LIPSCHITZ AND HÖLDER MAPPINGS INTO JET SPACE CARNOT GROUPS

BY

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DISSERTATION

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Abstract

For $k, n \geq 1$, the jet space $J^k(\mathbb{R}^n)$ is the set of k^{th}-order Taylor polynomials of functions in $C^k(\mathbb{R}^n)$. Warhurst constructs a Carnot group structure on $J^k(\mathbb{R}^n)$ such that the jets of functions in $C^{k+1}(\mathbb{R}^n)$ are horizontal [War05a]. Like in all Carnot groups, one can define a Carnot-Carathéodory metric on $J^k(\mathbb{R}^n)$ by minimizing lengths of horizontal paths. Unfortunately, exact forms or even the regularities of geodesics connecting generic pairs of points are not known for $J^k(\mathbb{R}^n)$. We will study the metric structure of $J^k(\mathbb{R}^n)$, focusing primarily on the model filiform jet spaces $J^k(\mathbb{R})$.

After describing the Carnot group structure of $J^k(\mathbb{R}^n)$, we will prove that there exists a biLipschitz embedding of $\mathbb{S}^n$ into $J^k(\mathbb{R}^n)$ that does not admit a Lipschitz extension to $\mathbb{B}^{n+1}$. This strengthens a result of Rigot and Wenger [RW10] and generalizes a result for $\mathbb{H}^n$ of Dejarnette, Hajłasz, Lukyanenko, and Tyson [DHLT14].

We will then consider a problem related to Gromov's conjecture on the Hölder equivalence of Carnot groups. We will prove that for all $m \geq 2$ and $\epsilon > 0$, there does not exist an injective, locally $(\frac{1}{2} + \epsilon)$-Hölder mapping $f : \mathbb{R}^m \to J^k(\mathbb{R})$ that is locally Lipschitz as a mapping into $\mathbb{R}^{k+2}$. This builds on a result of Balogh, Hajłasz, and Wildrick for $\mathbb{H}^n$ [BHW14].

We will conclude by proposing analogues of horizontal and vertical projections for $J^k(\mathbb{R})$. We prove Marstrand-type results for these mappings. This continues efforts of Balogh, Durand-Cartagena, Fässler, Mattila, and Tyson over the past decade to prove Marstrand-type theorems in a sub-Riemannian setting [BDCF$^+$13, BFMT12].

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Chapter 1

Introduction

Like many works of math before it, this thesis will begin with a relevant real-world example and then proceed with abstraction/fantasy gradually injected into the example until it is a direct analogue of the research problem/can no longer be classified as a real-world example. We will start by discussing one of the most familiar examples of a sub-Riemannian space, especially if you grew up near San Francisco like the author: the roto-translation group. This group models the motion of a car and it is how we will set this thesis in motion.

Why is parallel parking such a pain? Suppose your car is parallel to a parking spot and you want to pull into the spot. Assuming you're not Mr. Incredible, you have to perform this weird dance with your car of pulling past the spot, turning your wheel, backing up, and then straightening out. But why would you choose this procedure to parallel park? There are a dozen other ways you could move your car into the spot. You could drive in front of the spot then pull backwards. Or maybe you could drive backwards fifty yards and slowly merge into the spot. Or you could even make a spiral into the spot. What's with the weird dance? The reasoning behind it is that your car can only change its state in two ways: by pressing the pedal to move forwards or backwards and by turning the wheel to change its angle (and we guess also by crossing a border, but that won't be relevant for our purposes). It cannot move perpendicularly to the direction it is facing. Thus, to parallel park and move in a direction orthogonal to its current state, a car must perform a combination of turning and moving backwards or forwards. Besides not causing massive damage to other cars or possibly hurting pedestrians, the weird dance is optimal because it is the most energy-efficient way to get into the spot with only two methods of control.

For parallel parking, the problem boils down to determining the optimal way to change from one position and angle to another position and angle. Now suppose we attach a trailer to our car. Due to the strange configuration of parking spots, we now care about three variables: the position and angle of the car and the angle of the trailer. (We warned you that this example would get complex/fantastical. There is much more coming.) As you move the car forwards, the trailer will straighten out and slowly conform to the angle of the car. As you turn, the trailer will also turn. What is the best way to park your car/trailer combo, and more generally, what is the best way to transition from one state to another?

But oh wait, you have forgotten to take into account the environment you're driving in. If you're driving on ice, it will be much easier to drive forwards and turn. But if you're driving through cornfields, it will be much easier (and slightly less criminal) driving along rows compared to turning into stalks. And if you're driving up a mountain, it would be much easier to just turn so that you could drive on a level surface. Taking into account all of these things, what would be the best way to move your car and trailer from one state to another? If you were attempting to get to the other side of a mountain, it would probably be easiest to drive around the mountain. But if you needed to get to the other side of a long fence, it might be easier to ram the fence and go straight through rather than drive around. Now interpolate these examples and suppose there is a fence on a mountain. What's the best way to get past the fence now? (We're guessing though that the fence is on the mountain to keep out mountain lions, so we would highly recommend not damaging the fence.) If you think this example hasn't involved enough convolution, then we are happy to report we can do better. (Also you're probably a harmonic analyst and we must forewarn that this is probably not the thesis for you.)

Suppose instead of just one trailer, you had a hundred trailers attached to a single car. How would you properly change the position of the car and all of the angles of the trailers? (By the way, why do you have a hundred trailers? We think you have a problem. Your spouse and children miss you.) For every two states, there will be a (possibly not unique) most energy-efficient way to transition from one state to the other. We could define the distance of one state to another state to be the least energy it takes to transition between them. In this thesis, we will be interested in analyzing the metric structure resulting from defining this distance function.

Things look pretty poor from the onset. It isn't clear at all what the best path is between two generic states or how much energy this path will take. For example, changing the angle of a trailer by a radian will be much more difficult among the cornfields compared to on a mountainside. How is one supposed to say anything meaningful about the metric structure? Fortunately, we have two very crude tools for estimating distances in this space.

Our first tool will be a continuity condition: As states get closer together, the energy required to move between them will become small. Moreover, we have quantitative bounds on what it means to become small. For example, in an icy region, to change the angle of the fifth trailer by θ radians, we might know that it takes somewhere between $\frac{1}{100}\theta^{1/7}$ and $100\theta^{1/7}$ units of energy. While these bounds might seem rough (and possibly useless), they do come in handy when one is trying to prove a statement for which fixed constant factors can be disregarded. For our purposes, the bounding terms typically won't be as simple as multiples of $\theta^{1/7}$. Rather, the bounding terms will be polynomials in terms of the coordinates of the two states. The point, though, is that one can estimate a distance between two states by knowing how they interact with each other (or in this example, from knowing the layout of the environment).

Our second tool will be the existence of traversable paths in our state space. In our example with the trailers, you can travel along every road (assuming the road can accomodate a hundred trailers and an apparently very strong car). Thus, for any two states attained along the road, the energy needed to travel between them will be at most the energy required to travel between them while traveling along the road. This will provide us with bounds on distances between two states, despite the fact that the form of an actual geodesic is beyond us.

Now time to math. Before stating the general problem this thesis seeks to address, we need to describe the primary class of spaces to be studied: the model filiform jet space Carnot groups $J^k(\mathbb{R})$. Each space modeling the movement of a car pulling $k-1$ trailers is locally modeled by $J^k(\mathbb{R})$ (in the sense of tangent group approximations). Each group $J^k(\mathbb{R})$ is exactly the space modeling the movement of a car pulling $k-1$ tractors. For each $k \in \mathbb{N}$, $J^k(\mathbb{R})$ is defined to be the set of all k^{th}-order Taylor polynomials of functions in $C^k(\mathbb{R})$. As a k^{th} order Taylor polynomial is determined entirely by a basepoint and derivatives up to order k at the point, $J^k(\mathbb{R})$ can be identified with $\mathbb{R}^{k+2}$. Motivated by the collection of k differential equations that all functions in $C^k(\mathbb{R})$ satisfy (that is, $df^{(j)} = f^{(j+1)}dx$ for $j = 0, \ldots, k-1$), one can assign a two-dimensional plane to each point of $\mathbb{R}^{k+2}$ of permissible directions in which to move. As in the space modeling the movement of a car pulling trailers, one can define the distance (called the **Carnot-Carathéodory distance**) between any two points in $\mathbb{R}^{k+2}$ to be the length of the shortest connecting path that flows along these planes. Moreover, Warhurst constructed a group operation on $J^k(\mathbb{R})$ so that the planes and distance are both compatible with the group structure and $J^k(\mathbb{R})$ is a step $k+1$ Carnot group [War05a]. We remark that one can define more general jet spaces $J^k(\mathbb{R}^n)$ by considering Taylor polynomials of functions in $C^k(\mathbb{R}^n)$. In his thesis, Warhurst also equipped these general jet spaces with Carnot group structures [War05a].

These jet space are part of a larger collection of groups called Carnot groups, in which the Heisenberg groups are most well-known. In all of these Carnot groups, a plane is assigned to each point along which paths are allowed to flow. As in the jet space case, one then can define a distance based on the minimal length of connecting paths. For the Heisenberg groups $\mathbb{H}^n$, these planes are codimension-1 and the exact forms of geodesics connecting any pair of points are known. This enables one to calculate the distance between any two points and prove that the distance function measuring distance from the origin is analytic away from the vertical axis [HZ15].

The situation is far worse for jet space Carnot groups. Exact forms of geodesics connecting generic pairs of points or even the regularities of these geodesics are not even known. In fact, Le Donne, Pinamonti, and

Speight showed that the distance function measuring distance from the origin is not even Pansu differentiable in horizontal directions [LDPS17]. My research focused on better understanding the metric structure of these jet spaces.

My research sought to answer:

How Euclidean is $J^k(\mathbb{R})$ as a metric space?

For example, can one embed copies of Euclidean space into $J^k(\mathbb{R})$? With which regularity? What are examples of biLipschitz, Lipschitz, or Hölder mappings of Euclidean space into $J^k(\mathbb{R})$? What are examples of Lipschitz or Hölder mappings of $J^k(\mathbb{R})$ to itself? How does the dimension (topological or Hausdorff) of a set compare to the dimension of its image under a Lipschitz or Hölder mapping? Answering these questions will help us better understand the metric structure of $J^k(\mathbb{R})$ arising from navigating a possibly high-dimensional space using only two controls.

As in the car pulling trailers example, one has two tools with which to estimate distances in jet spaces: a continuity condition and the existence of many traversable paths. In 1985, Nagel, Stein, and Wainger proved that the identity map is a homeomorphism between $J^k(\mathbb{R})$ and $\mathbb{R}^{k+2}$, where the latter is now implicitly equipped with the standard Euclidean metric [NSW85]. In fact, their result applied in our context states that the identity map $\mathrm{id} : J^k(\mathbb{R}) \to \mathbb{R}^{k+2}$ is locally Lipschitz with its inverse locally $\frac{1}{k+1}$-Holder. One can then use the dilations and group operation of $J^k(\mathbb{R})$ to estimate the distance between *any* two points. This is commonly referred to as the Ball-Box Theorem.

In addition, for every function in $C^{k+1}(\mathbb{R})$, there exists a locally biLipschitz copy of $\mathbb{R}$ in $J^k(\mathbb{R})$. To explain what we mean by this, first define $j^k_x(f)$ to be the k^{th}-order Taylor polynomial of $f \in C^k(\mathbb{R})$ at $x \in \mathbb{R}$. Rigot and Wenger observed that the jet mapping $j^k(f) : \mathbb{R} \to J^k(\mathbb{R})$ is a horizontal path for all $f \in C^{k+1}(\mathbb{R})$. Thus one can locally estimate from above the distance between any two points in the image of $j^k(f)$. On the other hand, one can estimate this distance from below using the Ball-Box Theorem. Thus, $j^k(f) : \mathbb{R} \to J^k(\mathbb{R})$ is a locally biLipschitz mapping (see Proposition 2.13).

This thesis is based upon the following three papers:

[Jun17b] Derek Jung. *A variant of Gromov's problem on Hölder equivalence of Carnot groups.* J. Math. Anal. Appl., 456(1):251-273, 2017.

[Jun17a] Derek Jung. *BiLipschitz embeddings of spheres into jet space Carnot groups not admitting Lipschitz extensions.* Annales Academiæ Scientiarum Fennicæ (accepted). 2018.

[Jun18] Derek Jung. *Dimension results for mappings of jet space Carnot groups.* Available at arXiv:1804.09069 [math.GT]. April 2018.

In Chapter 3, we will consider the question: With what regularity can one embed a circle into $J^k(\mathbb{R})$, and more generally, a sphere $\mathbb{S}^n$ into $J^k(\mathbb{R}^n)$? It is well-known that there exists a biLipschitz embedding of the circle into $\mathbb{H}^1$, a group that is isomorphic and biLipschitz equivalent to $J^1(\mathbb{R})$. In fact, an explicit example is given by concatenating two geodesics that connect the origin to a point on the vertical axis. In [DHLT14], Dejarnette, Hajłasz, Lukyanenko, and Tyson provide examples of biLipschitz (in fact, smooth and horizontal) embeddings of $\mathbb{S}^n$ into $\mathbb{H}^n$ that do not admit Lipschitz extensions to $\mathbb{B}^{n+1}$. As $J^1(\mathbb{R}^n)$ is isomorphic and biLipschitz equivalent to $\mathbb{H}^n$ for all n, one could ask if it is possible to extend this result for all k. More specifically, for all $k, n \geq 1$, does there exist a biLipschitz embedding of $\mathbb{S}^n$ into $J^k(\mathbb{R}^n)$ that does not admit a Lipschitz extension to $\mathbb{B}^{n+1}$? We answer this question in the affirmative (Theorem 3.1). This strengthens a result of Rigot and Wenger in [RW10], where they prove that there exists a (non-injective) Lipschitz mapping $f : \mathbb{S}^n \to J^k(\mathbb{R}^n)$ that does not admit a Lipschitz extension $F : \mathbb{B}^{n+1} \to J^k(\mathbb{R}^n)$.

In Chapter 4, we ask: With what regularity can one embed a Euclidean space into $J^k(\mathbb{R})$? We study a problem related to Gromov's conjecture on Hölder equivalence of Carnot groups. Gromov asked: Given a Carnot group $(\mathbb{R}^n, \cdot)$, for which α does there exist a locally α-Hölder homeomorphism $f : \mathbb{R}^n \to (\mathbb{R}^n, \cdot)$? Using an isoperimetric inequality for Carnot groups [Val44], Gromov proved that any such α would need to satisfy $\alpha \leq \frac{n-1}{Q-1}$, where Q is the Hausdorff dimension of $(\mathbb{R}^n, \cdot)$. On the other hand, Nagel, Stein, and Wainger proved that if $(\mathbb{R}^n, \cdot)$ is step r, then the identity map $\mathrm{id} : \mathbb{R}^n \to (\mathbb{R}^n, \cdot)$ is locally $\frac{1}{r}$-Hölder [NSW85], hence $\alpha \geq \frac{1}{r}$. Even for the first Heisenberg group, these are the best known bounds for α in Gromov's conjecture. In 2014, Balogh, Hajłasz, and Wildrick considered a related problem. They proved that for all $m > n$ and $\epsilon > 0$, there does not exist an injective, locally $(\frac{1}{2}+\epsilon)$-Hölder mapping $f : \mathbb{R}^m \to \mathbb{H}^n$ that is locally Lipschitz as a mapping into $\mathbb{R}^{2n+1}$ [BHW14]. The key to their proof was noting that such a map would need to be differentiable a.e. by Rademacher's theorem, and at any point of differentiability, the map would have to be horizontal. The reason for requiring $m > n$ is that $\mathbb{H}^n$ is purely m-unrectifiable for all $m > n$ [Mag04], which essentially means that $\mathbb{H}^n$ does not see Lipschitz images of $\mathbb{R}^m$ from a metric standpoint. Using that $J^k(\mathbb{R})$ is purely 2-unrectifiable, we showed that one could prove an analogous result for $J^k(\mathbb{R})$. We prove that for all $m \geq 2$ and $\epsilon > 0$, there does not exist an injective, locally $(\frac{1}{2} + \epsilon)$-Hölder mapping $f : \mathbb{R}^m \to J^k(\mathbb{R})$ that is locally Lipschitz as a mapping into $\mathbb{R}^{k+2}$ (Corollary 4.4). As in the Heisenberg case, the key to the proof will be showing that at any points of differentiability, such a map would need to be horizontal (Proposition 4.15).

We conclude in Chapter 5 by considering the questions: On which subsets in $J^k(\mathbb{R})$ do the restriction of the Carnot-Carathéodory distance admit a simple form? Could one define projections (or at least analogues of projections) onto these sets? How will a set in $J^k(\mathbb{R})$ compare to its image by one of these projections? This is an attempt in a continued effort to prove Marstrand-type results in a sub-Riemannian setting. In 1954, Marstrand proved statements comparing the size of a set in Euclidean space to the size of its orthogonal projections to planes [Mar54]. In 2013, Balogh, Durand-Cartagena, Fässler, Mattila, and Tyson defined horizontal and vertical projections in $\mathbb{H}^1$ [BDCF$^+$13]. The authors defined projections onto the horizontal lines $V_\theta \subset \mathbb{R}^2 \times \{0\}$ passing through the angle at angle θ and projections onto the orthogonal complement $V_\theta^\perp$. While the mappings onto the orthogonal complements are not technically linear or projections in general, the Korányi distance restricts to a simple snow-flaked form on these planes. Motivated by this, we will consider analogues of projections onto the vertical hyperplanes with first coordinate fixed in $J^k(\mathbb{R})$. To complement these planes, we will take images of jets of smooth functions. Each plane with the jet of a smooth function provides a splitting of $J^k(\mathbb{R})$ and induces horizontal and vertical mappings of the group. We will investigate how the Hausdorff and topological dimension of sets are affected by these mappings.

And, of course, this introduction will be followed by an (obligatory) background section.

Chapter 2

Jet spaces as Carnot groups

Before diving into my work, we will provide general background material and notation that will be used throughout this thesis.

We begin by describing the structure of Carnot groups and how these groups can be identified with Euclidean spaces via coordinates of the first and second kind. We then define the Carnot-Carathéodory distance for Carnot groups and describe the resulting metric structure. We will conclude by focusing on a specific class of Carnot groups: jet space Carnot groups. This is my favorite class of Carnot groups and has been the focus of my research.

2.1 The Lie algebra structure of Carnot groups

A Lie algebra $\mathfrak{g}$ is said to admit an r-step **stratification** if

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r,$$

where $\mathfrak{g}_1 \subseteq \mathfrak{g}$ is a subspace, $\mathfrak{g}_{j+1} = [\mathfrak{g}_1, \mathfrak{g}_j]$ for all $j = 1, \ldots, r-1$, and $[\mathfrak{g}, \mathfrak{g}_r] = 0$. A **Carnot group** is a connected, simply-connected, nilpotent Lie group with stratified Lie algebra. If the Lie algebra of a Carnot group G has an r-step stratification, then we will say G is step r. This is well-defined [BLU07, Proposition 2.2.8]. We write $[\mathfrak{g}_1, \mathfrak{g}_j]$ above to denote the subspace generated by commutators of elements of $\mathfrak{g}_1$ with elements of $\mathfrak{g}_j$, and similarly with $[\mathfrak{g}, \mathfrak{g}_r]$. The subspaces $\mathfrak{g}_j$ are commonly referred to as the **layers** of $\mathfrak{g}$, with $\mathfrak{g}_1$ referred to as the **horizontal layer**. *Throughout, we will implicitly fix a stratification for each Carnot group.* In other words, we will view the stratification of $Lie(G)$ as data of a Carnot group G.

After combining bases of the subspaces $\mathfrak{g}_j$ to obtain a basis of $\mathfrak{g}$, we can define an inner product $g = \langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by declaring the combined basis to be orthonormal. Thus, we say that a basis $\mathcal{B} = \{X^1, \ldots, X^n\}$ of $\mathfrak{g}$ is **compatible with the stratification of** $\mathfrak{g}$ if

$$\{X^{h_{j-1}+1}, \ldots, X^{h_j}\}$$

is a basis of $\mathfrak{g}_j$ for each j, where $h_j = \sum_{i=1}^{j} \dim(\mathfrak{g}_i)$. As we discuss coordinates of the first and second kind, it will be implied that coordinates are being taken with respect to a basis compatible with the stratification of $\mathfrak{g}$. While choosing different bases may technically result in different group structures, we will see that the resulting Carnot groups are all isomorphic to G.

2.2 Identification of a Carnot group with Euclidean space

At first glance, Carnot groups may seem somewhat unwieldy. A Lie group is a Carnot group if it is Euclidean from a topological standpoint and the Lie algebra satisfies a bracket-generating condition. Fortunately, this nice structure can be leveraged to identify each Carnot group with a Euclidean space equipped with a group operation. In this section, we will describe coordinates of the first and second kind, two ways that we can make this identification. While the choices of coordinates and compatible basis may vary, each pair of choices results in a group isomorphic to the original Carnot group.

For every Carnot group G, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism [CG90, Page 13]. Hence we can define $\star : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$X \star Y = \exp^{-1}(\exp(X)\exp(Y)).$$

The Baker-Campbell-Hausdorff formula gives us an explicit formula for $X \star Y$:

$$X \star Y = \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{0<p_i+q_i} \frac{1}{C_{p,q}} (\mathrm{ad}X)^{p_1} (\mathrm{ad}Y)^{q_1} \cdots (\mathrm{ad}Y)^{q_{n-1}} W(p_n, q_n),$$

where

$$C_{p,q} = p_1! q_1! \cdots p_n! q_n! \sum_{i=1}^{n} (p_i + q_i)$$

and

$$W(p_n, q_n) = \begin{cases} (\mathrm{ad}X)^{p_n} (\mathrm{ad}Y)^{q_n - 1} Y, & \text{if } q_n \geq 1, \\ (\mathrm{ad}X)^{p_n - 1} X, & \text{if } q_n = 0. \end{cases}$$

The expansion of $X \star Y$ up to order 3 is given by

$$X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]).$$

Set n equal to the topological dimension of G, and let $\mathcal{B} \subset \mathfrak{g}$ be a basis compatible with the stratification of $\mathfrak{g}$. We can identify $\mathfrak{g}$ with $\mathbb{R}^n$ via coordinates of $\mathcal{B}$, and then $\star$ on $\mathfrak{g}$ translates into an operation on $\mathbb{R}^n$. With a slight abuse of notation, we will also denote this operation on $\mathbb{R}^n$ by $\star$. Then $(\mathbb{R}^n, \star)$ is a Carnot group isomorphic to G via the exponential map [BLU07, Proposition 2.2.22]. We say that $(\mathbb{R}^n, \star)$ is G equipped with **coordinates of the first kind with respect to** $\mathcal{B}$. Observe that if G is step r, each coordinate of $X \star Y$ is a polynomial of homogeneous degree at most r in the coordinates of X and Y.

We define the family of dilations $\{\mathfrak{d}_\epsilon\}_{\epsilon>0}$ to be the collection of isomorphisms of $\mathfrak{g}$ induced by $\mathfrak{d}_\epsilon(X_j) = \epsilon^j X_j$, $X_j \in \mathfrak{g}_j$. Each $\mathfrak{d}_\epsilon$ is a Lie group automorphism of $(\mathfrak{g}, \star)$ [BLU07, Remark 1.3.32], i.e.,

$$\mathfrak{d}_\epsilon(X \star Y) = (\mathfrak{d}_\epsilon(X)) \star (\mathfrak{d}_\epsilon(Y)) \quad \text{for all } X, Y \in \mathfrak{g}. \tag{2.1}$$

These dilations on $\mathfrak{g}$ are also commonly notated as δ_ϵ, but we will not do so here to avoid confusion with the dilations on G. As the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism, this induces a family of dilations δ_ϵ on G:

$$\delta_\epsilon := \exp \circ \mathfrak{d}_\epsilon \circ \exp_G . \tag{2.2}$$

We will now describe a second system of coordinates that one can equip Carnot groups with. The resulting Carnot group will also be isomorphic to G. We will first state a result that will allow us to define our other model.

Theorem 2.1. ([Var74, Theorem 2.10.1]) Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $\mathfrak{g}$ is the direct sum of linear subspaces $\mathfrak{h}_1, \ldots, \mathfrak{h}_s$. Then there are open neighborhoods B_i of 0 in $\mathfrak{h}_i$ $(1 \leq i \leq s)$ and U of 1 in G, such that the map

$$\Psi : (Z_1, \ldots, Z_s) \mapsto \exp Z_1 \cdots \exp Z_s$$

is an analytic diffeomorphism of $B_1 \times \cdots \times B_s$ onto U.

Fix a basis $\mathcal{B} = \{X^1, \ldots, X^n\}$ of $\mathfrak{g}$ compatible with the stratification, and define $\Phi : \mathfrak{g} \to G$ by

$$\Phi(a_1 X^1 + \cdots + a_n X^n) = \exp(a_1 X^1) \cdots \exp(a_n X^n).$$

By Theorem 2.1, the restriction $\Phi|_V : V \to U$ is a diffeomorphism for some open neighborhoods $V \subset \mathfrak{g}$ of 0 and $U \subset G$ of e. After noticing $\Phi(a_1 X^1 + \cdots + a_n X^n) = \exp(a_1 X^1 \star \cdots \star a_n X^n)$, it follows from (2.1) and (2.2) that Φ is a global diffeomorphism.

We can then define $\odot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$X \odot Y = \Phi^{-1}(\Phi(X)\Phi(Y)).$$

As for coordinates of the first kind, we can identify $\mathfrak{g}$ with $\mathbb{R}^n$ and define a corresponding operation $\odot$ on

$\mathbb{R}^n$, with a slight abuse of notation. We say $(\mathbb{R}^n, \odot)$ is G equipped with **coordinates of the second kind with respect to** $\mathcal{B}$. If $(\mathbb{R}^n, \star)$ is $\mathfrak{g}$ equipped with coordinates of the first kind via the same basis, observe that $\exp^{-1} \circ \Phi : (\mathbb{R}^n, \odot) \to (\mathbb{R}^n, \star)$ is a Lie group isomorphism. In particular, $(\mathbb{R}^n, \odot)$ is isomorphic to G. In this thesis, we will be primarily interested in this second system of coordinates. It is this system that Warhurst used to define a Carnot group structure on jet spaces [War05a].

2.3 Metric structure of Carnot groups

As described in the previous section, a Carnot group may be identified (isomorphically) with a Euclidean space equipped with an operation via coordinates of the first or second kind. Henceforth, we will consider Carnot groups of the form $(\mathbb{R}^n, \cdot)$.

Before we dive into defining the Carnot-Carathéodory metric on a Carnot group, we will give a heuristic description of the metric. As mentioned previously, every Carnot group is equipped with a horizontal bundle. This bundle defines a subspace of the full tangent space at each point and the subspaces all share the same dimension. The thing that can change is the actual subspace, which can be thought of as a plane assigned to each point. Possibly differing from point to point, these planes define a collection of "allowable" directions to move at each point. We will be interested in horizontal curves of the Carnot group, which can be thought as curves that flow along these planes. One can measure distance in a Carnot group by defining the distance between two points to be the length of the shortest connecting horizontal curve. It turns out that shortest curves actually exist, and this distance function satisfies properties related to the group's structure. We will now provide a more rigorous definition of this metric.

Let $\{X^1, \dots, X^{m_1}\}$ be a left-invariant frame for $Lie(\mathbb{R}^n, \cdot)$. The horizontal bundle $H(\mathbb{R}^n, \cdot)$ is defined fiberwise by

$$H_p(\mathbb{R}^n, \cdot) := \operatorname{span}\{X^1_p, \dots, X^{m_1}_p\}.$$

One can define an inner product on each fiber by declaring $\{X^1_p, \dots, X^{m_1}_p\}$ to be orthonormal. A path $\gamma : [a, b] \to (\mathbb{R}^n, \cdot)$ is said to be **horizontal** if it is absolutely continuous as a map into $\mathbb{R}^n$ and satisfies $\gamma'(t) \in H_{\gamma(t)}(\mathbb{R}^n, \cdot)$ for a.e. $t \in [a, b]$. The **length** of a horizontal path $\gamma : [a, b] \to (\mathbb{R}^n, \cdot)$ is defined to be

$$l(\gamma) := \int_a^b |\gamma'(t)|_H \, dt.$$

Chow proved that every Carnot group is horizontally path-connected [Cho39]. Hence, we may define a

Carnot-Carathéodory metric on $(\mathbb{R}^n, \cdot)$ by

$$d_{cc}(p,q) := \inf_{\gamma:[a,b]\to(\mathbb{R}^n,\cdot)} \{l(\gamma) : \gamma \text{ is horizontal}, \gamma(a) = p,\ \gamma(b) = q\}.$$

This forms a left-invariant metric that is in fact geodesic [BLU07, Theorem 5.15.5]. We will sometimes refer to the Carnot-Carathéodory metric as the Carnot-Carathéodory distance, CC-metric, or CC-distance.

For each $x \in (\mathbb{R}^n, \cdot)$, we may write $x = (\vec{x_1}, \ldots, \vec{x_r})$, where $\vec{x_j}$ is a $\dim(\mathfrak{g}_j)$-tuple. We define the dilations δ_ϵ, $\epsilon > 0$, on $(\mathbb{R}^n, \cdot)$ by

$$\delta_\epsilon(\vec{x_1}, \ldots, \vec{x_r}) = (\epsilon\vec{x_1}, \ldots, \epsilon^r \vec{x_r}).$$

These dilations coincide with the dilations defined in Section 2.2. A key property of the CC-distance is that it is one-homogeneous with respect to these dilations:

$$d_{cc}(\delta_\epsilon(x), \delta_\epsilon(y)) = \epsilon d_{cc}(x,y), \qquad x, y \in (\mathbb{R}^n, \cdot), \quad \epsilon > 0.$$

This property allows one to sometimes transform local statements on Carnot groups to global statements (see, for instance, Proposition 4.11).

In general Carnot groups, geodesics connecting points are not well-understood. For example, explicit formulas or even smoothness of geodesics are unknown in general. This results in it being difficult to make precise estimates on CC-distances.

Thankfully, the identity map of a Carnot group $(\mathbb{R}^n, \cdot)$ to the Euclidean space $\mathbb{R}^n$ is a homeomorphism. In fact, Nagel, Stein, and Wainger proved that that it is locally Lipschitz with locally Hölder inverse [NSW85].

Theorem 2.2. [NSW85] Let $(\mathbb{R}^n, \cdot)$ be a step r Carnot group. Then $\mathrm{id} : \mathbb{R}^n \to (\mathbb{R}^n, \cdot)$ is locally $\frac{1}{r}$-Hölder and $\mathrm{id} : (\mathbb{R}^n, \cdot) \to \mathbb{R}^n$ is locally Lipschitz.

While it may seem that the localness of the previous result will hinder its usefulness, left-invariance and homogeneity can be leveraged to obtain a global result in the form of the Ball-Box Theorem.

Theorem 2.3. (Ball-Box Theorem) Suppose $(\mathbb{R}^n, \cdot)$ is a step r Carnot group. Define $m_j := \dim(\mathfrak{g}_j)$ for each j. For $\epsilon > 0$ and $p \in (\mathbb{R}^n, \cdot)$, define

$$Box(\epsilon) := \prod_{j=1}^{r} [-\epsilon^j, \epsilon^j]^{m_j}$$

and

$$B_{cc}(p, \epsilon) := \{q \in J^k(\mathbb{R}^n) : d_{cc}(p,q) \le \epsilon\}.$$

There exists $C > 0$ such that for all $\epsilon > 0$ and $p \in (\mathbb{R}^n, \cdot)$,

$$B_{cc}(p, \epsilon/C) \subseteq p \cdot Box(\epsilon) \subseteq B_{cc}(p, C\epsilon).$$

The essence of the Ball-Box Theorem is that you can estimate the distance of a point from the origin by its algebraic coordinates. We will state this more clearly in the following corollary of the Ball-Box Theorem. This will serve as our primary tool for estimating distances in Carnot groups.

Corollary 2.4. Suppose $(\mathbb{R}^n, \cdot)$ is a step r Carnot group. There exists $C > 0$ such that for all $p = (\vec{x_1}, \ldots, \vec{x_r}) \in (\mathbb{R}^n, \cdot)$,

$$\frac{1}{C} \cdot d_{cc}(0, p) \leq \max\{|\vec{x_j}|^{1/j} : 1 \leq j \leq r\} \leq C d_{cc}(0, p).$$

By left-invariance, $d_{cc}(p, q) = d_{cc}(0, p^{-1} \cdot q)$ for all $p, q \in (\mathbb{R}^n, \cdot)$. The last corollary implies that we may estimate the CC-distance between p and q by the coordinates of $p^{-1} \cdot q$. Hence, if we can apply our knowledge of the group operation on $(\mathbb{R}^n, \cdot)$ to obtain the precise form for $p^{-1} \cdot q$, we will be able to (crudely) estimate the distance between p and q. This is the primary technique we use to perform distance estimates in Carnot groups.

We conclude this section by mentioning a topic related to the embeddability of subsets of Euclidean spaces into Carnot groups.

Definition 2.5. A Carnot group $(\mathbb{R}^n, \cdot)$ is said to be **purely k-unrectifiable** if for every $A \subseteq \mathbb{R}^k$ and Lipschitz map $f : A \to (\mathbb{R}^n, \cdot)$, we have

$$\mathcal{H}^k_{cc}(f(A)) = 0.$$

Here, $\mathcal{H}^k_{cc}(f(A))$ denotes the k-Hausdorff measure of $f(A)$ with respect to the CC-distance.

Ambrosio and Kirchheim proved that $\mathbb{H}^1$ is purely k-unrectifiable for $k = 2, 3, 4$ [AK00, Theorem 7.2]. More generally, Magnani proved that a Carnot group is purely k-unrectifiable if and only if its horizontal layer does not contain a Lie subalgebra of dimension k [Mag04, Theorem 1.1]. In particular, $\mathbb{H}^n$ is purely k-unrectifiable for all $k > n$. In 2014, Balogh, Hajłasz, and Wildrick provided a different proof of this last result by using approximate derivatives and a weak contact condition [BHW14, Theorem 1.1]. In the process, they prove that a Lipschitz mapping of an open subset of $\mathbb{R}^k$, $k > n$, into $\mathbb{H}^n$ has an approximate derivative that is horizontal almost everywhere. This is related to my work in Chapter 4.

2.4 The Heisenberg groups

The most well-known class of Carnot groups (besides the abelian Euclidean spaces) are the Heisenberg groups. Much of the work in this thesis involves expounding on and generalizing results proven for the Heisenberg groups to results for jet space Carnot groups. While we will not prove results specifically for $\mathbb{H}^n$, the metric structure of these groups will provide a nice contrast to the more complicated structures of jet spaces. We refer the reader to [CDPT07] for a more thorough discussion of these groups.

For $n \geq 1$, we denote the n^{th} Heisenberg group by $\mathbb{H}^n$. As a set, $\mathbb{H}^n$ equals $\mathbb{C}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$, and points are written in the form $(z_1, \ldots, z_n, t) = (x_1, y_1, \ldots, x_n, y_n, t)$. We equip $\mathbb{H}^n$ with the group structure

$$(z_1, \ldots, z_n, t) \cdot (w_1, \ldots, w_n, u) = \left(z_1 + w_1, \ldots, z_n + w_n, t + u - 2\sum_{j=1}^{n} \mathrm{Im}(z_j \overline{w_j}) \right).$$

We define

$$X_j = \frac{\partial}{\partial x_j} - 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + 2x_j \frac{\partial}{\partial t} \qquad \text{for } j = 1, \ldots, n.$$

These vector fields form a global left-invariant frame for $H\mathbb{H}^n$. Intuitively, each point of $\mathbb{H}^n$ is assigned a hyperplane of allowable directions of movement. We have $[X_j, Y_j] = 4\frac{\partial}{\partial t}$ for all j with all other bracket relations trivial. Hence, the Lie algebra of $\mathbb{H}^n$ admits a stratification

$$Lie(\mathbb{H}^n) = \langle \{X_j, Y_j : j = 1, \ldots, n\} \rangle \oplus \left\langle \frac{\partial}{\partial t} \right\rangle.$$

Geodesics connecting pairs of points in $\mathbb{H}^n$ have been well-studied. In fact, the forms and uniqueness of these geodesics are known. For each pair of points differing by a point on the t-axis, there is a $U(n)$-family of connecting geodesics. Otherwise, there exists a unique connecting geodesic. In 2015, Hajłasz and Zimmerman used Fourier analysis to prove an isoperimetric inequality for $\mathbb{R}^{2n}$ and provide a new proof of the uniqueness and forms of these geodesics [HZ15].

Proposition 2.6. [HZ15, Theorem 2.1 and Corollary 2.7]

(i) A horizontal curve

$$\gamma(s) = (z_1(s), \ldots, z_n(s), t(s)) : [0,1] \to \mathbb{H}^n$$

of constant speed, connecting $\gamma(0) = 0$ to a point $\gamma(1) = (0, \ldots, 0, \pm T)$, $T > 0$, is a geodesic if and only if

$$x_j(s) = A_j(1 - \cos(2\pi s)) \mp B_j \sin(2\pi s)$$

$$y_j(s) = B_j(1-\cos(2\pi s)) \pm A_j \sin(2\pi s)$$

for $j = 1, \dots, n$ and

$$t(s) = \pm T\left(s - \frac{\sin(2\pi s)}{2\pi}\right),$$

where $A_1, \dots, A_n, B_1, \dots, B_n$ are any real numbers such that $4\sum_{j=1}^n (A_j^2 + B_j^2) = T$.

(ii) For any point p not on the t-axis, there exists a unique geodesic connecting the origin to p.

By left-invariance, one can determine the set of geodesics connecting any pair of points. Hajlasz and Zimmerman used this result to prove that the function measuring CC-distance from the origin is smooth, in fact analytic, away from the vertical axis [HZ15].

Theorem 2.7. [HZ15, Theorem 3.1] The Carnot-Carathéodory distance $d_{cc} : \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \to \mathbb{R}$ is real analytic on the set

$$\{(p,q) \in \mathbb{H}^n \times \mathbb{H}^n : q^{-1} \cdot p \notin t\text{-axis}\}.$$

Thus, the CC-distance is very well-behaved in the Heisenberg groups. This will not be the case for jet space Carnot groups, where the forms or even regularities of geodesics connecting generic pairs of points are not known. This is related to the presence of abnormal geodesics in jet spaces of step at least three. We will need to employ cruder tools to estimate distances in these spaces.

2.5 Carnot group structure of jet spaces

We now define the Carnot group structure of jet spaces. We include this section to store all of the notation for jet space Carnot groups in one place. We follow Section 3 of [War05b] and we refer the reader there for more detail.

Fix $k, n \geq 1$. Given $x_0 \in \mathbb{R}^n$ and $f \in C^k(\mathbb{R}^n)$, the k^{th}-order Taylor polynomial of f at x_0 is given by

$$T^k_{x_0}(f) = \sum_{j=0}^{k} \sum_{I \in I(j)} \frac{\partial_I f(x_0)}{I!}(x - x_0)^I,$$

where $I(j)$ denotes the set of j-indices $(i_1, \dots, i_n)$ $(i_1 + \cdots + i_n = j)$. For a convenient shorthand, *we write* $\tilde{I}(j) := I(0) \cup \cdots \cup I(j)$, *the set of all indices of length at most* j.

Given $x \in \mathbb{R}^n$, we can define an equivalence relation $\sim_x$ on $C^k(\mathbb{R}^n)$ by $f \sim_x g$ if $T^k_x(f) = T^k_x(g)$. We call $[f]_{\sim_x}$ the k**-jet** of f at x and denote it by $j^k_x(f)$. We then define the jet space $J^k(\mathbb{R}^n)$ by

$$J^k(\mathbb{R}^n) := \bigcup_{x \in \mathbb{R}^n} C^k(\mathbb{R}^n)/_{\sim_x}.$$

Define

$$p : J^k(\mathbb{R}^n) \to \mathbb{R}^n, \quad p(j^k_x(f)) = x$$

and

$$u_I : J^k(\mathbb{R}^n) \to \mathbb{R}, \quad u_I(j^k_x(f)) := \partial_I f(x)$$

for $I \in \tilde{I}(k)$. We have a global chart

$$\psi : J^k(\mathbb{R}^n) \to \mathbb{R}^n \times \mathbb{R}^{d(n,k)} \times \mathbb{R}^{d(n,k-1)} \times \cdots \times \mathbb{R}^{d(n,0)}$$

given by $\psi = (p, u^{(k)})$, where

$$u^{(k)} := \{u_I : I \in \tilde{I}(k)\}.$$

Here, $d(n,j) = \binom{n+j-1}{j}$ denotes the number of distinct j-indices over n coordinates.

For all $f \in C^k(\mathbb{R}^n)$ and $I \in \tilde{I}(k-1)$,

$$d(\partial_I f) = \sum_{j=1}^{n} \partial_{I+e_j} f \cdot dx^j.$$

This motivates us to define the 1-forms

$$\omega_I := du_I - \sum_{j=1}^{n} u_{I+e_j} dx^j, \quad I \in \tilde{I}(k-1)$$

to serve as contact 1-forms for $J^k(\mathbb{R}^n)$. The horizontal bundle of $J^k(\mathbb{R}^n)$ is defined by

$$HJ^k(\mathbb{R}^n) := \bigcap_{I \in \tilde{I}(k-1)} \ker \omega_I.$$

A global frame for $HJ^k(\mathbb{R}^n)$ is given by

$$\left\{X_j^{(k)} : j = 1, \ldots, n\right\} \cup \left\{\frac{\partial}{\partial u_I} : I \in I(k)\right\},$$

where

$$X_j^{(k)} := \frac{\partial}{\partial x_j} + \sum_{I \in \tilde{I}(k-1)} u_{I+e_j} \frac{\partial}{\partial u_I}, \quad j = 1, \ldots, n.$$

We can extend this to a global frame of $TJ^k(\mathbb{R}^n)$ by including $\frac{\partial}{\partial u_I}$ for $I \in \tilde{I}(k-1)$. With respect to the group operation on $J^k(\mathbb{R}^n)$ (to be defined soon), this frame is left-invariant.

The nontrivial commutator relations are given by

$$\left[\frac{\partial}{\partial u_{I+e_j}}, X_j^{(k)}\right] = \frac{\partial}{\partial u_I}, \quad I \in \tilde{I}(k-1).$$

Here, $[\cdot,\cdot]$ denotes the standard Lie bracket of vector fields on Euclidean space. Thus, $Lie(J^k(\mathbb{R}^n))$ admits a $(k+1)$-step stratification:

$$Lie(J^k(\mathbb{R}^n)) = HJ^k(\mathbb{R}^n) \oplus \left\langle \frac{\partial}{\partial u_I} : I \in I(k-1) \right\rangle \oplus \cdots \oplus \left\langle \frac{\partial}{\partial u_0} \right\rangle.$$

One defines a group operation on $J^k(\mathbb{R}^n)$ by

$$(x, u^{(k)}) \odot (y, v^{(k)}) = (x + y, uv^{(k)}),$$

where

$$uv_I := v_I + \sum_{I \leq J} u_J \frac{y^{J-I}}{(J-I)!}, \quad I \in \tilde{I}(k).$$

We will now make jet spaces more grounded by explicitly writing out the Carnot group structure of the model filiform jet spaces $J^k(\mathbb{R})$. The k-jet of $f \in C^k(\mathbb{R})$ at a point x_0 is given by

$$j^k_{x_0}(f) = (x_0, f^{(k)}(x_0), \ldots, f(x_0)).$$

The horizontal bundle $HJ^k(\mathbb{R})$ is defined by the contact forms

$$\omega_j := du_j - u_{j+1}dx, \quad j = 0, \ldots, k-1,$$

and is framed by the left-invariant vector fields $X^{(k)} := \frac{\partial}{\partial x} + u_k \frac{\partial}{\partial u_{k-1}} + \cdots + u_1 \frac{\partial}{\partial u_0}$ and $\frac{\partial}{\partial u_k}$. A $(k+1)$-step stratification of $\mathrm{Lie}(J^k(\mathbb{R}))$ is given by

$$\mathrm{Lie}(J^k(\mathbb{R})) := \left\langle X^{(k)}, \frac{\partial}{\partial u_k} \right\rangle \oplus \left\langle \frac{\partial}{\partial u_{k-1}} \right\rangle \oplus \cdots \oplus \left\langle \frac{\partial}{\partial u_0} \right\rangle.$$

The group operation on $J^k(\mathbb{R})$ is given by

$$(x, u_k, \ldots, u_0) \odot (y, v_k, \ldots, v_0) = (z, w_k, \ldots, w_0),$$

where $z = x + y$, $w_k = u_k + v_k$, and

$$w_s = u_s + v_s + \sum_{j=s+1}^{k} u_j \frac{y^{j-s}}{(j-s)!}, \quad s = 0, \ldots, k-1.$$

Despite the much simpler appearance of $J^k(\mathbb{R})$ relative to that of $J^k(\mathbb{R}^n)$, $n \geq 2$, valuable intuition and methods can often be built up in the model filiform case, which can later be employed for higher dimensions.

2.6 Metric structure of jet space Carnot groups

We will conclude our background by expounding on section 2.3 for the special case of jet space Carnot groups.

For $\epsilon > 0$, the dilations $\delta_\epsilon : J^k(\mathbb{R}^n) \to J^k(\mathbb{R}^n)$ are given by

$$x(\delta_\epsilon j^k_{x_0}(f)) = \epsilon x_0$$

and

$$u_I(\delta_\epsilon j^k_{x_0}(f)) := \epsilon^{k+1-|I|} \partial_I f(x_0), \quad I \in \tilde{I}(k).$$

In the special case $n = 1$, these dilations take the form

$$\delta_\epsilon(x, u_k, u_{k-1}, \ldots, u_0) = (\epsilon x, \epsilon u_k, \epsilon^2 u_{k-1}, \ldots, \epsilon^{k+1} u_0).$$

As noted before, the CC-metric is one-homogeneous with respect to these dilations:

$$d_{cc}(\delta_\epsilon j^k_{x_0}(f), \delta_\epsilon j^k_{y_0}(g)) = \epsilon \cdot d_{cc}(j^k_{x_0}(f), j^k_{y_0}(g)).$$

The important corollary of the Ball-Box Theorem from Section 2.3 now takes the following form for jet spaces:

Corollary 2.8. Fix $k, n \geq 2$. There exists $C > 0$ such that for all $(x, u^{(k)}) \in J^k(\mathbb{R}^n)$,

$$\frac{1}{C} \cdot d_{cc}(0, (x, u^{(k)})) \leq \max\{|x|,\ |u_I|^{1/(k+1-|I|)} : I \in \tilde{I}(k)\} \leq C \cdot d_{cc}(0, (x, u^{(k)})).$$

This corollary will serve as our most important tool for showing that our embeddings are bounded from below in Chapter 3. This is primarily due to the upcoming set of identities concerning the group structure on $J^k(\mathbb{R}^n)$. It allows one to estimate CC-distances up to a constant factor via the group operation on $J^k(\mathbb{R})$.

We will first state the result for $J^k(\mathbb{R})$, where notation is much easier to work with. By left-invariance of the CC-distance,

$$d_{cc}((x,u_k,\ldots,u_0),(y,v_k,\ldots,v_0)) = d_{cc}(0,(x,u_k,\ldots,u_0)^{-1}\odot(y,v_k,\ldots,v_0))$$

for all $(x,u_k,\ldots,u_0),(y,v_k,\ldots,v_)\in J^k(\mathbb{R})$. Coupled with Corollary 2.8, this implies that we may estimate

$$d_{cc}((x,u_k,\ldots,u_0),(y,v_k,\ldots,v_0))$$

if we know the form of $(x,u_k,\ldots,u_0)^{-1}\odot(y,v_k,\ldots,v_0)\in J^k(\mathbb{R})$. The following proposition provides the form.

Proposition 2.9. For $(x,u_k,\ldots,u_0)$, $(y,v_k,\ldots,v_0)\in J^k(\mathbb{R})$,

$$((x,u_k,\ldots,u_0)^{-1}\odot(y,v_k,\ldots,v_0))_s = v_s - u_s - \sum_{j=s+1}^{k}\frac{(y-x)^{j-s}}{(j-s)!}\cdot u_j, \qquad s=0,\ldots,k.$$

Proof. Recall for $s=0,\ldots,k$,

$$((x,u_k,\ldots,u_0)^{-1})_s = -\sum_{j=s}^{k}\frac{(-x)^{j-s}}{(j-s)!}u_j$$

and

$$((x,u_k,\ldots,u_0)\odot(y,v_k,\ldots,v_0))_s = v_s + \sum_{t=s}^{k}\frac{y^{t-s}}{(t-s)!}u_t.$$

Thus,

$$\begin{aligned}((x,u_k,\ldots,u_0)^{-1}\odot(y,v_k,\ldots,v_0))_s &= v_s - \sum_{t=s}^{k}\sum_{j=t}^{k}\frac{y^{t-s}}{(t-s)!}\cdot\frac{(-x)^{j-t}}{(j-t)!}\cdot u_j\\ &= v_s - \sum_{j=s}^{k}\sum_{t=s}^{j}\frac{y^{t-s}}{(t-s)!}\cdot\frac{(-x)^{j-t}}{(j-t)!}\cdot u_j\\ &= v_s - \sum_{j=s}^{k}\sum_{t=s}^{j}\binom{j-s}{t-s}y^{t-s}(-x)^{j-t}\cdot\frac{u_j}{(j-s)!}\\ &= v_s - \sum_{j=s}^{k}\frac{(y-x)^{j-s}}{(j-s)!}\cdot u_j,\end{aligned}$$

where the last equality comes from the Binomial Theorem. ∎

We can easily generalize this result to jet space Carnot groups once we recall the definitions of powers of points in $\mathbb{R}^n$ and factorials of multi-indices:

$$(x_1,\ldots,x_n)^{(j_1,\ldots,j_n)} := x_1^{j_1}\cdots x_n^{j_n}, \quad (j_1,\ldots,j_n)! := j_1\cdots j_n$$

for $(x_1,\ldots,x_n)\in\mathbb{R}^n$ and $(j_1,\ldots,j_n)\in\mathbb{N}^n$. Also for two multi-indices $I=(i_1,\ldots,i_n), J=(j_1,\ldots,j_n)$, we say $I\geq J$ if $i_s\geq j_s$ for all s.

Proposition 2.10. For all $I\in\tilde{I}(k)$ and $(x,u^{(k)}),(y,v^{(k)})\in J^k(\mathbb{R}^n)$,

$$((x,u^{(k)})^{-1}\odot(y,v^{(k)}))_I = v_I - \sum_{J\geq I}\frac{(y-x)^{J-I}}{(J-I)!}\cdot u_J.$$

To illustrate the usefulness of combining Proposition 2.9 with Corollary 2.8, we will prove a small result about the norm of commutators and conjugates in a model filiform group. While it will not be used later in this document, we think it is good to introduce the reader here to the types of computations and arguments we will use throughout this paper. First, we define the Korányi distance on $J^k(\mathbb{R})$ by

$$d(p,q) := ||p^{-1}\odot q||, \qquad \text{where } ||(x,u_k,\ldots,u_0)|| := |x| + \sum_{s=0}^{k}|u_s|^{1/(k+1-s)}.$$

Note that by Corollary 2.8, d is uniformly equivalent to d_{cc}.

Proposition 2.11. For all $s=0,\ldots,k$ and $p=(x,u_k,\ldots,u_0), q=(y,v_k,\ldots,v_0)\in J^k(\mathbb{R})$,

$$(p^{-1}q^{-1}pq)_s = \sum_{j=s+1}^{k}\left(u_j\cdot\frac{y^{j-s}}{(j-s)!} - v_j\frac{x^{j-s}}{(j-s)!}\right).$$

In particular, there exists a global constant C such that

$$d(qp,pq) = ||p^{-1}q^{-1}pq|| \leq C||p||^{1/2}||q||^{1/2}$$

and

$$||q^{-1}pq|| \leq C\left(||p|| + ||p||^{1/2}||q||^{1/2}\right).$$

whenever $||p||\leq 1$ and $||q||\leq 1$.

Proof. We have

$$(ba)_s = u_s + \sum_{j=s}^{k} v_j\frac{x^{j-s}}{(j-s)!}$$

and

$$(ab)_s = v_s + \sum_{j=s}^{k} u_j \frac{y^{j-s}}{(j-s)!}.$$

Note $(ba)_x = x + y$ and $(ab)_x = x + y$, which means $(ab)_x - (ba)_x = 0$. Also $(ba)_k = u_k + v_k$ and $(ab)_k = u_k + v_k$, which means $(ab)_k - (ba)_k = 0$.

Suppose $s < k$. By (2.9),

$$\begin{aligned}
(a^{-1}b^{-1}ab)_s &= ((ba)^{-1}ab)_s \\
&= v_s + \sum_{j=s}^{k} u_j \frac{y^{j-s}}{(j-s)!} - u_s - \sum_{j=s}^{k} v_j \frac{x^{j-s}}{(j-s)!} \\
&= \sum_{j=s+1}^{k} \frac{1}{(j-s)!} \left(u_j y^{j-s} - v_j x^{j-s}\right).
\end{aligned}$$

We then estimate

$$\begin{aligned}
|(a^{-1}b^{-1}ab)_s| &\leq \sum_{j=s+1}^{k} \left| u_j \frac{y^{j-s}}{(j-s)!} \right| + \left| v_j \frac{x^{j-s}}{(j-s)!} \right| \\
&\leq \sum_{j=s+1}^{k} ||a||^{k-j+1} \cdot \frac{||b||^{j-s}}{(j-s)!} + ||b||^{k-j+1} \cdot \frac{||a||^{j-s}}{(j-s)!}.
\end{aligned}$$

Note for $s+1 \leq j \leq k$, $\frac{k-j+1}{k-s+1} \geq \frac{1}{2}$ and $\frac{j-s}{k-s+1} \geq \frac{1}{2}$. Hence,

$$\begin{aligned}
|(a^{-1}b^{-1}ab)_s|^{\frac{1}{k-s+1}} &\lesssim \sum_{j=s+1}^{k} ||a||^{\frac{k-j+1}{k-s+1}} ||b||^{\frac{j-s}{k-s+1}} + ||a||^{\frac{j-s}{k-s+1}} ||b||^{\frac{k-j+1}{k-s+1}} \\
&\leq 2k ||a||^{1/2} ||b||^{1/2}.
\end{aligned}$$

We conclude by estimating

$$||b^{-1}ab|| \lesssim d(0,a) + d(a, b^{-1}ab) \leq ||a|| + C||a||^{1/2}||b||^{1/2}.$$

□

With Corollary 2.8, our other main tool for estimating distances in jet spaces will be an observation of Rigot and Wenger [RW10]. This will be key to constructing Lipschitz mappings from spheres into jet spaces in Chapter 3 and to proving Marstrand-type theorems in Chapter 5. As it is so important, and for the purposes of keeping this document more self-contained, we will conclude this chapter by going over its proof.

Proposition 2.12. [RW10, pages 4-5] Fix $f \in C^{k+1}(\mathbb{R}^n)$. There exists $C > 0$ such that for all $x, y \in \mathbb{R}^n$,

$$\frac{1}{C} \cdot ||x-y|| \leq d_{cc}(j^k_x(f), j^k_y(f)) \leq \sup_{t\in[0,1]} \left(1 + \sum_{I\in I(k)} \sum_{j=1}^{n} (\partial_{I+e_j} f(x + t(y-x))^2\right)^{1/2} ||y-x||.$$

In particular, $j^k(f) : \mathbb{R}^n \to J^k(\mathbb{R}^n)$ defined by $j^k(f)(x) := j^k_x(f)$ is locally biLipschitz.

Proof. For $f \in C^{k+1}(\mathbb{R}^n)$, the jet map $j^k(f)$ is C^1 and horizontal with

$$\partial_{x_j}(j^k_x(f)) = X^{(k)}_j(j^k_x(f)) + \sum_{I\in I(k)} \partial_{I+e_j} f(x) \cdot \frac{\partial}{\partial u_I}.$$

For $x, y \in \mathbb{R}^n$, define $\gamma : [0,1] \to \mathbb{R}^n$, $\gamma(t) := x + t(y-x)$, to be the straight line path connecting x to y. The chain rule implies $j^k(f) \circ \gamma$ is a horizontal path connecting $j^k_x(f)$ to $j^k_y(f)$. Hence, by the definition of the CC-metric,

$$d_{cc}(j^k_x(f), j^k_y(f)) \leq \sup_{t\in[0,1]} \left(1 + \sum_{I\in I(k)} \sum_{j=1}^{n} (\partial_{I+e_j} f(x + t(y-x))^2\right)^{1/2} ||y-x||.$$

On the other hand,

$$j^k_x(f)^{-1} \odot j^k_y(f) = (y-x, \ldots).$$

By Corollary 2.8, this implies

$$\frac{1}{C}||y-x|| \leq d_{cc}(0, j^k_x(f)^{-1} \odot j^k_y(f)) = d_{cc}(j^k_x(f), j^k_y(f)),$$

where C is the constant from Corollary 2.8.

As $f \in C^{k+1}(\mathbb{R}^n)$, $\partial_{I+e_j} f$ is bounded on compact sets for each $I \in I(k)$ and $j = 1, \ldots, n$. It follows that the restriction of $j^k(f)$ to each compact set is biLipschitz. □

Note that in the case $n = 1$, this result takes the following simpler form.

Proposition 2.13. [RW10, Pages 4-5] Fix $f \in C^{k+1}(\mathbb{R})$. There exists $C > 0$ such that for all $x, y \in \mathbb{R}$,

$$\frac{1}{C} \cdot |x-y| \leq d_{cc}(j^k_x(f), j^k_y(f)) \leq \sup_{t\in[x,y]} \left(1 + (f^{(k+1)}(t))^2\right)^{1/2} |x-y|.$$

In particular, $j^k(f) : \mathbb{R} \to J^k(\mathbb{R})$ defined by $j^k(f) := j^k_x(f)$ is locally biLipschitz.

Chapter 3

BiLipschitz embeddings of spheres into jet space Carnot groups not admitting Lipschitz extensions

3.1 Introduction

Extend partially-defined maps rich area research. The reader may have interpreted the previous expression to mean: The question of whether one can suitably extend partially-defined maps forms a rich area of research. Or perhaps the reader wasnt familiar with this thesis's content and thought of: The earliest explorers sought to extend partially-defined maps in search of rich areas and research. And a third option would be: A random list of words is extend, partially-defined, maps, rich, area, research. Yet, this last choice is the least satisfying because it fails to maintain the structure inherent in the initial collection of words. In this chapter, we will study an analogous problem in math, whether one can fill in the gaps of a partially-defined mapping while preserving its regularity. More specifically, we will be interested in Lipschitz extensions of mappings into Carnot groups.

In 2010, Rigot and Wenger proved that there exists a Lipschitz mapping from $\mathbb{S}^n$ to $J^k(\mathbb{R}^n)$ that cannot be extended in a Lipschitz way to $\mathbb{B}^{n+1}$ [RW10, Theorem 1.2]. For their proof, they actually construct a Lipschitz mapping $f : \partial[0,1]^{n+1} \to J^k(\mathbb{R}^n)$ that does not admit a Lipschitz extension to $[0,1]^{n+1}$. Their mapping f is constant on each line $\{x\} \times [0,1]$, $x \in \partial[0,1]^n$, and, in particular, is not biLipschitz. In this paper, we provide an explicit construction of a biLipschitz embedding of $\mathbb{S}^n$ into $J^k(\mathbb{R}^n)$ that cannot be Lipschitz extended to $\mathbb{B}^{n+1}$.

Theorem 3.1. For all $k, n \geq 1$, there exists a biLipschitz embedding $\phi : \mathbb{S}^n \to J^k(\mathbb{R}^n)$ that does not admit a Lipschitz extension $\tilde{\phi} : \mathbb{B}^{n+1} \to J^k(\mathbb{R}^n)$.

We remark that the theorem's statement would be false if we replaced $\mathbb{S}^n$ with a lower dimensional sphere. Wenger and Young proved that every biLipschitz embedding of $\mathbb{S}^m$ into $J^k(\mathbb{R}^n)$, $m < n$, can be extended to $\mathbb{B}^{m+1}$ in a Lipschitz fashion [WY10, Theorem 1.1].

BiLipschitz embeddings of spheres into Carnot groups have been used to prove the nondensity of Lipschitz mappings in Sobolev spaces. In 2009, Balogh and Fässler provided an example of a horizontal embedding $\phi : \mathbb{S}^n \to \mathbb{H}^n$ that does not admit a Lipschitz extension $\tilde{\phi} : \mathbb{B}^{n+1} \to \mathbb{H}^n$ [BF09, Theorem 1]. Their example consisted of the Legendrian lift of a Lagrangian map $f : \mathbb{S}^n \to \mathbb{R}^{2n}$. Dejarnette, Hajłasz, Lukyanenko, and Tyson then proved in 2014 that every horizontal embedding $\phi : \mathbb{S}^n \to \mathbb{H}^n$ does not admit a Lipschitz extension to $\mathbb{B}^{n+1}$ [DHLT14, Proposition 4.7]. The last authors used such an embedding to prove that the collection of Lipschitz mappings $\mathrm{Lip}(\mathbb{B}^{n+1}, \mathbb{H}^n)$ is not dense in the Sobolev space $W^{1,p}(\mathbb{B}^{n+1}, \mathbb{H}^n)$ for $n \leq p < n+1$ [DHLT14, Proposition 1.3]. Hajłasz, Schikorra, and Tyson have also horizontal embedding to prove the non-density of Lipschitz mappings in Heisenberg group-valued Sobolev spaces [HST14, Theorem 1.9]. Theorem 3.1 is a step towards proving the following non-approximation result for $J^k(\mathbb{R}^n)$:

Conjecture 3.2. Lipschitz mappings $\mathrm{Lip}(\mathbb{B}^{n+1}, J^k(\mathbb{R}^n))$ are not dense in $W^{1,p}(\mathbb{B}^{n+1}, J^k(\mathbb{R}^n))$, when $n \leq p < n+1$.

All smooth horizontal embeddings of $\mathbb{S}^n$ into $\mathbb{H}^n$ are biLipschitz [DHLT14, Theorem 3.1]. The difficulty of proving that our embedding $\phi : \mathbb{S}^n \to J^k(\mathbb{R}^n)$ is biLipschitz will stem from the fact that it is not smooth along the equator of $\mathbb{S}^n$. In fact, ϕ will not even be differentiable at these points. Fortunately, ϕ will be horizontal when restricted to the lower and upper hemispheres, which will imply that our embedding is biLipschitz when restricted to either of these halves. Still, the lack of differentiability begs the following question:

Question 3.3. For $n \geq 2$, does there exist a smooth, horizontal embedding $\psi : \mathbb{S}^n \hookrightarrow J^k(\mathbb{R}^n)$ that does not admit a Lipschitz extension to $\mathbb{B}^{n+1}$?

In Section 3.2, we prove Theorem 3.1 for $n = 1$ and observe that $\pi_m^{Lip}(J^k(\mathbb{R})) = 0$ for all $m \geq 2$ and $k \geq 1$. In Section 3.4, we generalize the construction and prove our main theorem for $n \geq 2$. We treat the case $n = 1$ separately because in this case, the function f serving as the body of the embedding is an explicit polynomial and there are no mixed partial derivatives to deal with. Also, the proof that the embedding lacks a Lipschitz extension will be simpler.

3.2 BiLipschitz embedding $\mathbb{S}^1 \hookrightarrow J^k(\mathbb{R})$

Definition 3.4. Fix $k \geq 1$. Define the polynomial $f_k : \mathbb{R} \to \mathbb{R}$ by $f_k(\theta) := \theta^{k+1}(\pi - \theta)^{k+1}$.

As f_k is smooth on $\mathbb{R}$, Proposition 2.12 implies that $j^k(f_k) : [0, \pi] \to J^k(\mathbb{R})$ is Lipschitz. In addition, as

$$j_\theta^k(f_k)^{-1} \odot j_\eta^k(f_k) = (\eta - \theta, f_k^{(k)}(\eta) - f_k^{(k)}(\theta), \ldots),$$

Corollary 2.8 and left-invariance of d_{cc} imply

$$|\eta - \theta| \lesssim d_{cc}(0, j^k_\theta(f_k)^{-1} \odot j^k_\eta(f_k)) = d_{cc}(j^k_\theta(f_k), j^k_\eta(f_k)).$$

Here, we write $\lesssim$ to denote that the left quantity is bounded above by the right quantity up to a positive factor depending only on k.

We have proven

Lemma 3.5. The map $j^k(f_k) : [0, \pi] \to J^k(\mathbb{R})$ is biLipschitz.

Gluing together two copies of $[0, \pi]$ at the endpoints, we can construct a continuous map of $\mathbb{S}^1$ into $J^k(\mathbb{R})$.

Definition 3.6. Define $\phi : \mathbb{S}^1 \to J^k(\mathbb{R})$ by

$$\phi(e^{i\theta}) := \begin{cases} j^k_\theta(f_k) & \text{if } 0 \le \theta \le \pi \\ j^k_{2\pi-\theta}(-f_k) & \text{if } \pi \le \theta \le 2\pi. \end{cases}$$

This map is well-defined because $f^{(j)}_k(0) = f^{(j)}_k(\pi) = 0$ for $j = 0, \dots, k$. A more intuitive expression of ϕ (which matches the original definition) is $\phi(e^{i\theta}) = j^k_\theta(f_k)$ and $\phi(e^{-i\theta}) = j^k_\theta(-f_k)$ for $0 \le \theta \le \pi$. In this subsection, we will prove:

Theorem 3.7. The map $\phi : \mathbb{S}^1 \to J^k(\mathbb{R})$ is a biLipschitz embedding.

Denote the upper and lower semicircles by $\mathbb{S}^1_+ := \{e^{i\theta} : 0 \le \theta \le \pi\}$ and $\mathbb{S}^1_- := \{e^{i\theta} : \pi \le \theta \le 2\pi\}$, respectively. As $e^{i\theta} : [0, \pi] \to \mathbb{S}^1_+$ and $e^{-i\theta} : [0, \pi] \to \mathbb{S}^1_-$ are biLipschitz, the restrictions $\phi|_{\mathbb{S}^1_+}$ and $\phi|_{\mathbb{S}^1_-}$ are biLipschitz. It remains to prove that

$$d_{cc}(\phi(e^{i\theta}), \phi(e^{i\eta})) \approx d_{\mathbb{S}^1}(e^{i\theta}, e^{i\eta}) \quad \text{for } e^{i\theta} \in \mathbb{S}^1_+, \ e^{i\eta} \in \mathbb{S}^1_-.$$

By $d_{\mathbb{S}^1}$, we mean the geodesic path metric on $\mathbb{S}^1$. We write $A \approx B$ to denote that there exists a single constant C such that

$$\frac{1}{C} \cdot A \le B \le C \cdot A,$$

for all relevant choices of A and B. *We will use this notation throughout this paper.* Note that since we are merely showing maps are biLipschitz and not caring about the actual Lipschitz constants, we can allow for positive constant factors in our comparisons.

Proving that ϕ is Lipschitz follows easily from the triangle inequality combined with the fact that ϕ is biLipschitz when restricted to the upper and lower semicircles. Indeed, if the geodesic connecting $e^{i\theta} \in \mathbb{S}^1_+$

to $e^{i\eta} \in \mathbb{S}^1_-$ passes through e^{i0}, then

$$\begin{aligned} d_{cc}(\phi(e^{i\theta}), \phi(e^{i\eta})) &\leq d_{cc}(\phi(e^{i\theta}), \phi(e^{i0})) + d_{cc}(\phi(e^{i0}), \phi(e^{i\eta})) \\ &\approx d_{\mathbb{S}^1}(e^{i\theta}, e^{i0}) + d_{\mathbb{S}^1}(e^{i0}, e^{i\eta}) \\ &= d_{\mathbb{S}^1}(e^{i\theta}, e^{i\eta}). \end{aligned}$$

The same reasoning works if the geodesic passes through $e^{i\pi}$. We have shown

Proposition 3.8. $\phi : \mathbb{S}^1 \to J^k(\mathbb{R})$ is Lipschitz.

We are now halfway towards proving that ϕ is biLipschitz.

Definition 3.9. A map $g : X \to Y$ between metric spaces is said to be **co-Lipschitz** if there exists a constant $C > 0$ such that

$$d_Y(g(x_1), g(x_2)) \geq \frac{1}{C} \cdot d_X(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

If a map is co-Lipschitz, we say it has the **co-Lipschitz property**.

It remains to show that ϕ is co-Lipschitz. Before we prove this, we will observe that the k^{th} derivative of f_k is approximately linear near 0 and near π. This behavior was the primary reason for our choice of f_k.

Lemma 3.10. There exists a constant $0 < \epsilon < 1$ such that

$$f_k^{(k)}(\theta) \geq \frac{\pi^{k+1}(k+1)!}{2} \cdot \theta \quad \text{if } 0 \leq \theta \leq \epsilon$$

and

$$\begin{cases} f_k^{(k)}(\theta) \geq \frac{\pi^{k+1}(k+1)!}{2} \cdot (\pi - \theta) & \text{if } \pi - \epsilon \leq \theta \leq \pi \text{ and } k \text{ is even} \\ f_k^{(k)}(\theta) \leq -\frac{\pi^{k+1}(k+1)!}{2} \cdot (\pi - \theta) & \text{if } \pi - \epsilon \leq \theta \leq \pi \text{ and } k \text{ is odd.} \end{cases}$$

Proof. By induction,

$$f_k^{(k)}(\theta) = (k+1)!\theta(\pi - \theta)^{k+1} + \theta^2 p(\theta)$$

and

$$f_k^{(k)}(\theta) = (k+1)!(-1)^k \theta^{k+1}(\pi - \theta) + (\pi - \theta)^2 q(\theta),$$

for some polynomials p, q. This implies

$$\lim_{\theta\to 0}\frac{f_k^{(k)}(\theta)}{\theta} = \lim_{\theta\to\pi}\frac{(-1)^k f_k^{(k)}(\theta)}{\pi-\theta} = (k+1)!\cdot\pi^{k+1}.$$

The lemma follows. □

We can now finish the proof of Theorem 3.7, proving that ϕ is biLipschitz.

Proof of Theorem 3.7. We proved in Proposition 3.8 that ϕ is Lipschitz. It remains to show ϕ is co-Lipschitz, i.e., that there exists a constant $C > 0$ such that

$$d_{cc}(\phi(e^{i\theta}), \phi(e^{-i\eta})) \geq \frac{1}{C}\cdot d_{\mathbb{S}^1}(e^{i\theta}, e^{-i\eta})$$

for all $e^{i\theta}\in\mathbb{S}^1_+$ and $e^{-i\eta}\in\mathbb{S}^1_-$.

Let $0 < \epsilon < 1$ be the constant from Lemma 3.10. To prove the co-Lipschitz property, it suffices to consider three arrangements of pairs of points $e^{i\theta}\in\mathbb{S}^1_+$ and $e^{-i\eta}\in\mathbb{S}^1_-$, where $0\leq\theta,\eta\leq\pi$:

(i) $0\leq\theta,\eta\leq\epsilon$, or $\pi-\epsilon\leq\theta,\eta\leq\pi$ (points are close to each other and the x-axis).

(ii) $\epsilon\leq\theta\leq\pi-\epsilon$ or $\epsilon\leq\eta\leq\pi-\epsilon$ (one of the points is far from the x-axis).

(iii) $|\theta-\eta|\geq\pi-2\epsilon$ (arguments are far from each other).

(Readers should convince themselves that these cases handle all possible pairs of a point on the upper semicircle and a point on the lower semicircle.)

Case (i): Fix $0\leq\theta,\eta\leq\epsilon$. By Corollary 2.8 and Lemma 3.10,

$$\begin{aligned} d_{cc}(\phi(e^{i\theta}), \phi(e^{-i\eta})) &= d_{cc}(j^k_\theta(f_k), j^k_\eta(-f_k)) \\ &\gtrsim |f_k^{(k)}(\theta) + f_k^{(k)}(\eta)| \\ &\geq \frac{(k+1)!}{2}\cdot(\theta+\eta) \\ &= \frac{(k+1)!}{2}\cdot d_{\mathbb{S}^1}(e^{i\theta}, e^{-i\eta}). \end{aligned}$$

A similar calculation shows

$$d_{cc}(\phi(e^{i\theta}), \phi(e^{-i\eta})) \gtrsim \frac{(k+1)!}{2}\cdot(2\pi-\theta-\eta) = \frac{(k+1)!}{2}\cdot d_{\mathbb{S}^1}(e^{i\theta}, e^{i\eta})$$

for $\pi-\epsilon\leq\theta,\eta\leq\pi$. This handles case (i).

Case (ii): Suppose $\epsilon \leq \theta \leq \pi - \epsilon$ and $0 \leq \eta \leq \pi$. Then $f_k(\theta) > 0$ while $-f_k(\eta) \leq 0$. Hence, $j^k_\theta(f_k) \neq j^k_\eta(-f_k)$, so that

$$0 < d_{cc}(j^k_\theta(f_k), j^k_\eta(-f_k)) = d_{cc}(\phi(e^{i\theta}), \phi(e^{-i\eta})).$$

This implies that the restriction of d_{cc} on the compact set

$$\{\phi(e^{i\theta}) : \epsilon \leq \theta \leq \pi - \theta\} \times \{\phi(e^{-i\eta}) : 0 \leq \eta \leq \pi\}$$

is strictly positive. By the Extreme Value Theorem, there must exist $\delta_1 > 0$ such that

$$d_{cc}(\phi(e^{i\theta}), \phi(e^{-i\eta})) > \delta_1$$

whenever $\epsilon \leq \theta \leq \pi - \epsilon$ and $0 \leq \eta \leq \pi$. By the same argument, there also exists $\delta_2 > 0$ such that

$$d_{cc}(\phi(e^{i\theta}), \phi(e^{-i\eta})) > \delta_2$$

whenever $0 \leq \theta \leq \pi$ and $\epsilon \leq \eta \leq \pi - \epsilon$. As $\mathbb{S}^1$ is bounded, this handles case (ii).

Case (iii): This case is handled in the same way as case (ii) was. We need only observe that $\{(e^{i\theta}, e^{-i\eta}) \in \mathbb{S}^1_+ \times \mathbb{S}^1_- : |\theta - \eta| \geq \pi - 2\epsilon,\ 0 \leq \theta, \eta \leq \pi\}$ is compact and $j^k_\theta(f_k) \neq j^k_\eta(-f_k)$ whenever $\theta \neq \eta$.

This concludes the proof that ϕ is co-Lipschitz, hence biLipschitz. □

3.3 Embedding of circle does not admit a Lipschitz extension

In this section, we will prove that the embedding from Definition 3.6 does not admit a Lipschitz extension. The author originally proved this by modifying an argument of Hajłasz, Schikorra and Tyson for $\mathbb{H}^1$ [HST14]. Then a reviewer provided a much simpler, clearer proof. The author wants to reiterate his appreciation to the reviewer for this. We will also prove that each of the Lipschitz homotopy groups of $J^k(\mathbb{R})$ is trivial. These proofs will rely on a result of Wenger and Young [WY14, Theorem 5], which states in particular that every Lipschitz map from $\mathbb{B}^2$ to $J^k(\mathbb{R})$ factors through a metric tree.

In [WY14], Wenger and Young prove that every Lipschitz mapping from $\mathbb{S}^m$, $m \geq 2$, to $\mathbb{H}^1$ factors through a metric tree. A *metric tree* (or $\mathbb{R}$-tree) is a geodesic metric space for which every geodesic triangle is isometric to a tripod, or equivalently, is 0-hyperbolic in the sense of Gromov. Metric trees are CAT(κ) spaces for all $\kappa \leq 0$ and are uniquely geodesic (see Proposition 1.4(1) and Example 1.15(5) of Chapter II.1 in [BH99]). We note that in his book on the more general Λ-trees [Chi01], Chiswell defines metric trees in a manner equivalent to as above (see Lemmata 2.1.6 and 2.4.13 of [Chi01]). For a much greater discussion on metric trees, we refer the reader to this book [Chi01]. The first property of metric trees below is usually cited without proof while the second was stated without proof in [WY14]. We will provide justification here.

Lemma 3.11. For every metric tree (Z, d), its completion $(\hat{Z}, \hat{d})$ is a metric tree and is Lipschitz contractible.

Proof. Let (Z, d) be a metric tree. Chiswell proved that the completion of a metric tree $(\hat{Z}, \hat{d})$ is still a metric tree [Chi01, Theorem 2.4.14] (we note that this result is usually attributed to Imrich at [Imr77], but the author was unable to track down this work). Then since metric trees are CAT(κ) spaces for all $\kappa \leq 0$, a version of Kirszbraun's theorem proven by Lang and Schroeder [LS97, Theorem B] implies that $(\hat{Z}, \hat{d})$ is Lipschitz contractible. □

A metric space X is *quasi-convex* if there exists a constant C such that every two points $x, y \in X$ can be connected by a path of length at most $Cd(x, y)$. For example, each sphere $\mathbb{S}^n$ is quasi-convex. In 2014, Wenger and Young proved a factorization result for mappings into purely 2-unrectifiable spaces.

Theorem 3.12. [WY14, Theorem 5] Let X be a quasi-convex metric space with $\pi_1^{Lip}(X) = 0$. Let furthermore Y be a purely 2-unrectifiable metric space. Then every Lipschitz map from X to Y factors through a metric tree. That is, there exist a metric tree Z and Lipschitz maps $\varphi : X \to Z$ and $\psi : Z \to Y$ such that $f = \psi \circ \varphi$.

Wenger and Young used this result to prove that $\pi_m^{Lip}(\mathbb{H}^1) = 0$ for all $m \geq 2$ [WY14, Corollary 4]. We can easily modify their proof to prove the triviality of $\pi_m^{Lip}(J^k(\mathbb{R}))$ for $m \geq 2$. We only include a proof to help keep this paper self-contained. We note that Lipschitz homotopy groups $\pi_m^{Lip}(J^k(\mathbb{R}))$ are defined in the same way as typical homotopy groups are, except the maps and homotopies are required to be Lipschitz (see Section 4 of [DHLT14] for a greater discussion).

Corollary 3.13. For $m \geq 2$ and $k \geq 1$, $\pi_m^{Lip}(J^k(\mathbb{R})) = 0$.

Proof. Fix $m \geq 2$ and $k \geq 1$. Suppose $f : \mathbb{S}^m \to J^k(\mathbb{R})$ is Lipschitz. By a theorem of Magnani, $J^k(\mathbb{R})$ is purely 2-unrectifiable [Mag04, Theorem 1.1]. Hence, by Theorem 3.12, there exist a metric tree Z and Lipschitz maps $\varphi : \mathbb{S}^m \to Z$ and $\psi : Z \to J^k(\mathbb{R})$ such that $f = \psi \circ \varphi$. Lemma 3.11 combined with the fact that $J^k(\mathbb{R})$ is complete imply that we may assume Z is complete. Furthermore, by Lemma 3.11, there exists a Lipschitz homotopy $h : Z \times [0, 1] \to Z$ of the identity map to a constant map. Then $\alpha : \mathbb{S}^m \times [0, 1] \to J^k(\mathbb{R})$ defined by $\alpha(x, t) = (\psi \circ h)(\varphi(x), t)$ is a Lipschitz homotopy of f to a constant map. □

Proof of Theorem 3.1 for $n = 1$. Suppose, for contradiction, that the biLipschitz embedding $\phi : \mathbb{S}^1 \to J^k(\mathbb{R})$ from Theorem 3.7 admits a Lipschitz extension $\tilde{\phi} : \mathbb{B}^2 \to J^k(\mathbb{R})$. Since $J^k(\mathbb{R})$ is purely 2-unrectifiable, Wenger and Young's result (Theorem 3.12) implies that $\tilde{\phi}$, and hence ϕ, factors through a metric tree. However, any two topological embeddings of $[0,1]$ into a metric tree that share common endpoints must have the same image. This leads to a contradiction that ϕ is injective. $\square$

3.4 BiLipschtz embedding $\mathbb{S}^n \hookrightarrow J^k(\mathbb{R}^n)$

In this section, we will prove our main theorem, Theorem 3.1, for $n \geq 2$. We begin by stating the section's assumptions and notation. We will assume $n \geq 2$. Whenever we write $|x|$, we will mean the norm of $x \in \mathbb{R}^n$ with respect to the standard Euclidean metric. On the other hand, when we are calculating distances between points and write $\rho(\cdot,\cdot)$, we will be referring to the *Manhattan metric* on Euclidean space. Explicitly, for $x, y \in \mathbb{R}^n$,

$$\rho(x,y) := \sum_{i=1}^{n} |x_i - y_i|.$$

In Proposition 3.20, we will use the geodesic path metric on $\mathbb{S}^n$ and denote it by $d_{\mathbb{S}^n}(\cdot,\cdot)$. Of course, there are no problems switching between these three metrics since they are all equivalent (see Theorem 3.1 of [DHLT14] for equivalence of path metric and Euclidean metric).

For the case $n = 1$, we implicitly used that the exponential $e^{i\theta} : [0,\pi] \to \mathbb{S}^1$ is biLipschitz. This allowed us to view the upper and lower semicircles as copies of $[0,\pi]$. We then employed a smooth function $f_k : [0,\pi] \to \mathbb{R}$ to define our biLipschitz map $\phi : \mathbb{S}^1 \to J^k(\mathbb{R})$. We will follow a similar strategy in higher dimensions.

We begin with some notation.

Definition 3.14. Define the *upper hemisphere*

$$\mathbb{S}^n_+ := \{(x,t) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} : |x|^2 + t^2 = 1,\ t \geq 0\}$$

and the *lower hemisphere*

$$\mathbb{S}^n_- := \{(x,t) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} : |x|^2 + t^2 = 1,\ t \leq 0\}.$$

Note $\mathbb{S}^n = \mathbb{S}^n_+ \cup \mathbb{S}^n_-$ with $\mathbb{S}^n_+ \cap \mathbb{S}^n_- = \mathbb{S}^{n-1} \times \{0\}$. I will later refer to this last set as the **equator** of $\mathbb{S}^n$.

Our first step will be to determine how to lift the n-ball to the upper hemisphere in a biLipschitz way. We will accomplish this via polar coordinates.

Proposition 3.15. The map $L : \mathbb{B}^n \to \mathbb{S}^n_+$ defined by

$$L(\theta \cdot x) = (x \cdot \sin(\pi\theta/2), \cos(\pi\theta/2)), \quad \theta \in [0,1],\ x \in \mathbb{S}^{n-1}$$

is well-defined and biLipschitz.

Proof. It isn't hard to see that L is well-defined.

Via a rotation, it suffices to assume we have two points $(\eta, 0)$, $\theta \cdot (x, y) \in \mathbb{B}^n$, where $0 < \eta \leq 1$, $0 \leq \theta \leq \eta$, and $(x, y) \in \mathbb{S}^{n-1} \subset \mathbb{R} \times \mathbb{R}^{n-1}$.

First note

$$\rho_{\mathbb{B}^n}((\eta, 0), \theta \cdot (x, y)) = (\eta - \theta x) + \theta \sum_{i=2}^{n} |y_i|$$

(recall we are using the Manhattan metric). We have (with justification below)

$$\begin{aligned}
&\rho_{\mathbb{S}^n_+}\left((\sin\frac{\pi\eta}{2}, 0, \cos\frac{\pi\eta}{2}), (x\sin\frac{\pi\theta}{2}, y\sin\frac{\pi\theta}{2}, \cos\frac{\pi\theta}{2})\right) \\
&\quad = \left(\sin\frac{\pi\eta}{2} - x\sin\frac{\pi\theta}{2}\right) + \sin\left(\frac{\pi\theta}{2}\right)\sum_{i=2}^{n}|y_i| + \left(\cos\frac{\pi\theta}{2} - \cos\frac{\pi\eta}{2}\right) \\
&\quad = \left(\sin\frac{\pi\eta}{2} - \sin\frac{\pi\theta}{2}\right) + \sin\left(\frac{\pi\theta}{2}\right)(1-x) + \sin\left(\frac{\pi\theta}{2}\right)\sum_{i=2}^{n}|y_i| + \left(\cos\frac{\pi\theta}{2} - \cos\frac{\pi\eta}{2}\right) \\
&\quad \approx |e^{i\pi\eta/2} - e^{i\pi\theta/2}| + \theta(1-x) + \theta\sum_{i=2}^{n}|y_i| \\
&\quad \approx (\eta - \theta x) + \theta\sum_{i=2}^{n}|y_i| \\
&\quad = \rho_{\mathbb{B}^n}((\eta, 0), \theta \cdot (x, y)).
\end{aligned}$$

For the first approximation above, we used the fact that the Manhattan metric and standard Euclidean metric are uniformly equivalent. We also used that $\sin\theta \approx \theta$ on $[0, \pi/2]$. For the second approximation, we used that the Euclidean metric and geodesic path metric are uniformly equivalent on the upper half circle. □

Recalling the strategy used to embed a circle, we now find a smooth function on $\mathbb{R}^n$ to serve as the "body of our jet." For the circle, the main difficulty was finding a positive function f_k that satisfied

$$f_k^{(k)}(\theta) \approx \theta = \rho_{\mathbb{S}^1}(e^{i\theta}, e^{i0}) \quad \text{for } \theta \text{ near } 0$$

and similar behavior for θ near π. For general n, the natural choice would be $f(x) := (1 - |x|)^{k+1}$. However,

f has a singularity at 0. Fortunately, we only need f to equal $(1-|x|)^{k+1}$ near the boundary of $\mathbb{B}^n$. We encapsulate the necessary conditions of f in the following lemma.

Lemma 3.16. There exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying:

(a) $f(x) = (1-|x|)^{k+1}$ for $\frac{1}{2} \le |x| \le \frac{3}{2}$; and

(b) $f(x) > 0$ for $|x| < 1$.

Proof. Choose a smooth function $\alpha : \mathbb{R}^n \to [0,1]$ satisfying $\alpha = 1$ on $\{x : \frac{1}{2} \le |x| \le \frac{3}{2}\}$ and $\alpha = 0$ on $\{x : |x| \le \frac{1}{4}\}$. Then $\alpha(x) \cdot (1-|x|)^{k+1}$ satisfies property (a). To satisfy (b) as well, we merely need to add a smooth, non-negative function that is zero on $\{x : \frac{1}{2} \le |x| \le \frac{3}{2}\}$ and is positive where $\alpha = 0$ in $\mathbb{B}^n$. But $1-\alpha$ clearly satisfies these conditions. Hence, $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) := \alpha(x) \cdot (1-|x|)^{k+1} + (1-\alpha(x))$$

works. □

Definition 3.17. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function satisfying properties (a) and (b) of Lemma 3.16. We define $\phi : \mathbb{S}^n \to J^k(\mathbb{R}^n)$ by

$$\phi(x \sin(\pi\theta/2), t) := \begin{cases} j^k_{\theta \cdot x}(f) & \text{if } x \in \mathbb{S}^{n-1},\ 0 \le \theta \le 1,\ t \ge 0 \\ j^k_{\theta \cdot x}(-f) & \text{if } x \in \mathbb{S}^{n-1},\ 0 \le \theta \le 1,\ t \le 0. \end{cases}$$

Observe that ϕ is well-defined since $\partial_I f(x) = 0$ whenever $|x| = 1$ and $|I| \le k$. We will prove:

Theorem 3.18. The map $\phi : \mathbb{S}^n \to J^k(\mathbb{R}^n)$ is a biLipschitz embedding.

As in the circle case, proving that ϕ is Lipschitz is easier than proving that ϕ is co-Lipschitz (see Definition 3.9), so we will do the former first. Before this, we need to prove that ϕ is biLipschitz when restricted to the upper and lower hemispheres.

Lemma 3.19. The restrictions $\phi|_{\mathbb{S}^n_+}$ and $\phi|_{\mathbb{S}^n_-}$ are biLipschitz.

Proof. By Proposition 2.12 and the Ball-Box Theorem, $j^k(f) : \mathbb{B}^n \to J^k(\mathbb{R}^n)$ is biLipschitz.

Let $L : \mathbb{B}^n \to \mathbb{S}^n_+$ be the biLipschitz map defined in Proposition 3.15. Then the restriction $\phi|_{\mathbb{S}^n_+} = j^k(f) \circ L^{-1}$ is biLipschitz. As the reflection $R : \mathbb{S}^n \to \mathbb{S}^n$ given by $(x,t) \mapsto (x,-t)$, $x \in \mathbb{B}^{n-1}$, $t \in \mathbb{R}$ is an isometry, the restriction $\phi|_{\mathbb{S}^n_-} = \phi|_{\mathbb{S}^n_+} \circ R$ is also biLipschitz. □

It remains to consider the application of ϕ to points on opposite halves of $\mathbb{S}^n$. More precisely, we need to prove

$$d_{cc}(j^k_{\eta\cdot x}(f), j^k_{\theta\cdot y}(-f)) \approx \rho_{\mathbb{S}^n}((x\sin(\pi\eta/2), \cos(\pi\eta/2)), (y\sin(\pi\theta/2), -\cos(\pi\theta/2)))$$

for $x, y \in \mathbb{S}^{n-1}$, $0 \le \eta, \theta \le 1$.

Proving that ϕ is Lipschitz will be proven in the same way here as it was for $n = 1$ (see Proposition 3.8).

Proposition 3.20. $\phi : \mathbb{S}^n \to J^k(\mathbb{R}^n)$ is Lipschitz.

Proof. It remains to prove

$$d_{cc}(j^k_{\eta\cdot x}(f), j^k_{\theta\cdot y}(-f)) \lesssim \rho_{\mathbb{S}^n}((x\sin(\pi\eta/2), \cos(\pi\eta/2)), (y\sin(\pi\theta/2), -\cos(\pi\theta/2)))$$

for $x, y \in \mathbb{S}^{n-1}$ and $0 \le \eta, \theta < 1$.

Let $(x\sin(\pi\eta/2), \cos(\pi\eta/2)) \in \mathbb{S}^n_+$, $(y\sin(\pi\theta/2), -\cos(\pi\theta/2)) \in \mathbb{S}^n_-$, $\gamma : [0,1] \to \mathbb{S}^n$ the geodesic connecting them, and $r \in [0,1]$ such that $\gamma(r)$ is on the equator. Note that $j^k_z(f) = \gamma(r) = j^k_z(-f)$ if $\gamma(r) = (z, 0)$. By Lemma 3.19,

$$\begin{aligned}
&d_{cc}(j^k_{\eta\cdot x}(f), j^k_{\theta\cdot y}(-f)) \\
&\qquad \le d_{cc}(j^k_{\eta\cdot k}(f), j^k_z(f)) + d_{cc}(j^k_z(-f), j^k_{\theta\cdot y}(-f)) \\
&\qquad \approx \rho_{\mathbb{S}^n}((x\sin(\pi\eta/2), \cos(\pi\eta/2)), (z,0)) + d_{\mathbb{S}^n}((z,0), (y\sin(\pi\theta/2), -\sin(\pi\eta/2))) \\
&\qquad \approx d_{\mathbb{S}^n}((x\sin(\pi\eta/2), \cos(\pi\eta/2)), (z,0)) + \tilde{d}_{\mathbb{S}^n}((z,0), (y\sin(\pi\theta/2), -\sin(\pi\eta/2))) \\
&\qquad = d_{\mathbb{S}^n}((x\sin(\pi\eta/2), \cos(\pi\eta/2)), (y\sin(\pi\theta/2), -\cos(\pi\theta/2))) \\
&\qquad \approx \rho_{\mathbb{S}^n}((x\sin(\pi\eta/2), \cos(\pi\eta/2)), (y\sin(\pi\theta/2), -\cos(\pi\theta/2))),
\end{aligned}$$

where $d_{\mathbb{S}^n}$ denotes the geodesic path metric on $\mathbb{S}^n$. □

It remains to prove that $\phi : \mathbb{S}^n \to J^k(\mathbb{R}^n)$ is co-Lipschitz. As in the initial case $n = 1$, we first need to prove that certain k^{th}-order derivatives of $(1 - |x|)^{k+1}$ are approximately linear near the boundary of $\mathbb{B}^n$.

Lemma 3.21. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function satisfying properties (a)-(b) of Lemma 3.16. There exist constants $0 < \epsilon < \frac{1}{2} < C$ satisfying the following: For all $i = 1, \ldots, n$ and $x \in \mathbb{R}^n$ satisfying $1 - \epsilon \le |x| \le 1$

and $|x_i| > \frac{1}{4\sqrt{n}}$, we have

$$\begin{cases} \frac{1-|x|}{C} \leq \frac{\partial^k f}{\partial x_i^k}(x) \leq C(1-|x|) & \text{if } k \text{ is even} \\ \frac{1-|x|}{C} \leq \frac{\partial^k f}{\partial x_i^k}(x) \leq C(1-|x|) & \text{if } k \text{ is odd and } x_i < 0 \\ \frac{1-|x|}{C} \leq -\frac{\partial^k f}{\partial x_i^k}(x) \leq C(1-|x|) & \text{if } k \text{ is odd and } x_i > 0. \end{cases}$$

Proof. Fix $i = 1, \ldots, n$. By condition (a), $f(x) = (1-|x|)^{k+1}$ for $\frac{1}{2} < |x| < \frac{3}{2}$. We have

$$\frac{\partial f}{\partial x_i}(x) = (1-|x|)^k \cdot \frac{-(k+1)x_i}{\sqrt{x_1^2 + \cdots + x_n^2}} \quad \text{for } \frac{1}{2} < |x| < \frac{3}{2}.$$

By induction, there exists a smooth function $g_i : \{x \in \mathbb{R}^n : \frac{1}{2} \leq |x| \leq \frac{3}{2}\} \to \mathbb{R}$ such that

$$\frac{\partial^k f}{\partial x_i^k}(x) = (1-|x|) \cdot \frac{(-1)^k (k+1)! x_i^k}{(x_1^2 + \cdots + x_n^2)^{\frac{k}{2}}} + (1-|x|)^2 g_i(x)$$

for $\frac{1}{2} < |x| < \frac{3}{2}$. Restricting to x with $|x_i| \geq \frac{1}{4\sqrt{n}}$, the second term becomes relatively neglible as $|x| \to 1$. The lemma follows. □

We can now prove that $\phi : \mathbb{S}^n \to J^k(\mathbb{R}^n)$ is co-Lipschitz, hence biLipschitz by Proposition 3.20.

Proof of Theorem 3.18. It remains to prove that ϕ is co-Lipschitz, i.e., that exists a constant D such that

$$d_{cc}(j^k_{\eta \cdot x}(f), j^k_{\theta \cdot y}(-f)) \geq \frac{1}{D} \cdot \rho_{\mathbb{S}^n}((x \sin(\pi\eta/2), s), (y \sin(\pi\theta/2), -t)).$$

for all points $(x\sin(\pi\eta/2), s)$, $(y\sin(\pi\theta/2), -t) \in \mathbb{S}^n$ with $x, y \in \mathbb{S}^{n-1}$, $s, t > 0$, and $0 \leq \eta, \theta \leq 1$.

Let ϵ, C be the constants from Lemma 3.21. Consider the following three properties:

(A) $\eta \geq 1 - \epsilon$.

(B) $\theta \geq 1 - \epsilon$.

(C) $|\eta \cdot x - \theta \cdot y| \leq \frac{1}{4\sqrt{n}}$.

First suppose that at least one of properties (A)-(C) is not satisfied. None of the pairs in the compact sets

- $\{(\phi(x\sin(\pi\eta/2), s), \phi(y\sin(\pi\theta/2), -t)) \in \mathbb{S}^n \times \mathbb{S}^n : s, t \geq 0,\ 1-\epsilon \leq \eta \leq 1,\ 0 \leq \theta \leq 1\}$;
- $\{(\phi(x\sin(\pi\eta/2), s), \phi(y\sin(\pi\theta/2), -t)) \in \mathbb{S}^n \times \mathbb{S}^n : s, t \geq 0,\ 1-\epsilon \leq \theta \leq 1,\ 0 \leq \eta \leq 1\}$;
- $\{(\phi(x\sin(\pi\eta/2), s), \phi(y\sin(\pi\theta/2), -t)) \in \mathbb{S}^n \times \mathbb{S}^n : s, t \geq 0,\ 0 \leq \theta, \eta \leq 1,\ |\eta \cdot x - \theta \cdot y| \geq \frac{1}{4\sqrt{n}}\}$

are of the form (x, x) for $x \in \mathbb{S}^n$. By the Extreme Value Theorem, it follows that there exists $\delta > 0$ such that

$$d_{cc}(z_1, z_2) > \delta$$

for each pair (z_1, z_2) in the above compact sets.

Now suppose that properties (A)-(C) are satisfied. By Proposition 3.15,

$$\rho_{\mathbb{S}^n}((x\sin(\pi\eta/2), s), (y\sin(\pi\theta/2), t)) = \rho_{\mathbb{S}^n}(L(\eta \cdot x), L(\theta \cdot y)) \approx |\eta \cdot x - \theta \cdot y|.$$

In particular,

$$|x\sin(\pi\eta/2) - y\sin(\pi\theta/2)| \lesssim |\eta \cdot x - \theta \cdot y|.$$

As $p(j^k_{\eta\cdot x}(f)^{-1} \odot j^k_{\theta\cdot y}(-f)) = \theta \cdot y - \eta \cdot x$, we then have

$$\rho_{\mathbb{B}^n}(x\sin(\pi\eta/2), y\sin(\pi\theta/2)) \approx |x\sin(\pi\eta/2) - y\sin(\pi\theta/2)| \lesssim d_{cc}(j^k_{\eta\cdot x}(f), j^k_{\theta\cdot y}(-f)) \tag{3.1}$$

by Corollary 2.8.

As

$$\rho_{\mathbb{S}^n}((x\sin(\pi\eta/2), s), (y\sin(\pi\theta/2), -t)) = \rho_{\mathbb{B}^n}(x\sin(\pi\eta/2), y\sin(\pi\theta/2)) + |s + t|,$$

it remains to bound $|s+t|$ from above by (a multiple of) $d_{cc}(j^k_{\eta\cdot x}(f), j^k_{\theta\cdot y}(-f))$. Note $s = \cos(\pi\eta/2)$ and $t = \cos(\pi\theta/2)$. Via the Taylor series expansion of cosine at $\pi/2$,

$$\cos\nu = \pi/2 - \nu + O((\pi/2 - \nu)^3) \quad \text{as } \nu \to \pi/2.$$

It follows that

$$\cos\frac{\pi\nu}{2} \lesssim \frac{\pi}{2}(1 - \nu) \quad \text{for } 1 - \epsilon \le \nu \le 1.$$

Since $|\eta \cdot x| \ge \frac{1}{2}$, we must have $\eta \cdot |x_i| \ge \frac{1}{2\sqrt{n}}$ for some i. Since $|\eta \cdot x - \theta \cdot y| \le \frac{1}{4\sqrt{n}}$, we must have $\theta \cdot y_i \ge \frac{1}{4\sqrt{n}}$ if $x_i > 0$ and $\theta \cdot y_i \le -\frac{1}{4\sqrt{n}}$ if $x_i < 0$. Since $1 - \epsilon \le \eta, \theta \le 1$, Lemma 3.21 shows that

$$\left|\frac{\partial^k f}{\partial x_i^k}(\eta \cdot x) + \frac{\partial^k f}{\partial x_i^k}(\theta \cdot y)\right| \ge \frac{1}{C} \cdot (1 - \eta) + \frac{1}{C} \cdot (1 - \theta) \gtrsim \frac{2}{\pi C}\left(\cos\frac{\pi\eta}{2} + \cos\frac{\pi\theta}{2}\right) = \frac{2}{\pi C}(s + t).$$

Let J be the the k-index with $j_i = k$ and $j_l = 0$ for $l \neq i$. By Corollary 2.8,

$$d_{cc}(j^k_{\eta\cdot x}(f), j^k_{\theta\cdot y}(-f)) \gtrsim u_J(j^k_{\eta\cdot x}(f)^{-1} \odot j^k_{\theta\cdot y}(-f)) = \left|\frac{\partial^k f}{\partial x_i^k}(\eta\cdot x) + \frac{\partial^k f}{\partial x_i^k}(\theta\cdot y)\right| \gtrsim \frac{2}{\pi C}(s+t).$$

From (3.1), we may conclude

$$d_{cc}(j^k_{\eta\cdot x}(f), j^k_{\theta\cdot y}(-f)) \gtrsim \rho_{\mathbb{S}^n}((x\sin(\pi\eta/2), s), (y\sin(\pi\theta/2), -t)).$$

3.5 The embedding does not admit a Lipschitz extension

In this subsection, we will finish the proof of Theorem 3.1. For the aid of the reader, we outline the remaining steps of the proof:

Step 1: Define the cylinder $C^{n+1} := \mathbb{B}^n \times [1,1]$ and construct a Lipschitz map $P : C^{n+1} \to \mathbb{B}^{n+1}$.
Step 2: We define the map λ that shrinks $[-1,1]^n$ onto $\mathbb{B}^n$ by scaling line segments passing through the origin. Show that λ is invertible and Lipschitz. Then define $\Lambda : [-1,1]^{n+1} \to C^{n+1}$ by $\Lambda(x,t) = (\lambda(x), t)$.
Step 3: Make sure that f satisfies an integral condition, which may require slightly modifying f.
Step 4: Suppose that ϕ admitted a Lipschitz extension $\tilde{\phi}$ and consider the Lipschitz constants of dilates of $\tilde{\phi} \circ P \circ \Lambda$ to arrive at a contradiction.

We first define a Lipschitz map that maps the cylinder $C^{n+1} := \mathbb{B}^n \times [-1,1]$ onto $\mathbb{B}^{n+1}$. For some intuition, this map projects $\mathbb{S}^{n-1} \times [-1,1]$ onto $\mathbb{S}^{n-1} \times \{0\}$ and fixes $\{0\}^n \times [-1,1]$.

Definition 3.22. Define $P : C^{n+1} \to \mathbb{B}^{n+1}$ by

$$P(\theta\cdot x, t) := (x\sin(\pi\theta/2), t\cos(\pi\theta/2)),$$

where $x \in \mathbb{S}^{n-1}, \theta \in [0,1]$, and $-1 \leq t \leq 1$.

Lemma 3.23. The map $P : C^{n+1} \to \mathbb{B}^{n+1}$ is Lipschitz.

Proof. Via a rotation, it suffices to prove

$$\begin{aligned}\rho_{\mathbb{B}^{n+1}}((\sin(\pi\eta/2), 0, t\cos(\pi\eta/2)), (x\sin(\pi\theta/2), y\sin(\pi\theta/2), s\cos(\pi\theta/2)))\\ \lesssim |(\eta, 0, t) - (\theta x, \theta \cdot y, s)|,\end{aligned}$$

or equivalently

$$\begin{aligned}\left|\sin(\pi\eta/2) - x\sin(\pi\theta/2)\right| + |y\sin(\pi\theta/2)| + |t\cos(\pi\eta/2) - s\cos(\pi\theta/2)|\\ \lesssim \rho_{\mathbb{B}^n}((\eta, 0), (\theta x, \theta \cdot y)) + |t - s|,\end{aligned}$$

where $-1 \leq x \leq 1$, $(x, y) \in \mathbb{S}^{n-1}$, $0 \leq \theta \leq \eta$, $0 < \eta$, and $-1 \leq s, t \leq 1$.

From the estimates performed in the proof of Proposition 3.15,

$$\left|\sin(\pi\eta/2) - x\sin(\pi\theta/2)\right| + \sin\left(\frac{\pi\theta}{2}\right)\sum_{i=2}^{n}|y_i| \lesssim \rho_{\mathbb{B}^n}((\eta, 0), \theta \cdot (x, y))$$

and

$$\begin{aligned}|t\cos(\pi\eta/2) - s\cos(\pi\theta/2)| &\leq |t\cos(\pi\eta/2) - s\cos(\pi\eta/2)| + |s\cos(\pi\eta/2) - s\cos(\pi\theta/2)|\\ &\leq |t - s| + |\cos(\pi\eta/2) - \cos(\pi\theta/2)|\\ &\lesssim |t - s| + \rho_{\mathbb{B}^n}((\eta, 0), \theta \cdot (x, y)).\end{aligned}$$

It follows that P is Lipschitz. □

We now consider the invertible map that shrinks $[-1, 1]^{n+1}$ to $\mathbb{B}^{n+1}$ by scaling lines passing through the origin.

Definition 3.24. For $i = 1, \ldots, n$, define

$$S_i := \{x \in [-1, 1]^n : |x_i| \geq |x_j| \text{ for all } j \neq i\}.$$

Define $\lambda : [-1, 1]^n \to \mathbb{B}^n$ by

$$\lambda(x) := \begin{cases} \frac{|x_i|}{|x|} \cdot x & \text{if } x \in S_i \setminus \{0\}\\ 0 & \text{if } x = 0.\end{cases}$$

Note that $[-1, 1]^n$ is the union of the S_i. Also each S_i is the disjoint union of two convex sets, the subset of x with $x_i \geq 0$ and the subset with $x_i \leq 0$.

We now show that λ is biLipschitz.

Proposition 3.25. The map λ is invertible with $\lambda^{-1} : \mathbb{B}^n \to [-1,1]^n$ given by

$$\lambda^{-1}(u) = \begin{cases} \frac{|u|}{|u_i|} \cdot u & \text{if } u \neq 0 \text{ and } |u_i| \geq |u_j| \text{ for all } j \neq i \\ 0 & \text{if } u = 0. \end{cases}$$

Moreover, λ is biLipschitz with

$$\frac{1}{3(n+1)}|x-y| \leq |\lambda(x) - \lambda(y)| \leq 3|x-y|, \quad x, y \in [-1,1]^{n+1}.$$

Proof. We leave it to the reader to confirm that λ is invertible with its inverse having the form as in the statement.

We show that λ is Lipschitz. Consider the case $x, y \in S_i$ for some common i. If $y = 0$ or $x = 0$, then

$$|\lambda(x) - \lambda(y)| \leq |x-y|.$$

If x, $y \neq 0$ are given with $|x| \leq |y|$,

$$\begin{aligned} |\lambda(x) - \lambda(y)| &\leq \left| \frac{|x_i|}{|x|} \cdot x - \frac{|y_i|}{|y|} x \right| + \left| \frac{|y_i|}{|y|} x - \frac{|y_i|}{|y|} \cdot y \right| \\ &\leq \left| \frac{|x_i|}{|y|} \cdot (|y| - |x|) + \frac{|x|}{|y|} \cdot |x_i| - \frac{|x|}{|y|} \cdot |y_i| \right| + |x-y| \\ &\leq \frac{|x_i|}{|y|} \cdot |y-x| + \frac{|x|}{|y|} \cdot |x_i - y_i| + |x-y| \\ &\leq 3|x-y|. \end{aligned}$$

For general $x, y \in [-1,1]^n$, let $\gamma : [0,1] \to [-1,1]^n$ be the straight line path connecting x to y. Fix a partition $0 = t_0 < t_1 < \ldots < t_m = 1$ such that each restriction $\gamma|_{[t_j, t_{j+1}]}$ is contained in some S_{i_j}. This is possible because each S_i is the disjoint union of two convex sets. Then

$$|\lambda(x) - \lambda(y)| \leq \sum_{i=0}^{m-1} |\lambda(\gamma(t_{i+1})) - \lambda(\gamma(t_i))| \leq \sum_{i=0}^{m-1} 3|\gamma(t_{i+1}) - \gamma(t_i)| = 3|x-y|.$$

This proves that λ is 3-Lipschitz. The proof that λ^{-1} is $\frac{1}{3n}$-Lipschitz is similar. □

This enables us to define a map that stretches C^{n+1} horizontally to $[-1,1]^{n+1}$ via λ. Note that this map will be biLipschitz since λ is.

Definition 3.26. Define $\Lambda : [-1,1]^{n+1} \to C^{n+1} = \mathbb{B}^n \times [-1,1]$ by $\Lambda(x,t) = (\lambda(x), t)$.

We take a moment to note that why we choose to use $P \circ \Lambda$ to map a cube onto $\mathbb{B}^{n+1}$. Note that $P \circ \Lambda$ maps the boundary of $[-1,1]^{n+1}$ onto the boundary of $\mathbb{B}^{n+1}$. This will set us up to replicate Rigot and Wenger's proof of Theorem 1.2 in [RW10] for the lack of a Lipschitz extension. We could have used spherical coordinates to map a cube onto $\mathbb{B}^{n+1}$, but that would have been more delicate since one would not have the "mapping of boundaries".

The trickiest part of this proof will be ensuring that the smooth mapping $f : \mathbb{R}^n \to \mathbb{R}$ serving as the "body" of the embedding satisfies a nonzero integral condition. Before, we need to define integrals of Lipschitz forms on cubes and on the boundaries of cubes.

Definition 3.27. Let $g_1, \ldots, g_{n+1} : [-1,1]^{n+1} \to \mathbb{R}$ be Lipschitz functions. We define

$$\int_{[-1,1]^{n+1}} dg_1 \wedge \cdots \wedge dg_{n+1} := \int_{[-1,1]^{n+1}} \det(\partial_{x_j} g_i) dx_1 \cdots dx_{n+1}$$

and

$$\int_{\partial[-1,1]^{n+1}} g_1 dg_2 \wedge \cdots \wedge dg_{n+1}$$
$$:= \sum_{l=1}^{n+1} \int_{[-1,1]^n} \hat{g}_1^{l,1} \det(\partial_{x_j} \hat{g}_i^{l,1})_{\substack{i\geq 2\\ j\neq l}}\, d\hat{x}_l - \int_{[0,1]^n} \hat{g}_1^{l,0} \det(\partial_{x_j} \hat{g}_i^{l,0})_{\substack{i\geq 2\\ j\neq l}}\, d\hat{x}_l,$$

where $\hat{x}_l := (x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{n+1}) \in \mathbb{R}^n$ and $\hat{g}_i^{l,m}(\hat{x}_l) := g_i(x_1, \ldots, x_{l-1}, m, x_{l+1}, \ldots, x_{n+1})$ for $m = -1, 1$.

Rigot and Wenger's proof in [RW10] relies on a version of Stokes' Theorem for Lipschitz forms.

Lemma 3.28. [RW10, Lemma 3.3] For all Lipschitz functions $g_1, \ldots, g_{n+1} : [-1,1]^{n+1} \to \mathbb{R}$,

$$\int_{[-1,1]^{n+1}} dg_1 \wedge \cdots \wedge dg_{n+1} = \int_{\partial[-1,1]^{n+1}} g_1 dg_2 \wedge \cdots \wedge dg_{n+1}.$$

For the next proof, it will be helpful (to avoid repetition) if we set up notation for a function on $\partial[-1,1]^{n+1}$ obtained from a function on $\mathbb{R}^n$.

Notation 3.29. For each smooth function $g : \mathbb{R}^n \to \mathbb{R}$, define $\bar{g} : \partial[-1,1]^{n+1} \to \mathbb{R}$ by

$$\bar{g}(x,t) = \begin{cases} g(\lambda(x)) & \text{if } x \in [-1,1]^n \text{ and } t = 1 \\ -g(\lambda(x)) & \text{if } x \in [-1,1]^n \text{ and } t = -1 \\ g(\lambda(x)) & \text{if } x \in \partial[-1,1]^n \text{ and } t \in (-1,1). \end{cases}$$

Note that if $g \equiv 0$ on $\mathbb{S}^{n-1}$, then $\bar{g}$ admits the Lipschitz extension $(x,t) \mapsto tg(\lambda(x))$ to $[-1,1]^{n+1}$.

We now state the extra property we need our function f to satisfy.

Proposition 3.30. There exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying:

(a) $f(x) = (1-|x|)^{k+1}$ for $\frac{1}{2} \leq |x| \leq \frac{3}{2}$; and

(b) $f(x) > 0$ for $|x| < 1$.

(c) $\int_{\partial[-1,1]^{n+1}} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d\bar{f} \neq 0$, where $\lambda_1, \ldots, \lambda_n$ are the components of λ.

Proof. By Lemma 3.16, there exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying properties (a) and (b). If $\int_{\partial[-1,1]^{n+1}} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d\bar{f} \neq 0$, then f works, so assume otherwise.

Suppose β is a smooth function supported in a cube inside $\{x \in S_1 : x_1 > 0,\ |x| < \frac{1}{2}\}$ (recall $S_1 = \{x \in [-1,1]^n : |x_1| \geq |x_j| \text{ for all } j > 1\}$). By linearity,

$$\begin{aligned}
&\int_{\partial[-1,1]^{n+1}} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d\overline{(f+\beta)} \\
&\qquad = \int_{\partial[-1,1]^{n+1}} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d\bar{f} + \int_{\partial[-1,1]^{n+1}} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d\bar{\beta} \\
&\qquad = \int_{\partial[-1,1]^{n+1}} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d\bar{\beta}.
\end{aligned}$$

Thus if we show the last integral is nonzero, then $f + \beta$ will work.

As $\bar{\beta} \equiv 0$ on $\partial[-1,1]^n \times [-1,1]$, $\int_{\partial[-1,1]^n \times [-1,1]} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d\bar{\beta} = 0$. We can simplify

$$\int_{\partial[-1,1]^{n+1}} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d\bar{\beta} = 2\int_{[-1,1]^n} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d(\beta \circ \lambda).$$

Note that λ^{-1} is smooth on $\operatorname{int}(S_1) \cap \mathbb{B}^n$, where $\operatorname{int}(S_1)$ is the interior of S_1. Hence,

$$\begin{aligned}
2\int_{[-1,1]^n} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d(\beta \circ \lambda) &= 2\int_{\{x \in \operatorname{int}(S_1) : x_1 > 0\}} \lambda^*(u_1 du_2 \wedge \cdots \wedge du_n \wedge d\beta)\ dx \\
&= 2(-1)^{n+1} \int_{\{u \in \operatorname{int}(S_1) \cap \mathbb{B}^n : u_1 > 0\}} u_1 \frac{\partial \beta}{\partial u_1} \cdot J(\lambda^{-1})\ du,
\end{aligned}$$

where we used that β is supported in $\{u \in \operatorname{int}(S_1) \cap \mathbb{B}^n : u_1 > 0\}$ for the first equality and change of coordinates for the second equality. Integrating by parts,

$$\int_{\{u \in \operatorname{int}(S_1) \cap \mathbb{B}^n : u_1 > 0\}} u_1 \frac{\partial \beta}{\partial u_1} \cdot J(\lambda^{-1})\ du = -\int_{\{u \in \operatorname{int}(S_1) \cap \mathbb{B}^n : u_1 > 0\}} \frac{\partial (u_1 \cdot J(\lambda^{-1}))}{\partial u_1} \cdot \beta\ du.$$

It remains to define β carefully to ensure that the last integral is nonzero.

For $u \in \mathrm{int}(S_1)$ with $u_1 > 0$ and $|u| < 1$, one can calculate

$$\frac{\partial \lambda^{-1}}{\partial u_1}(u) = -\frac{1}{u_1^2} \cdot |u| u + \frac{1}{u_1} \cdot \left(\frac{u_1}{|u|} \cdot u + |u| \cdot e_1 \right)$$

and

$$\frac{\partial \lambda^{-1}}{\partial u_i}(u) = \frac{1}{u_1} \cdot \left(\frac{u_i}{|u|} \cdot u + |u| \cdot e_i \right), \quad i = 2, \dots, n.$$

In particular, $\frac{\partial \lambda^{-1}}{\partial u_j}(u_1, 0, \dots, 0) = e_j$ for $j = 1, \dots, n$ and $0 < u_1 < \frac{1}{2}$. Since $J(\lambda^{-1})(u_1, 0, \dots, 0) = 1$ for $0 < u_1 < \frac{1}{2}$,

$$\frac{\partial (u_1 \cdot J(\lambda^{-1}))}{\partial u_1}(1/4, 0, \dots, 0) = 1.$$

By smoothness, there exists a cube $C \subset \{u \in S_1 : |u| < \frac{1}{2}\}$ centered at $(1/4, 0, \dots, 0)$ on which $\frac{\partial (u_1 \cdot J(\lambda^{-1}))}{\partial u_1} > 0$. If $\beta : \mathbb{R}^n \to [0,1]$ is supported on C and $\beta(1/4, 0, \dots, 0) = 1$, then

$$\int_{\partial[-1,1]^{n+1}} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d\overline{\beta} = 2(-1)^n \int_{\{u \in \mathrm{int}(S_1) \cap \mathbb{B}^n : u_1 > 0\}} \frac{\partial (u_1 \cdot J(\lambda^{-1}))}{\partial u_1} \cdot \beta \; du \neq 0$$

as desired and $f + \beta$ works. □

Let d_0 be the Riemannian metric distance arising from defining an inner product on $Lie(J^k(\mathbb{R}^n))$ that makes the layer of the stratification orthogonal. Define $\iota : (J^k(\mathbb{R}^n), d_{cc}) \to (J^k(\mathbb{R}^n), d_0)$ to be the identity map, which is 1-Lipschitz. With the extra integral condition on f, we can prove that the corresponding embedding of $\mathbb{S}^n$ into $J^k(\mathbb{R}^n)$ does not admit a Lipschitz extension.

Proof of Theorem 3.1. Fix a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying properties (a)-(c) of Proposition 3.30, and let $\phi : \mathbb{S}^n \to J^k(\mathbb{R}^n)$ be the corresponding biLipschitz embedding (see Definition 3.17 and Theorem 3.18).

Suppose, for contradiction, that ϕ admits a Lipschitz extension $\tilde{\phi} : \mathbb{B}^{n+1} \to J^k(\mathbb{R}^n)$. Let λ equal the Lipschitz constant $\mathrm{Lip}(F)$ of the Lipschitz map $F := \tilde{\phi} \circ P \circ \Lambda$. We show that for all $M > 0$,

$$M^{1+\frac{k}{n+1}} \left| \int_{\partial[-1,1]^{n+1}} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d\bar{f} \right|^{1/(n+1)} \leq \mathrm{Lip}(\iota \circ \delta_M \circ F) \leq M\lambda. \tag{3.2}$$

Letting $M \to \infty$, we will arrive at a contradiction.

The right inequality is clear since δ_M is M-Lipschitz and ι is 1-Lipschitz.

For the other inequality, let h_i denote the x_i-coordinate of F for $i = 1, \dots, n$ and h_{n+1} the u_0-coordinate of $\iota \circ \delta_M \circ F$. For $(x,t) \in \partial[-1,1]^{n+1}$, $h_i(x,t) = M\lambda_i(x)$ for $i = 1, \dots, n$ and $h_{n+1}(x,t) = M^{k+1}\bar{f}(x)$. This implies

$$\int_{\partial[-1,1]^{n+1}} h_1 dh_2 \wedge \cdots \wedge dh_{n+1} = M^{n+k+1} \int_{\partial[-1,1]^{n+1}} \lambda_1 d\lambda_2 \wedge \cdots \wedge d\lambda_n \wedge d\bar{f} \neq 0. \tag{3.3}$$

By Lemma 3.28,

$$\int_{\partial[-1,1]^{n+1}} h_1 dh_2 \wedge \cdots \wedge dh_{n+1} = \int_{[-1,1]^{n+1}} dh_1 \wedge dh_2 \wedge \cdots \wedge dh_{n+1}.$$

Define the $(n+1)$-form $\omega := dx_1 \wedge \cdots \wedge dx_n \wedge du_0$ on $J^k(\mathbb{R}^n)$. By Lemma 3.2 of [RW10],

$$|\omega_p(v_1, \cdots, v_{n+1})| \leq 1$$

for all $p \in J^k(\mathbb{R}^n)$ and $v_1, \ldots, v_{n+1} \in T_p J^k(\mathbb{R}^n)$ with $||v_i||_{g_0} \leq 1$. We have

$$\left| \int_{[-1,1]^{n+1}} dh_1 \wedge dh_2 \wedge \cdots \wedge dh_{n+1} \right| = \left| \int_{[-1,1]^{n+1}} (\iota \circ \delta_M \circ F)^* \omega \right| \leq Lip(\iota \circ \delta_M \circ F)^{n+1}.$$

The left inequality of (3.2) follows from (3.3). We may conclude that ϕ does not admit a Lipschitz extension to $\mathbb{B}^{n+1}$. $\square$

Chapter 4

A variant of Gromov's problem on Hölder equivalence of Carnot groups

4.1 Introduction

It is natural to ask the following general question:

When are two Carnot groups equivalent?

Pansu proved that two Carnot groups are biLipschitz homeomorphic if and only if they are isomorphic [Pan89]. With the problem of biLipschitz equivalence somewhat well-understood, we can go on to ask when two Carnot groups are Hölder equivalent.

Gromov considered the problem of Hölder equivalence of Carnot groups: If a Carnot group G is identified with $\mathbb{R}^n$ equipped with a group operation, for which α does there exist a locally α-Hölder homeomorphism $f : \mathbb{R}^n \to G$ [Gro96]? If such α exist, what is the supremum of the set of such α? As noted in Chapter 1, if $(\mathbb{R}^n, \cdot)$ is a step r Carnot group of Hausdorff dimension Q, then it must be that

$$\frac{1}{r} \leq \alpha \leq \frac{n-1}{Q-1}.$$

This is a consequence of the result by Nagel-Stein-Wainger [NSW85] and an isoperimetric inequality for Carnot groups [Val44]. Beyonds these bounds, little more is known, even for the first Heisenberg group. Here, we do not require any regularity of f^{-1} beyond continuity.

We comment on the notation that will be used throughout this chapter. We will simply write $\mathbb{R}^n$ to denote Euclidean space equipped with addition and the standard Euclidean metric. *We will write $(\mathbb{R}^n, \cdot)$ to denote a Carnot group equipped with coordinates of the first or second kind and with the Carnot-Carathéodory metric (recall the content of Sections 2.2 and 2.3).* When we equip a Carnot group with coordinates of the first or second kind, it is implied that we are taking coordinates with respect to a basis compatible with the stratification of its Lie algebra.

In this chapter, we will consider a related problem. We first define a class of maps related to the class $C^{0,\alpha}(X; Y)$ of α-Hölder maps $f : X \to Y$.

Definition 4.1. Fix metric spaces $(X, d_X), (Y, d_Y)$ and $\alpha > 0$. We say a map $f : X \to Y$ is of class $C^{0,\alpha+}(X;Y)$ if there exists a homeomorphism $\beta : [0,\infty) \to [0,\infty)$ such that

$$d_Y(f(a), f(b)) \leq d_X(a,b)^\alpha \beta(d_X(a,b)) \quad \text{for all } a, b \in X. \tag{4.1}$$

We will sometimes simply write $C^{0,\alpha+}$ if the domain and target are clear.

Remark 4.2. Suppose X, Y are metric spaces with X bounded. It is easy to check that

$$C^{0,\eta}(X;Y) \subseteq C^{0,\alpha+}(X;Y) \subseteq C^{0,\alpha}(X;Y).$$

whenever $0 < \alpha < \eta$. Thus, $C^{0,\alpha+}(X;Y)$ can thought of as a right limit of Hölder spaces.

Motivated by Gromov's Hölder equivalence problem, Balogh, Hajłasz, and Wildrick prove that one cannot embed $\mathbb{R}^k$, $k > n$, into $\mathbb{H}^n$ via a sufficiently regular ($\alpha+$)-Hölder mapping. More specifically, they prove that if $k > n$ and $\Omega \subseteq \mathbb{R}^k$ is open, then there is no injective mapping of class $C^{0,\frac{1}{2}+}(\Omega, \mathbb{H}^n)$ that is locally Lipschitz as a mapping into $\mathbb{R}^{2n+1}$ [BHW14, Theorem 1.11]. The main key to their proof is showing that if such a map existed, then it would have to be horizontal almost everywhere. Notice that Remark 4.2 combined with the identity map $\mathrm{id} : \mathbb{R}^3 \to \mathbb{H}^1$ being locally $\frac{1}{2}$-Hölder suggest that this result is sharp except for the extra local Lipschitz assumption.

In this chapter, we will extend the result in the previous paragraph the jet spaces $J^k(\mathbb{R})$. In these groups, there are few nontrivial bracket relations relative to the step, making them ideal settings to generalize the result from the previous paragraph. The proofs for these Carnot groups will again boil down to showing the almost everywhere horizontality of certain $C^{0,\frac{1}{2}+}$ mappings into these groups.

Theorem 4.3. Fix $\alpha \geq \frac{1}{2}$ and positive integers n, k with $n > 1$. Let Ω be an open subset of $\mathbb{R}^n$. Then there is no injective mapping in the class $C^{0,\alpha+}(\Omega; J^k(\mathbb{R}))$ that is also locally Lipschitz when considered as a map into $\mathbb{R}^{k+2}$.

We will prove this result in the case $\alpha = \frac{1}{2}$, and the cases for $\alpha > \frac{1}{2}$ will follow from the fact

$$C^{0,\alpha+}(\Omega; J^k(\mathbb{R})) \subset C^{0,\frac{1}{2}+}(\Omega; J^k(\mathbb{R}))$$

for $\alpha > \frac{1}{2}$.

Using the same reasoning, one can show the same lack of existence for the more conventional class of Hölder mappings (see Proposition 4.15).

Corollary 4.4. Fix $\epsilon > 0$ and positive integers n, k with $n > 1$. Let Ω be an open subset of $\mathbb{R}^n$. Then there is no injective, locally $(\frac{1}{2} + \epsilon)$-Hölder mapping $f : \Omega \to J^k(\mathbb{R})$ that is also locally Lipschitz when considered as a map into $\mathbb{R}^{k+2}$.

The identity map $\mathbb{R}^{k+2} \to J^k(\mathbb{R})$ is locally $\frac{1}{k+1}$-Hölder. From the Heisenberg case, one may expect for it to be unknown whether there exist locally α-Hölder, injective maps $f : \mathbb{R}^n \to J^k(\mathbb{R})$ for $\alpha > \frac{1}{k+1}$. However, we will give an example of a locally $\frac{1}{2}$-Hölder, injective map $f : \mathbb{R}^2 \to J^k(\mathbb{R})$ that is locally Lipschitz as a map into $\mathbb{R}^{k+2}$ (Example 4.16). Comparing with Remark 4.2 and Corollary 4.4, this suggests that our result is sharp, at least in the case $n = 2$.

We will first prove Theorem 4.3 for when $J^k(\mathbb{R})$ is equipped with coordinates of the second kind. We will then prove at the end of the section 4.3 that this implies that the theorem holds for first kind coordinates as well.

We can also prove analogous results for Carnot groups of small step. We will refer the reader to [Jun17b] for proofs.

Theorem 4.5. Fix $\alpha \geq \frac{1}{2}$ and an open subset $\Omega \subseteq \mathbb{R}^k$. Suppose $(\mathbb{R}^n, \cdot)$ is a Carnot group of step at most three that is purely k-unrectifiable. Then there is no injective mapping in the class $C^{0,\alpha+}(\Omega; (\mathbb{R}^n, \cdot))$ that is also locally Lipschitz when considered as a map into $\mathbb{R}^n$.

Corollary 4.6. Fix $\epsilon > 0$ and an open subset $\Omega \subseteq \mathbb{R}^k$. Suppose $(\mathbb{R}^n, \cdot)$ is a Carnot group of step at most three that is purely k-unrectifiable. Then there is no injective mapping in the class $C^{\frac{1}{2}+\epsilon}(\Omega; (\mathbb{R}^n, \cdot))$ that is also locally Lipschitz when considered as a map into $\mathbb{R}^n$.

These theorems and corollaries will be proven in a similar fashion, implied by the following result:

Proposition 4.7. Fix an open subset $\Omega \subseteq \mathbb{R}^k$. Let $(\mathbb{R}^n, \cdot)$ be a Carnot group that is purely k-unrectifiable. Then there is no injective mapping $f : \Omega \to (\mathbb{R}^n, \cdot)$ that is weakly contact and locally Lipschitz when considered as a map into $\mathbb{R}^n$.

Thus, to prove Theorems 4.3 and 4.5, it suffices to show that if a map in $C^{0,\frac{1}{2}+}(\Omega, (\mathbb{R}^n, \cdot))$ is locally Lipschitz as a map into $\mathbb{R}^n$, then it is weakly contact. We will prove the stronger result that these mappings are horizontal at points of differentiability. We prove this for $J^k(\mathbb{R})$ in Proposition 4.13 and we refer the reader to [Jun17b] for proofs of this for Carnot groups of step at most three.

4.2 Weakly contact Lipschitz mappings

Fix an open set $\Omega \subseteq \mathbb{R}^k$ and a Carnot group $(\mathbb{R}^n, \cdot)$. If $f : \Omega \to (\mathbb{R}^n, \cdot)$ is Lipschitz, f is locally Lipschitz as a map into $\mathbb{R}^n$ by Theorem 2.2. By Rademacher's Theorem, then f is differentiable almost everywhere in

Ω. We say a locally Lipschitz map $f : \Omega \to \mathbb{R}^n$ is **weakly contact** if

$$\operatorname{im} df_x \subset H_{f(x)}(\mathbb{R}^n, \cdot) \quad \text{for } \mathcal{H}^k - \text{almost every } x \in \Omega.$$

Here, we write df_x to denote the **differential** or **total derivative** of f at x. Observe that by Theorem 9.18 of [Rud76], if f is differentiable at $x \in \Omega$, then

$$\operatorname{im} df_x \subset H_{f(x)}(\mathbb{R}^n, \cdot) \qquad \Longleftrightarrow \qquad \partial_i f(x) \in H_{f(x)}(\mathbb{R}^n, \cdot) \text{ for all } i = 1, \ldots, k.$$

Balogh, Hajłasz, and Wildrick proved for $\mathbb{H}^n$ that if a Lipschitz map $f : [0,1]^k \to \mathbb{R}^{2n+1}$ is weakly contact, then it is actually Lipschitz as a map into $\mathbb{H}^n$ [BHW14, Proposition 8.2]. Their proof easily converts into a statement for all Carnot groups. To keep this chapter as self-contained as possible, we will repeat the argument here.

Proposition 4.8. Let k be a positive integer. If $f : [0,1]^k \to \mathbb{R}^n$ is Lipschitz and weakly contact, then $f : [0,1]^k \to (\mathbb{R}^n, \cdot)$ is Lipschitz.

Proof. Fix a weakly contact map $f : [0,1]^k \to \mathbb{R}^n$ that is L-Lipschitz. Fubini's Theorem implies the restriction of f to almost every line segment parallel to a coordinate axis is horizontal. On bounded sets, the lengths with respect to the sub-Riemannian metrics and to the Euclidean metrics are equivalent for horizontal vectors. As $f[0,1]^k$ is bounded and the Euclidean speed of f is bounded by L on line segments, it follows that the restriction of f on almost every line segment parallel to a coordinate axis is CL-Lipschitz as a map into $(\mathbb{R}^n, \cdot)$. Hence the restriction of f on *each* line segment parallel to a coordinate axis is CL-Lipschitz as a map into $(\mathbb{R}^n, \cdot)$, and the result follows. □

This enables us to prove Proposition 4.7, a result fundamental to this chapter. The proof of Theorem 1.11 in [BHW14] for the Heisenberg group translates into a result for all Carnot groups.

Proof of Proposition 4.7. Assume that there is an injective map $f : \Omega \to (\mathbb{R}^n, \cdot)$ that is locally Lipschitz as a map into $\mathbb{R}^n$. Restricting f, we may assume Ω is a closed cube and f is Lipschitz as a map into $\mathbb{R}^n$. If f is weakly contact, $f : \Omega \to (\mathbb{R}^n, \cdot)$ is Lipschitz, which implies $\mathcal{H}^k_{(\mathbb{R}^n,\cdot)}(f(\Omega)) = 0$. As the identity map from $(\mathbb{R}^n, \cdot)$ to $\mathbb{R}^n$ is locally Lipschitz (by Theorem 2.2), $\mathcal{H}^k_{\mathbb{R}^n}(f(\Omega)) = 0$. It follows from Theorem 8.15 of [Hei01] that the topological dimension of $f(\Omega)$ is at most $k-1$. Since $f|_\Omega$ is a homeomorphism, $f(\Omega)$ is of the same topological dimension as Ω, which is a contradiction. □

The main theorems of this chapter thus reduce to showing locally Lipschitz maps f into $\mathbb{R}^n$ that are of class $C^{0,\frac{1}{2}+}(\Omega,(\mathbb{R}^n,\cdot))$, are weakly contact. Balogh, Hajłasz, and Wildrick proved this for the Heisenberg group [BHW14, Proposition 8.1]. In this chapter, we will prove it for $J^k(\mathbb{R})$.

4.3 Strata-preserving isomorphisms

Fix a Carnot group G with Lie algebra $\mathfrak{g}$. Recall in Section 2.2, we defined $\mathfrak{d}_\epsilon$, $\epsilon > 0$, on $\mathfrak{g}$ by dilating coordinates in the j^{th} layer of $\mathfrak{g}$ by ϵ^j. This induced dilations $\delta_\epsilon : G \to G$ via the exponential map (see (2.2)).

Suppose H is a Carnot group isomorphic to G, with stratification

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r.$$

A Lie group isomorphism $\varphi : G \to H$ induces a Lie algebra isomorphism $\varphi_* : \mathfrak{g} \to \mathfrak{h}$ that satisfies the following identity:

$$\exp_H \circ \varphi_* = \varphi \circ \exp_G . \tag{4.2}$$

We say that a Lie group isomorphism $\varphi : G \to H$ **commutes with dilations** if

$$\varphi(\delta^G_\epsilon g) = \delta^H_\epsilon \varphi(g) \quad \text{for all } g \in G,\ \epsilon > 0,$$

where $\delta^G_\epsilon, \delta^H_\epsilon$ denote the dilations on G, H, respectively. If we say that a Lie algebra isomorphism $f : \mathfrak{g} \to \mathfrak{h}$ commutes with dilations if

$$f(\mathfrak{d}^G_\epsilon X) = \mathfrak{d}^H_\epsilon f(X) \quad \text{for all } X \in \mathfrak{g},\ \epsilon > 0,$$

it is easy to check using (2.2) and (4.2) that an isomorphism $\varphi : G \to H$ commutes with dilations if and only if $\varphi_* : \mathfrak{g} \to \mathfrak{h}$ commutes with dilations.

Example 4.9. Let G be a Carnot group. Suppose $\mathcal{B} \subset \mathfrak{g}$ is a basis compatible with the stratification of $\mathfrak{g}$. Let $(\mathbb{R}^n, \odot)$ and $(\mathbb{R}^n, \star)$ be G equipped with coordinates of the second and first kind, respectively, with respect to $\mathcal{B}$. Then $(\mathbb{R}^n, \odot)$ is isomorphic to $(\mathbb{R}^n, \star)$ via $\exp^{-1} \circ \Phi$ and coordinates. Moreover, this isomorphism commutes with dilations.

We say that an isomorphism $\varphi : G \to H$ is **strata-preserving** if

$$\varphi_*(\mathfrak{g}_j) = \mathfrak{h}_j \quad \text{for all } j = 1, \ldots, r.$$

Note that φ is strata-preserving if and only if φ^{-1} is strata-preserving.

The next result follows from the use of dilations:

Lemma 4.10. Let G, H be isomorphic Carnot groups. An isomorphism $\varphi : G \to H$ commutes with dilations if and only if φ is strata-preserving.

In fact, if we say that an isomorphism $\varphi : G \to H$ is *contact* if $\varphi_*(\mathfrak{g}_1) = \mathfrak{h}_1$, it's easy to check from the stratifications of $\mathfrak{g}$ and $\mathfrak{h}$ that φ is a contact map if and only if it is strata-preserving.

We will show weakly contact mappings are invariant under isomorphisms that commute with dilations. We first prove that such isomorphisms are biLipschitz.

Proposition 4.11. Let $\varphi : (\mathbb{R}^n, \cdot) \to (\mathbb{R}^n, *)$ be an isomorphism between Carnot groups, that commutes with dilations. Then φ is biLipschitz, i.e., there exists a constant C such that

$$\frac{1}{C} d_{cc}^{(\mathbb{R}^n,\cdot)}(g,h) \leq d_{cc}^{(\mathbb{R}^n,*)}(\varphi(g),\varphi(h)) \leq C d_{cc}^{(\mathbb{R}^n,\cdot)}(g,h) \quad \text{for all } g,h \in (\mathbb{R}^n,\cdot).$$

Proof. As φ commutes with dilations and the cc-metrics on $(\mathbb{R}^n, \cdot)$ and $(\mathbb{R}^n, *)$ are one-homogeneous, it suffices to show φ is biLipschitz when restricted to $B_{cc}(e, 1)$.

Let $\{X^1, \ldots, X^{m_1}\}$, $\{Y^1, \ldots, Y^{m_1}\}$ be left-invariant frames for $H(\mathbb{R}^n, \cdot)$, $H(\mathbb{R}^n, *)$, respectively. For each $g \in (\mathbb{R}^n, \cdot)$, define the linear isomorphism $S_g : H_{\varphi(g)}(\mathbb{R}^n, *) \to H_{\varphi(g)}(\mathbb{R}^n, *)$ induced by $(\varphi_* X^j)_{\varphi(g)} \mapsto Y^j_{\varphi(g)}$. The function $g \mapsto ||S_g||$ is continuous, and hence, is bounded on $B_{cc}(e, 2)$, say by C. This implies for all $g \in B_{cc}(e, 2)$ and $v \in H_g(\mathbb{R}^n, \cdot)$, we have $|d\varphi_g(v)|_{\varphi(g)} \leq C|v|_g$. It then follows from Lemma 4.10 that

$$d_{cc}^{(\mathbb{R}^n,*)}(\varphi(g),\varphi(h)) \leq C d_{cc}^{(\mathbb{R}^n,\cdot)}(g,h)$$

for all $g, h \in B_{cc}(e, 1)$. Applying this argument to φ^{-1}, the lemma follows. $\square$

It follows from the chain rule that weak contactness is preserved by strata-preserving isomorphisms.

Corollary 4.12. Fix $\Omega \subseteq \mathbb{R}^k$ an open subset. Let $\varphi : (\mathbb{R}^n, \cdot) \to (\mathbb{R}^n, *)$ be an isomorphism between Carnot groups, that commutes with dilations. If $f : \Omega \to (\mathbb{R}^n, \cdot)$ is locally Lipschitz and weakly contact, then $\varphi \circ f : \Omega \to (\mathbb{R}^n, *)$ is also locally Lipschitz and weakly contact.

4.4 Non-existence of certain $C^{0,\alpha+}$ mappings into $J^k(\mathbb{R})$

In this section, we will prove a horizontality condition for $J^k(\mathbb{R})$, from which Theorem 4.3 will follow.

Proposition 4.13. Let k, n be positive integers with $\Omega \subseteq \mathbb{R}^n$ an open set. Suppose that $f = (f^x, f^{u_k}, \ldots, f^{u_0}) : \Omega \to J^k(\mathbb{R})$ is of class $C^{0,\frac{1}{2}+}$. If the component f^x is differentiable at a point $p_0 \in \Omega$, then the components $f^{u_{k-1}}, f^{u_{k-2}}, \ldots, f^{u_0}$ are also differentiable at p_0 with

$$df^{u_j}_{p_0} = f^{u_{j+1}}(p_0) df^x_{p_0}$$

for all $j = 0, \ldots, k-1$. In particular, if f^{u_k} is also differentiable at p_0, then the image of df_{p_0} lies in the horizontal space $H_{f(p_0)} J^k(\mathbb{R})$.

Proof. We prove this result by induction on $k \geq 1$. Below, p is a point in Ω.

Let $f = (f^x, f^{u_1}, f^{u_0}) : \Omega \to J^1(\mathbb{R})$ be given of class $C^{0,\frac{1}{2}+}$. Choose a map β for f satisfying (4.1). By Proposition 2.9,

$$(f(p_0)^{-1} f(p))_0 = f^{u_0}(p) - f^{u_0}(p_0) - f^{u_1}(p_0)(f^x(p) - f^x(p_0)).$$

Thus by Corollary 2.8, there exists $C > 0$ such that

$$\begin{aligned} |f^{u_0}(p) - f^{u_0}(p_0) - f^{u_1}(p_0)(f^x(p) - f^x(p_0))|^{1/2} &\leq C d_{cc}(f(p), f(p_0)) \\ &\leq C\beta(|p - p_0|) \cdot |p - p_0|^{1/2}. \end{aligned}$$

We have

$$\begin{aligned} &|f^{u_0}(p) - f^{u_0}(p_0) - f^{u_1}(p_0) df^x_{p_0}(p - p_0)| \\ &\quad \leq C^2 \beta^2(|p - p_0|) \cdot |p - p_0| + |f^{u_1}(p_0)(f^x(p) - f^x(p_0)) - f^{u_1}(p_0) df^x_{p_0}(p - p_0)| \\ &\quad = o(|p - p_0|), \end{aligned}$$

where we used the differentiability of f^x at p_0 for the last equality.

Suppose we have proven the result up to k. Let $f = (f^x, f^{u_{k+1}}, \ldots, f^{u_0}) : \Omega \to J^{k+1}(\mathbb{R})$ be given of class $C^{0,\frac{1}{2}+}$ with f^x differentiable at p_0. Let $\tilde{\beta}$ be a map satisfying (4.1) for f. Define the projection $\pi : J^{k+1}(\mathbb{R}) \to J^k(\mathbb{R})$ by

$$\pi(x, u_{k+1}, \ldots, u_0) = (x, u_{k+1}, \ldots, u_1).$$

As π maps horizontal curves to horizontal curves of the same length, it's not hard to see that π is a contraction. This implies $\pi \circ f = (f^x, f^{u_{k+1}}, \ldots, f^{u_1})$ is of class $C^{0,\frac{1}{2}+}(\Omega, J^k(\mathbb{R}))$. By induction, $f^{u_k}, \ldots, f^{u_1}$ are differentiable at p_0 with

$$df^{u_j}_{p_0} = f^{u_{j+1}}(p_0) df^x_{p_0}$$

for all $j = 1, \ldots, k$.

It remains to show f^{u_0} is also differentiable at p_0 with

$$df^{u_0}_{p_0} = f^{u_1}(p_0) df^x_{p_0}.$$

Corollary 2.8 and Proposition 2.9 combine to imply

$$\left| f^{u_0}(p) - f^{u_0}(p_0) - \sum_{j=1}^{k+1} \frac{f^{u_j}(p_0)}{j!} (f^x(p) - f^x(p_0))^j \right|^{1/(k+1)} \leq C\tilde{\beta}(|p - p_0|) \cdot |p - p_0|^{1/2}.$$

Moreover, as f^x is differentiable at p_0,

$$f^x(p) - f^x(p_0) = O(|p - p_0|),$$

and hence

$$|f^x(p) - f^x(p_0)|^j = o(|p - p_0|) \quad \text{for all } j \geq 2.$$

It follows

$$\begin{aligned}
&|f^{u_0}(p) - f^{u_0}(p_0) - f^{u_1}(p_0) df^x_{p_0}(p - p_0)| \\
&\quad \leq C^{k+1} \tilde{\beta}^{k+1}(|p - p_0|) \cdot |p - p_0|^{\frac{k+1}{2}} + |f^{u_1}(p_0)(f^x(p) - f^x(p_0)) - f^{u_1}(p_0) df^x_{p_0}(x - x_0)| \\
&\qquad + \sum_{j=2}^{k+1} \left| \frac{f^{u_j}(p_0)}{j!} (f^x(p) - f^x(p_0)) \right|^j \\
&\quad = o(|x - x_0|).
\end{aligned}$$

This proves f^{u_0} is differentiable at p_0 with

$$df^{u_0}_{p_0} = f^{u_1}(p_0) df^x_{p_0},$$

and the proposition follows. □

Remark 4.14. In the above proof, we needed f to lie in $C^{0,\frac{1}{2}+}(\Omega, J^k(\mathbb{R}))$ in order to ensure $f^{u_{k-1}}$ was differentiable at the point with the desired form. To prove the differentiability of the components of f corresponding to higher layers, one can assume lower regularity. In fact, the above proof shows the following:

Assume $\Omega \subseteq \mathbb{R}^n$ is open and $j \geq 2$. Suppose $f = (f^x, f^{u_k}, \ldots, f^{u_0}) : \Omega \to J^k(\mathbb{R})$ is of class $C^{0,\frac{1}{j}+}$. If f^x

is differentiable at a point $p_0 \in \Omega$, then $f^{u_{k+1-j}}, f^{u_{k-j}}, \ldots, f^{u_0}$ are also differentiable at p_0 with

$$df^{u_l}_{p_0} = f^{u_{l+1}}(p_0) df^x_{p_0}, \quad l = k+1-j, \ldots, 0.$$

While $C^{0,\frac{1}{2}+\epsilon}(X;Y)$ is not contained in $C^{0,\frac{1}{2}+}(X;Y)$ in general, the same proof works to show that differentiable $C^{0,\frac{1}{2}+\epsilon}$ are horizontal.

Proposition 4.15. Let k, n be positive integers with $\Omega \subseteq \mathbb{R}^n$ an open set. Suppose that $f = (f^x, f^{u_k}, \ldots, f^{u_0}) : \Omega \to J^k(\mathbb{R})$ is locally $(\frac{1}{2}+\epsilon)$-Hölder for some $\epsilon > 0$. If the component f^x is differentiable at a point $p_0 \in \Omega$, then the components $f^{u_{k-1}}, f^{u_{k-2}}, \ldots, f^{u_0}$ are also differentiable at p_0 with

$$df^{u_j}_{p_0} = f^{u_{j+1}}(p_0) df^x_{p_0}$$

for all $j = 0, \ldots, k-1$. In particular, if f^{u_k} is also differentiable at p_0, then the image of df_{p_0} lies in the horizontal space $H_{f(p_0)} J^k(\mathbb{R})$.

Before we prove Theorem 4.3, we will give an example of a locally $\frac{1}{2}$-Hölder map $f : \mathbb{R}^2 \to J^k(\mathbb{R})$ that is Lipschitz as a map into $\mathbb{R}^{k+2}$. Comparing with Remark 4.2 and Proposition 4.15, this suggests that our result is sharp in the case $n = 2$.

Example 4.16. Define $f : \mathbb{R}^2 \to J^k(\mathbb{R})$ by

$$f(x, y) = (0, x, y, 0, \ldots, 0).$$

Then f is Lipschitz (in fact, is an isometry) as a map into $\mathbb{R}^{k+2}$.

To show f is locally $\frac{1}{2}$-Hölder, first note in $J^k(\mathbb{R})$,

$$(0, -x_1, -y_1, 0, \ldots, 0) \odot (0, x_2, y_2, 0, \ldots, 0) = (0, x_2 - x_1, y_2 - y_1, 0, \ldots, 0).$$

By Corollary 2.8, there exists a constant C such that

$$d_{cc}(f(x_1, y_1), f(x_2, y_2)) \leq C \max\{|x_2 - x_1|, |y_2 - y_1|^{1/2}\}$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. By considering cases, one can then show

$$d_{cc}(f(x_1, y_1), f(x_2, y_2)) \leq \sqrt{2M} C |(x_1, y_1) - (x_2, y_2)|^{1/2}$$

whenever $(x_1,y_1),(x_2,y_2) \in [-M,M]^2$ with $M>1$. We see that this example is uniformly Hölder in the sense that for every $R>0$, there exists a uniform constant C_R such that for every Euclidean ball B of radius R in $\mathbb{R}^2$, we have

$$d_{cc}(f(x_1,y_1), f(x_2,y_2)) \le C_R |(x_1,y_1)-(x_2,y_2)|^{1/2}$$

for all $(x_1,y_1),(x_2,y_2) \in B$.

Proof of Theorem 4.3. Fix positive integers n,k with $n \ge 2$. Suppose $f : \Omega \to J^k(\mathbb{R})$ is of class $C^{0,\frac{1}{2}+}$ and is locally Lipschitz as a map into $\mathbb{R}^{k+2}$. By Rademacher's Theorem, each of the components of f is differentiable almost everywhere, and in particular, f^x is differentiable almost everywhere. Proposition 4.13 then implies that f is weakly contact. Since $J^k(\mathbb{R})$ is purely n-unrectifiable [Mag04, Theorem 1.1], Theorem 4.3 in the case of second kind coordinates follows from Proposition 4.8. Example 4.9 with Proposition 4.11 then proves the result for coordinates of the first kind. □

Remark 4.17. Observe that $J^k(\mathbb{R}^m,\mathbb{R}^n)$ is purely j-unrectifiable if $j > \binom{m+k-1}{k}$ [Mag04, Theorem 1.1]. Hence, from Proposition 2.10, one can use similar reasoning to show the following generalization:

Fix a jet space $J^k(\mathbb{R}^m,\mathbb{R}^n)$ and equip it with the group structure from Section 2.5. Suppose $j > \binom{m+k-1}{k}$ and Ω is an open subset of $\mathbb{R}^j$. If N is the topological dimension of $J^k(\mathbb{R}^m,\mathbb{R}^n)$, there is no injective mapping in the class $C^{0,\frac{1}{2}+}(\Omega; J^k(\mathbb{R}^m,\mathbb{R}^n))$ that is also locally Lipschitz when considered as a map into $\mathbb{R}^N$.

Remark 4.18. Theorem 4.3 has an easier proof if we assume $n < \frac{1}{2}\left(1+\frac{(k+1)(k+2)}{2}\right)$. Making this assumption, suppose that $f : \Omega \to J^k(\mathbb{R})$ is injective and of class $C^{0,\frac{1}{2}+}$. Let $B(x,r)$ be an open ball with $\overline{B(x,r)} \subseteq \Omega$. Then the restriction $f|_{\overline{B(x,r)}}$ is injective and of class $C^{0,\frac{1}{2}+}(\overline{B(x,r)}), J^k(\mathbb{R}))$. Since $\overline{B(x,r)}$ is bounded, it follows that $f|_{\overline{B(x,r)}}$ is a $\frac{1}{2}$-Hölder homeomorphism. In particular, $f(B(x,r))$ is open in $J^k(\mathbb{R})$, which implies

$$\dim_{\mathrm{Hau}} f(B(x,r)) = \dim_{\mathrm{Hau}} J^k(\mathbb{R}) = 1 + \frac{(k+1)(k+2)}{2}.$$

But as f is $\frac{1}{2}$-Hölder,

$$\dim_{\mathrm{Hau}} f(B(x,r)) \le 2 \cdot \dim_{\mathrm{Hau}} B(x,r) = 2n,$$

which is a contradiction.

We will conclude this section by further considering Hölder mappings into jet spaces. We will prove that a mapping into $J^k(\mathbb{R})$ is Hölder if and only if each of its components is Hölder to various degrees.

Hajłasz and Mirra proved that the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is α-Hölder in $\mathbb{H}^1$ and if and only if f is $\alpha/2$-Hölder continuous [HM13, Proposition 3.1]. Since $\mathbb{H}^1$ is isomorphic to $J^1(\mathbb{R})$, one might suspect that a similar statement holds true for all model filiform jet spaces. We will spend the rest of this section proving that this is in fact true.

Proposition 4.19. Let $\Omega \subset \mathbb{R}^2$ be a bounded subset. Suppose $u_{k-1}, \ldots, u_0$ are functions of $(x, u_k) \in \Omega$. Define $\Phi(x, u_k) = (x, u_k, u_{k-1}(x, u_k), \ldots, u_0(x, u_k))$. Fix $\alpha \in [0, \frac{1}{k+1}]$. Then Φ is α-Hölder continuous if and only if each component u_s is $(k+1-s)\alpha$-Hölder, $s = 0, \ldots, k-1$.

Reworded, the last statement says that u_{k-1} is 2α-Hölder, u_{k-2} is 3α-Hölder,... and u_0 is $(k+1)\alpha$-Hölder. Before we prove this result, we make a small observation. This essentially allows us to ignore the summands $\frac{u_j(x,u_k)}{(j-s)!}(y-x)^{j-s}$ in the the proof of Proposition 4.19.

Lemma 4.20. Let the assumptions be as in Proposition 4.19. If Φ is α-Hölder continuous, then u_s is bounded for all $s = 0, \ldots, k-1$.

Proof. As Φ is Hölder, $\Phi(\Omega)$ is bounded in $J^k(\mathbb{R})$. In particular, the restriction of $id : J^k(\mathbb{R}) \to \mathbb{R}^{k+2}$ to $\Phi(\Omega)$ is bounded. Post-composing with the projection π_s, $s = 0, \ldots, k-1$, the lemma follows. □

Proof of Proposition 4.19. (⇒) Recall that for each $s = 0, \ldots, k-1$,

$$(\Phi(x, u_k)^{-1} \odot \Phi(y, v_k))_s = u_s(y, v_k) - u_s(x, u_k) - \sum_{j=s+1}^{k} \frac{u_j(x, u_k)}{(j-s)!}(y-x)^{j-s}. \tag{4.3}$$

Also recall by Corollary 2.8,

$$d_{cc}(\Phi(x, u_k), \Phi(y, v_k)) \approx \max\left\{|(x, u_k) - (y, v_k)|, |(\Phi(x, u_k)^{-1} \odot \Phi(y, v_k))_s|^{\frac{1}{k+1-s}} : s = 0, \ldots, k-1\right\}. \tag{4.4}$$

Fix $s = 0, \ldots, k-1$. We have

$$\begin{aligned} |u_s(y, v_k) - u_s(x, u_k)|^{\frac{1}{k+1-s}} &\lesssim |(\Phi(x, u_k)^{-1} \odot \Phi(y, v_k))_s|^{\frac{1}{k+1-s}} + \left|\sum_{j=s+1}^{k} \frac{u_j(x, u_k)}{(j-s)!}(y-x)^{j-s}\right|^{\frac{1}{k+1-s}} \\ &\lesssim |(x, u_k) - (y, v_k)|^{\alpha} + \sum_{j=s+1}^{k} |y-x|^{\frac{j-s}{k+1-s}}, \end{aligned}$$

where we used Lemma 4.20 for the last estimate. Since Ω is bounded,

$$|y-x|^{\frac{j-s}{k+1-s}} \lesssim |y-x|^{\alpha}, \quad j \geq s+1.$$

Indeed, since $\alpha \leq \frac{1}{k+1}$,

$$\frac{j-s}{k+1-s} - \alpha \geq \frac{1}{k+1-s} - \alpha \geq 0.$$

Continuing,

$$|u_s(y,v_k) - u_s(x,u_k)|^{\frac{1}{k+1-s}} \lesssim |(x,u_k)-(y,v_k)|^\alpha + \sum_{j=s+1}^{k} |y-x|^\alpha \lesssim |(x,u_k)-(y,v_k)|^\alpha.$$

This shows u_s is $(k+1-s)\alpha$-Hölder.

($\Leftarrow$) Now suppose u_s is $(k+1-s)\alpha$-Hölder for all $s = 0, \ldots, k-1$. By (4.4) above, we need to show

$$|(x,u_k)-(y,v_k)| \lesssim |(x,u_k)-(y,v_k)|^\alpha$$

and

$$|(\Phi(x,u_k)^{-1} \odot \Phi(y,v_k))_s| \lesssim |(x,u_k)-(y,v_k)|^{(k+1-s)\alpha}, \quad s = 0, \ldots, k-1.$$

The first estimate is clear since $\alpha < 1$ and Ω is bounded. Next, for each $s = 0, \ldots, k-1$ (with estimates explained after)

$$\begin{aligned} |(\Phi(x,u_k)^{-1} \odot \Phi(y,v_k))_s| &\leq |u_s(y,v_k) - u_s(x,u_k)| + \sum_{j=s+1}^{k} \frac{|u_j(x,u_k)|}{(j-s)!} |y-x|^{j-s} \\ &\lesssim |(x,u_k)-(y,v_k)|^{(k+1-s)\alpha} + \sum_{j=s+1}^{k} |y-x|^{j-s} \\ &\lesssim |(x,u_k)-(y,v_k)|^{(k+1-s)\alpha}. \end{aligned}$$

In the first line, we used the identities at (4.3) above. At the second line, we used the Hölder assumption of the u_s's and Lemma 4.20. For the final estimate, we used that

$$|y-x|^{j-s} \lesssim |y-x|^{(k+1-s)\alpha}$$

since Ω is bounded and $\alpha \leq \frac{1}{k+1}$. It follows from (4.4) that Φ is α-Hölder. □

4.5 Almost Lipschitz surfaces in Carnot groups

Gromov's problem is based upon the general problem of determining how smoothly one can embed Euclidean space into a Carnot group. Related to this problem, Hajłasz and Mirra proved in 2013 that there exist almost Lipschitz surfaces in the Heisenberg group that are horizontal a.e. [HM13]. We will conclude this chapter by generalizing their construction to all Carnot groups.

Hajłasz and Mirra proved:

Theorem 4.21. [HM13, Theorem 3.2] Let $\mu : [0,\infty) \to [0,\infty)$ be a continuous function such that $\mu(0) = 0$ and $\mu(t) = O(t)$ as $t \to \infty$. Then there is a continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ such that

1. $|u(x) - u(y)| \leq |x-y|/\mu(|x-y|)$ for all $x, y \in \mathbb{R}^2$;

2. u is differentiable a.e.;

3. the tangent plane to the graph of u is horizontal in $\mathbb{H}^1$ for almost all $(x,y) \in \mathbb{R}^2$.

Defining μ appropriately, e.g., taking

$$\mu(t) := \begin{cases} 0 & \text{if } t = 0 \\ |\ln t|^{-1} & \text{if } 0 < t < e^{-1} \\ et & \text{if } t \geq e^{-1}, \end{cases} \tag{4.5}$$

one immediately obtains

Corollary 4.22. [HM13] There is a continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ such that

- u is α-Hölder continuous for all $\alpha \in (0,1)$;
- u is differentiable a.e.;
- the tangent plane to the graph of u is horizontal in $\mathbb{H}^1$ for almost all $(x,y) \in \mathbb{R}^2$.

This stems from the fact that for all $\lambda \in (0,1)$, there is $C_\lambda > 0$ such that

$$\mu(t) > C_\lambda t^{1-\lambda}, \quad t > 0.$$

The proof of Theorem 4.21 relied on first proving a Lusin-type theorem. They proved that given any measurable functions f_α, $\alpha \in \tilde{I}(m)$, there exists a function $g \in C^{m-1}$ that is almost Lipschitz and satisfies $D^\alpha g = f_\alpha$ a.e.

Theorem 4.23. [HM13, Theorem 1.1] Let $\Omega \subset \mathbb{R}^n$ be open, and let $f_1, \ldots, f_n$ be measurable functions on Ω. Let $\sigma > 0$ and $\mu : [0,\infty) \to [0,\infty)$ be a continuous function with $\mu(0) = 0$ and $\mu(t) = O(t)$ as $t \to \infty$. Then there is a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ that is differentiable a.e. such that

1. $\frac{\partial g}{\partial x_i} = f_i$ a.e. on Ω for $i = 1, \ldots, n$;

2. $||g||_{L^\infty(\mathbb{R}^n)} < \sigma$;

3. $|g(x) - g(y)| \le \frac{|x-y|}{\mu(|x-y|)}$ for all $x, y \in \mathbb{R}^n$.

In particular, one can choose μ so that g is α-Hölder continuous for all $\alpha \in (0,1)$.

Before we prove Theorem 4.21 in general, we will need to dive a little deeper into the structure of Carnot groups than we did in Sections 2.1 and 2.2.

Let $(\mathbb{R}^n, \cdot)$ be a Carnot group equipped with coordinates of the first kind. For each i, let $m_i := \dim(\mathfrak{g}_i)$, where

$$Lie(\mathbb{R}^n, \cdot) = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r.$$

Also define $h_0 := 0$ and $h_j = m_1 + \cdots + m_j$ for $j \ge 1$.

Cassano proved the following about the group structure of $(\mathbb{R}^n, \cdot)$:

Proposition 4.24. [Cas16, Proposition 2.3] The group product has the form

$$x \cdot y = x + y + \mathcal{Q}(x, y),$$

where $\mathcal{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and each $\mathcal{Q}_j$ is a degree α_j homogeneous polynomial, that is

$$\mathcal{Q}_j(\delta_\lambda x, \delta_\lambda y) = \lambda^{\alpha_j} \mathcal{Q}_j(x, y), \quad \forall x, y \in \mathbb{G}.$$

Moreover,

$$\mathcal{Q}_1(x, y) = \cdots = \mathcal{Q}_{m_1}(x, y) = 0 \quad \forall x, y \in \mathcal{G}$$

and

$$\mathcal{Q}_j(x, y) = \mathcal{Q}_j(x_1, \ldots, x_{h_{i-1}}, y_1, \ldots, y_{h_{i-1}}), \quad \text{if } 1 < l \le k \quad \text{and} \quad h_{l-1} < j \le h_l.$$

Finally,

$$\mathcal{Q}_j(x, y) = \sum_{k,h} \mathcal{R}^j_{k,h}(x, y)(x_k y_h - x_h y_k),$$

where the functions $\mathcal{R}^j_{k,h}$ are polynomials, homogeneous of degree $\alpha_j - \alpha_k - \alpha_h$ with respect to group dilations, and the sum is extended to all h, k such that $\alpha_h + \alpha_k \le \alpha_j$.

The key point of the proposition is that $\mathcal{Q}_j$ $(h_{l-1} < j \le h_l)$ depends only on variables of degree less than j.

The unique left-invariant vector fields X_j that satisfy $(X_j)_0 = \frac{\partial}{\partial x_j}$ form a global frame of $Lie(\mathbb{R}^n, \cdot)$. One can leverage the form of the group operation to obtain global formulas for these vector fields.

Proposition 4.25. [Cas16, Proposition 2.4] The vector fields X_j have polynomial coefficients and if $h_{l-1} < j \leq h_l$, $1 \leq l \leq k$, then

$$X_j(x) = \partial_j + \sum_{i>h_l}^{n} q_{i,j}(x)\partial_i,$$

where $q_{i,j}(x) = \frac{\partial Q_i}{\partial y_j}(x,y)|_{y=0}$, so that if $h_{l-1} < j \leq h_l$, then $q_{i,j}(x) = q_{i,j}(x_1, \ldots, x_{h_{l-1}})$ and $q_{i,j}(0) = 0$.

Note that $q_{i,j}$ depends only on variables of degree lower than x_j. In the proof of Theorem 4.26, we will play with the fact that each $q_{i,j}$ can be viewed both as a function on $\mathbb{R}^n$ and a function on the variables of lower degree.

We can now prove our main theorem.

Theorem 4.26. Let $\mu : [0,\infty) \to [0,\infty)$ be a continuous function such that $\mu(0) = 0$ and $\mu(t) = O(t)$ as $t \to \infty$. There exist continuous functions $u_i : \mathbb{R}^{m_1} \to \mathbb{R}$, $i > m_1$, that are differentiable a.e. and satisfy

$$|u_i(x) - u_i(y)| \leq \frac{|x-y|}{\mu(|x-y|)} \quad \text{for all } x, y \in \mathbb{R}^{m_1}.$$

Moreover, if we define $\Phi : \mathbb{R}^{m_1} \to \mathbb{R}^n$ by

$$\Phi(x) = (x, u_{m_1+1}(x), \ldots, u_n(x)),$$

then the tangent plane to the image of Φ is horizontal for almost all $x \in \mathbb{R}^{m_1}$. In fact,

$$\frac{\partial \Phi}{\partial x_j}(x) = X_j(\Phi(x)) \quad \text{a.e.} \quad \text{for all } j = 1, \ldots, m_1.$$

Proof. Fix $h_1 < i \leq h_2$. By Theorem 4.23, there exists a continuous function $u_i : \mathbb{R}^{m_1} \to \mathbb{R}$ that is differentiable a.e. such that

$$\frac{\partial u_i}{\partial x_j}(x) = q_{i,j}(x) \quad \text{a.e.,} \quad j = 1, \ldots, m_1,$$

and

$$|u_i(x) - u_i(y)| \leq \frac{|x-y|}{\mu(|x-y|)} \quad \text{for all } x, y \in \mathbb{R}^{m_1}. \tag{4.6}$$

Note that the first set of equalities make sense since $q_{i,j}$ is a polynomial depending only on the horizontal variables.

Now suppose that we have defined u_i for all $h_1 < i \leq h_l$, some $l < k$, and fix $h_l < i < h_{l+1}$. By Theorem

4.23, there exists a continuous function $u_i : \mathbb{R}^{m_1} \to \mathbb{R}$, differentiable a.e., satisfying (4.6) and

$$\frac{\partial u_i}{\partial x_j}(x) = q_{i,j}(x, u_{h_1+1}(x), \dots, u_{h_l}(x)) \quad \text{a.e.}, \qquad j = 1, \dots, m_1.$$

Once again, we are using that $q_{i,j}$ depends only on $x_1, \dots, x_{h_l}$.

The map Φ is differentiable a.e. with

$$\frac{\partial \Phi}{\partial x_j}(x) = \partial_j + \sum_{i>m_1} q_{i,j}(\Phi(x))\partial_i = X_j(\Phi(x)) \quad \text{a.e.}, \qquad j = 1, \dots, m_1.$$

□

Letting μ be as at (4.5), we immediately obtain

Corollary 4.27. There exist functions $u_i : \mathbb{R}^{m_1} \to \mathbb{R}$, $m_1 < i \leq n$, such that

- each u_i is α-Hölder continuous simultaneously for all $\alpha \in (0, 1)$;
- each u_i is differentiable a.e.; and
- if we define $\Phi : \mathbb{R}^{m_1} \to \mathbb{R}^n$ by $\Phi(x) = (x, u_{m_1+1}(x), \dots, u_n(x))$, then

$$\frac{d\Phi}{dx_j}(x) = X_j(\Phi(x)) \quad \text{a.e.} \quad \text{for all } j = 1, \dots, m_1.$$

In particular, Φ is horizontal a.e.

Chapter 5

Dimension results for mappings of jet space Carnot groups

5.1 Introduction

We will conclude by proposing analogues of horizontal and vertical projections for jet space Carnot groups. This continues work over the past decade to obtain Marstrand-type results in the setting of sub-Riemannian geometry [BDCF+13, BFMT12].

The effect of projections in Euclidean space on Hausdorff dimension has been studied for decades. Marstrand began the study in 1954, essentially proving that the projection of a set onto almost every plane has large Hausdorff dimension relative to the size of the set [Mar54]. Later research in this area was performed by Kaufman in 1968 [Kau68], Mattila in 1975 [Mat75], and Peres and Schlag in 2000 [PS00] among many others. For a greater discussion on this topic, we refer to the reader to Mattila's recent survey [Mat18].

In 2012, Balogh, Durand-Cartagena, Fässler, Mattila, and Tyson defined analogues of horizontal and vertical projections in the first Heisenberg group $\mathbb{H}^1$ [BDCF+13]. Each horizontal line in $\mathbb{R}^2 \times \{0\}$ passing through the origin paired with its orthogonal complement provides a semidirect group splitting of $\mathbb{H}^1$, which induces horizontal and vertical projections on $\mathbb{H}^1$. The main objective of their work was to prove Marstrand-type theorems for these projections. It should be noted that while each $P_{\mathbb{V}_\theta}$ is linear, a projection, and a group homomorphism, the mappings $P_{\mathbb{V}_\theta^\perp}$ are none of these in general. This illustrates the increased difficulty as one studies projection theorems in the sub-Riemannian setting. The situation will be even worse for the mappings in our present study.

The choice of the vertical planes in [BDCF+13] was motivated by the fact that the restriction of the gauge metric is a snowflaked metric, which makes calculations much easier. This will motivate the choice of our vertical planes. By the identities proven in Proposition 2.9, we observe that the gauge metric restricted to vertical hyperplanes with first coordinate fixed, is snowflaked as well. Hence, we will take these as the images of our vertical projections.

For the horizontal sets, the Carnot group structure of $J^k(\mathbb{R})$ supplies us with a rich family of sets to complement these planes, in fact a $C^\infty(\mathbb{R})$-worth of such sets! For each $t \in \mathbb{R}$, $f \in C^\infty(\mathbb{R})$, and $p \in J^k(\mathbb{R})$, there exist unique points in the plane $\{x = t\} := \{(x, u_k, \dots, u_0) \in J^k(\mathbb{R}) : x = t\}$ and the image of $j^k(f)$, which we denote by $V_{f,t}(p)$, $J_{f,t}(p)$, respectively, such that $p = V_{f,t}(p) \odot J_{f,t}(p)$. We first show that each $V_{f,t}$ is biLipschitz when restricted to a hyperplane $\{x = t\}$. The map $V_{f,t}$ has a complicated definition involving right multiplication that makes it difficult to analyze from a projection of sets viewpoint. The mappings $V_{f,t}$ and $J_{f,t}$ are rarely idempotent, hence it would be wrong to call them projections. Nevertheless, it is helpful (at least for the author) to think them of as projections. We show in Section 5.4 that the mappings share some regularity. For all $f \in C^\infty(\mathbb{R})$ and $t \in \mathbb{R}$, $V_{f,t}$ is locally $\frac{1}{k+1}$-Hölder (Proposition 5.9) while $J_{f,t}$ is locally Lipschitz (Proposition 5.6).

The main questions we ask are: Given $t \in \mathbb{R}$, $f \in C^\infty(\mathbb{R})$, and a Borel set $E \subset J^k(\mathbb{R})$, how do the dimensions (topological and Hausdorff) of $J_{f,t}(E)$ and $V_{f,t}(E)$ compare to the dimensions of E? Which triples $(\dim E, \dim J_{f,t}(E), \dim V_{f,t}(E))$ can be attained as E varies over Borel sets, where dim denotes topological or Hausdorff dimension? Can E be chosen independently of t or f?

We show that if t and f are fixed, all pairs $(\dim_{\text{Hau}} J_{f,t}(E), \dim_{\text{Hau}} V_{f,t}(E))$, $(\dim_{\text{Top}} J_{f,t}(E), \dim_{\text{Top}} V_{f,t}(E))$ and $(\dim_{\text{Hau}} E, \dim_{\text{Hau}} J_{f,t}(E))$ are possible after taking into account the dimensions of $\text{im}(j^k(f))$ and the hyperplanes $\{x = t\}$ (Theorem 5.22, Proposition 5.28, and Proposition 5.23). We show that in some cases, the Borel set E can be chosen independently of f (Corollary 5.16 and Proposition 5.29). The question of which pairs $(\dim_{\text{Hau}} E, \dim_{\text{Hau}} V_{f,t}(E))$ are possible is much more difficult, and we only partially answer this question when f is a nonconstant linear function (Theorem 5.26).

5.2 Horizontal and vertical mappings

Before we define our mappings, we will first step back and consider how Balogh, Durand-Cartagena, Fässler, Mattila, and Tyson defined horizontal and vertical projections in $\mathbb{H}^1$ in [BDCF+13]. This is, after all, the motivation behind how we will define our horizontal and vertical mappings. See section 2.4 for a brief description of the Heisenberg groups.

The authors of [BDCF+13, BFMT12] equip $\mathbb{H}^1$ with the **Korányi metric**:

$$d_K(p, p') := ||p^{-1} * p'||, \quad \text{where } ||p||_K := (|z|^4 + t^2)^{1/4}.$$

It is easy to see that this is equivalent to the gauge metric on $\mathbb{H}^1$:

$$d_G(p, p') := ||p^{-1} * p'||, \quad \text{where } ||p|| := |z| + |t|^{1/2}.$$

We also note that by the result of Nagel, Stein, and Wainger [NSW85, Proposition 1.1], the Korányi metric is equivalent to the Carnot-Carathéodory metric on $\mathbb{H}^n$.

The authors of [BDCF$^+$13] considered projections onto horizontal and vertical subgroups in $\mathbb{H}^1$. Let V_θ be the (horizontal) line in $\mathbb{R}^2 \times \{0\}$ passing through the origin at angle θ, and note that the Korányi metric agrees with the standard Euclidean metric on V_θ. Moreover, the Korányi metric takes on a simple form on the orthogonal complement $V_\theta^\perp$:

$$d_K((aie^{i\theta}, t), (a'ie^{i\theta}, t')) = (|a - a'|^4 + |t - t'|^2)^{1/4}.$$

Finally, for each θ, we have a semidirect group splitting $\mathbb{H}^1 = V_\theta \rtimes V_\theta^\perp$. In particular, for all $p \in \mathbb{H}^1$, there exist unique $P_{V_\theta}(p) \in V_\theta$ and $P_{V_\theta^\perp}(p) \in V_\theta^\perp$ such that

$$p = P_{V_\theta}(p) * P_{V_\theta^\perp}(p).$$

They then proceed to consider the effect of P_{V_θ} and $P_{V_\theta^\perp}$ on Hausdorff dimension.

We will seek to find a similar splitting of our jet spaces. Rather than a Korányi metric, we will use a gauge distance on $J^k(\mathbb{R})$.

Definition 5.1. (Gauge distance of $J^k(\mathbb{R})$) For all $p, q \in J^k(\mathbb{R})$, define the **gauge distance**

$$d(p, q) := ||p^{-1} \odot q||, \qquad \text{where } ||(x, u_k, \ldots, u_0)|| := |x| + \sum_{j=0}^{k} |u_j|^{1/(k+1-j)}.$$

While the gauge distance isn't an actual metric in the metric space sense, it is equivalent to the CC-metric thanks to Corollary 2.8.

Proposition 5.2. There exists a constant $C > 0$ such that for all $p, q \in J^k(\mathbb{R})$,

$$\frac{d(p, q)}{C} \leq d_{cc}(p, q) \leq C d(p, q).$$

In hoping to replicate the construction for $\mathbb{H}^1$, we will first find a family of vertical sets on which the restriction of the gauge distance takes on a simple form. Suppose $(x, u_k, \ldots, u_0), (y, v_k, \ldots, v_0)$ are two points such that

$$((x, u_k, \ldots, u_0)^{-1} \odot (y, v_k, \ldots, v_0))_s = v_s - u_s \tag{5.1}$$

for all s. Then

$$d((x, u_k, \dots, u_0)^{-1} \odot (y, v_k, \dots, v_0)) = |x - y| + \sum_{j=0}^{k} |v_j - u_j|^{1/(k+1-j)}.$$

(We think that this would be a pretty simple form for the distance!) From Proposition 2.9, we see that (5.1) holds if $x = y$. Thus, for our vertical sets, we will choose the planes

$$\{x = t\} := \{(x, u_k, \dots, u_0) \in J^k(\mathbb{R}) : x = t\}, \qquad t \in \mathbb{R}.$$

We now seek a horizontal set that induces a splitting of $J^k(\mathbb{R})$ when coupled with each of the planes $\{x = t\}$. Fortunately, by the Carnot group structure of $J^k(\mathbb{R})$, we have a whole $C^{k+1}(\mathbb{R})$-family of such sets- images of jets of functions in $C^{k+1}(\mathbb{R})$! For simplicity, we will primarily consider functions in $C^\infty(\mathbb{R})$ in this chapter, but it would be interesting to explore if anything would change if we allowed all functions in $C^{k+1}(\mathbb{R})$.

Fix $f \in C^\infty(\mathbb{R})$. For all $p = (x, u_k, \dots, u_0) \in J^k(\mathbb{R})$,

$$p = (p \odot j^k_{x-t}(f)^{-1}) \odot j^k_{x-t}(f), \tag{5.2}$$

where $p \odot j^k_{x-t}(f)^{-1} \in \{x = t\}$ and $j^k_{x-t}(f) \in \mathrm{im}(j^k(f)) := \{j^k_y(f) : y \in \mathbb{R}\}$. Moreover, it isn't hard to see that $p \odot j^k_{x-t}(f)^{-1}, j^k_{x-t}(f)$ are the unique points in $\{x = t\}, \mathrm{im}(j^k(f))$, respectively, for which (5.2) holds. Indeed, suppose

$$(x, u_k, \dots, u_0) = q \odot j^k_s(f)$$

for some $q \in \{x = t\}$ and $s \in \mathbb{R}$. Then

$$s = (j^k_s(f))_x = (q^{-1} \odot (x, u_k, \dots, u_0))_x = x - t.$$

From there,

$$q = (x, u_k, \dots, u_0) \odot j^k_{x-t}(f)^{-1}.$$

For each $f \in C^\infty(\mathbb{R})$ and $t \in \mathbb{R}$, we define the **vertical mapping** $V_{f,t} : J^k(\mathbb{R}) \to \{x = t\}$ by

$$V_{f,t}(x, u_k, \dots, u_0) := (x, u_k, \dots, u_0) \odot j^k_{x-t}(f)^{-1}$$

and the **horizontal mapping** $J_{f,t} : J^k(\mathbb{R}) \to \mathrm{im}(j^k(f))$ by

$$J_{f,t}(x, u_k, \ldots, u_0) := j^k_{x-t}(f).$$

Then

$$p = V_{f,t}(p) \odot J_{f,t}(p)$$

for all $p \in J^k(\mathbb{R})$.

The planes $\{x = t\}$ are clearly not vertical subspaces or closed under $\odot$ unless $t = 0$. Also, the horizontal sets $\mathrm{im}(j^k(f))$ are not subgroups of $J^k(\mathbb{R})$ in general. However, by left-invariance, the left-cosets $p \odot \mathrm{im}(j^k(f))$ are isometrically equivalent with respect to the CC-distance.

We emphasize that these mappings are not linear projections much less idempotent in general. More specifically, it is not the case that $V_{f,t} = V_{f,t} \circ V_{f,t}$ or $J_{f,t} = J_{f,t} \circ J_{f,t}$ in general. It is true that

$$\mathrm{im}(V_{f,t}) = \{x = t\} \qquad \text{and} \qquad \mathrm{im}(J_{f,t}) = \mathrm{im}(j^k(f)).$$

However, for all $p \in \{x = t\}$,

$$V_{f,t}(p) = p \odot j^k_0(f)^{-1} = p$$

if and only if $j^k_0(f) = 0$. Also, for all $j^k_x(f) \in \mathrm{im}(j^k(f))$,

$$J_{f,t}(j^k_x(f)) = j^k_{x-t}(f) = j^k_x(f)$$

if and only if $t = 0$. It follows that $V_{f,t} = V_{f,t} \circ V_{f,t}$ if and only if $j^k_0(f) = 0$, and $J_{f,t} = J_{f,t} \circ J_{f,t}$ if and only if $t = 0$.

As in the Heisenberg group, we will be interested in the effect of $V_{f,t}$ and $J_{f,t}$ on dimensions of Borel sets, both topological and Hausdorff. We will also be interested in the possibilities of pairs $(\dim_{\mathrm{Hau}} V_{f,t}(E), \dim_{\mathrm{Hau}} J_{f,t}(E))$ as E ranges over Borel sets. The beauty of studying the effect of these mappings on sets is that we now have three parameters to play with: the Borel set E, the hyperplane parameter t, and the smooth function f.

5.3 Simple examples

Before we dive too deeply into studying the maps $J_{f,t}$ and $V_{f,t}$, we will consider a couple of examples that, at first thought, should be simple in terms of studying the Hausdorff dimensions of their images under $J_{f,t}$ and $V_{f,t}$.

Example 5.3. Fix $t \in \mathbb{R}$. For $f \in C^\infty(\mathbb{R})$, we will consider the horizontal image $J_{f,t}(\{x = t\})$ and the vertical image $V_{f,t}(\{x = t\})$ of the plane $\{x = t\}$. In this example, we will obtain dimension results that are independent of f.

For all $p \in \{x = t\}$, $J_{f,t}(p) = j_0^k(f)$. Hence,

$$J_{f,t}(\{x = t\}) = \{j_0^k(f)\},$$

and

$$\dim_{\text{Hau}}(J_{f,t}(\{x = t\}) = \dim_{\text{Top}}(J_{f,t}(\{x = t\})) = 0.$$

By Lemma 5.8, for all $p \in \{x = t\}$,

$$V_{f,t}(p) = p \odot j_0^k(f)^{-1} = p - j_0^k(f).$$

Here, $p - j_0^k(f)$ represents the vector difference of p and $j_0^k(f)$ when both are viewed as elements of $\mathbb{R}^{k+2}$. This implies

$$V_{f,t}(\{x = t\}) = \{x = t\} - j_0^k(f) = \{x = t\}.$$

We may conclude

$$\dim_{\text{Top}}(V_{f,t}(\{x = t\})) = \dim_{\text{Top}}(\{x = t\}) = k + 1$$

and

$$\dim_{\text{Hau}}(V_{f,t}(\{x = t\})) = \dim_{\text{Hau}}(\{x = t\}) = \frac{(k+1)(k+2)}{2},$$

where both equalities are independent of the function f.

A careful examination of our work shows something remarkable: the restriction of $V_{f,t}$ to the plane $\{x = t\}$ is given by subtraction by a fixed vector. This is a homeomorphism of $\{x = t\}$, which implies that $V_{f,t}$ preserves the topological dimension of subsets of $\{x = t\}$. Moreover, the gauge distance behaves very well with respect to subtraction by a fixed element:

$$d(V_{f,t}(p), V_{f,t}(q)) = d(p - j_0^k(f), q - j_0^k(f)) = d(p, q), \quad p, q \in \{x = t\}.$$

By Proposition 5.2, the fact that $V_{f,t}|_{\{x=t\}}$ is d-isometric implies that $V_{f,t}|_{\{x=t\}}$ is d_{cc}-biLipschitz. We mark all of this in a proposition.

Proposition 5.4. Fix $t \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R})$. Then

$$\dim_{\mathrm{Top}}(V_{f,t}(E)) = \dim_{\mathrm{Top}}(E)$$

for all $E \subset \{x = t\}$. Moreover, the restriction $V_{f,t}|_{\{x=t\}} : \{x = t\} \to \{x = t\}$ is biLipschitz when $\{x = t\}$ is equipped with the restriction of the Carnot-Carathéodory distance.

We make the remark that it won't make a difference whether we use d or d_{cc} to compute Hausdorff dimension as they are biLipschitz equivalent by Proposition 5.2. It will be useful for the reader to keep this in mind throughout this chapter (and possibly in life in general).

We conclude this section with the complementary example of a set: the image of a jet.

Example 5.5. Fix $f \in C^\infty(\mathbb{R})$ and $t \in \mathbb{R}$. In this example, we will consider applying our mappings to the image $\mathrm{im}(j^k(f))$ of $j^k(f)$. At first glance, it might seem like things will be similar to the previous example and $V_{f,t}(\mathrm{im}(j^k(f)))$ and $J_{f,t}(\mathrm{im}(j^k(f)))$ will be simple to study from a dimension standpoint. And, in fact, $J_{f,t}(\mathrm{im}(j^k(f)))$ is pretty easy to study. For all $x \in \mathbb{R}$,

$$J_{f,t}(j^k_x(f)) = j^k_{x-t}(f),$$

so that

$$J_{f,t}(\mathrm{im}(j^k(f))) = \mathrm{im}(j^k(f)).$$

However, the study of $V_{f,t}(\mathrm{im}(j^k(f)))$ is a bit more complicated than one would first expect. One has

$$V_{f,t}(j^k_x(f)) = j^k_x(f) \odot j^k_{x-t}(f)^{-1}, \qquad x \in \mathbb{R}.$$

By Lemma 5.8 (to be proven in the next section),

$$(j^k_x(f) \odot j^k_{x-t}(f)^{-1})_s = \sum_{j=s}^{k} \frac{(t-x)^{j-s}}{(j-s)!}(f^{(j)}(x) - f^{(j)}(x-t)), \qquad s = 0, \dots, k.$$

Unless f is a constant function, $V_{f,t}(\mathrm{im}(j^k_x(f)))$ will be a smooth, nonconstant curve, hence have topological dimension 1. However, it isn't clear what its Hausdorff dimension is.

5.4 Regularity of $J_{f,t}$ and $V_{f,t}$

Now that we have seen a couple examples and played around a little with the maps, we will prove our first result. We will show that the horizontal mappings and vertical mappings share some regularity amongst themselves. As one might expect, each of the mappings $J_{f,t}$ is locally Lipschitz and each of the $V_{f,t}$ is locally $\frac{1}{k+1}$-Hölder.

The proof for $J_{f,t}$ is much simpler, so we will begin there. And in fact, it should be expected that the proof will be easier for $J_{f,t}$ since each maps to a 1-dimensional subset of $J^k(\mathbb{R})$ and is given by essentially shifting the x-coordinate of a point by t.

Proposition 5.6. For all $t \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R})$, $J_{f,t} : J^k(\mathbb{R}) \to \operatorname{im}(j^k(f))$ is locally Lipschitz.

Proof. Fix $t \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R})$, and let $(x, u_k, \ldots, u_0), (y, v_k, \ldots, v_0) \in J^k(\mathbb{R})$ be given. By Proposition 2.13 and Corollary 2.8,

$$
\begin{aligned}
&d_{cc}(J_{f,t}(x, u_k, \ldots, u_0), J_{f,t}(y, v_k, \ldots, v_0)) \\
&\qquad = d_{cc}(j^k_{x-t}(f), j^k_{y-t}(f)) \\
&\qquad \le \sup_{s\in[x-t,y-t]} \left(1 + (f^{(k+1)}(s))^2\right)^{1/2} |x-y| \\
&\qquad \le C \sup_{s\in[x-t,y-t]} \left(1 + (f^{(k+1)}(s))^2\right)^{1/2} d_{cc}((x, u_k, \ldots, u_0), (y, v_k, \ldots, v_0)),
\end{aligned}
$$

where C is the constant from Corollary 2.8. As $f^{(k+1)}$ is bounded on compact sets, $J_{f,t}$ is locally Lipschitz. □

As an immediate corollary, we obtain

Corollary 5.7. For all Borel sets $E \subset J^k(\mathbb{R})$, $t \in \mathbb{R}$, and $f \in C^\infty(\mathbb{R})$,

$$\dim_{\mathrm{Hau}}(J_{f,t}(E)) \le \min\{\dim_{\mathrm{Hau}}(E), 1\}.$$

We see that the proof for $J_{f,t}$ being locally Lipschitz follows pretty easily from Proposition 2.13 and Corollary 2.8. However, things get a bit more difficult when we shift to analyzing $V_{f,t}$. $V_{f,t}$ maps to a hyperplane as opposed to a curve, and also $V_{f,t}$ involves a right-translation, which is notorious for being unwieldy. Fortunately, the simple form of the group operation on $J^k(\mathbb{R})$ will save us. We first prove the form of $p \odot q^{-1}$ for $p, q \in J^k(\mathbb{R})$, with the motivation of doing so being the particular form of $V_{f,t}$.

Lemma 5.8. For all $(x, u_k, \dots, u_0), (y, v_k, \dots, v_0) \in J^k(\mathbb{R})$,

$$((x, u_k, \dots, u_0) \odot (y, v_k, \dots v_0)^{-1})_s = \sum_{j=s}^{k} \frac{(-y)^{j-s}}{(j-s)!}(u_j - v_j), \qquad s = 0, \dots, k.$$

Proof. First,

$$((y, v_k, \dots, v_0)^{-1})_s = -\sum_{j=s}^{k} \frac{(-y)^{j-s}}{(j-s)!} v_j, \qquad s = 0, \dots, k.$$

We can calculate

$$\begin{aligned} ((x, u_k, \dots, u_0) \odot (y, v_k, \dots v_0)^{-1})_s &= u_j - \sum_{j=s}^{k} \frac{(-y)^{j-s}}{(j-s)!} v_j + \sum_{j=s+1}^{k} \frac{(-y)^{j-s}}{(j-s)!} u_j \\ &= \sum_{j=s}^{k} \frac{(-y)^{j-s}}{(j-s)!}(u_j - v_j). \end{aligned}$$

□

We can now prove that the $V_{f,t}$ are locally Hölder.

Proposition 5.9. For all $t \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R})$, $V_{f,t} : J^k(\mathbb{R}) \to \{x = t\}$ is locally $\frac{1}{k+1}$-Hölder.

Proof. Let $t \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R})$ be given. It suffices to prove that for all $M > 1$, there exists a constant C_M such that

$$d_{cc}(V_{f,t}(x, u_k, \dots, u_0), V_{f,t}(y, v_k, \dots, v_0)) \le C_M d_{cc}((x, u_k, \dots, u_0), (y, v_k, \dots, v_0))^{1/(k+1)}$$

for all $(x, u_k, \dots, u_0), (y, v_k, \dots, v_0) \in [-M, M]^{k+2} \subset J^k(\mathbb{R})$.

Fix $M > 1$ and $(x, u_k, \dots, u_0), (y, v_k, \dots, v_0) \in [-M, M]^{k+2}$. We have

$$\begin{aligned} & d_{cc}(V_{f,t}(x, u_k, \dots, u_0), V_{f,t}(y, v_k, \dots, v_0)) \\ & \quad = d_{cc}(0, V_{f,t}(x, u_k, \dots, u_0)^{-1} \odot V_{f,t}(y, v_k, \dots, v_0)) \\ & \quad \le C\Big(|(V_{f,t}(x, u_k, \dots, u_0))_x - (V_{f,t}(y, v_k, \dots, v_0))_x| \\ & \qquad + \sum_{s=0}^{k} |(V_{f,t}(x, u_k, \dots, u_0)^{-1} \odot V_{f,t}(y, v_k, \dots, v_0))_s|^{1/(k+1-s)} \Big), \end{aligned} \tag{5.3}$$

where C is the constant from Proposition 5.2. Thus, it suffices to bound the coordinates of $V_{f,t}(x, u_k, \dots, u_0)^{-1} \odot V_{f,t}(y, v_k, \dots, v_0)$.

By definition of $V_{f,t}$,

$$(V_{f,t}(x, u_k, \ldots, u_0))_x - (V_{f,t}(y, v_k, \ldots, v_0))_x = t - t = 0.$$

By Lemma 5.8,

$$((x, u_k, \ldots, u_0) \odot j^k_{x-t}(f)^{-1})_s = \sum_{j=s}^{k} \frac{(t-x)^{j-s}}{(j-s)!}(u_j - f^{(j)}(x-t))$$

and

$$((y, v_k, \ldots, v_0) \odot j^k_{y-t}(f)^{-1})_s = \sum_{j=s}^{k} \frac{(t-y)^{j-s}}{(j-s)!}(v_j - f^{(j)}(y-t))$$

for $s = 0, \ldots, k$. By Proposition 2.9, this implies

$$\begin{aligned}\Big(((x, u_k, \ldots, u_0) \odot j^k_{x-t}(f)^{-1})^{-1} \odot ((y, v_k, \ldots, v_0) \odot j^k_{x-t}(f)^{-1})\Big)_s \\ = \sum_{j=s}^{k} \frac{(t-y)^{j-s}}{(j-s)!}\Big(v_j - f^{(j)}(y-t)\Big) - \sum_{j=s}^{k} \frac{(t-x)^{j-s}}{(j-s)!}\Big(u_j - f^{(j)}(x-t)\Big)\end{aligned}$$

(Note that the x-coordinates of $V_{f,t}(x, u_k, \ldots, u_0)$ and $V_{f,t}(y, v_k, \ldots, u_0)$ agree, hence most terms drop out.) Adding and subtracting terms, the last expression can be rewritten as

$$\begin{aligned}&\sum_{j=s}^{k} \frac{(t-y)^{j-s}}{(j-s)!}\Big(v_j - f^{(j)}(y-t)\Big) - \sum_{j=s}^{k} \frac{(t-x)^{j-s}}{(j-s)!}\Big(v_j - f^{(j)}(y-t)\Big) \\ &\quad + \sum_{j=s}^{k} \frac{(t-x)^{j-s}}{(j-s)!}\Big(v_j - f^{(j)}(y-t)\Big) - \sum_{j=s}^{k} \frac{(t-x)^{j-s}}{(j-s)!}\Big(u_j - f^{(j)}(x-t)\Big).\end{aligned} \tag{5.4}$$

By the Binomial Theorem,

$$|a^n - b^n| \leq n(M + |t|)^{n-1}|a - b|$$

for $-M - t \leq a, b \leq M - t,\ n \in \mathbb{N}$, which implies

$$|(t-y)^{j-s} - (t-x)^{j-s}| \leq (j-s)(M + |t|)^{j-s-1}|x - y|, \qquad j = s+1, \ldots, k.$$

If we define

$$A := \max\{|f^{(j)}(a)| : a \in [-M - t, M - t],\ j = s+1, \ldots, k+1\},$$

then

$$\begin{aligned}
&\sum_{j=s}^{k} \frac{(t-y)^{j-s}}{(j-s)!}\left(v_j - f^{(j)}(y-t)\right) - \sum_{j=s}^{k} \frac{(t-x)^{j-s}}{(j-s)!}\left(v_j - f^{(j)}(y-t)\right) \\
&\quad \le \sum_{j=s+1}^{k} \frac{(j-s)(M+|t|)^{j-s-1}|x-y|}{(j-s)!} \cdot (M+A) \\
&\quad \le (k-s)^2 (M+|t|)^{k-s-1}(M+A)\cdot|x-y| \\
&\quad \le k^2(M+|t|)^{k-1}(M+A)Dd_{cc}((x,u_k,\dots,u_0),(y,v_k,\dots,v_0)),
\end{aligned} \tag{5.5}$$

where D is the constant from Corollary 2.8. This bounds the first expression of (5.4).

For the second expression, by the Mean Value Theorem,

$$|f^{(j)}(y-t) - f^{(j)}(x-t)| \le A|x-y| \le ADd_{cc}((x,u_k,\dots,u_0),(y,v_k,\dots,v_0)) \tag{5.6}$$

for $j = s,\dots,k$. Moreover, by the theorem of Nagel-Stein-Wainger [NSW85, Proposition 1.1], the identity map id : $J^k(\mathbb{R}) \to \mathbb{R}^{k+2}$ is locally Lipschitz. In particular, there exists a constant D_M such that for all $p, q \in [-M, M]^{k+2} \subset J^k(\mathbb{R})$,

$$|\mathrm{id}(p) - \mathrm{id}(q)| \le D_M d_{cc}(p,q).$$

This implies

$$\begin{aligned}
|v_j - u_j| &\le |\mathrm{id}(x,u_k,\dots,u_0) - \mathrm{id}(y,v_k,\dots,v_0)| \\
&\le D_M d_{cc}((x,u_k,\dots,u_0),(y,v_k,\dots,v_0)).
\end{aligned}$$

We can combine this with (5.6) to obtain

$$|(v_j - f^{(j)}(y-t)) - (u_j - f^{(j)}(x-t))| \le (AD + D_M)d_{cc}((x,u_k,\dots,u_0),(y,v_k,\dots,v_0)).$$

By (5.4) and (5.5),

$$\begin{aligned}
&\left(((x,u_k,\dots,u_0)\odot j^k_{x-t}(f)^{-1})^{-1} \odot ((y,v_k,\dots,v_0)\odot j^k_{x-t}(f)^{-1})\right)_s \\
&\quad \le k^2(M+|t|)^{k-1}(M+A)Dd_{cc}((x,u_k,\dots,u_0),(y,v_k,\dots,v_0)) \\
&\qquad + \sum_{j=s}^{k} \frac{(|t|+M)^{j-s}}{(j-s)!}(AD+D_M)d_{cc}((x,u_k,\dots,u_0),(y,v_k,\dots,v_0)) \\
&\quad \le \widetilde{C_M} d_{cc}((x,u_k,\dots,u_0),(y,v_k,\dots,v_0)),
\end{aligned}$$

where

$$\widetilde{C_M} := k^2(M+|t|)^{k-1}(M+A)D + (k+1)(|t|+M)^k(AD+D_M) + 1.$$

(We included the extra term of 1 at the end just to be secure later when we consider roots of $\widetilde{C_M}$.) By (5.3), we have

$$\begin{aligned} &d_{cc}(V_{f,t}(x,u_k,\ldots,u_0), V_{f,t}(y,v_k,\ldots,v_0)) \\ &\qquad \le C\sum_{s=0}^{k}(\widetilde{C_M}d_{cc}((x,u_k,\ldots,u_0),(y,v_k,\ldots,v_0)))^{1/(k+1-s)} \\ &\qquad \le C(k+1)\widetilde{C_M}\operatorname{diam}([-M,M]^{k+2})\cdot d_{cc}((x,u_k,\ldots,u_0),(y,v_k,\ldots,v_0)))^{1/(k+1)}. \end{aligned}$$

This proves that $V_{f,t}$ is locally $\frac{1}{k+1}$-Hölder. For the last inequality, we used that

$$\begin{aligned} &d_{cc}((x,u_k,\ldots,u_0),(y,v_k,\ldots,v_0)))^{1/(k+1-s)} \\ &\qquad \le \operatorname{diam}([-M,M]^{k+2})\cdot d_{cc}((x,u_k,\ldots,u_0),(y,v_k,\ldots,v_0)))^{1/(k+1)} \end{aligned}$$

for all $s=0,\ldots,k$. □

As a consequence, $V_{f,t}$ cannot increase Hausdorff dimension by more than a factor of $k+1$.

Corollary 5.10. For all $t\in\mathbb{R}$, $f\in C^\infty(\mathbb{R})$, and Borel sets $E\subset J^k(\mathbb{R})$,

$$\dim_{\mathrm{Hau}}(V_{f,t}(E)) \le \min\left\{(k+1)\dim_{\mathrm{Hau}}(E), \frac{(k+1)(k+2)}{2}\right\}.$$

5.5 Possible pairs for Hausdorff dimensions of images

For all $f\in C^\infty(\mathbb{R})$, $t\in\mathbb{R}$, and Borel sets E,

$$0 \le \dim_{\mathrm{Hau}}(J_{f,t}(E)) \le \dim_{\mathrm{Hau}}(\mathrm{im}(j^k(f))) = 1$$

and

$$0 \le \dim_{\mathrm{Hau}}(V_{f,t}(E)) \le \dim_{\mathrm{Hau}}(\{x=t\}) = \frac{(k+1)(k+2)}{2}.$$

In this section, we will prove that for all $f \in C^\infty(\mathbb{R})$, $t \in \mathbb{R}$ and pairs $(\alpha, \beta) \in [0,1] \times \left[0, \frac{(k+1)(k+2)}{2}\right]$, there exists a Borel set $E \subset \mathbb{R}$ such that

$$\dim_{\text{Hau}}(J_{f,t}(E)) = \alpha \quad \text{and} \quad \dim_{\text{Hau}}(V_{f,t}(E)) = \beta.$$

(Theorem 5.22). We will first prove the result for $\beta = 0$ (Proposition 5.12) and then prove for $\alpha = 0$ (Corollary 5.15). The desired set will be given by their union.

5.5.1 Sets that are $J_{f,t}$-large and $V_{f,t}$-null

In this section, $t \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R})$ will be fixed throughout. Next section, we will construct sets independent of f that are large, null after being mapped by $V_{f,t}$, $J_{f,t}$, respectively.

As $j^k(f) : \mathbb{R} \to J^k(\mathbb{R})$ is locally biLipschitz, it preserves Hausdorff dimension. Hence, for all Borel sets $E \subset J^k(\mathbb{R})$,

$$0 \leq \dim_{\text{Hau}}(J_{f,t}(E)) \leq \dim_{\text{Hau}}(\text{im}(j^k(f))) = \dim_{\text{Hau}}(\mathbb{R}) = 1.$$

Moreover, by the argument in Example 5.5, if $E_\alpha \in \mathbb{R}$ has Hausdorff dimension $\alpha \in [0,1]$ (see Theorem 5.11 below), then

$$\dim_{\text{Hau}}(J_{f,t}(j^k(f)(E_\alpha))) = \dim_{\text{Hau}}(j^k(f)(E_\alpha)) = \dim_{\text{Hau}}(E_\alpha) = \alpha.$$

This shows that the full range of values for $\dim_{\text{Hau}}(J_{f,t}(E))$ is possible as E varies. We can actually prove an even stronger statement.

We first recall a well-known result about the existence of Cantor-like sets of every dimension in $\mathbb{R}$.

Theorem 5.11. (see for instance [Mat95, Section 4.10]) For all $\alpha \in [0,1]$, there exists a compact set $E_\alpha \subset \mathbb{R}$ with $\dim_{\text{Hau}}(E_\alpha) = \alpha$.

We can now prove our main result of the section.

Proposition 5.12. Fix $t \in \mathbb{R}$, $f \in C^\infty(\mathbb{R})$, and $0 \leq \alpha \leq 1$. Let $E_\alpha \subset \mathbb{R}$ be a compact set with $\dim_{\text{Hau}}(E_\alpha) = \alpha$. For all $p \in \{x = t\}$, the compact set

$$p \odot j^k(f)(E_\alpha) := \{p \odot j^k_x(f) : x \in E_\alpha\} \subset J^k(\mathbb{R})$$

satisfies

$$\dim_{\text{Hau}} J_{f,t}(p \odot j^k(f)(E_\alpha)) = \alpha \qquad \text{and} \qquad \dim_{\text{Hau}} V_{f,t}(p \odot j^k(f)(E_\alpha)) = 0.$$

Proof. The proof is quite simple compared to what we have seen thus far.

Let t, f, E_α, and p be as in the statement. For all $x \in E_\alpha$,

$$J_{f,t}(p \odot j^k_x(f)) = j^k_x(f),$$

which implies

$$J_{f,t}(p \odot j^k(f)(E_\alpha)) = \{j^k_x(f) : x \in E_\alpha\}.$$

By Proposition 2.13,

$$\dim_{\text{Hau}} J_{f,t}(p \odot j^k(f)(E_\alpha)) = \dim_{\text{Hau}} j^k(f)(E_\alpha) = \dim_{\text{Hau}}(E_\alpha) = \alpha.$$

On the other hand,

$$V_{f,t}(p \odot j^k_x(f)) = p$$

for all $x \in E_\alpha$. This implies

$$V_{f,t}(p \odot j^k(f)(E_\alpha)) = \{p\},$$

hence $\dim_{\text{Hau}} V_{f,t}(p \odot j^k(f)(E_\alpha)) = 0$. □

This proposition begs a couple of questions.

Question 5.13. Fix $t \in \mathbb{R}$ and $\alpha \in (0,1)$. Does there exist a set $F_\alpha \subset J^k(\mathbb{R})$ such that

$$\dim_{\text{Hau}} J_{f,t}(F_\alpha) = \alpha \qquad \text{and} \qquad \dim_{\text{Hau}} V_{f,t}(F_\alpha) = 0$$

for all $f \in C^\infty(\mathbb{R})$?

Question 5.14. Fix $f \in C^\infty(\mathbb{R})$ and $\alpha \in (0,1)$. Does there exist a set $G_\alpha \subset J^k(\mathbb{R})$ such that

$$\dim_{\text{Hau}} J_{f,t}(G_\alpha) = \alpha \qquad \text{and} \qquad \dim_{\text{Hau}} V_{f,t}(G_\alpha) = 0$$

for all $t \in \mathbb{R}$?

5.5.2 Sets that are $J_{f,t}$-null and $V_{f,t}$-large

We now consider the complementary problem to the one considered in the previous section. Fix $t \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R})$. We will show in this section that $\dim_{\text{Hau}}(\{x = t\}) = \frac{(k+1)(k+2)}{2}$. Assuming this for now,

$$0 \le \dim_{\text{Hau}}(V_{f,t}(E)) \le \dim_{\text{Hau}}(\{x = t\}) = \frac{(k+1)(k+2)}{2}$$

for all $E \subset J^k(\mathbb{R})$. We will show that the full range of values for $\dim_{\text{Hau}}(V_{f,t}(E))$ is possible. In fact, the sets E will be null sets when mapped by $J_{f,t}$ to $\text{im}(j^k(f))$. Moreover, the sets E we construct in this section will be independent of f, unlike what was constructed in the previous section. We will prove the following two results.

Theorem 5.15. Fix $t \in \mathbb{R}$ and $0 \leq \beta \leq \frac{(k+1)(k+2)}{2}$. There exists a compact set $E_{t,\beta} \subset \{x = t\}$ such that

$$\dim_{\text{Hau}} E_{t,\beta} = \beta.$$

Corollary 5.16. Fix $t \in \mathbb{R}$ and $0 \leq \beta \leq \frac{(k+1)(k+2)}{2}$. There exists a compact set $E_{t,\beta} \subset \{x = t\}$ such that for all $f \in C^\infty(\mathbb{R})$,

$$\dim_{\text{Hau}} V_{f,t}(E_{t,\beta}) = \beta \quad \text{and} \quad \dim_{\text{Hau}} J_{f,t}(E_{t,\beta}) = 0.$$

Note that the corollary follows easily from the theorem. Indeed, for all $f \in C^\infty(\mathbb{R})$, the restriction $V_{f,t}|_{\{x=t\}} : \{x = t\} \to \{x = t\}$ is biLipschitz (Proposition 5.4) and $J_{f,t}(\{x = t\}) = \{j_0^k(f)\}$.

Given a set $E \subset \mathbb{R}$, our intuition based on the form of the dilations on $J^k(\mathbb{R})$ says

$$\dim_{\text{Hau}}(\{t\} \times [0,1]^{j-1} \times E \times \{0\}^{k+1-j}) = \frac{j(j-1)}{2} + j \dim_{\text{Hau}}(E), \qquad j \leq k+1. \tag{5.7}$$

Then it's simple algebra to figure out what we should take j and E to be to construct the set $E_{t,\beta}$ in Theorem 5.15. The main purpose of this section is to validate our intution and prove (5.7) holds true.

To prove Theorem 5.15, we first state a variant of the Mass Distribution Principle. This result is well-known, so we will not prove it here and refer the reader to [BP17, Lemma 1.2.8].

Lemma 5.17. (Mass Distribution Principle) Fix $s > 0$ and a metric space A. Suppose there exists a Borel measure μ on A and constants $\delta, C > 0$ such that

$$\mu(B(x,r)) \leq Cr^s$$

for all $x \in A$ and $r < \delta$. Then $\mathcal{H}^s(A) > 0$.

Frostman is also credited with proving that the converse holds for compact metric spaces [Fro35].

Theorem 5.18. [Fro35] Suppose A is a compact metric space with $\mathcal{H}^s(A) > 0$. There exists $\delta > 0$ and a Radon measure μ on A satisfying $\mu(A) > 0$ and

$$\mu(E) \leq \text{diam}(E)^s \quad \text{for all } E \subset A \text{ with } \text{diam}(E) < \delta.$$

We now have a lower bound on the Hausdorff dimension of products of sets. As the proof is identical as the Euclidean case (now using Lemma 5.17 and Theorem 5.18), we will simply refer the reader to the proof of Theorem 8.10 in [Mat95].

Corollary 5.19. Let (A, d_A), (B, d_B) be compact metric spaces. Equip $A \times B$ with the metric

$$d_{A\times B}((a_1, b_1), (a_2, b_2)) := d_A(a_1, a_2) + d_B(b_1, b_2).$$

Then

$$\dim_{\text{Hau}}(A) + \dim_{\text{Hau}}(B) \leq \dim_{\text{Hau}}(A \times B).$$

The opposite inequality does not hold in general. For example, Hatano constructed compact sets $E_1, E_2 \subset \mathbb{R}^2$ of Hausdorff dimension 0 for which $\dim_{\text{Hau}}(E_1 \times E_2) = 1$ [Hat71, Theorem 4]. Fortunately, we will not need the statement in full generality.

Lemma 5.20. Let (A, d_A) be a compact metric space. For $1 \leq \alpha < \infty$, let $[0,1]_\alpha$ be the interval $[0,1]$ equipped with the metric $d_\alpha(a_1, a_2) := |a_1 - a_2|^{1/\alpha}$. Then

$$\dim_{\text{Hau}}([0,1]_\alpha \times A) = \alpha + \dim_{\text{Hau}}(A).$$

Proof. By Corollary 5.19,

$$\dim_{\text{Hau}}([0,1]_\alpha \times A) \geq \alpha + \dim_{\text{Hau}}(A).$$

It remains to show the reverse inequality, which is just a standard exercise in measure theory. We will include the proof here for completeness.

Fix $\epsilon, \delta > 0$. As $\mathcal{H}_\delta^{\dim_{\text{Hau}}(A)+\epsilon}(A) = 0$, there exists a covering $\{E_i\}$ of A satisfying $r_i := \text{diam}(E_i) < \delta$ for all i and

$$\sum_i r_i^{\dim_{\text{Hau}}(A)+\epsilon} < \delta.$$

We may assume $r_i > 0$ for all i.

For each i, cover $[0,1]$ with approximately $\frac{1}{r_i^\alpha}$ intervals I_{ij} of length r_i^α (so each of these intervals will have diameter r_i in $[0,1]_\alpha$). Then $\{I_{ij} \times E_i\}_{i,j}$ is a covering of $[0,1]_\alpha \times A$ with

$$\text{diam}(I_{ij} \times E_i) = 2r_i < 2\delta.$$

Moreover,

$$\begin{aligned}\sum_{i,j} \operatorname{diam}(I_{ij} \times E_i)^{\alpha+\dim_{\mathrm{Hau}}(A)+\epsilon} &= \sum_{i,j} (2r_i)^{\alpha+\dim_{\mathrm{Hau}}(A)+\epsilon} \\ &\approx \sum_i \frac{1}{r_i^\alpha} \cdot (2r_i)^{\alpha+\dim_{\mathrm{Hau}}(A)+\epsilon} \\ &< 2^{\alpha+\dim_{\mathrm{Hau}}(A)+\epsilon}\delta,\end{aligned}$$

which shows

$$\mathcal{H}_{2\delta}^{\alpha+\dim_{\mathrm{Hau}}(A)+\epsilon}([0,1]_\alpha \times A) \lesssim 2^{\alpha+\dim_{\mathrm{Hau}}(A)+\epsilon}\delta.$$

As $\delta, \epsilon > 0$ are arbitary, we may conclude

$$\dim_{\mathrm{Hau}}([0,1]_\alpha \times A) = \alpha + \dim_{\mathrm{Hau}}(A).$$

□

As a consequence, we can use induction to calculate the Hausdorff dimension of a product of a box with a compact set.

Proposition 5.21. For $j \in \mathbb{N}_{\geq 2}$, equip $\mathbb{R}^j$ with the metric

$$d_j((x_1, \ldots, x_j), (y_1, \ldots, y_j)) = \sum_{i=1}^{j} |x_i - y_i|^{1/i}.$$

For all compact sets $E \subset \mathbb{R}$,

$$\begin{aligned}\dim_{\mathrm{Hau}}([0,1]^{j-1} \times E) &= 1 + \cdots + (j-1) + j \dim_{\mathrm{Hau}}(E) \\ &= \frac{j(j-1)}{2} + j \dim_{\mathrm{Hau}}(E),\end{aligned}$$

where $[0,1]^j \times E$ is equipped with the restriction of d_j.

We can now prove our main result of the section, Theorem 5.15.

Proof of Theorem 5.15. Fix $t \in \mathbb{R}$ and $0 \leq \beta \leq \frac{k(k+1)}{2}$. If $\beta = 0$, define $E_\beta = \emptyset$. Hence assume otherwise.

For each nonnegative integer j, define $h_j = \frac{j(j+1)}{2}$. Choose the unique index j such that $h_{j-1} < \beta \leq h_j$. As $0 < \frac{\beta - h_{j-1}}{j} \leq 1$, there exists a compact set $\widetilde{E_\beta} \subset \mathbb{R}$ with $\dim_{\mathrm{Hau}}(\widetilde{E_\beta}) = \frac{\beta - h_{j-1}}{j}$ (the existence of which is guaranteed by Theorem 5.11). By Proposition 5.21, $[0,1]^{j-1} \times \widetilde{E}_\beta \subset (\mathbb{R}^j, d_j)$ has Hausdorff dimension equal to

$$\frac{(j-1)j}{2} + j\left(\frac{\beta - h_{j-1}}{j}\right) = \beta.$$

As shown in Section 5.2, the gauge distance takes the form

$$d((t,u_k,\dots,u_0),(t,v_k,\dots,v_0)) = \sum_{j=0}^{k} |x_j - y_j|^{1/(k+1-j)}$$

on $\{x=t\}$. If we define $E_{t,\beta} := \{t\} \times [0,1]^{j-1} \times \widetilde{E}_\beta \times \{0\}^{k-j+1} \subset \{x=t\}$, then

$$\dim_{\text{Hau}}(E_{t,\beta}) = \beta.$$

□

Now fix $f \in C^\infty(\mathbb{R})$ and $t \in \mathbb{R}$. Also let $0 \le \alpha \le 1$ and $0 \le \beta \le \frac{(k+1)(k+2)}{2}$ be given. By Proposition 5.12, there exists $F_\alpha \subset J^k(\mathbb{R})$ satisfying

$$\dim_{\text{Hau}} J_{f,t}(F_\alpha) = \alpha \quad \text{and} \quad \dim_{\text{Hau}} V_{f,t}(F_\alpha) = 0.$$

By Corollary 5.16, there exists $E_{t,\beta} \subset J^k(\mathbb{R})$ for which

$$\dim_{\text{Hau}} V_{f,t}(E_{t,\beta}) = \beta \quad \text{and} \quad \dim_{\text{Hau}} J_{f,t}(E_{t,\beta}) = 0.$$

Observe then that

$$\dim_{\text{Hau}} J_{f,t}(F_\alpha \cup E_{t,\beta}) = \alpha \quad \text{and} \quad \dim_{\text{Hau}} V_{f,t}(F_\alpha \cup E_{t,\beta}) = \beta.$$

We have proven the following satisfying result.

Theorem 5.22. Fix $f \in C^\infty(\mathbb{R})$ and $t \in \mathbb{R}$. For all $E \subset J^k(\mathbb{R})$,

$$0 \le \dim_{\text{Hau}} J_{f,t}(E) \le 1 \quad \text{and} \quad 0 \le \dim_{\text{Hau}} V_{f,t}(E) \le \frac{(k+1)(k+2)}{2}.$$

Moreover, for all pairs $(\alpha,\beta) \in [0,1] \times \left[0, \frac{(k+1)(k+2)}{2}\right]$, there exists a compact set $E_{\alpha,\beta} \subset J^k(\mathbb{R})$ (depending on f and t) satisfying

$$\dim_{\text{Hau}} J_{f,t}(E_{\alpha,\beta}) = \alpha \quad \text{and} \quad \dim_{\text{Hau}} V_{f,t}(E_{\alpha,\beta}) = \beta.$$

5.6 Effects of $J_{f,t}$ and $V_{f,t}$ on Hausdorff dimension

In Section 5.4, we briefly touched on the topic of how $J_{f,t}$ and $V_{f,t}$ affect the Hausdorff dimension of sets. We showed that for all $t \in \mathbb{R}$, $f \in C^{\infty}(\mathbb{R})$, and $E \subset J^k(\mathbb{R})$,

$$\dim_{\mathrm{Hau}}(J_{f,t}(E)) \leq \min\{\dim_{\mathrm{Hau}}(E), 1\}$$

and

$$\dim_{\mathrm{Hau}}(V_{f,t}(E)) \leq \min\left\{(k+1)\dim_{\mathrm{Hau}}(E), \frac{(k+1)(k+2)}{2}\right\}$$

(Corollaries 5.7 and 5.10, respectively).

In the previous two sections, we considered the dimensions of images of sets by $J_{f,t}$ and by $V_{f,t}$, but we didn't emphasize the dimensions of the sets themselves. We will do that here. We first restate Theorem 5.22, showing that nearly all possibilities for the pairs $\dim_{\mathrm{Hau}}(E), \dim_{\mathrm{Hau}}(J_{f,t}(E))$ are possible after taking into account Corollary 5.7. Note $\dim_{\mathrm{Hau}} J^k(\mathbb{R}) = 1 + \frac{(k+1)(k+2)}{2}$.

Proposition 5.23. Fix $t \in \mathbb{R}$ and $f \in C^{\infty}(\mathbb{R})$. For all $0 \leq \alpha \leq 1$ and $\alpha \leq \mu \leq \frac{(k+1)(k+2)}{2}$, there exists a set $E = E_{\alpha,\mu,f,t} \subset J^k(\mathbb{R})$ with $\dim_{\mathrm{Hau}} J_{f,t}(E) = \alpha$ and $\dim_{\mathrm{Hau}} E = \mu$.

Proof. The set $E_{\alpha,\mu}$ from Theorem 5.22 works. □

The question of whether the full range of possibilities for pairs $(\dim_{\mathrm{Hau}} E, \dim_{\mathrm{Hau}} V_{f,t}(E))$ is much more difficult, as Hausdorff dimension may increase after mapping by $V_{f,t}$. To illustrate this, we will consider the problem when f is a nonconstant linear function.

Example 5.24. Suppose $f(x) = mx + b$ for $m \neq 0$. By the calculation performed in the proof of Proposition 5.9,

$$(V_{f,t}(j_x^k(f))^{-1} \odot V_{f,t}(j_y^k(f)))_s = 0$$

for $x, y \in \mathbb{R},\ s = 1, \ldots, k$. We also have

$$(V_{f,t}(j_x^k(f))^{-1} \odot V_{f,t}(j_y^k(f)))_0 = (t-y)m - (t-x)m = m(x-y).$$

This implies

$$d(V_{f,t}(j_x^k(f)), V_{f,t}(j_y^k(f))) = \sqrt[k+1]{|m(x-y)|}.$$

By Propositions 5.2 and 2.13, for each compact set $K \subset \mathbb{R}$, there is a constant $C = C(K) > 0$ such that

$$d_{cc}(j^k_x(f), j^k_y(f))^{1/(k+1)} \approx d_{cc}(V_{f,t}(j^k_x(f)), V_{f,t}(j^k_y(f)))$$

for all $x, y \in K$. This implies for all Borel sets $\tilde{E} \subset \mathbb{R}$ (in particular, Cantor-like sets) and $t \in \mathbb{R}$,

$$\dim_{\text{Hau}}(V_{f,t}(j^k(f)(\tilde{E}))) = (k+1)\dim_{\text{Hau}}(j^k(f)(\tilde{E})) = (k+1)\dim_{\text{Hau}}(\tilde{E}).$$

We have shown the following:

Proposition 5.25. For all linear functions $f(x) = mx + b$, $m \neq 0$, and $0 \leq \mu \leq 1$, there exists a compact set $E \subset J^k(\mathbb{R})$ such that for all $t \in \mathbb{R}$,

$$\dim_{\text{Hau}}(E) = \mu \quad \text{and} \quad \dim_{\text{Hau}}(V_{f,t}(E)) = (k+1)\mu.$$

By appending a set to E in the previous proposition, we can show that even more pairs $(\dim_{\text{Hau}}(E), \dim_{\text{Hau}}(V_{f,t}(E)))$ are possible (where now t is fixed).

Theorem 5.26. For all $t \in \mathbb{R}$, linear functions $f(x) = mx + b$, $m \neq 0$, $0 \leq \mu \leq 1$, and $0 \leq \beta \leq (k+1)\mu$, there exists a compact set $E_{t,f,\mu,\beta} \subset J^k(\mathbb{R})$ such that

$$\dim_{\text{Hau}}(E_{t,f,\mu,\beta}) = \mu \quad \text{and} \quad \dim_{\text{Hau}}(V_{f,t}(E_{t,f,\mu,\beta})) = \beta.$$

Proof. Fix t, f, μ, β as in the statement. By Proposition 5.25, there exists a set $E \subset J^k(\mathbb{R})$ such that

$$\dim_{\text{Hau}}(E) = \frac{\beta}{k+1} \quad \text{and} \quad \dim_{\text{Hau}}(V_{f,t}(E)) = \beta.$$

For a compact set $\tilde{F} \subset \mathbb{R}$ with $\dim_{\text{Hau}}(\tilde{F}) = \mu$, define

$$F := (t, 0, \ldots, 0) \odot j^k(f)(\tilde{F}) = \{(t, 0, \ldots, 0) \odot j^k_x(f) : x \in \tilde{F}\}.$$

We showed in Proposition 5.12 that

$$\dim_{\text{Hau}}(V_{f,t}(F)) = 0.$$

Moreover, by left-invariance and Proposition 2.13,

$$\dim_{\text{Hau}}(F) = \dim_{\text{Hau}} j^k(f)(\tilde{F}) = \dim_{\text{Hau}}(\tilde{F}) = \mu.$$

This implies

$$\dim_{\text{Hau}}(E \cup F) = \max\{\mu, \beta/(k+1)\} = \mu \quad \text{and} \quad \dim_{\text{Hau}}(V_{f,t}(E \cup F)) = \beta,$$

and $E_{t,f,\mu,\beta} := E \cup F$ works. □

It's not clear if one could prove a similar result if μ is allowed to vary between 0 and $\frac{k+2}{2}$. It also isn't clear what happens when one considers more general functions or even constant functions. The nonconstant linear function case is relatively simple because many terms drop out when calculating the coordinates of $V_{f,t}(j_x^k(f))^{-1} \odot V_{f,t}(j_y^k(f))$ for $x, y \in \mathbb{R}$. We will conclude this section by stating the problem in general.

Question 5.27. Fix $t \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R})$. For which $0 < \mu \leq \frac{k+2}{2}$ and $0 < \beta \leq (k+1)\mu$ does there exist a set $F = F_{\mu,\beta,t,f}$ satisfying $\dim_{\text{Hau}} F = \mu$ and $\dim_{\text{Hau}} V_{f,t}(F) = \beta$? What if f is assumed to be a constant linear function?

5.7 Mappings and topological dimension

In Example 5.3, we broached the question of how the mappings $J_{f,t}$ and $V_{f,t}$ affect topological dimension. We will continue that study in this section by proving that all of the possible pairs $(\dim_{\text{Top}} J_{f,t}(E), \dim_{\text{Top}} V_{f,t}(E))$ can be attained as E varies over Borel sets while t and f are fixed. In fact, the set E can be chosen independently of f (if we ask for $\dim_{\text{Top}} V_{f,t}(E) \geq 1$).

In Example 5.3, we saw that for all $t \in \mathbb{R}$ and $p \in \{x = t\}$,

$$V_{f,t}(p) = p - j_0^k(f).$$

In particular, for all $E \subset \{x = t\}$,

$$\dim_{\text{Top}} V_{f,t}(E) = \dim_{\text{Top}}(E) \quad \text{and} \quad \dim_{\text{Top}} J_{f,t}(E) = 0$$

for all $f \in C^\infty(\mathbb{R})$. Moreover, for all $p \in \{x = t\}$,

$$\dim_{\text{Top}} V_{f,t}(p \odot \text{im}(j^k(f))) = 0 \quad \text{and} \quad \dim_{\text{Top}} J_{f,t}(p \odot \text{im}(j^k(f))) = 1$$

(compare with set constructed in proof of Proposition 5.12).

By taking the union of these two sets, we have the following:

Proposition 5.28. Fix $f \in C^\infty(\mathbb{R})$ and $t \in \mathbb{R}$. For all pairs $(a, b) \in \{0, 1\} \times \{0, \ldots, k+1\}$, there exists a set $E_{a,b}$ satisfying

$$\dim_{\mathrm{Top}} J_{f,t}(E_{a,b}) = a \quad \text{and} \quad \dim_{\mathrm{Top}} V_{f,t}(E_{a,b}) = b.$$

This set is highly dependent on f and t. Hence, we could ask if the set could be constructed to have the desired dimensions independent of f or t. It turns out that we can construct the set to be independent of f (at least for $\dim_{\mathrm{Top}} V_{f,t}(E) \geq 1$).

Proposition 5.29. For all $t \in \mathbb{R}$ and pairs $(a, b) \in \{0, 1\} \times \{1, \ldots, k+1\}$, there exists a compact set $F_{a,b}$ satisfying

$$\dim_{\mathrm{Top}} J_{f,t}(F_{a,b}) = a \quad \text{and} \quad \dim_{\mathrm{Top}} V_{f,t}(F_{a,b}) = b$$

for all $f \in C^\infty(\mathbb{R})$. Moreover, $\dim_{\mathrm{Top}}(F_{a,b}) = b$.

Proof. Fix $t \in \mathbb{R}$ and a pair $(a, b) \in \{0, 1\} \times \{1, \ldots, \frac{(k+1)(k+2)}{2}\}$. Let $E \subset \{x = t\}$ be a set with $\dim_{\mathrm{Top}}(E) = b$. Then $\dim_{\mathrm{Top}} V_{f,t}(E) = b$ and $\dim_{\mathrm{Top}} J_{f,t}(E) = 0$ (see Example 5.3). If $a = 0$, we are done.

Now suppose $a = 1$. Consider $F := \{(x, x, 0, \ldots, 0) : x \in (t+1, t+2)\}$. We will show $\dim_{\mathrm{Top}} J_{f,t}(F) = 1$ and $\dim_{\mathrm{Top}} V_{f,t}(F) = 1$ for all $f \in C^\infty(\mathbb{R})$. Then $F_{a,b} := E \cup F$ will work to prove the proposition.

It suffices to show $J_{f,t}(F)$ and $V_{f,t}(F)$ are both nonconstant smooth curves for all $f \in C^\infty(\mathbb{R})$. We see that $J_{f,t}(F)$ is such a curve for all $f \in C^\infty(\mathbb{R})$ since $J_{f,t}(x, x, 0, \ldots, 0) = j^k_{x-t}(f)$ for all $x \in (t+1, t+2)$. To see $V_{f,t}(F)$ is a smooth curve, note that

$$V_{f,t}(x, x, 0, \ldots 0) = (x, x, 0, \ldots, 0) \odot j^k_{x-t}(f)^{-1}$$

for all $x \in (t+1, t+2)$.

Suppose, for contradiction, that $x \mapsto (x, x, 0, \ldots, 0) \odot j^k_{x-t}(f)^{-1}$, $x \in (t+1, t+2)$, is constant for some $f \in C^\infty(\mathbb{R})$. Then there is $c \in \mathbb{R}$ such that

$$c = ((x, x, 0, \ldots, 0) \odot j^k_{x-t}(f)^{-1})_k = x - f^{(k)}(x - t)$$

for all $x \in (t+1, t+2)$. This implies for some $d \in \mathbb{R}$,

$$f^{(k-1)}(x - t) = \frac{1}{2}x^2 - cx + d$$

for all $x \in (t+1, t+2)$. Then by Proposition 5.8,

$$\begin{aligned}((x, x, 0, \ldots, 0) \odot j^k_{x-t}(f)^{-1})_{k-1} &= -f^{(k-1)}(x-t) - (x-t)(x - f^{(k)}(x-t)) \\ &= -\frac{1}{2}x^2 + cx - d - c(x-t)\end{aligned}$$

must also be constant, but it is clearly not by the presence of the $-\frac{1}{2}x^2$ term. This proves that $V_{f,t}$ is a nonconstant smooth curve, and the proposition follows. □

The question of whether we can construct the set independently seems much more difficult, and we will leave it open.

Question 5.30. For a fixed $f \in C^\infty(\mathbb{R})$ and pair $(a, b) \in \{0, 1\} \times \{0, \ldots, \frac{(k+1)(k+2)}{2}\}$, does there exist a set $G_{a,b}$ satisfying

$$\dim_{\text{Top}} J_{f,t}(G_{a,b}) = a \quad \text{and} \quad \dim_{\text{Top}} V_{f,t}(G_{a,b}) = b$$

for all $t \in \mathbb{R}$?

5.8 Open questions and mappings of general jet spaces

We conclude with a few open questions related to fixing a Borel set and letting t or f vary. We also note that an analogous construction could be performed to define mappings of general jet space Carnot groups.

We have not considered how $\dim_{\text{Hau}} V_{f,t}(E)$ changes as t varies while f and E are fixed. For fixed $t \in \mathbb{R}$, $p \in \{x = t\}$, and $f \in C^\infty(\mathbb{R})$, we showed

$$\dim_{\text{Hau}} V_{f,t}(p \odot \operatorname{im} j^k(f)) = 0.$$

For all $p \in \{x = t\}$ and $s \neq t$, the map $x \mapsto V_{f,s}(p \odot j^k_x(f))$ will be a nonconstant curve if f isn't a constant function. This implies

$$\dim_{\text{Hau}} V_{f,s}(p \odot \operatorname{im} j^k(f)) \geq 1.$$

This motivates the following general question.

Question 5.31. Fix $f \in C^\infty(\mathbb{R})$ and a Borel set $E \subset J^k(\mathbb{R})$. What is the behavior of the function $t \mapsto \dim_{\text{Hau}} V_{f,t}(E)$? For example, does it attain a finite number of values? What is its regularity? Is there a value that it attains almost everywhere?

The final set of questions we pose seems to be the most difficult one. What happens if we let the function f vary as t and E are fixed? Recall the notion of prevalence in the sense of Hunt, Sauer, and Yorke [HSY92, OY05, SY97]. Prevalence provides a notion of "almost everywhere" in an infinite-dimensional Banach space, such as $C^{k+1}(\mathbb{R})$. In 2013, Balogh, Tyson, and Wildrick showed that the set of Newtonian-Sobolev functions that maximally increase Hausdorff dimension is prevalent within a certain Newtonian-Sobolev space [BTW13, Theorem 1.2]. One could study an analogous problem in our setting.

Question 5.32. Fix $t \in \mathbb{R}$ and a Borel set $E \subset J^k(\mathbb{R})$. What are the behaviors of the functions $f \mapsto \dim_{\text{Hau}} J_{f,t}(E)$ and $f \mapsto \dim_{\text{Hau}} V_{f,t}(E)$, where f ranges over $C^{k+1}(\mathbb{R})$? Are there topologies on $C^{k+1}(\mathbb{R})$ under which these two functions are continuous? Do there exist $\alpha, \beta \in \mathbb{R}$ for which the set of $f \in C^{k+1}(\mathbb{R})$ satisfying $\dim_{\text{Hau}} J_{f,t}(E) = \alpha$ and $\dim_{\text{Hau}} V_{f,t}(E) = \beta$ is prevalent?

We conclude by noting that one could define vertical and horizontal mappings in the same way in general jet space Carnot groups $J^k(\mathbb{R}^n)$ (see [War05b, Section 4.4] for discussion and notation of these groups). For the vertical planes, one would take the codimension-n planes $\{x = t\} := \{(x, u^{(k)}) \in J^k(\mathbb{R}^n) : x = t\}$ for $t \in \mathbb{R}^n$. Then for every $f \in C^\infty(\mathbb{R}^n)$, $t \in \mathbb{R}^n$, and $p \in J^k(\mathbb{R}^n)$, there exist uniquely $p_V \in \{x = t\}$ and $p_H \in \text{im}(j^k(f))$ such that

$$p = p_V \odot p_H.$$

One could then prove analogues of every result in this chapter for general jet space Carnot groups.

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