A NEW BOUND ON QUBIT ENTANGLEMENT DISTILLATION

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Abstract

In this thesis we study the theory of entanglement among qubits, and derive a new limit on the rate at which we may convert different forms of entanglement in an optimal way. In particular, we prove that for an arbitrary three qubit state held by Alice, Bob and Charlie, the optimal probability to distill a two qubit Bell pair held by Alice and Bob is the smaller of two numbers: the entanglement between Alice and the Bob+Charlie system, and the entanglement between the Bob and Alice+Charlie system. If one desires two qubit entanglement for a particular information processing task where Bell pairs are needed (for example quantum state teleportation), but has for whatever reason a source of three qubit entanglement, this result can help predict the feasibility of carrying out the task.

Subject Keywords: Qubit; LOCC; Entanglement of Assistance; Entanglement Distillation

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Chapter 1

Introduction

Quantum information science is still a relatively new field of study; therefore, much exciting work is currently being done to develop both the theoretical tools and the technology that may well revolutionize the way we process information. The strange properties of quantum mechanics like entanglement and superposition have captured the interest of scientists, philosophers, and popular science fans alike. Scientists and mathematicians, for example, have realized that entanglement is a useful resource for processing and communicating information. Examples of emergent technology in various degrees of realization include quantum key distribution for secure information transfer, quantum computing with exponentially faster run-times, efficient simulation of quantum chemistry and teleportation-assisted telescopes with resolutions far exceeding the current standard.

This project focuses on entanglement. To understand it, we must note that it is often admissible to characterize a quantum system as being composed of multiple smaller systems called A,B,C,... with which we associate experimenters Alice, Bob, Charlie,... who hold in their laboratories parts of this global quantum system. For example, suppose Alice holds a helium atom at a very low temperature in the ground state, where the spins of the two electrons are entangled. If she could send one of the electrons to Bob in a distant laboratory then, to a very good approximation, Alice and Bob would each hold a single qubit system in their labs, which are part of a global two-qubit system. After generating many entangled electron pairs in this way, they may each perform certain measurements many times, whose outcome statistics cannot be predicted without considering the global quantum system.

Because of the remarkable experimental success that quantum theory has enjoyed throughout the past century, we can expect that continuing to make theoretical predictions will provide useful guidance in developing devices to perform the tasks we would like to do. In particular, if quantum theory suggests a certain information processing protocol is impossible, then we should not expend resources in attempting it. To this end, we derive an upper limit of exactly this type. We deal exclusively with qubits, the mathematical abstractions of two-level noiseless quantum systems, because these objects are the simplest way to gain insight into quantum phenomena. From a practical perspective, however, the restriction to qubits is not so problematic, as most current and proposed device architectures operate on qubits.

Chapter 2

Mathematical Background

A **quantum state** is a linear operator $\rho : \mathcal{H} \to \mathcal{H}$ such that $\rho \geq 0$ and $\text{Tr}\rho = 1$, where $\mathcal{H}$ is a Hilbert space over $\mathbb{C}$. In particular, a **pure state** is a rank-one operator, meaning it may be written as $\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle \in \mathcal{H}$ and $\langle\psi| \in \mathcal{H}^*$, the dual space of $\mathcal{H}$. We also simply call $|\psi\rangle$ a pure state. If ρ has rank larger than one, it is called a **mixed state**. This is because a mixed state can always be written as $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $\{p_i\}_i$ is a probability distribution and $\{|\psi_i\rangle\}_i$ is some set of pure states. These sets form an **ensemble** for ρ, which is a "mixture" of the $|\psi_i\rangle$. In some instances we may think of p_i as the probability of finding the system in the state $|\psi_i\rangle$, though this is in general problematic because ρ does not have a unique ensemble. If the eigenvectors and eigenvalues of ρ are $\{\lambda_i, |e_i\rangle\}_i$, then ρ has spectral decomposition $\rho = \sum_i \lambda_i |e_i\rangle\langle e_i|$ which is indeed an ensemble for ρ. Any other ensemble $\rho = \sum_j q_j |\phi_j\rangle\langle\phi_j|$ must satisfy the constraint $\sqrt{q_j}|\phi_j\rangle = \sum_i u_{ij}\sqrt{\lambda_i}|e_i\rangle$, with u_{ij} the elements of a unitary matrix. A qubit in a mixed or pure state may be represented as a point $\vec{r} = (r, \theta, \phi)$ either on the surface (pure) or inside (mixed) of a unit sphere in $\mathbb{R}^3$. There exists such a picture because the set of Pauli matrices $\{\mathbb{I}, \sigma_1, \sigma_2, \sigma_3\}$ forms an orthonormal basis for linear operators on qubits, with respect to the Hilbert-Schmidt inner product $(A, B) = \text{Tr } B^\dagger A$, allowing us to write $\rho = \frac{1}{2}(\mathbb{I} + \vec{r} \cdot \vec{\sigma})$ for $|\vec{r}| \leq 1$. This is called the **Bloch sphere representation** of a qubit, which we will later make significant use of. A more complete but accessible introduction to the mathematics used in quantum information theory is given in the standard text by Nielsen and Chuang [1].

Chapter 3

Quantifying Entanglement

EPR (Einstein, Podolsky, Rosen) were the first to notice entanglement in quantum theory. Bell later provided a methodology for testing the existence of entanglement. If one thinks of entanglement as a resource which can be expended to assist in information processing protocols, it makes sense to quantify it. However, doing this for a general distributed quantum state $\rho^{ABCD\cdots}$ has proven to be quite difficult. At the same time, work over the past two decades towards this goal has exposed a rich and diverse mathematical structure that underlies the space of quantum states and the space of operations that can be performed on them. We will do quite an injustice to this field and describe only how entanglement can be quantified for two qubit pure states. For a more comprehensive review of the subject, we point the reader to [5], [6].

To begin measuring entanglement, we need to consider at a minimum the tensor product space $\mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B$, which represents the composite system of Alice and Bob. An **entangled** state is any state that is not separable across this partitioning of $\mathcal{H}^{AB}$ into $\mathcal{H}^A$ and $\mathcal{H}^B$. For pure states, **separable** means that $|\psi\rangle^{AB} = |\alpha\rangle^A \otimes |\beta\rangle^B$. People often omit the tensor product symbol and write this state as $|\alpha\rangle^A|\beta\rangle^B$. One can then easily conjure up an entangled state, for example, the infamous EPR state or Bell pair $|\Phi^+\rangle^{AB} = \frac{1}{\sqrt{2}}(|0\rangle^A|0\rangle^B + |1\rangle^A|1\rangle^B)$. This turns out to be a **maximally entangled** state. Our entanglement measure should at least reflect the above properties; we want a function $E : \mathcal{H}_2^A \otimes \mathcal{H}_2^B \to [0, 1]$, which vanishes for separable states and equals unity for states with maximal entanglement. We should demand that E can only be increased on average when Alice and Bob are distributed more entanglement. Even when they are allowed to exchange classical information (via phone calls for example, as opposed to quantum information like sending photons or electrons in a coherent fashion) and operate locally on their quantum systems, they cannot with certainty increase E. The set of all transformations that Alice and Bob can effect under these restrictions is termed **LOCC** for local operation and classical communication. Additionally, E should remain invariant under **LU** transformations, for local unitary transformations. These are of the form $T^{AB} = U^A \otimes V^B$ with U^A a unitary operator on the Alice Hilbert space, V^B a unitary operator on the Bob Hilbert space, and T^{AB} by definition a unitary operator on the Alice–Bob Hilbert space. Note that this is not the most general unitary operator on $\mathcal{H}^{AB}$; there are also non-local unitaries which physically are generated by at-range interactions via some global Hamiltonian. They will in general be entangling, and as such may increase E. To be very careful, one should further constrain E depending on the problem at hand, but the above will suffice for our purposes. When E satisfies the above conditions we call it an LOCC **entanglement monotone**. We now show two examples and make some final remarks before presenting our work.

Consider the most general two-qubit pure state $|\psi\rangle^{AB} = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$. This state is separable if and only if $|\alpha\gamma - \beta\delta| = 0$. That is, the determinant of the matrix M_ψ that can be associated with this state tells us about its separability. This is called the **concurrence** $\mathcal{C}(\psi) = 2\sqrt{|\det M_\psi|}$. There is another useful entanglement monotone for two qubit pure states: $E(\psi) = 2\lambda_{min}(\rho^A) = 2\lambda_{min}(\sigma^B)$, where $\lambda_{min}(\rho^A)$ is the minimum eigenvalue of the **reduced density operator** $\rho^A = (\mathbb{I} \otimes \mathrm{Tr})|\psi\rangle\langle\psi|^{AB} = \mathrm{Tr}_B|\psi\rangle\langle\psi|^{AB}$. In words, this quantity is twice the minimum eigenvalue as seen by Alice, and equivalently as seen by Bob. It turns out that concurrence and any other bipartite LOCC entanglement monotone are in one-to-one correspondence with λ_{min} [6]. For qubits in particular, if $\mu : \mathcal{H}_2^A \otimes \mathcal{H}_2^B \to [0,1]$ is another entanglement monotone which vanishes for pure states and is maximized to unity for Bell pairs, then $\mu = f(\lambda_{min})$, where $f : [0, \frac{1}{2}] \to [0,1]$ is a concave function. For three qubits, the problem becomes much more complicated. As such, three qubit states admit some very interesting properties, such as having two distinct maximally entangled states: $|W\rangle^{ABC} = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$ and $|GHZ\rangle^{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Because we now have three parties, one can ask about the amount of entanglement across different partitions: $E_{A(BC)}$, $E_{B(AC)}$, and $E_{(AB)C}$, but it turns out that these three numbers are in general not enough to fully characterize the entanglement present in a three qubit state. The famous "entanglement monogamy" relations were discovered by studying this problem [2].

Chapter 4

Bounding Entanglement Distillation

Let $|\psi\rangle^{ABC} \in \mathcal{H}_2^A \otimes \mathcal{H}_2^B \otimes \mathcal{H}_n^C$. Given a bipartite entanglement measure E (not necessarily an LOCC monotone), one can define the **entanglement of assistance** EoA with a predetermined assistant, say party C, as

$$EoA(|\psi\rangle^{ABC}) = \max_{\mathcal{E}} \sum_k p_k E(|\psi_k\rangle^{AB}), \tag{4.1}$$

with the maximization over all pure state decompositions or ensembles $\mathcal{E} = \{p_k, |\psi_k\rangle^{AB}\}_k$ of $\rho^{AB} = \mathrm{Tr}_C |\psi\rangle\langle\psi|^{ABC}$ [3]. Because various ensembles for ρ^{AB} correspond exactly to various projective measurements that party C can perform, the entanglement of assistance quantifies the ability of the assistant C to leave the joint AB system with the most entanglement possible after measurement. If $E_{A(BC)}$ and $E_{B(AC)}$ are the bipartite entanglements in the bipartitions A(BC)and B(AC), then because neither can increase on average under LOCC,

$$EoA(|\psi\rangle^{ABC}) \leq \min\,(E_{A(BC)}, E_{B(AC)}). \tag{4.2}$$

We would like to consider $2\lambda_{min}(\rho^A)$ as the monotone E. This can be interpreted as the optimal probability to transform in any LOCC protocol $|\phi\rangle^{AB} \rightarrow |\Phi^+\rangle^{AB}$ where the latter state is a maximally entangled state [6]. The EoA with respect to this measure can then be interpereted as the optimal probability in any 1-shot (single copy) LOCC protocol to **distill** a maximally entangled state on AB. Call this probability $\mathcal{P}_{max}$. We now state our main result.

Theorem. Continuing the work of [4] we prove that for an arbitrary three qubit state,

$$\mathcal{P}_{max} = \min\,(E_{A(BC)}, E_{B(AC)}). \tag{4.3}$$

Proof. Consider first $|\psi\rangle^{ABC} \in \mathcal{H}_2^A \otimes \mathcal{H}_2^B \otimes \mathcal{H}_n^C$. Due to a theorem in [4] and the well known isomorphism between linear operators and bipartite state vectors, it follows that there always exists a basis $\{|k\rangle^C\}_{1\leq k\leq n}$ for Charlie's system such that the state may be written as $|\psi\rangle^{ABC} = \sum_k A^k \otimes \mathbb{I}^{BC}|\Phi^+\rangle^{AB}|k\rangle^C$ with the operators $A^k A^{k\dagger}$ simultaneously diagonalizable for $1 \leq k \leq n$. There also exists another basis $\{|k'\rangle^C\}_{1\leq k\leq n}$ for which we may write $|\psi\rangle^{ABC} = \sum_k A'^k \otimes \mathbb{I}^{BC}|\Phi^+\rangle^{AB}|k'\rangle^C$ with the $A'^{k\dagger}A'^k$ now simultaneously diagonalizable. We restrict now $n = 2$, so that all indices take values in the set $\{1, 2\}$. Let $\{p_k, |\psi_k\rangle^{AB}\}_k$ be the ensemble resulting from Charlie's projective measurement in the basis $\{|k\rangle^C\}_k$. If $p_k = 0$ for some k, then ρ^{AB} is pure, the state is a product state across the partition (AB)C, and $\mathcal{P}_{max} = E_{AB} = E_{A(BC)} = E_{B(AC)}$. Assuming $p_k \neq 0\ \forall k$, define

$$\rho_k^A = \mathrm{Tr}_B|\psi_k\rangle\langle\psi_k| = \frac{1}{p_k}A^k A^{k\dagger} = \frac{1}{2}(\mathbb{I} + \vec{r_k}\cdot\vec{\sigma}) \tag{4.4}$$

$$\sigma_k^B = \mathrm{Tr}_A|\psi_k\rangle\langle\psi_k| = \frac{1}{p_k}(A^{k\dagger}A^k)^* = \frac{1}{2}(\mathbb{I} + \vec{s_k}\cdot\vec{\sigma}). \tag{4.5}$$

With these definitions, observe that the following commutators evaluate to

$$[A^l A^{l\dagger}, A^k A^{k\dagger}] = i\frac{p_k p_l}{2}(\vec{r_k}\times\vec{r_l})\cdot\vec{\sigma}, \tag{4.6}$$

$$[A^{l\dagger}A^l, A^{k\dagger}A^k]^* = i\frac{p_k p_l}{2}(\vec{s_k}\times\vec{s_l})\cdot\vec{\sigma}, \tag{4.7}$$

which vanish for $k \neq l$ if and only if their corresponding Bloch vectors are parallel or anti-parallel to each other. Let $\uparrow\uparrow$ denote parallel and $\uparrow\downarrow$ denote anti-parallel, so that either $\vec{r_1} \uparrow\uparrow \vec{r_2}$ or $\vec{r_1} \uparrow\downarrow \vec{r_2}$ (we consider the case of $\vec{r_k} = \vec{0}$ to be anti-parallel). This calculation could be repeated for the ensemble $\{p'_k, |\psi'_k\rangle^{AB}\}_k$ resulting from Charlie's projective measurement in the basis $\{|k'\rangle^C\}_k$. It would be found that $\vec{s_k}' \times \vec{s_l}' = 0$ and either $\vec{s_1}' \uparrow\uparrow \vec{s_2}'$ or $\vec{s_1}' \uparrow\downarrow \vec{s_2}'$. We propose the following procedure to show there is always exists an optimal measurement saturating inequality (4.2). Let the reduced density operators be

$$\rho^A = \mathrm{Tr}_{BC}|\psi\rangle\langle\psi|^{ABC} = \frac{1}{2}(\mathbb{I} + \vec{R}\cdot\vec{\sigma}), \quad \sigma^B = \mathrm{Tr}_{AC}|\psi\rangle\langle\psi|^{ABC} = \frac{1}{2}(\mathbb{I} + \vec{S}\cdot\vec{\sigma}). \tag{4.8}$$

The bipartite measure E is twice the minimum eigenvalue of the reduced density operators:

$$E_{A(BC)} = 1 - |\vec{R}|, \quad E_{B(AC)} = 1 - |\vec{S}|. \tag{4.9}$$

Note that if $\{p_k, |\psi_k\rangle^{AB}\}_k$ is any ensemble for ρ^{AB} then

$$\vec{R} = \sum_k p_k \vec{r_k}, \quad \vec{S} = \sum_k p_k \vec{s_k}, \tag{4.10}$$

$$\langle E_{AB}\rangle := \sum_k p_k E_{AB}(|\psi_k\rangle^{AB}) = \sum_k p_k(1 - |\vec{r_k}|) = \sum_k p_k(1 - |\vec{s_k}|). \tag{4.11}$$

Suppose Charlie measures in the basis $\{|k\rangle^C\}_k$, and we find $\vec{r_1} \uparrow\uparrow \vec{r_2}$. Then,

$$\langle E_{AB}\rangle = 1 - \sum_k p_k |\vec{r_k}| = 1 - |\vec{R}| = E_{A(BC)}. \tag{4.12}$$

Consequently $E_{A(BC)} = \langle E_{AB}\rangle \leq \mathcal{P}_{max} \leq \min(E_{A(BC)}, E_{B(AC)})$ so $\mathcal{P}_{max} = E_{A(BC)}$, and $|k\rangle^C$ was the optimal measurement. If instead we find that $\vec{r_1} \uparrow\downarrow \vec{r_2}$, we can look at the $\vec{s_k}$. In this ensemble, the $\vec{s_k}$ are not guaranteed to satisfy $\vec{s_1} \times \vec{s_2} = \vec{0}$, but if by chance $\vec{s_1} \uparrow\uparrow \vec{s_2}$, then $\langle E_{AB}\rangle = E_{B(AC)}$ with $|k\rangle^C$ still optimal. At the other extreme, if by chance $\vec{s_1} \uparrow\downarrow \vec{s_2}$ then we show in the lemma below that the optimizing measurement is $|\widetilde{\pm}\rangle^C = \frac{1}{\sqrt{2}}(|1\rangle \pm i|2\rangle)$. Finally if $\vec{s_1} \times \vec{s_2} \neq \vec{0}$, then neither $|k\rangle^C$ nor $|\widetilde{\pm}\rangle^C$ attains the optimal probability. To remedy this case, Charlie should attempt a measurement in the basis $|k'\rangle^C$. In this new ensemble we are now guaranteed that $\vec{s_1}' \times \vec{s_2}' = \vec{0}$, so either $|k'\rangle^C$ or $|\widetilde{\pm}'\rangle^C$ will optimize unless both $\vec{s_1}' \uparrow\downarrow \vec{s_2}'$ and $\vec{r_1}' \times \vec{r_2}' \neq \vec{0}$. Therefore, there always exists an optimizing measurement unless we have encountered a state in which all four of the following conditions hold:

$$\begin{aligned} \vec{r_1} \uparrow\downarrow \vec{r_2}, \qquad & \vec{s_1} \times \vec{s_2} \neq \vec{0}, \\ \vec{s_1}' \uparrow\downarrow \vec{s_2}', \qquad & \vec{r_1}' \times \vec{r_2}' \neq \vec{0}. \end{aligned} \tag{4.13}$$

However all hope is not lost, as we claim here that there cannot exist such a state. Note again that $r_k = |\vec{r_k}| = |\vec{s_k}|$ and $r'_k = |\vec{r_k}'| = |\vec{s_k}'|$. Let $|\vec{R}| = R$ and $|\vec{S}| = S$. Suppose (4.13) are satisfied. Then on the one hand, we have

$$S = |p_1\vec{s_1} + p_2\vec{r_2}| > |p_1 s_1 - p_2 s_2| = |p_1 r_1 - p_2 r_2| = R, \tag{4.14}$$

due to the triangle inequality, but with strict inequality because $\vec{s_1} \times \vec{s_2} \neq \vec{0}$ and the previous assumption of p_k nonzero. On the other hand we have by a similar reasoning:

$$R = |p'_1\vec{r_1}' + p'_2\vec{r_2}'| > |p'_1 r'_1 - p'_2 r'_2| = |p'_1 s'_1 - p'_2 s'_2| = S, \tag{4.15}$$

which is a contradiction. Therefore there are no states of this type, so one of the projective measurements prescribed above will attain the optimal probability. ■

Lemma. For any state with $\vec{r_1} \uparrow\downarrow \vec{r_2}$ and $\vec{s_1} \uparrow\downarrow \vec{s_2}$, the optimizing measurement basis is $|\widetilde{\pm}\rangle^C = \frac{1}{\sqrt{2}}(|1\rangle \pm i|2\rangle)$. Likewise, for any state with $\vec{s_1}' \uparrow\downarrow \vec{s_2}'$ and $\vec{r_1}' \uparrow\downarrow \vec{s_2}'$, it is $|\widetilde{\pm}'\rangle^C = \frac{1}{\sqrt{2}}(|1'\rangle \pm i|2'\rangle)$

Proof. We first purify the reduced density operators as follows. Let the spectral decompositions be

$$\rho_1^A = \left(\frac{1+r_1}{2}\right)|e_0\rangle\langle e_0|^A + \left(\frac{1-r_1}{2}\right)|e_1\rangle\langle e_1|^A \tag{4.16}$$

$$\rho_2^A = \left(\frac{1+r_2}{2}\right)|g_0\rangle\langle g_0|^A + \left(\frac{1-r_2}{2}\right)|g_1\rangle\langle g_1|^A \tag{4.17}$$

$$\sigma_1^B = \left(\frac{1+r_1}{2}\right)|f_0\rangle\langle f_0|^B + \left(\frac{1-r_1}{2}\right)|f_1\rangle\langle f_1|^B \tag{4.18}$$

$$\sigma_2^B = \left(\frac{1+r_2}{2}\right)|h_0\rangle\langle h_0|^B + \left(\frac{1-r_2}{2}\right)|h_1\rangle\langle h_1|^B. \tag{4.19}$$

If we require $\vec{r_1} \uparrow\downarrow \vec{r_2}$ and $\vec{s_1} \uparrow\downarrow \vec{s_2}$, then $\langle e_0|g_0\rangle^A = 0$, $\langle f_0|h_0\rangle^B = 0$, and

$$|g_1\rangle^A = |e_0\rangle^A, \quad |g_0\rangle^A = |e_1\rangle^B, \quad |h_0\rangle^B = |f_1\rangle^B, \quad |h_1\rangle^B = |f_0\rangle^B. \tag{4.20}$$

The unique purifications are then

$$\begin{aligned} |\psi_1\rangle^{AB} &= \sqrt{\frac{1+r_1}{2}}|e_0\rangle^A|f_0\rangle^B + \sqrt{\frac{1-r_1}{2}}|e_1\rangle^A|f_1\rangle^B \\ |\psi_2\rangle^{AB} &= \sqrt{\frac{1-r_2}{2}}|e_0\rangle^A|f_0\rangle^B + \sqrt{\frac{1+r_2}{2}}|e_1\rangle^A|f_1\rangle^B. \end{aligned} \tag{4.21}$$

We can further purify ρ^{AB} to obtain, up to a unitary on C,

$$|\psi\rangle^{ABC} = \sqrt{p_1}|\psi_1\rangle^{AB}|1\rangle^C + \sqrt{p_2}|\psi_2\rangle^{AB}|2\rangle^C. \tag{4.22}$$

This implies after measurement in the basis $|\widetilde{\pm}\rangle^C$ the equally likely states held by AB:

$$\begin{aligned} |\psi_+\rangle &= \left(\sqrt{p_1\frac{1+r_1}{2}} + i\sqrt{p_2\frac{1-r_2}{2}}\right)|e_0\rangle|f_0\rangle + \left(\sqrt{p_1\frac{1-r_1}{2}} + i\sqrt{p_2\frac{1+r_2}{2}}\right)|e_1\rangle|f_1\rangle \\ |\psi_-\rangle &= \left(\sqrt{p_1\frac{1+r_1}{2}} - i\sqrt{p_2\frac{1-r_2}{2}}\right)|e_0\rangle|f_0\rangle + \left(\sqrt{p_1\frac{1-r_1}{2}} - i\sqrt{p_2\frac{1+r_2}{2}}\right)|e_1\rangle|f_1\rangle. \end{aligned} \tag{4.23}$$

The equal bipartite entanglements are

$$E_{AB}^{\pm} = \langle E_{AB}\rangle = 2\min\left(\left|\sqrt{p_1\frac{1+r_1}{2}} \pm i\sqrt{p_2\frac{1-r_2}{2}}\right|^2, \left|\sqrt{p_1\frac{1-r_1}{2}} \pm i\sqrt{p_2\frac{1+r_2}{2}}\right|^2\right). \tag{4.24}$$

Because $\vec{r_1} \uparrow\downarrow \vec{r_2}$ and $\vec{s_1} \uparrow\downarrow \vec{s_2}$,

$$E_{A(BC)} = E_{B(AC)} = \min\{p_1(1+r_1) + p_2(1-r_2), p_1(1-r_1) + p_2(1+r_2)\}, \tag{4.25}$$

so this is the optimizing ensemble. The same could be done for the primed ensemble. ■

Chapter 5

Conclusion

In this thesis we have provided a straightforward proof that the natural bound in (4.2) saturates when measured with respect to $2\lambda_{min}$. We have shown it can always be saturated with a simple protocol: Charlie performs a projective measurement in one of the bases we have described above and broadcasts his outcome to Alice and Bob, who then convert with some optimal probability to a Bell pair. It is, however, difficult to compute the optimizing measurement basis when given the coefficients c_{ijk} describing a general three qubit state, we have only discussed the existence of these bases. In the future we hope to learn whether this feature of the EoA when measured with respect to $2\lambda_{min}$ also holds for other measures of entanglement.

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