On Deriving the Upper Bound of α-Lifetime for Large Sensor Networks

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Abstract

In this paper, we explore the fundamental limits of sensor network lifetime that all algorithms can possibly achieve. Specifically, under the assumptions that nodes are deployed as a Poisson point process with density λ in a square region with side length ℓ and each sensor can cover a unit-area disk, we first derive the necessary and sufficient condition of the node density in order to maintain complete k-coverage with probability approaching 1. With this result, we obtain that if $\lambda = \log \ell^2 + (k+2)\log\log \ell^2 + c(\ell), c(\ell) \to -\infty$, as $\ell \to +\infty$, the sensor network lifetime (for maintaining complete coverage) is upper bounded by kT with probability approaching 1 as $\ell \to +\infty$, where T is the lifetime of each sensor. Second, we derive, given a fixed node density in a finite (but reasonably large) region, the upper bounds of lifetime when only α-portion of the region is required to be covered at any time. We also carry out simulations to validate the derived results. Simulation results indicate that the derived upper bounds apply not only to networks of large sizes and homogeneous nodal distributions but also to small-size networks with clustering nodal distributions.

1 Introduction

Driven by advances in MEMS micro-sensors, wireless networking, and embedded processing, ad-hoc networks of sensors have become increasingly available for commercial and military applications such as environmental monitoring (e.g., traffic, habitat, and security), industrial sensing and diagnostics (e.g., factory and appliances), monitoring critical infrastructures (e.g., power grids, water distribution and waste disposal), and information collecting for battlefield awareness [1, 9, 13, 14].

Sensor nodes in such a network usually have limited on-board processing and wireless communication capabilities, and are equipped with batteries with limited power. Moreover, it is impractical or infeasible to replenish energy via replacing batteries on these sensors in most applications. As a result, it is well perceived that a sensor network should be deployed with reasonable density (up to 20 nodes/m^3 [19]) in order to prolong the network lifetime.

In a high-density network with energy-constrained sensors, it is neither necessary nor desirable to have all nodes operate simultaneously in the active mode. *Density control* — the function that controls the density of working sensors at a desirable level — becomes an important issue [24]. Specifically, density control ensures only a subset of sensor nodes operate in the active mode, while meeting the following two requirements: (i) *coverage:* the region that can be monitored is not smaller than the region that can be monitored by a full set of sensors; and (ii) *connectivity:* the sensor network remains connected so that the information collected by sensor nodes can be relayed back to data sinks or controllers. In addition, to prolong network lifetime, it is desirable that nodes wake up on a rotational basis and at any time only a *minimum* set of sensors wakes up to maintain coverage and connectivity. Several authors have proposed several algorithms for this purpose [25, 21, 26, 11, 22].

In this paper, instead of proposing another algorithm for selecting the set of working sensor nodes, we explore the fundamental limit of sensor network lifetime that all algorithms can possibly achieve. The derivation is based on the theory of coverage processes [12] and made under the assumptions that the locations of the deployed sensors form a Poisson point process in a square region and that sensor nodes only fail because of power depletion (but not of malicious destruction). First, we prove that a necessary and sufficient condition of complete k-coverage of a square region with side length ℓ (in the almost surely sense) is the density of the nodes $\lambda = \log \ell^2 + (k+1) \log\log \ell^2 + c(\ell)$ where $c(\ell) \to +\infty$ as $\ell \to \infty$. And consequently, given the density $\lambda = \log \ell^2 + (k+2) \log\log \ell^2 + c(\ell)$ and $c(\ell) \to -\infty$, the lifetime upper bound is kT with probability approaching 1 as $\ell \to +\infty$, where T is a single sensor lifetime. Second, we derive, given a certain density in a finite (but still large) region, the upper bounds of the lifetime under the scenario that only α-portion of the region is required to be covered. In this scenario, we derive two upper bounds: one holds universally for any possible algorithm, and the other is targeted for algorithms that attempt to completely cover the region initially and gradually reduce the coverage ratio, until it drops below a certain threshold α.

We also carry out a simulation study to validate the derived upper bounds of network lifetime and to study to what extent they can be applied to networks in which the assumptions do not hold. Simulation results indicate that the derived upper bounds apply not only to sensor networks of large sizes and with homogeneous nodal distributions, but also to networks of small sizes and with non-homogeneous (e.g., clustering) nodal distributions, although in the latter cases the derived upper bounds may not be tight.

With our derivation and simulation results, we will be able to answer several important questions, e.g., given the lifetime T of a single sensor node, how many sensor nodes (or the sensor density) have to be deployed in a region, in order to continuously monitor the region for a period of $k \cdot T$. We also observe that although it is, in general, desirable to deploy a sensor network of high density to achieve a large lifetime per unit of nodal density, the increase in the lifetime per unit of nodal density becomes marginal when the density exceeds certain threshold. The overhead incurred in maintaining coverage in a distributed manner dominates when the sensor density becomes high.

Several research efforts have been made to derive the upper bounds of network lifetime in wireless ad hoc networks and sensor networks [3, 4, 5]. We will give a detailed summary of existing work in Section 6. Our work is different from existing works in two aspects. First, we consider as the network lifetime the time interval during which at least α-portion of the region can be continuously monitored. Second, the lifetime upper bounds derived in this paper are independent of power-saving schemes used.

The rest of the paper is organized as follows. In Section 2, we state the assumptions we make on the system model, define what we mean by network lifetime in sensor networks. Then we present our derivation in Section 3 and 4. Following that, we carry out simulations in Section 5 to validate the derived results. Finally we summarize the related work in Section 6 and conclude the paper in Section 7.

2 Preliminary

To facilitate the derivation, in this section we state the assumptions we make on the system model, and define the network lifetime in sensor networks.

2.1 Assumptions on the System Model

We assume the region R to be monitored is a square region with side length ℓ. We further assume torus convention (a.k.a. the toroidal model [16]) described in [12], page 23, i.e., each disk that protrudes one side of the region R enters R again from the opposite side (Fig. 1). This eliminates consideration of boundary effects. Each sensor node can detect an event of interest within a distance of r, and this distance is termed as the *sensing range*. The disk centered at a sensor node and with a radius of r is termed as the *coverage disk* of this node. Without loss of generality, we assume that each sensor node has a sensing range of $r = \frac{1}{\sqrt{\pi}}$ and thus each sensor node can cover a disk of unit area. We assume $\ell >> r$. We assume that each sensor has the same lifetime of T. This assumption is generally made when to analyze the network lifetime, for example, in [5].

We assume the deployed sensor nodes in the square region R form a (homogeneous) Poisson point process with density λ. There are several ways of defining a Poisson point process, one of which is stated below. First, for any subset A of the region R, the distribution of the number of nodes in the set is Poisson with mean $\lambda||A||$, where $||A||$ is the area of A. Second, given that the number of nodes in such a set A is m, the node locations in A are m mutually independent random variables, each uniformly distributed over A. It is well known that n nodes whose locations are mutually independent random variables, each with uniform distribution in R, are essentially a Poisson point process with density $\lambda = n/\ell^2$ if R is large ([12], page 39).

As analytically proved in [26], if the radio transmission range is at least twice as large as the sensing range, network coverage implies connectivity. That is, as long as the set of working nodes completely covers the monitored region, the network is connected. We make this assumption so as to facilitate derivation. As tabulated in Tables 1–2, this assumption holds for most commercially available sensor devices. A study on the network lifetime when the above assumption does not hold (and hence one has to consider both coverage and connectivity in selecting the working set) is a subject of future investigation.

2.2 Definition of Sensor Network Lifetime

We define the α-lifetime as the entire interval in which at least α portion of the region R is covered by at least one sensor node, where α is a tunable parameter.

A discussion on how the α-lifetime defined above compares against the

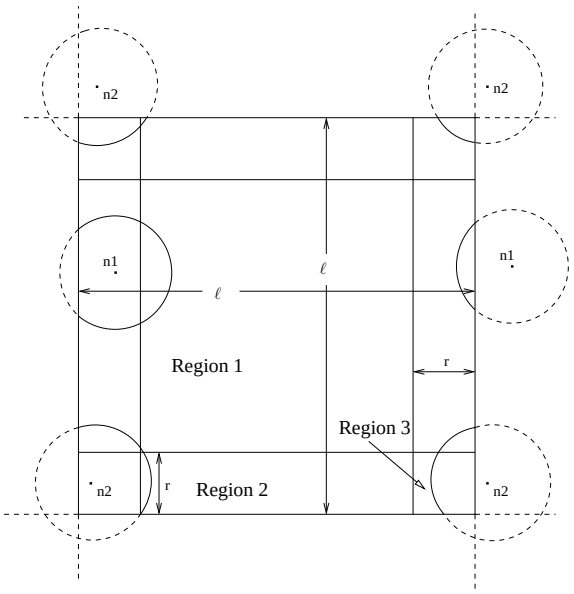


Figure 1: The toroidal model. The model can be interpreted by considering R as simply one member of a lattice of squares and assuming that all nodes are repeated in precisely the same relative positions in all squares).

lifetime defined in [5] and [24] is in order. Blough and Santi [5] defined the lifetime of sensor networks as $\min\{t_1, t_2, t_3\}$, where t_1 is the time it takes for the cardinality of the largest connected components to drop below $c_1 \cdot n(t)$, where $n(t)$ is the number of alive nodes at time t, t_2 is the time it takes for $n(t)$ to drop below $c_2 \cdot n(0)$, and t_3 is the time it takes for the area covered to drop below $c_3 \cdot \ell^2$. Here $0 \leq c_1, c_2, c_3 \leq 1$. If we set $c_1 = 0, c_2 = 0$ and $c_3 = \alpha$, then the network lifetime is exactly the same as the α-lifetime defined in this paper. Under the assumption that the radio range is at least twice as large as the sensing range (and thus network coverage implies connectivity), it makes sense to ignore the connectivity requirement imposed by t_1 and set $c_1 = 0$. The requirement imposed by t_2 is not really necessary in sensor networks, since one is usually concerned with how many sensors remain alive but with whether or not the remaining sensors can perform certain functions such as monitoring and relaying information back to data sinks.

Ye *et al.* [24] defined the lifetime as the time it takes for the coverage (defined as the ratio of the area covered by working nodes to the total area) to drop below, and never exceed again a pre-determined threshold. Due to the network dynamics, the coverage may occasionally drop below a

threshold and come back again. They take into account of the time interval when the coverage temporarily drops below the threshold in the network lifetime, while we do not.

3 Asymptotic Upper Bound of 1-Lifetime

In this section, we investigate the asymptotic lower bound on the density λ required to guarantee full coverage ($\alpha = 1$) for time kT as the monitored region $\ell \to +\infty$. This result can also be interpreted as the asymptotic 1-lifetime upper bound given the density λ of the sensor nodes. For completeness of the paper, we succinctly summarize several results of coverage processes [12] that pertain to our derivation in Appendix A.

The problem of deriving the upper bound of the asymptotic 1-lifetime is highly related to the k-coverage problem, where by k-coverage we mean every point in the monitored region is covered by at least k nodes. Let the *coverage number* denote the maximum value of k such that the sensor network has k-coverage in the monitored region. It is obvious that the coverage number k times the lifetime T of a single sensor gives a strict upper bound of the sensor network 1-lifetime.

Let the k-vacancy V_k denote the area that is covered by at most $k-1$ nodes. We need to determine the required density λ such that $P(V_k > 0) \to 0$ as $\ell \to +\infty$. Let $\chi_k(x)$ denote the indication function of whether a point x is covered by at most $k-1$ sensor nodes, i.e.,

$$\chi_k(x) = \begin{cases} 1, & \text{if at most } k-1 \text{ nodes cover} \\ & \text{the point } x, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The fact that a point x is covered by at most $k-1$ sensor nodes indicates that there are at most $k-1$ sensor nodes within the unit-area disk centered at x (recall that each sensor can cover a unit-area disk centered at itself). Under the assumption that the deployed sensors form a Poisson point process, we have

$$P(\chi_k(x) = 1) = e^{-\lambda} \left(\sum_{i=0}^{k-1} \frac{\lambda^i}{i!} \right). \tag{2}$$

Now the random variable V_k can be expressed as

$$V_k = \int_R \chi_k(x) dx. \tag{3}$$

To calculate its expectation, we use Fubini's theorem [17] and exchange the order of integral and expectation, i.e.,

$$\begin{aligned} E(V_k) &= \int_R E(\chi_k(x))dx \\ &= \int_R P(\chi_k(x)=1)\ dx \\ &= ||R||P(\chi_k(x)=1) \\ &= \ell^2 e^{-\lambda}\left(\sum_{i=0}^{k-1}\frac{\lambda^i}{i!}\right), \end{aligned} \tag{4}$$

where the third equality results from the fact that $P(\chi_k(x)=1)$ is a constant for all x.

In order to ensure complete coverage for the duration of kT, each point should be covered by at least k nodes, which implies $V_k = 0$. As nodes form a Poisson point process in the region R, it cannot be guaranteed that this always occurs with a finite density λ, no matter how large λ is. However, with $\lambda \to +\infty$ as $\ell \to +\infty$ we can ensure this occurs almost surely, i.e. $P(V_k = 0) \to 1$ as $\ell \to +\infty$. This ensures complete k-coverage almost surely (see Remark 3 after the proof of the theorem below).

In what follows, we establish a tight bound on the density λ that ensures asymptotic complete k-coverage.

Theorem 1 *Let* $\lambda = \log \ell^2 + (k+1)\log\log \ell^2 + c(\ell)$. *If* $c(\ell) \to +\infty$ *as* $\ell \to +\infty$, *then* $P(V_k > 0) \to 0$. *If* $c(\ell) \le C < +\infty$, *then* $P(V_k > 0) \ge \epsilon$ *as* $\ell \to +\infty$, *where* $\epsilon = 1/(1 + 4e^C(k+1)!)$.

Proof. First we prove if $c(\ell) \to +\infty$ as $\ell \to +\infty$, $P(V_k > 0) \to 0$. Clearly if the value of λ increases, $P(V_k > 0)$ will decrease. Hence we assume that $c(\ell) = o(\log \ell^2)$.

Define a *crossing* to be either an intersection point of the boundaries of two disks or an intersection point of the boundary of a disk and the boundary of region R. A crossing is said to be *k-covered* if it is an interior point of at least k disks. By Theorem 4 in [22], region R is completely k-covered if there exist crossing points and every crossing point is k-covered. Equivalently, if R is not completely k-covered and there exist crossings, some of the crossings are not k-covered.

With $\lambda \to +\infty$ as $\ell \to +\infty$ and $\pi r^2 = 1$, we can write

$$P(V_k > 0) = p_1 + p_2 + p_3, \tag{5}$$

where

$$\begin{aligned} p_1 &\equiv P(\text{no disk is centered within } R) \\ &= \exp(-\lambda\ell^2) \to 0, \end{aligned} \tag{6}$$

$$\begin{aligned} p_2 &\equiv P(\text{at least one disk is centered within } R, \text{ but} \\ &\quad \text{none of the disks intersects any other disk and} \\ &\quad \text{none of the disks intersect the boundary of } R) \\ &\leq P(\text{at least one disk is centered within } R) \\ &\quad \times P(\text{a given disk intersects no other disks}) \\ &= (1 - \exp(-\lambda\ell^2)) \times \exp(-\lambda\pi(2r)^2) \\ &\leq \exp(-4\lambda) \to 0, \end{aligned} \tag{7}$$

and

$$\begin{aligned} p_3 &\equiv P(R \text{ is not completely } k\text{-covered, at least} \\ &\quad \text{one disk is centered within } R, \text{ and at least} \\ &\quad \text{two disks intersect each other or at least} \\ &\quad \text{one disk intersects the boundary of } R). \end{aligned} \tag{8}$$

Therefore

$$P(V_k > 0) \to p_3 \text{ as } \ell \to +\infty. \tag{9}$$

Next we derive an upper bound of p_3.

If R is not completed k-covered, if one or more disks are centered within R, and if there exist crossings in R, then at least one of the disks has two or more un-k-covered crossings on its boundary. Let M_k denote the number of crossings that are not k-covered. Then we have

$$p_3 \leq P(M_k \geq 2) \leq E(M_k)/2. \tag{10}$$

We first consider crossings created by two disks intersecting each other. The expected number, D, of nodes in region R is $\lambda\ell^2$. If any two nodes are within a distance of $2r$ from each other, their coverage disks intersect. Hence, the expected number of crossings created by a given node is $2\lambda\pi(2r)^2$. Since each crossing is counted twice, the expected value of the total number, N_1, of crossings created by two disks intersecting each other is given by

$$E(N_1) = \lambda\ell^2 \cdot \lambda\pi(2r)^2 = 4\lambda^2\ell^2. \tag{11}$$

Now we consider crossings created by a disk intersecting the boundary of region R. If a node is within a distance of r to the boundary of region R,

at most two crossings will be created, except when the node is located on the corner of region R (e.g., Region 3 in Fig. 1). In that case, at most 4 crossings will be created. Hence the expected value of the total number, N_2, of crossings created by a disk intersecting the boundary of region R is given by

$$E(N_2) \leq 8\lambda\ell r. \tag{12}$$

Recall that the probability that a given crossing is not k-covered is $e^{-\lambda}\sum_{i=0}^{k-1}\lambda^i/i!$ (Eq. (2)). By Eqs. (11) and (12), we have

$$\begin{aligned} E(M_k) &= (E(N_1)+E(N_2))\cdot e^{-\lambda}\sum_{i=0}^{k-1}\frac{\lambda^i}{i!} \\ &\leq (4\lambda^2\ell^2+8\lambda\ell r)e^{-\lambda}\sum_{i=0}^{k-1}\frac{\lambda^i}{i!} \\ &= 4\lambda^2\ell^2(1+o(1))\cdot e^{-\lambda}\sum_{i=0}^{k-1}\frac{\lambda^i}{i!}. \end{aligned} \tag{13}$$

Since $\lambda \to +\infty$ as $\ell \to +\infty$, by Eqs. (10) and (13) we have

$$\begin{aligned} p_3 &\leq 2\lambda^2\ell^2(1+o(1))e^{-\lambda}\sum_{i=0}^{k-1}\frac{\lambda^i}{i!} \\ &= 2\ell^2 e^{-\lambda}\frac{\lambda^{k+1}}{(k-1)!}(1+o(1)). \end{aligned} \tag{14}$$

If $\lambda = \log\ell^2 + (k+1)\log\log\ell^2 + c(\ell)$, then $\lambda_1 \equiv \lambda - \log\ell^2 = (k+1)\log\log\ell^2 + c(\ell)$. By the reasoning at the beginning of the proof, we can assume $c(\ell) = o(\log\ell^2)$ as $\ell \to +\infty$, and hence $\lambda_1 = o(\log\ell^2)$. This gives $\lambda^{k+1} = (\log\ell^2)^{k+1}(1+o(1))$, and hence

$$\begin{aligned} p_3 &\leq \frac{2\lambda^{k+1}}{(\log\ell^2)^{k+1}e^{c(\ell)}(k-1)!}(1+o(1)) \\ &= \frac{2(1+o(1))}{e^{c(\ell)}(k-1)!}. \end{aligned} \tag{15}$$

Since $c(\ell) \to +\infty$ as $\ell \to +\infty$, $p_3 \to 0$ as $\ell \to +\infty$. The first part is proved.

Now we prove that if $c(\ell) \leq C$ for some finite C as $\ell \to +\infty$, $P(V_k > 0) \geq \epsilon$ for $\epsilon = 1/(1+4e^C(k+1)!)$. By the Cauchy-Schwartz inequality, we have

$$E(V_k) = E(V_k I(V_k > 0))$$

$$\begin{aligned} &\leq \left(E(V_k^2)E(I^2(V_k>0))\right)^{1/2} \\ &= \left(E(V_k^2)P(V_k>0)\right)^{1/2}, \end{aligned} \tag{16}$$

and

$$P(V_k>0) \geq \frac{(EV_k)^2}{E(V_k^2)}. \tag{17}$$

Eq. (4) gives the expression of $E(V_k)$. We now derive the bound of $E(V_k^2)$. By definition, we have

$$\begin{aligned} E(V_k^2) &= E\left(\int\int_{R^2} \chi_k(x_1)\chi_k(x_2)dx_1dx_2\right) \\ &= \int\int_{R^2} E\left(\chi_k(x_1)\chi_k(x_2)\right)\, dx_1dx_2 \\ &\equiv I_1 + I_2, \end{aligned} \tag{18}$$

where

$$\begin{aligned} I_1 &\equiv \int\int_{R^2\cap\{|x_1-x_2|>2r\}} E\left(\chi_k(x_1)\chi_k(x_2)\right) dx_1dx_2 \\ I_2 &\equiv \int\int_{R^2\cap\{|x_1-x_2|\leq 2r\}} E\left(\chi_k(x_1)\chi_k(x_2)\right) dx_1dx_2. \end{aligned}$$

For $|x_1 - x_2| > 2r$,$\chi_k(x_1)$ and $\chi_k(x_2)$ are independent, and $E(\chi_k(x)) = e^{-\lambda}\sum_{i=0}^{k-1}\lambda^i/i!$ for all x. Hence,

$$\begin{aligned} I_1 &\equiv \int\int_{R^2\cap\{|x_1-x_2|>2r\}} E\left(\chi_k(x_1)\chi_k(x_2)\right) dx_1dx_2 \\ &= \int\int_{R^2\cap\{|x_1-x_2|>2r\}} E\chi_k(x_1)E\chi_k(x_2)dx_1dx_2 \\ &\leq \int\int_{R^2} E\chi_k(x_1)E\chi_k(x_2)dx_1dx_2 \\ &= \left(\ell^2 e^{-\lambda}\sum_{i=0}^{k-1}\frac{\lambda^i}{i!}\right)^2 = \left(E(V_k)\right)^2. \end{aligned} \tag{19}$$

What is left is the derivation of I_2. Let B_1 and B_2 denote the unit-area disks centered at x_1 and x_2, respectively. If $|x_1 - x_2| = x \leq 2r$ and x_1 and x_2 are given, then

$$\begin{aligned} &E\left(\chi_k(x_1)\chi_k(x_2)\right) \\ &\quad = P(\text{Both } B_1 \text{ and } B_2 \text{ contain less than } k \text{ nodes}) \end{aligned}$$

$$
\begin{aligned}
&\leq P(B_1 \text{ contains less than } k \text{ nodes}, B_2 - B_1 \\
&\quad \text{contains less than } k \text{ nodes}) \\
&= P(B_1 \text{ contains less than } k \text{ nodes}) \times \\
&\quad P(B_2 - B_1 \text{ contains less than } k \text{ nodes}) \qquad (20)
\end{aligned}
$$

The last equality results from the fact that B_1 and $B_2 - B_1$ are disjoint and thus the number of nodes that are located in them are independent (under the Poisson point process assumption).

$$
P(B_1 \text{ contains less than } k \text{ nodes}) = e^{-\lambda} \sum_{i=0}^{k-1} \frac{\lambda^i}{i!}. \qquad (21)
$$

Let $B(u)$ denote the intersection area of the two unit-area disks whose centers are $2u$ apart. Then,

$$
B(u) = 4 \int_u^1 (1 - y^2)^{1/2} dy = \pi - 4 \int_0^u (1 - y^2)^{1/2} dy \qquad (22)
$$

Now the second term of Eq. (22) can be expressed as

$$
\begin{aligned}
\int_0^u (1 - y^2)^{1/2} dy &= (u/2)\{u^{-1} \arcsin u + (1 - u^2)^{1/2}\} \\
&\geq (u/2) \arcsin 1 = (\pi/4)u, \qquad (23)
\end{aligned}
$$

since $u^{-1} \arcsin u + (1 - u^2)^{1/2}$ is decreasing on (0,1). Hence the area of $B_2 - B_1$ is

$$
\begin{aligned}
||B_2 - B_1|| &= r^2(\pi - B(x/(2r))) \\
&\geq r^2 \cdot 4(\pi/4) \cdot x/(2r) = x/(2r).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&P(B_2 - B_1 \text{ contains less than } k \text{ nodes}) \\
&= e^{-\lambda ||B_2 - B_1||} \sum_{i=0}^{k-1} \frac{(\lambda ||B_2 - B_1||)^i}{i!} \\
&\leq e^{-\lambda x/(2r)} \sum_{i=0}^{k-1} \frac{(\lambda x/(2r))^i}{i!}, \qquad (24)
\end{aligned}
$$

since $e^{-x} \sum_{i=0}^{k-1} x^i/i!$ is decreasing on $[0, +\infty)$.

By Eqs. (20), (21) and (24), we can express I_2 as

$$
I_2 \equiv \int\int_{R^2 \cap \{|x_1 - x_2| \leq 2r\}} E\{\chi_k(x_1)\chi_k(x_2)\} dx_1 dx_2
$$

$$
\begin{aligned}
&\leq \int_R dx_1 \int_0^{2r} \left(e^{-\lambda} \sum_{i=0}^{k-1} \frac{\lambda^i}{i!} \right) \\
&\quad \cdot \left(e^{-\lambda x/(2r)} \sum_{i=0}^{k-1} \frac{(\lambda x/(2r))^i}{i!} \right) 2\pi x dx \\
&= \ell^2 \left(e^{-\lambda} \sum_{i=0}^{k-1} \frac{\lambda^i}{i!} \right) \left(\int_0^1 e^{-\lambda u} \sum_{i=0}^{k-1} \frac{(\lambda u)^i}{i!} 8u du \right), \qquad (25)
\end{aligned}
$$

where the last equality is obtained by changing variable $u = x/(2r)$. The third factor in Eq. (25) can be further simplified as follows.

$$
\begin{aligned}
&\int_0^1 e^{-\lambda u} \sum_{i=0}^{k-1} \frac{(\lambda u)^i}{i!} \cdot 8u du \\
&\leq \int_0^{+\infty} e^{-\lambda u} \sum_{i=0}^{k-1} \frac{(\lambda u)^i}{i!} \cdot 8u du \\
&= \int_0^{+\infty} e^{-\lambda u} \sum_{i=0}^{k-1} \frac{\lambda^i u^{i+1}}{i!} \cdot 8 du \\
&= \sum_{i=0}^{k-1} \frac{\lambda^{-2}\Gamma(i+2)}{i!} \cdot 8 \\
&= \lambda^{-2} \sum_{i=0}^{k-1} (i+1) \cdot 8 \\
&= 4k(k+1)\lambda^{-2}. \qquad (26)
\end{aligned}
$$

Hence we have

$$
I_2 \leq 4k(k+1)\lambda^{-2}\ell^2 \left(e^{-\lambda} \sum_{i=0}^{k-1} \frac{\lambda^i}{i!} \right). \qquad (27)
$$

Combining Eqs. (17), (18), (19) and (27), we have

$$
\begin{aligned}
&P(V_k > 0) \\
&= \frac{E(V_k)^2}{E(V_k^2)} \\
&\geq \frac{(EV_k)^2}{(EV_k)^2 + 4k(k+1)\lambda^{-2}\ell^2 \left(e^{-\lambda} \sum_{i=0}^{k-1} \frac{\lambda^i}{i!} \right)} \\
&\equiv \frac{1}{1+\beta}, \qquad (28)
\end{aligned}
$$

where

$$\begin{aligned}
\beta &\equiv \frac{4k(k+1)\lambda^{-2}\ell^2\left(e^{-\lambda}\sum_{i=0}^{k-1}\frac{\lambda^i}{i!}\right)}{(EV_k)^2} \\
&= \frac{4k(k+1)\lambda^{-2}\ell^2\left(e^{-\lambda}\sum_{i=0}^{k-1}\frac{\lambda^i}{i!}\right)}{\left(\ell^2 e^{-\lambda}\sum_{i=0}^{k-1}\frac{\lambda^i}{i!}\right)^2} \\
&= \frac{4k(k+1)\lambda^{-2}}{\ell^2 e^{-\lambda}\sum_{i=0}^{k-1}\frac{\lambda^i}{i!}} \\
&\leq \frac{4k(k+1)}{\ell^2 e^{-\lambda}\lambda^{k+1}/(k-1)!}.
\end{aligned}$$

Let $\lambda_1 \equiv \lambda - \log \ell^2 = (k+1)\log\log \ell^2 + c(\ell)$. By the assumption $c(\ell) \leq C$, with sufficiently large ℓ, we have $\lambda_1 > 0$, and

$$\beta \leq \frac{4e^{c(\ell)}(\log \ell^2)^{k+1}(k+1)!}{(\log \ell^2 + \lambda_1)^{k+1}} \leq 4e^C(k+1)!. \tag{29}$$

It then follows from Eqs. (28) and (29) that

$$P(V_k > 0) \geq \frac{(EV_k)^2}{E(V_k^2)} \geq \frac{1}{1+4e^C(k+1)!}. \tag{30}$$

This completes the proof. □

Remark 1 If we let $c(\ell) \to -\infty$, and $\lambda = \log \ell^2 + (k+1)\log\log \ell^2 + c(\ell)$, we can conclude $P(V_k > 0) \to 1$ as $\ell \to +\infty$.

Remark 2 If we let $-(k+1)\log\log \ell^2 \leq c(\ell) \leq C$, then $P(V_k > 0) \geq 1/(1+4e^C(k+1)!)$ is true for any finite ℓ, since the second part of the proof does not require any asymptotic property in this case.

Remark 3 The terms "complete k-coverage" and "$V_k = 0$" have been used interchangeably, as it has been proved (for the case of $k = 1$) in [12] that the probability of their difference goes to 0 if the region is open and the coverage shape (e.g., the disk in the paper) is closed. It has also been stated in [12] that the same conclusion holds for any regular region and shape. (The interested reader is referred to the discussions following Theorem 3.3 in [12]). The proof can also be extended to the case of any finite k.

The following corollary is an obvious consequence of Theorem 1 and Remark 1.

Corollary 1 *If $\lambda = \log \ell^2 + (k+2)\log\log \ell^2 + c(\ell)$, and $c(\ell) \to -\infty$ as $\ell \to +\infty$, then the upper bound of the 1-lifetime is kT with probability approaching 1, where T is the lifetime of each sensor.*

It is interesting to observe from Corollary 1 that the node density required to achieve a 1-lifetime of kT is not equal to k times the required density for asymptotic coverage. As a matter of fact, the former is much smaller than the latter. This is because with a larger node density, one can make better use of coverage areas of sensor nodes. This trend will be confirmed again in the following sections.

In many cases, it may not be necessary to require $P(V_k > 0) \to 0$. One way of relaxing the requirement is to derive the density requirement for $E(V_k) \to 0$ as $\ell \to +\infty$. We give a tight lower bound for this in the following theorem.

Theorem 2 *Let $\lambda = \log \ell^2 + (k-1)\log\log \ell^2 + c(\ell)$. If $c(\ell) \to +\infty$ as $\ell \to +\infty$, then $E(V_k) \to 0$; if $c(\ell) \le C < +\infty$, then $E(V_k) \ge e^{-C}/(k-1)!$ as $\ell \to +\infty$.*

Proof. Since $E(V_k)$ decreases as λ increases, we can assume $c(\ell) = o(\log\log \ell^2)$ in the first case ($\ell \to +\infty$) and $c(\ell) = C$ in the second case ($c(\ell) \le C$). Thus in both cases we have $\lambda \to +\infty$ as $\ell \to +\infty$ and $\lambda^i = o(\lambda^{i+1})$. Let $\lambda_1 \equiv \lambda - \log \ell^2 = (k-1)\log\log \ell^2 + c(\ell)$. When ℓ is sufficiently large, we have $\lambda_1 > 0$ and $\lambda_1 = o(\log \ell^2)$. Therefore,

$$\begin{aligned}
E(V_k) &= \ell^2 \exp(-\lambda)\left(\sum_{i=0}^{k-1}\frac{\lambda^i}{i!}\right) \\
&= \ell^2 \exp(-\lambda)\left(\frac{\lambda^{(k-1)}}{(k-1)!}(1+o(1))\right) \\
&= \ell^2 \exp(-\lambda_1 - log\ell^2)\left(\frac{\lambda^{(k-1)}}{(k-1)!}(1+o(1))\right) \\
&= \exp(-\lambda_1)\left(\frac{\lambda^{(k-1)}}{(k-1)!}(1+o(1))\right) \\
&= \exp(-((k-1)\log\log \ell^2 + c(\ell))) \\
&\quad \left(\frac{(\log \ell^2)^{k-1}(1+o(1))}{(k-1)!}(1+o(1))\right) \\
&= \exp(-c(\ell))\left(\frac{1+o(1)}{(k-1)!}\right). \qquad (31)
\end{aligned}$$

If $c(\ell) \to +\infty$ as $\ell \to +\infty$, we have $E(V_k) \to 0$ since k is finite. If $c(\ell) \le C < +\infty$ as $\ell \to +\infty$, we have $E(V_k) \ge e^{-C}/(k-1)!$ as $\ell \to +\infty$. □.

Remark If we let $c(\ell) \to -\infty$, we have $E(V_k) \to +\infty$ as $\ell \to +\infty$.

4 Upper Bound of α-Lifetime in Finite Regions

The asymptotic upper bound of the 1-lifetime derived in Section 3 gives the required node density in order to achieve complete coverage as the monitored area grows to infinity ($\ell \to \infty$). However in practice one may be more interested in knowing how many nodes should be deployed (or, equivalently, what is the node density) in order to achieve the α-lifetime in a finite region. Results derived in Section 3 cannot be directly applied to answer this question, as they are derived for complete coverage for infinitely large regions.

In this section, we consider the α-lifetime in a finite region with a finite density of sensor nodes, where $0 < \alpha < 1$ and usually α is close to 1. We derive two bounds: (i) an upper bound of α-lifetime for a special family of algorithms in which the entire region is completely covered initially, and the coverage ratio is gradually reduced until it drops below a certain threshold α; and (ii) an upper bound of α-lifetime that applies to algorithms that maintain the coverage ratio of α from the beginning of network deployment. The second bound applies to any algorithm.

4.1 Upper Bound of α-Lifetime for a Special Family of Algorithms

We first derive the upper bound of α-lifetime for the family of algorithms that intend to completely cover the region initially and gradually reduce the coverage ratio, until it drops below a certain threshold α.

We can divide the entire region R into several sub-regions $R_0, R_1, \cdots, R_n$, where all points in R_i are exactly covered by i sensor nodes (Fig. 2). Thus $V_k = \sum_{i=0}^{k-1} ||R_i||$ and $1 - V_k/\ell^2$ is the portion of the region in which each point is covered by at least k nodes. We can also divide the network lifetime into rounds with the duration of each round set to T. In each round, a minimum set of nodes which are not chosen in previous rounds and have maximum coverage is chosen to operate. Thus after k rounds, the maximum possible coverage ratio is at most $1 - V_{k+1}/\ell^2$. Clearly, if $\alpha > 1 - V_{k+1}/\ell^2$, the sensor network can not provide coverage ratio α any more. Thus the upper bound of α-lifetime is

$$L(\lambda, \alpha) = \max\{k : \alpha \leq 1 - V_k/\ell^2\} \cdot T. \tag{32}$$

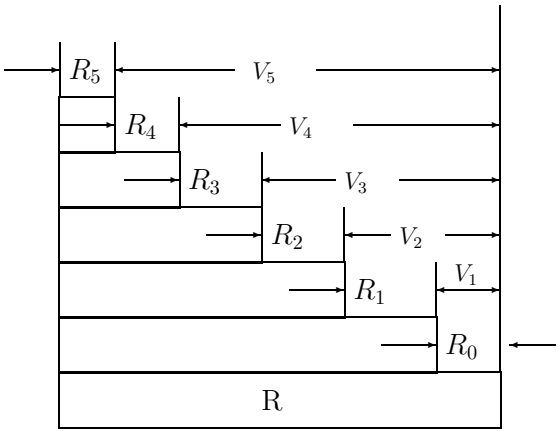


Figure 2: The entire region R can be divided into different sub-regions: $R_0, R_1, \cdots, R_n$, where all points in R_i are exactly covered by i nodes.

As V_k's are random variables whose distributions are difficult, if not impossible, to obtain. For analysis tractability, we use $E(V_k)$ to approximate V_k and hence the resulting α-lifetime can be regarded as the average α-lifetime:

$$G(\lambda, \alpha) = \max\{k : \alpha \leq 1 - F(k, \lambda)\} \cdot T, \tag{33}$$

where

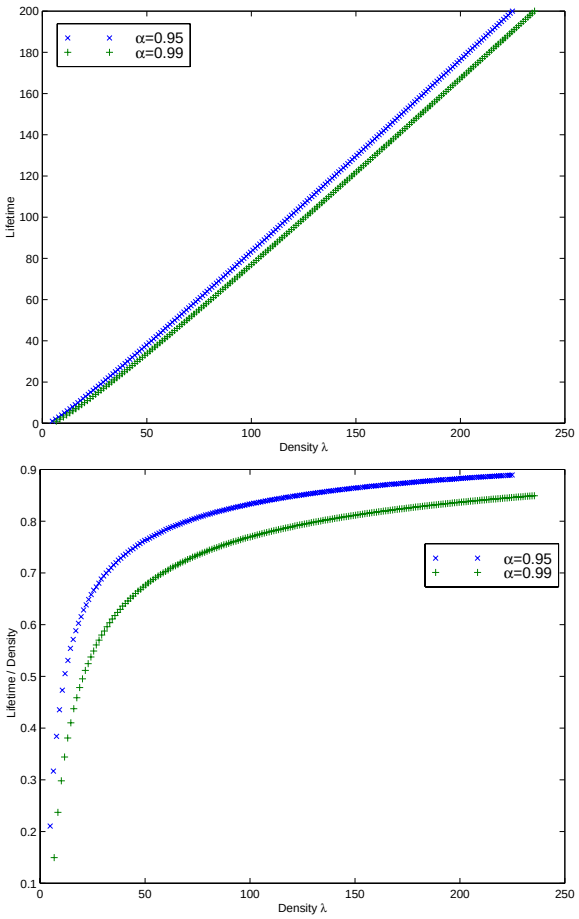
$$F(k, \lambda) = E(V_k)/\ell^2 = \exp(-\lambda) \left(\sum_{i=0}^{k-1} \frac{\lambda^i}{i!} \right). \tag{34}$$

As a matter of fact, $G(\lambda, \alpha)$ is not the expectation of $L(\lambda, \alpha)$. However, we prove in the following theorem that it suffices to approximate V_k with $E(V_k)$ in regions of large area.

Theorem 3 *As $\ell \to +\infty$ and $n/\ell^2 \to \lambda$, $V_k/\ell^2 \to F(k, \lambda)$ almost surely, where $F(k, \lambda)$ is defined in Eq. (34).*

Proof. Refer to Appendix B for the proof.

Numerical examples Figure 3 depicts the average α-lifetime $G(\lambda, \alpha)$ and the average α-lifetime per unit density $G(\lambda, \alpha)/\lambda$ versus the minimum density λ required to achieve the average α-lifetime of kT under the cases of $\alpha = 0.95$ and 0.99, where we let $T = 1$. The average α-lifetime per unit



(a) Average α-lifetime v.s. density (b) Average α-lifetime per unit of density v.s. the density

Figure 3: Average α-lifetime and α-lifetime per unit of density versus node density.

density increases as the node density λ increases. This coincides with our earlier observation in the asymptotic case that a higher density of sensors leads to better utilization of coverage areas.

4.2 Upper Bound of Lifetime for All Algorithms

Several sensor network applications do not require that full coverage be maintained. Instead it is sufficient to maintain the coverage ratio above a certain threshold α throughout the network lifetime. In this case, energy can be saved by maintaining α-coverage since the network is initialized. In this section, we derive the upper bound of the network lifetime in this case. Note that this upper bound can be applied to all algorithms that maintain α-coverage. For analysis tractability, again we use $E(V_k)$ to approximate V_k. The following theorem establishes the upper bound of the lifetime.

Theorem 4 *Let $\gamma_i \triangleq 1 - E(V_i)/l^2$ and $\beta_i \triangleq \gamma_i - \gamma_{i+1}$. Then the upper bound of α-lifetime for a sensor network with density λ is*

$$\left\lfloor \min_{k:\alpha>\gamma_k} H(k,\alpha) \triangleq \frac{\sum_{i=1}^{k-1} i\beta_i}{\alpha - \gamma_k} \right\rfloor \cdot T, \tag{35}$$

where $\lfloor x \rfloor$ is the maximum integer that is less than or equal to x.

Proof. We still divide the entire region R into different sub-regions $R_0, R_1, \cdots$, where all points in R_i are exactly covered by i nodes. By definition, γ_k represents the portion of region R that is covered by at least k nodes and β_k represents the portion of region R that is covered by exactly k nodes. Thus $\beta_k = ||R_k||/||R||$.

For each k such that $\gamma_k < \alpha$, in each round of time T, the working nodes must cover α portion of the region R, among which at least $\alpha - \gamma_k$ portion must come from $R_1 \cup \cdots \cup R_{k-1}$ since $\cup_{i \geq k} R_i$ can provide at most γ_k coverage (and R_0 is not covered by any node). On the other hand, for each $i < k$, the total coverage contribution of region R_i throughout the lifetime is at most $i\beta_i$ (since it can provide β_i portion of coverage for i rounds). Hence, the total amount of coverage $R_1, R_2, \cdots, R_{k-1}$ can contribute throughout the lifetime over all rounds is $\sum_{i=1}^{k-1} i\beta_i$. Therefore, the maximum lifetime is upper bounded by

$$\left\lfloor \frac{\sum_{i=1}^{k-1} i\beta_i}{\alpha - \gamma_k} \right\rfloor \cdot T. \tag{36}$$

Since this is true for every k such that $\alpha > \gamma_k$, the α-lifetime is upper bounded by Eq. (35). □

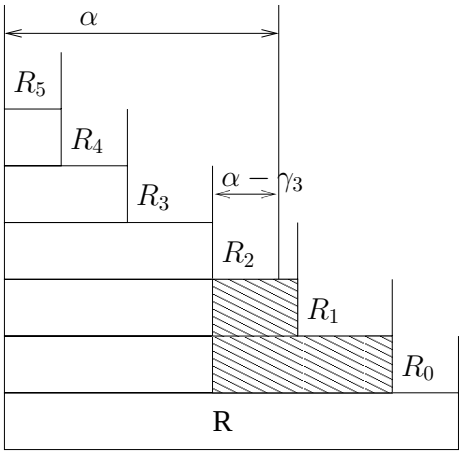


Figure 4: In each round, $\alpha-\gamma_3$ portion of the region must come from region R_2 and R_1 to ensure α-coverage. The total lifetime "contribution" R_1 and R_2 can make over all rounds is $\beta_1 + 2\beta_2$. Hence the α-lifetime is upper bounded by $(\beta_1 + 2\beta_2)/(\alpha - \gamma_3)$.

As an example, as shown in Figure 4, in each round, $\alpha-\gamma_3$ portion of the region must come from region R_2 and R_1 to ensure α-coverage since $\alpha > \gamma_3$. The total lifetime "contribution" R_1 and R_2 can make over all rounds is $\beta_1 + 2\beta_2$. Hence the α-lifetime is upper bounded by $(\beta_1 + 2\beta_2)/(\alpha - \gamma_3)$.

Recall in the proof of Theorem 4, in each round we divide the entire region into two sub-regions. In the first sub-region, each point is covered by at least k nodes and in the second sub-region, each point is covered by at most $k-1$ nodes. The proof of Theorem 4 only considers the limit implied by the second sub-region. In what follows, we prove that the first sub-region can always provide γ_k portion coverage for at least $\lfloor H(k,\alpha) \rfloor$ rounds for the k that minimizes $H(k,\alpha)$.

Theorem 5 *Let* $k = \arg\min_{i:\alpha>\gamma_i} H(i,\alpha)$*, then*

$$k \geq H(k,\alpha).$$

Proof. To facilitate the proof, we first give several nice properties of $H(k,\alpha)$ in the next lemma.

Lemma 1 *For all* k *such that* $\alpha > \gamma_k$*,* $H(k,\alpha)$ *given in Eq. (35) has the following properties:*

(i) *If* $H(k,\alpha) > k$*,* $H(k,\alpha)$ *monotonically decreases as* k *increases;*

(ii) *If $H(k,\alpha) < k$, $H(k,\alpha)$ monotonically increases as k increases;*

(iii) *If $H(k,\alpha) = k$, then $H(k,\alpha) = H(k+1,\alpha)$;*

(iv) *If $H(k,\alpha) > k$, then $H(k+1,\alpha) > k$;*

(v) *If $H(k,\alpha) = k$, then $H(k+1,\alpha) = k$;*

(vi) *If $H(k,\alpha) < k$, then $H(k+1,\alpha) < k$.*

Proof. Refer to Appendix C.

Now since $k = \arg\min_{i:\alpha>\gamma_i} H(i,\alpha)$, if $H(k,\alpha) > k$, by property (i) in Lemma 1, $H(k,\alpha) > H(k+1,\alpha)$. Since $\gamma_{k+1} < \gamma_k < \alpha$, this contradicts our assumption that $k = \arg\min_{i:\alpha>\gamma_i} H(i,\alpha)$. So $H(k,\alpha) \leq k$. □

The above theorem can be used to prove that the lifetime upper bound given in Theorem 4 is tight. This is given in the next Corollary.

Corollary 2 *The upper bound in Theorem 4 is tight if we can arbitrarily select the region(s) to be covered to provide exactly α-coverage in each round.*

Proof. In each round, we select all the regions that are covered by at least k nodes (which is γ_k portion of the region R). In addition, we select $\alpha - \gamma_k$ portion in the regions that are covered by less than k nodes. Then in each round α-coverage is achieved and at least $\lfloor H(k,\alpha) \rfloor$ rounds can be supported because $H(k,\alpha) \leq k$. □

Let $H_o(\lambda,\alpha) \triangleq \lfloor \min_{k:\alpha\geq\gamma_k} H(k,\alpha) \rfloor$, where $H(k,\alpha)$ is given in Eq. (35). Lemma 1 also suggests a quick method for finding H_o, which is given in the following Corollary.

Corollary 3 **(i)** $H_o(\lambda,\alpha) = \max_{\alpha>\gamma_k}\{k : H(k,\alpha) \geq k\}$,

(ii) $H_o(\lambda,\alpha) = \min_{\alpha>\gamma_k}\{k : H(k,\alpha) < k\} - 1$.

Proof. (i) By Lemma 1 (i), (v) and (vi), $j \equiv \max\{k : H(k,\alpha) \geq k\}$ exists and is unique. We consider two cases: $H(j,\alpha) = j$ and $H(j,\alpha) > j$.

If $H(j,\alpha) = j$, by the definition of j, $H(k,\alpha) < k$ for any $k > j$. By Lemma 1 (ii), (iii), and induction on k, $H(k,\alpha) \geq H(j,\alpha)$ for all $k > j$. For any $k < j$ such that $\alpha > \gamma_k$, we claim that $H(k,\alpha) > k$. Otherwise $H(k,\alpha) \leq k$. By Lemma 1 (v), (vi), and induction on all $m \geq k$, $H(m+1,\alpha) \leq m$. Thus $H(j,\alpha) < j$ and this contradicts the definition of j. Since $H(k,\alpha) > k$ for any $k < j$ such that $\alpha > \gamma_k$, by Lemma 1 (i) and induction, $H(k,\alpha) > H(j,\alpha)$. Thus, $H(j,\alpha) = j$ is the minimum of $H(k,\alpha)$ for all k such that $\alpha > \gamma_k$.

If $H(j,\alpha) > j$, by the definition of j, $H(k,\alpha) < k$ for all $k > j$. By Lemma 1 (ii) and induction on k, $H(k,\alpha) > H(j+1,\alpha)$ for all $k > j+1$. For any $k \leq j$ such that $\alpha > \gamma_k$, we claim that $H(k,\alpha) > k$. Otherwise, $H(k,\alpha) \leq k$. By Lemma 1 (v), (vi) and induction on all $m \geq k$, $H(m+1,\alpha) \leq m$. Hence $H(j,\alpha) < j$ and contradicts the definition of j again. So we can see that $H(j+1,\alpha)$ is the minimum of $H(k,\alpha)$ for all k such that $\alpha > \gamma_k$. In addition since $H(j,\alpha) > j$, by Lemma 1 (iv), $H(j+1,\alpha) > j$. By the definition of j, $H(j+1,\alpha) < j+1$. As a result, $H_o(\lambda,\alpha) = \lfloor H(j+1,\alpha) \rfloor = j$.

The proof for (ii) is similar, and is thus omitted. □

Numerical examples Figure 5 gives the upper bound of the lifetime derived in Section 4.1 and that in this subsection, and their respective lifetime per unit of density. As compared with the upper bound of the lifetime derived in Section 4.1, the "universal" upper bound of the lifetime increases by 15% for 99%-coverage and over 20% for 95%-coverage. The upper bound of the lifetime per unit density increases as the density increases in general, and slightly decreases at certain density values. This is because the upper bound of the lifetime does not change for the slight increase in the node density λ. It is not surprising to observe that the lifetime per unit density can be more than 1 in some cases, because less than 100%-coverage is required in each round.

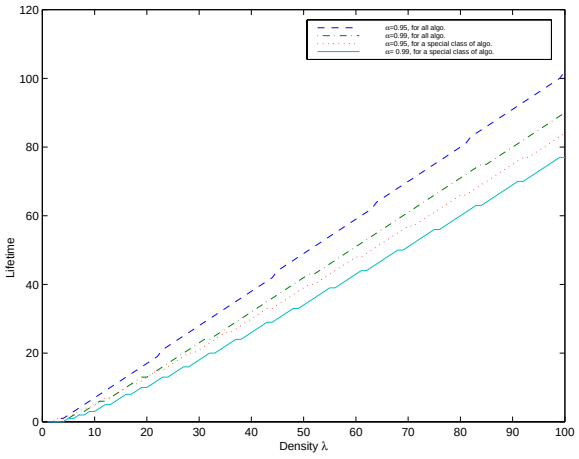
Another interesting finding is that although it is, in general, desirable to deploy sensors with high density to achieve a large lifetime per unit of nodal density, the increase in the lifetime per unit of nodal density levels off when the density exceeds certain threshold. The overhead incurred in maintaining coverage in a distributed manner dominates when the sensor density becomes very high.

5 Simulation Study

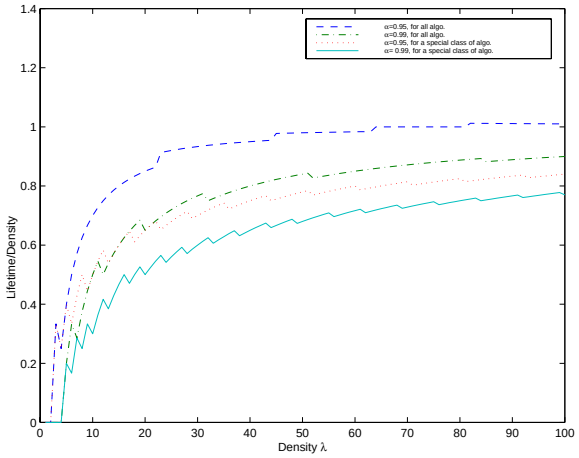
In this section, we carry out several sets of simulations to validate the theoretical lifetime upper bound. Since it is difficult, if not impossible, to simulate 1-lifetime for the infinite dimension case, our simulation study focuses on validating the upper bound of α-lifetime in a reasonably large area.

5.1 Simulation Methodology

We use N independently and randomly distributed disks with uniform distribution to approximate a Poisson point process. In a square region with



(a) Average α-lifetime v.s. density



(b) Average α-lifetime per unit of density v.s. the density

Figure 5: Average α-lifetime and α-lifetime per unit of density derived in Section 4.1 and in Section 4.2.

1000×1000 pixels, we randomly generate N disks, each with radius r. The centers of the N disks are independently and randomly distributed in the square region with uniform distribution. Then for each pixel we count the number of disks that cover it. We term this as the *coverage number* of each pixel. For each value of α, we calculate V_i as the number of pixels that have *coverage number* less than i. With the simulated value of V_i, we calculate the upper bound of the lifetime for the special class of algorithms using Eq. (32), and that for all algorithms using Eq. (35). These upper bounds thus calculated correspond to the theoretical upper bounds of the lifetime in Section 4.1 and in Section 4.2 respectively. The lifetime of a single sensor T is set to 1. The network density is evaluated as

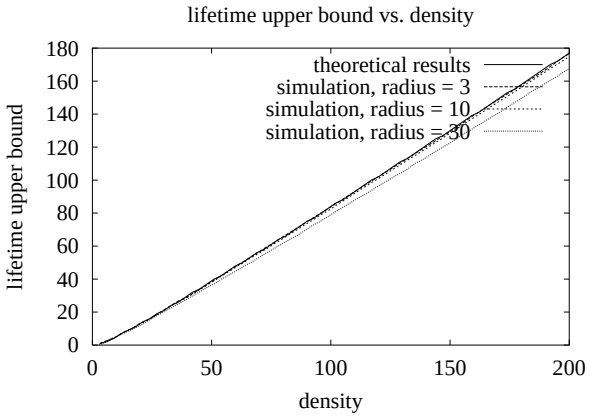
$$\frac{N\pi r^2}{1000000}.$$

All the results reported below are averages of 50 simulation runs.

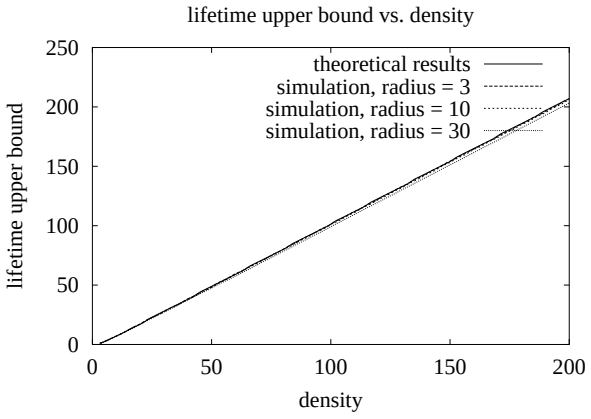
Note that decreasing the disk radius with the side length of the square area fixed has the same effect of increasing the side length of the square area with the disk radius fixed. For each value of α we vary the disk radii over different runs (but keep the radii of all disks constant in each run) to investigate how the area size of the region affects the upper bound of the lifetime. For each value of α and disk radius, we vary the number of sensors to change the node density. As the same trend has been observed for different values of α, in what follows we only report results for $\alpha = 0.95$.

5.2 Simulation Results

Figure 6 shows the theoretical and simulated upper bound of the network lifetime. In particular, Fig. 6 (a) shows the upper bound of the lifetime for a special class of algorithms, derived in Section 4.1, while Fig. 6 (b) shows the upper bound of the lifetime for all algorithms, derived in Section 4.2. When the disk radius decreases (which is equivalent to increasing the side length of the area but keeping the disk radius fixed), the simulated upper bounds become closer to the theoretical upper bounds. This is exactly what Theorem 3 states: when the size of the monitored region increases, V_k will asymptotically converge to $E(V_k)$, and thus we can use $E(V_k)$ to approximate V_k for large sensor networks. In addition, Fig. 6 also suggests that even when the monitored region is small, the theoretical upper bound is still an upper bound, although it may not be tight.



(a) Average 95%-lifetime derived for a special class of algorithms



(b) Average 95%-lifetime derived for all algorithms

Figure 6: Average 95%-lifetime derived in Section 4.1 and in Section 4.2 respectively.

Clustering Sensor Distribution As sensors may not be uniformly distributed in reality, in this set of simulations we study if the theoretical upper bound of network lifetime still serves as an upper bound in the case of clustering nodal distribution. We randomly place sensor nodes using the following clustering method: we first generate K clusters with centers randomly distributed in the 1000×1000 square area R. The radius of each cluster is r_c. Then in each cluster, we randomly generate S sensors. The radius of the coverage area of each sensor is still r. With this configuration, the overall sensor density is evaluated as

$$\frac{K \cdot S \cdot \pi r^2}{1000000}.$$

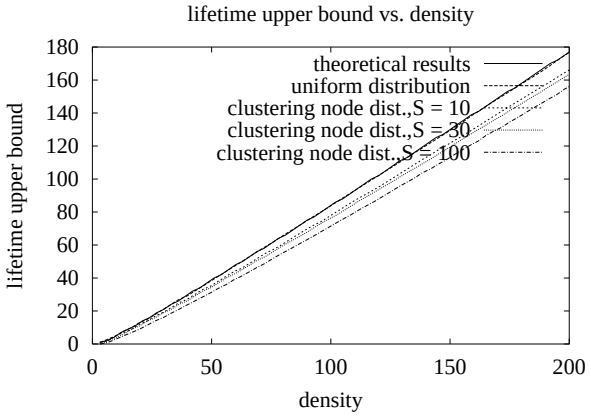
As we conjecture that with a fixed overall density, the sensor density within each cluster (which is defined as $\frac{S \cdot \pi r^2}{\pi r_c^2}$) will affect the lifetime upper bound, we fix the overall density by fixing r, r_c, and $K \cdot S$, but vary S to evaluate the upper bound of the lifetime. We evaluate three different values of S: 10, 30, 100, while fixing $r = 3$ and $r_c = 30$.

Figure 7 (a) and (b) show, respectively, both simulation results and theoretical results (derived in Section 4.1 and in Section 4.2) of the upper bounds of 95%-lifetime under the clustering node distribution. The topmost curve is the theoretical result (under the uniform nodal distribution), while the second curve, which is almost indistinguishable from the topmost curve, is the simulation result under the uniform nodal distribution. The bottom three curves are simulation results under the clustering node distribution with respect to different values of S. As the node density within each cluster increases (but the average density over the whole area R is kept the same), the upper bound of α-lifetime decreases.

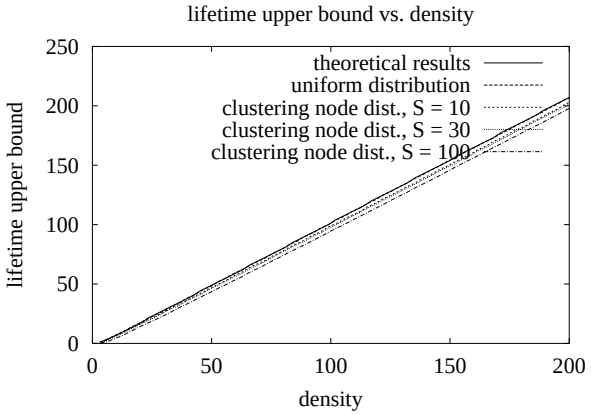
6 Related Work

Several researchers [23, 6, 20, 24, 25, 21, 26, 11, 22] have addressed various methods of minimizing energy consumption and prolonging network lifetime in sensor networks. For example, GAF [23] conserves energy by dividing a region into rectangular grids, ensuring that the maximum distance between any pair of nodes in adjacent grids is within the transmission range of each other, and electing a leader in each grid to stay awake and relay packets (while putting all the other nodes into sleep). The leader election scheme in each grid takes into account of battery usage at each node.

SPAN [6], on the other hand, decides if a node should be working or sleeping based on connectivity among its neighbors.



(a) 95%-lifetime derived in Section 4.1 under clustering node distribution v.s. density



(b) 95%-lifetime derived in Section 4.2 under clustering node distribution v.s. density

Figure 7: 95%-lifetime derived in Section 4.1 and in Section 4.2 respectively under the clustering node distribution.

Slijepcevic *et al.* [20] address the problem of finding the maximal number of covers in a sensor network, where a cover is defined as a set of nodes that can completely cover the monitored area. They prove the NP completeness of this problem, and provide a centralized heuristic solution which approaches the upper bound of the solution under most cases.

Ye *et al.* [24, 25] present PEAS, a distributed, probing-based density control algorithm for robust sensing coverage. In this work, a subset of nodes operate in the active mode to maintain coverage while others are put into sleep. It ensures no two active nodes are in the proximity of each other but does not preserve complete coverage.

Tian *et al.* [21] devise an algorithm that ensures complete coverage using the concept of "sponsored area." Whenever a sensor node receives a packet from one of its working neighbors, it calculates its sponsored area (defined as the maximal sector covered by the neighbor). If the union of all the sponsored areas of a sensor node covers the coverage disk of the node, the node turns itself off.

Zhang and Hou [26] analyze the relationship between complete coverage and connectivity, develop some optimal conditions of maintaining coverage, and devise a localized method to maintain coverage and connectivity based on the optimal conditions.

Wang *et al.* [22] analyze the relationship between k-coverage and j-connectivity, prove a sufficient condition for satisfying k-coverage and propose an algorithm (combined with SPAN [6]) to maintaining coverage and connectivity based on the sufficient condition.

Gupta *et al.* [11] devise both a centralized and a distributed algorithm to find a subset of nodes that ensure both coverage and connectivity. The centralized algorithm guarantees that the size of the formed subset is within $O(\log n)$ factor of the optimal size, where n is the network size.

Although all the above methods are targeted for prolonging the network lifetime, most of them do not perform any analysis on the network lifetime. Recently, research efforts have also been made to analyze the upper bound of the lifetime for ad hoc and/or sensor networks [3, 4, 5, 7]. Bhardwaj *et al.* [3, 4] study the upper bound of the lifetime of data gathering sensor networks. They assume the data source is randomly distributed in a region with a certain p.d.f function and the data sink is located at a fixed point. They calculate the minimum power required to transmit a bit from the source to the sink and then compute the upper bound of the network lifetime based on the minimum power consumption. In [4] they do not consider network topology or the effect of data aggregation of data streams. In [3] they extend the work in [4] by taking into account of these factors and deriving the

upper bound of the network lifetime for networks with arbitrarily complex capabilities. However, their model only considers the power consumption when sensor nodes sense active events, process, transmit and/or receive data, but not when sensor nodes are monitoring but sense no active events. As shown in the empirical study in [19, 10], energy is consumed not only by active communications, but also by wireless devices in the idle and/or sensing state. As a matter of fact, the energy consumed by wireless devices in the idle and/or monitoring state is only a little less than that in the transmitting or receiving states. Thus it makes more sense to derive the network lifetime under the scenario that only a minimum set of sensors are turned on, while the other sensors operate in the low-power mode (or sleep mode).

Blough and Santi [5] study the upper bound of the network lifetime for cell-based energy conservation techniques. While the bound derived does consider energy consumption both in the transmitting/receiving state and in the idle state, it is restricted to the GAF scheme proposed in [23]. In contrast, the lifetime derived in this paper is independent of power-saving schemes used.

Coleri *et al.* [7] investigate the lifetime of networked sensor nodes where sensors are organized in a tree-based multi-hop networks. They analyze the lifetime of nodes in four different groups based on their distances to the data sink using the finite automata technique. However, their analysis is primarily on the lifetime of individual nodes instead of that of the network.

7 Conclusions

In this paper we have investigated the upper bound of α-lifetime for large scale sensor networks. We first derive the asymptotic node density required to ensure full coverage for the duration of k times the lifetime of a single sensor (in the almost surely sense) in large sensor networks, as the network size approaches infinity. Then we derive the upper bound of α-lifetime in a finite region with a finite density of nodes. In particular, we derive two bounds: (i) an upper bound of α-lifetime for a special family of algorithms in which the entire region is completely covered initially, and the coverage ratio is gradually reduced until it drops below a certain threshold α; and (ii) an upper bound of α-lifetime that applies to algorithms that maintain the coverage ratio of α from the beginning of network deployment. We carry out several sets of simulations to validate the derived results. The simulation results indicate that the derived upper bounds of the network lifetime apply

not only to extremely large areas, but also to small areas and areas in which sensor nodes are clustered (rather than uniformly distributed as assumed in the derivation), although the derived upper bounds for the latter two cases may not be tight.

With our derivation, we are able to determine, given the lifetime T of a single sensor node, how many sensor nodes have to be deployed in a region, in order to continuously monitor the region for a period of $k \cdot T$. Also, we observe that although it is, in general, desirable to deploy sensors with high density to achieve a large lifetime per unit of nodal density, the increase in the lifetime per unit of nodal density becomes marginal when the density exceeds certain threshold. The overhead incurred in maintaining coverage in a distributed manner dominates when the sensor density becomes high.

We have identified several research avenues. In particular, we will relax the assumption that the radio transmission range is at least twice as large as the sensing range, and derive the upper bound of the network lifetime by considering jointly coverage and connectivity. Note, however, that the upper bounds derived in this paper serve as upper bounds (although not as tight) even when the assumption does not hold.

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A Results of Coverage Processes

For completeness of the paper, we summarize some of the results on asymptotic coverage drawn from [12] that pertain to our derivation in Section 3.

Let the vacancy $V(\ell, \lambda)$ denote the area that is not covered by any node, S_i the coverage disk of node i, and $\chi(x)$ an indication function of whether a point x is covered by any coverage disk, i.e.,

$$\chi(x) = \begin{cases} 1, & \text{if for all } i,\ x \notin S_i, \\ 0, & \text{otherwise.} \end{cases}$$

When all nodes are randomly placed on the region R, V is a random variable that can be expressed as

$$V = V(R) = \int_R \chi(x)dx. \tag{37}$$

To calculate the expectation of V, we use Fubini's theorem and take the expectation within the integral in Eq. (37). That is,

$$\begin{aligned} E(V) &= \int_R E(\chi(x))dx \\ &= ||R||E(\chi(x))dx \\ &= \ell^2 \exp(-\lambda). \end{aligned} \tag{38}$$

We have used interchangeably the terms complete coverage and vacancy area is 0 throughout the paper. This is supported by the following theorem (Theorem 3.3 in [12]).

Theorem 6 *Let C be a Boolean model in R^k in which covering shapes are distributed as S. If R is an open subset of R^k, S is a random closed set with $E(||S||) < +\infty$, and V is the vacancy area, then*

$$P(V = 0; R \textit{ is not completed covered}) = 0.$$

Although the theorem requires R be an open subset, it can be generalized to the case that R is closed and regular. The interested reader is referred to the discussions after the theorem in [12].

B Proof of Theorem 3

Without loss of generality, we assume l is an integer. We can divide the region $R = [0, l] \times [0, l]$ into unit grids: $R = \cup_{0 \le i,j \le l-1,} D(i, j)$, where $D(i, j) = [i, i + 1] \times [j, j + 1]$. Now, let $V_k(i, j) = V_k \cap D(i, j)$. Since each disk is of radius $r = 1/\sqrt{\pi}$, if two grids $D(i, j)$ and $D(i', j')$ are separated by at least $2r$, then $V_k(i, j)$ and $V_k(i', j')$ are independent variables by the assumption of Poisson point process. Thus we can divide the $V_k(i, j)$'s into (finite) m groups $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_m$, and V_k's in each group are independent of each other. As such, we can write

$$\sum_{0 \le i,j \le l-1} V_k(i, j) = \sum_{(i,j) \in \mathcal{I}_1} V_k(i, j) + \cdots$$

$$+ \sum_{(i,j)\in\mathcal{I}_m} V_k(i,j), \tag{39}$$

where for each p the variables $\{V_k(i,j) : (i,j) \in \mathcal{I}_p\}$ are stochastically independent. The number n_p of elements in each $\mathcal{I}_p$ go to $+\infty$ while the number m of groups is a finite constant as $l \to +\infty$, and $\cup_p \mathcal{I}_p = [0,l] \times [0,l]$. By the strong law of large numbers,

$$n_p^{-1} \sum_{(i,j)\in\mathcal{I}_p} V_k(i,j) \to F(k,\lambda) \tag{40}$$

almost surely as $l \to +\infty$ for $1 \le p \le m$. Hence,

$$\begin{aligned}
&\lim_{l\to+\infty} \frac{V_k}{l^2} \\
&= \lim_{l\to+\infty} \frac{\sum_p \sum_{(i,j)\in\mathcal{I}_p} V_k(i,j)}{\sum_p n_p} \\
&= \lim_{l\to+\infty} \sum_p \frac{\sum_{(i,j)\in\mathcal{I}_p} V_k(i,j)}{n_p} \cdot \frac{n_p}{\sum_p n_p} \\
&= \lim_{l\to+\infty} \sum_p \frac{n_p}{\sum_p n_p} \cdot F(k,\lambda) \\
&= F(k,\lambda)
\end{aligned} \tag{41}$$

almost surely. This completes the proof. □

C Proof of Lemma 1

First we prove (i) $H(k,\alpha)$ monotonically decreases as k increases if $H(k,\alpha) > k$. We need to show that $H(k,\alpha) > H(k+1,\alpha)$. Since we only consider k such that $\alpha > \gamma_k$,

$$\begin{aligned}
& H(k,\alpha) > H(k+1,\alpha) \\
\Leftrightarrow\quad & \frac{\sum_{i=1}^{k-1} i\beta_i}{\alpha-\gamma_k} > \frac{\sum_{i=1}^{k} i\beta_i}{\alpha-\gamma_{k+1}} \\
\Leftrightarrow\quad & (\alpha-\gamma_{k+1})(\sum_{i=1}^{k-1} i\beta_i) > (\alpha-\gamma_k)(\sum_{i=1}^{k} i\beta_i) \\
\Leftrightarrow\quad & (\gamma_k-\gamma_{k+1})(\sum_{i=1}^{k-1} i\beta_i) > k\beta_k\alpha - k\gamma_k\beta_k \\
\Leftrightarrow\quad & \beta_k(\sum_{i=1}^{k-1} i\beta_i) > k\beta_k\alpha - k\gamma_k\beta_k
\end{aligned}$$

$$\Leftrightarrow \quad \sum_{i=1}^{k-1} i\beta_i > k(\alpha - \gamma_k)$$
$$\Leftrightarrow \quad H(k,\alpha) > k. \tag{42}$$

So the first part of the Lemma is proved. In order to prove (ii), we only need to reverse the inequality directions in the above proof. For (iii), we only need to change the inequality sign to equality sign in (i). Next we prove (iv). Since $H(k,\alpha) > k$, we have $\sum_{i=1}^{k-1} i\beta_i > k(\alpha - \gamma_k)$. Hence,

$$\begin{aligned} H(k+1,\alpha) &= \frac{\sum_{i=1}^{k} i\beta_i}{\alpha - \gamma_{k+1}} \\ &= \frac{\sum_{i=1}^{k-1} i\beta_i + k\beta_k}{(\alpha - \gamma_k) + \beta_k} \\ &> \frac{k(\alpha - \gamma_k) + k\beta_k}{(\alpha - \gamma_k) + \beta_k} \\ &= k. \end{aligned} \tag{43}$$

In order to prove (v) and (vi) we only need to change the ">" sign in (iv) to "=" and "<" sign respectively. □

Table 1: Radio transmission range of Berkeley Motes [15]

Product	Transmission Range
MPR300*	30m
MPR400CB	150m
MPR410CB	300m
MPR420CB	300m
MPR500CA	150m
MPR510CA	300m
MPR520CA	300m

∗ MPR300 is the second generation sensors, while the rest are the third generation sensors.

Table 2: Sensing range of several typical sensors

Product	Sensing Range	Functions
HMC1002 Magnetometer sensor[8]	5m	Detecting disturbance from Automobiles
Reflective type photoelectric sensor [2]	1m	Detecting targets of virtually any material
Thrubeam type photoelectric sensor [2]	10m	Detecting targets of virtually any material
Pyroelectric infrared sensor (RE814S) [18]	30m	Detecting moving objects
Acoustic sensor Berkeley Motes * [8]	$\sim$ 1m	Detecting acoustic on sound sources

$*$ This result is based on our own measurement on Berkeley motes [8].