HYBRID ADAPTIVE SEQUENTIAL SAMPLING FOR RELIABILITY-BASED DESIGN OPTIMIZATION

BY

YANWEN XU

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Industrial Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2019

Urbana, Illinois

Adviser:

Associate Professor Pingfeng Wang

ABSTRACT

To reduce the cost for reliability analysis and optimization of complex engineering systems, surrogate models can be used to replace expensive physical models. One of the critical tasks of employing surrogate models for system design optimization is to develop an accurate surrogate model cost-effectively. In this thesis, a new hybrid adaptive sequential sampling strategy has been developed to substantially improve the efficiency of the surrogate model development process. The developed sampling strategy combines local sampling that focuses on regional model fidelity improvements with global sampling that ensures effective design updates in the design optimization process. Specifically, a confidence-guided sequential sampling scheme is developed for local sampling, which identifies most useful sample points along the descending direction of the objective function as well as the constraints to improve the regional model fidelity. Similarly, a constraint boundary sampling scheme is adopted for the global sampling purpose, which efficiently locates the constraint boundaries and balances the efforts devoted to global sampling and local sampling processes. The efficacy of the developed hybrid adaptive sequential sampling technique for reliability-based design optimization using surrogate models is assessed with several numerical case studies, through comparisons with existing approaches that have been reported in the literature. The case study results have demonstrated that the developed new sampling strategy can significantly reduce the number of sample points required in updating the surrogate model along with the design optimization process. By using the developed adaptive sequential sampling strategy for surrogate modeling, the design processes become more efficient and cost-effective.

ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere gratitude to my advisor Prof. Pingfeng Wang at the University of Illinois. As a student, great guidance I received from Prof. Wang is what makes this thesis possible. His quick and sharp observations always give me a valuable source of ideas and breakthroughs. Moreover, his patience and insightful comments enabled me to step out of a specific problem to have a big picture and to thoroughly understand the problem. Under his friendly guidance, I become to recognize how a research idea is brought up and developed. And I learn to keep my interests open and wide.

I would also like to thank my lab mates for the fun and their continued support.

Last but not the least, I would like to thank my parents for their unconditional support and encouragement.

To my parents, for their love and support.

TABLE OF CONTENTS

Chapter 1: Introduction

1.1 Background and Motivation

With a growing demand for higher reliability but the lower cost of complex systems, efficient reliability assessment techniques become increasingly important for engineering system design and optimization. Traditional deterministic design optimization models [1, 2] have already been successfully applied in engineering design to reduce development costs and improve performances systematically. However, uncertainties widely exist in system design and manufacturing processes [3], which shows the necessity to systematically take into account these uncertainties in the system design process, such as using a reliability-based design optimization (RBDO) model, in order to generate robust yet cost-effective designs. Generally, deterministic design optimization methods obtain an optimal design often located at the constraint boundary in the design space, leading to low-reliability values while considering various sources of uncertainties that may be introduced from material properties, operating conditions, and the manufacturing process. Moreover, many of these uncertainties could be irreducible, resulting in a significant level of variability in the system performance, and may cause failures if not addressed appropriately in the system design process. Therefore, it is necessary to take such uncertainties into account in the design process to ensure high reliability [4].

The RBDO model can be used to generate designs systematically that minimize design costs while ensuring a target reliability level can be satisfied. Traditionally, the RBDO model can be formulated as a stochastic optimization problem under probabilistic constraints. The primary challenge of solving the RBDO problem is to effectively evaluate all probabilistic constraints for reliability estimation and sensitivity analysis. While solving the RBDO problem, the system design optimization can be conducted in an outer design loop, and accordingly, the evaluations of probabilistic constraints for reliability and design sensitivity assessments can be performed using different reliability methods in the inner loop. This technique usually leads to a nested optimization problem with high computational cost, referred to be as the double loop approach [5]. The two commonly used methods for reliability analysis are the Reliability Index Approach (RIA) [6] in which the optimization problem is solved to estimate the probability of failure, and the Performance Measure Approach (PMA) [7] where the probability estimation has been transformed into a performance measure by solving the inverse problem.

To improve the efficiency of solving the RBDO problem, surrogate models have been used to alleviate the computational burden in evaluating the probabilistic constraints. A surrogate model can be developed with a limit number of training samples and then used to predict the performance function values at unknown sample points in the design space, thus offering an efficient approach to solve the RBDO problem by substantially reducing the cost of function evaluations in reliability analysis and design optimization. As one of the surrogate modeling techniques, the Kriging model was introduced in solving RBDO problem, and different sampling method such as the Latin Hypercube sampling (LHS) [8, 9] have also been used to generate random sample points for developing the Kriging model. Various efficient methods can be applied to solving the RBDO problem and reducing the number of function evaluations which could improve efficiency and accuracy of the reliability analysis thus reduce system design costs. As reported in the literature [10], surrogate models can be combined with the PMA technique to solve the RBDO problem in a sequential optimization and reliability analysis (SORA) [11] structure. The SORA formulation separates the reliability analysis from the main optimization loop and transforms the RBDO problem into the sequences of deterministic optimization and reliability analysis cycles to reduce the computational cost, whereby the PMA technique is used for reliability assessment based on the Kriging models.

1.2 Objective and Challenge

While the accuracy of predicting the limit state function value is critical for solving the RBDO problem using surrogate models, several sampling techniques have been developed in the literature to identify critical points that are most valuable for improving the surrogate model fidelity and reduce the computational cost [12-19]. There are in general two different ways to construct the surrogate model. The input and output training samples have been collected in advance, and the surrogate model can be built based on those collected training samples. The size of the training samples can be increased until the surrogate model reaches the desired accuracy. This type of experimental design methods is commonly used because it is straightforward to implement [12]. However, it is challenging to determine the number of sample points in advance for many black-box problems. To overcome this difficulty, sample points are selected iteratively by some criterions in the sequential sampling design. The main question for sequential sampling design methods is how to find a trade-off between exploring the region with high uncertainty and the area of interest, in order to select new sample points efficiently at each iteration [13].

Several adaptive sequential sampling methods have been reported in the literature in recent years. Based upon the response surface approach, an active learning method using an expected improvement measure was proposed by Jones et al. [14]. Bichon et al. [15] proposed an efficient global reliability analysis (EGRA) method for structural systems design. An expected feasibility function was developed to balance the effort between a regional search near the response surface and a global search in the whole design space. Lee and Jung [16] proposed a global sampling method, referred to as the constraint boundary sampling (CBS) method, for solving the RBDO problem, which performs well in terms of accuracy and efficiency since it approximates the limit state boundaries in the global region only. Based upon the CBS method, a local adaptive sampling (LAS) [17] method and an important boundary sampling (IBS) method [18] were also introduced to solve the RBDO problem. Echard, B., Gayton, N., and Lemaire, M. [19] proposed a new sampling strategy, where an active learning reliability method was combined with the Kriging and Monte Carlo simulation, referred to as the AK-MCS method. The AK-MCS is a local sampling strategy, which focuses on an MCS population generated from a given design point instead of approximating the limit state function in the entire system input space, thus saves the efforts of evaluating expensive performance functions for those sample points with very low failure probabilities. Wang and Wang [20] introduced a maximum confidence enhancement (MCE) based sequential sampling approach that use the cumulative confidence level (CCL) as a sampling criterion to select sample points with the maximum value of the estimated CCL improvement successively.

However, most of existing sampling methods focus on either global or local sampling strategy. While global sampling strategy often approximates the limit state in the entire system input space, it generally requires a large number of evaluations of performance functions at different sampling points in the areas that may not be important for solving the RBDO problem. On the other hand, local sampling strategy generally focuses on selecting the most useful sample points to improve the model fidelity at a particular local region. For example, a sampling strategy based upon the Monte Carlo Simulation (MCS) method is used for reliability analysis at a specific region. Although local sampling strategy can be efficient computationally, it often has a poor convergence performance for the optimization process in the RBDO and could also yield an infeasible solution.

1.3 Developed Solutions in This Thesis

To alleviate this difficulty and balance the local and global sampling needs in solving the RBDO problem with adaptive surrogate model, in this thesis, we have successfully combined the local sampling strategy that focuses on improving regional model fidelity with the global sampling technique that ensures effective design updates in the design optimization process, and developed a hybrid adaptive sequential sampling (HASS) methodology. The HASS method integrates two sampling strategies: the global searching approach where a constraint boundary sampling scheme is adopted to efficiently locates the constraint boundaries, and the local searching approach where a confidence-guided sequential sampling scheme is developed to identifies most useful sample points along the descending direction of the objective function as well as the constraints to improve the regional model fidelity. A sampling procedure has been developed in HASS to alternate these two sampling schemes during the RBDO optimization process based on the specific surrogate modeling needs.

Moreover, most of existing active learning functions developed in the literature have only focused on the estimated uncertainties concerning the probabilistic constraints and the relative distance of new candidate sampling points to existing ones in the system input space. However, it is generally true that one sampling region would be more important than others if the objective function has smaller values in that sampling region, and thus new sampling points should be accordingly selected from that sampling region. To take into account the impact of the optimization process such as the objective function minimization on sampling, a new sampling criterion is introduced in this thesis and integrated in the HASS method, which also takes the objective function value and the trajectory of design points into consideration and combines local sampling strategy that focuses on regional model fidelity improvements with global sampling approach that ensures effective design updates in the design optimization process. Specifically, for local sampling, the developed criterion identifies the most useful sample points along the descending direction of the objective function as well as the constraints to improve the regional model fidelity.

1.4 Summary of Results and Thesis Outline

The numerical experiments show that the developed hybrid adaptive sequential sampling method can effectively identify the most useful sample points sequentially to enhance the fidelity of the surrogate models, thereby improving the efficiency of the modeling process. Also, based on the HASS strategy, the developed adaptive sensitivity analysis method has also shown to be capable of reducing the computational cost further.

The rest of this thesis has been organized as follows.

Chapter 2 reviews the basic theory of the reliability-based design optimization problem and existing methods that have been used to solve the problem. Section 2.1 gives a review of the reliability analysis studies model in the literature. Section 2.2 introduces the RBDO problem and the design formulation to be used throughout this thesis.

Chapter 3 introduces the Kriging surrogate modeling technique and existing sampling methods used to develop Kriging surrogate models. Section 3.1 introduces the Kriging surrogate model. Commonly used sequential sampling techniques and the relevant literature are presented in Section 3.2. The adaptive surrogate modeling strategy used to efficiently solve the reliability optimization problem is then introduced in Section 3.3.

Chapter 4 presents the developed HASS methodology, including the existing sampling criterions that have been integrated into the developed HASS method. Section 4.1 gives the global sampling criterion used in the surrogate development processes. The developed adaptive confidence-guided sequential sampling approach (CGSS) technique is analyzed in Section 4.2. Based on the above criterion, the developed HASS strategy is presented in Section 4.3. Section 4.4 gives the developed adaptive sensitivity analysis (ASA) method. Section 4.5 combines the developed HASS criterion and the ASA method with the RBDO problem.

Chapter 5 provides two case studies to demonstrate the developed HASS methodology together with the new adaptive sensitivity analysis approach. Section 5.1 compares the developed HASS criterion with existing sampling methods, and Section 5.2 examines the developed ASA methods with existing sensitivity analysis techniques based on the developed HASS methodology.

Finally, the last chapter concludes the thesis and summarizes the directions for future researches.

Chapter 2: Reliability-Based Design Optimization

Traditional deterministic design models have been successfully used in many engineering designs to systematically reduce system development costs and improve performances. But these models often lead a final product with a high probability of failure due to the uncertainties from various sources such as the manufacturing processes, material properties, and others related to system operating conditions. Reliability analysis, therefore, plays a significant role in the system design process. However, accurate reliability analysis could be very challenging in many engineering problems, primarily due to the overwhelmingly high cost, experimentally or computationally, in evaluating the probabilistic performance functions. To resolve these difficulties, a variety of numerical and simulation techniques have been developed in the literature to conduct the reliability analysis and design of engineering systems.

This chapter first gives a brief review of the reliability analysis and then introduces the reliability-based design optimization (RBDO) formulation for system design. Section 2.1 provides a review of reliability analysis and two numerical methods, namely the Monte Carlo simulation (MCS) method and the first-order reliability method (FORM), that have been commonly used in the literature for reliability analysis. Section 2.2 then presents the RBDO problem formulation and three different problem-solving procedures.

2.1 Reliability Analysis

Considering the performance function $G(\mathbf{x})$ with random input variables $\mathbf{x}$ of a system, the system failure can then be described generally as the performance function $G(\mathbf{x})$ is greater than zero, and accordingly, a negative value of $G(\mathbf{x})$ indicates that the system is safe. The boundary between system failure and safe domains is generally called the limit state.

With these notations, the probability of failure can be defined mathematically as:

$$P_f = P(G(\mathbf{x}) > 0) = \int \cdots \int_{\Omega:\{\mathbf{x}|\ G(\mathbf{x})>0\}} f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \tag{1}$$

where **x** represents a vector of random input variables, $G(\mathbf{x})$ is the system performance function, and $f_{\mathbf{x}}(\mathbf{x})$ is the joint probability density function of the system random inputs. Moreover, Ω: $\{\mathbf{x}\ |\ G(\mathbf{x})>0\}$ is defined as the failure region over the system random input space.

For the reliability analysis, the multidimensional integration as described in Eq. (1) must be evaluated considering the joint probability density function over the failure region. The exact probability integration is extremely complicated to compute in most engineering cases, and alternatively different numerical techniques have been developed. In the following, two commonly used numerical methods will be briefly introduced.

2.1.1 The MCS Method

It is generally challenging to evaluate the probability of failure using the above Eq. (1), since a multidimensional integration of the joint probability density function over the system failure domain is involved, where the failure domain in the system random input space is usually unknown. To overcome this difficulty, the Monte Carlo Simulation [22] can be used to provide a convenient approximate of the multidimensional integration through sampling the random input space with relatively large sample size. The MCS method is very straightforward and easy to implement.

Generally, a large number of sample points over the random input space will be evaluated when using the MCS for reliability analysis, and accordingly, the probability of failure can be approximated using the ratio of the total number of sample points found in the failure region over the total number of sample points being used. Mathematically, the indicator function can often be used to indicate if a sample point has been found in the failure region, as

$$P_f = \Pr(G(\mathbf{x}) > 0) = \int_{G(\mathbf{x})>0} I_f(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} = E[I_f(\mathbf{x})] \tag{2}$$

where $I_f(\mathbf{x})$ represents an indicator function, which can be defined as

$$I_f(\mathbf{x}) = \begin{cases} 1, & \text{if } G(\mathbf{x}) > 0 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

With the use of the indicator function as defined in Eq. (3), the expectation of the indicator function over the random input space can be used to represent the probability of system failure. Accordingly, this expectation can be approximated by using a large number of sample points of **x**, randomly sampled in the random input space. It is clear that a large number of performance function evaluations are often required when using the MCS method for reliability analysis, leading to a prohibitively high experimental or computational cost. To alleviate this difficulty, surrogate models have usually been developed over a limited number of sample points, and the MCS can be used to estimate the system reliability more efficiently based on the developed surrogate models.

2.1.2 The First Order Reliability Method

The first-order reliability method (FORM) is commonly used in reliability analysis due to its good computational efficiency performance. The basic idea of the FORM is to use the first-order Taylor series expansion to approximate system performance function, $G(\mathbf{x})$, and the expansion is usually done at the most probable failure point (MPP). Thus, by using the FORM method, the reliability analysis problem can be converted into the search of the MPP, which can often be formulated as an optimization problem.

To linearize the performance function G(X) at the MPP on the limit-state surface G(X) = 0, the random input variables are usually transformed into a standard normal space, namely the **U**-space. For example, for a normal random input variable X_i, the transformation can be expressed as follows [21]:

$$U_i = T\left(X_i\right) = \frac{X - \mu_{X_i}}{\sigma_{X_i}}, \quad i = 1, 2, ..., N \tag{4}$$

where μ_{Xi} and σ_{Xi} are the mean and standard deviation of the random variable X_i, respectively.

The reliability can be approximated by the standard normal tail distribution. The transformation of random variables x is expressed as:

$$F_{X_i}\left(X_i\right) = \Phi\left(U_i\right) \tag{5}$$

where F_{Xi} and Φ are the CDFs of X_i and U_i, respectively.

In the **U**-space, the MPP $\mathbf{u}^*$ has the shortest distance from the origin to the failure surface, which denoted by the reliability index β, and can be expressed as:

$$\beta = \left\|\mathbf{u}^*\right\| = \left[\sum_{i=1}^{N}\left(\mathbf{u}_i^*\right)^2\right]^{1/2} \tag{6}$$

Therefore, the reliability R can be computed as (when $R \geq 0.5$)

$$R = \Phi\left(\beta\right) \tag{7}$$

where the reliability index β is the distance from the origin to the MPP in the standard normal space.

2.2 Reliability-based Design Optimization

The reliability-based design optimization (RBDO) model has been widely used to generate designs that minimize system design costs while ensuring a target reliability level can be satisfied. In the RBDO model, the mean values of the random design parameters are often used as design variables so that the system design can be optimized to minimize the cost function while satisfying all reliability constraints. Thus, the design solution obtained by solving the RBDO model generally provides a high level of reliability as compared with the one obtained with the deterministic design model.

Mathematically, the RBDO model [23-29] can generally be formulated as

$$\begin{aligned} & \textit{find} \quad d \\ & \min \quad \mathrm{Cost}(\mathrm{d}) \\ & \mathrm{s.t.} \quad \mathrm{P}_r(\mathrm{G}_i(\mathrm{x},\mathrm{d})<0) \le 1-\Phi(\beta_{ti}) \\ & \qquad d^L \le d \le d^U,\, d \in R^{nd} \ \textit{and} \ x \in R^{nr} \end{aligned} \tag{8}$$

where **d** is a vector of design variables, *Cost*(**d**) is the objective function, *nc* is the number of probabilistic constraints; *nd* is the design variables; *nr* is the random variables. Superscripts 'L' and 'U' represent the lower bound and upper bound, respectively.

The primary challenge in solving the RBDO model as shown in Eq. (8) lies in the accurate evaluation of the probabilistic constraints during the iterative design optimization process, since considerable computational or experimental efforts are normally required. Based upon different processes in evaluating the probabilistic constraints during the optimization process, there are three different categories of approaches in solving the RBDO problem [30], namely the double-loop approaches, the single-loop approaches, and the decoupled approaches.

2.2.1 Double-Loop Approach

The double-loop approach (DLA) for solving the RBDO problem often consists of a nested structure of the optimization problem with outer and inner loops, where the outer loop optimizes the design variables in the original design variable space (**X**-space) and the inner loop conducts reliability assessment at a given design point [31]. Depending on different reliability analysis methods using, the inner loop problem can generally be solved in the transformed standard normal space (**U**-space). The above two steps, design optimization and reliability analysis, are conducted alternatively until the optimum design of the system is found, which minimizes the design cost as well as satisfies the probabilistic constraints. While the double loop approach for solving the RBDO problem can be straightforward for implementation, it however often leads to a high computational cost, primarily due to the fact that at every system design iteration a nested optimization problem must be solved at the inner loop for each probabilistic constraints in order to evaluate the reliability.

2.2.2 Single-Loop Approach

Different from the double loop approaches, another strategy is to decouple the nested problem-solving structure and eliminate the inner loop for reliability analysis by transforming the probabilistic constraints into deterministic ones. Two approaches have been commonly used to approximating the probabilistic constraints. One is the Karush-Kuhn-Tucker (KKT) optimality condition where the probabilistic constraints are replaced by KKT optimality conditions of the first-order reliability method. However, the KKT method has been reported [32] that it has weak stability and high computational cost as compared to the double loop method due to the increased number of equality constraints. The second method, called the single-loop single vector (SLSV) approach, has been developed [30], by which the limit state function is only evaluated at the point in the **U**-space that is away from the current design point, with the distance of the target reliability index along the direction of the most probable point (MPP). As a result, the inner loop for reliability analysis is removed, and the RBDO problem becomes a deterministic optimization one accordingly.

Once the probabilistic constraints are approximated into deterministic ones, a simple deterministic optimization problem can be solved without additional computational costs for reliability analysis. Figure 1 below shows a flowchart of the single-loop RBDO problem.

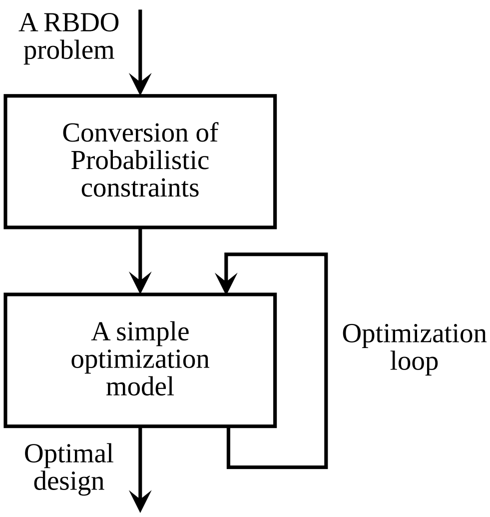


Figure 1. Flowchart of single-loop RBDO. Ref.[33].

2.2.3 Sequential Optimization and Reliability Assessment

To reduce the computational cost, the Sequential Optimization and Reliability Assessment (SORA) technique has been developed. The main idea of SORA is to separate the reliability assessment from the optimization loop and therefore transform the RBDO problem into a sequence of deterministic optimization and reliability assessment cycles [11]. The SORA technique for solving the RBDO problem conducts reliability analysis once when the deterministic optimization problem is solved and a deterministic optimum has been obtained, thus it could largely reduce computational cost.

Although the SORA technique can largely improve the computational efficiency, solving the RBDO problem could still be challenging, especially when a very high dimension problem with a large number of random design variables is addressed. To further reduce the expensive computational cost arose mostly from the function evaluations needed for the reliability assessment, the surrogate model can be used to replace the high-fidelity simulations or physical experiments and conduct the reliability analysis more efficiently together with the Monte Carlo simulation (MCS) method. In this thesis, the Kriging surrogate modeling technique has been employed for solving the RBDO problem more efficiently, which has been introduced in the next chapter.

Chapter 3: Kriging Surrogate Modeling

The Kriging has become one of the most commonly used methods to develop computationally efficient surrogate models in various engineering applications, including simulation-based design optimization and uncertainty quantification. To develop Kriging surrogate models efficiently, different sequential sampling methods have been reported in recent years for engineering design problems with expensive system performance functions. This chapter introduces the Kriging surrogate modeling technique, in which Section 3.1 reviews the Kriging surrogate model technique and Section 3.2 then presents two different sequential sampling techniques for the development of Kriging models. Then, the adaptive surrogate modeling with reliability optimization is introduced in Section 3.3.

3.1 Kriging Surrogate Model

The Kriging surrogate model can be developed based upon a set of training sample points and then used to predict system performances at unobserved design points. A general Kriging model can be expressed as

$$G_K(\mathbf{x}) = f(\mathbf{x}) + S(\mathbf{x}) \tag{9}$$

where $G_K(\mathbf{x})$ represents the prediction result of performance function at point $\mathbf{x}$ using the Kriging model, $f(\mathbf{x})$ is a polynomial term of $\mathbf{x}$ that interpolates the input sample points, and $S(\mathbf{x})$ is a Gaussian stochastic process with zero mean and variance σ^2. The polynomial term $f(\mathbf{x})$ can be replaced by a constant value, the mean response value μ, leading to an ordinary Kriging model, which can be described as

$$G_K(\mathbf{x}) = \mu + S(\mathbf{x}) \tag{10}$$

The covariance function between arbitrary two input points x_i and x_j can be defined as

$$Cov[\, S(\mathbf{x}_i),\ S(\mathbf{x}_j)\,] = \sigma^2\, \mathbf{R}\,(\mathbf{x}_i, \mathbf{x}_j) \tag{11}$$

where $\mathbf{R}(\mathbf{x}_i, \mathbf{x}_j)$ denotes the correlation function matrix and can be defined as

$$\mathbf{R}(\mathbf{x}_i, \mathbf{x}_j) = Corr(\mathbf{x}_i, \mathbf{x}_j) = \exp\left[-\sum_{p=1}^{N} a_p \,|\, x_i^p - x_j^p \,|^{b_p} \right] \tag{12}$$

where a_p and b_p are parameters of the Kriging model.

With n observations O= [X, G], where X is the input data set and G is the input performance, the log-likelihood function of the Kriging model can be expressed as

$$\ln L = -\frac{1}{2}\left[n\ln(2\pi) + n\ln\sigma^2 + \ln\left|\mathbf{R}\right| + \frac{1}{2\sigma^2}(\mathbf{G} - \mathbf{A}\mu)^T \mathbf{R}^{-1}(\mathbf{G} - \mathbf{A}\mu) \right] \tag{13}$$

where A is an n×1 unit vector. Then μ and σ^2 can be solved by maximizing the likelihood function respectively, as

$$\mu = \left[\mathbf{A}^T \mathbf{R}^{-1} \mathbf{A} \right]^{-1} \mathbf{A}^T \mathbf{R}^{-1} \mathbf{G} \tag{14}$$

$$\sigma^2 = \frac{(\mathbf{G} - \mathbf{A}\mu)^T \mathbf{R}^{-1} (\mathbf{G} - \mathbf{A}\mu)}{n} \tag{15}$$

With the Kriging model, the response for any given new point **x**' can be estimated as

$$G_K(\mathbf{x}') = \mu + \mathbf{r}^T \mathbf{R}^{-1} (\mathbf{G} - \mathbf{A}\mu) \tag{16}$$

where **r** is the correlation vector between **x**' and the sampled points X. The mean square error e(**x'**) can be computed by

$$e(\mathbf{x}') = \sigma^2 \left[1 - \mathbf{r}^T \mathbf{R}^{-1} \mathbf{r} + \frac{(1 - \mathbf{A}^T \mathbf{R}^{-1} \mathbf{r})^2}{\mathbf{A}^T \mathbf{R}^{-1} \mathbf{A}} \right] \tag{17}$$

Thus, the prediction of the response value at the new sample point **x**' using the Kriging model can be considered as a random variable that follows a normal distribution with mean $G_K(\mathbf{x}')$ and variance $e(\mathbf{x}')$.

In the thesis, the Kriging model has been used to reduce the cost of function evaluations in the reliability analysis part of the RBDO model. Grid sampling has been used to build the initial Kriging model, and new sample points selected by the developed new hybrid adaptive sequential sampling (HASS) approach have been used to improve the fidelity of the surrogate model until the target accuracy level is achieved. The HASS strategy will be detailed in chapter 4. Finally, based on the Kriging prediction results, the inner loop of the RBDO model returns reliability sensitivity information to the outer loop and continues the optimization process of the RBDO problem.

3.2 Sequential Sampling Techniques

To improve the efficiency of the Kriging surrogate model development process, sequential sampling approaches have been used to iteratively select new training sample points and refine the Kriging model until it reaches a target accuracy level. For sequential sampling, the primary question to be answered is how to determine the locations of new training sample points at each iteration. Essentially, for different sequential sampling approaches, different sampling criterion or so-called active learning function can be used to guide this sample selection process, and the goal is to select the most valuable training sample points to refine the surrogate model and improve its prediction accuracy.

3.2.1 Adaptive Kriging with Monte Carlo Simulation

Echard et al. [19] proposed a learning function (i.e., U function) to select new training sample points for developing Kriging surrogate models sequentially for reliability analysis, and an active learning reliability method combining the Kriging and Monte Carlo simulation was developed, referred to as the AK-MCS method. The AK-MCS is a local sampling strategy, which focuses on an MCS population generated from a given design point instead of approximating the limit state function in the entire system input space. This AK-MCS method is useful in using Kriging surrogate models for reliability analysis, especially for saving the efforts of evaluating expensive performance functions for those sample points with very low failure probabilities.

The idea behind this U learning function is to choose the sample point with a high potential risk of crossing the predicted failure surface and add the selected point into the training data set at every iteration to improve the prediction accuracy of the Kriging model. Due to the uncertainties, these points identified at each sequential sampling iteration often are more likely to be misclassified by the MCS method and thus affect the reliability estimation. The learning function U can be expressed as

$$U(\mathrm{x}) = \frac{|G(\mathrm{x})|}{\sigma_G(\mathrm{x})} \tag{18}$$

where $G(\mathbf{x})$ is the Kriging mean prediction, and $\sigma(\mathbf{x})$ is the prediction variance. The learning function shown in Eq. (18) describes the distance in predicted standard deviation between the predicted response value and the estimated limit state. This sequential sampling strategy could help us focus on the random input spaces where the sample points are more likely to be within the failure domain.

3.2.2 Constraint Boundary Sampling Method

The Constraint Boundary Sampling (CBS) method has been developed by Lee and Jung [16] to effectively allocate sample points that are close to the limit state functions, therefore generate accurate constraint boundary predictions for solving the RBDO problem. The CBS method aims to locate training sample points for developing Kriging surrogate models around the limit state functions for the RBDO problem (i.e., the limit state G(x)=0). In the Kriging model, mean μ and standard deviation σ have been used to predict the response function. Moreover, if **Ω**: {x | G(x)>=0} defines the safe region in the random input space, the probability that the Kriging model would classify a sample point **x** into the "safe" category when using the MCS method for reliability analysis, or in orther words the probability that Kriging model prediction of G(**x**) at the sample point **x** would satisfy the constraint G(**x**)>=0, can be expressed as

$$\Pr(\mathbf{x}) = 1 - \Phi(\frac{0-\mu}{\sigma}) = \Phi(\frac{\mu}{\sigma}) \tag{19}$$

where Φ(.) denotes the cumulative distribution function (CDF) of the standard normal distribution. Based on Eq. (19), it is clear that the higher the probability value, the farther the sample point x is from the constraint boundary. Thus, the standard normal probability density function (PDF) value can be used to measure the distance between a given sample point and the limit state surface where $G(\mathbf{x}) = 0$ (the constraint boundary). The CBS criterion based upon the standard normal PDF can be defined as

$$CBS = \begin{cases} \sum_{i=1}^{N} \phi(\frac{\mu_i(\mathbf{x})}{\sigma_{\mu_i}}) \cdot d_{\min}, & \mu_i(\mathbf{x}) \geq 0 \\ 0, & otherwise \end{cases} \tag{20}$$

where $\phi(.)$ is the standard normal PDF, d_{min} represents the minimum distance from the candidate sample point **x** to all existing sample points in the training datasets, and N is the number of constraints.

From the CBS sampling rule as shown in Eq. (20), it is clear that if the candidate sample point has been predicted to be in the failure domain for the i^{th} performance function, as indicated by $\mu_i(\mathbf{x}) < 0$, the value of CBS sampling criterion will equal to zero. Accordingly, by maximizing the CBS sampling criterion while considering all candidate sample points, the sample points that are near or along the constraint boundaries but are not close to existing sample points will be more likely to be selected, as they are generally more significant than other points which are located well within the failure region or far away from the constraint boundary.

3.3 Adaptive Surrogate Modeling with Reliability Optimization

This section introduces an approach that integrates a sequential sampling method with the RBDO model. Section 3.3.1 introduces a sequential sampling approach, referred to as the Maximum Confidence Enhancement (MCE) based sequential sampling method, that has been used to select new sampling points adaptively along with the iterative RBDO process to carry out reliability analysis. Moreover, the sensitivity analysis technique, which calculates the design sensitivity of the probabilistic constraints with respect to system design variables based upon the adaptively constructed surrogate models for solving the RBDO problem, is also introduced in Section 3.3.2.

3.3.1 Maximum Confidence Enhancement (MCE) Sampling

The MCE-based sequential sampling approach developed by Wang and Wang [20] uses the cumulative confidence level (CCL) as a sampling criterion to successively select sample points with the maximum value of the estimated CCL improvement, along with each RBDO iteration when carrying out the reliability analysis based on the adaptively constructed Kriging surrogate models.

Since the MCS method has been used for reliability analysis based on Kriging models of performance functions during the iterative RBDO process, surrogate models should be accurate enough for the reliability analysis needs at a given region where the current system design point is located. Thus, the CCL measure [20] has been used to quantify the accuracy of the surrogate model, particularly for the reliability analysis using the MCS method together with the Kriging surrogate models.

First, an indicator function has been introduced to classify samples:

$$I_{RF}(\mathbf{x}_{m,i}) = \begin{cases} 1, & G_k(\mathbf{x}_{m,i}) < 0 \\ 0, & otherwise \end{cases} \tag{21}$$

The confidence level of the classification for the sample point **x** can be generally estimated by the probability that G(x) >0, if G_k(x) is positive; or the probability that G(x) <0, if G_k(x) is negative. The confidence level of the above classification can be shown as [20]

$$CL(\mathbf{x}_{m,i}) = \begin{cases} \Pr(G(\mathbf{x}_{m,i}) < 0), & G_k(\mathbf{x}_{m,i}) < 0 \\ \Pr(G(\mathbf{x}_{m,i}) > 0), & G_k(\mathbf{x}_{m,i}) > 0 \end{cases} = \Phi\left(\frac{|G_k(\mathbf{x}_{m,i})|}{\sqrt{e(\mathbf{x}_{m,i})}}\right) \tag{22}$$

where |.| is the absolute operator, Φ is a standard normal cumulative distribution function, and CL(.) is a positive value within [0.5,1].

Then based on the predicted response from the Kriging model and the number of MCS points, the reliability can be calculated by

$$R = 1 - P_f = 1 - \frac{\sum_{i=1}^{N} I_{RF}(\mathbf{x}_{m,i})}{N} \tag{23}$$

As the number of MCS sample points are known, the expectation value of the confidence level can be used to measure the confidence level of the whole system which called the cumulative confidence level (CCL). The CCL of the design system can be defined as

$$CCL(\mathbf{M}, \mathbf{X}_m) = \frac{\sum_{i=1}^{N} CL(\mathbf{x}_{m,i})}{N} \tag{24}$$

where the CCL(.) is also a positive value within [0.5,1].

The MCE sampling method uses the CCL as the criterion to measure the accuracy of the Kriging model for reliability analysis within a given local region where the current system design point is located. If the surrogate model satisfies the prescribed CCL target, it is considered to be accurate if used to conduct reliability analysis for the current system design and no more sample point is further needed to improve the model fidelity at current local region. On the other hand, if the Kriging model is not considered to be accurate enough, the following active learning rule that maximizes the estimated improvement can be used to select new sample points to refine the Kriging model.

The estimated improvement (EI) measure aims to assess the potential improvement by adding a new sample point $\mathbf{x}^*$, which can be expressed mathematically as

$$EI(\mathbf{x}^*) = (1 - CL(\mathbf{x}^*)) \times f_x(\mathbf{x}^*) \times \sqrt{e(\mathbf{x}^*)} \tag{25}$$

where $CL(\mathbf{x}^*)$ denotes the confidence level of classification at $\mathbf{x}^*$ using current Kriging model; so, the term (1- $CL(\mathbf{x}^*)$) represents the maximum amount that can be potentially improved by adding new sample points. Therefore, sample points with low classification confidence levels based upon existing Kriging models are more likely to be selected as new training sampling points. Moreover, $f_x(\mathbf{x}^*)$ is the joint probability density function value of random input variable at the newly selected input sample point $\mathbf{x}^*$; $e(\mathbf{x}^*)$ is the mean square error of the performance prediction at $\mathbf{x}^*$ returned by the Kriging model, and we can directly use this value from section 3.1.

To find valuable sample points, we compute the estimated improvements for the MCS population in the local area. Accordingly, a high *EI* value from one candidate sample point generally indicates a high potential to obtain a good improvement of the surrogate model fidelity if the candidate sample point is chosen. Thus, sample points with the largest *EI* values will be chosen to update the Kriging model as necessarily along the RBDO process at different system designs. At a given RBDO iteration, the MCE sampling approach will be used to select adaptive sample points and these newly selected points can be added to the training data set iteratively to improve the fidelity of the surrogate models, and the updating process will thus be repeated until a target level of accuracy for reliability analysis is achieved.

3.3.2 Design Sensitivity Analysis

Design sensitivity information is essential for the RBDO in the iterative design processes, as it affects not only the accuracy of reliability analysis in each design iteration but also the convergence rate to the optimum design.

While computational models are used for the RBDO problem and the system performance functions are treated as black box functions, the finite difference method has been widely used for design sensitivity analysis, with moving step size being set generally as a constant or a coefficient of design variables. The main concept behind the finite difference scheme is related to the definition of the derivative of a smooth function u at a point $x \in R$. For example, the one dimension case can be generally expressed as

$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \tag{26}$$

where h should be sufficiently small to yield a good sensitivity approximation.

Chapter 4: Hybrid Adaptive Sequential Sampling Strategy

Most of existing adaptive sampling methods focus on either the global sampling strategy or the local sampling strategy. While the global sampling strategy often approximates the limit state in the entire system input space, it generally requires a large number of evaluations of performance functions at different sampling points in the areas that may not be important for solving the RBDO problem. On the other hand, the local sampling strategy generally focuses on selecting the most useful sample points to improve the model fidelity at a particular local region. Although the local sampling strategy can be efficient computationally, it often has a poor convergence performance for the optimization process in the RBDO model and could also produce infeasible solutions. To alleviate this difficulty and balance the local and global sampling needs in solving the RBDO problem with adaptive surrogate models, in this thesis the global sampling strategy that ensures effective design updates in the design optimization process has been combined with a novel local sampling approach, and consequently a new hybrid adaptive sequential sampling (HASS) methodology is developed.

This chapter presents the developed HASS methodology, including the global and local sampling criterions that have been integrated into the developed HASS method. Section 4.1 introduces the global sampling criterion used in the surrogate development processes. A novel local sampling approach developed in this study, referred to as the Confidence Guided Sequential Sampling (CGSS) technique, is presented in Section 4.2. Based on the above sampling criterions, the developed HASS methodology is then detailed in Section 4.3. Section 4.4 presents a new adaptive sensitivity analysis (ASA) method developed in this thesis for the RBDO problem using surrogate models. Section 4.5 combines the developed HASS approach and the ASA method with the RBDO problem.

4.1 Global Sampling Criterion

In this thesis, the constraint boundary sampling (CBS) criterion has been adopted for the global sampling purpose, which approximates the limit state boundaries in the global design region only and has good performances in terms of accuracy and efficiency.

The learning function of the CBS aims to select sample points around the limit state because the points near the limit state have a high potential risk of crossing the predicted boundaries of constraints. The uncertainty on these points is more likely to affect the predicted value and the probability of failure.

Then, we use an optimization problem to demonstrate the constraint boundary sampling.

$$\begin{aligned} &\textit{Minimize} && (x_1-3.7)^2+(x_2-2) \\ &\textit{Subject to} && G_1(x)=-x_1\sin(4x_1)-1.1x_2\sin(2x_2) \\ & && G_2(x)=x_1+x_2-3 \\ & && 0\le x_1\le 3.7,\ 0\le x_2\le 4 \end{aligned} \tag{27}$$

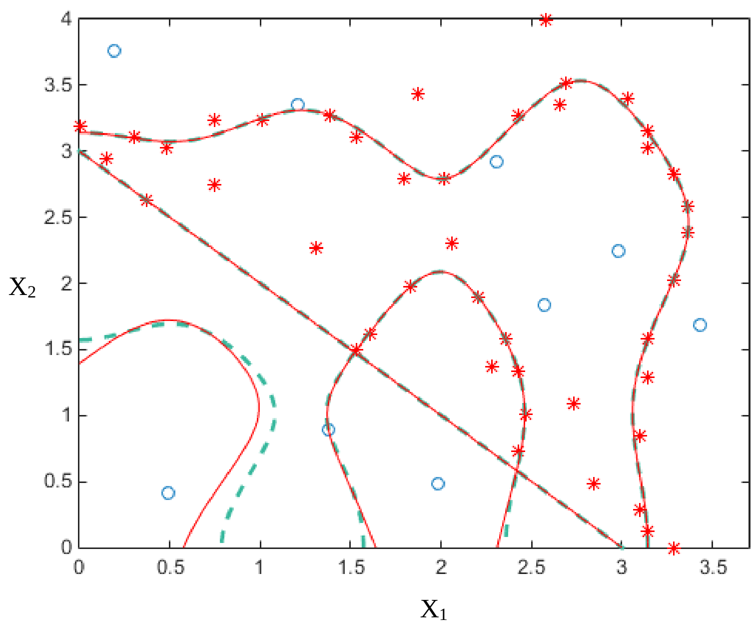


Figure 2. CGSS sampling process for RBDO problem

In Fig. 2, dotted lines represent real constraint functions, and red lines are boundaries predicted by the surrogate model using the CBS approach. The blue circles are initial sample points that have been used to build the initial Kriging model, and the red stars are the sample points adaptively selected by the CBS criterion.

From Fig. 2, we can find that the points selected by the CBS criterion have been largely located very close to the constraint boundaries.

4.2 Local Sampling Criterion

In this thesis, a novel sampling approach, referred to as the Confidence Guided Sequential Sampling (CGSS) technique, has been developed and used as the local sampling strategy in the HASS method. It is generally true that one sampling region would be more important than others if the objective function has smaller values in that sampling region when the goal is to minimize the objective function, and thus new sample points should be accordingly selected from that sampling region. To take into account the impact of the optimization process, the developed CGSS sampling scheme not only considers the expective improvement value of the candidate sample points for the fidelity of surrogate models, but also takes the gradient information of the objective function as well as the constraint functions into consideration, in order to identify the most useful sample points and improve the regional model fidelity. For example, the sample point with a smaller cost function value and a more significant estimated improvement value has a higher opportunity to be chosen. Also, design points need to satisfy the reliability required in the RBDO problem, then the descending direction of the constraint function should also be considered.

The developed local sampling method CGSS focuses on the Monte Carlo population, which means that we use the same Monte Carlo population around the current design point as candidates and calculate the sampling criterion values of these candidate sample points at each adaptive sampling iteration. $\mathbf{X}_{mcs}$-$\mathbf{X}_{d}$ has been used to calculate the distances between the candidate sample points to the current design point at a given RBDO design iteration, and the inner product has been used to measure the contribution of candidate points on the descending direction of the constraint function and the objective function.

The contribution of new sample points on the descending direction of the constraint function G_i is indicated as

$$(\mathbf{x}_{mcs} - \mathbf{x}_d) \cdot \nabla G_i \tag{28}$$

Similarly, the contribution of new points on the descending direction of the objective function F is denoted as

$$(\mathbf{x}_{mcs} - \mathbf{x}_d) \cdot \nabla F \tag{29}$$

Combining the above two parts with the estimated improvement, the active learning function for the CGSS local sampling approach can be defined as

$$CGSS = \frac{(\mathbf{x}_{mcs} - \mathbf{x}_d) \cdot \nabla G_i + (\mathbf{x}_{mcs} - \mathbf{x}_d) \cdot \nabla F}{\|\nabla G\| \cdot \|\nabla F\| \cdot \|\mathbf{x}_{mcs} - \mathbf{x}_d\|^2} \cdot EI \tag{30}$$

where $\mathbf{X}_{mcs}$ denotes the Monte Carlo population, $\mathbf{X}_{d}$ is the current design point, ∇G_i represents the descending direction of the constraint function G_i, and ∇F represents the descending direction of the objective function F.

The sample point chosen from the set of alternatives is the one that maximizes the CGSS criterion. Eq. (30) shows that the CGSS criterion aims to identify the most valuable sample points that not only have high improvement potentials but also are located in important feasible regions to improve the fidelity of surrogate models as well as the efficiency of the modeling process.

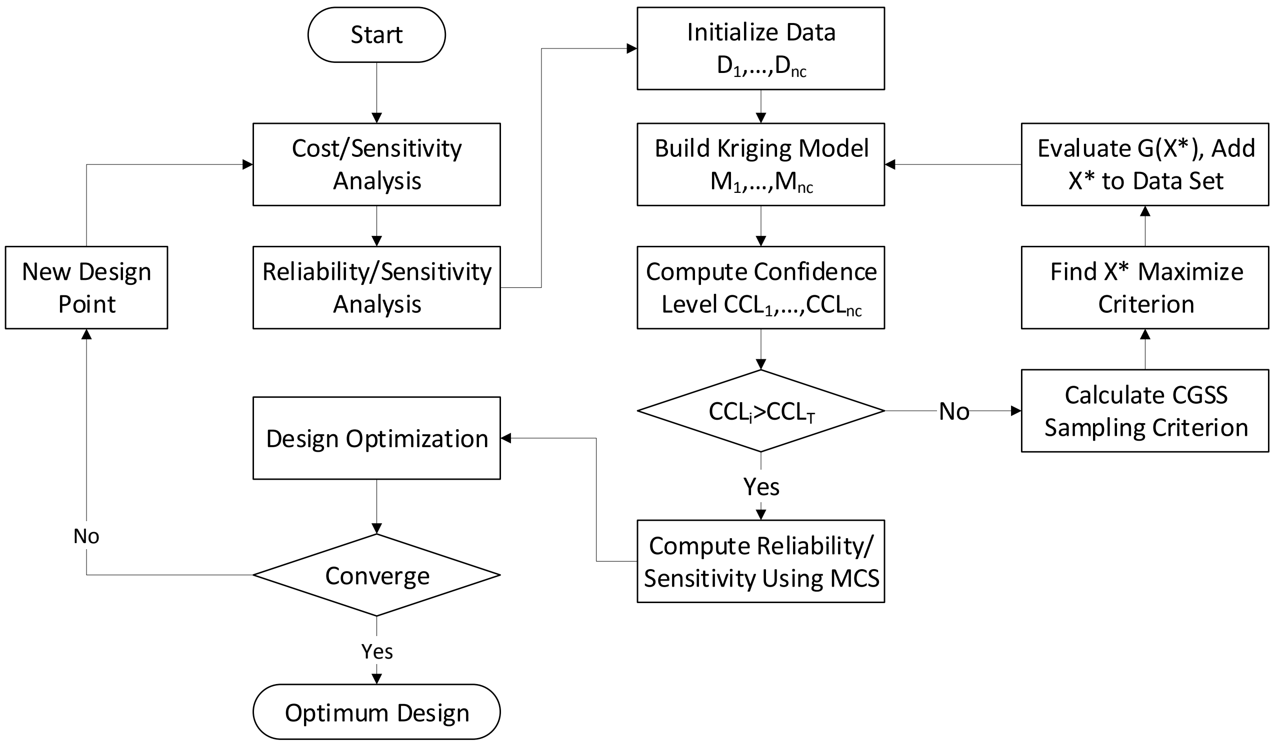


Figure 3. CGSS sampling process for RBDO problem

The flowchart shown in Fig. 3 summarizes the procedure for the adaptive CGSS sampling method integrated with the double loop RBDO problem solving process. The outer loop is the design optimization process for RBDO and the inner loop involves reliability analysis as well as the sensitivity analysis using the Kriging surrogate model in each optimization iteration.

The right-hand-side of the above flowchart shows the reliability analysis process based on the proposed CGSS sampling criterion. First, an initial dataset can be generated and also the target confidence level, CCL_T, can be set accordingly. By using the dataset that is currently available to build initial Kriging models, the reliability R of the system and the cumulative confidence level CCL_i for each surrogate model can be computed and compared with the prescribed target values. If the current value of CCL_i larger than CCL_T, the accuracy of current surrogate models are good enough and no more sample point would be further needed to refine Kriging models in the current local design region. Thus in this scenario, the algorithm would return the reliability and sensitivity information to the outer loop design optimization process. Otherwise, the CGSS sampling criterion will be used and applied to all candidate sample points in the MCS population, in order to find the sample point $\mathbf{x}^*$ that maximizes the CGSS criterion, and accordingly the selected $\mathbf{x}^*$ will be included to the initial dataset to further update Kriging models iteratively until the accuracy level reaches the target.

4.3 Hybrid Sampling Criterion

With the global and local sampling approaches introduced, in the thesis a new hybrid adaptive sequential sampling (HASS) methodology is developed for solving the RBDO problem to substantially improve the efficiency of the surrogate model development process, which combines the local searching strategy that focuses on improving regional fidelity with the global sampling technique that ensures effective design updates in the design optimization process. To balance the use of those two strategies, the searching factor has been used as a pointer to determine if the global or local search is needed. The CCL measure from the MCE based sequential sampling method has been used to measure the accuracy of the Kriging model and control the switch between sampling to enhance surrogate model fidelity and using the surrogate model for reliability analysis in the design optimization process.

To implement those two searching strategies, a new variable named searching factor (SF) has been introduced, which can be defined as

$$SF = \begin{cases} 0, & if\ \ R - R_t < 0 \\ 1, & if\ \ R - R_t > 0 \end{cases} \tag{31}$$

where R denotes the reliability of the system, and R_t represents the reliability target.

When the current reliability is much lower than the target reliability, the value of *SF* will be set to zero, which indicates that the global sampling strategy will be chosen. Similarly, while the reliability from current design is higher than the reliability target, *SF* will be set to one and accordingly the local sampling strategy using the CGSS criterion will be implemented to improve the regional fidelity of the surrogate model.

With the definition of the SF, the local searching criterion (LSC) based on the Monte Carlo population using the CGSS criterion can be accordingly updated as

$$LSC = SF \cdot CGSS = SF \cdot \frac{(x_{mcs} - x_d) \cdot \nabla G + (x_{mcs} - x_d) \cdot \nabla F}{\|\nabla G\| \cdot \|\nabla F\| \cdot \|x_{mcs} - x_d\|^2} \cdot EI \tag{32}$$

Similarly, the global searching criterion (GSC) which selects new sample points from the grid sampling pool using the CBS criterion can be expressed as

$$GSC = (1 - SF) \cdot CBS \tag{33}$$

As shown in the above two equations, the global sampling strategy and the local sampling strategy can be switched according to binary SF. If the *SF* equals to zero, the R-R_t is less than zero, which indicates that the reliability estimated by Monte Carlo population around the current design point is smaller than the target reliability prescribed. Thus, (1-*SF*) equals to 1 and the global sampling strategy can be chosen to update the design effectively. Similarly, when the value of R-R_t is greater than zero, the reliability at the current design point satisfies the reliability requirement. As a result, the local sampling strategy will be chosen to improve regional model fidelity.

Therefore, the sampling criterion for the developed HASS methodology can be defined concisely as

$$HASS = LSC + GSC \tag{34}$$

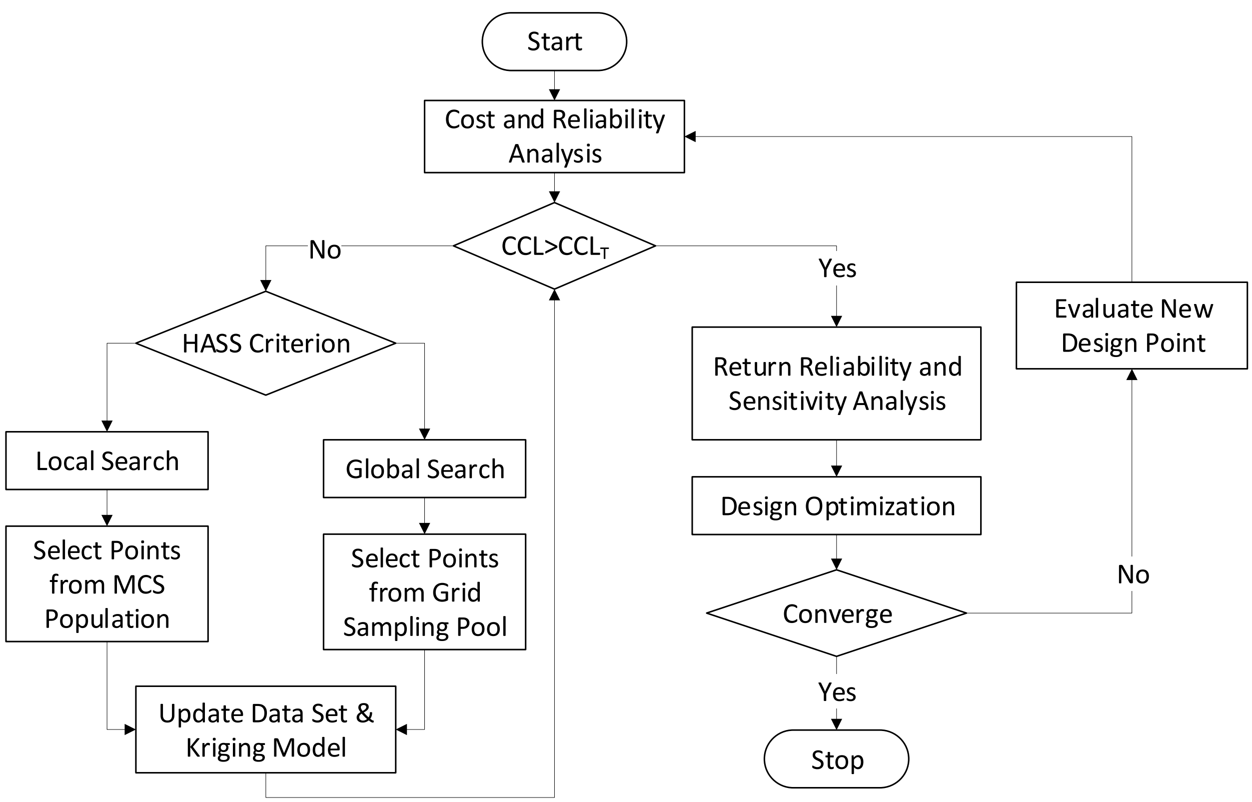


Figure 4. HASS criterion for RBDO problem

The flowchart shown in Fig. 4 illustrates the procedure of the developed HASS methodology for solving the RBDO problem using Kriging surrogate models. By using the sampling criterion for the developed HASS methodology, the local sampling and the global sampling strategies can be alternated, depending on the comparison between the estimated reliability level of the current design based upon the Kriging model and the reliability target.

To further reduce the computational cost, the trajectory of the searching and moving process of the design points in each iteration has also been taken into consideration for the developed HASS methodology. The high fidelity model output could be used for the evaluation at the current design point if it is far away from the previous one, and accordingly this new design point can be included to the training dataset to refine the surrogate model and improve its fidelity. Otherwise, the Kriging model is used directly to estimate reliability performance at the design point without surveying the high fidelity model to reduce the computational cost caused by expensive function evaluations.

4.4 Adaptive Sensitivity Analysis Method

Although the finite difference method has been widely used for design sensitivity analysis, the approximation of the sensitivity information could be inaccurate, which could potentially lead to divergence of the optimization problem or more design iterations with higher computational cost in the design process. In this thesis, an adaptive sensitivity analysis (ASA) approach is developed to accelerate the optimization process and reduce computational cost.

From an accuracy perspective, when multiple design points resulted from successive design iterations are found to be close to each other with very small differences, it is likely that the RBDO problem approaches to a local minimum. In this scenario, more accurate sensitivity information is usually needed to provide a smooth convergence performance in the optimization process, and consequently the step size used in the finite difference method can be adjusted based on the history and current information from the RBDO model. Thus, the reliability information, the distance between two successive design points, and trajectory of the optimization processes can all be used as measures to adjust the step size for the finite difference method adaptively. With the step size to be adaptively adjusted based upon the RBDO process needs, it enables us to determine when and where to reduce the moving step size so that the desired balance between the efficiency of the RBDO process and the accuracy of the reliability analysis during the intermediate design iterations can be achieved.

To measure the distance between the current design point with the previous design point, we transform the design space into a standard normal space. Then the reliability index, β, corresponds to the target reliability becomes the distance measure, and the distance between the current design point and the previous design point can be calculated and compared with the reliability index β. Therefore, when the distance between the current design point and the previous design point is less than β, more accurate gradient information is needed for searching a new moving direction in order to improve the design and continue the optimization process. Hence, the step size used to calculate the gradient information of reliability by the finite difference method needs to be adjusted down. Also, with the increase of the number of design iterations, design points gradually move closer to the optimal design, and accordingly a smaller step size needs to be chosen.

The ASA approach is thus to adjust the step size used in the finite difference based on the information of the trajectory of design points, the reliability index, the moving distance and the number of iterations, which can be defined as

$$stepsize = \frac{dist}{iteration + K} \tag{35}$$

where K is a constant number, here we choose K equals to 15 in our numerical examples.

4.5 RBDO with HASS Strategy and ASA Method

In this section, the developed HASS methodology and the ASA approach are combined with the RBDO model. The procedure has been summarized in Fig. 5 below.

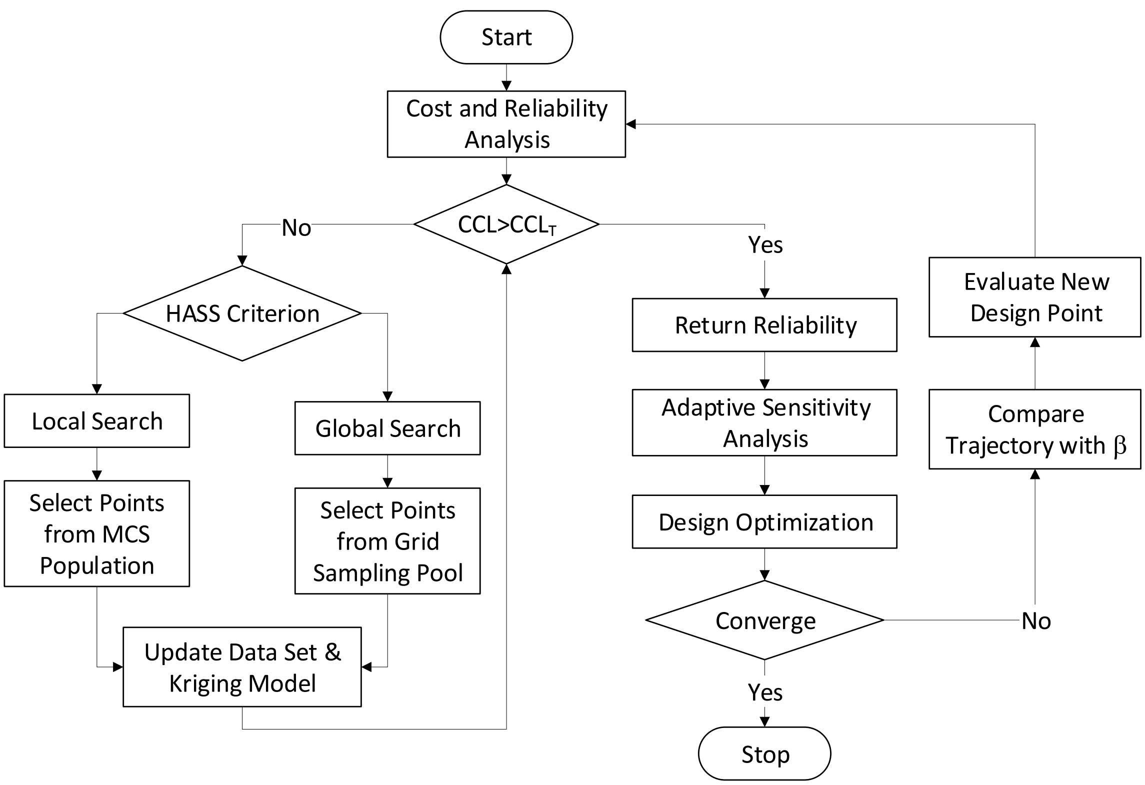


Figure 5. Flowchart of developed HASS methodology and the ASA method for RBDO

As the flowchart shown in Fig. 5, the RBDO problems are solved using the double loop formulation, where the outer loop is the optimization loop that optimizes design variables and minimizes the objective function, and the inner loop in charges of the reliability analysis which uses the developed HASS methodology to select new sample points to refine the Kriging model and use the ASA approach to calculate reliability sensitivity information.

Following the procedure as outlined in Fig. 5, random sample points will first be generated and used to build initial Kriging models. With the initial Kriging models, the CCL measures are calculated to measure the accuracy of Kriging models. If $CCL < CCL_T$, the developed HASS sampling criterion will be used to choose new sample points to update the Kriging models until the target CCL_T is achieved. Then, the constructed Kriging surrogate models can be used for reliability analysis with the MCS method and the design sensitivity analysis by the developed ASA approach, providing the required reliability and sensitivity information for the optimization process. In the optimization loop, based on the sensitivity information calculated by the ASA method, the optimizer will search for a new design point to improve the system design. The procedure can be repeated until the optimization process is converged to an optimal design point.

In the following, the detail steps for using the developed HASS methodology together with the ASA approach for RBDO are detailed below.

1. Set the target confidence level CCL_T; Use a random sampling method (e.g., the Latin Hypercube Sampling) to generate an initial set of training sample points; Build initial Kriging models; Generate candidate sample points with a grid sampling pool for global sampling and a Monte Carlo population for local sampling.

2. Calculate the distance between the current design point and the previous design point. If the distance is larger than the reliability index β, the design point will then be selected, the high-fidelity performance functions will be evaluated, and the new data point can then be added to the training data set. Otherwise, the design point is evaluated by the Kriging model.
3. Compute reliability R, confidence level, and CCL. Then Compare CCL with CCL_T. If CCL > CCL_T, go to the optimization process and calculate reliability and design sensitivity information.
4. Check convergence performance. If the optimization process converges to the optimum design, output the results. Otherwise, move to a new design point. If the distance between the current design point and the previous design point is smaller than β, the step size used in the finite difference for reliability sensitivity information can then be adjusted by the developed ASA method.
5. Check the accuracy of surrogate models. If CCL < CCL_T, compute the HASS criterion and determines which type of searching strategy should be taken at this iteration. If we go to the global searching process, the new point will be selected from the grid sampling pool; if the local searching strategy has been taken, the new point will be chosen from the Monte Carlo population. Moreover, the new sample points x* with the maximum value of the criterion can be selected.
6. Evaluate G(x*) and update datasets by adding the new data (x*, G(x*)). Then rebuild the Kriging model and calculate reliability and CCL. Then, go to step 5.

Chapter 5: Case Study and Results

In this chapter, two case studies are provided to demonstrate the accuracy and efficiency of the developed HASS methodology. The first case study focuses on different sequential sampling strategies only and compares the developed CGSS and HASS strategies with several existing sampling methods. The second case study examines different step sizes used in the finite difference calculation for reliability analysis based on the Kriging model using the HASS approach and the developed ASA method.

5.1 Numerical Case Study I

The first case study considers the following mathematical design optimization problem with two random design variables X_1 and X_2. Both random variables are normally distributed as $X_1 \sim N(\mu_1, 0.3464^2)$ and $X_2 \sim N(\mu_2, 0.3464^2)$, where the design variables $\mathbf{d} = [d_1, d_2]^T = [\mu(X_1), \mu(X_2)]^T$. The initial design point is set as $d_0 = [2.1, 1.59]$. We set the target reliability level equals to $R_t = 0.98$ and the cumulative confidence level target equals to $CCL_T = 0.998$.

The RBDO problem is formulated as:

$$
\begin{aligned}
&\textit{Minimize}: \ \cos t = 10 - d_1 + d_2 \\
&\textit{subject to}: P_r[G_i(x) < 0] \le 1 - R_t, \ i = 1 \sim 3 \\
&\qquad\qquad 0 \le d_1 \ \& \ 0 \le d_2 \\
&G_1 = -0.052x_1^3 + 0.8637x_1^2 - 4.02x_1 + 5.88 - x_2 + 3 \\
&G_2 = -0.0096x_1^4 + 0.24x_1^3 - 1.94x_1^2 + 5.86x_1 - 4.86 + x_2 - 2 \\
&G_3 = (x_1 - 10)^2 + (x_2 - 1.5)^2 - 0.5
\end{aligned}
\tag{36}
$$

The contours of the constraint boundaries have been shown in Fig. 6. Five initial points have been generated by the Latin Hypercube Sampling method to build the initial Kriging model. Then, the MCS population and grid sampling pool have been made. The developed HASS approach has been compared with CBS, MCE and CGSS methods, and the comparison results are summarized in Table 1. The information is given as the optimal design point, the corresponding reliability, the optimum value and the number of calls to the performance function.

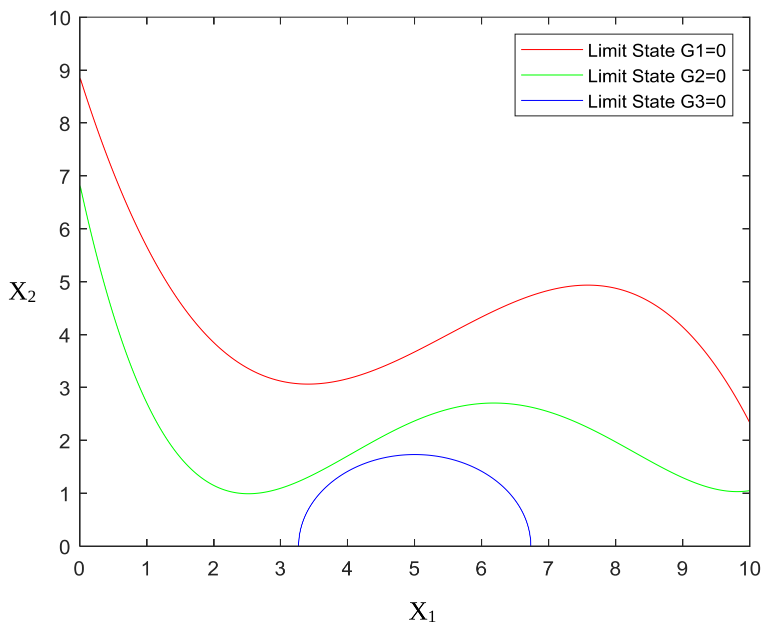
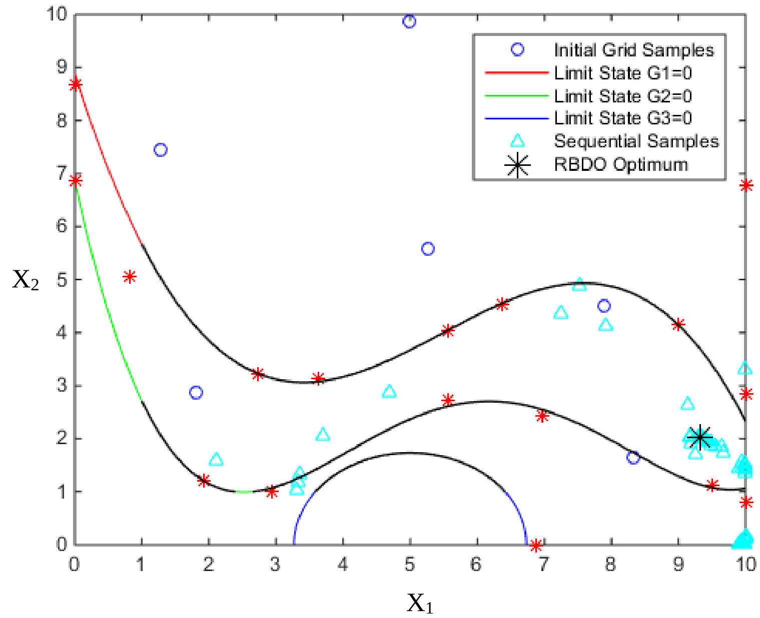
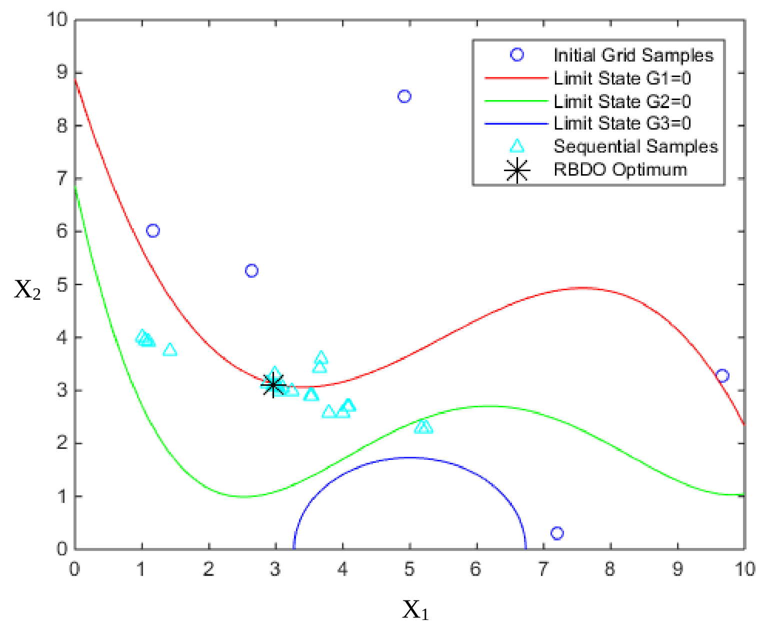


Figure 6. Contours of the limit state functions for the numerical case study

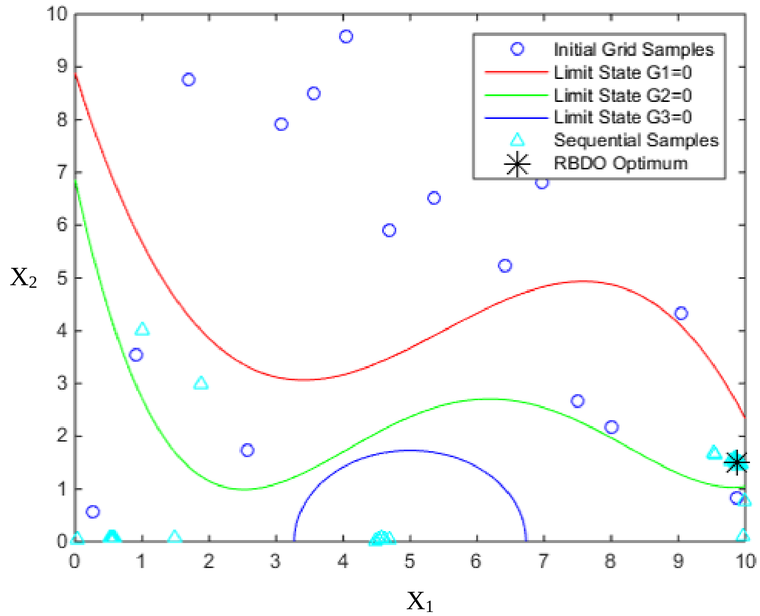


(a) The RBDO problem solving process with the CBS strategy

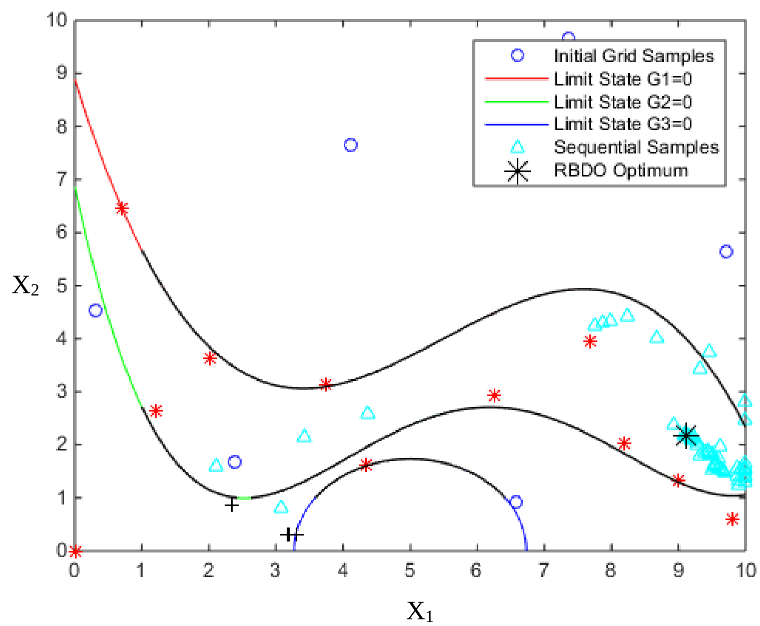


(b) The RBDO problem solving process with the MCE strategy

Figure 7. Sampling points used in the design process for the numerical case study I



(c) The RBDO problem solving process with the CGSS strategy



(d) The RBDO problem solving process with HASS

Figure 7 (Cont.). Sampling points used in the design process for the numerical case study I

Fig. 7 (a) shows the RBDO result for case study I that the global sampling strategy using the CBS criterion has been employed. The red stars represent sample points that are selected adaptively by only using the CBS sampling criterion. The optimal design can be found by using this global sampling criterion.

Fig. 7 (b) shows the RBDO result by using the local sampling method based on the MCE method, and the right optimal design cannot be found by only using this local sampling criterion. The optimization searching process finally converges to an infeasible point, which is largely caused by the low fidelity of the surrogate model and ill-suited sampling criterion. So, it is necessary to combine the global sampling strategy with local sampling approaches instead of only focusing on the regional model fidelity.

Fig. 7 (c) shows the RBDO result by using the developed local sampling method, CGSS. An optimum design has not been found that satisfies all the constraints.

Fig. 7 (d) shows the RBDO result by using the developed HASS methodology, which combines the developed CGSS method with the CBS based global sampling strategy. Red stars represent sample points that have been selected adaptively by CBS process. And the optimal design can be found under the reliability requirement with a lower number of function evaluations compared with the CBS method.

Table 1 shows the comparison result of four sampling methods. The MCE and the CGSS are two local sampling methods, and in these cases, the optimal design could not be found by using the local sampling approaches only. The result obtained by using the MCE method shows that the final design point converges to an infeasible point, and the searching processes stop before the optimum point has been reached. The result obtained by using the CGSS method shows that the final design point is near the optimum design point but located in the infeasible region. Nonetheless, the CBS method and the developed HASS methodology perform well in this case study. For the global sampling method CBS, the optimum design point under the reliability requirement can be found, but it requires a large number of function evaluations. The result of using the developed HASS method, however, indicates that using this criterion can significantly reduce the computational cost and the number of function evaluations.

Table 1. Comparison Results of Case Study I

Sampling Methods	**Optimum**		**Cost**	**Reliability**			**Number of evaluations**		
CBS	9.31	2.04	2.73	0.98	0.98	1	29	29	29
MCE	2.96	3.09	10.13	0.59	0.99	1	6	5	6
CGSS	9.87	1.51	1.63	0.88	0.87	1	15	15	15
HASS	9.12	2.16	3.04	0.99	0.98	1	23	23	23

5.2 Numerical Case Study II

Design sensitivity information is essential in iterative design processes. It affects not only the efficiency of sensitivity analysis but also the convergence rate to the optimum design. In this case study, the same numerical example has been used. Based on the developed HASS sampling criterion, the developed ASA method has been used for reliability sensitivity analysis using the Kriging model. To better localize the constraint boundaries, design points far away from other design points are evaluated by the high-fidelity model and will be added to datasets to develop surrogate models. Otherwise, the low-fidelity surrogate model will be used to evaluate design points. The high-fidelity model and low-fidelity model have been used alternatively depending on the trajectory of the design points and the reliability index.

When multiple design points locate close to each other, the developed ASA method has been used to adaptively adjust the step size used in the finite difference method based on the distance between design points and the number of iterations.

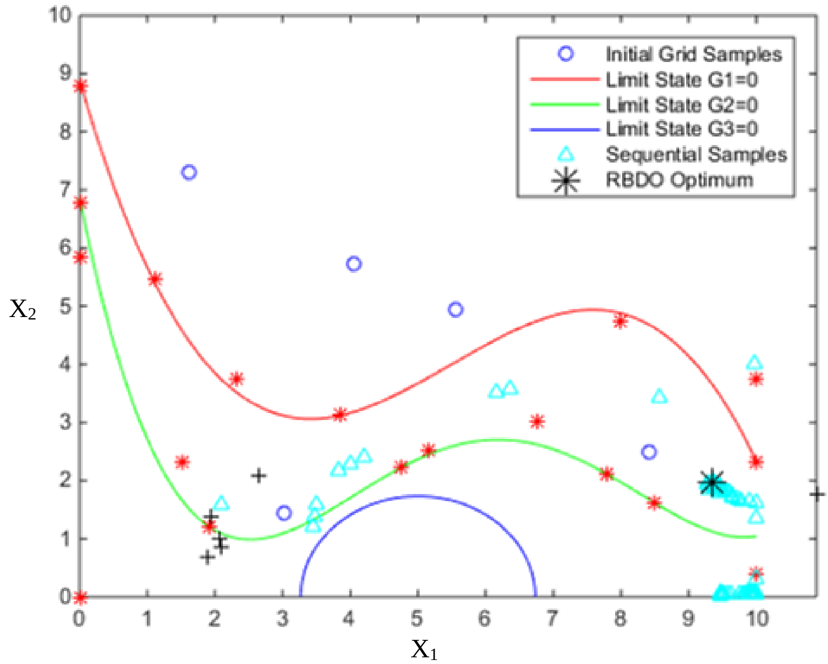
Here, we compare the following four different combinations of the HASS approach, the trajectory of design points, and the ASA approach.

(a) **HASS**: Use HASS criterion with the sensitivity analysis method of finite difference with standard step size by a factor of 30;

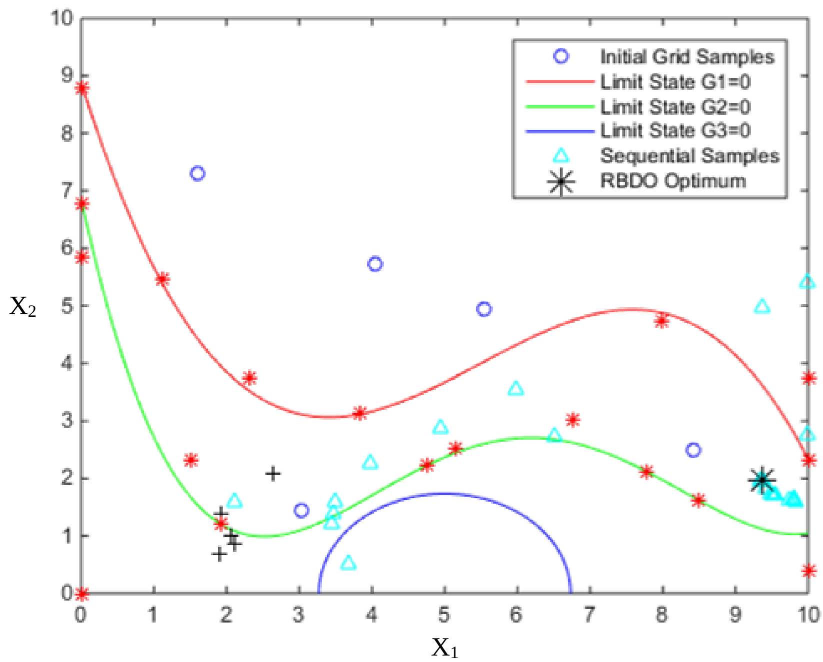
(b) **HASS and FDM**: Use HASS criterion with the sensitivity analysis method of finite difference with reduced step size by a factor of 50.

(c) **HASS, DTI and FDM**: Use HASS criterion and the design trajectory information (DTI) to choose high/low fidelity model with the sensitivity analysis method of finite difference with reduced step size by a factor of 50.

(d) **HASS, DTI and ASA**: Use HASS and trajectory information with developed ASA approach.

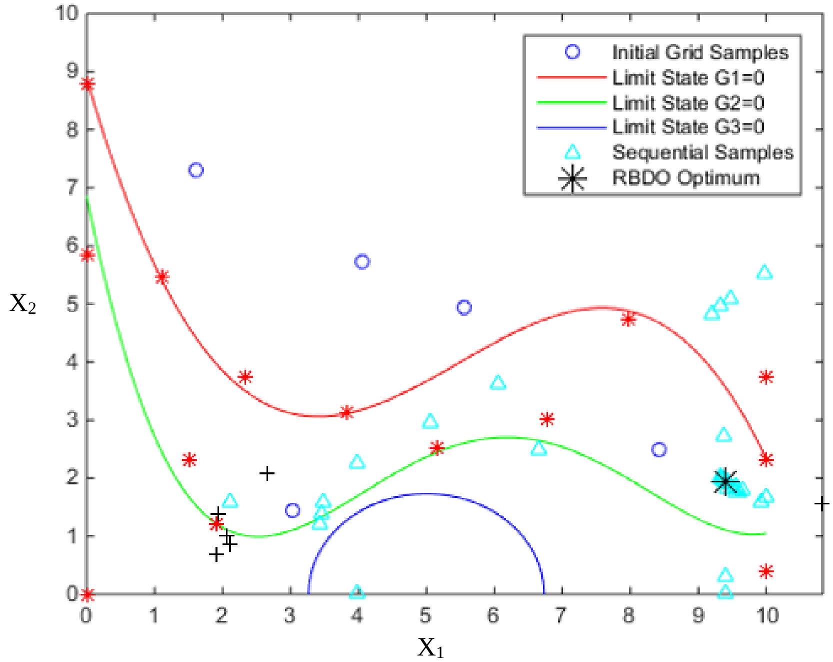


(a) The RBDO problem solving process with HASS

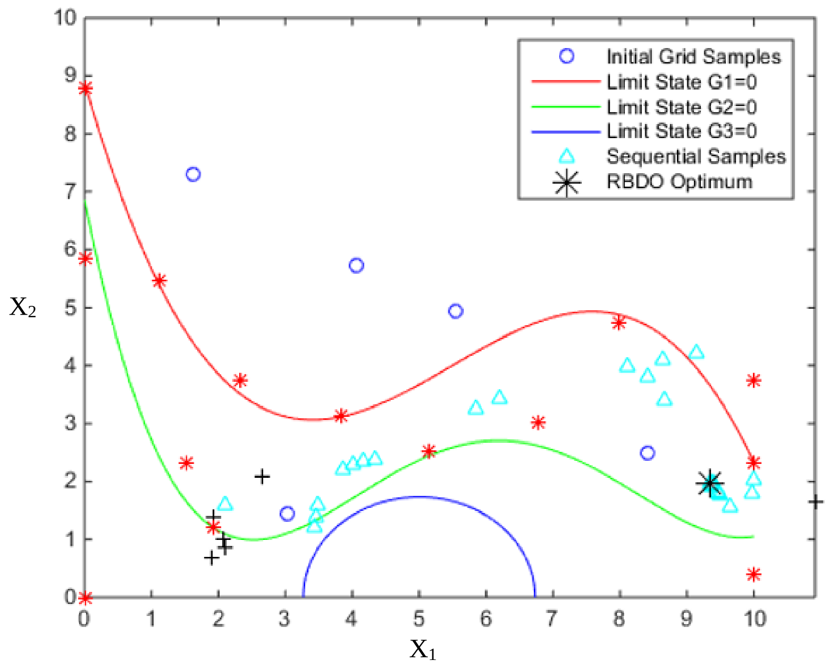


(b) The RBDO problem solving process with HASS and FDM

Figure 8. Sampling points used in the design process for the numerical case study II



(c) The RBDO problem solving process with HASS, DTI and FDM



(d) The RBDO problem solving process with HASS, DTI and ASA

Figure 8 (Cont.). Sampling points used in the design process for the numerical case study II

Table 2. Comparison Results of Case Study II

Sampling Methods	**Opt**	**Cost**				**R**	**Num. of Eval.**			**Iter.**
HASS	9.34	1.96	2.62	0.983	0.983	1	28	29	28	71
HASS / FDM	9.36	1.95	2.59	0.981	0.983	1	28	28	28	30
HASS /DTI / FDM	9.39	1.93	2.54	0.979	0.982	1	24	24	24	52
HASS /DTI /ARA	9.35	1.95	2.60	0.982	0.982	1	24	25	24	36

Table 2 shows the comparison result of the above four methods with the numerical case study II. Comparing the first two methods, the result shows that the number of iterations can be largely reduced by diminishing the step size used in the finite difference method. Also, comparing the second and the third method, we can find that taking the trajectory information into consideration can significantly reduce the number of expensive function evaluations, especially when the design point approaches the optimal design or moves close to each other.

To balance the accuracy of the sensitivity information and the efficiency of the optimization process, our advanced ASA method adjusts the step size adaptively based on the distance between design points and the number of iterations. The last two rows of the table show that the developed ASA method performs better than the finite difference method with a constant step size by using a less number of design iterations.

Chapter 6: Conclusion and Future Work

6.1 Conclusion

In this thesis, we develop new sampling techniques based on the Kriging model for the RBDO problem. And the cumulative confidence level has been used to measure the accuracy of the Kriging model since the Monte Carlo Simulation has been used to estimate the reliability of the system. A new hybrid adaptive sequential sampling strategy has been developed, which combines the developed CGSS method with the global sampling strategy, and those two strategies can be used alternatively. For the local sampling criterion, CGSS scheme is developed, which identifies the most useful sample points along the descending direction of the objective function as well as the constraints to improve the regional fidelity. The developed HASS technique, therefore, can identify the most useful sample points effectively. Moreover, points selected by the HASS criterion will both have a high probability to failure and a low objective function value and can be used to improve the fidelity of the surrogate models regionally and globally, thereby improve the efficiency of the modeling process.

To further reduce the computational cost and shorten the optimization process, design points in the optimization process are evaluated by the high and low fidelity model alternatively to refine the surrogate model based on the moving trajectory of design points. Also, the step size used in reliability sensitivity analysis has been adjusted by the developed ASA method based on the reliability index, the number of iterations, and distances between the design points.

The case studies on a mathematics-based design problem show that the developed HASS approach and the ASA method can obtain better results than other methods, which can reduce the number of function evaluations and the size of our data sets dramatically. The first case study indicates that the developed HASS method can significantly reduce the time and cost in the design process. In the second case study, the efficiency of the RBDO problem has been improved dramatically in terms of the number of function evaluations as well as the number of iterations by using the developed HASS criterion with the developed ASA method.

6.2 Future Work

The work presented in this thesis has investigated the hybrid adaptive sequential sampling methodology for selecting useful sample points to improve the fidelity of Kriging surrogate models for the reliability-based design optimization. Currently, Kriging surrogate models have been developed individually for each constraint functions, which could be computationally expensive. In future work, work can also be done with the focus on advanced sampling methods that can sample points for all constraint functions simultaneously.

References

[1] J. S. Arora, "Introduction to Optimum Design, McGraw-Hill, New York, NY.," 1989.

[2] R. T. a. G. Haftka, Z., "Elements of Structural Optimization," Dordrecht, Netherlands, 1991.

[3] P. W. Yumeng Li, and Weirong Xiao, "Uncertainty Quantification of Atomistic Materials Simulation with Machine Learning Potentials," presented at the 2018 AIAA Non-Deterministic Approaches Conference, 2018.

[4] E. H. A. Mohsine A, "A robust study of reliability-based optimization methods under eigenfrequency," *Computer Methods in Applied Mechanics and Engineering*, vol. 199 (17–20):1006-1018, 2010.

[5] S. E. Gano, J. E. Renaud, H. Agarwal, and A. Tovar, "Reliability-based design using variable-fidelity optimization," *Struct Infrastruct E,* vol. 2, no. 3-4, pp. 247-260, Sep-Dec 2006.

[6] E. Nikolaidis and R. Burdisso, "Reliability Based Optimization - a Safety Index Approach," *Comput Struct,* vol. 28, no. 6, pp. 781-788, 1988.

[7] J. Tu, K. K. Choi, and Y. H. Park, "A new study on reliability-based design optimization," *J Mech Design,* vol. 121, no. 4, pp. 557-564, Dec 1999.

[8] J. S. Park, "Optimal Latin-Hypercube Designs for Computer Experiments," *J Stat Plan Infer,* vol. 39, no. 1, pp. 95-111, Apr 1 1994.

[9] Z. Hu and X. P. Du, "First order reliability method for time-variant problems using series expansions," *Struct Multidiscip O,* vol. 51, no. 1, pp. 1-21, Jan 2015.

[10] X. T. Zhuang and R. Pan, "A Sequential Sampling Strategy to Improve Reliability-Based Design Optimization With Implicit Constraint Functions," *J Mech Design,* vol. 134, no. 2, Feb 2012.

[11] X. P. Du and W. Chen, "Sequential optimization and reliability assessment method for efficient probabilistic design," *J Mech Design,* vol. 126, no. 2, pp. 225-233, Mar 2004.

[12] D.-T. L. Crombecq K, Gorissen D, et al. , "A novel sequential design strategy for global surrogate modeling Simulation conference," proceedings of the 2009 winter Austin, USA, 2009.

[13] J. Eason and S. Cremaschi, "Adaptive sequential sampling for surrogate model generation with artificial neural networks," *Comput Chem Eng,* vol. 68, pp. 220-232, Sep 4 2014.

[14] D. R. Jones, M. Schonlau, and W. J. Welch, "Efficient global optimization of expensive black-box functions," *J Global Optim,* vol. 13, no. 4, pp. 455-492, Dec 1998.

[15] E. M. Bichon BJ, Swiler LP, et al. , "Efficient global reliability analysis for non- linear implicit performance functions," *AIAA J,* vol. 2008;46(10):2459–68, 2008.

[16] T. H. Lee and J. J. Jung, "A sampling technique enhancing accuracy and efficiency of metamodel-based RBDO: Constraint boundary sampling," *Comput Struct,* vol. 86, no. 13-14, pp. 1463-1476, Jul 2008.

[17] Z. Z. Chen, H. B. Qiu, L. Gao, X. K. Li, and P. G. Li, "A local adaptive sampling method for reliability-based design optimization using Kriging model," *Struct Multidiscip O*, vol. 49, no. 3, pp. 401-416, Mar 2014.

[18] Z. Z. Chen *et al.*, "An important boundary sampling method for reliability-based design optimization using kriging model," *Struct Multidiscip O,* vol. 52, no. 1, pp. 55-70, Jul 2015.

[19] B. Echard, N. Gayton, and M. Lemaire, "AK-MCS: An active learning reliability method combining Kriging and Monte Carlo Simulation," *Struct Saf,* vol. 33, no. 2, pp. 145-154, 2011.

[20] Z. Q. Wang and P. F. Wang, "A Maximum Confidence Enhancement Based Sequential Sampling Scheme for Simulation-Based Design," *Proceedings of the Asme International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, 2013, Vol 3b,* 2014.

[21] B. D. Y. Chao Hu, Pingfeng Wang, "Fundamentals of Reliability Analysis," in *Engineering Design under Uncertainty and Health Prognostics*: Springer, Cham, 2019, pp. 83-100.

[22] R. Y. Rubinstein, "Simulation and the Monte Carlo Method," presented at the John Wiley and Sons, New York, NY, 1981.

[23] I. Enevoldsen and J. D. Sorensen, "Reliability-Based Optimization in Structural-Engineering," *Struct Saf,* vol. 15, no. 3, pp. 169-196, Sep 1994.

[24] V. L. Swamy, C. Grandi, and R. V. Grandhi, "General-Purpose Procedure for Reliability-Based Structural Optimization under Parametric Uncertainties," *Adv Eng Softw,* vol. 23, no. 1, pp. 7-14, 1995.

[25] K. K. Choi, Yu, X., and Chang, K.H., "A Mixed Design Approach for Probabilistic Structural Durability," presented at the Sixth AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization, Bellevue, WA, 1996.

[26] Y. T. Wu and W. Wang, "A new method for efficient reliability-based design optimization," *Probabilistic Mechanics & Structural Reliability*, pp. 274-277, 1996.

[27] X. Yu, K. K. Choi, and K. H. Chang, "A mixed design approach for probabilistic structural durability," *Struct Optimization*, vol. 14, no. 2-3, pp. 81-90, Oct 1997.

[28] X. M. Yu, K. H. Chang, and K. K. Choi, "Probabilistic structural durability prediction," *Aiaa Journal*, vol. 36, no. 4, pp. 628-637, Apr 1998.

[29] R. V. Grandhi and L. P. Wang, "Reliability-based structural optimization using improved two-point adaptive nonlinear approximations," *Finite Elem Anal Des*, vol. 29, no. 1, pp. 35-48, May 15 1998.

[30] A. Chakri, X. S. Yang, R. Khelif, and M. Benouaret, "Reliability-based design optimization using the directional bat algorithm," *Neural Comput Appl*, vol. 30, no. 8, pp. 2381-2402, Oct 2018.

[31] B. D. Y. Chao Hu, Pingfeng Wang, "Reliability-Based Design Optimization," in *Engineering Design under Uncertainty and Health Prognostics*: Springer, Cham, 2019, pp. 187-231.

[32] Y. Aoues and A. Chateauneuf, "Benchmark study of numerical methods for reliability-based design optimization," *Struct Multidiscip O*, vol. 41, no. 2, pp. 277-294, Mar 2010.

[33] S. Q. Shan and G. G. Wang, "Reliable design space and complete single-loop reliability-based design optimization," *Reliab Eng Syst Safe*, vol. 93, no. 8, pp. 1218-1230, Aug 2008.