Two Elements of a Mathematical Sensemaking Approach to Teaching Introductory Physics

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We describe two elements of a mathematical sensemaking curriculum that has been used in university-level, large-lecture, introductory physics courses. The goals of this curriculum are to foster physics students' mathematical sensemaking—that is, their leveraging of the coherence between conceptual understanding and formal mathematics in their reasoning. The learning outcomes of this course are described elsewhere [1].

1) Explicit strategies to build coherence between mathematics and conceptual understanding

One focus in the mathematical sensemaking curriculum is to explicitly help students engage in coherence between mathematics and conceptual understanding. For example, when introducing equations, the instructor will often give a symbolic form interpretation of the equation, explicitly stating the conceptual schema that fits with the mathematical structure. Additionally, problems are designed to elicit both conceptual and calculation approaches, especially on problems for which we predicted these two modes of reasoning would not be consistent. These questions further prompt students to resolve inconsistencies between the two modes of reasoning, to further help students refine their problem-solving strategies and make them aware that they should seek this coherence in physics. As an example, consider this problem from any early assignment in introductory mechanics:

> *Standing on a cliff, I take one rock and throw it straight up at a speed of 30 m/s. I take another rock and throw it straight down at 30 m/s. Suppose the cliff is 50 meters high.* (see Fig. 1)
>
> a) *Just based on common sense, which rock would be moving faster when it hits the ground, 50 meters below? What's the reasoning for that?*
>
> b) *Now find an answer based on the kinematics of constant acceleration: Find x(t) and v(t) for each of the rocks, and find their respective speeds when they hit the ground 50 m below the point of release.*
>
> c) *Did your answer to (b) agree with your answer to (a)? If not, try to reconcile the contradiction: Figure out what it is about the reasoning in part (a) or part (b) that doesn't work. Get it all to make sense!*

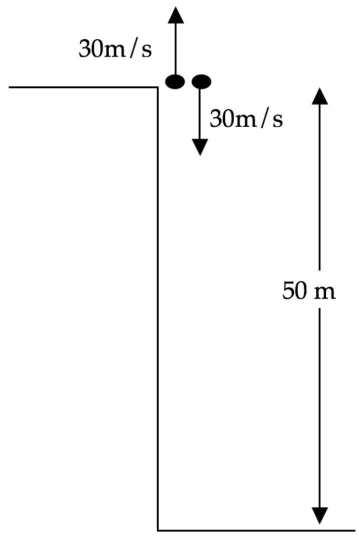

Figure 1. Diagram for “Rocks and Cliff” problem.

Part (a) asks students for their intuitive, conceptual reasoning. Students commonly respond that the rock thrown downward will hit the ground faster. One intuition is that the initial downward motion of the rock adds on to the effects of gravity, whereas the initial motion of the rock thrown upward opposes gravity, so the rock thrown down will hit with a greater speed. In part (b), students use kinematic equations for $x(t)$ and $v(t)$ to calculate the speed of each rock at the bottom of the cliff. The correct calculations show that the rocks land with the same speed. Part (c) asks students to resolve potential disagreements in parts (a) and (b). One conceptual resolution here is that motion under gravity is symmetric: even though a ball tossed upward at 30 m/s is moving away from the final destination, when it returns to its initial height, that ball will be traveling 30 m/s downward. By demonstrating and addressing a common disagreement between conceptual reasoning and calculations, this question aims to help students bring these two forms of reasoning into alignment by developing their conceptual understanding and a tendency to seek and value such alignment.

The theme of aligning intuitions with calculation serves as a persistent thread in the mathematical sensemaking course. Even in the early weeks of the course when this homework question is given, students have seen this type of resolution modeled in class and have attempted it in class and on homework. Through this repeated practice, seeking such resolutions becomes the normal problem-solving activity in the class. Although novice students may have difficulties generating the correct physics resolution in part (c), the homework solutions present a resolution after students have tried to reach one on their own. This three-part problem structure is a good example of the type of reasoning the mathematical sensemaking instruction aims to teach. As the course goes on and explicit scaffolding fades, the hope is that students will routinely seek this type of coherence while learning physics.

2) Framing physics as independent thinking, not standard procedures: an example from the lecture

In this section, we briefly present a snippet of physics instruction from an implementation of the mathematical sensemaking instruction by an instructor who is experienced with the approach. The course was a calculus-based introductory physics course of about 65 students. The excerpt shows how a clicker question in lecture expanded to a larger discussion around students’ ideas, in a way that, we argue, supported students’ mathematical sensemaking.

The question presented Xena and Yara, standing 30 meters apart on frictionless ice. Yara has twice the mass of Xena. Xena pulls on a rope that Yara has tied around her waist. On an in-class clicker question and a previous homework question, the students addressed questions about the relative speeds of the two people after Xena tugs the rope, and where they would collide—all assuming a massless rope. They had found that Xena would move twice as fast: $V_{Xena} = 2V_{Yara.}$

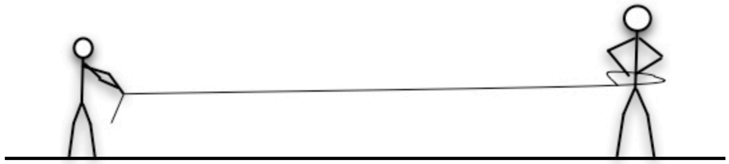

Figure 2. Xena and Yara standing on frictionless ice

At the start of the excerpt, after the clicker question and subsequent discussion, a student, Brian, asked, "In these types of problems they always say the mass of the rope doesn't matter. Why would the mass of the rope make a difference?" While it is a standard problem-solving assumption in introductory physics that is often glossed over, since the mass of the rope is often much less than the other objects in the problem, the instructor took the opportunity to promote this student's question rather than provide an immediate explanation. The instructor hadn't planned to address this scenario, but he decided to make a new clicker question on-the-fly so that the class could consider Brian's issue:

How do the post-tug speeds of Xena and Yara compare, assuming a rope of non-negligible mass?

(A) $V_{Xena} = 2V_{Yara}$
(B) $V_{Xena} > 2V_{Yara}$
(C) $V_{Xena} < 2V_{Yara}$

After students discussed the question in small groups and voted with their clickers, the conversation resumed:

Instructor: You've already talked to people, so you've heard arguments. 50% say B and then there are answers in other categories. So, let's hear arguments for B. Why would you say B? Somebody who believes B.
Oona: Well, when the rope's mass didn't matter, the force that got to Y is the same as the other [inaudible], but now the weight matters, so some of the force goes into accelerating the rope, so there's less force that gets to Y. So, the acceleration is [inaudible].
Instructor: So, some of the force that Xena exerts doesn't get to Y because some of that force goes into accelerating the rope, was that argument for B.

In his revoicing, the instructor ignores Oona's conflation of weight and mass to highlight the mechanism she expresses: because some of Xena's pulling force "goes into accelerating the rope," less force is available to act on Yara. The instructor then opens the class the further arguments, and Michael responds, building on Oona's idea of considering the rope's motion:

Michael: Um, I put C because if you look at what happens to the rope with now, non-negligible mass, as Xena and Yara move close to each other, the rope is going to move towards the point where they meet--wherever that may now be. So, whatever force is on Xena and whatever force is on Yara, doesn't just move them. It moves them and some portion of the rope.
Instructor: Ok...
Michael: And so, I'm assuming that they're still going to end up closer to Yara's starting point than Xena's. But even if they don't, even if they end up right smack dab in the middle,

if you add the mass of half of the rope to Xena, it has a greater effect than adding the mass of half the rope to Yara. Because Yara is more massive to begin with. So, the rope is less a percent of her mass. So if you divide the force on them by that new mass, it will result in a greater change in Xena's acceleration, which means that her final velocity will be less than two times Yara's final velocity.

Instructor: Awesome. Your argument, I think, if I understand your argument is--let's suppose, let's think of the rope as in two halves. And let's just assign this half of the rope to Xena and this half of the rope to Yara and let's consider these as two objects. Adding half of the mass to Yara doesn't make as much of a difference to Yara's mass as adding half of the rope to Xena makes to Xena's mass. Is your argument. So we should have more of an effect on Xena's mass, therefore more of an effect on her velocity. Wow, all right. This is good. This is good. So, so...yeah, I'm torn over what to do with this right now. But I, so, so...can you respond to an argument that you disagree with?

In these few minutes of classroom interaction, the instructor makes several instructional choices aimed to support mathematical sensemaking. First, instead of simply answering Brian's question about why assigning the rope a non-negligible mass makes a difference in these sorts of problems, he treated the question as one that could lead to productive conceptual and/or mathematical sensemaking, "promoting" it to the status of clicker question. Next, in revoicing Oona's idea, he focused attention not on the correctness or incorrectness of her account, but on her intuitive story of why the rope's mass could affect Yara's motion. This may have helped create space for Michael to speak up and express his reasoning, which "plays the same game" of considering the rope's motion and how it takes away from Yara's motion—and Xena's motion as well, by Michael's argument. Then, while revoicing Michael's argument, the instructor offers positive comments ("Awesome…Wow, all right. This is good."). This sends the message that Michael's reasoning, which blends cause-and-effect conceptual reasoning about the rope's motion with proportion-based reasoning about why half-a-rope's worth of mass added to Xena has a bigger effect (percentage-wise) than half-a-rope's worth of mass added to Yara, is valued in the class. Note that Michael's reasoning was incorrect: the center-of-mass of Xena's (or Yara's) "piece" of the rope does not accelerate at the same rate as Xena (or Yara), and hence the mass of that piece cannot simply be added to Xena's (or Yara's) mass. But supporting Michael's mathematical sensemaking took precedence in this moment, over promotion only of fully-correct physics.

This example illustrates a more general pattern in the lecture instruction under the mathematical sensemaking approach. This instruction includes not only pre-planned opportunities for mathematical sensemaking, but also (i) in-the-moment recognition of emergent opportunities for mathematical sensemaking and (ii) the broadcasting of messages that such reasoning is valued. These moves could help create a classroom climate supportive of the epistemological stance that seeking coherence between calculations and concepts is possible and productive.

References

[1] E. Kuo, M.M. Hull, A. Elby, and A. Gupta, *Assessing mathematical sensemaking in physics through calculation-concept crossover,* Physical Review Physics Education Research (Accepted).