-\neg \psi_k]\}$. If $[-\neg \psi_k]$ has the form $\exists x.\phi$, we need special effort to satisfy condition (2). Without loss of generality and to ease our notation, let us assume that for every $1 \leq i \leq n$, the pattern $[-\neg \psi_k]$ has the same form $\exists x.\phi$. We are going to find for each index $i$ a variable $z_i$ such that

$$\sigma(\pi_1^{(k)} \land \exists x.\phi \land \exists x.\phi \rightarrow \phi(z_i/x), \ldots, \pi_n^{(k)} \land \exists x.\phi \land \phi(z_n/x)) \in \Gamma$$

This will allow us to define $\Gamma_i^{(k+1)} = \Gamma_i^{(k)} \cup \{\exists x.\phi\} \cup \{\exists x.\phi \rightarrow \phi(z_i/x), \ldots, \exists x.\phi \land \phi(z_n/x)) \in \Gamma$.

We find these variables $z_i$’s by Lemma 78 and the fact that $\Gamma$ is a witnessed set. Let $\Phi_i \equiv \pi_1^{(k)} \land \exists x.\phi$ for $1 \leq i \leq n$. By construction, $\sigma(\Phi_1, \ldots, \Phi_n) \in \Gamma$. Hence, by Lemma 78 and Proposition 67 for any distinct variables $y_1, \ldots, y_n \notin \text{VF}(\phi) \cup \text{VF}(\Phi_i)$,

$$\exists y_1 \ldots \exists y_n. \sigma(\Phi_1 \land (\exists x.\phi \rightarrow \phi(y_1/x)), \ldots, \Phi_n \land (\exists x.\phi \rightarrow \phi(y_n/x))) \in \Gamma$$

The set $\Gamma$ is a witnessed set, so there exist variables $z_1, \ldots, z_n$ such that

$$\sigma(\Phi_1 \land (\exists x.\phi \rightarrow \phi(z_1/x)), \ldots, \Phi_n \land (\exists x.\phi \rightarrow \phi(z_n/x))) \in \Gamma$$

This justifies our construction $\Gamma_i^{(k+1)} = \Gamma_i^{(k)} \cup \{\exists x.\phi\} \cup \{\exists x.\phi \rightarrow \phi(z_i/x)}$.

So far we have proved our construction of the sequences $\Gamma_i^{(0)} \subseteq \Gamma_i^{(1)} \subseteq \Gamma_i^{(2)} \subseteq \ldots$ for $1 \leq i \leq n$ satisfy the conditions (1)–(5). Let $\Gamma_i = \bigcup_{k \geq 0} \Gamma_i^{(k)}$ for $1 \leq i \leq n$. By construction, $\Gamma_i$ is a witnessed MCS. It remains to prove that $\Gamma \in \sigma W(\Gamma_1, \ldots, \Gamma_n)$. To prove it, assume $\phi_i \in \Gamma_i$ for $1 \leq i \leq n$. By construction, there exists $K > 0$ such that $\phi_i \in \Gamma_i^{(K)}$ for all $1 \leq i \leq n$. Therefore, $\pi_i^{(K)} \rightarrow \phi_i$. By
Lemma 79 (Truth Lemma). Let $\Gamma$ be a witnessed MCS, $M$ be its completed model, and $\rho$ be the completed valuation. For any witnessed MCS $\Delta \in M$ and any pattern $\varphi$ such that $\Delta$ and $\varphi$ have the same sort,

$$\varphi \in \Delta \quad \text{if and only if} \quad \Delta \in \bar{\rho}(\varphi)$$

Proof: The proof is by induction on the structure of $\varphi$. If $\varphi$ is a variable the conclusion follows by Definition 70. If $\varphi$ has the form $\psi_1 \land \psi_2$ or $\neg \psi_1$, the conclusion follows from Proposition 67. If $\varphi$ has the form $\sigma(\varphi_1, \ldots, \varphi_n)$, the conclusion from left to right is given by Lemma 71. The conclusion from right to left follows from Definition 70.

Now assume $\varphi$ has the form $\exists x. \psi$. If $\exists x. \psi \in \Delta$, since $\Delta$ is a witnessed set, there is a variable $y$ such that $\exists x. \psi \rightarrow \psi[y/x] \in \Delta$, and thus $\psi[y/x] \in \Delta$. By induction hypothesis, $\Delta \in \bar{\rho}(\psi[y/x])$, and thus by the semantics of the logic, $\Delta \in \bar{\rho}(\exists x. \psi)$.

Consider the other direction. Assume $\Delta \in \bar{\rho}(\exists x. \psi)$. By definition there exists a witnessed set $\Delta' \in M$ such that $\Delta \in p(\Delta'/x)(\psi)$. By Definition 77 every element in $M$ (no matter if it is an MCS or $\ast$) has a variable that is assigned to it by the completed valuation $\rho$. Let us assume that variable $y$ is assigned to $\Delta'$, i.e., $\rho(y) = \Delta'$. By Lemma 64 $\Delta \in \bar{\rho}(\psi[y/x])$. By induction hypothesis, $\psi[y/x] \in \Delta$. Finally notice that $\forall \psi[y/x] \rightarrow \exists x. \psi[y/x]$. By Proposition 67,

$$\exists x. \psi[y/x] \in \Delta, \text{i.e., } \exists x. \psi \in \Delta.$$ 

Theorem 80. For any consistent set $\Gamma$ and a valuation $\rho$ such that for all patterns $\varphi \in \Gamma$, $\bar{\rho}(\varphi) \neq \emptyset$.

Proof: Use Lemma 69 and extend $\Gamma$ to a witnessed MCS $\Gamma^+$. Let $M$ and $\rho$ be the completed model and valuation generated by $\Gamma^+$ respectively. By Lemma 79 for all patterns $\varphi \in \Gamma \subseteq \Gamma^+$, we have $\Gamma^+ \subseteq \bar{\rho}(\varphi)$, so $\bar{\rho}(\varphi) \neq \emptyset$.

Now we are ready to prove Theorem 16.

Proof of Theorem 79. Assume the opposite. If $\emptyset \notin \varphi$, then $\neq \varphi$ is consistent by Definition 66. Then there is a model $M$ and an valuation $\rho$ such that $\bar{\rho}(\neq \varphi) \neq \emptyset$, i.e., $\bar{\rho}(\neq \varphi) \neq \emptyset$. This contradicts the fact that $\emptyset \notin \varphi$.

We point out that Lemma 79 in fact gives us the following stronger completeness result of $H$. In literature, Theorem is called weak local completeness theorem while Theorem 81 is called strong local completeness theorem.

Theorem 81. For any set $\Gamma$ and any pattern $\varphi$, $\Gamma \vdash_{\text{loc}} \varphi$ implies $\Gamma \vdash \varphi$, where $\Gamma \vdash_{\text{loc}} \varphi$ means that for all models $M$, all valuations $\rho$, and all elements $a \in M$, if $a \in \bar{\rho}(\varphi)$ for all $\psi \in \Gamma$ then $a \in \bar{\rho}(\varphi)$.

Proof: Assume the opposite that $\Gamma \notin \varphi$, which implies that $\Gamma \cup \{\neg \varphi\}$ is consistent. Extend it to a witnessed MCS $\Gamma^+$ and let $M, \rho$ be the completed model and completed valuation generated by $\Gamma^+$. By Lemma 79 $\Gamma^+ \in \bar{\rho}(\varphi)$ for all $\psi \in \Gamma$, and $\Gamma^+ \in \bar{\rho}(\neg \varphi)$, i.e., $\Gamma^+ \notin \bar{\rho}(\varphi)$. This contradicts with $\Gamma \vdash_{\text{loc}} \varphi$.

Appendix E

Proof of Proposition 20

Proof of Proposition 20. Trivial. Note that MMIL coincides with ML on the fragment without $\mu$.

Appendix F

Proof of Proposition 22 and 23

We prove that the theory $\Gamma_{\text{term}}^\ast$ captures precisely term algebras, up to isomorphism. The proof is mainly a feast of inductive reasoning.

Proof: Let us fix a $\Sigma^\ast$-model $M$ such that $M \models \Gamma_{\text{term}}^\ast$. By axiom (Function), the interpretation $c_M : M \times \cdots \times M \rightarrow \mathcal{P}(M)$ must be a function, where $c \in \Sigma_{\text{Term}...\text{Term Term}}$, meaning that for all $a_1, \ldots, a_n \in M$, $c_M(a_1, \ldots, a_n)$ contains exactly one element. By abuse of language, we denote that element as $c_M(a_1, \ldots, a_n)$ and regard $c_M : M \times \cdots \times M \rightarrow M$ as really a function.

To prove the proposition, it suffices to establish an isomorphism between the two algebras $(M, \{c_M\}_{c \in \Sigma})$ and $(\mathcal{P}_{\text{Term}}^\ast\{\phi \times c \in \Sigma\})$.

Let us define a subset $M_0 \subseteq M$ inductively as follows (in which we separate the cases of constant constructs from non-constant constructors for clarity):

- $c_M \in M_0$, if $c \in \Sigma_{\text{Term}}$;
- $c_M(a_1, \ldots, a_n)$, if $c \in \Sigma_{\text{Term}...\text{Term Term}}$ and $a_1, \ldots, a_n \in M_0$.

We claim that for all valuation $\rho$,

$$\bar{\rho}(\mu D. \bigvee_{c \in \Sigma} c(D, \ldots, D)) = M_0.$$ 

We prove the equation by proving set containment for both directions. Notice that by definition,

$$\bar{\rho}(\mu D. \bigvee_{c \in \Sigma} c(D, \ldots, D)) = \bigcap_{c \in \Sigma} \{A \subseteq M | \bigcup_{c \in \Sigma} c_M(A, \ldots, A) \subseteq A\}.$$ 

Denote the above set $M_1$ and we prove $M_0 = M_1$.

(Case $M_1 \subseteq M_0$). Notice that $M_0$ is defined inductively, so we carry out induction. The base case is $c \in \Sigma_{\text{Term}}$ and $c_M \in M_0$. We aim to prove $c_M \in M_1$. For that purpose, assume a set $A \subseteq M$ such that $\bigcup_{c \in \Sigma} c_M(A, \ldots, A) \subseteq A$ and try to prove $c_M \in A$. This is trivial, because $c_M$ is in the big-union set on the left. The induction case is $c \in \Sigma_{\text{Term}...\text{Term Term}}$ and $a_1, \ldots, a_n \in M_0$ and $c_M(a_1, \ldots, a_n) \in M_0$. We aim to prove $c_M(a_1, \ldots, a_n) \in M_1$. Similarly, we assume a set $A \subseteq M$ such that $\bigcup_{c \in \Sigma} c_M(A, \ldots, A) \subseteq A$ and try to prove $c_M(a_1, \ldots, a_n) \in M_0$. By induction hypothesis, $a_1, \ldots, a_n \in M_1$, which implies that $c_M(a_1, \ldots, a_n)$ is in the big-union on the left, and thus in $A$. Done.

(Case $M_1 \subseteq M_0$). We just need to prove that $M_1$ satisfies the condition that $\bigcup_{c \in \Sigma} c_M(M_0, \ldots, M_0) \subseteq M_0$, which follows directly by the construction of $M_0$.

Hence we conclude that $M_0 = M_1$. By axiom (Inductive Domain), $M_1 = M$ is the total set, and thus $M = M_0$. Note that (Inductive Domain) forces the model $M$ to be an inductive one (i.e., $M_0$), and thus admits inductive reasoning.
We now define the isomorphism:

\[(M, \{c_M\}_c) \overset{i}{\mapsto} (\mathcal{F}, \{c_f\}_c)\]

inductively as follows:

- \(i(c_M) = c\), for \(c \in \Sigma_{\text{Term}}\);
- \(i(c_M(a_1, \ldots, a_n)) = c(i(a_1), \ldots, i(a_n))\), for \(c \in \Sigma_{\text{Term}}\);
- \(j(c) = c_M\), for \(c \in \Sigma_{\text{Term}}\);
- \(j(c(t_1, \ldots, t_n)) = c_M(j(t_1), \ldots, j(t_n))\), for \(c \in \Sigma_{\text{Term}}\).

It is then straightforward to verify that \(i \circ j = \text{id}\) and \(j \circ i = \text{id}\) are both identity function, by induction. In addition, they are isomorphic to each other.

Proposition 23 is a direct corollary of Theorem 22. Theorem 22 states that the four axioms form a valid uniqueness of the interpretations of \(\Sigma\), as they clearly satisfy the axioms. But the interpretation in \(\Sigma\) is isomorphic to another interpretation, \(\rho\), by definition.

\[\rho(\varphi) = M\] for all \(\rho\). Our goal is to show \(\Gamma \vdash \varphi[\mu X. \varphi]/X\). Let \(\rho\) be any valuation. We have \(\rho(\varphi[\mu X. \varphi]/X) = \rho(\rho(\varphi)/X)[\varphi]\). Note that \(\rho(\varphi)/X\) is just another valuation, so \(\rho(\varphi)/X\) by assumption.

(Pre-Fixpoint). Let \(\rho\) be any valuation. Our goal is to prove \(\rho(M[X]. \varphi)/X = M\). By definition, \(\rho(M[X]. \varphi)/X = \rho(\rho(\varphi)/X)[\varphi]\). By Knaster-Tarski theorem, \(\rho(M[X]. \varphi)/X\) is a fixpoint of \(\rho(A[X])(\varphi) = A\). Therefore, \(\rho(\varphi(X)/X)(\varphi) = \rho(M(X). \varphi)/X\).

Let \(\rho(\varphi[\mu X. \varphi]/X) = \psi\). Our goal is to prove \(\Gamma \vdash \varphi[\mu X. \varphi]/X\). Let \(\rho\) be any valuation. We need to prove \(\rho(M(X). \varphi) \subseteq \varphi(\psi)/X\).

Lemma 83. \(\vdash \mu X. \varphi \iff \varphi[\mu X. \varphi]/X\).

Proof: We prove both directions.

- (Case “\(\text{\small{\rightarrow}}\)”). Apply (Knaster-Tarski), and we prove \(\vdash \varphi[\mu X. \varphi]/X \rightarrow \varphi[\mu X. \varphi]/X\).
- (Case “\(\text{\small{\leftarrow}}\)”) is exactly (Pre-Fixpoint).

The following propositions hold:

- Pre-Fixpoint: \(\vdash \mu X. \varphi \rightarrow \varphi[\mu X. \varphi]/X\).
- Knaster-Tarski: \(\vdash \varphi \rightarrow \varphi[\mu X. \varphi]/X\) implies \(\vdash \varphi \rightarrow \varphi[\mu X. \varphi]/X\).

We prove these standard proofs as in modal \(\mu\)-logic.

Lemma 85. \(\Gamma \vdash \varphi_1 \rightarrow \varphi_2\) implies \(\Gamma \vdash \mu X. \varphi_1 \rightarrow \mu X. \varphi_2\).

Proof: Use (Knaster-Tarski), and then (Set Variable Substitution).

Lemma 86. For any context \(C\), we have \(\Gamma \vdash \varphi_1 \leftrightarrow \varphi_2\) if and only if \(\Gamma \vdash C[\varphi_1] = C[\varphi_2]\).

Proof: Carry out induction on the structure of \(C\). Except the case \(C = \mu X.C\), all other cases have been proved in Proposition 44. While the \(\mu\)-case is proved by Lemma 85.

Note that Lemma 86 along with Lemma 83 allow us to “unfold” a least fixpoint pattern \(\mu X. \varphi\) and replace it, in-place in any context, by \(\varphi[\mu X. \varphi]/X\).

Lemma 87. A context \(C\) is positive if it is positive in \(\Box\): otherwise, it is negative. Let \(\Gamma \vdash \varphi_1 \rightarrow \varphi_2\). We have

\[
\begin{align*}
\Gamma &\vdash C[\varphi_1] \rightarrow C[\varphi_2] \quad \text{if } C \text{ is positive}, \\
\Gamma &\vdash C[\varphi_2] \rightarrow C[\varphi_1] \quad \text{if } C \text{ is negative}.
\end{align*}
\]

Proof: Carry out induction on the structure of \(C\). The cases when \(C\) is a propositional/FOL context are trivial. The
case when \( C \) is a symbol application is proved by (FRAMING). The case when \( C \) is a \( \mu \)-binder is proved by Lemma 85.

**Lemma 88.** Let \( \psi \) be a predicate pattern and \( C \) be a context where \( \Box \) is not under any \( \mu \)-binder. We have \( \vdash \psi \land C[\psi] \iff \psi \land C[\psi \land \psi] \) for all \( \psi \).

**Proof:** Carry out induction on the structure of \( C \). The cases when \( C \) is a propositional/FOL context are trivial. The case when \( C \) is a symbol application is proved using the fact that predicate patterns propagate through symbols. Since \( \Box \) does not occur under any \( \mu \)-binder, that is all cases.

**Lemma 89.** Let \( \psi \) be a predicate pattern and \( \varphi \) be a pattern. Let \( X \) be a set variable that does not occur under any \( \mu \)-binder in \( \varphi \), and \( X \not\in \text{FV}(\psi) \). We have \( \vdash \psi \land \mu X.\varphi \iff \mu X.(\psi \land \varphi) \).

**Proof:** Note that \( \vdash \) is proved by Lemma 55. We only need to prove \( \vdash \). By propositional reasoning, the goal becomes \( \vdash \mu X.\varphi \iff \psi \land \mu X.(\psi \land \varphi) \) and we apply (KNASTER-TARSKI). We obtain \( \vdash \psi \land \varphi \iff \mu X.(\psi \land \varphi)/X \).

By (PRE-FIXPOINT), we just need to prove \( \vdash \psi \land \varphi \iff \mu X.(\psi \land \varphi)/X \). By Lemma 89, we just need to prove \( \vdash \psi \land \varphi \iff \mu X.(\psi \land \varphi)/X \) \iff \( \psi \land \varphi \mu X.(\psi \land \varphi)/X \) \iff \( \psi \land \varphi \mu X.(\psi \land \varphi)/X \), which then by Lemma 87 becomes \( \vdash \psi \land \varphi \iff \mu X.(\psi \land \varphi)/X \) which follows naturally.

We now obtain a version of deduction theorem for \( \mathcal{H}_\mu \), which we believe is not in its strongest form, but it is good enough to prove other theorems in this paper.

**Theorem 90** (Deduction Theorem of \( \mathcal{H}_\mu \)). Let \( \Gamma \) be an axiom set containing definedness axioms \( \psi \) and \( \varphi \) are patterns. If \( \Gamma \cup \{\psi\} \vdash \varphi \) and the proof (1) does not use (UNIVERSAL GENERALIZATION) on free element variables in \( \psi \); (2) does not use (KNASTER-TARSKI), unless set variable \( X \) does not occur under any \( \mu \)-binder in \( \varphi \) and \( X \not\in \text{FV}(\psi) \); (3) does not use (SET VARIABLE SUBSTITUTION) on free set variables in \( \psi \), then \( \Gamma \vdash \psi \iff \varphi \).

**Proof:** Carry out induction on the length of the proof \( \Gamma \cup \{\psi\} \vdash \varphi \). (Base Case) and (Induction Case) for Modus Ponens and (Universal Generalization) are proved as in Theorem 90. We only need to prove (Induction Case) for (KNASTER-TARSKI) and (SET VARIABLE SUBSTITUTION).

(KNASTER-TARSKI). Suppose \( \varphi \equiv \mu X.\varphi_1 \to \varphi_2 \). We should prove that \( \Gamma \vdash [\psi] \to (\mu X.\varphi_1 \to \varphi_2) \), i.e., \( \Gamma \vdash [\psi] \land \mu X.\varphi_1 \to \varphi_2 \). Note that \( [\psi] \) is a predicate pattern. By Lemma 89, our goal becomes \( \Gamma \vdash \mu X.([\psi] \land \varphi_1) \to \varphi_2 \). By (KNASTER-TARSKI), we need to prove \( \Gamma \vdash ([\psi] \land \varphi_1)/X \to \varphi_2 \). Note that \( X \not\in \text{FV}([\psi]) \), so the above becomes \( \Gamma \vdash [\psi] \land [\varphi_1]/X \to \varphi_2 \), i.e., \( \Gamma \vdash [\psi] \to \varphi_1/[\varphi_2/X] \to \varphi_2 \), which is our induction hypothesis.

(SET VARIABLE SUBSTITUTION). Trivial. Note that \( X \not\in \text{FV}(\psi) \).
Notice that the last step is by \( \Gamma \vdash \varphi / \psi \rightarrow \psi \).

By (Key-Value) and definition of \( \alpha_0 \), the above becomes

\[
\Gamma \vdash z_1 \in \varphi_1 \land \cdots \land z_n \in \varphi_n \land \psi [z_1 / x_1 \cdots z_n / x_n] \\
\rightarrow \psi [\varphi_1 / x_1 \cdots \varphi_n / x_n]
\]

which holds by assumption. Done.

What is interesting in the above proof is that we used only (Key-Value) and did not use (Injectivity) and (Product Domain). The last two axioms are used in the proof of Theorem 30 where we need to establish an isomorphism between models of LFP and MinL. In there, the two axioms are needed to constrain MinL models.

**Appendix J**

**Proof of Theorem 30**

We first show that the theory of products (see Definition 27) capture precisely the product set \( M_s \times M_t \). We denote the theory of products as \( T_{\text{product}} \), consisting of the three axioms (Injectivity), (Key-Value), and (Product Domain).

**Lemma 92.** For any signature \( \Sigma \) consisting two sorts \( s, t \) and their product sort \( s \otimes t \), there exists an isomorphism

\[
M_{s \otimes t} \cong M_s \times M_t.
\]

Under the above isomorphism, we adopt the following abbreviations for all \( a \in M_s, b \in M_s, p \in M_s \times M_t \):

\[
(a, b) \equiv (\langle \_x, \_y \rangle)_M(a, b) \\
p(v) \equiv (\langle \_y, \_y \rangle)_M(p, v)
\]

Then for all \( f : M_s \rightarrow P(M_t) \) and \( a \subseteq P(M_s \times M_t) \), we have

\[
f(a) = \text{uncurry}(f)(a) \\
curry(a)(a) = \alpha(a).
\]

**Proof:** By (Product Domain), \( M_{s \otimes t} = \bar{\rho}(\exists x \forall y.(k, v)) = \cup_{a \in M_s, b \in M_t}(a, b) \). Define the \((i, j)\)-isomorphism such that \( i(a, b) = (a, b) \) and \( j(a, b) = (b, a) \). Note that \( i, j \) is well-defined because of (Injectivity). Clearly, \( i, j \) form an isomorphism between \( M_s \times M_t \) and \( M_t \times M_s \).

Now we prove the two equations. They are straightforward. Note that \( \text{uncurry}(f)(a) = \{ (a, b) | b \in f(a) \} = \{ b | b \in f(a) \} = f(a). \) Similarly, \( \text{curry}(a)(a) = \{ b | (a, b) \in a \} = \alpha(a) \) by definition. Done.

**Corollary 93.** For any signature \( \Sigma \) containing sorts \( s_1, \ldots, s_n, t \) and their product sorts \( s_1 \otimes \cdots \otimes s_n \otimes t \), there exists an isomorphism between \( M_{s_1 \otimes \cdots \otimes s_n \otimes t} \) and \( M_{s_1} \times \cdots \times M_{s_n} \times M_t \). And for any function \( f : M_{s_1} \times \cdots \times M_{s_n} \rightarrow P(M_t) \) and sets \( \alpha \subseteq M_{s_1} \times \cdots \times M_{s_n} \times M_t \), we have

\[
f(a_1, \ldots, a_n) = \text{uncurry}(f)(a) \\
curry(a)(a_1, \ldots, a_n) = \alpha(a_1, \ldots, a_n)
\]

where we abbreviate \( \alpha(a_1, \ldots, a_n) \equiv \alpha(a_1) \ldots \alpha(a_n) \) is a composition of projections.

We now review the syntax and semantics of LFP, slightly adapted to fit the best with our setting.
Definition 94. Let \((S, \Sigma, \Pi)\) be a FOL signature. LFP extends FOL formulas by the following additional rule:

\[
\varphi ::= \cdots \mid \langle \forall \varphi, \varphi(t_1, \ldots, t_n) \rangle
\]

where \(R\) is an \(n\)-ary predicate variable and \(\varphi\) is positive in \(R\). LFP valuations also extend FOL that map every \(n\)-ary predicate variable \(R\) to and \(n\)-ary relation \(\rho(R) \subseteq \mathcal{P}(M^n)\).

Given a FOL model \(M\) and a valuation \(\rho\), LFP extends the semantics of FOL by adding the following valuation rule for least fixpoint formulas:

\[
M, \rho \vDash LFP \langle \forall \varphi, \varphi(t_1, \ldots, t_n) \rangle \iff (\rho(t_1), \ldots, \rho(t_n)) \in \bigcap \{ \{ \alpha \subseteq M_{s_1} \times \cdots \times M_{s_n} \mid \text{for all } a_i \in M_{s_i}, 1 \leq i \leq n, M, \rho[a/R, \vec{a}/\vec{x}] \vDash LFP \varphi \implies (a_1, \ldots, a_n) \in \alpha \} \}
\]

LFP formula \(\varphi\) is valid, denoted \(\vDash LFP \varphi\), if \(M, \rho \vDash LFP \varphi\) for all \(M\) and \(\rho\).

Proof of Theorem 30. The proof is mainly based on the isomorphism between LFP models and MmL \(\Gamma\)-LFP-models. Notice that the (Function) axioms forces symbols in all \(\Gamma\)-LFP-models are functions. In addition, the axiom \(\forall x : \text{Pred} x = y\) forces the carrier set of \(\text{Pred}\) must be a singleton set, say, \{\star\}.

The "if" direction. We follow the same idea as we prove that ML captures faithfully FOL (see [1]), we construct the from an FOL model \(\langle (M_n^{\text{LFP}})_s \subseteq \Sigma^{\text{LFP}} \cup \Pi^{\text{LFP}} \rangle\) a corresponding MmL \(\Gamma\)-LFP model, denoted \(\langle (M_n^{\text{MmL}})_s \subseteq \Sigma^{\text{MmL}} \cup \Sigma^{\text{Mmm}} \rangle\) with \(M_n^{\text{MmL}} = M_n^{\text{LFP}}, M_{\text{Pred}}^{\text{MmL}} = \{\star\}\), and \(\Sigma^{\text{Mmm}}\) defined as in Section II-D consisting of symbols that are all functions. An LFP valuation \(\rho^{\text{MmL}}\) derives a corresponding MmL valuation \(\rho^{\text{MmL}}\) such that \(\rho^{\text{MmL}}(x) = \rho^{\text{LFP}}(x)\) for all LFP (element) variables \(x\) and \(\rho^{\text{MmL}}(\langle \forall \varphi, \varphi(t_1, \ldots, t_n) \rangle) = \rho^{\text{LFP}}(\langle \forall \varphi, \varphi(t_1, \ldots, t_n) \rangle)\times\{\star\}\). Our goal is to prove that for all LFP formulas \(\varphi\), we have \(M, \rho^{\text{LFP}} \vDash LFP \varphi\) if and only if \(\rho^{\text{MmL}}(\varphi) = \{\star\}\). Firstly, notice that as shown in [1],

\[
\rho^{\text{MmL}}(t) = \{\rho^{\text{LFP}}(t)\} \text{ for all terms } t.
\]

Therefore, to simplify our notation we uniformly use \(\rho(t)\) in both LFP and MmL settings. Carry out induction on the structure of \(\varphi\). The only additional cases (compared with FOL) are \(\varphi \equiv R(t_1, \ldots, t_n)\) and \(\varphi \equiv \langle \forall \varphi, \varphi(t_1, \ldots, t_n) \rangle\). The first case is easy, as shown in the following reasoning: \(M, \rho^{\text{LFP}} \vDash \langle \forall \varphi, \varphi(t_1, \ldots, t_n) \rangle\) if and only if \(\rho^{\text{MmL}}(\langle \forall \varphi, \varphi(t_1, \ldots, t_n) \rangle) = \{\star\}\) where \(\varphi = \langle \forall \varphi, \varphi(t_1, \ldots, t_n) \rangle\).

*This is where we are different from the classic LFP. In classic LFP, formulas cannot contain predicate variables that occur free. And the semantics of predicate variables, which is needed when we define the semantics of \(\langle \forall \varphi, \varphi(t_1, \ldots, t_n) \rangle\), are given by an extended model \(M'\) that takes \(R\) as an \(n\)-ary predicate symbol and interprets it as a relation \(\alpha \subseteq M_1 \times \cdots \times M_n\). Here, we allow predicate variables to occur free in a formula, and we extend valuations to give them semantics, instead of modifying the model. This slightly modified presentation is obviously the same as the classic one, but fits better in our setting and looks more similar and uniform to MmL.*

Proof of Theorem 31. Theorem 31 shows that our definition of modal \(\mu\)-logic in MmL is faithful. We have shown a proof sketch in the main
paper. We give the complete detailed proof in this subsection. The main purpose is to give an example, as the proofs of the corresponding theorems for LTL/CTL/DL have similar forms.

**Lemma 95.** \( r_\mu \varphi \) implies \( \Gamma^\mu \vdash \varphi \).

**Proof:** We need to prove that all modal \( \mu \) logic proof rules are provable in matching \( \mu \) logic. Recall that modal \( \mu \) logic contains all propositional tautologies and (Modus Ponens), plus the following four rules:

(K) \( \omega(\varphi_1 \to \varphi_2) \to (\omega \varphi_1 \to \omega \varphi_2) \)  \( (N) \ varphi/\varphi \to (\varphi/\psi) \to \varphi \)  \( (\mu_1) \ [\mu \varphi \land [\mu X . \varphi] X \to \mu X . \varphi \)  \( (\mu_2) \[\mu X . \varphi] \to \mu X . \varphi \)

Notice that (K) and (N) are proved by Proposition 12 and (\( \mu_1 \)) and (\( \mu_2 \)) are exactly (Pre-Fixpoint) and (Knaster-Tarski).

**Lemma 96.** For all \( S = (S, R) \) and all valuations \( V : \text{PVAR} \to \mathcal{P}(S) \), we have \( s \in \{ \xi \}^{S \cdot \psi}_V \) if and only if \( s \in \bar{V}(\varphi) \).

**Proof:** Carry out structural induction on \( \varphi \).

(Case \( \varphi \equiv X \).) We have \([X]^V = V(X) = \bar{V}(X) \). Proved.

(Case \( \varphi \equiv \varphi_1 \land \varphi_2 \).) We have \([\varphi_1 \land \varphi_2]^V = [\varphi_1]^V \cap [\varphi_2]^V = \bar{V}(\varphi_1) \land \bar{V}(\varphi_2) \). Proved.

(Case \( \varphi \equiv \neg \varphi_1 \).) We have \([\neg \varphi_1]^V = S \setminus \{ \varphi_1 \}^V = S \setminus \bar{V}(\varphi_1) = S \setminus (S \setminus \bar{V}(\varphi_1)) \). Proved.

(Case \( \varphi \equiv \omega \varphi_1 \).) By Proposition 32, we have \([\omega \varphi_1]^V = \{ s \in S \mid \text{R } t \text{ implies } t \in \{ \varphi_1 \}^V \text{ for all } t \in S \} = \{ s \in S \mid s \in \bar{V}(\omega \varphi_1) \} = \bar{V}(\omega \varphi_1) \). Proved.

(Case \( \varphi \equiv [\mu X . \varphi] \).) We have \([\mu X . \varphi]^V = \bigcap \{ A \subseteq S \mid \{ [\varphi_1]^V X \subseteq A \} \subseteq A \} = \mu X . \varphi \). Proved.

Induction is finished and lemma is proved.

**Corollary 97.** \( \Gamma^\mu \vdash \varphi \) implies \( r_\mu \varphi \).

**Proof:** Assume the opposite. Then there exist \( S = (S, R) \), \( \rho : \text{PVAR} \to \mathcal{P}(S) \), and \( S \in S \) such that \( s /\in [\mu \varphi]^V \). By Lemma 96, \( s /\in \bar{V}(\varphi) \). Since \( S \equiv \Gamma^\mu \), we have \( \Gamma^\mu \varphi \). Contradiction.

Now we have completed the proof of Theorem 31 where (2) \( \Rightarrow \) (3) is given by Lemma 95 and (5) \( \Rightarrow \) (6) is given by Corollary 97.

**APPENDIX L**

**Proof of Proposition 32** We simply apply definition. Recall that \( s \in \omega_2(t) \) if \( s \in R t \).

(Case \( \omega \)). \( s \in \bar{\rho}(\varphi) \) if there exists \( t \in \bar{\rho}(\varphi) \) such that \( s \in \omega_2(t) \) and \( s \in \bar{\rho}(\varphi) \).

(Case \( \omega' \)). \( s \in \bar{\rho}(\varphi) \) if \( s \in \bar{\rho}(\omega \varphi) \) if \( s \in \bar{\rho}(\varphi) \).

(Case \( \omega'' \)). \( s \in \bar{\rho}(\varphi) \) if \( s \in \bar{\rho}(\omega \varphi) \).

Note that \( \bar{\rho}(\varphi) = \bigcap \{ A \subseteq S \mid \bar{\rho}(\varphi) \subseteq A \} \). On the other hand, \( \{ s \in S \mid s \in \bar{\rho}(\varphi) \} = \Omega_{\bar{\rho}(\varphi)} \).

**APPENDIX M**

**Proof of Theorem 33** As a review, we formally define the semantics of infinite-trace LTL and present in Fig. 4 its sound and complete proof system. There are different notions of semantics of infinite-trace LTL. We here review the one that fits best in our setting.

Let us first formally define some characteristic subclasses of transition systems.

**Definition 98.** A transition system \( S = (S, R) \) is:

- **well-founded** if for all \( s \in S \), there is no infinite path from \( s \);
- **non-terminating**, if for all \( s \in S \) there is \( t \in S \) such that \( s R t \);
- **linear**, if for all \( s \in S \) and \( t_1, t_2 \in S \) such that \( s R t_1 \) and \( s R t_2 \), then \( t_1 = t_2 \).

**Definition 99.** Infinite-trace LTL formulas \( \varphi \) is interpreted over a transition system \( S = (S, R) \) that is non-terminating and linear. We use \( s_k \) to denote the unique state such that \( s R s_k \) for \( k \geq 0 \). When \( k = 0 \), we let \( s_0 = s \). Given a valuation \( V : \text{PVAR} \to \mathcal{P}(S) \), semantics of infinite-trace LTL is inductively defined for all \( s \in S \) and \( \varphi \) as follows:

- \( s \in \text{FinLTL} X \) if \( s \in V(X) \);
- \( s \in \text{FinLTL} \varphi_1 \lor \varphi_2 \) if \( s \in \text{FinLTL} \varphi_1 \) or \( s \in \text{FinLTL} \varphi_2 \);
- \( s \in \text{FinLTL} \neg \varphi \) if \( s \in \text{FinLTL} \varphi \);
- \( s \in \text{FinLTL} \varphi_1 \land \varphi_2 \) if \( s \in \text{FinLTL} \varphi_1 \) and \( s \in \text{FinLTL} \varphi_2 \);
- \( s \in \text{FinLTL} \varphi \) if \( s \in \text{FinLTL} \varphi \) for all \( 0 \leq i < k \), \( s_i \in \text{FinLTL} \varphi \), and \( s_k \).

**Lemma 100.** \( \text{FinLTL} \varphi \) implies \( \text{FinLTL} \varphi \).

**Proof:** We just need to prove that all proof rules in Fig. 4 can be proved in \( \text{FinLTL} \).

(Taut) and (MP). Trivial.
(Kₐ) and (Nₐ). By Proposition 12, (Kₐ) and (Nₐ) hold in MmL, and the transition system C = (S, R) that is well-founded and linear. One can show that S = {s₁, ..., sₙ} is finite, and the transition relation of S is of the linear structure s₁ R ... R sₙ. Given a valuation V: PVAR → P(S), semantics of infinite-trace LTL is inductively defined for all sᵢ ∈ S and ϕ as follows:

- si ∈ finLTL X if si ∈ V(ϕ);
- si ∈ finLTL ϕ₁ ∧ ϕ₂ if si ∈ finLTL ϕ₁ and si ∈ finLTL ϕ₂;
- si ∈ finLTL ¬ϕ if si ∈ finLTL ϕ;
- si ∈ finLTL ϕ if si = s₀ or si+1 ∈ finLTL ϕ;
- si ∈ finLTL ϕ₁ Uₖ ϕ₂ if either sj ∈ finLTL ϕ₁ for all j ≥ i, or there exists i ≤ k ≤ n such that sk ∈ finLTL ϕ₂ and for all i ≤ j < k, sj ∈ finLTL ϕ₁.

Lemma 104. finLTL ϕ implies finLTL ⊨ ϕ.

Proof: We just need to prove all proof rules in Fig. 5 can be proved by axioms (Fin) and (Lin) in MmL. We skip the ones that have shown in Lemma 100 and 101. Axiom (Fin). Proved by axiom (Lin).

Lemma 105. si ∈ finLTL ϕ if and only if si ∈ V(ϕ).

Proof: As in Lemma 101, we just need to prove the “only if” part, by showing that s ∈ V(ϕ) satisfies the five conditions in Definition 103. This is easily followed by Proposition 32. The case ϕ₁ Uₖ ϕ₂ shall be proved by directly applying MmL semantics.

Corollary 106. finLTL ⊨ ϕ implies finLTL ϕ.

Proof: Assume the opposite and use Lemma 105. Now we can prove Theorem 34. Proof of Theorem 34 Use Lemma 104 and Corollary 106 as well as the soundness of MmL proof system and the completeness of finite-trace LTL proof system.
We review the semantics of CTL as well as its sound and complete proof system presented in Fig. 6.

Definition 107. CTL formulas are interpreted on a transition system $S = (S, R)$ that is non-terminating, and a valuation $V$: PVAR $\rightarrow$ $\mathcal{P}(S)$. We call an (infinite) sequence of states $s_0 s_1 \ldots$ a path if $s_i R s_{i+1}$ for all $i \geq 0$. CTL semantics is defined inductively for all $s_0 \in S$ and $\varphi$ as follows:

\begin{itemize}
  \item $s_0 \VdashCTL X$ if $s_0 \in V(X)$;
  \item $s_0 \VdashCTL \varphi_1 \land \varphi_2$ if $s_0 \VdashCTL \varphi_1$ and $s_0 \VdashCTL \varphi_2$;
  \item $s_0 \VdashCTL \neg \varphi$ if $s_0 \not\VdashCTL \varphi$;
  \item $s_0 \VdashCTL EX \varphi$ if there exists $s_1$ such that $s_0 R s_1, s_1 \VdashCTL \varphi$;
  \item $s_0 \VdashCTL AX \varphi$ if for all $s_1$ such that $s_0 R s_1, s_1 \VdashCTL \varphi$;
  \item $s_0 \VdashCTL E\chi \varphi_1 EU \varphi_2$ if there exists a path $s_0 s_1 \ldots$ and $k \geq 0$ such that $s_k \VdashCTL \varphi_2$ and $s_0, s_0, \ldots, s_{k-1} \VdashCTL \varphi_1$;
  \item $s_0 \VdashCTL \varphi_1 AU \varphi_2$ if for all paths $s_0 s_1 \ldots$ there exists $k \geq 0$ such that $s_k \VdashCTL \varphi_2$ and $s_0, s_0, \ldots, s_{k-1} \VdashCTL \varphi_1$.
\end{itemize}

We write $\VdashCTL \varphi$ if for all $S = (S, R)$, all valuations $\rho$, and all $s \in S$, $s \VdashCTL \varphi$.

Lemma 108. $\VdashCTL \varphi$ implies $\GammaCTL \models \varphi$.

Proof: We just need to prove all CTL rules from the axiom (INF) in MmL. We skip the first 7 rules as they are simple. The rest 3 rules can be proved by applying (KNASTER-TARSKI) and use properties in Properties 115.

Lemma 109. $s \VdashCTL \varphi$ if and only if $s \in \tilde{V}(\varphi)$.

Proof: As in Lemma 101, we just need to prove the “only if” part by showing that $s \in V(\varphi)$ satisfies all 7 conditions in Definition 107. The first 5 of them are simple. We show the last two ones about “EU” and “AU”.

(Case EU). Assume there exists a path $s_0 s_1 \ldots$ and $k \geq 0$ such that $s_k \in \tilde{V}(\varphi_2)$ and $s_0, s_0, \ldots, s_{k-1} \in \tilde{V}(\varphi_1)$. Our goal is to show $s_0 \in \tilde{V}(\varphi_1) EU \varphi_2$. By semantics of MmL, $\tilde{V}(\varphi_1) EU \varphi_2 = \tilde{V}(\mu X. \varphi_2 \lor (\varphi_1 \land X) = \bigcap \{ A \subseteq S | \tilde{V}(\varphi_2) \cup \tilde{V}(\varphi_1) \cap \bullet_{\mathcal{E}}(A) \subseteq A \}$. Therefore, it suffices to prove that $s_0 \in A$ for all $A \subseteq S$ such that $\tilde{V}(\varphi_2) \subseteq A$ and $\tilde{V}(\varphi_1) \cap \bullet_{\mathcal{E}}(A) \subseteq A$. This is easy, $s_k \in \tilde{V}(\varphi_2) \subseteq A$ implies $s_{k-1} \in \bullet_{\mathcal{E}}(s_k)$. Also, $s_{k-1} \in \tilde{V}(\varphi_1)$ by assumption. Then $s_{k-1} \in \tilde{V}(\varphi_1) \cap \bullet_{\mathcal{E}}(s_k) \subseteq A$. Repeat this procedure for $k$ times and we obtain $s_0 \in A$. Done.

(Case AU). Let us denote $\circ_{\mathcal{E}}(A) = \{ s \in S | \text{ for all } t \in S \text{ such that } s R t, t \in A \}$ to be the “interpretation” of all-path next $\circ$ in $S$. Prove by contradiction. Assume the opposite statement that $s_0 \not\in \tilde{V}(\varphi_1) AU \varphi_2 = \tilde{V}(\mu X. \varphi_2 \lor (\varphi_1 \land \circ_{\mathcal{E}} X)) = \bigcap \{ A \subseteq S | \tilde{V}(\varphi_2) \cup \tilde{V}(\varphi_1) \cap \bullet_{\mathcal{E}}(A) \subseteq A \}$. This means that there exists $A \subseteq S$ such that $\tilde{V}(\varphi_2) \subseteq A$ and $\tilde{V}(\varphi_1) \cap \bullet_{\mathcal{E}}(A) \subseteq A$, and $s_0 \not\in A$. This is going to cause contradiction. Firstly by $\tilde{V}(\varphi_2) \subseteq A$, $s_0 \not\in \tilde{V}(\varphi_2)$, which implies that $s_0 \not\in \tilde{V}(\varphi_2)$. Secondly by $\tilde{V}(\varphi_1) \cap \bullet_{\mathcal{E}}(A) \subseteq A$, we know that $\{ A \subseteq S \subseteq \tilde{V}(\varphi_2) \cup \tilde{V}(\varphi_1) \cap \bullet_{\mathcal{E}}(A) \} \subseteq A$. Since $s_0 \not\in A$, we know either $s_0 \not\in \tilde{V}(\varphi_1)$ or $s_0 \not\in \bullet_{\mathcal{E}}(S \setminus A)$. If it is the first case, then we have a contradiction as any path starting from $s_0$ contradicts with the condition. If it is the second case, then there exists a state, say $s_1$, such that $s_0 R s_1$ and $s_1 \not\in A$, which also implies $s_1 \not\in \tilde{V}(\varphi_2)$. Repeat this process and obtain a sequence of state $s_0 s_1 \ldots$ If the sequence is finite, say $s_0 s_1 \ldots s_n$, then by construction $s_0, \ldots, s_n \not\in \tilde{V}(\varphi_2)$ and $s_n \in \tilde{V}(\varphi_1)$, which is a contradiction to the condition. If the sequence is infinite, then by construction $s_0 s_1 \ldots$ satisfies that $s_0, s_1 \not\in \tilde{V}(\varphi_2)$, which also contradicts to the condition. Done.

Corollary 110. $\GammaCTL \models \varphi$ implies $\VdashCTL \varphi$.

Proof: Use Lemma 109 and prove by contradiction. Note that it is easy to verify that $\tilde{S} \models \GammaCTL$ if $\tilde{S}$ is non-terminating.

Now we are ready to prove Theorem 35.

Proof of Theorem 35. Use Lemma 108 and Corollary 110 as well as soundness of MmL and completeness of CTL.

Appendix P

Proof of Theorem 36

We review the semantics of DL as well as its sound and complete proof system presented in Fig. 7.

Definition 111. Let $S = (S, \{ A_a \}_{a \in \text{APcm}})$ be an APcm-labeled transition system where $R_a \subseteq S \times S$ is the transition relation for atomic program $a$. Let $V$: PVAR $\rightarrow$ $\mathcal{P}(S)$ be a valuation. DL semantics is inductively defined as follows where state
formulas are evaluated to subsets of $S$ and program formulas are evaluated to relations of $S$:

- $[\alpha]^S_V = V(\rho)$;
- $[\alpha_1 \land \alpha_2]^S_V = [\alpha_1]^S_V \cap [\alpha_2]^S_V$;
- $[\neg \alpha]^S_V = S \setminus [\alpha]^S_V$;
- $[\exists \alpha][\bar{\gamma}]^S_V = \{ s \in S \mid \text{ for all } t \in S \text{ such that } (s, t) \in [\alpha]^S_V \}$;
- $[\alpha][\bar{\gamma}]^S_V = \{(s, s) \mid s \in [\alpha]^S_V \}$.

where "$R_1 \circ R_2$" is the composition of two relations $R_1, R_2$ defined as $R_1 \circ R_2 = \{(s_1, s_3) \mid \text{ there exists } s_2 \text{ such that } (s_1, s_2) \in R_1 \text{ and } (s_2, s_3) \in R_2 \}$. We write $\varphi_{DL}$ if $\varphi$ for all $S$ and $V$.

**Lemma 112.** $\varphi_{DL}$ implies $\Gamma_{DL} \vdash \varphi$.

**Proof:** We just need to prove that all proof rules in Fig. [7] can be proved in $\Gamma_{DL}$. First of all, rules (DL1) to (DL6) follow from (syntactic sugar) definitions. Rules (TAUT) and (MP) are trivial. We only prove (DL1), (DL2), (DL7), and (GEN).

Notice that $[\alpha][\bar{\gamma}]$ is defined a syntactic sugar based on the structure of $\alpha$. Therefore, we carry out structure induction on $\alpha$. We should be careful to prevent circular reasoning. Our proving strategy is to prove (GEN) first, and then prove (DL1) and (DL2) simultaneously, and finally prove (DL7).

(GEN). Carry out induction on $\alpha$. All cases are trivial. Notice the case when $\alpha \equiv \beta^*$ is proved by proving $\Gamma_{DL} \vdash \psi \rightarrow [\alpha^*][\varphi]$ using (KNASTER-TARSKI). After simplification, the goal becomes $\Gamma_{DL} \vdash \psi \rightarrow [\beta^*][\varphi]$. This is proved by applying induction hypothesis, which shows $\Gamma_{DL} \vdash [\beta]^S_{V}$. ($\text{DL1}$) and ($\text{DL2}$). We prove both rules simultaneously by induction on $\alpha$. We discuss only interesting cases and skip the trivial ones. ($\text{DL1}$, $\alpha \equiv \beta_1 \; \beta_2$) is proved from induction hypothesis, by applying (GEN) on $[\beta_1]$. ($\text{DL1}, \alpha \equiv \beta^*$) is proved by applying (KNASTER-TARSKI), followed by applying (DL2, "\rightarrow") on $[\beta]$. ($\text{DL2}, \alpha \equiv \beta^*, \rightarrow$) is proved by (KNASTER-TARSKI), followed by (DL2) on $[\beta]$.

($\text{DL7}$) is proved directly by (KNASTER-TARSKI), followed by ($\text{DL2}, \rightarrow$) on $[\alpha]$.

We now connect the semantics of DL with the semantics of MmL. First of all, we observe that the transition system $S = (S, (R_a)_{a \in APgm})$ can be regarded as a $\Sigma_{mmlgm}$-model, where $S$ is the carrier set of $State$ and $APgm$ (the set of atomic programs) is the carrier set of $Pgm$. The "one-path next $\bullet \in \Sigma_{Pgm}$State,State is interpreted according to DL semantics for all $t \in S$ and $a \in APgm$:

$$\omega_2(a, t) = \{ s \in S \mid (s, t) \in R_a \}.$$  

In addition, valuation $V$: $PVAR \rightarrow P(S)$ can be regarded as a matching $\mu$-logic valuation (where we safely ignore the valuations of element variables because they do not appear in DL syntax).

**Lemma 113.** Under the above notations, $[\varphi]^S_V = \bar{V}(\varphi)$.

**Proof:** As in Lemma [101], we just need to prove that $[\varphi]^S_V \subseteq \bar{V}(\varphi)$ by showing that $\bar{V}(\varphi)$ satisfies the conditions in Definition [111].

The only interesting case is to show $\bar{V}([\alpha]) = \{ s \in S \mid \text{ for all } t \in S, (s, t) \in [\alpha]^S_V \}$. We prove it by carrying out structural induction on the DL program formula $\alpha$. The case when $\alpha \equiv a$ for $a \in APgm$ is easy. The cases when $\alpha \equiv \beta_1 ; \beta_2$, $\alpha \equiv \beta_1 \cup \beta_2$, and $\alpha \equiv \psi$ follows directly by basic analysis about sets and using definition of the semantics of DL program formulas.

The interesting case is when $\alpha \equiv \beta^*$. In this case we should prove $\bar{V}([\beta^*][\varphi]) = \bar{V}(\nu X. \varphi \land [\beta]X) = \bigcup \{ A \mid A \subseteq \bar{V}(\varphi) \cap \bar{V}[A/X]([\beta]X) = \bigcup \{ A \mid A \subseteq \bar{V}(\varphi) \cap \{ s \mid \text{ for all } t, (s, t) \in [\beta]^S_V \} \} \} \supseteq \{ s \mid \text{ for all } t, (s, t) \in [\beta]^S_V \} \implies t \in \bar{V}(\varphi) \}$.

We denote the left-hand side of "$\supseteq"$ as $(\eta)$ and the right-hand side as $(\xi)$.

To prove $(\eta) = (\xi)$, we prove containment from both directions.

(Case $(\eta) \subseteq (\xi)$). This is proved by considering an $s \in (\eta)$ and show $s \in (\xi)$. By construction of $(\eta)$, there exists $A \subseteq S$ such that $A \subseteq \bar{V}(\varphi) \cap \{ s \mid \text{ for all } t, (s, t) \in [\beta]^S_V \} \implies t \in A$, and that $s \in A$. In order to prove $s \in (\xi)$, assume $s \in (\xi)$, we prove $t \in S$ such that $(s, t) \in ([\beta]^S_V)^*$ and try to prove $t \in \bar{V}(\varphi)$. By definition, there exists $k \geq 0$ and $s_0, s_1, \ldots, s_k$ such that $s = s_0$, $t = s_k$, and $(s_i, s_{i+1}) \in [\beta]^S_V$ for all $0 \leq i < k$. By induction and the property of $A$, and that $s_0 \in A$, we can prove that $s_0, s_1, \ldots, s_k \in \bar{V}(\varphi)$, and thus $t \in \bar{V}(\varphi)$. Done.

(Case $(\xi) \subseteq (\eta)$). Notice that the set $(\eta)$ is defined as a greatest fixpoint, so it suffices to show that $(\xi)$ satisfies the condition, i.e., to prove that $(\xi) \subseteq \bar{V}(\varphi) \cap \{ s \mid \text{ for all } t, (s, t) \in [\beta]^S_V \} \implies t \in (\xi)$). This can be easily proved by the definition of $(\xi)$. Done.

**Corollary 114.** $\Gamma_{DL} \vdash \varphi$ implies $\varphi_{DL}$.

**Proof:** Use Lemma [113] and for the sake of contradiction, assume the opposite. Suppose there exists $S = (S, (R_a)_{a \in APgm})$ and a valuation $V$ and a state $s$ such that $s \not\in [\varphi]^S_V$. We then know $s \not\in \bar{V}(\varphi)$, which implies that $S \not\models \varphi$. Obviously $S \models \Gamma_{DL}$ as the theory $\Gamma_{DL}$ contains no addition axioms. This means that $\Gamma\not\models \varphi$. Done.

We are ready to prove Theorem [36].

**Proof of Theorem [36].** Use Lemma [112] and Corollary [114] as well as soundness of MmL and completeness of DL.
Proposition 115. The following propositions hold:

1) \( \vdash \bot \rightarrow \bot \)
2) \( \vdash (\forall x. \varphi) \leftrightarrow \exists x. \exists \varphi \)
3) \( \vdash \square T \leftrightarrow T \)
4) \( \vdash \circ T \leftrightarrow T \)
5) \( \vdash \circ (\varphi \land \varphi) \leftrightarrow \varphi \land \varphi \)
6) \( \vdash \circ (\forall x. \varphi) \leftrightarrow \forall x. \varphi \)
7) \( \vdash \varphi \rightarrow \exists x. \varphi \) and \( \vdash \circ \varphi \rightarrow \varphi \)
8) \( \vdash \varphi \rightarrow \varphi \) and \( \vdash \circ \varphi \rightarrow \circ \varphi \)
9) \( \vdash \varphi \leftrightarrow \circ \varphi \) and \( \vdash \circ \varphi \rightarrow \circ \varphi \)
10) \( \Gamma \vdash \varphi_1 \rightarrow \varphi_2 \) implies \( \Gamma \vdash \star \varphi_1 \rightarrow \star \varphi_2 \) when \( \star \in \{\circ, \square, \circ, \square \} \)

Proof: We prove them in order.

(1–3) follows from (Propagation), and (Framing).

(4–6) are proved from (1–3) and that \( \circ \varphi \equiv \neg \circ \neg \varphi \).

(7) is proved by (Pre-Fixpoint).

(8) is proved by (Pre-Fixpoint).

(9) is proved by (Knaster-Tarski).

(10, when \( \star \) is \( \circ \)) requires us to prove \( \Gamma \vdash \circ \varphi_1 \rightarrow \varphi_2 \).

By (Knaster-Tarski), it suffices to prove \( \Gamma \vdash \circ \varphi_1 \rightarrow \varphi_1 \lor \circ \varphi_2 \), which is proved by (Pre-Fixpoint).

(11, \( \rightarrow \)) is proved by (Knaster-Tarski).

(11, \( \neg \rightarrow \)) is trivial.

(12, \( \rightarrow \)) is proved by (Knaster-Tarski), followed by (2) to propagate \( \circ \varphi \) through \( \lor \), and finished with (7).

(12, \( \neg \rightarrow \)) is proved by (10, when \( \star \) is \( \circ \)).

(13, \( \rightarrow \)) is proved by (Knaster-Tarski), followed by (3) to propagate \( \circ \varphi \) through \( \rightarrow \), and finished with (7).

(13, \( \neg \rightarrow \)) is proved by (10, when \( \star \) is \( \circ \)).

(14–16) are proved similar to (11–13).

(17, both directions) are proved by (Knaster-Tarski) followed by (Pre-Fixpoint).

(18) is proved by \( \circ \varphi \equiv \neg \circ \neg \varphi \) and (Propagation).

(19) is proved by (18) followed by (10).

(20, \( \rightarrow \)) is proved by proving \( \vdash \varphi \rightarrow \varphi \lor \circ \varphi \) which is proved by (Knaster-Tarski) followed by (19).

(20, \( \rightarrow \)) is proved by (Knaster-Tarski), followed by (2) to propagate \( \circ \varphi \) through \( \lor \). After some additional propositional reasoning, we obtain two more proof goals: \( \vdash \varphi \rightarrow \varphi \lor \circ \varphi \) and \( \vdash \varphi \rightarrow \varphi \lor \circ \varphi \). The former is proved by (Knaster-Tarski) and the latter is exactly (Pre-Fixpoint).

(21, \( \rightarrow \)) is proved by applying (10) everywhere followed by (12).

(21, \( \rightarrow \)) is proved by (10, when \( \star \) is \( \circ \)).

(22, \( \rightarrow \)) is proved by (20) everywhere followed by (13).

(22, \( \rightarrow \)) is proved by (10, when \( \star \) is \( \circ \)).

(23, when \( \star \) is \( \circ \), \( \rightarrow \)) is proved by (7) and (10).

(23, when \( \star \) is \( \circ \), \( \rightarrow \)) is proved by (8) and (10).

(23, when \( \star \) is \( \circ \), \( \rightarrow \)) is proved by (Knaster-Tarski) followed by (8).

(23, when \( \star \) is \( \circ \), \( \rightarrow \)) is proved by applying (Knaster-Tarski) first. Then we need to prove \( \vdash \circ \varphi \rightarrow \varphi \lor \circ \varphi \).

By (Pre-Fixpoint), we know \( \vdash \circ \varphi \rightarrow \varphi \lor \circ \varphi \).

Thus, it suffices to prove \( \vdash \circ \varphi \rightarrow \varphi \lor \circ \varphi \).

By propositional reasoning, we just need to prove \( \vdash \circ \varphi \rightarrow \varphi \lor \circ \varphi \).

By (Knaster-Tarski), we know \( \vdash \circ \varphi \rightarrow \varphi \lor \circ \varphi \). Again by propositional reasoning, it suffices to prove \( \vdash \varphi \lor \circ \varphi \).

Note it is sufficient to prove (24) only for the case \( k = 1 \).

(24, \( \neg \rightarrow \)) is proved by (Knaster-Tarski) and (Pre-Fixpoint) first. Then we need to prove \( \vdash \mu X. \circ X \rightarrow \circ \mu X. \circ X \).

(24, \( \neg \rightarrow \)) is proved by applying (Knaster-Tarski) and (Pre-Fixpoint) first. Then we need to prove \( \vdash \mu X. \circ X \rightarrow \circ \mu X. \circ X \).

(25, \( \rightarrow \)) is proved by applying (Knaster-Tarski) followed by (Pre-Fixpoint).

(25, \( \neg \rightarrow \)) is proved by applying (Knaster-Tarski) followed by (Pre-Fixpoint).

We obtain \( \vdash \mu X. \circ X \rightarrow \circ \mu X. \circ X \).
Above:

Axiom: 
ϕ ⊢ ϕ' ∈ A
A ⊢ C ϕ ⇒ ϕ'

Reflexivity: 
A ⊢ ϕ ⇒ ϕ

Transitivity: 
A ⊢ C ϕ₁ ⇒ ϕ₂, A ∪ C ⊢ ϕ₂ ⇒ ϕ₃
A ⊢ C ϕ₁ ⇒ ϕ₃

Logic Framing: 
A ⊢ C ϕ ⇒ ϕ' ψ is a FOL formula
A ⊢ C ϕ ∧ ψ ⇒ ϕ' ∧ ψ

Consequence: 
M^{c[ψ]} ⊢ ϕ₁ ⇒ ϕ'₁, A ⊢ C ϕ₁' ⇒ ϕ₂' M^{c[ψ]} ⊢ ϕ₂' ⇒ ϕ₂
A ⊢ C ϕ₁ ⇒ ϕ₂

Case Analysis: 
A ⊢ C ϕ₁ ⇒ ϕ
A ⊢ C ϕ₂ ⇒ ϕ
A ⊢ C ϕ₁ ∨ ϕ₂ ⇒ ϕ

Abstraction: 
A ⊢ C ϕ ⇒ ϕ', X ∈ FV(ϕ') = Ø
A ⊢ C ∃X. ϕ ⇒ ϕ'

Circularity: 
A ⊢ C (∃ϕ'₁ → ϕ) ⇒ ϕ'
A ⊢ C ϕ ⇒ ϕ'

Fig. 8. Reachability logic proof system

Apply (Knaster-Tarski) on ⊢, and we obtain ⊢ μX.○X → ○(|X.○X|), finished by (Pre-Fixpoint).

(25, “→”) is proved by (8), (10), and then apply Lemma 85.
(26) is proved by applying (Knaster-Tarski) firstly. After propositional reasoning, we obtain two goals: ⊢ □ϕ₁ ∧ ↓ wϕ₂ → ϕ₁ ∧ •(□wϕ₁ ∧ ↓ wϕ₂) and ⊢ □ϕ₁ ∧ ↓ wϕ₂ → ϕ₂ ∧ •(□wϕ₁ ∧ ↓ wϕ₂).
The first goal is easily proved by (8). The second goal is by unfolding “○wϕ₂” and “□ϕ₁”, and then use (18).
(27) is proved by (8).

Lemma 116. A ⊢ C ϕ₁ ⇒ ϕ₂ implies Γ^{R} ⊢ RL2MmL(A ⊢ C ϕ₁ ⇒ ϕ₂).

Proof: We need to prove that all reachability logic proof rules in Fig. 8 are provable in μ-logic.

(Axiom). We prove for the case when C ≠ Ø. The case when C = Ø is the same. Our goal, after translation, is Γ^{R} ⊢ ∀A ∧ ∀C → (ϕ₁ → ○wϕ₂). By assumption, ϕ₁ ⇒ ϕ₂ ∈ A, and thus we just need to prove Γ^{R} ⊢ ∀(ϕ₁ → ○wϕ₂) → (ϕ₁ → ○wϕ₂), which is trivial by FOL reasoning.

(Relexivity). Notice that C = Ø in this rule. Our goal, after translation, is Γ^{R} ⊢ ∀A → (ϕ₁ → ○wϕ₂), which is true by Proposition 115.

(Transitivity, C ≠ Ø). Our goal, after translation, is Γ^{R} ⊢ ∀A → (ϕ₁ → ○wϕ₂). Our two assumptions are Γ^{R} ⊢ ∀A → (ϕ₁ → ○wϕ₂) and Γ^{R} ⊢ ∀A → (ϕ₂ → ○wϕ₂).
From the latter assumption and Proposition 115 we have Γ^{R} ⊢ ∀A → (○wϕ₂ → ○wϕ₂), and then by propositional reasoning and the former assumption we have Γ^{R} ⊢ ∀A → (ϕ₁ → ○wϕ₂). Finally, by Proposition 115, we have Γ^{R} ⊢ ∀A → (ϕ₁ → ○wϕ₂), which is what we want to prove.

(Transitivity, C ≠ Ø). Our goal, after translation, is Γ^{R} ⊢ ∀A ∧ ∀C → (ϕ₁ → ○wϕ₂). Our two assumptions are Γ^{R} ⊢ ∀A ∧ ∀C → (ϕ₁ → ○wϕ₂) and Γ^{R} ⊢ ∀A ∧ ∀C → (ϕ₂ → ○wϕ₂). From the first assumption, we have Γ^{R} ⊢ ∀A ∧ ∀C → ϕ₁ → ○wϕ₂, and thus by propositional reasoning, it suffices to prove that Γ^{R} ⊢ ∀A ∧ ∀C ∧ ϕ₁ → (ϕ₁ → ○wϕ₂). From the second assumption and Proposition 115(10), we know that Γ^{R} ⊢ ○wϕ₂(∀A ∧ ∀C ∧ ϕ₂) → ○wϕ₂, which by Proposition 115(23), implies Γ^{R} ⊢ ○wϕ₂(∀A ∧ ∀C ∧ ϕ₂) → ○wϕ₂. Then, it suffices to prove Γ^{R} ⊢ ∀A ∧ ∀C ∧ ϕ₁ → ○wϕ₂. From the rest is easy, since by Proposition 115(8), we just need to prove Γ^{R} ⊢ ∀A ∧ ∀C ∧ ϕ₁ → ○wϕ₂, which then by Proposition 115(18) becomes Γ^{R} ⊢ ○(∀A ∧ ∀C ∧ ϕ₁) ∧ ○wϕ₂, and then by Proposition 115(10) becomes Γ^{R} ⊢ ∀A ∧ ∀C ∧ ϕ₁ → ○wϕ₂, which is proved by Proposition 115(26).

(Logic Framing). We prove for the case C ≠ Ø. The case when C = Ø is the same. Our goal, after translation, is Γ^{R} ⊢ ∀A ∧ ∀C → (ϕ₁ ∧ ϕ₂ → ○wϕ₂). Our assumption is Γ^{R} ⊢ ∀A ∧ ∀C → (ϕ₁ → ○wϕ₂). Notice that FOL formula ϕ is a predicate pattern, so ⊢ ○wϕ₂ ∧ ϕ → (ϕ₁ ∧ ϕ₂) ∧ ϕ, and the rest is by propositional reasoning.
The condition that ψ is a FOL formula (and thus a predicate pattern) is crucial to propagate ψ throughout its context.


(Circularity). We prove for the case when C ≠ Ø, as the case when C = Ø is the same. Our goal, after translation, is Γ^{R} ⊢ ∀A ∧ ∀C → (ϕ₁ → ○wϕ₂). By FOL reasoning and Proposition 115(20,25,225), the goal becomes Γ^{R} ⊢ μX.○X → ∀A ∧ ∀C → ϕ₁ → ○wϕ₂. By (Knaster-Tarski) and some FOL reasoning, it suffices to prove Γ^{R} ⊢ ○(∀A ∧ ∀C → (ϕ₁ → ○wϕ₂)) ∧ ∀A ∧ ∀C → ϕ₁ → ○wϕ₂. Our assumption, after translation, is Γ^{R} ⊢ ∀A ∧ ∀C → (ϕ₁ → ○wϕ₂) → (ϕ₁ → ○wϕ₂), so it suffices to prove Γ^{R} ⊢ ○(∀A ∧ ∀C → (ϕ₁ → ○wϕ₂)) ∧ ∀A ∧ ∀C → (ϕ₁ → ○wϕ₂), which by some propositional reasoning becomes Γ^{R} ⊢ ○(∀A ∧ ∀C → (ϕ₁ → ○wϕ₂)) ∧ ∀A ∧ ∀C → (ϕ₁ → ○wϕ₂) ∧ ○(∀A ∧ ∀C → (ϕ₁ → ○wϕ₂)), which then by some propositional reasoning becomes Γ^{R} ⊢ ○(∀A ∧ ∀C → (ϕ₁ → ○wϕ₂)) ∧ ∀A ∧ ∀C → (ϕ₁ → ○wϕ₂) ∧ ○(∀A ∧ ∀C → (ϕ₁ → ○wϕ₂)). By Proposition 115(8), it becomes Γ^{R} ⊢ ○(∀A ∧ ∀C → (ϕ₁ → ○wϕ₂)) ∧ ∀A ∧ ∀C → (ϕ₁ → ○wϕ₂), and by Proposition 115(5,6,10), it becomes Γ^{R} ⊢ ○(∀A ∧ ∀C → (ϕ₁ → ○wϕ₂)) ∧ ∀A ∧ ∀C → (ϕ₁ → ○wϕ₂), which
is proved by Proposition 115(27).

**Corollary 117.** $S \vdash \emptyset \varphi_1 \Rightarrow \varphi_2$ implies $\Gamma^{RL} \vdash RL2MmL(S \vdash \emptyset \varphi_1 \Rightarrow \varphi_2)$.

**Proof:** Let $A = S$ and $C = \emptyset$ in Lemma 116.

**Lemma 118.** $\Gamma^{RL} \models RL2MmL(S \vdash \emptyset \varphi_1 \Rightarrow \varphi_2)$ implies $S \models RL \varphi_1 \Rightarrow \varphi_2$.

**Proof:** Let $S = (M_{cfg}^{cfg}, R)$ be the transition system that is yielded by $S$. We tactically use the same letter $S$ to mean the extended $\Sigma^{RL}$-model $M^{c} \models \varphi$ with $\bullet \in \Sigma_{cfg}$ be interested as the transition relation $R$. Then $S \models \Gamma^{RL}$, because all axioms in $\Gamma^{RL}$ are about only the configuration model $M_{cfg}$ and says nothing about the transition relation $R$. Since $M^{c} \models \varphi$ (by definition), then $S \models \Gamma^{c}$. By condition of the lemma, $S \models RL2MmL(S \vdash \emptyset \varphi_1 \Rightarrow \varphi_2)$, i.e., $S \models \forall \square S \rightarrow \varphi_1 \rightarrow \diamond w \varphi_2$. By construction of $S$, for all rules $\psi_1 \Rightarrow \psi_2 \in S$, we have $S \models \psi_1 \rightarrow \bullet \psi_2$ (in $MmL$), which implies $S \models \forall \square (\psi_1 \rightarrow \psi_2)$, meaning that $S \models \forall \square S$. Therefore, $S \models \forall \square \varphi_1 \rightarrow \diamond w \varphi_2$ (in $MmL$), which is exactly the same meaning as $S \models RL \varphi_1 \Rightarrow \varphi_2$ (in $RL$).

Finally, we are ready to prove Theorem 39.

**Proof of Theorem 39.** Following the same roadmap as in the proof of Theorem 31, where (2) $\Rightarrow$ (3) is given by Corollary 117 and (5) $\Rightarrow$ (6) is given by Lemma 118. The rest is by the sound and (relative) completeness of $RL$. Notice that technical assumptions of [2] are needed for the completeness result of $RL$. 

\[\blacksquare\]