L-FUNCTIONS AND J-SPECTRA

BY

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DISSERTATION

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Abstract

The relation between Eisenstein series and the J-homomorphism is an important topic in chromatic homotopy theory at height 1. Both sides are related to the special values of the Riemann ζ-function. Number theorists have studied the twistings of the Riemann ζ-functions and Eisenstein series by Dirichlet characters.

We first explain congruences of these twisted Eisenstein series of level $\Gamma_1(N)$ and character χ via the Dieudonné theory of height 1 formal groups and formal A-modules and their finite subgroups. Our approach is based on Katz's algebro-geometric explanation of p-adic congruences of normalized Eisenstein series E_{2k} of level 1. The crucial step is to translate the Dirichlet character χ to the Galois descent data of formal A-modules.

We further connect congruences of modular forms in the Eisenstein subspace $\mathcal{E}_k(\Gamma_1(N),\chi)$ with certain group cohomology involving the Dirichlet character χ. When χ is trivial, this group cohomology is on the E_2-page of a spectral sequence to compute homotopy groups of the $K(1)$-local sphere, which is the p-completion of the J-spectra. This gives a new explanation of the connection between congruences of E_{2k} and the image of the stable J-homomorphism in the stable homotopy groups of spheres.

Following our analysis of congruences of Eisenstein series, we introduce the Dirichlet J-spectra. The homotopy groups of the Dirichlet J-spectra are related to the special values of the Dirichlet L-functions, and thus to congruences of the twisted Eisenstein series. Moreover, the pattern of these homotopy groups suggests a possible Brown-Comenetz duality of the Dirichlet J-spectra, which resembles the functional equations of the Dirichlet L-functions. In this sense, the Dirichlet J-spectra constructed in this paper are analogs of Dirichlet L-functions in chromatic homotopy theory.

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Contents

Notations and Conventions

- Denote the Teichmüller character by the Greek letter ω and denote the sheaf of invariant differentials on various stacks by the boldface version of the same Greek letter $\boldsymbol{\omega}$.
- We will suppress the subscript c in the group of continuous homomorphisms and group cohomology of profinite groups.
- Denote the suspension spectrum $\Sigma^\infty X_+$ of a based space X_+ also by X_+.
- X_E is the Bousfield localization of a spectrum X at a homology theory E. Also, we write S_p^0 for the p-complete sphere spectrum.
- By a G-equivariant spectrum, we mean a naïve G-spectrum, i.e. a spectrum with a G-action.
- C_n is the cyclic group of order n and σ is the sign representation of C_2.
- $\mathbb{C}_p$ is the analytic completion of $\overline{\mathbb{Q}_p}$, the algebraic closure of the rational p-adics.
- We write $\underline{G}$ for the constant G-group scheme.
- μ_N is the N-torsion finite subgroup scheme of $\widehat{G}_m$.
- Let M be a G-representation in an R-modules and $\chi : G \to R^\times$ be a character. Write M^χ for the χ-eigensubspace of M.
- We will suppress the $\mathbb{Z}_p$ in $M \otimes_{\mathbb{Z}_p} N$ when M and N are both $\mathbb{Z}_p$-modules.
- Let χ be a Dirichlet character of conductor N. Write $N = p^v N'$, where $p \nmid N'$. Then there is a unique decomposition $\chi = \chi_p \chi'$, where the conductors of χ_p and χ' are p^v and N', respectively. We fix the meanings of N, N', v, χ_p, and χ' throughout the paper.

Introduction

Bernoulli numbers show up in many seemingly unrelated areas of mathematics, as observed in [Maz08]. They are the special values of the Riemann ζ-function at negative integers:

$$\zeta(1-k) = -\frac{B_k}{k}.$$

Another two such occasions are the q-expansion of normalized Eisenstein series in number theory:

$$E_{2k}(q) = 1 - \frac{4k}{B_{2k}} \sum_{n\geq 1} \sigma_{2k-1}(n) q^n,$$

and the images of the J-homomorphisms in the stable homotopy groups of spheres in algebraic topology:

$$\mathrm{Im}(J_{4k-1}) \simeq \mathbb{Z}/D_{2k}, \ D_{2k} = \text{the denominator of } B_{2k}/4k.$$

The connections between the congruences of E_{2k} and images of the J_{4k-1} have been explained in [Bak99; Lau99; Hop02; Beh09] in different ways since the invention of elliptic cohomology and topological modular forms (TMF).

Number theorists have studied the twistings of the Riemann ζ-functions and Eisenstein series by Dirichlet characters. Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character of conductor N. Leopoldt defined generalized Bernoulli numbers $B_{k,\chi}$ associated to χ (1.1) in [Leo58]. These numbers are algebraic numbers in $\mathbb{Q}(\chi)$. Moreover, they are related to the special values of the Dirichlet L-functions $L(s,\chi)$ at negative integers:

$$L(1-k;\chi) = -\frac{B_{k,\chi}}{k}.$$

As in the classical case, $B_{k,\chi}$ appears in the q-expansion of $E_{k,\chi}$ (1.8), a normalized Eisenstein series of weight k, level $\Gamma_1(N)$, and character χ where $(-1)^k = \chi(-1)$:

$$E_{k,\chi}(q) = 1 - \frac{2k}{B_{k,\chi}} \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n.$$

Denote the ideal of $\mathbb{Z}[\chi] \coloneqq \mathbb{Z}[\operatorname{Im}\chi]$ generated by the denominator of $\frac{B_{k,\chi}}{2k}$ by $\mathcal{D}_{k,\chi}$ when $(-1)^k = \chi(-1)$.[1] One may now wonder what is the object in homotopy theory that completes the analogy below:

L-functions	Modular forms	Homotopy theory
$\zeta(1-2k) = -\frac{B_{2k}}{2k}$	$E_{2k} \equiv 1 \mod D_{2k}$	$\operatorname{Im} J_{4k-1} \simeq \mathbb{Z}/D_{2k}$
$L(1-k;\chi) = -\frac{B_{k,\chi}}{k}$	$E_{k,\chi} \equiv 1 \mod \mathcal{D}_{k,\chi}$	?

Table 1: Analogy of L-functions, modular forms and homotopy theory

In this paper, we construct analogs of Dirichlet L-functions in chromatic homotopy theory, called the **Dirichlet J-spectra**, that fit in the table above. The construction of these Dirichlet J-spectra in Part III is based on an algebro-geometric explanation of congruences of Eisenstein series $E_{k,\chi}$ in Part II.

In [Kat73b], Katz gave an algebro-geometric explanation of the p-adic congruences of normalized Eisenstein series E_{2k}. Using a Riemann-Hilbert type correspondence (Theorem 3.1) and a theorem of Igusa, Katz showed:

Theorem. [Kat73b, Corollary 4.4.1] *The followings are equivalent:*

1. $E_{2k}(q) \equiv 1 \mod p^m$.
2. *The $2k$-th power representation $\mathbb{Z}_p^{\otimes 2k}$ of $\mathbb{Z}_p^\times$ is trivial mod p^m.*

The first goal of this paper is to adapt Katz method's to study congruences of modular forms in the Eisenstein subspace

$$\mathcal{E}_k(\Gamma_1(N),\chi) \subseteq M_k(\Gamma_1(N),\chi) \coloneqq M_k(\Gamma_1(N))^{\chi^{-1}}.$$

The strategy is to study a p-adic version of this problem and then assemble the congruence at each prime. As we will be working integrally and p-adically at level $\Gamma_1(N)$ where $p \mid N$,[2] it is necessary to specify meaning

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[1] A priori, the denominator of $\frac{B_{k,\chi}}{2k}$ is not well-defined since the ring $\mathbb{Z}[\chi]$ is in general not a unique factorization domain and has non-trivial unit group. But since $\mathbb{Z}[\chi]$ is a Dedekind domain, its fractional ideals have unique factorizations. As a result, the principal fractional ideal generated by $\frac{B_{k,\chi}}{2k}$ can be uniquely written as the difference of two actual ideals of $\mathbb{Z}[\chi]$. Thus the "denominator ideal" makes sense.

[2] N is either odd or divisible by 4.

of level structures. Let $\mathcal{M}_{ell}(\mu_N)$ be a stack over $\mathbb{Z}$ whose R points are:

$$\mathcal{M}_{ell}(\mu_N)(R) := \left\{ (C/R, \eta : \mu_N \hookrightarrow C) \;\middle|\; \begin{array}{c} C \text{ is an elliptic curve over } R, \\ \eta \text{ is an embedding of group schemes} \end{array} \right\}.$$

When N is invertible in R, a μ_N-level structure on an elliptic curve is (non-canonically) equivalent to a classical $\Gamma_1(N)$-level structure. Write $N = p^v N'$ where $p \nmid N'$. The p-adic version of $\mathcal{M}_{ell}(\mu_N)$ we will consider is $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$, whose R points are:

$$\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))(R) = \left\{ (C/R, \eta_p, \eta') \;\middle|\; \begin{array}{c} C \text{ is a } p\text{-ordinary elliptic curve over } R, \\ \eta_p : \mu_{p^v} \xrightarrow{\sim} \widehat{C}[p^v], \quad \eta' : \underline{\mathbb{Z}/N'} \hookrightarrow C[N'] \end{array} \right\}.$$

$\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')) = \mathcal{M}_{ell}(\mu_N)_p^\wedge$ when $p \mid N$ and is an open substack otherwise. Now let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ be a p-adic primitive Dirichlet character of conductor N. Write $\mathbb{Z}_p[\chi] := \mathbb{Z}_p[\operatorname{Im}\chi]$. χ is uniquely factorized as product $\chi = \chi_p \cdot \chi'$ where χ_p and χ' have conductors p^v and N', respectively. Let k be an integer such that $(-1)^k = \chi(-1)$. Denote by $\mathbb{Z}_p^{\otimes k}[\chi]$ the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representation, whose underlying module is $\mathbb{Z}_p[\chi]$ and where $(a,b) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$ acts by multiplication by $a^k \cdot \chi_p(a) \cdot \chi'(b)$. Denote the Eisenstein subspace of χ^{-1}-eigensubspace of $H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])$ by $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$. The first main result of this paper is:

Theorem (3.3)**.** *Let $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ be an ideal. The followings are equivalent:*

(i). *There is an Eisenstein series f in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ with q-expansion $f(q) \in 1 + \mathcal{I}q[\![q]\!]$.*

(v). *The $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representation $\mathbb{Z}_p^{\otimes k}[\chi]$ is trivial modulo $\mathcal{I}$.*

The main theorem implies the maximal congruence of Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ is equal to that of the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representation $\mathbb{Z}_p^{\otimes k}[\chi]$. The latter is further related to the group cohomology of $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$.

Corollary (4.1)**.** *The followings are equivalent:*

1. $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ is the maximal congruence of Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$.

2. $H_c^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes k}[\chi]) \simeq \mathbb{Z}_p[\chi]/\mathcal{I}$.

The maximal congruences of the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representations $\mathbb{Z}_p^{\otimes k}[\chi]$ are easy to compute since the group is topologically finitely generated. The result of this computation is recorded in Theorem 4.1. From there, we want to find explicit formulas of Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ that match the congruence of $\mathbb{Z}_p^{\otimes k}[\chi]$.

Write $\chi = \chi_p \cdot \chi'$ as above. When $\mathbb{Q}_p(\chi')$ is *not* a totally ramified extension of $\mathbb{Q}_p$, the maximal congruence of Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ is realized by $E_{k,\chi}$. The argument in paper is therefore a cohomological explanation of the *denominator* of $\frac{B_{k,\chi}}{2k}$, whose arithmetic properties were described in [Car59]. When $\mathbb{Q}_p(\chi')$ is a totally ramified extension of $\mathbb{Q}_p$, the maximal congruence is realized as a linearly combination of $E_{k,\chi}$ with some other basis in the Eisenstein subspace $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$. In this case, the group cohomology $H_c^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes k}[\chi])$ sheds light on the *numerator* of $\frac{B_{k,\chi}}{2k}$. One such example is:

Proposition (4.2 and 4.1)**.** *Let $p > 2$ be a prime and $\chi : (\mathbb{Z}/\ell)^\times \to \mathbb{C}_p^\times$ be a Dirichlet character of conductor ℓ such that $\ell \neq p$ is a prime number and $\mathbb{Q}_p(\chi') = \mathbb{Q}_p(\chi)$ is a totally ramified extension of $\mathbb{Q}_p$. Denote the maximal ideal of $\mathbb{Z}_p[\chi]$ by $\mathfrak{m}$. Assume $(-1)^k = \chi(-1)$, $\frac{B_{k,\chi}}{2k} \in \mathbb{Z}_p[\chi]$ by* [Car59, Theorem 1]. *We then have*

$$\frac{B_{k,\chi}}{2k} \in \mathfrak{m} \iff (p-1) \nmid k.$$

This relation is reflected in the cohomological computation that

$$H_c^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/\ell)^\times; \mathbb{Z}_p^{\otimes k}[\chi]) = \begin{cases} \mathbb{Z}_p[\chi]/\mathfrak{m}, & (p-1) \mid k; \\ 0, & \text{otherwise.} \end{cases}$$

When the character χ is trivial, the group cohomology $H_c^1(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes k})$ is on the E_2-page of a **homotopy fixed point spectral sequence**:

$$E_2^{s,2t} = H^s(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes t}) \Longrightarrow \pi_{2t-s}(S^0_{K(1)}),$$

where $S^0_{K(1)}$ is the Bousfield localization of the sphere spectrum S^0 at the Morava K-theory $K(1)$ of height 1 at prime p. The homotopy groups $\pi_{4k-1}(S^0_{K(1)})$ is the p-completion of the image of the stable J-homomorphism in $\pi_{4k-1}(S^0)$. Our analysis on congruences of Eisenstein series E_{2k} therefore gives a new explanation of its relations with image of J. Following [Ada66], we review the constructions and computations of the J-homomorphism in Chapter 2.

Now we want to construct a Dirichlet J-spectrum for each character χ such that its homotopy groups are related to generalized Bernoulli numbers $\frac{B_{k,\chi}}{2k}$. From the trivial character case, we want to construct a spectrum such that its p-completion is computed by a spectral sequence with E_2-page:

$$E_2^{s,2t} = H^s(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times ; \mathbb{Z}_p^{\otimes t}[\chi]).$$

In this way, the 1-line on the E_2-page carries the information of maximal congruence of Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$, which is given by the denominators of $\frac{B_{k,\chi}}{2k}$ in many cases.

The starting point of our construction is the Dirichlet equivariance of the Eisenstein series $E_{k,\chi}$:

$$E_{k,\chi} \in M_k(\Gamma_0(N), \chi) := \mathrm{Hom}_{(\mathbb{Z}/N)^\times\text{-rep}}(\mathbb{C}_{\chi^{-1}}, H^0(\mathcal{M}_{ell}(\Gamma_1(N))_{\mathbb{C}}, \boldsymbol{\omega}^{\otimes k})).$$

Imitating this formula, we define the Dirichlet J-spectrum in Construction 5.3 by

$$J(N)^{h\chi} := \mathrm{Map}\left(M(\mathbb{Z}[\chi]), J(N)\right)^{h(\mathbb{Z}/N)^\times}.$$

In this formula,

- The notation $(-)^{h\chi}$ stands for the "homotopy χ-eigen-spectrum".

- $\mathbb{Z}[\chi]$ is the $\mathbb{Z}$-subalgebra of $\mathbb{C}$ generated by the image of χ. The character χ induces a $(\mathbb{Z}/N)^\times$-action on $\mathbb{Z}[\chi]$ where $a \in (\mathbb{Z}/N)^\times$ acts by multiplication by $\chi(a)$. $M(\mathbb{Z}[\chi])$ is the Moore spectrum of $\mathbb{Z}[\chi]$ with a $(\mathbb{Z}/N)^\times$-action such that the induced $(\mathbb{Z}/N)^\times$-action on π_0 is equivalent to that on $\mathbb{Z}[\chi]$. The existence of such actions on the Moore spectra is non-trivial since the taking Moore spectra is NOT functorial. In Section 5.3, we give an explicit construction of $M(\mathbb{Z}[\chi])$ with $(\mathbb{Z}/N)^\times$-action suggested by Charles Rezk.

- $J(N)$ is the "J-spectrum with μ_N-level structure". It is defined as the homotopy pullback of the arithmetic fracture square (5.3):

$$\begin{tikzcd} J(N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p S^0_{K/p}\left(p^{v_p(N)}\right) \arrow[d, "\text{Rationalization}"] \\ S^0_{\mathbb{Q}} \arrow[r, "\text{Hurewicz}"] & \left(\prod_p S^0_{K/p}\left(p^{v_p(N)}\right)\right)_{\mathbb{Q}} \end{tikzcd}$$

  Here, $S^0_{K/p}(p^v) := \left(K_p^\wedge\right)^{h(1+p^v\mathbb{Z}_p)}$ is a $(\mathbb{Z}/p^v)^\times$-Galois extension of the $K(1)$-local sphere $S^0_{K/p}$. $J(N)$ is endowed with a $(\mathbb{Z}/N)^\times$-action by assembling the Galois actions of $\left(\mathbb{Z}/p^{v_p(N)}\right)^\times$ for each prime $p \mid N$.

  In particular, $J := J(1)$ is equivalent to S^0_K, the Bousfield localization of the sphere spectrum at K, as discussed in [Bou79]. We call it the J-spectrum, because its Hurewicz map detects the image of the stable J-homomorphism. The details of this construction are explained in Section 5.2.

Proposition. (5.10) *There is a variant of the HFPSS to compute $\pi_*(J(N)^{h\chi})$:*

$$E_2^{s,t} \simeq \mathrm{Ext}^s_{\mathbb{Z}[(\mathbb{Z}/N)^\times]}(\mathbb{Z}[\chi], \pi_t(J(N))) \Longrightarrow \pi_{t-s}\left(J(N)^{h\chi}\right).$$

As the E_2-page consists of derived χ-eigenspaces of $\pi_*(J(N))$, it is appropriate to call this spectral sequence the "homotopy eigen(-spectrum) spectral sequence".

This computation is carried out p-adically. For a p-adic Dirichlet character $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$, we construct the Dirichlet $K(1)$-local sphere $S^0_{K/p}(p^v)^{h\chi}$ in a similar fashion. We show in Proposition 5.12 that the p-completion of $J(N)^{h\chi}$ decomposes into a wedge sum of Dirichlet $K(1)$-local spheres. When $N = p > 2$ or 4, the summands in this decomposition represent elements of finite order in the $K(1)$-local Picard group, first defined in [HMS94]. Moreover, we notice the definitions of the Dirichlet J-spectra and $K(1)$-local spheres depend on the group actions on the Moore spectra. In the case when $N = 4$ and $p = 2$, we observe in Remark 6.5 that the Dirichlet $K(1)$-local spheres constructed using different group actions on the Moore spectra are differed by the the exotic element in the $K(1)$-local Picard group at $p = 2$.

The homotopy groups of these Dirichlet $K(1)$-local spheres are computed by a homotopy fixed point spectral sequence (HFPSS), whose E_2-page consists of continuous group cohomology of $\mathbb{Z}_p^\times$.

Proposition. (6.8) *Let χ be a p-adic Dirichlet character of conductor $N = p^v \cdot N'$. There is a spectral sequence*

$$E_2^{s,2t} = H^s_c(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes t}[\chi^{-1}]) \Longrightarrow \pi_{2t-s}\left(S^0_{K(1)}(p^v)^{h\chi}\right),$$

This spectral sequence collapses at the E_2-page if $p > 2$ and $\mathbb{Q}_p(\chi')$ is an unramified extension of $\mathbb{Q}_p$. In particular, when $N = p^v$ and $(-1)^k = \chi(-1)$, the following holds for all primes p:

$$H^1_c(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes k}[\chi^{-1}]) \simeq \pi_{2k-1}\left(S^0_{K(1)}(p^v)^{h\chi}\right).$$

Assembling the computations of homotopy groups of the Dirichlet $K(1)$-local spheres, we observe the homotopy groups of the Dirichlet J-spectra are related to the special values of the corresponding Dirichlet L-functions.

Theorem. (6.2) *Assume $N = p^v > 1$. For all integers k satisfying $(-1)^k = \chi(-1)$, we have*

$$\pi_{2k-1}\left(J(p^v)^{h\chi}\left[\frac{1}{p-1}\right]\right) \simeq \mathbb{Z}[\chi]/\mathcal{D}_{|k|,\chi^{-1}},$$

where $\mathcal{D}_{k,\chi}$ is the denominator ideal of $\frac{B_{k,\chi}}{2k}$ in $\mathbb{Z}[\chi]$.

Moreover we observe in Remark 6.8 that the homotopy groups of the Dirichlet J-spectra and $K(1)$-local spheres suggest possible Brown-Comenetz duality of these spectra. This possible duality phenomena resemble the functional equations of the Dirichlet L-functions.

It is because of these observations that the Dirichlet J-spectra constructed in this paper are analogs of Dirichlet L-functions in chromatic homotopy theory. We conclude the introduction by filling in question mark in Table 1 of analogy between L-functions, Eisenstein series, and K-local spectra:

L-functions	**Eisenstein series**	**K-local spectra**
$\zeta(1-2k)$	E_{2k}	$\pi_{4k-1}(J)$
$L(1-k;\chi)$	$\mathcal{E}_k(\Gamma_1(N),\chi)$	$\pi_{2k-1}\left(J(N)^{h\chi^{-1}}\right)$

Table 2: Analogy of L-functions, Eisenstein series and K-local spectra

Part I

Background

Chapter 1

Dirichlet characters and modular forms

1.1 Dirichlet L-functions

Except for the last two theorems, definitions and statements in this section are from [Iwa72, §1, §2].

Definition 1.1. A multiplicative map $\chi : \mathbb{Z} \to \mathbb{C}$ is called a **Dirichlet character** of modulus N if it is nonzero only at integers coprime to N and it only depends on the residue class modulo N. Alternatively, a Dirichlet character is equivalent to a group homomorphism $\chi : (\mathbb{Z}/N)^{\times} \to \mathbb{C}^{\times}$. A Dirichlet character $\chi : \mathbb{Z} \to \mathbb{C}$ of modulus N is said to be **primitive** if it is not of modulus M for any $M < N$. This N is called the **conductor** of χ. Denote the trivial Dirichlet character that maps every nonzero integer to 1 by χ^0.

The **Dirichlet L-function** associated to χ is defined to be the series:

$$L(s;\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

By definition, $L(s;\chi^0) = \zeta(s)$. Like the Riemann ζ-function, $L(s;\chi)$ has a Euler factorization:

$$L(s;\chi) = \prod_{p} (1 - \chi(p)p^{-s})^{-1}.$$

As a function of s, $L(s,\chi)$ converges absolutely for all s with $\mathrm{Re}(s) > 0$ and non-absolutely for $\mathrm{Re}(s) > 0$ when $\chi \neq \chi^0$. Thus $L(s;\chi)$ defines a holomorphic function on the half plane $\mathrm{Re}(s) > 0$ ($\mathrm{Re}(s) > 1$ if $\chi = \chi^0$) and it admits an analytic continuation to the whole complex plane (minus $s = 1$ if $\chi = \chi^0$). Just as the Riemann ζ function, $L(s;\chi)$ takes special values at negative integers. These values are related to the **generalized Bernoulli numbers**.

Definition 1.2. The ordinary Bernoulli numbers are defined to by

$$F(t) = \frac{te^t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Let χ be a Dirichlet character with conductor N. We define the generalized Bernoulli numbers associated to χ by setting

$$F_\chi(t) = \sum_{a=1}^{N} \frac{\chi(a)te^{at}}{e^{Nt}-1} = \sum_{n=0}^{\infty} B_{k,\chi}\frac{t^k}{k!}. \tag{1.1}$$

Remark 1.1. Notice that the conductor of the trivial character χ^0 is 1. So we have $F_{\chi^0}(t) = F(t)$ and $B_{k,\chi^0} = B_k$.

Proposition 1.1. *$B_{k,\chi} = 0$ unless $(-1)^k = \chi(-1)$. In particular, $B_k = 0$ when k is odd.*

Proposition 1.2. *Let k be a positive integer. For any Dirichlet character $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$, we have*

$$L(1-k;\chi) = -\frac{B_{k,\chi}}{k}.$$

It now follows from (1.1) that $L(1-k;\chi) \in \mathbb{Q}(\chi)$, where $\mathbb{Q}(\chi)$ is the field extension of $\mathbb{Q}$ by the image of χ. In particular, $\zeta(1-k) \in \mathbb{Q}$.

Arithmetic properties of B_k and $B_{k,\chi}$ are summarized below:

Theorem 1.1 (Clausen-von Staudt, von-Staudt)**.** [MS74, Theorem B.3, B.4]

1. *The denominator of B_k, expressed as a fraction in the lowest term is equal to the product of all primes p with $(p-1) \mid 2k$.*
2. *A prime divides the denominator of $\frac{B_k}{2k}$ if and only if it divides the denominator of B_k.*

Theorem 1.2. [Car59, Theorem 1 and 3] *Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character of conductor N.*

1. *If N is divisible by at least two distinct prime numbers, then $\frac{B_{k,\chi}}{k}$ is an algebraic integer. When $N = p^v$, the ideal of $\mathbb{Z}[\chi]$ generated by the denominator of $\frac{B_{k,\chi}}{k}$ contains only prime ideal factors of (p).*
2. *If $N = p^v, p > 2$, let g be a primitive $\phi(N)$-th root of unity mod p. $\frac{B_{k,\chi}}{k}$ is integral unless $\mathfrak{p} = (p, 1-\chi(g)g^k) \neq (1)$. In this case, when $v = 1$,*

$$pB_{k,\chi} \equiv p-1 \mod \mathfrak{p}^{v_p(k)+1}; \tag{1.2}$$

when $v > 1$,

$$(1-\chi(1+p))\frac{B_{k,\chi}}{k} \equiv 1 \mod \mathfrak{p}. \tag{1.3}$$

3. *If $N = 4$, then*

$$\frac{B_{k,\chi}}{k} \equiv \frac{k}{2} \mod 1. \tag{1.4}$$

If $N = 2^v, v > 2$, then $\frac{B_{k,\chi}}{k}$ is an algebraic integer.

1.2 Eisenstein series

One way to study the Dirichlet L-functions is through modular forms, more precisely the Eisenstein series. Here, we give a brief review of the basic theory of modular forms from [Sil94].

Definition 1.3. A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a **congruence subgroup** if it contains all matrices congruent to NI_2 in $\mathrm{SL}_2(\mathbb{Z})$ for some integer $N > 0$. Examples of congruence subgroups are

- $\Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \mod N\right\}$,
- $\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N\right\}$,
- $\Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1, c \equiv 0 \mod N\right\}$.

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ when $N = 1$. A modular form of level Γ and weight k is a holomorphic function over the complex upper half plane $\mathfrak{h}$ satisfying the functional equation:

$$f(\gamma z) = (cz+d)^k f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \mathrm{Im}\, z > 0. \tag{1.5}$$

and is holomorphic at all cusps. The space of such modular forms is denoted by $M_k(\Gamma)$, where Γ is omitted if it is $\mathrm{SL}_2(\mathbb{Z})$.

Recall that the classical Eisenstein series of weight k attached to a lattice $\Lambda \subseteq \mathbb{C}$ is defined by

$$G_k(\Lambda) = \sum_{w \in \Lambda \setminus \{0\}} \frac{1}{w^k}.$$

This formal power series is absolutely convergent when $k > 2$. Let $z \in \mathfrak{h}$ be a complex number in the upper half plane and denote the lattice $(z\mathbb{Z} \oplus \mathbb{Z}) \subseteq \mathbb{C}$ by $\Lambda(z)$. Define

$$G_k(z) := G_k(\Lambda(z)) = \sum_{(m,n) \neq (0,0)} \frac{1}{(mz+n)^k}.$$

This is a modular function of weight k and level $\mathrm{SL}_2(\mathbb{Z})$. It is easy to see $G_k(z) = 0$ when k is odd. As $G_{2k}(z+1) = G_{2k}(z)$ by (1.5), G_{2k} is a function of $q = e^{2\pi i z}$:

$$G_{2k}(q) = 2\zeta(2k) + \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \text{ where } \sigma_m(n) = \sum_{0<d|n} d^m.$$

This is the **q-expansion** of G_{2k}. As $G_{2k}(q)$ is a power series of q, it is holomorphic at the only cusp $q = 0$ and thus a modular form. Dividing G_{2k} by the constant term in its q-expansion, we get the **normalized Eisenstein series** E_{2k} of weight $2k$:

$$E_{2k}(q) := \frac{G_{2k}(q)}{2\zeta(2k)} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character of conductor N. We are now going to introduce the Eisenstein series of level $\Gamma_1(N)$ and character χ, following [Hid93, §5.1] and [Ste07, Chapter 5].

Definition 1.4. Let $\Gamma \le \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. Let $\mathbb{T} \subseteq \mathrm{End}(M_k(\Gamma))$ be the subring generated by the Hecke operators. Then there is decomposition of $\mathbb{T}$-modules:

$$M_k(\Gamma) = \mathcal{E}_k(\Gamma) \oplus \mathcal{S}_k(\Gamma), \tag{1.6}$$

where $\mathcal{S}_k(\Gamma)$ is subspace of cusp forms, i.e. modular forms that vanish at all cusps. The subspace $\mathcal{E}_k(\Gamma)$ is the **Eisenstein subspace** of weight k and level Γ.

Example 1.1. One family of Eisenstein series associated χ of weight k and level $\Gamma_1(N)$ is

$$G_k(z;\chi) := \sum_{(m,n)\neq(0,0)} \frac{\chi^{-1}(n)}{(mNz+n)^k}.$$

This series is nonzero only when $\chi(-1) = (-1)^k$. It is not hard to see $G_k(z;\chi) \in M_k(\Gamma_1(N))$. Moreover, it also satisfies an automorphic equation for $\gamma \in \Gamma_0(N)$:

$$G_k(\gamma \cdot z;\chi) = \chi(d)(cz+d)^k G_k(z;\chi), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \tag{1.7}$$

Definition 1.5. $M_k(\Gamma_1(N),\chi) = \{f \in M_k(\Gamma_1(N)) \mid f \text{ satisfies } (1.7)\}$. In particular, $M_k(\Gamma_1(N),\chi^0) = M_k(\Gamma_0(N))$. When $\Gamma = \Gamma_1(N)$ the decomposition (1.6) is compatible with $(\mathbb{Z}/N)^\times$. As a result, we get $\mathcal{E}_k(\Gamma_1(N),\chi) \subseteq M_k(\Gamma_1(N),\chi)$, the Eisenstein subspace of character χ.

Proposition 1.3. *Set $q = e^{2\pi i z}$ and assume $(-1)^k = \chi(-1)$. The q-expansion of $G_{k,\chi}$ is*

$$G_{k,\chi}(q) = 2L(k,\chi^{-1}) + 2N^{-k}\left(\sum_{l=1}^{N} \chi^{-1}(l) e^{\frac{2\pi i l}{N}}\right) \frac{(-2\pi i)^k}{(k-1)!} \left(\sum_{\substack{m\ge0,n\ge0 \\ (n,N)=1}} \chi(n) n^{k-1} q^{nm} \right).$$

When χ is primitive or $\chi = \chi^0$, one can use the functional equation of $L(s;\chi^{-1})$ to normalize the constant term of $G_{k,\chi}(z)$. We define

$$E_{k,\chi}(q) := \frac{G_{k,\chi}(z;q)}{2L(k,\chi^{-1})} = 1 - \frac{2k}{B_{k,\chi}} \sum_{n=1}^{\infty} \sigma_{k-1,\chi}(n) q^n, \text{ where } \sigma_{m,\chi}(n) = \sum_{0<d|n} \chi(d) d^m. \tag{1.8}$$

Remark 1.2. E_{2k} and $E_{k,\chi}$ can be expressed in terms of z as:

$$E_{2k}(z) = \sum_{(m,n)=1, m>0} \frac{1}{(mz+n)^{2k}}, \quad E_k(z;\chi) = \sum_{(m,n)=1,m>0} \frac{\chi^{-1}(n)}{(mNz+n)^k}.$$

It is straight forward to check from these formulas that

$$G_{2k}(z) = 2\zeta(2k) E_{2k}(z), \quad G_k(z;\chi) = 2L(k,\chi^{-1}) E_k(z;\chi).$$

Example 1.2. There are other families of Eisenstein series in $\mathcal{E}_k(\Gamma_1(N),\chi)$. Let $\chi_1 : (\mathbb{Z}/N_1)^\times \to \mathbb{C}^\times$ and $\chi_2 : (\mathbb{Z}/N_2)^\times \to \mathbb{C}^\times$ be two primitive Dirichlet characters of conductors N_1 and N_2 such that $N_1 N_2 \mid N$. Define the Eisenstein series:

$$G_{k,\chi_1,\chi_2}(z) := \sum_{(n,m)\neq(0,0)} \frac{\chi_1(m)\chi_2^{-1}(n)}{(mNz+n)^k}$$

Theorem 1.3. *Let $N > 1$ be a positive integer. $\{G_{k,\chi_1,\chi_2}(tz) \mid (N_1 N_2 t)|N, \chi_2/\chi_1 = \chi\}$ forms a basis of $\mathcal{E}_k(\Gamma_1(N),\chi)$.*

1.3 Moduli interpretations of modular forms

Modular forms are closely related to moduli stacks of elliptic curves with level structures over $\mathbb{C}$.

Definitions 1.1. Let $\mathcal{M}_{ell}$ be the moduli stack of **generalized elliptic curves** over $\mathbb{C}$. That is, cubic curves with possible nodal singularities. Let N be a positive integer. Define the following moduli stacks:

- $\mathcal{M}_{ell}(\Gamma_0(N))$ is the moduli stack for the pairs (C,H), where C is a generalized elliptic curve and $H \subseteq C$ is a subgroup of order N.
- $\mathcal{M}_{ell}(\Gamma_1(N))$ is the moduli stack for the triples (C,H,η), where C is a generalized elliptic curve, $H \subseteq C$ is a subgroup of order N, and $\eta : \mathbb{Z}/N \xrightarrow{\sim} H$ is an isomorphism.

We define $\mathcal{M}_{ell}(\Gamma_0(1)) = \mathcal{M}_{ell}(\Gamma_1(1)) = \mathcal{M}_{ell}$.

Proposition 1.4. *For the stacks above, denote the sheaves of invariant differentials by* $\boldsymbol{\omega}$*. Then we have*

$$M_k(\Gamma) \simeq H^0(\mathcal{M}_{ell}(\Gamma), \boldsymbol{\omega}^{\otimes k}).$$

The forgetful map $\mathcal{M}_{ell}(\Gamma_1(N)) \to \mathcal{M}_{ell}(\Gamma_0(N))$ is a $(\mathbb{Z}/N)^{\times}$-torsor where $g \in (\mathbb{Z}/N)^{\times} \simeq \mathrm{Aut}(\mathbb{Z}/N)$ acts by $(C, H, \eta) \mapsto (C, H, \eta \circ g)$. As a result, there is a natural action of $(\mathbb{Z}/N)^{\times}$ on $M_k(\Gamma_1(N)) \simeq H^0(\mathcal{M}_{ell}(\Gamma_1(N)), \boldsymbol{\omega}^{\otimes k})$.

Proposition 1.5. *Let* $\chi : (\mathbb{Z}/N)^{\times} \to \mathbb{C}^{\times}$ *be a Dirichlet character.* $M_k(\Gamma_0(N), \chi)$ *defined in Definition 1.5 is isomorphic to* $\mathrm{Hom}_{(\mathbb{Z}/N)^{\times}\text{-rep}}(\mathbb{C}_{\chi^{-1}}, M_k(\Gamma_1(N)))$.

Proof. It suffices to rephrase the automorphic equation (1.7) in terms of the $(\mathbb{Z}/N)^{\times}$-action on the moduli stack $\mathcal{M}_{ell}(\Gamma_1(N))$. Consider the lattice $\Lambda(z) = z\mathbb{Z} \oplus \mathbb{Z}$. There is a triple (C, H, η) associated to $\Lambda(z)$:

$$C = \mathbb{C}/\Lambda(z), H = \Lambda(z/N)/\Lambda(z) \subseteq C, \eta : (\mathbb{Z}/N) \xrightarrow{\sim} H, 1 \mapsto z/N.$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, its actions on the lattices are:

$$\begin{aligned} \Lambda(z) &\mapsto \mathbb{Z}(az+b) \oplus \mathbb{Z}(cz+b) = \Lambda(z), \\ \Lambda(z/N) &\mapsto \mathbb{Z}(az/N+b) \oplus \mathbb{Z}(cz/N+b) \equiv \Lambda(az/N) \equiv \Lambda(z/N) && \mod \Lambda(z), \\ z/N &\mapsto az/N + b \equiv az/N && \mod \Lambda(z). \end{aligned}$$

Here the second line uses the facts $c \equiv 0 \mod N$ and a is invertible mod N. From this formula, the action of γ is trivial when $a \equiv 1 \mod N$, i.e. $\gamma \in \Gamma_1(N)$. For $[\gamma] \in \Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N)^{\times}$, its action on the triple (C, H, η) is:

$$(C, H, \eta : 1 \mapsto z/N) \longmapsto (C, H, \eta \circ [\gamma] : 1 \mapsto a \mapsto az/N).$$

Thus for $f(z) \in M_k(\Gamma_0(N), \chi) \simeq \mathrm{Hom}_{(\mathbb{Z}/N)^{\times}\text{-rep}}(\mathbb{C}_{\chi^{-1}}, M_k(\Gamma_1(N)))$, we have

$$f(\gamma \cdot z) = \chi^{-1}(a)(cz+d)^k f(z) = \chi(d)(cz+d)^k f(z).$$

□

1.4 μ_N-level structures

As we will be working integrally and p-adically at levels divisible by p, it is necessary to specify the meaning of $\Gamma_1(N)$-level structures.

Definition 1.6. A μ_N-level structure on an elliptic curve C is an embedding of group schemes $\eta : \mu_N \hookrightarrow C$. Denote by $\mathcal{M}_{ell}(\mu_N)$ the moduli stack of elliptic curves with μ_N-level structures. Let R be a ring. The R points of $\mathcal{M}_{ell}(\mu_N)$ are

$$\mathcal{M}_{ell}(\mu_N)(R) = \left\{ (C/R, \eta) \;\middle|\; \begin{array}{c} C \text{ is an elliptic curve over } R \text{ and} \\ \eta : \mu_N \hookrightarrow C \text{ is an embedding of group schemes} \end{array} \right\}.$$

Define the space of modular forms of weight k and level μ_N by

$$M_k(\mu_N) := H^0(\mathcal{M}_{ell}(\mu_N), \boldsymbol{\omega}^{\otimes k}), \quad M_k(\mu_N, \chi) := M_k(\mu_N)^{\chi^{-1}},$$

where χ is a Dirichlet character of conductor N.

Lemma 1.1. *$M_k(\Gamma_1(N), \chi) = M_k(\mu_N, \chi)$ over $\mathbb{C}$.*

Proof. This is because $\mathcal{M}_{ell}(\Gamma_1(N))(R) \simeq \mathcal{M}_{ell}(\mu_N)(R)$ when R contains a primitive N-th root of unity. ☐

Proposition 1.6. *When $N \geq 4$, $\mathcal{M}_{ell}(\mu_N)$ is represented by a smooth affine curve over $\mathbb{Z}$.*

Proof. By [KM85, Corollary 4.7.1], it suffices to show:

(1) The forgetful map $\mathcal{M}_{ell}(\mu_N) \to \mathcal{M}_{ell}$ is relatively representable, affine, and étale.

(2) $\mathcal{M}_{ell}(\mu_N)$ is rigid, meaning that there is no non-trivial automorphism of the pair $(C, \eta : \mu_N \hookrightarrow C)$.

(1) is proved in [KM85, Section 4.9, 4.10]. (2) is proved in the [KM85, Corollary 2.7.4] when $N \geq 4$. ☐

1.5 The q-expansion principle

Let $\mathcal{M}_{ell}(\Gamma)_R$ be moduli stack of generalized elliptic curves over R-schemes with Γ-level structures.

Definition 1.7. A **cusp** in $\mathcal{M}_{ell}(\Gamma)_R$ is an embedding $\operatorname{Spf} R[\![q]\!] \to \mathcal{M}_{ell}(\Gamma)_R$ that classifies a Γ-level structure on the Tate curve $T(q)$. The **q-expansion** of a modular form $f \in H^0(\mathcal{M}_{ell}(\Gamma)_R, \boldsymbol{\omega}^{\otimes k})$ at a cusp is its image under restriction map to the said cusp.

Proposition 1.7 (The q-expansion principle)**.** *A modular form $f \in H^0(\mathcal{M}_{ell}(\Gamma)_R, \boldsymbol{\omega}^{\otimes k})$ is zero iff its restriction to all cusps are zero. Furthermore, when $\mathcal{M}_{ell}(\Gamma)_R$ is connected, the restriction map to any cusp is injective.*

It follows that congruences of modular forms are determined by their q-expansions at any cusp when $\mathcal{M}_{ell}(\Gamma)_R$ is connected. By [Con07, Theorem 1.2.1], this is indeed the case when $\Gamma = \Gamma_1(N)$ and $R = \mathbb{Z}$ (so works for any ring R).

Now normalize E_{k,χ_1,χ_2} so that its coefficients are algebraic integers.

Definition 1.8 (Normalization of G_{k,χ_1,χ_2})**.** When χ_2 is non-trivial,

$$E_{k,\chi_1,\chi_2}(q) = \sum_{n\geq 1}\left(\sum_{0<d|n} \chi_2(d)\chi_1(n/d)d^{k-1}\right)q^n.$$

When χ_1 is the trivial character χ^0, recall that we defined $E_{k,\chi}$ in (1.8). Now define $E_{k,\chi^0,\chi}(q) = c \cdot E_{k,\chi}(q)$ for some constant $c \neq 0$ so that $E_{k,\chi^0,\chi}(q) \in \mathbb{Z}[\chi][\![q]\!]$.

Remark 1.3. As $\mathbb{Z}[\chi]$ has non-trivial unit group, the constant c is not unique in general.

Proposition 1.8. $E_{k,\chi_1,\chi_2}(q) \in (H^0(\mathcal{M}_{ell}(\mu_N), \boldsymbol{\omega}^{\otimes k}) \otimes_{\mathbb{Z}} \mathbb{Z}[\chi_1,\chi_2])^{\chi_1/\chi_2}$.

Proof. By Lemma 1.1, $E_{k,\chi_1,\chi_2} \in M_k(\mu_N)$. It is in the χ_1/χ_2-eigensubspace by Theorem 1.3. As the coefficients of $E_{k,\chi_1,\chi_2}(q)$ are all in $\mathbb{Z}[\chi_1,\chi_2]$ by Definition 1.8, the q-expansion principle Proposition 1.7 implies that

$$E_{k,\chi_1,\chi_2} \in H^0(\mathcal{M}_{ell}(\mu_N) \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[\chi_1,\chi_2], \boldsymbol{\omega}^{\otimes k}).$$

When the conductors of χ_1 and χ_2 are 3, their images are $\{\pm 1\}$ and $\mathbb{Z}[\chi_1,\chi_2] = \mathbb{Z}$. When the conductors of χ_1 and χ_2 are at least 4, the claim follows from Proposition 1.6. ☐

1.6 p-adic modulis

We will study congruences of Eisenstein series in $\mathcal{E}_k(\mu_N,\chi)$ p-adically.

Definition 1.9. An elliptic curve C over a p-complete ring is called (p-)**ordinary** if it has nodal singularity, or its reduction mod p is ordinary, i.e. the formal group $\widehat{C}$ associated to C has height 1 reduction mod p.

Denote the p-completed moduli stack of p-ordinary elliptic curve by $\mathcal{M}_{ell}^{ord}$. This is an open substack of $\mathcal{M}_{ell}$, since it is the non-vanishing locus of the Hasse invariant.

Restricted to $\mathcal{M}_{ell}^{ord}$, the μ_{p^v}-level structures on an elliptic curve C are identified with the corresponding level structures on the height 1 formal group $\widehat{C}$. As formal groups of height 1 are étale locally isomorphic to $\widehat{G}_m$, the multiplicative formal group, there is a tower of stacks:

$$\mathcal{M}_{ell}^{triv} \longrightarrow \cdots \longrightarrow \mathcal{M}_{ell}^{ord}(p^2) \longrightarrow \mathcal{M}_{ell}^{ord}(p) \longrightarrow \mathcal{M}_{ell}^{ord},$$

where $\mathcal{M}_{ell}^{ord}(p^v)$ and $\mathcal{M}_{ell}^{triv}$ are the moduli stacks of the pairs $(C, \eta : \mu_{p^v} \xrightarrow{\sim} \widehat{C}[p^v])$ and $(C, \eta : \widehat{G}_m \xrightarrow{\sim} \widehat{C})$ respectively, where C is an ordinary elliptic curves. The forgetful map $\mathcal{M}_{ell}^{ord}(p^v) \to \mathcal{M}_{ell}^{ord}$ is a $(\mathbb{Z}/p^v)^\times$-torsor and $\mathcal{M}_{ell}^{triv} \to \mathcal{M}_{ell}^{ord}$ is a $\mathbb{Z}_p^\times$-torsor. There is a pullback diagram of towers of stacks:

$$\begin{tikzcd}
\mathcal{M}_{ell}^{triv} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \cdots \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}_{ell}^{ord}(p^2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}_{ell}^{ord}(p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}_{ell}^{ord} \arrow[d] \\
\operatorname{Spf} \mathbb{Z}_p \arrow[r] & \cdots \arrow[r] & B(1+p^2\mathbb{Z}_p) \arrow[r] & B(1+p\mathbb{Z}_p) \arrow[r] & B\mathbb{Z}_p^\times
\end{tikzcd} \tag{1.9}$$

Proposition 1.9. [Kat75; Beh14] *When $p > 2$ or $p = 2$ and $v > 1$, $\mathcal{M}_{ell}^{ord}(p^v)$ and $\mathcal{M}_{ell}^{triv}$ are affine formal schemes. In particular, $\mathcal{M}_{ell}^{triv} \simeq \operatorname{Spf} D_p$ where D_p is the ring of divided congruences of p-adic modular forms.*

The strategy now is to relate congruences of $E_{k,\chi}$ to finite subgroups of the formal groups and formal A-modules associated to p-ordinary elliptic curves. Below are some facts about needed in the study of formal group of a p-ordinary elliptic curve.

Proposition 1.10. *Let C be a p-ordinary elliptic curve over a $\mathbb{Z}_p$-algebra. Denote its formal group by $\widehat{C}$.*

1. *C has a canonical subgroup H of order p, where $H = \widehat{C}[p]$.*
2. *The quotient map $\varphi : C \mapsto C/H$ is the relative Frobenius map on $\mathcal{M}_{ell}^{ord}$.*
3. *Let $f(q)$ be the q-expansion of a modular form over $\mathcal{M}_{ell}^{ord}$, then $\varphi^* f(q) = f(q^p)$.*
4. *There is an isomorphism of invertible sheaves $F : \boldsymbol{\omega} \xrightarrow{\sim} \varphi^*\boldsymbol{\omega}$ over $\mathcal{M}_{ell}^{ord}$, where $\boldsymbol{\omega}$ is the sheaf of invariant differentials of C.*

We conclude by comparing the integral and p-adic moduli problems.

Lemma 1.2. *If an elliptic curve C admits a μ_N-level structure, then it is p-ordinary for all primes $p \mid N$.*

Proof. As μ_p is a subgroup scheme of μ_N when $p \mid N$, it suffices to prove the case when $N = p$. Notice μ_p is p-torsion, any embedding of μ_p into an elliptic curve C must factor through $C[p]$. When C is p-supersingular, $C[p] = \widehat{C}[p]$. Thus it reduces to showing that there is no embedding of μ_p into a height 2 formal group.

Using Dieudonné theory of finite groups schemes, we can show the only finite subgroup scheme of rank p in a height 2 formal group is étale locally isomorphic to α_p, which is not étale locally isomorphic to μ_p. ☐

Definition 1.10. Let $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$ be the stack whose R-points are

$$\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))(R) = \left\{ (C/R, \eta_p, \eta') \,\middle|\, \begin{array}{l} C \text{ is a } p\text{-ordinary elliptic curve over } R, \\ \eta_p : \mu_{p^v} \xrightarrow{\sim} \widehat{C}[p^v], \quad \eta' : \mathbb{Z}/N' \hookrightarrow C[N] \end{array} \right\}.$$

Proposition 1.11. *Write $N = p^v \cdot N'$, where $p \nmid N'$. Then we have*

$$(\mathcal{M}_{ell}(\mu_N))_p^\wedge \simeq \begin{cases} \mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), & \text{if } p \mid N; \\ (\mathcal{M}_{ell})_p^\wedge(\Gamma_1(N)), & \text{if } p \nmid N. \end{cases}$$

Proof. Canonical subgroups and Lemma 1.2. ☐

Proposition 1.12. *The forgetful map $\xi : \mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ is a $(\mathbb{Z}/N)^\times$-torsor of stacks.*

Proof. Let R be a $\mathbb{Z}_p$-algebra. Abbreviate the stack $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$ by X and the stack $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ by Y. An object in $Y(R)$ is a pair (C, H), where H is a finite subgroup scheme of $C[N]$ that is isomorphic to $\underline{\mathbb{Z}/N'}$. Recall that the 2-categorical pullback $X(R) \times_{Y(R)} X(R)$ is a groupoid with objects $\{(x_1, x_2, \alpha : \xi(\times_1) \xrightarrow{\sim} \xi(x_2))) \mid x_i = (C_i, \eta_i) \in X(R)\}$. A morphism $\tau : (x_1, x_2, \alpha) \to (x_1', x_2', \alpha')$ consists of a pair of morphisms $\tau_i : x_i \to x_i'$ in $X(R)$ such that the following diagram commutes in $Y(R)$:

$$\begin{tikzcd} \xi(x_1) \arrow[r, "\xi(\tau_1)"] \arrow[d, "\alpha"] & \xi(x_1') \arrow[d, "\alpha'"] \\ \xi(x_2) \arrow[r, "\xi(\tau_2)"] & \xi(x_2') \end{tikzcd} \tag{1.10}$$

By [Ols16, 4.5.1], we need to check that:

- For each object $(C, H) \in Y(R)$, there is a finite étale extension R' of R such that $\xi^{-1}(R')(C, H) \neq \varnothing$.
- Think of the group $(\mathbb{Z}/N)^\times$ as a discrete groupoid. The morphism of groupoids is an equivalence:

$$\begin{aligned} F : (\mathbb{Z}/N)^\times \times X(R) &\longrightarrow X(R) \times_{Y(R)} X(R) \\ (g, x) &\longmapsto (x, g \cdot x, \mathrm{id}_{\xi(x)}) \\ \alpha : (g, x) \to (g, x') &\Longrightarrow (\alpha, \alpha) : (x, g \cdot x, \mathrm{id}_{\xi(x)}) \to (x', g \cdot x', \mathrm{id}_{\xi(x')}) \end{aligned}$$

The first part follows from the facts that $\widehat{C}[p^v]$ is isomorphic to μ_{p^v} after a finite étale extension and that the subgroup scheme H is isomorphic to $\underline{\mathbb{Z}/N'}$ by definition.

For the second point, we need to check F is fully faithful and essentially surjective.

- **Faithful** This is obvious since $F(\alpha) = (\alpha, \alpha)$ is the identity iff α is the identity.

- **Full** Let $\tau = (\tau_1, \tau_2) : (x, g\cdot x, \mathrm{id}_{\xi(x)}) \to (x', g\cdot x', \mathrm{id}_{\xi(x')})$ be a morphism. We need to show $\tau_1 = \tau_2$. (1.10) implies that $\xi(\tau_1) = \xi(\tau_2)$. It follows from the faithfulness of ξ as the forgetful functor that $\tau_1 = \tau_2$. As a result, F is full.

- **Essentially surjective** Given an object (x_1, x_2, α) in $X(R) \times_{Y(R)} X(R)$, we need to find a pair $(g, x) \in G \times X(R)$ such that $F(g, x) \simeq (x_1, x_2, \alpha)$. Write $x_i = (C_i, \eta_i) \in X(R)$ as above. Consider the morphism:

  $$\tau = (\mathrm{id}, \alpha^{-1}) : ((C_1, \eta_1), (C_2, \eta_2), \alpha) \to ((C_1, \eta_1), (C_1, \eta_1 \circ g), \mathrm{id})$$

  where g is determined by the relation $\eta_1 \circ g = \alpha^{-1} \circ \eta_2$. Then $\tau : (x_1, x_2, \alpha) \xrightarrow{\sim} F(x_1, g)$ is an isomorphism and F is essentially surjective.

□

Proposition 1.13. *The stack $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$ is represented by a smooth formal affine curve over $\mathbb{Z}_p$ in the following cases:*

- $N = p^v \cdot N' \geq 4$ *for any* p.

- $N = p = 3$.

- $N = N' = 3$ *and* $p \equiv 2 \mod 3$.

Proof. By Proposition 1.11, $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$ is the p-completion (when $p \mid N$), or a distinguished open substack of the p-completion (when $p \nmid N$) of $\mathcal{M}_{ell}(\mu_N)$. As the latter is represented by a smooth affine curve over $\mathbb{Z}$ by Proposition 1.6, the first case of the claim follows.

When $N = p = 3$, $\mathcal{M}_{ell}^{ord}(3)$ is affine by Proposition 1.9.

When $N = 3$ and $p \neq 3$, it suffices to show the moduli problem is rigid as in the proof of Proposition 1.6. Let ε be a nontrivial automorphism of C that preserves a $\Gamma_1(3)$-level structure $\eta' : \underline{\mathbb{Z}/3} \hookrightarrow C[3]$. Adapting the proof of [KM85, Corollary 2.7.3] to the $N = 3$ case, we can show ε must satisfy $\varepsilon^2 + \varepsilon + 1 = 0$. This implies ε is an element of order 3 in $\mathrm{Aut}(C)$. By [Sil09, Proposition A.1.2.(c)], $\mathrm{Aut}(C)$ has an element of order 3 iff its j-invariant is 0. By [Sil09, Example V.4.4, Exercise 5.7], the $j = 0$ elliptic curve is p-supersingular when

$p \equiv 2 \mod 3$. As a result, when $p \equiv 2 \mod 3$, there is no non-trivial automorphism of a p-ordinary elliptic C that preserves a $\Gamma_1(3)$-structure. This shows the moduli problem $\mathcal{M}_{ell}^{ord}(\Gamma_1(3))$ is rigid at such primes, and hence represented by a smooth formal affine curve over $\mathbb{Z}_p$. ☐

Remark 1.4 (Patrick Allen). The moduli problem $\mathcal{M}_{ell}^{ord}(\Gamma_1(3))$ is NOT rigid when $p \equiv 1 \mod 3$. For such primes, the $j = 0$ elliptic curve C is p-ordinary. Take an automorphism ε of order 3 of C. As $C[3] \simeq \underline{\mathbb{Z}/3}^{\oplus 2}$, ε restricts to an element of order 3 in $\mathrm{GL}_2(\mathbb{Z}/3)$. From the identity $0 = \varepsilon^3 - 1 = (\varepsilon - 1)^3$ in $\mathrm{End}(C[3]) \simeq M_2(\mathbb{Z}/3)$, ε is unipotent. Then there is a basis $\{P, Q\}$ of $C[3]$ under which ε acts by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $\eta' : \underline{\mathbb{Z}/3} \hookrightarrow C[3]$ that sends $1 \in \underline{\mathbb{Z}/3}$ to $P \in C[3]$. The matrix representations of ε shows it is an automoprhism of the pair (C, η'). Consequently, $\mathcal{M}_{ell}^{ord}(\Gamma_1(3))$ is the not rigid and is therefore not represented by a scheme.

Proposition 1.14. *Let χ be a Dirichlet character of conductor N, where $N = p^v N'$ with $p \nmid N'$. Denote the Eisenstein subspace in the χ^{-1}-eigensubspace in $H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])$ by $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$. Then we have a decomposition:*

$$\mathcal{E}_k(\mu_N, \chi)_p^\wedge \simeq \bigoplus_{[\sigma] \in \mathrm{Coker}\, \iota^*} \mathcal{E}_k(p^v, \Gamma_1(N'), \iota \circ \sigma \circ \chi),$$

where $\iota : \mathbb{Q}(\chi) \hookrightarrow \mathbb{C}_p$ is a field extension and $\iota^* : \mathrm{Gal}(\iota(\mathbb{Q}(\chi))/\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ is the induced map of ι on Galois groups.

Proof. This follows from Corollary B.2. ☐

Corollary 1.1. *Let χ_1 and χ_2 be p-adic Dirichlet characters of conductor N_1 and N_2 respectively. Then the normalized Eisenstein series E_{k,χ_1,χ_2} in Definition 1.8 defines a p-adic Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi_2/\chi_1)$, where $N = N_1 N_2 = p^v N'$ and $p \nmid N'$.*

Chapter 2

From the J-homomorphism to the $K(1)$-local sphere

2.1 The J-homomoprhism and the e-invariant

The J-homomorphism is a group homomorphism $J_{k,n} : \pi_k(\mathrm{SO}(n)) \to \pi_{n+k}(S^n)$. This map passes to a stable J-homomorphism $J_k : \pi_k(\mathrm{SO}) \to \pi_k(S^0)$.

Definitions 2.1. The (unstable) J-homomoprhism is defined in the following ways:

1. Loop spaces. An linear isometry of $\mathbb{R}^n$ restricts to a boundary preserving isometry of the unit ball D^n and thus induces a selfmap $S^n \to S^n$. From this, we get a continuous map $g_n : \mathrm{SO}(n) \to \Omega^n S^n$. We define

$$J_{k,n} := \pi_k(g_n) : \pi_k(\mathrm{SO}(n)) \longrightarrow \pi_k(\Omega^n S^n) \simeq \pi_{n+k}(S^n).$$

2. Framed cobordism. Geometrically, the image of the J-homomorphism identifies the framed k-dimensional submanifolds of S^{n+k} whose underlying submanifolds are S^k. As the normal bundle of $S^k \hookrightarrow S^{n+k}$ is trivial, a framing of this embedding is equivalent a map $f : S^k \to O(n)$. One can further show two framings of the embedding $S^k \hookrightarrow S^{n+k}$ are equivalent iff the associated maps are homotopical. Thus we get a map $J_{k,n} : \pi_k(O(n)) \to \pi_{n+k}(S^n)$.

3. Thom space. A map $f \in \pi_k(\mathrm{SO}(n)) \simeq \pi_{k+1}(B\mathrm{SO}(n))$ induces a n-dimensional oriented vector bundle ξ_f over S^{k+1}. The Thom space of ξ_f is a two-cell complex $\mathrm{Th}(\xi_f) = S^n \cup e^{n+k+1}$. Define $J_{k,n}(f)$ to be the gluing map of $\mathrm{Th}(\xi_f)$, i.e.

$$S^{n+k} = \partial e^{n+k+1} \xrightarrow{J_{k,n}(f)} S^n \longrightarrow \mathrm{Th}(\xi_f).$$

Proposition 2.1. *The definitions above are equivalent up to a sign.*

Proposition 2.2. *The J-homomorphisms $J_{k,n}$ are compatible under stabilization. More precisely, let i_n : $\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+1)$ be the map that sends an $n \times n$ orthogonal matrix A to $\begin{pmatrix} A & \\ & 1 \end{pmatrix}$. The following diagram* *commutes:*

$$\begin{tikzcd}
\pi_k(\mathrm{SO}(n)) \arrow[r, "J_{k,n}"] \arrow[d, "\pi_k(i_n)"] & \pi_{n+k}(S^k) \arrow[d, "\Sigma"] \\
\pi_k(\mathrm{SO}(n+1)) \arrow[r, "J_{k,n+1}"] & \pi_{n+k+1}(S^{k+1})
\end{tikzcd}$$

Definition 2.1. We define the stable J-homomorphism to be the colimit:

$$J_k \coloneqq \operatorname*{colim}_n J_{k,n} : \pi_k(\mathrm{SO}) \longrightarrow \pi_k(S^0)$$

Remark 2.1. $J_{k,n}$ stabilizes when $n > k+1$.

Remark 2.2. The definitions of the J-homomorphism above can be phrased stably:

1. The colimit of the maps g_n in the first definition is a map $g : \mathrm{SO} \longrightarrow \Omega^\infty S^\infty$. The induced map

$$\pi_k(g) : \pi_k(\mathrm{SO}) \longrightarrow \pi_k\left(\Omega^\infty S^\infty\right) \simeq \pi_k(S^0)$$

   is then the k-th stable J-homomorphism.

2. In terms of framed cobordism, the stable homotopy group $\pi_k(S^0)$ classifies the framed-cobordism classes of k-dimensional manifolds with a framing on its stable normal bundle, when embedded in $\mathbb{R}^\infty$. A framing on the stable normal bundle of S^k is then a map $f : S^k \to \mathrm{SO}$. Again if $f_1, f_2 : S^k \to \mathrm{SO}$ are homotopic, then the corresponding stably framed k-dimensional manifolds are framed cobordant. From this point view we get the stable J-homomorphism $J_k : \pi_k(\mathrm{SO}) \to \pi_k(S^0)$.

3. $f \in \pi_k(\mathrm{SO}) \simeq \pi_{k+1}(B\mathrm{SO})$ induces a virtual vector bundle ξ_f of dimensional 0 on S^{k+1}. The Thom space of ξ_f is a two-cell complex $\mathrm{Th}(\xi_f) = e^0 \cup e^{k+1}$. Again, $J(f)$ is defined to be the gluing map of the stable two-cell complex $\mathrm{Th}(\xi_f)$.

Remark 2.3. The three definitions of the J-homomorphisms above lead to different directions in homotopy theory. (1) leads to the units of ring spectra, studied in [ABG+14]. (2) is related to the work of Kervaire and Milnor in [KM63]. (3) leads to the computation of the image of the J-homomorphism by Adams in [Ada66], which we explain below.

Define the e-invariant of a stable map $f : S^{2k-1} \to S^0$ as below. Consider the cofiber sequence:

$$S^0 \longrightarrow S^0 \cup_f e^{2k} \longrightarrow S^{2k}.$$

Apply complex K-theory homology to this sequence. As K_* is concentrated in even degrees, we get a short exact sequence:

$$0 \longrightarrow K_0(S^0) \longrightarrow K_0(S^0 \cup_f e^{2k}) \longrightarrow K_0(S^{2k}) \longrightarrow 0.$$

This is not only an extension of abelian groups, but also of K_0K-comodules. As such, this short exact sequence corresponds to an element

$$e(f) \in \mathrm{Ext}^1_{K_0K}(K(S^0), K(S^{2k})).$$

This is the e**-invariant** of $f : S^{2k-1} \to S^0$.

Remark 2.4. K_*K is computed in [AHS71, Theorem 2.3]:

$$K_*K \simeq \left\{ f(u,v) \in \mathbb{Q}((u,v)) \,\middle|\, f(ht,kt) \in \mathbb{Z}\left[t, t^{-1}, \frac{1}{hk}\right], \forall h,k \in \mathbb{Z} \right\},$$

where $t \in K_2(K)$. In particular,

$$K_0K \simeq \{f(w) \in \mathbb{Q}((w)) \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}.$$

Theorem 2.1. [Ada66, Theorem 1.1–1.6] *The image of the stable J-homomorphism $J_k : \pi_k(\mathrm{SO}) \to \pi_k(S^0)$ are described below:*

1. *J_k is injective when $k \equiv 0, 1 \mod 8$.*

2. *The image of J_{8k+3} is a cyclic group of order D_{4k+2}, the denominator of $\frac{B_{4k+2}}{8k+4}$. The image of J_{8k-1} is a cyclic group of order D_{4k} or $2D_{4k}$.*

3. *The image of J_{4k-1} in $\pi_{4k-1}(S^0)$ is a direct summand. The direct sum splitting is accomplished by the homomorphism $e' \circ J_{4k-1} : \pi_{4k-1}(\mathrm{SO}) \twoheadrightarrow \mathbb{Z}/D_{2k}$ associated to the e-invariant.*

2.2 K-theory and formal groups of height 1

In this section, we will discuss the relation between complex K-theory and formal groups of height 1. In the end, we will identify $\mathrm{Ext}^1_{K_0K}(K(S^0), K(S^{2k}))$ to a group cohomology. A more general reference on formal groups and chromatic homotopy theory can be found in [Ada95; Hop99; Lur10].

Definition 2.2. A cohomology theory E is called **complex oriented** if it is multiplicative and it satisfies the Thom isomorphism theorem for complex vector bundles. It is **even periodic** if E_* is concentrated in even degrees and there is a $\beta \in E^{-2}(\mathrm{pt})$ such that β is invertible in E_*.

Proposition 2.3. *Let E be a complex oriented evenly periodic cohomology theory, then*

1. $E^*(\mathbb{CP}^\infty) \simeq E_*[\![t]\!]$ *where* $t \in E^2(\mathbb{CP}^\infty)$ *is the first Chern class of the tautological line bundle* ξ *over* $\mathbb{CP}^\infty$.
2. *Let* $p_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ *be the projection map of the* i*-th component for* $i = 1, 2$. *Then* $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \simeq E_*[\![t_1, t_2]\!]$, *where* $t_i = p_i^* c_1(\xi)$.
3. *The tensor product of line bundles over* $\mathbb{CP}^\infty$ *induces a* E_0*-**formal group** structure on* $\operatorname{Spf} E(\mathbb{CP}^\infty)$. *Denote this formal group associated to a complex-oriented cohomology theory* E *by* $\widehat{G}_E$.
4. $E(S^{2k})$ *is identified with* $\boldsymbol{\omega}^{\otimes k}$, *the* k*-th tensor power of the sheaf of invariant differentials on* $\widehat{G}_E$.

Examples 2.1. Here are two examples of complex oriented cohomology theories and their associated formal groups:

1. For ordinary cohomology theory H, $\widehat{G}_H \simeq \widehat{G}_a$ is the additive formal group.
2. For complex K-theory, $\widehat{G}_K \simeq \widehat{G}_m$ is the multiplicative formal group.

Theorem 2.2 (Quillen)**.** *The formal group associated to the periodic complex cobordism* MP *is the universal formal group. More precisely, the pair* $(MP_0, MP_0(MP))$ *classifies formal groups and isomorphisms between formal groups.*

As $\widehat{G}_{MP}$ is the universal formal group, one might wonder given a formal group over a ring R classified by a map $MP_0 \to R$, is $MP_*(-) \otimes_{MP_0} R$ a cohomology theory? The answer is yes when the map $MP_0 \to R$ satisfies certain flatness conditions. In particular, we have

Theorem 2.3 (Conner-Floyd)**.** *Let* $\theta : MP_0 \to K_0$ *be the map that classifies* $\widehat{G}_m$. *Then* $K_*(X) \simeq MP_0(X) \otimes_{MP_0} K_*$ *and*

$$K_0K \simeq K_0 \otimes_{MP_0} MP_0(MP) \otimes_{MP_0} K_0.$$

The map of Hopf algebroids $\theta : (MP_0, MP_0(MP)) \to (K_0, K_0K)$ induces a map of comodule ext-groups:

$$\theta_* : \operatorname{Ext}^1_{MP_0MP}(MP(S^0), MP(S^{2k})) \to \operatorname{Ext}^1_{K_0K}(K(S^0), K(S^{2k}))$$

The e-invariant lives in the target and the source is on the E_2-page of the **Adams-Novikov spectral sequence** (ANSS):

$$E_2^{s,t} = \operatorname{Ext}^s_{MP_0MP}(MP(S^0), MP(S^t)) \Longrightarrow \pi_{t-s}(S^0).$$

Theorem 2.4. *The* e*-invariant map* $e : \pi_{2k-1}(S^0) \to \operatorname{Ext}^1_{K_0K}(K(S^0), K(S^{2k}))$ *factors through* θ_*. *Moreover,* θ_* *is an isomorphism when restricted the image of the* J*-homomorphism.*

Remark 2.5. The computation of the 1-line in the ANSS and its comparison with the images of the J-homomorphisms can be found in [Rav86, Section 5.3].

Thus, the image of the J-homomorphism is computed by its image under the e-invariant map in the K_0K-Ext groups. Completed at a prime p, these Ext-groups are identified with group cohomology.

Corollary 2.1. *As $MP_0(MP)$ classifies isomorphisms between formal group,* $\operatorname{Spec} K_0K$ *is isomorphic to the group scheme* $\operatorname{Aut}(\widehat{G}_m)$ *over* $\mathbb{Z}$.

Theorem 2.5. [Hov02] *Let (A,Γ) be a Hopf algebroid.*

1. $(\operatorname{Spec} A, \operatorname{Spec}\Gamma)$ *is a groupoid scheme.*
2. *There is an equivalence of abelian categories between (A,Γ)-comodules and quasicoherent sheaves over the quotient stack* $\operatorname{Spec} A // \operatorname{Spec}\Gamma$.

Corollary 2.2. *The stack associated to the pair (K_0, K_0K) is the classifying stack*

$$B\operatorname{Aut}(\widehat{G}_m) \coloneqq \operatorname{Spec}\mathbb{Z}//\operatorname{Aut}(\widehat{G}_m).$$

As a result, the e-invariant lives in

$$\operatorname{Ext}^1_{K_0K}(K(S^0), K(S^{2k})) \simeq R^1\operatorname{Hom}_{\operatorname{Qcoh}(B\operatorname{Aut}(\widehat{G}_m))}(\mathcal{O}, \boldsymbol{\omega}^{\otimes k}) \simeq H^1(B\operatorname{Aut}(\widehat{G}_m), \boldsymbol{\omega}^{\otimes k}).$$

The group scheme $\operatorname{Aut}(\widehat{G}_m)$ is not a constant group scheme over $\mathbb{Z}$. However, it becomes one when restricted to the closed points $\operatorname{Spec}\mathbb{F}_p \in \operatorname{Spec}\mathbb{Z}$. This is even true over $\operatorname{Spf}\mathbb{Z}_p$, the formal neighborhood of $\operatorname{Spec}\mathbb{F}_p$ in $\operatorname{Spec}\mathbb{Z}$.

Lemma 2.1. *Over $\mathbb{F}_p$ or $\mathbb{Z}_p$,* $\operatorname{Aut}(\widehat{G}_m) \simeq \underline{\mathbb{Z}_p^\times}$ *as a constant pro-group scheme.*

Thus for the p-adic e-invariant, it suffices to compute

$$e \in H^1(B\operatorname{Aut}(\widehat{G}_m)^\wedge_p, \boldsymbol{\omega}^{\otimes k}) \simeq H^1\left(B\mathbb{Z}_p^\times, \boldsymbol{\omega}^{\otimes k}\right) \simeq H^1\left(\mathbb{Z}_p^\times; \pi_{2k}\left(K_p^\wedge\right)\right), \tag{2.1}$$

where $K_p^\wedge$ is the p-completion of the complex K-theory and $\mathbb{Z}_p^\times$ acts on $(K_p^\wedge)_{2k}$ by the k-th power map.

2.3 The homotopy fixed point spectral sequence

Let G be a finite group. Recall that the group cohomology of G is the derived functor of G-fixed points. If G acts on a spectrum E, then the group cohomology of G with coefficients in $\pi_*(E)$ computes homotopy groups of E^{hG}, the **homotopy fixed point spectrum** of E under the G-action.

Definition 2.3. Let $G_+^{\wedge\bullet} \wedge E$ be the group action cosimpicial spectrum. The homotopy fixed points of this action is defined to be the totalization of this cosimplicial spectrum:

$$E^{hG} := \mathrm{Map}(\Sigma^\infty EG_+, E)^G \simeq \left(\mathrm{Tot}\left[\mathrm{Map}(G_+^\bullet, E)\right]\right)^G.$$

The Bousfield-Kan spectral sequence associated to this cosimpicial spectrum is called the **homotopy fixed point spectral sequence** (HFPSS), whose E_2-page is identified with

$$E_2^{s,t} = H^s(G; \pi_t(E)) \Longrightarrow \pi_{t-s}(E^{hG}). \tag{2.2}$$

In (2.1), we showed that the p-adic e-invariant is in $H^1\left(\mathbb{Z}_p^\times; \left(K_p^\wedge\right)_{2k}\right)$, where $\mathbb{Z}_p^\times$ acts on the p-adic K-theory spectrum by the Adams operations. In [DH04], Devinatz and Hopkins defined E^{hG} for *pro-finite* groups and showed that the E_2-page of the associated HFPSS consists of *continuous* group cohomology of G. Moreover, they proved

Theorem 2.6. *Let $\mathbb{Z}_p^\times$ acts on the p-adic K-theory spectrum by Adams operation. Then the homotopy fixed points $\left(K_p^\wedge\right)^{h\mathbb{Z}_p^\times}$ is equivalent to $S^0_{K(1)}$, the $K(1)$-**local sphere**. Here, $S^0_{K(1)}$ is the Bousfield localization of the sphere spectrum S^0 at the Morava K-theory $K(1) := K/p$.*

For a purpose of this paper, we need to study finite Galois extensions of $S^0_{K(1)}$ in the sense of [Rog08].

Definition 2.4. Define $S^0_{K(1)}(p^v)$ to be the homotopy fixed point spectrum $\left(K_p^\wedge\right)^{h(1+p^v\mathbb{Z}_p)}$ under the Adams operations. This notation was used in [LN12, Definition 5.10].

$S^0_{K(1)}(p^v)$ is a $(\mathbb{Z}/p^v)^\times$-Galois extension of $S^0_{K(1)}$. This shows that there is a Galois correspondence between open subgroups of $\mathbb{Z}_p^\times$ and finite Galois extensions of $S^0_{K(1)}$. We consider the following family of open subgroups of $\mathbb{Z}_p^\times$ nested in a descending chain when $p > 2$:

$$\mathbb{Z}_p^\times \supsetneq 1 + p\mathbb{Z}_p \supsetneq 1 + p^2\mathbb{Z}_p \supsetneq 1 + p^3\mathbb{Z}_p \supsetneq \cdots,$$

and when $p = 2$:

$$\mathbb{Z}_2^\times = 1 + 2\mathbb{Z}_p \supsetneq 1 + 2^2\mathbb{Z}_p \supsetneq 1 + 2^3\mathbb{Z}_p \supsetneq \cdots.$$

Now we are going to compute $\pi_*\left(S^0_{K(1)}(p^v)\right)$ using HFPSS, whose E_2-page is

$$E_2^{s,t} = H^s\left(1 + p^v\mathbb{Z}_p; \left(K_p^\wedge\right)_t\right) \Longrightarrow \pi_{t-s}\left(S^0_{K(1)}(p^v)\right). \tag{2.3}$$

One reference of this computation (and also the HFPSS at height n) is [Hen17]. There are two cases.

Case I: $p > 2$ or $p = 2$ and $v \geq 2$. In this case, $\mathbb{Z}_p^\times$ and $1+4\mathbb{Z}_2$ are pro-cyclic. Let g be a topological generator in $\mathbb{Z}_p^\times$ for $p > 2$ and in $1+4\mathbb{Z}_2$ for $p = 2$. Then for $p > 2$, $1+p^v\mathbb{Z}_p = \left\langle g^{(p-1)p^{v-1}} \right\rangle$ and for $p = 2$, $1+2^v\mathbb{Z}_2 = \left\langle g^{2^{v-2}} \right\rangle$. Let $n = 1$ if $G = \mathbb{Z}_p^\times$ and $n = (p-1)p^{v-1}$ if $G = 1+p^v\mathbb{Z}_p$ for $p > 2$, and $n = 2^{v-2}$ if $G = 1+2^v\mathbb{Z}_2$. The minimal continuous projective resolution for $\mathbb{Z}_p$ in $\mathbb{Z}_p[\![G]\!]$ is

$$0 \longrightarrow \mathbb{Z}_p[\![G]\!] \xrightarrow{1-g^n} \mathbb{Z}_p[\![G]\!] \xrightarrow{g^n \mapsto 1} \mathbb{Z}_p \longrightarrow 0. \tag{2.4}$$

Since the length of the resolution is 1, the HFPSS collapses on E_2-page. The p-adic Adams operations on $K_p^\wedge$ realize $\left(K_p^\wedge\right)_{2t}$ as the t-th power representation of G. From this we get when $G = \mathbb{Z}_p^\times$ for $p > 2$:

$$H^s(\mathbb{Z}_p^\times; \left(K_p^\wedge\right)_t) = \begin{cases} \mathbb{Z}_p, & s = 0, 1 \text{ and } t = 0; \\ \mathbb{Z}/p^{v_p(t')+1}, & s = 1 \text{ and } t = 2(p-1)t'; \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

$$\Longrightarrow \pi_i\left(S^0_{K(1)}\right) = \begin{cases} \mathbb{Z}_p, & i = 0, -1; \\ \mathbb{Z}/p^{v_p(t')+1}, & i = 2(p-1)t' - 1; \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

and when $G = 1 + p^v\mathbb{Z}_p$ ($v > 1$ if $p = 2$):

$$H^s(1+p^v\mathbb{Z}_p; \left(K_p^\wedge\right)_t) = \begin{cases} \mathbb{Z}_p, & s = 0, 1 \text{ and } t = 0; \\ \mathbb{Z}/p^{v_p(t')+v}, & s = 1 \text{ and } t = 2t' \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$\Longrightarrow \pi_i\left(S^0_{K(1)}(p^v)\right) = \begin{cases} \mathbb{Z}_p, & i = 0, -1; \\ \mathbb{Z}/p^{v_p(t')+v}, & i = 2t' - 1 \neq -1; \\ 0, & \text{otherwise.} \end{cases}$$

Case II: $p = 2$ and $G = \mathbb{Z}_2^\times$. In this case, $\mathbb{Z}_2^\times$ is not pro-cyclic. Rather, we have

$$\mathbb{Z}_2^\times \simeq \{\pm 1\} \times (1 + 4\mathbb{Z}_2).$$

Notice $(K_2^\wedge)^{h\mathbb{Z}/2} \simeq KO_2^\wedge$, where $\mathbb{Z}/2$ acts by complex conjugation on $K_2^\wedge$. The homotopy groups of $KO_2^\wedge$ are given by:

$i \mod 8$	0	1	2	3	4	5	6	7
$\pi_i(KO_2^\wedge)$	$\mathbb{Z}_2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}_2$	0	0	0

(2.7)

Let $g \in 1 + 4\mathbb{Z}_2$ be a topological generator. g acts on π_{4l} by multiplication by g^{2l} and on π_{8l+1} and π_{8l+2} by identity. The E_2-page of the HFPSS is

$$E_2^{s,t} = H^s(1 + 4\mathbb{Z}_2; \pi_t(KO_2^\wedge)) = \begin{cases} \mathbb{Z}_2, & s = 0, 1 \text{ and } t = 0; \\ \mathbb{Z}/2, & s = 0, 1 \text{ and } t \equiv 1, 2 \mod 8; \\ \mathbb{Z}/2^{v_2(t')+3}, & s = 1 \text{ and } t = 4t' \neq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

Proposition 2.4. *The extension problems of this spectral sequence are trivial.*

Proof. We need to solve the extension problems when $t - s = 0$ or $t - s \equiv 1 \mod 8$. The following explanation is from Mark Behrens.

The extension when $t - s = 0$ is trivial, because there is no non-trivial extension of $\mathbb{Z}/2$ by $\mathbb{Z}_2$.

When $t - s \equiv 1 \mod 8$, we recall that the Hopf element $\eta \in \pi_1(S^0)$ has order 2. η is represented in (2.8) by the non-zero element of $H^0(1 + 4\mathbb{Z}_2; \pi_1(KO_2^\wedge)) = \mathbb{Z}/2$. If the extension at $t - s = 1$ were nontrivial, then $\pi_1\left(S^0_{K(1)}\right) \simeq \mathbb{Z}/4$. From the short exact sequence

$$0 \longrightarrow H^1(1 + 4\mathbb{Z}_2; \pi_0(KO_2^\wedge)) \longrightarrow \pi_1\left(S^0_{K(1)}\right) \longrightarrow H^0(1 + 4\mathbb{Z}_2; \pi_1(KO_2^\wedge)) \longrightarrow 0,$$

η would then have order 4 in $\pi_1\left(S^0_{K(1)}\right)$. This contradicts the fact that the order of $\eta \in \pi_1(S^0)$ is 2.

For the general $t - s = 8k + 1$ case, replace η by $\beta^k \cdot \eta \in \pi_{8k+1}(KO)$ in the argument above, where $\beta \in \pi_8(KO)$ is the Bott element. ☐

In conclusion, we get when $p = 2$,

$$\pi_i\left(S^0_{K(1)}\right) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/2, & i = 0; \\ \mathbb{Z}_2, & i = -1; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 1 \mod 8; \\ \mathbb{Z}/2, & i \equiv 0, 2 \mod 8 \text{ and } i \neq 0; \\ \mathbb{Z}/2^{v_2(t')+3}, & i = 4t' - 1 \neq -1; \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

Alternatively, we can apply HFPSS on $G = \mathbb{Z}_2^\times$ directly. The E_2-page is computed using the **Hochschild-Serre spectral sequence** (HSSS) whose E_2-page is

$$E_2^{p,q} = H^p(1+4\mathbb{Z}_2; H^q(\mathbb{Z}/2; (K_2^\wedge)_t)) \Longrightarrow H^{p+q}(\mathbb{Z}_2^\times; (K_2^\wedge)_t). \tag{2.10}$$

This spectral sequence collapses on the E_2-page and we have

$$H^s(\mathbb{Z}_2^\times; (K_2^\wedge)_t) = \begin{cases} \mathbb{Z}_2, & s = 0, 1 \text{ and } t = 0; \\ \mathbb{Z}/2^{v_2(t')+3}, & s = 1 \text{ and } t = 4t' \neq 0; \\ \mathbb{Z}/2, & s = 1 \text{ and } t = 4t' + 2; \\ \mathbb{Z}/2, & s \geq 2 \text{ and } t \text{ even}; \\ 0, & \text{otherwise.} \end{cases}$$

Part II

Congruences of Eisenstein series

Chapter 3

Eisenstein series and Galois representations

In this chapter, we adapt Katz's explanation of congruences of E_{2k} as p-adic modular forms in [Kat73b] to study the congruences of p-adic Eisenstein series with level $(\mu_{p^v}, \Gamma_1(N'))$.

Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ be a Dirichlet character of conductor N. Write $N = p^v N'$, where $p \nmid N'$. Then χ is uniquely factorized as a product $\chi = \chi_p \cdot \chi'$, where χ_p and χ' have conductors p^v and N', respectively. Let $\mathbb{Z}_p^{\otimes k}[\chi]$ be the p-adic $(\mathbb{Z}/N)^\times$-representation, whose underlying module is $\mathbb{Z}_p[\chi]$ and where $(a,b) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$ acts on $\mathbb{Z}_p[\chi]$ by multiplication by $a^k \cdot \chi_p(a) \cdot \chi'(b)$. Throughout this chapter, we abbreviate the Eisenstein subspace in $H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])^{\chi^{-1}}$ by $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$.

Theorem (Main Theorem 3.3)**.** *Let $\mathcal{I}$ be an ideal of $\mathbb{Z}_p[\chi]$. The followings are equivalent:*

(i). *There is an Eisenstein series f in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ with q-expansion $f(q) \in 1 + \mathcal{I}q[\![q]\!]$.*

(v). *The $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representation $\mathbb{Z}_p^{\otimes k}[\chi]$ is trivial modulo $\mathcal{I}$.*

The proof of the Main Theorem relies heavily on the Dieudonné theory of formal groups and formal A-modules, which will be briefly reviewed in the next section. We give a more detailed account of the Dieudonné modules of formal groups in Appendix A.

3.1 Review of Dieudonné modules and Galois descent of formal groups

Let R be a smooth $\mathbb{Z}_p$-algebra such that R/p is an integrally closed domain and R admits an endomorphism $\varphi : R \to R$ that lifts the p-th power map on R.

The **Dieudonné module** $\mathbb{D}(\widehat{G})$ of a formal group $\widehat{G}_0$ over R/p is a triple

$$\mathbb{D}(\widehat{G}) = (M, F : M \longrightarrow \varphi^* M, V : \varphi^* M \longrightarrow M),$$

where $M = PH^1_{\mathrm{dR}}(\widehat{G}/R)$ is the primitives in the de-Rham cohomology for some lift $\widehat{G}$ of $\widehat{G}_0$ to R and

$FV = p = VF$ on the respective domains. Formal groups of the same height $h < \infty$ over R/p are étale locally isomorphic to each other. It follows that their isomorphism classes are classified by the Galois cohomology $H^1(\pi_1^{\acute{e}t}(R/p); \mathrm{Aut}(\Gamma_h))$, where Γ_h is the height h Honda formal group. The Galois cohomology class $[\rho] \in H^1(\pi_1^{\acute{e}t}(R/p); \mathrm{Aut}(\Gamma_h))$ that corresponds to $\widehat{G}_0$ is called the **Galois descent data** of $\widehat{G}_0$.

When $\widehat{G}$ has height (slope) 1, $PH^1_{\mathrm{dR}}(\widehat{G}/R) = \boldsymbol{\omega}(\widehat{G})$ is the sheaf of invariant differentials of $\widehat{G}$ and $F : M \longrightarrow \varphi^* M$ is an isomorphism. As a result, the Verschiebung V is determined by F. In this case, we will write $\mathbb{D}(\widehat{G}) = (\boldsymbol{\omega}(\widehat{G}), F : \boldsymbol{\omega}(\widehat{G}) \xrightarrow{\sim} \varphi^*\boldsymbol{\omega}(\widehat{G}))$. The Galois descent data of height 1 formal groups are described by the following:

Proposition 3.1. $\mathrm{Hom}(\pi_1^{\acute{e}t}(R), \mathbb{Z}_p^\times)$ *is an abelian group and classifies isomorphism classes of formal groups over R with height 1 reductions modulo p.*

Proof. When $h = 1$, $\Gamma_1 = \widehat{G}_m$ and $\mathrm{Aut}(\widehat{G}_m) \simeq \mathbb{Z}_p^\times$ is an abelian group. Also since p is (topologically) nilpotent in R, $\pi_1^{\acute{e}t}(R) \simeq \pi_1^{\acute{e}t}(R/p)$. This implies $H^1(\pi_1^{\acute{e}t}(R); \mathbb{Z}_p^\times) \simeq H^1(\pi_1^{\acute{e}t}(R/p); \mathbb{Z}_p^\times)$ classifies isomorphism classes of formal groups of height 1 over R/p. This Galois cohomology is an abelian group since $\mathbb{Z}_p^\times$ is an abelian group. As the étale fundamental group acts trivially on $\mathbb{Z}_p^\times$, we have $H^1(\pi_1^{\acute{e}t}(R); \mathbb{Z}_p^\times) \simeq \mathrm{Hom}(\pi_1^{\acute{e}t}(R), \mathbb{Z}_p^\times)$.

By Lubin-Tate deformation theory of formal groups, height 1 formal groups over R/p have unique deformations to R. This shows $\mathrm{Hom}(\pi_1^{\acute{e}t}(R), \mathbb{Z}_p^\times) \simeq H^1(\pi_1^{\acute{e}t}(R); \mathbb{Z}_p^\times)$ classifies isomorphism classes of formal groups over R with height 1 reductions modulo p. ☐

This suggests a closed symmetric monoidal structure in the category of 1-dimensional formal groups of height 1, which is described in Proposition A.8. Let $\rho_i : \pi_1^{\acute{e}t}(R) \to \mathbb{Z}_p^\times$ be the Galois descent data for the height 1 formal groups $\widehat{G}_i$, $i = 1, 2$. Then the Galois descent data for $\widehat{G}_1 \otimes \widehat{G}_2$ is $\rho_1 \cdot \rho_2$. In terms of Dieudonné modules, this monoidal structure is described by

$$\mathbb{D}(\widehat{G}_1 \otimes \widehat{G}_2) = (\boldsymbol{\omega}_1 \otimes_R \boldsymbol{\omega}_2, \quad F_1 \otimes F_2 : \boldsymbol{\omega}_1 \otimes_R \boldsymbol{\omega}_2 \xrightarrow{\sim} \varphi^*\boldsymbol{\omega}_1 \otimes_{\varphi^* R} \varphi^*\boldsymbol{\omega}_2 \simeq \varphi^*(\boldsymbol{\omega}_1 \otimes_R \boldsymbol{\omega}_2)),$$

where $\mathbb{D}(\widehat{G}_i) = (\boldsymbol{\omega}_i, F_i, V_i)$. The relevant example in this paper is:

Example 3.1. Let C be the universal elliptic curve over $\mathcal{M}_{ell}^{ord}$ and $\widehat{C}$ be its formal group. $\widehat{C}$ is a height 1 formal group since C is a p-ordinary elliptic curve. Denote the Galois descent data for $\widehat{C}$ by $\rho^1 : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}) \to \mathbb{Z}_p^\times$. The pair $(\boldsymbol{\omega}, F : \boldsymbol{\omega} \xrightarrow{\sim} \varphi^*\boldsymbol{\omega})$ described in Proposition 1.10 is the Dieudonné module of $\widehat{C}$, where $F(f(q)) = f(q^p)$ on q-expansions of modular forms. Denote of the k-th monoidal power of $\widehat{C}$ by $\widehat{C}^{\otimes k}$. The Galois descent data for $\widehat{C}^{\otimes k}$ is

$$\rho^k : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}) \xrightarrow{\rho^1} \mathbb{Z}_p^\times \xrightarrow{(-)^k} \mathbb{Z}_p^\times,$$

The Dieudonné module of $\widehat{C}^{\otimes k}$ is

$$\mathbb{D}(\widehat{C}^{\otimes k}) = (\boldsymbol{\omega}^{\otimes k}, F^{\otimes k} : \boldsymbol{\omega}^{\otimes k} \xrightarrow{\sim} \varphi^* \boldsymbol{\omega}^{\otimes k}),$$

where $F^{\otimes k}(f(q)) = f(q^p)$ on q-expansions.

As the Eisenstein series we study in this paper have coefficients in $\mathbb{Z}_p[\chi]$, it is necessary to work with formal $\mathbb{Z}_p[\chi]$-modules. Let A be an algebra. A formal A-module is a formal group $\widehat{G}$ together with an embedding of algebras $i : A \hookrightarrow \mathrm{End}_{FG}(\widehat{G})$ such that the composite

$$A \hookrightarrow \mathrm{End}_{FG}(\widehat{G}) \longrightarrow \mathrm{End}(\boldsymbol{\omega}(\widehat{G}))$$

realizes $\boldsymbol{\omega}(\widehat{G})$ as an A-module. We will write the power series representation of $i(a)$ by $[a]$. Any formal group $\widehat{G}$ comes with a unique formal $\mathbb{Z}$-module structure. When $\widehat{G}$ is defined over a p-complete ring R, this formal $\mathbb{Z}$-module structure extends (uniquely) to a formal $\mathbb{Z}_p$-module structure, since $\lim\limits_{v\to\infty}[p^v](t) = 0$ in $R[\![t]\!]$.

Construction 3.1. When A is $\mathbb{Z}_p$-algebra that is a finite free $\mathbb{Z}_p$-module, we define a formal A-module $\widehat{G} \otimes A$ out of a 1-dimensional formal group $\widehat{G}$. The underlying formal group of $\widehat{G} \otimes A$ is $\widehat{G}^{\oplus r}$, where r is the rank of A as a free $\mathbb{Z}_p$-module. The A-action on $\widehat{G} \otimes A = \widehat{G}^{\oplus r}$ is given by

$$A = \mathrm{End}_A(A) \hookrightarrow \mathrm{End}_{\mathbb{Z}_p}(\mathbb{Z}_p^{\oplus r}) \hookrightarrow \mathrm{End}_{FG}(\widehat{G}^{\oplus r}).$$

where the first map is induced by $A \simeq \mathbb{Z}_p^{\oplus r}$. Write $\mathbb{D}(\widehat{G}) = (\boldsymbol{\omega}(\widehat{G}), F, V)$. The Dieudonné module of $\widehat{G} \otimes A$ is

$$\mathbb{D}(\widehat{G} \otimes A) = \mathbb{D}(\widehat{G}) \otimes A = (\boldsymbol{\omega}(\widehat{G}) \otimes A, F \otimes 1, V \otimes 1).$$

If the height of $\widehat{G}$ is h, let $[\rho] \in H^1(\pi_1^{ét}(R); \mathrm{Aut}(\Gamma_h))$ be the Galois descent data for $\widehat{G}$. $\widehat{G} \otimes A$ is étale locally isomorphic to $\Gamma_h \otimes A$ as a formal A-module. Then we have an embedding of algebras:

$$i : \mathrm{End}(\Gamma_h) \hookrightarrow \mathrm{End}_{\text{formal } A\text{-mod}}(\Gamma_h \otimes A) \simeq \mathrm{End}(\Gamma_h) \otimes A \quad g \longmapsto g \otimes 1.$$

i restricts to a group homomorphism on the units (automorphisms). The Galois descent data for $\widehat{G} \otimes A$ is then the image of $[\rho]$ under the induced map of i in Galois cohomology.

3.2 Sketch of the proof

The proof of Theorem 3.3 has three steps, which will be explained in details in the rest of this chapter. Here is a sketch:

I. By viewing the Dirichlet character χ as a Galois cohomology class, we construct a formal $\mathbb{Z}_p[\chi]$-module $\widehat{C}^{k,\chi}$ of height 1 over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ such that

$$H^0(\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N')),\boldsymbol{\omega}^{\otimes k}\otimes\mathbb{Z}_p[\chi])^{\chi^{-1}}\simeq H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')),\boldsymbol{\omega}(\widehat{C}^{k,\chi})).$$

In this way, we translate congruences of Eisenstein series in $\mathcal{E}_k(p^v,\Gamma_1(N'),\chi)$ to those of elements in the Dieudonné module of $\widehat{C}^{k,\chi}$.

II. By reformulating a Riemann-Hilbert type correspondence in [Kat73b] using the Dieudonné theory of height 1 formal A-modules and their finite subgroups, we relate the congruence of the Dieudonné module $\mathbb{D}(\widehat{C}^{k,\chi})$ with that of the Galois descent data $[\rho^{k,\chi}]$ for $\widehat{C}^{k,\chi}$.

III. The Galois cohomology class $[\rho^{k,\chi}]\in H^1(\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')));(\mathbb{Z}_p[\chi])^\times)$ is represented by a group homomorphism that factorizes as

$$\rho^{k,\chi}:\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))\xrightarrow{\rho^1\times\lambda_{N'}}\mathbb{Z}_p^\times\times(\mathbb{Z}/N')^\times\xrightarrow{(a,b)\mapsto\chi_p(a)\chi'(b)a^k}(\mathbb{Z}_p[\chi])^\times.$$

Here $\rho^1:\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))\to\mathbb{Z}_p^\times$ is the Galois descent data for $\widehat{C}$ described in Example 3.1 and $\lambda_{N'}:\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))\to(\mathbb{Z}/N')^\times$ classifies the $(\mathbb{Z}/N')^\times$-torsor $\mathcal{M}_{ell}^{ord}(\Gamma_1(N'))\to\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. The theorem then follows from the surjectivity of $\rho^1\times\lambda_{N'}$.

3.3 Step I: Dirichlet characters and Galois descent

The first step in the proof of the Main Theorem is to view the Dirichlet character $\chi:(\mathbb{Z}/N)^\times\to\mathbb{C}_p^\times$ as the Galois descent data for a formal A-module $\widehat{C}^{k,\chi}$ of height 1 over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ along the $(\mathbb{Z}/N)^\times$-torsor $\xi:\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N'))\longrightarrow\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. (See Proposition 1.12 for a proof that ξ is a $(\mathbb{Z}/N)^\times$-torsor.)

Construction 3.2. Let (C,η_p,η') be the universal elliptic curve with the given level structures over $\mathcal{M}_{ell}^{ord}(p^v,\Gamma_1(N'))$ and $\widehat{C}$ be its formal group. Then $\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi]$ is a formal $\mathbb{Z}_p[\chi]$-module of height 1. Notice that:

- The automorphism group of $\widehat{C}^{\otimes k}\otimes\mathbb{Z}_p[\chi]$ as a formal $\mathbb{Z}_p[\chi]$-module is $(\mathbb{Z}_p[\chi])^\times$.

- The forgetful map $\xi : \mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ is a $(\mathbb{Z}/p^v)^\times \times (\mathbb{Z}/N')^\times \simeq (\mathbb{Z}/N)^\times$-torsor.

The Dirichlet character $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ then represents a cohomology class

$$\begin{aligned}[\chi] \in & H^1((\mathbb{Z}/N)^\times ; (\mathbb{Z}_p[\chi])^\times) \\ \simeq & H^1(\mathrm{Aut}_{\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))}(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))); \mathrm{Aut}_{\text{formal } \mathbb{Z}_p[\chi]\text{-mod}}(\widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi])),\end{aligned}$$

where $(\mathbb{Z}/N)^\times$ acts on $(\mathbb{Z}_p[\chi])^\times$ trivially. This cohomology group classifies isomorphism classes of formal $\mathbb{Z}_p[\chi]$-modules $\widehat{G}$ over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ such that $\xi^* \widehat{G} \simeq \widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$ over $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$. In this way, the cohomology class $[\chi]$ corresponds to a formal $\mathbb{Z}_p[\chi]$-module $\widehat{C}^{k,\chi}$ over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. More precisely, fix an isomorpshim $\eta : \xi^* \widehat{C}^{k,\chi} \xrightarrow{\sim} \widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$, then for any $\sigma \in (\mathbb{Z}/N)^\times \simeq \mathrm{Aut}_{\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))}(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')))$, we have a commutative diagram of isomorphisms:

$$\begin{tikzcd}
\xi^* \widehat{C}^{k,\chi} \arrow[r, "\sigma\otimes 1"] \arrow[d, "\eta"'] & \sigma^* \xi^* \widehat{C}^{k,\chi} \arrow[r, equal] \arrow[d, "\sigma^*\eta"'] & \xi^* \widehat{C}^{k,\chi} \arrow[d, "\sigma^*\eta"'] \\
\widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi] \arrow[r] \arrow[rr, bend right, "{[\chi(\sigma)]}"'] & \sigma^*(\widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]) \arrow[r, equal] & \widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]
\end{tikzcd}$$

In this diagram,

- $[\chi(\sigma)]$ is defined in Construction 3.1.
- $\sigma^* \eta = \eta$ since $(\mathbb{Z}/N)^\times$ acts on $(\mathbb{Z}_p[\chi])^*$ trivially.
- The correspondence between $\widehat{C}^{k,\chi}$ and χ is independent of the choice of the isomorpshism η, since $\mathrm{Aut}_{\mathbb{Z}_p[\chi]}(\widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]) = (\mathbb{Z}_p[\chi])^\times$ is abelian.

Let $\boldsymbol{\omega}^{k,\chi} := \boldsymbol{\omega}(\widehat{C}^{k,\chi})$ be the sheaf of invariant differentials of $\widehat{C}^{k,\chi}$. $\boldsymbol{\omega}^{k,\chi}$ is a locally free finitely generated sheaf over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$, since it is the cotangent sheaf of a formal scheme that is étale locally isomorphic to $\widehat{\mathbb{A}}^r$, where r is the rank of $\mathbb{Z}_p[\chi]$ as a $\mathbb{Z}_p$-module.

Proposition 3.2. *$\xi^* \boldsymbol{\omega}^{k,\chi} \simeq \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$ over $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$. The sheaf cohomology of $\boldsymbol{\omega}^{k,\chi}$ is computed as follows:*

(1). *$H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi}) \simeq H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])^{\chi^{-1}}$ for all $N > 1$.*

(2). *When $N > 3$ or $N = 3$ and $p \not\equiv 1 \mod 3$, we have for all $s \geq 0$:*

$$H^s(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi}) \simeq H^s((\mathbb{Z}/N)^\times ; H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])).$$

(3). *When $p \nmid \phi(N) = |(\mathbb{Z}/N)^\times|$, we have for all $t \geq 0$:*

$$H^t(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi}) \simeq H^t(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])^{\chi^{-1}}.$$

(4). *In particular, when N and p satisfy both conditions above, we further have:*

$$H^s(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi}) = \begin{cases} H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])^{\chi^{-1}}, & s = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The functor $\boldsymbol{\omega}$ is compatible with pullbacks, yielding

$$\xi^* \boldsymbol{\omega}^{k,\chi} = \xi^* \boldsymbol{\omega}(\widehat{C}^{k,\chi}) \simeq \boldsymbol{\omega}(\xi^* \widehat{C}^{k,\chi}) \simeq \boldsymbol{\omega}(\widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]) = \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi].$$

To compute $H^s(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi})$, we use the Hochschild-Serre spectral sequence [Mil80, Theorem 2.20]:

$$E_2^{s,t} = H^s((\mathbb{Z}/N)^\times; H^t(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \xi^* \boldsymbol{\omega}^{k,\chi})) \Longrightarrow H^{s+t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi}), \tag{3.1}$$

where $\sigma \in (\mathbb{Z}/N)^\times$ acts on $\xi^* \boldsymbol{\omega}^{k,\chi} \simeq \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$ by the Galois descent data $1 \otimes \chi(\sigma)$. As the spectral sequence is concentrated in the first quadrant, its $E_2^{0,0}$-term receives and supports no differentials. This implies (1).

By Proposition 1.13, the stack $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$ is a formal affine scheme when $N \geq 4$ or $N = 3$ and $p \not\equiv 1$ mod 3. It follows that (3.1) is concentrated in the $t = 0$ line in those cases. As a result, the spectral sequence collapses on the E_2-page and we have proved (2).

When $p \nmid \phi(N) = |(\mathbb{Z}/N)^\times|$, the group cohomology of $(\mathbb{Z}/N)^\times$ with coefficients in $\mathbb{Z}_p$-modules vanishes in positive degrees. It follows that (3.1) is concentrated in the $s = 0$ line in this case and thus collapses on the E_2-page. This implies (3).

(4) is the intersection of (2) and (3). □

Remark 3.1. Note that 2 is the only prime p dividing $\phi(3) = 2$. The spectral sequence (3.1) collapses on the E_2-page for all $N \geq 3$ and p.

We have proved in Proposition 3.2:

$$H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])^{\chi^{-1}} \simeq H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi}). \tag{3.2}$$

Write $\mathbb{D}(\widehat{C}^{k,\chi}) = (\boldsymbol{\omega}^{k,\chi}, F^{k,\chi} : \boldsymbol{\omega}^{k,\chi} \xrightarrow{\sim} \varphi^*\boldsymbol{\omega}^{k,\chi})$. The Frobenius homomorphism $F^{k,\chi}$ of $\widehat{C}^{k,\chi}$ descends from that of $\xi^*\widehat{C}^{k,\chi} \simeq \widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$. By Example 3.1 and Construction 3.1, we have

$$\xi^* F^{k,\chi} = F^{\otimes k} \otimes 1 : \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi] \xrightarrow{\sim} \varphi^*\boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi].$$

Notice $F^{\otimes k} \otimes 1$ commutes with the Galois descent data $1 \otimes \chi(\sigma)$ for $\sigma \in (\mathbb{Z}/N)^\times$, we have shown

Proposition 3.3 (Step I). *Let $f \in H^0(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')), \boldsymbol{\omega}^{\otimes k} \otimes \mathbb{Z}_p[\chi])^{\chi^{-1}} \simeq H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi})$ be an Eisenstein series, then $F^{k,\chi}(f(q)) = (F^{\otimes k} \otimes 1)(f(q)) = f(q^p)$. Let $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ be an ideal. Then the followings are equivalent:*

(i). *There is an Eisenstein series $f \in \mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ such that $f(q) \in 1 + \mathcal{I}q[\![q]\!]$.*

(ii). *There is a generator $\gamma \in H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi})$ as an $H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \mathcal{O}) \otimes \mathbb{Z}_p[\chi]$-module such that $F^{k,\chi}(\gamma) \equiv \gamma \mod \mathcal{I}$.*

This concludes step I in Section 3.2.

3.4 Step II: From Dieudonné modules to Galois representations

One major tool Katz used in [Kat73b, Chapter 4] to explain the congruences of the normalized Eisenstein series E_{2k} of level 1 is a Riemann-Hilbert type correspondence. In this section, we reformulate the correspondence in terms of formal A-modules and their finite subgroup schemes, and then apply it to the formal $\mathbb{Z}_p[\chi]$-module $\widehat{C}^{k,\chi}$ over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ we constructed in Construction 3.2.

Let κ be a perfect field of characteristic p containing $\mathbb{F}_q$ and $\mathbb{W}_m(\mathbb{F}_q)$ be the ring of Witt vectors of length m on $\mathbb{F}_q$. Let S_m be a flat affine $\mathbb{W}_m(\kappa)$-scheme whose special fiber is normal, reduced, and irreducible. Assume S_m admits an endomorphism $\varphi : S_m \to S_m$ that lifts the q-th power map on S_m/p. Then Katz proved

Theorem 3.1. [Kat73b, Proposition 4.1.1, Remark 4.1.2.1]
There is an equivalence of closed symmetric monoidal categories:

$$\left\{ \begin{array}{c} \textit{Finite locally free sheaves } \mathscr{F} \textit{ on } S_m \\ \textit{with an isomorphism } F : \varphi^*\mathscr{F} \xrightarrow{\sim} \mathscr{F} \end{array} \right\} \cong \left\{ \begin{array}{c} \textit{Finite free } \mathbb{W}_m(\mathbb{F}_q)\textit{-modules} \\ \textit{with continuous } \pi_1^{\acute{e}t}(S_m)\textit{-actions} \end{array} \right\}.$$

Proposition 3.4. [Kat73a, Remark 5.5] *Theorem 3.1 holds for affine formal schemes S over $\mathbb{W}(\kappa)$ under* *the same assumption. That is, there is an equivalence of closed symmetric monoidal category:*

$$\left\{\begin{array}{c}\textit{Finite locally free sheaves } \mathscr{F} \textit{ on } S\\ \textit{with an isomorphism } F:\varphi^*\mathscr{F}\xrightarrow{\sim}\mathscr{F}\end{array}\right\}\cong\left\{\begin{array}{c}\textit{Finite free } \mathbb{W}(\mathbb{F}_q)\textit{-modules}\\ \textit{with continuous } \pi_1^{\acute{e}t}(S)\textit{-actions}\end{array}\right\}.$$

This equivalence of Katz is essentially an equivalence of Dieudonné module and Galois descent data of a formal group and its finite subgroups. Let A be a $\mathbb{Z}_p$-algebra that is finite free as a $\mathbb{Z}_p$-module and $\widehat{G}$ be formal A-module of height 1. Let $\mathcal{I}\trianglelefteq A$ be an ideal.

Definition 3.1. Define $\widehat{G}[\mathcal{I}]$ to be the kernel of all the endomorphisms in $\mathcal{I}\trianglelefteq A\hookrightarrow \operatorname{End}(\widehat{G})$. If $\widehat{G}=\operatorname{Spf} R[\![t]\!]$ has a coordinate, then $\widehat{G}[\mathcal{I}]=\operatorname{Spf} R[\![t]\!]/([a](t)\mid a\in\mathcal{I})$ as a finite flat scheme. When $\mathcal{I}=(a)$ is a principal ideal, $\widehat{G}[\mathcal{I}]=\widehat{G}[a]=\operatorname{Spf} R[\![t]\!]/([a](t))$.

Proposition 3.5. *Let $\widehat{G}$ be a formal A-module. Write the Dieudonné module of $\widehat{G}$ as $\mathbb{D}(\widehat{G})=(M,F,V)$. Then M has an A-module structure and the homomorphisms F and V are A-linear. The Dieudonné module of $\widehat{G}[\mathcal{I}]$ is $\mathbb{D}(\widehat{G})/\mathcal{I}:=(M/\mathcal{I}M, F:M/\mathcal{I}M\to\varphi^*(M/\mathcal{I}M), V:\varphi^*(M/\mathcal{I}M)\to M/\mathcal{I}M)$.*

Proposition 3.6. *Let $\widehat{G}$ be a formal A-module over R that is isomorphic to $\widehat{G}'$ over the separable closure R^{sep} of R. Let the cohomology class $[\rho]\in H^1(\pi_1^{\acute{e}t}(R);\operatorname{Aut}_A(\widehat{G}'))$ be the Galois descent data for $\widehat{G}$. $[\rho]$ is represented by some crossed homomorphism $\rho:\pi_1^{\acute{e}t}(R)\to\operatorname{Aut}_A(\widehat{G}')$. Then the Galois descent data for the finite flat group scheme $\widehat{G}[\mathcal{I}]$ is represented by the crossed homomorphism:*

$$\rho_{\mathcal{I}}:\pi_1^{\acute{e}t}(R)\xrightarrow{\rho}\operatorname{Aut}_A(\widehat{G}')\longrightarrow\operatorname{Aut}(\widehat{G}'[\mathcal{I}]),$$

where the last map $\operatorname{Aut}_A(\widehat{G}')\longrightarrow\operatorname{Aut}(\widehat{G}'[\mathcal{I}])$ is the restriction of the quotient map

$$\operatorname{End}_A(\widehat{G}')\longrightarrow\!\!\!\!\rightarrow\operatorname{End}_A(\widehat{G}')/(\mathcal{I}\otimes_A\operatorname{End}_A(\widehat{G}'))\simeq\operatorname{End}_A(\widehat{G}'[\mathcal{I}])$$

to the units.

In the view of Proposition 3.5 and Proposition 3.6, Katz's Riemann-Hilbert correspondence (Theorem 3.1) can be generalized as:

Theorem 3.2. *Let $\widehat{G}$ be a formal A-module of height 1 over R, where $\operatorname{Spf} R$ satisfies the same assumptions as in Theorem 3.1. Let $\mathbb{D}(\widehat{G})=(M,F:M\xrightarrow{\sim}\varphi^*M)$ and $\rho:\pi_1^{\acute{e}t}(R)\to A^\times$ be the Dieudonné module and Galois descent data for $\widehat{G}$, respectively. Then the followings are equivalent:*

1. *There is a generator γ of M as an $R\otimes A$-module such that $F\gamma\equiv\gamma \mod \mathcal{I}$.*

2. $\widehat{G}[\mathcal{I}] \simeq (\widehat{G}_m \otimes A)[\mathcal{I}]$.

3. The composition homomorphism $\rho_{\mathcal{I}} : \pi_1^{ét}(R) \xrightarrow{\rho} A^\times \twoheadrightarrow (A/\mathcal{I})^\times$ *is trivial.*

Proof. This follows from the computation of the Dieudonné module and the Galois descent data of $\widehat{G}_m$ in Example A.2, as well as Proposition 3.5 and Proposition 3.6. ☐

Remark 3.2. Katz's Theorem 3.1 is the $\mathcal{I} = (p^m) \trianglelefteq A = \mathbb{W}\mathbb{F}_q$ case of Theorem 3.2.

Remark 3.3. We can generalize Theorem 3.1 and Proposition 3.4 in terms of formal groups and formal A-modules of height $h > 1$. In that case, we need to study the Dieudonné module of the height h Honda formal group Γ_h and its finite subgroup schemes.

Now apply Theorem 3.2 to the formal $\mathbb{Z}_p[\chi]$-module $\widehat{C}^{k,\chi}$ over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ constructed in Construction 3.2, we have established Step II in Section 3.2:

Corollary 3.1 (Step II)**.** *Let* $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ *be an ideal. The followings are equivalent:*

(ii). *There is a generator* $\gamma \in H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi})$ *such that* $F^{k,\chi}(\gamma) \equiv \gamma \mod \mathcal{I}$.

(iii). $\widehat{C}^{k,\chi}[\mathcal{I}] \simeq (\widehat{G}_m \otimes \mathbb{Z}_p[\chi])[\mathcal{I}]$.

(iv). *The Galois descent data* $\rho^{k,\chi} : \pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')) \to (\mathbb{Z}_p[\chi])^\times$ *of* $\widehat{C}^{k,\chi}$ *is trivial modulo* $\mathcal{I}$.

3.5 Step III: Factorizations of the Galois descent data

The final step is to study the Galois descent data $\rho^{k,\chi}$ for $\widehat{C}^{k,\chi}$. Recall from Construction 3.2, $\widehat{C}^{k,\chi}$ is constructed using the following data:

- $\xi^* \widehat{C}^{k,\chi} \simeq \widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$ over $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$, where $\xi : \mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ is the forgetful map.
- $\widehat{C}^{k,\chi}$ corresponds to the character $[\chi] \in H^1((\mathbb{Z}/N)^\times; (\mathbb{Z}_p[\chi])^\times)$.

Proposition 3.7. $\rho^{k,\chi} : \pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')) \to (\mathbb{Z}_p[\chi])^\times$ *factorizes as*

$$\rho^{k,\chi} : \pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \xrightarrow{\rho^1 \times \lambda_\xi} \mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times \xrightarrow{(-)^k \cdot \chi(-)} (\mathbb{Z}_p[\chi])^\times,$$

where $\lambda_\xi : \pi_1^{ét}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \to (\mathbb{Z}/N)^\times$ *is the character that classifies the* $(\mathbb{Z}/N)^\times$*-torsor* ξ.

Proof. Recall in Construction 3.2, we used the following correspondence to construct $\widehat{C}^{k,\chi}$ from the character χ:

$$H^1((\mathbb{Z}/N)^\times; (\mathbb{Z}_p[\chi])^\times) \simeq \left\{ \begin{array}{c} \text{Formal } \mathbb{Z}_p[\chi]\text{-modules } \widehat{G} \text{ over } \mathcal{M}_{ell}^{ord}(\Gamma_0(N')) \\ \text{such that } \xi^*\widehat{G} \simeq \widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi] \text{ over } \mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')) \end{array} \right\} \Big/ \sim \tag{3.3}$$

Here, the constant group homomorphism on the left hand side corresponds to the formal $\mathbb{Z}_p[\chi]$-module $\widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$ over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. Now we need to describe this correspondence in terms of the Galois descent data $\rho_{\widehat{G}}$ of $\widehat{G}$. On the one hand, since $\xi^*\widehat{G} \simeq \widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$, the composition

$$\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))) \xrightarrow{\pi_1^{\acute{e}t}(\xi)} \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \xrightarrow{\rho_{\widehat{G}}} (\mathbb{Z}_p[\chi])^\times \tag{3.4}$$

is the same as the Galois descent data for the formal $\mathbb{Z}_p[\chi]$-module $\widehat{C}^{\otimes k} \otimes \mathbb{Z}_p[\chi]$ over $\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))$. On the other hand, by Example 3.1 and Construction 3.1, this Galois descent data also factorizes as

$$\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))) \xrightarrow{\pi_1^{\acute{e}t}(\xi)} \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \xrightarrow{\rho^1} \mathbb{Z}_p^\times \xrightarrow{(-)^k} \mathbb{Z}_p^\times \xrightarrow{i} (\mathbb{Z}_p[\chi])^\times. \tag{3.5}$$

Denote the composition $i \circ (-)^k \circ \rho^1$ in (3.5) by ρ^k. Since the first maps in (3.4) and (3.5) are both $\pi_1^{\acute{e}t}(\xi)$ and the compositions are the same, the difference of $\rho_{\widehat{G}}$ and ρ^k must factor through the cokernel of $\pi_1^{\acute{e}t}(\xi)$. We have the following diagram:

$$\begin{tikzcd}
\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N'))) \arrow[r, "\pi_1^{\acute{e}t}(\xi)"] & \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \arrow[r, "\lambda_\xi"] \arrow[d, shift right, "\rho_{\widehat{G}}"'] \arrow[d, shift left, "\rho^k"] & (\mathbb{Z}/N)^\times \arrow[r] \arrow[dl, dashed, "\exists!\ \chi_{\widehat{G}}"] & 1 \\
 & (\mathbb{Z}_p[\chi])^\times & &
\end{tikzcd}$$

As the cokernel of $\pi_1^{\acute{e}t}(\xi)$, λ_ξ classifies the $(\mathbb{Z}/N)^\times$-torsor $\xi : \mathcal{M}_{ell}^{ord}(p^v, \Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. It follows the that there exists a unique character $\chi_{\widehat{G}} : (\mathbb{Z}/N)^\times \to \mathbb{Z}_p[\chi]$ such that for any $\sigma \in \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))$, $\rho_{\widehat{G}}(\sigma) = (\rho^1(\sigma))^k \cdot (\chi_{\widehat{G}} \circ \lambda_\xi)(\sigma)$.

This $\chi_{\widehat{G}}$ is the character corresponding to $\widehat{G}$ in (3.3). Since $\widehat{C}^{k,\chi}$ is constructed using χ, we have

$$\rho^{k,\chi}(\sigma) = (\rho^1(\sigma))^k \cdot (\chi \circ \lambda_\xi)(\sigma) = ((-)^k \cdot \chi(-)) \circ (\rho^1 \times \lambda_\xi)(\sigma)$$

for all $\sigma \in \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')))$. □

Now we need to find the image of $\rho^1 \times \lambda_\xi$.

Proposition 3.8. *$\rho^1 \times \lambda_\xi : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \longrightarrow \mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times$ factorizes as:*

$$\rho^1 \times \lambda_\xi : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \xrightarrow{\rho^1 \times \lambda_{N'}} \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \xrightarrow{(a,b)\mapsto(a,[a],b)} \mathbb{Z}_p^\times \times (\mathbb{Z}/p^v)^\times \times (\mathbb{Z}/N')^\times \simeq \mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times,$$

where $\lambda_{N'} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \to (\mathbb{Z}/N')^\times$ classifies the $(\mathbb{Z}/N')^\times$-torsor $\mathcal{M}_{ell}^{ord}(\Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$.

Proof. We prove the factorization by translating Galois representations into torsors over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$.

Lemma 3.1. *The character $\rho^1 : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \to \mathbb{Z}_p^\times$ classifies the $\mathbb{Z}_p^\times$-torsor $\mathcal{M}_{ell}^{triv}(\Gamma_0(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$, where $\mathcal{M}_{ell}^{triv}(\Gamma_0(N'))$ is a stack whose R-points are*

$$\mathcal{M}_{ell}^{triv}(\Gamma_0(N'))(R) := \{(C/R, \eta : \widehat{G}_m \xrightarrow{\sim} \widehat{C}, H \subseteq C[N']) \mid H \simeq \underline{\mathbb{Z}/N'}\}.$$

Proof of the Lemma. Recall that $[\rho^1] \in H^1(\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))); \mathbb{Z}_p^\times)$ is the Galois descent data for $\widehat{C}$, the formal group of the universal elliptic curve over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. The character ρ^1 then corresponds to a $\mathbb{Z}_p^\times$-torsor over $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ such that its fiber over the an R-point $(C/R, H \subseteq C[N'])$ is the set of triples $(C/R, \eta : \widehat{G}_m \xrightarrow{\sim} \widehat{C}, H \subseteq C[N'])$. □

Lemma 3.1 implies that the character $\rho^1 \times \lambda_\xi$ classifies the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times$-torsor $\mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$, where $\mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N'))$ is a stack whose R-points are

$$\mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N'))(R) := \{(C/R, \eta, \eta_p, \eta') \mid \eta : \widehat{G}_m \xrightarrow{\sim} \widehat{C}, \eta_p : \mu_{p^v} \xrightarrow{\sim} \widehat{C}[p^v], \eta' : \underline{\mathbb{Z}/N'} \hookrightarrow C[N']\}.$$

Sitting in between $\mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N'))$ and $\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$ is the stack $\mathcal{M}_{ell}^{triv}(\Gamma_1(N'))$, whose R-points are

$$\mathcal{M}_{ell}^{triv}(\Gamma_1(N'))(R) := \{(C/R, \eta, \eta') \mid \eta : \widehat{G}_m \xrightarrow{\sim} \widehat{C}, \eta' : \underline{\mathbb{Z}/N'} \hookrightarrow C[N']\}.$$

In the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times$-torsor

$$\mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N')) \longrightarrow \mathcal{M}_{ell}^{triv}(\Gamma_1(N')) \longrightarrow \mathcal{M}_{ell}^{ord}(\Gamma_0(N')),$$

the first map $\mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N')) \longrightarrow \mathcal{M}_{ell}^{triv}(\Gamma_1(N'))$ is a $(\mathbb{Z}/p^v)^\times$-torsor that admits a section:

$$s : \mathcal{M}_{ell}^{triv}(\Gamma_1(N')) \longrightarrow \mathcal{M}_{ell}^{triv}(p^v, \Gamma_1(N')), \quad (C/R, \eta, \eta') \longmapsto (C/R, \eta, \eta|_{\widehat{C}[p^v]}, \eta').$$

The existence of this section implies that $\rho^1 \times \lambda_\xi$ must factor through $\rho^1 \times \lambda_{N'} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \to$

$\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$, the character corresponding to the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-torsor $\mathcal{M}_{ell}^{triv}(\Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$. The formula of s then yields a commutative diagram:

$$\begin{tikzcd}
\pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \arrow[r, "\rho^1\times\lambda_\xi"] \arrow[d, "\rho^1\times\lambda_{N'}"'] & \mathbb{Z}_p^\times \times (\mathbb{Z}/N)^\times \arrow[d, equal] \\
\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \arrow[r, "{(a,b)\mapsto(a,[a],b)}"] & \mathbb{Z}_p^\times \times (\mathbb{Z}/p^v)^\times \times (\mathbb{Z}/N')^\times
\end{tikzcd}$$

□

Combining Proposition 3.7 and Proposition 3.8, we have shown

Corollary 3.2. *$\rho^{k,\chi} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')) \to (\mathbb{Z}_p[\chi])^\times$ factorizes as*

$$\rho^{k,\chi} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \xrightarrow{\rho^1\times\lambda_{N'}} \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \xrightarrow{(a,b)\mapsto\chi_p(a)\chi'(b)a^k} (\mathbb{Z}_p[\chi])^\times. \tag{3.6}$$

To relate the congruence of $\rho^{k,\chi}$ with that of the second map in (3.6), it remains to show:

Proposition 3.9. *$\rho^1 \times \lambda_{N'} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \longrightarrow \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$ is surjective.*

Proof. By [Sza09, Theorem 5.4.2], the surjectivity of $\rho^1 \times \lambda_{N'}$ is equivalent to the connectivity of the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-torsor it classifies. As $\rho^1 \times \lambda_{N'}$ classifies the torsor $\mathcal{M}_{ell}^{triv}(\Gamma_1(N')) \to \mathcal{M}_{ell}^{ord}(\Gamma_0(N'))$, we need to show $\mathcal{M}_{ell}^{triv}(\Gamma_1(N'))$ is connected.

By a relative version of Igusa's theorem in [KM85, Corrollary 12.6.2.(2)], $\mathcal{M}_{ell}^{triv}(\Gamma_1(N'))$ is connected whenever $\mathcal{M}_{ell}^{ord}(\Gamma_1(N'))$ is. The integral stack $\mathcal{M}_{ell}(\Gamma_1(N'))$ has geometrically connected fiber by [Con07, Theorem 1.2.1]. It is also smooth by [KM85, Corollary 4.7.1]. It follows that $\mathcal{M}_{ell}(\Gamma_1(N'))$ is irreducible and so is its p-completion $\mathcal{M}_{ell}(\Gamma_1(N'))_p^\wedge$. From this we conclude $\mathcal{M}_{ell}^{ord}(\Gamma_1(N'))$ is irreducible (hence connected), since it is an open substack of an irreducible stack. □

Now by Corollary 3.2 and Proposition 3.9, we have proved:

Corollary 3.3 (Step III)**.** *Let $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ be an ideal. The followings are equivalent:*

(iv). *The composition $\rho^{k,\chi} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \longrightarrow (\mathbb{Z}_p[\chi])^\times \twoheadrightarrow (\mathbb{Z}_p[\chi]/\mathcal{I})^\times$ is trivial.*

(v). *The composition $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \xrightarrow{(a,b)\mapsto\chi_p(a)\chi'(b)a^k} (\mathbb{Z}_p[\chi])^\times \twoheadrightarrow (\mathbb{Z}_p[\chi]/\mathcal{I})^\times$ is trivial.*

3.6 Restatement of the Main Theorem

Combining Proposition 3.3, Corollary 3.1, and Corollary 3.3, we now restate the Main Theorem:

Theorem 3.3 (Main Theorem, restated)**.** *Let $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ be an ideal. Then the followings are equivalent:*

(i). *There is an Eisenstein series $f \in \mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ such that $f(q) \in 1 + \mathcal{I}q[\![q]\!]$.*

(ii). *There is a generator $\gamma \in H^0(\mathcal{M}_{ell}^{ord}(\Gamma_0(N')), \boldsymbol{\omega}^{k,\chi})$ such that $F^{k,\chi}(\gamma) \equiv \gamma \mod \mathcal{I}$.*

(iii). $\widehat{C}^{k,\chi}[\mathcal{I}] \simeq (\widehat{G}_m \otimes \mathbb{Z}_p[\chi])[\mathcal{I}]$.

(iv). *The Galois descent data $\rho^{k,\chi} : \pi_1^{\acute{e}t}(\mathcal{M}_{ell}^{ord}(\Gamma_0(N'))) \to (\mathbb{Z}_p[\chi])^\times$ of $\widehat{C}^{k,\chi}$ is trivial modulo $\mathcal{I}$.*

(v). *The character $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \xrightarrow{(a,b)\mapsto \chi_p(a)\chi'(b)a^k} (\mathbb{Z}_p[\chi])^\times$ is trivial modulo $\mathcal{I}$.*

Remark 3.4. When the character χ is trivial, we recover Katz's algebro-geometric explanation of congruences of p-adic Eisenstein series of level 1 in [Kat73b, Corollary 4.4.1]. In that case, Step I in the proof above is not needed.

Chapter 4

The maximal congruence

Theorem 3.3 identifies the maximal congruence of Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ with that of $\mathbb{Z}_p^{\otimes k}[\chi]$ as a $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representation in $\mathbb{Z}_p[\chi]$-modules. In this chapter, we first compute the maximal congruence of $\mathbb{Z}_p^{\otimes k}[\chi]$ and then find explicit examples of Eisenstein series that realize this congruence in certain cases. In the final section, we relate the maximal congruence of $\mathbb{Z}_p^{\otimes k}[\chi]$ to the group cohomology of $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$.

4.1 Congruences of p-adic representations

Definition 4.1. Let R be a p-complete local ring and M be a torsion-free R-module with a continuous R-module action by a profinite group G. M is said to be a trivial G-representation modulo an ideal $\mathcal{I} \trianglelefteq R$ if G acts on $M/\mathcal{I}M$ trivially, or equivalently $(M/\mathcal{I}M)^G = M/\mathcal{I}M$. The maximal congruence of M as a G-representation is the smallest ideal $\mathcal{I}$ such that $M/\mathcal{I}M$ is a trivial G-representation.

Remark 4.1. The G-action on the quotient $M/\mathcal{I}M$ is well defined since G acts by R-linear maps. Otherwise, we need to assume $\mathcal{I} \trianglelefteq R$ is a G-invariant ideal, i.e. $g\mathcal{I} = \mathcal{I}$ for all $g \in G$.

Lemma 4.1. *When the underlying R-module of the G-representation M is R, the G-action of M is then associated to a character $\chi : G \to R^\times$. Let $\{g_i \mid i \in I\}$ be a set of generators of G. The maximal congruence of M is the ideal $(1 - \chi(g_i) \mid i \in I)$.*

Proof. The maximal congruence of M is by definition the ideal $(1 - \chi(g) \mid g \in G)$. Notice that

$$(1 - \chi(gg')) = (1 - \chi(g) + \chi(g) - \chi(gg')) \subseteq (1 - \chi(g)) + (\chi(g) - \chi(gg')) = (1 - \chi(g)) + (1 - \chi(g')).$$

and that $(1 - \chi(g^{-1})) = (\chi(g) - 1)$, we have $(1 - \chi(g) \mid g \in G) = (1 - \chi(g_i) \mid i \in I)$. □

When $p > 2$, $\mathbb{Z}_p^\times$ is topologically cyclic. When $p = 2$, $\mathbb{Z}_2^\times = \{\pm 1\} \times (1 + 4\mathbb{Z}_2)$ and $1 + 4\mathbb{Z}_2$ is topologically cyclic. Let g be a topological generator of $\mathbb{Z}_p^\times$ when $p > 2$ and a topological generator of $1 + 4\mathbb{Z}_2$ when $p = 2$.

Theorem 4.1. *The congruences of $\mathbb{Z}_p^{\otimes k}[\chi]$ have seven cases:*

I. *$p > 2$ and the conductor of χ is p or 1. In this case, $\chi = \omega^a$ for some integer $0 \le a \le p-2$, where $\omega : (\mathbb{Z}/p)^\times \to \mathbb{Z}_p^\times$ is the p-**adic Teichmüller character**. The image of χ is contained in $\mathbb{Z}_p^\times$. Then the maximal congruence of $\mathbb{Z}_p^{\otimes k}[\omega^a]$ is the following ideal in $\mathbb{Z}_p = \mathbb{Z}_p[\omega^a]$:*

$$(1 - g^k\chi(g)) = (1 - g^k\omega^a(g)) = \begin{cases} (p^{v_p(k)+1}), & (p-1) \mid (k+a); \\ (1) & \text{otherwise.} \end{cases}$$

II. *$p = 2$ and the conductor of χ is 4 or 1. In this case $\chi = \omega^a$ for $a = 0$ or 1, where $\omega : (\mathbb{Z}/4)^\times \to \mathbb{Z}_2^\times$ is the 2-adic Teichmüller character. As $g \in 1 + 4\mathbb{Z}_2$, $\omega(g) = 1$. Again the image of χ is contained in $\mathbb{Z}_2^\times$. Then the maximal congruence of $\mathbb{Z}_2^{\otimes k}[\omega^a]$ is the following ideal in $\mathbb{Z}_2 = \mathbb{Z}_2[\omega^a]$:*

$$(1 - g^k\omega^a(g), 1 - (-1)^k\omega^a(-1)) = \begin{cases} (2^{v_p(k)+2}), & 2 \mid (k+a); \\ (2), & \text{otherwise.} \end{cases}$$

III. *$p > 2$ and the conductor of χ is $p^v > p$. In this case, $(\mathbb{Z}/p^v)^\times \simeq (\mathbb{Z}/p)^\times \times C_{p^{v-1}}$ and As χ is primitive of conductor p^v, $\chi|_{C_{p^{v-1}}}$ is injective. As a result, $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\zeta_{p^{v-1}}]$. $\mathbb{Z}_p[\zeta_{p^{v-1}}]$ is a p-complete local ring with uniformizer $1 - \zeta_{p^{v-1}}$. Write $\chi|_{(\mathbb{Z}/p)^\times} = \omega^a$ for some $0 \le a \le p-2$. Then the maximal congruence of $\mathbb{Z}_p^{\otimes k}[\omega^a]$ is the following ideal in $\mathbb{Z}_p[\zeta_{p^{v-1}}] = \mathbb{Z}_p[\chi]$:*

$$(1 - g^k\chi(g)) = (1 - \zeta_{p^{v-1}} g^k\omega^a(g)) = \begin{cases} (1 - \zeta_{p^{v-1}}), & (p-1) \mid (k+a); \\ (1), & \text{otherwise.} \end{cases}$$

IV. *$p = 2$ and the conductor of χ is $2^v > 4$. In this case, $(\mathbb{Z}/2^v)^\times \simeq (\mathbb{Z}/4)^\times \times C_{2^{v-2}}$. As χ is primitive of conductor 2^v, $\chi|_{C_{2^{v-2}}}$ is injective. As a result, $\mathbb{Z}_2[\chi] = \mathbb{Z}_2[\zeta_{2^{v-2}}]$. $\mathbb{Z}_2[\zeta_{2^{v-2}}]$ is a 2-complete local ring with uniformizer $1 - \zeta_{2^{v-2}}$. Write $\chi|_{(\mathbb{Z}/4)^\times} = \omega^a$ for $a = 0$ or 1. Then the maximal congruence of $\mathbb{Z}_2^{\otimes k}[\chi]$ is the following ideal in $\mathbb{Z}_2[\zeta_{2^{v-2}}] = \mathbb{Z}_2[\chi]$:*

$$(1 - \zeta_{2^{v-2}} g^k\omega^a(g), 1 - (-1)^k\omega^a(-1)) = (1 - \zeta_{2^{v-2}}) \quad \textit{for all } k \textit{ and } a.$$

V. *$N' \neq 1$ and $\mathbb{Q}_p(\chi')$ is not a totally ramified extension of $\mathbb{Q}_p$. In this case, the image of χ' contains of a root of unity $\zeta_{n'}$ whose order n' is coprime to p. As $1 - \zeta_{n'}$ is invertible in $\mathbb{Z}_p[\zeta_{n'}] \subseteq \mathbb{Z}_p[\chi]$, we have the maximal congruence of $\mathbb{Z}_p^{\otimes k}[\chi]$ is the ideal (1) in $\mathbb{Z}_p[\chi]$.*

VI. *$p > 2$, $N' \neq 1$ and $\mathbb{Q}_p(\chi')$ is a totally ramified extension of $\mathbb{Q}_p$. In the case, the image of χ' is generated* *by $\zeta_{p^{v'}}$ for some $v' \geq 1$. We have $\mathbb{Z}_p^{\otimes k}[\chi] = \mathbb{Z}_p[\zeta_{p^{\max(v-1,v')}}]$. Write $\chi_p|_{(\mathbb{Z}/p)^\times} = \omega^a$ for some $0 \leq a \leq p-2$. Then the maximal congruence of $\mathbb{Z}_p^{\otimes k}[\chi]$ is the following ideal in $\mathbb{Z}_p^{\otimes k}[\chi] = \mathbb{Z}_p[\zeta_{p^{\max(v-1,v')}}]$:*

$$(1 - g^k\chi(g), 1 - \zeta_{p^{v'}}) = (1 - \zeta_{p^{v-1}} g^k \omega^a(g), 1 - \zeta_{p^{v'}}) = \begin{cases} (1 - \zeta_{p^{\max(v-1,v')}}), & (p-1) \mid (k+a); \\ (1), & \text{otherwise}. \end{cases}$$

VII. *$p = 2$, $N' \neq 1$ and $\mathbb{Q}_2(\chi')$ is a totally ramified extension of $\mathbb{Q}_2$. In the case, the image of χ' is generated by $\zeta_{2^{v'}}$ for some $v' \geq 1$. We have $\mathbb{Z}_2^{\otimes k}[\chi] = \mathbb{Z}_2[\zeta_{2^{\max(v',v-2)}}]$. Write $\chi_2|_{(\mathbb{Z}/4)^\times} = \omega^a$ for $a = 0, 1$. Then the maximal congruence of $\mathbb{Z}_2^{\otimes k}[\chi]$ is the following ideal in $\mathbb{Z}_2[\zeta_{2^{\max(v',v-2)}}] = \mathbb{Z}_2[\chi]$:*

$$(1 - \zeta_{2^{v'}}, 1 - \zeta_{2^{v-2}} g^k \omega^a(g), 1 - (-1)^k \omega^a(-1)) = (1 - \zeta_{2^{\max(v',v-2)}}) \quad \text{for all } k \text{ and } a.$$

4.2 Realizations of the maximal congruence

Having computed the maximal congruence of the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representation $\mathbb{Z}_p^{\otimes}[\chi]$, now we give explicit examples of Eisenstein series in $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ whose q-expansions realize the maximal congruence.

Let k be an integer such that $(-1)^k = \chi(-1)$. Recall from Theorem 1.3 and Corollary 1.1 that Eisenstein subspace $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi) \otimes \mathbb{Q}_p$ is spanned by Eisenstein series of the forms:

$$E_{k,\chi^0,\chi}(q^t) = c \cdot E_{k,\chi}(q^t) = c \cdot \left(1 - \frac{2k}{B_{k,\chi}} \sum_{n\geq 1} \left(\sum_{0<d|n} \chi(d) d^{k-1} \right) q^{nt} \right),$$

$$E_{k,\chi_1,\chi_2}(q^t) = \sum_{n\geq 1} \left(\sum_{0<d|n} \chi_1^{-1}(n/d)\chi_2(d) d^{k-1} \right) q^{nt},$$

where

- c is $\mathbb{Z}_p[\chi]$ with the smallest valuation so that $E_{k,\chi^0,\chi}(q^t) \in \mathbb{Z}_p[\chi][\![q]\!]$.
- χ_1 and χ_2 are characters of conductors N_1 and N_2 satisfying $\chi_1/\chi_2 = \chi^{-1}$ and $(N_1 N_2 t) \mid N$.

By the q-expansion principle Proposition 1.7, an element of $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$ is a $\mathbb{Q}_p$-linear combination $f(q)$ of these $E_{k,\chi_1,\chi_2}(q)$ such that $f(q) \in \mathbb{Z}_p[\chi][\![q]\!]$. Write $E_{k,\chi^0,\chi}$ and E_{k,χ_1,χ_2} for $E_{k,\chi^0,\chi}(q)$ and $E_{k,\chi_1,\chi_2}(q)$, respectively. Using the arithmetic properties of generalized Bernoulli numbers in [Car59, Theorem 1, 3], we can check

Proposition 4.1. *In Cases I-V in Theorem 4.1, the Eisenstein series $E_{k,\chi}$ realizes the maximal congruence predicted in Theorem 3.3.*

Again by Theorem 1.2, $\frac{B_{k,\chi}}{k}$ is an algebraic p-adic integer when N is not a power of p. As a result $E_{k,\chi}(q)$ does not realize the maximal congruence in Cases VI and VII in Theorem 4.1. Instead, we need to consider linear combinations of basis in the Eisenstein subspace. In general, it is hard to write down the explicit formulas of Eisenstein series that satisfies the maximal congruence predicted in Theorem 3.3 and Theorem 4.1 in cases VI and VII. Here, we work out one of the simplest cases in the rest of this section.

Example 4.1. Consider the character $\chi : (\mathbb{Z}/\ell)^\times \to \mathbb{C}_p^\times$, where ℓ is a prime different from $p > 2$ and $\mathbb{Q}_p(\chi)$ is a totally ramified extension of $\mathbb{Q}_p$. In this case, $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\zeta_{p^m}]$ for some $m \geq 1$ is a p-complete local ring with uniformizer $\varpi = 1 - \zeta_{p^m}$. Write the maximal ideal of $\mathbb{Z}_p[\chi]$ by $\mathfrak{m}$. By Theorem 1.2, $\frac{B_{k,\chi}}{2k}$ is an algebraic p-adic integer. As a result, we can take $E_{k,\chi^0,\chi}$ to be:

$$E_{k,\chi^0,\chi} = \frac{B_{k,\chi}}{2k} - \sum_{n\geq 1}\left(\sum_{0<d|n} \chi(d)d^{k-1}\right)q^n.$$

Comparing Theorem 3.3 and Case VI in Theorem 4.1, we should expect to find Eisenstein series of weight k, level $\Gamma_1(\ell)$, and character χ that is congruent to 1 modulo $\mathfrak{m} = (\varpi)$ only when $(p-1) \mid k$, and there is no Eisenstein series of level $\Gamma_1(\ell)$ and character χ that is congruent to 1 modulo $\mathfrak{m}^2$. The Eisenstein subspace in this case $\mathcal{E}_k(\Gamma_1(\ell), \chi)$ is spanned by $E_{k,\chi^0,\chi}(q)$ and $E_{k,\chi^{-1},\chi^0}(q)$.

When $(p-1) \mid k$, the maximal congruence is the realized by a linear combination of the two basis Eisenstein series, since neither of them satisfies the maximal congruence relation. As $\frac{B_{k,\chi}}{2k} \in \mathbb{Z}_p[\chi]$, we have $\frac{B_{k,\chi}}{2k} \cdot E_{k,\chi} \in \mathbb{Z}_p[\chi][\![q]\!]$. Notice that the coefficients of q in $E_{k,\chi^0,\chi}$ and E_{k,χ^{-1},χ^0} are -1 and 1, respectively. Consider the q-expansion of their sum:

$$E_{k,\chi^0,\chi} + E_{k,\chi^{-1},\chi^0} = \frac{B_{k,\chi}}{2k} + \sum_{n\geq 1} a_n q^n = \frac{B_{k,\chi}}{2k} + \sum_{n\geq 1}\left(\sum_{0<d|n} (\chi^{-1}(n/d) - \chi(d))d^{k-1}\right)q^n. \tag{4.1}$$

Lemma 4.2. *$E_{k,\chi^0,\chi} + E_{k,\chi^{-1},\chi^0} \equiv \frac{B_{k,\chi}}{2k} \mod \mathfrak{m}q[\![q]\!]$ for all k with $(-1)^k = \chi(-1)$.*

Proof. We need to show the coefficient a_n of q^n in (4.1) is in $\mathfrak{m}$ for all $n \geq 1$. Write $n = \ell^v n'$ where $\ell \nmid n'$.

Since the conductor of χ is the prime number ℓ, $\chi(a) = 0$ iff $\ell \mid a$. As a result, we have

$$\begin{aligned}
a_n = \sum_{0<d\mid n} (\chi^{-1}(n/d) - \chi(d))d^{k-1} &= \sum_{0<d\mid n} \chi^{-1}(n/d)d^{k-1} - \sum_{0<d\mid n} \chi(d)d^{k-1} \\
&= \sum_{\ell^v \mid d \mid n} \chi^{-1}(n/d)d^{k-1} - \sum_{0<d\mid n'} \chi(d)d^{k-1} \\
\text{(set } d = \ell^v d' \text{ in the first summation)} \quad &= \sum_{0<d'\mid n'} \chi^{-1}(n'/d')(\ell^v d')^{k-1} - \sum_{0<d\mid n'} \chi(d)d^{k-1} \\
&= \sum_{0<d'\mid n'} \chi^{-1}(n')\chi(d')\ell^{v(k-1)}d'^{k-1} - \sum_{0<d\mid n'} \chi(d)d^{k-1} \\
&= (\chi^{-1}(n')\ell^{v(k-1)} - 1) \sum_{0<d\mid n'} \chi(d)d^{k-1}. \qquad (4.2)
\end{aligned}$$

Since $(\mathbb{Z}/\ell)^\times$ surjects onto C_{p^m} by assumption, $\ell \equiv 1 \mod p$. This implies $\ell^{v(k-1)} \equiv 1 \mod p$. Also, as $\chi^{-1}(n') \neq 0$ is a p-power root of unity, $1 - \chi^{-1}(n') \in \mathfrak{m}$. Combining these two facts, we conclude

$$\chi^{-1}(n')\ell^{v(k-1)} - 1 = \chi^{-1}(n') - 1 + \chi^{-1}(n')(\ell^{v(k-1)} - 1) \in \mathfrak{m}$$

for all n' not divided by ℓ. This shows $a_n \in \mathfrak{m}$ for all n. From this, we conclude $E_{k,\chi^0,\chi} + E_{k,\chi^{-1},\chi^0} \equiv \frac{B_{k,\chi}}{2k} \mod \mathfrak{m}q[\![q]\!]$. ☐

Proposition 4.2. *The algebraic p-adic integer* $\frac{B_{k,\chi}}{2k}$ *is in* $\mathfrak{m}$ *iff* $(p-1) \nmid k$.

Proof. When $(p-1) \nmid k$, there is no Eisenstein series in $\mathcal{E}_k(\Gamma_1(\ell), \chi)$ whose q-expansion is in $1 + \mathfrak{m}q[\![q]\!]$ by Theorem 3.3 and Case VI in Theorem 4.1. In Lemma 4.2, we showed all the a_n's in (4.1) are in $\mathfrak{m}$. This implies its constant term $\frac{B_{k,\chi}}{2k}$ must also be in $\mathfrak{m}$ so that there is a common factor.

When $(p-1) \mid k$, Theorem 3.3 and Case VI in Theorem 4.1 predicts an Eisenstein series in $\mathcal{E}_k(\Gamma_1(\ell), \chi)$ whose q-expansion is in $1 + \mathfrak{m}q[\![q]\!]$, or equivalently $(\mathbb{Z}_p[\chi])^\times + \mathfrak{m}q[\![q]\!]$. We can write this Eisenstein series as $\varpi^{-v}(aE_{k,\chi^0,\chi} + bE_{k,\chi^{-1},\chi^0})$ for some $v \geq 0$ and $a, b \in \mathbb{Z}_p[\chi]$ such that one of them is in $(\mathbb{Z}_p[\chi])^\times$. Notice the coefficients of q in the q-expansions of this Eisenstein series is $\varpi^{-v}(-a+b)$. We have $b-a$ is in $\mathfrak{m}^{v+1} \subseteq \mathfrak{m}$. This implies $v_p(a) = v_p(b) = 0$. Without loss of generality, we can then assume $a = 1$ and $b \equiv 1 \mod \mathfrak{m}^{v+1} \subseteq \mathfrak{m}$. It now suffices to prove $v = 0$, for that implies $\frac{B_{k,\chi}}{2k}$, the constant term of $E_{k,\chi^0,\chi} + bE_{k,\chi^{-1},\chi^0}$ is in $(\mathbb{Z}_p[\chi])^\times$.

Suppose $v > 0$. Following (4.2), we have

$$\begin{aligned} E_{k,\chi^0,\chi} + bE_{k,\chi^{-1},\chi^0} &= E_{k,\chi^0,\chi} + E_{k,\chi^{-1},\chi^0} + (b-1)E_{k,\chi^{-1},\chi^0} && (4.3)\\ &= \frac{B_{k,\chi}}{2k} + \sum_{n\geq 1} b_n q^n && (4.4)\\ (\text{set } n = \ell^v n') \quad &= \frac{B_{k,\chi}}{2k} + \sum_{n\geq 1}\left((b\chi^{-1}(n')\ell^{v(k-1)} - 1) \sum_{0<d|n'} \chi(d)d^{k-1}\right) q^n \end{aligned}$$

Lemma 4.2 and (4.3) imply $E_{k,\chi^0,\chi} + bE_{k,\chi^{-1},\chi^0} \equiv \frac{B_{k,\chi}}{2k} \mod \mathfrak{m}q[\![q]\!]$. Now we want to find a b_n in (4.4) that is not in $\mathfrak{m}^2$. Let p' be a prime number such that $\chi(p')$ is a primitive p^m-th root of unity and that $p' \not\equiv (-1)$ mod p. This assumption implies $p' \neq \ell$. If $p' = p$ does not satisfy the assumption (i.e. $\chi(p)$ is not a primitive p^m root of unity), then there are infinitely many choices of the prime p'. This is because the conditions on p' depend only on its residual class modulo $p \cdot \ell$. In this case, we have by (4.4),

$$b_{p'} = (b\chi^{-1}(p') - 1)(1 + \chi(p')p'^{k-1}) = ((b-1)\cdot\chi^{-1}(p') + \chi^{-1}(p') - 1)(1 + \chi(p')p'^{k-1}).$$

Notice:

- $1 - \chi^{-1}(p')$ is a uniformizer in $\mathbb{Z}_p[\chi]$, since $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\zeta_{p^m}] = \mathbb{Z}_p[\chi(p')]$.
- $b - 1 \in \mathfrak{m}^{v+1} \subseteq \mathfrak{m}^2$, since $v > 0$ by assumption. We have $(b\chi^{-1}(p') - 1)$ is also a uniformizer.
- $p'^{k-1} \equiv p'^{-1} \not\equiv (-1) \mod p$, since $(p-1) \mid k$.

We have $1 + \chi(p')p'^{k-1} \notin \mathfrak{m}$. As a result, $b_{p'} \in \mathfrak{m} - \mathfrak{m}^2$ and $\varpi^{-v} b_{p'} \notin \mathfrak{m}$. This contradicts the assumption that $\varpi^{-v}(E_{k,\chi^0,\chi} + bE_{k,\chi^{-1},\chi^0}) \in (\mathbb{Z}_p[\chi])^\times + \mathfrak{m}q[\![q]\!]$ is an Eisenstein series that realizes the maximal congruence. Consequently, we have $v = 0$, $\frac{B_{k,\chi}}{2k} \in (\mathbb{Z}_p[\chi])^\times$ when $(p-1) \mid k$. □

Remark 4.2. When $(p-1) \nmid k$, it is possible that $\frac{B_{k,\chi}}{2k} \in \mathfrak{m}^s$ for some $s > 1$.

It follows that when $(p-1) \nmid k$, the maximal congruence in $\mathcal{E}_k(\Gamma_1(\ell), \chi)$ is realized by:

$$\frac{2k \cdot \left(E_{k,\chi^0,\chi} + bE_{k,\chi^{-1},\chi^0}\right)}{B_{k,\chi}} = 1 + \frac{2k}{B_{k,\chi}} \sum_{n\geq 1} \left(\sum_{0<d|n} (b\chi^{-1}(n/d) - \chi(d))d^{k-1}\right) q^n \in 1 + \mathfrak{m}q[\![q]\!],$$

where $b \in 1 + \mathfrak{m} \subseteq \mathbb{Z}_p[\chi]$.

4.3 Congruence and group cohomology

Let $\mathbb{Z}_p^{\otimes k}[\chi]$ be the $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representation associated to the character $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times \xrightarrow{(a,b)\mapsto\chi_p(a)\chi'(b)a^k} (\mathbb{Z}_p[\chi])^\times$. The maximal congruence of $\mathbb{Z}_p^{\otimes k}[\chi]$ as a $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$-representation is closely related to its group cohomology.

Lemma 4.3. *Suppose $(R, \mathfrak{m})$ is a p-complete discrete valuation ring and let $\varpi \in \mathfrak{m}$ be a uniformizer. The maximal congruence of M is the ideal $\mathfrak{m}^k$ such that*

$$\operatorname*{colim}_m((M/\varpi^m)^G) = \operatorname*{colim}_m((M/\mathfrak{m}^m)^G) = M/\mathfrak{m}^k.$$

Proposition 4.3. *When G is topologically finitely generated, the natural map*

$$\operatorname*{colim}_m((M/\varpi^m)^G) \longrightarrow \left(\operatorname*{colim}_m(M/\varpi^m)\right)^G =: (M/\varpi^\infty)^G$$

is an isomorphism.

Proof. When G is topologically finitely generated, taking G-fixed points is a equivalent to a finite limit (in the 1-categorical sense). As a result, $(-)^G$ commutes with the sequential colimit $(-/\varpi^\infty)$ by [Mac71, Theorem 1 in Section IX.2]. ☐

As M is a torsion-free R-module, the total quotient module M/ϖ^∞ can also be obtained from a short exact sequence of G-representations in R-modules:

$$0 \longrightarrow M \longrightarrow \varpi^{-1}M \longrightarrow M/\varpi^\infty \longrightarrow 0. \tag{4.5}$$

Remark 4.3. (4.5) is the colimit of the following tower of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\varpi"] \arrow[d, equal] & M \arrow[r] \arrow[d, "\varpi"] & M/\varpi \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "\varpi^2"] \arrow[d, equal] & M \arrow[r] \arrow[d, "\varpi"] & M/\varpi^2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "\varpi^3"] \arrow[d, equal] & M \arrow[r] \arrow[d, "\varpi"] & M/\varpi^3 \arrow[r] \arrow[d] & 0 \\
& \cdots & \cdots & \cdots &
\end{tikzcd}$$

Proposition 4.4. *Assumptions and notations as above. When $M^G = 0$, there is a natural injection $\delta : (M/\varpi^\infty)^G \to H^1(G; M)$ is injective.*

Proof. Apply $H^*(G;-)$ on (4.5), we get a long exact sequence of G cohomology that start with:

$$0 \longrightarrow M^G \longrightarrow (\varpi^{-1}M)^G \longrightarrow (M/\varpi^\infty)^G \xrightarrow{\delta} H^1(G;M) \longrightarrow H^1(G;\varpi^{-1}M) \longrightarrow \cdots$$

The fixed points $(\varpi^{-1}M)^G = 0$ since

$$(\varpi^{-1}M)^G = \left(\operatorname{colim}(M \xrightarrow{\varpi} M \xrightarrow{\varpi} \cdots)\right)^G \simeq \operatorname{colim}(M^G \xrightarrow{\varpi} M^G \xrightarrow{\varpi} \cdots) = \varpi^{-1}(M^G) = 0.$$

This shows δ is injective. ☐

Theorem 4.2. *The connecting homomorphism* $\delta : (\mathbb{Z}_p^{\otimes k}[\chi]/\varpi^\infty)^{\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times} \to H^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes k}[\chi])$ *is an isomorphism.*

Proof. In this case, $G = \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$ is topologically finitely generated, $R = \mathbb{Z}_p[\chi]$ and $M = \mathbb{Z}_p^{\otimes k}[\chi]$. $R = \mathbb{Z}_p[\chi]$ is a p-complete discrete valuation ring since it is isomorphic to the form $\mathbb{Z}_p[\zeta_n]$ for some n. As $\mathbb{Z}_p[\zeta_n]$ is an integral domain, the action of an non-identity element $(a,b) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$ on $\mathbb{Z}_p[\zeta_n]$ by multiplication by $\chi_p(a)\chi'(b)a^k$ has no fixed points. By Proposition 4.4, the connecting homomorphism δ is injective.

As p is a power of the uniformizer ϖ, $\varpi^{-1}M = p^{-1}M = \mathbb{Q}_p^{\otimes k}(\chi)$. We now show $H^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Q}_p^{\otimes k}(\chi)) = 0$, which would imply δ is surjective. Write $G = \mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times = G_{fin} \times G_{pro}$, where

- G_{fin} is the maximal finite subgroup of G.
- $G_{pro} = 1 + p\mathbb{Z}_p$ when $p > 2$ and $G_{pro} = 1 + 4\mathbb{Z}_p$ when $p = 2$.

Since $\mathbb{Q}_p^{\otimes k}(\chi)$ is a $\mathbb{Q}_p$-module, we have $H^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Q}_p^{\otimes k}(\chi)) \simeq H^0(G_{fin}; H^1(G_{pro}; \mathbb{Q}_p^{\otimes k}(\chi)))$. This is because the associated Hochschild-Serre spectral sequences is concentrated in the 0-line. Let g be a pro-generator of G_{pro}. From the continuous resolution in (2.4), we have

$$H^1\left(G_{pro}; \mathbb{Q}_p^{\otimes k}(\chi)\right) \simeq \mathbb{Q}_p(\chi) \big/ \left(1 - \chi_p(g)g^k\right),$$

where g is viewed as an element of $\mathbb{Q}_p(\chi)$ via $g \in G_{pro} = 1 + 2p\mathbb{Z}_p \subseteq \mathbb{Q}_p(\chi)$. The quotient is zero since $1 - \chi_p(g)g^k \neq 0$ and $\mathbb{Q}_p(\chi) = \mathbb{Q}_p(\zeta_n)$ is a field. This implies $H^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Q}_p^{\otimes k}(\chi)) = 0$. Consequently, the connecting homomorphism δ is surjective. ☐

Remark 4.4 (Patrick Allen). Unlike the finite group case, it is in general NOT true that $H^s(G;M) = 0$ for $s > 0$ when G is profinite and M is a $\mathbb{Q}_p$-module. Using Kummer theory, one can construct explicit examples G and M when the group cohomology $H^1(G;M)$ is non-zero.

However, it is true that $H^s(G;M) = 0$ when $s > 0$, if $M = \bigcup_{H \le G} M^H$ where H ranges over all open subgroups of G. Such an M is called a discrete G-module. In this case, we have by [Ser97, Corollary 1 in §2.2]:

$$H^s(G;M) \simeq \underset{H \le G \text{ open}}{\operatorname{colim}} H^s(G/H; M^H).$$

This is group cohomology is zero when $s > 0$, since G/H is finite for any open subgroup H of G and $M^H \subseteq M$ is a $\mathbb{Q}_p$-module. It is straight forward to check that $\mathbb{Z}_p[\chi]^{\otimes k}$ is not a discrete $\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times$. That is why we have to explicitly compute $H^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Q}_p^{\otimes k}(\chi)) = 0$ in Theorem 4.2.

Now combining Theorem 3.3 and Theorem 4.2 yields:

Corollary 4.1. *The followings are equivalent:*

1. $\mathcal{I} \trianglelefteq \mathbb{Z}_p[\chi]$ *is the maximal congruence of Eisenstein series in* $\mathcal{E}_k(p^v, \Gamma_1(N'), \chi)$.

2. $H^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes k}[\chi]) \simeq \mathbb{Z}_p[\chi]/\mathcal{I}$.

Remark 4.5. Comparing Corollary 4.1 with Proposition 4.1 and Proposition 4.2, the group cohomology $H^1(\mathbb{Z}_p^\times \times (\mathbb{Z}/N')^\times; \mathbb{Z}_p^{\otimes k}[\chi])$ computes the *denominator* of $\frac{B_{k,\chi}}{2k} \in \mathbb{Q}_p(\chi)$ under the assumptions in Cases I-V in Theorem 4.1. In Cases VI and VII, this cohomological computation sheds light on the *numerator* of $\frac{B_{k,\chi}}{2k}$ (still does not determine the valuation in general).

Remark 4.6. When the character χ is trivial, we have related the congruences of Eisenstein series E_{2k} of level 1 with the group cohomology $H^1(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes 2k})$. Recall from (2.1), this group cohomology is the target of the p-adic e-invariant and thus computes the p-completion of the image of the J-homomorphism $J_{4k-1} : \pi_{4k-1}(\mathrm{SO}) \to \pi_{4k-1}(S^0)$. In way we have given a new explanation of the connection between the congruences of E_{2k} and the image of J_{4k-1} in Corollary 4.1.

Part III

The Dirichlet J-spectra

Chapter 5

The construction of the Dirichlet J-spectra

Let $\chi : (\mathbb{Z}/N)^{\times} \to \mathbb{C}^{\times}$ be a primitive Dirichlet character of conductor N. In this chapter, we construct $J(N)^{h\chi}$, the Dirichlet J-spectrum in three steps:

1. Identify an integral model of the J-spectrum, a ring spectrum whose Hurewicz map detects the image of the J-homomorphism in $\pi_*(S^0)$.
2. Define $J(N)$, "the J-spectrum with μ_N-level structure" using local structures of the finite group scheme μ_N and the Hopkins-Miller theorem. $J(N)$ comes with a natural $(\mathbb{Z}/N)^{\times}$-action by assembling the $(\mathbb{Z}/p^v)^{\times}$-Galois action at each prime.
3. Construct a Moore spectrum $M(\mathbb{Z}[\chi])$ with a $(\mathbb{Z}/N)^{\times}$-action that lifts the $(\mathbb{Z}/N)^{\times}$-action on $\mathbb{Z}[\chi]$ induced by χ. Here $\mathbb{Z}[\chi]$ is the subalgebra of $\mathbb{C}$ generated by the image of χ. This construction is non-trivial since taking Moore spectrum is not functorial. We give an explicit construction of the Moore spectra with group actions suggested by Charles Rezk.

From these data, we define the Dirichlet J-spectrum associated to χ by

$$J(N)^{h\chi} := \operatorname{Map}\left(M(\mathbb{Z}[\chi]), J(N)\right)^{h(\mathbb{Z}/N)^{\times}}.$$

This definition leads to a spectral sequence whose E_2-page consists of derived χ-eigenspaces of $\pi_*(J(N))$:

$$E_2^{s,t} \simeq \operatorname{Ext}^s_{\mathbb{Z}[(\mathbb{Z}/N)^{\times}]}\left(\mathbb{Z}[\chi], \pi_t(J(N))\right) \Longrightarrow \pi_{t-s}\left(J(N)^{h\chi}\right).$$

The actual computation of $J(N)^{h\chi}$ is carried out by studying its local structures. Rationally, the Dirichlet J-spectra are contractible unless χ is trivial. Completed at each prime, the $J(N)^{h\chi}$ splits into a wedge sum of Dirichlet $K(1)$-local spheres. The Dirichlet $K(1)$-local spheres are constructed in a similar way as the Dirichlet J-spectra, but the p-adic Moore spectra with a prescribed $(\mathbb{Z}/N)^{\times}$-action induced χ is constructed by Cooke's obstruction theory in [Coo78]. This splitting of p-completion of integral Moore spectra uses the uniqueness part of Cooke's obstruction theory.

5.1 An integral model of the J-spectrum

In Chapter 2, we have explained the relations between the images of the stable J-homomorphisms and the $K(1)$-local spheres:

$$\mathrm{Im}(J_{4k-1})_p^{\wedge} \simeq \pi_{4k-1}\left(S^0_{K/p}\right), k > 0.$$

We are now going to define an integral J-spectrum by assembling the K/p-local spheres at each prime.

Theorem 5.1. [Bou79, Corollary 4.5, 4.6] *Let $J = S^0_K$, the Bousfield localization of the sphere spectrum S^0 at complex K-theory.*

1. *The J-spectrum and the K/p-local spheres are related by the arithmetic fracture square:*

$$\begin{tikzcd} J := S^0_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p S^0_{K/p} \arrow[d, "L_{\mathbb{Q}}"] \\ S^0_{\mathbb{Q}} \arrow[r, "h_{\mathbb{Q}}"'] & \left(\prod_p S^0_{K/p}\right)_{\mathbb{Q}} \end{tikzcd} \tag{5.1}$$

   Here $h_{\mathbb{Q}}$ is the rational Hurewicz map and $L_{\mathbb{Q}}$ is the rationalization map.

2. *Denote the denominator of $B_{2k}/4k$ by D_{2k}. We have:*

$$\pi_i(J) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2, & i = 0; \\ \mathbb{Q}/\mathbb{Z}, & i = -2; \\ \mathbb{Z}/D_{|2k|}, & i = 4k-1 \neq -1; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 1 \mod 8; \\ \mathbb{Z}/2, & i \equiv 0, 2 \mod 8 \text{ and } i \neq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

Corollary 5.1. *$J_p^{\wedge} \simeq S^0_{K/p}$ and $J_{(p)} \simeq S^0_{E(1)}$ is the Bousfield localization of S^0 at $E(1) := BP\langle 1\rangle$.*

Remark 5.1. $J := S^0_K$ is an $\mathbb{E}_\infty$-ring spectrum since it is the localization of an $\mathbb{E}_\infty$-ring spectrum by [EKM+97].

Proof. (5.1) is the almost same homotopy pullback diagram for S^0_K as in the proof of [Bou79, Corollary 4.7], except for the lower left corner – the rationalization of S^0_K is a priori $S^0_{K\mathbb{Q}}$, where $K\mathbb{Q} := K \wedge M\mathbb{Q}$ is the rational K-spectrum. Now it remains to show $K\mathbb{Q}$ and $H\mathbb{Q}$ are Bousfield equivalent. This follows from the facts that $K\mathbb{Q}$ and the periodic $HP\mathbb{Q} := \bigvee_i \Sigma^{2i} H\mathbb{Q}$ are equivalent cohomology theories via the Chern character map and that $HP\mathbb{Q}$ is Bousfield equivalent to $H\mathbb{Q}$.

The computation of $\pi_*(J)$ is the integral version of that of the $\pi_*\left(S^0_{E(1)}\right)$ in [Lur10, Theorem 6, Lecture 35]. The arithmetic fracture square (5.1) induces a long exact sequence of homotopy groups:

$$\cdots \to \pi_i(J) \longrightarrow \pi_i\left(S^0_{\mathbb{Q}}\right) \oplus \prod_p \pi_i\left(S^0_{K/p}\right) \longrightarrow \left(\prod_p \pi_i\left(S^0_{K/p}\right)\right) \otimes \mathbb{Q} \longrightarrow \pi_{i-1}(J) \to \cdots$$

Notice that $\left(\prod_p \pi_i\left(S^0_{K/p}\right)\right) \otimes \mathbb{Q} = 0$ unless $i = 0$ or -1 and $\pi_i\left(S^0_{\mathbb{Q}}\right) = 0$ unless $i = 0$, we have $\pi_i(J) \simeq \prod_p \pi_i\left(S^0_{K/p}\right)$ unless $i \in \{-2,-1,0\}$. In those three cases, there is an exact sequence:

$$0 \to \pi_0(J) \to \mathbb{Q} \oplus \prod_p \mathbb{Z}_p \oplus \mathbb{Z}/2 \xrightarrow{\pi_0(h_{\mathbb{Q}})} \prod_p \mathbb{Q}_p \to \pi_{-1}(J) \to \prod_p \mathbb{Z}_p \xrightarrow{\pi_{-1}(h_{\mathbb{Q}})} \prod_p \mathbb{Q}_p \to \pi_{-2}(J) \to 0.$$

As $\pi_0(h_{\mathbb{Q}})$ is surjective and $\pi_{-1}(h_{\mathbb{Q}})$ is injective, we have

$$\pi_0(J) \simeq \mathbb{Z} \oplus \mathbb{Z}/2, \quad \pi_{-1}(J) = 0, \quad \pi_{-2}(J) \simeq \mathbb{Q}/\mathbb{Z}.$$

For $i \neq 0,-1,-2$, we recover $\pi_i(J)$ from Section 2.3 and Theorem 1.1. ☐

Remark 5.2. We call S^0_K the J-spectrum because the Hurewicz map (also the K-localization map) $S^0 \longrightarrow S^0_K$ detects the image of J_{4k-1}. But $\pi_k(J)$ is not the same as the image of the stable J-homomorphism in general. The spectrum J is non-connective and has an extra $\mathbb{Z}/2$-summand in $\pi_0(J)$ and $\pi_{8k+1}(J)$ for $k > 0$. For details, see [Ada66].

5.2 J-spectra with level structures

We will now add level structures to the J-spectrum. Let μ_N be the N-torsion sub-group scheme of $\widehat{G}_m$. Define $\mathcal{M}_{mult}(N)$ to be the moduli stack of globally height 1 formal groups with μ_N-level structures. R-points of $\mathcal{M}_{mult}(N)$ are given by:

$$\mathcal{M}_{mult}(N)(R) := \left\{ \left(\widehat{G}, \eta : \mu_N \xrightarrow{\sim} \widehat{G}[N]\right) \middle| \begin{array}{c} \widehat{G} \text{ is a formal group over } R \\ \text{that has height 1 at all primes} \end{array} \right\}.$$

The local structures of $\mathcal{M}_{mult}(N)$ are determined by the local behaviors of μ_N.

Lemma 5.1. *$\widehat{G}_m$ has no non-trivial finite subgroup over $\mathbb{Q}$. Over $\mathbb{Z}_p$, finite subgroups of $\widehat{G}_m$ are of the form μ_{p^v} for some $v \geq 0$. As a result, $(\mu_N)_{\mathbb{Q}} \simeq 0$ for all N and $(\mu_N)^\wedge_p \simeq \mu_{p^v}$, where $v = v_p(N)$.*

Proof. This follows from the facts that $\mathrm{End}_{\mathbb{Q}}(\widehat{G}_m) \simeq \mathbb{Q}$ and $\mathrm{End}_{\mathbb{Z}_p}(\widehat{G}_m) \simeq \mathbb{Z}_p$. ☐

Proposition 5.1. *$(\mathcal{M}_{mult}(N))_{\mathbb{Q}} \simeq (\mathcal{M}_{mult})_{\mathbb{Q}}$. Fix a prime p and let $v = v_p(N)$, we have*

$$\mathcal{M}_{mult}(N)^{\wedge}_{p} \simeq \mathcal{M}_{mult}(p^v)^{\wedge}_{p} \simeq B(1 + p^v\mathbb{Z}_p).$$

Corollary 5.2. *$\mathcal{M}_{mult}(N) \simeq \mathcal{M}_{mult}(2N)$ for any odd number N.*

Proof. This follows from the fact $(\mathbb{Z}/2N)^{\times}$ is canonically isomorphic $(\mathbb{Z}/N)^{\times}$ if N is odd. ☐

Theorem 5.2 (Hopkins-Miller, Goerss-Hopkins)**.** [Rez98, Theorem 2.1] *Let $\mathcal{FG}$ denote the category whose objects are pairs (κ, Γ) where Γ is a finite height formal group over a finite field k of characteristic p and whose morphisms are pairs of maps $(i, f) : (\kappa_1, \Gamma_1) \to (\kappa_2, \Gamma_2)$, where $i : \kappa_1 \to \kappa_2$ is a ring homomorphism and $f : \Gamma_1 \xrightarrow{\sim} i^*\Gamma_2$ is an isomorphism of formal groups.*

Then there exists a functor $(\kappa, \Gamma) \to E_{\kappa,\Gamma}$ from $\mathcal{FG}^{\mathrm{op}}$ to the category of $\mathbb{E}_\infty$-ring spectra, such that

1. *$E_{\kappa,\Gamma}$ is a commutative ring spectra.*
2. *There is a unit in $\pi_2(E_{\kappa,\Gamma})$.*
3. *$\pi_{\mathrm{odd}}E_{\kappa,\Gamma} = 0$, which implies $E_{\kappa,\Gamma}$ is complex-oriented.*
4. *The formal group associated to $E_{\kappa,\Gamma}$ is the universal deformation of (κ, Γ).*

Proposition 5.2. *There is a sheaf $\mathcal{O}^{top}_{K(1)}$ of $K(1)$-local $\mathbb{E}_\infty$-ring spectra over the stack $\widehat{\mathcal{H}(1)} \simeq B\mathbb{Z}_p^{\times} :=$ $\operatorname{Spf}\mathbb{Z}_p//\mathbb{Z}_p^{\times}$ such that*

$$\Gamma(\mathcal{O}^{top}_{K(1)}, B\mathbb{Z}_p^{\times}) \simeq S^0_{K(1)}, \quad \Gamma(\mathcal{O}^{top}_{K(1)}, B(1 + p^v\mathbb{Z}_p)) \simeq S^0_{K(1)}(p^v) := \left(K_p^{\wedge}\right)^{h(1+p^v\mathbb{Z}_p)}.$$

Remark 5.3. Let $\widehat{\mathcal{H}(h)}$ be the moduli stack of formal groups over p-complete local rings with height h reductions modulo the maximal ideal. The Hopkins-Miller theorem and the Goerss-Hopkins theorem imply there is a sheaf of $K(h)$-local $\mathbb{E}_\infty$-ring spectra $\mathcal{O}^{top}_{K(h)}$ over $\widehat{\mathcal{H}(h)}$ whose global section is the $K(h)$-local sphere $S^0_{K(h)}$. For the algebro-geometric properties of the stack $\widehat{\mathcal{H}(h)}$, see [Goe08, Chapter 7].

Corollary 5.3 implies $\mathcal{M}_{mult}(N)^{\wedge}_{p} \simeq \mathcal{M}_{mult}(p^v)^{\wedge}_{p} \to (\mathcal{M}_{mult})^{\wedge}_{p}$ is a $(\mathbb{Z}/p^v)^{\times}$-torsor for each prime p. Thus by Proposition 5.2 we can define $J(N)$, the **J-spectrum with μ_N-level structure** by setting $J(N)^{\wedge}_{p} :=$ $\mathcal{O}^{top}_{K(1)}(\mathcal{M}_{mult}(p^v)) \simeq S^0_{K/p}(p^v)$ and $J(N)_{\mathbb{Q}} = S^0_{\mathbb{Q}}$ as follows:

Construction 5.1. $J(N)$ is the homotopy pullback of the following arithmetic fracture square as in (5.1):

$$\begin{tikzcd}
J(N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p S^0_{K/p}\left(p^{v_p(N)}\right) \arrow[d, "L_{\mathbb{Q}}"] \\
S^0_{\mathbb{Q}} \arrow[r, "h_{\mathbb{Q}}"'] & \left(\prod_p S^0_{K/p}\left(p^{v_p(N)}\right)\right)_{\mathbb{Q}}
\end{tikzcd} \tag{5.3}$$

Here $h_{\mathbb{Q}}$ is the rational Hurewicz map and $L_{\mathbb{Q}}$ is the rationalization map. $h_{\mathbb{Q}}$ exists because the lower right corner in the diagram is a rational ring spectrum.

The $J(N)$ defined above indeed satisfies the prescribed local properties:

Corollary 5.3. *$J(N)_{\mathbb{Q}} \simeq S^0_{\mathbb{Q}}$ for all N and $J(N)^{\wedge}_p \simeq S^0_{K(1)}(p^v)$, where $v = v_p(N)$. Moreover, $J(N) \simeq J(2N)$ for any odd number N.*

Proposition 5.3. *$J(N)$ admits a natural $(\mathbb{Z}/N)^{\times}$-action such that*

- *$(\mathbb{Z}/N)^{\times}$ acts on $J(N)_{\mathbb{Q}}$ trivially.*
- *$(\mathbb{Z}/N)^{\times}$ acts on $J(N)^{\wedge}_p \simeq S^0_{K(1)}(p^v)$ by the Galois action of its quotient group $(\mathbb{Z}/p^v)^{\times}$.*

Proof. Since the spectrum $S^0_{K(1)}(p^v)$ is a $(\mathbb{Z}/p^v)^{\times}$-Galois extension of $S^0_{K(1)}$, it admits a natural $(\mathbb{Z}/p^v)^{\times}$-action. As a result the product $\prod_p S^0_{K/p}\left(p^{v_p(N)}\right)$ admits a natural $(\mathbb{Z}/N)^{\times} \simeq \prod_{p|N} (\mathbb{Z}/p^v)^{\times}$-action. (When $p \nmid N$, $(\mathbb{Z}/N)^{\times}$ acts on $S^0_{K/p}$ trivially). The spectrum $\left(\prod_p S^0_{K/p}\left(p^{v_p(N)}\right)\right)_{\mathbb{Q}}$ in the lower right corner of (5.3) then inherits a $(\mathbb{Z}/N)^{\times}$-action from that on $\prod_p S^0_{K/p}\left(p^{v_p(N)}\right)$.

We now need to check the rational Hurewicz map $h_{\mathbb{Q}}$ in (5.3) is $(\mathbb{Z}/N)^{\times}$-equivariant. As both spectra are rational, it suffices to check the induced maps on homotopy groups are equivariant by Cooke's obstruction theory (see Section 5.3). Since $\pi_*(S^0_{\mathbb{Q}})$ is concentrated in π_0 and $(\mathbb{Z}/N)^{\times}$ acts on it trivially, it reduces to checking $(\mathbb{Z}/N)^{\times}$ acts $\pi_0\left(S^0_{K/p}\left(p^{v_p(N)}\right)_{\mathbb{Q}}\right)$ trivially. Recall from Definition 2.4, $S^0_{K/p}(p^v) := \left(K^{\wedge}_p\right)^{h(1+p^v\mathbb{Z}_p)}$. The HFPSS in Section 2.3 shows

$$\pi_0\left(S^0_{K/p}\left(p^v\right)_{\mathbb{Q}}\right) \simeq H^0\left(1+p^v\mathbb{Z}_p; \pi_0\left(K^{\wedge}_p\right)\right) \otimes \mathbb{Q}.$$

As the Adams operation ψ^a acts on $\pi_0\left(K^{\wedge}_p\right)$ trivially for all $a \in \mathbb{Z}_p^{\times}$, the residual $(\mathbb{Z}/p^v)^{\times}$-action on the group cohomology $H^*\left(1+p^v\mathbb{Z}_p; \pi_0\left(K^{\wedge}_p\right)\right)$ is also trivial. Hence $(\mathbb{Z}/p^v)^{\times}$ acts trivially on $\pi_0\left(S^0_{K/p}\left(p^v\right)_{\mathbb{Q}}\right)$.

We have shown the rational Hurewicz map $h_{\mathbb{Q}}$ is $(\mathbb{Z}/N)^{\times}$-equivariant. Then $J(N)$ as the homotopy pullback in (5.3) of a diagram of $(\mathbb{Z}/N)^{\times}$-equivariant maps of spectra has a natural $(\mathbb{Z}/N)^{\times}$-action with the prescribed local properties. ☐

Proposition 5.4. *$J(N)$ is a K-local $\mathbb{E}_\infty$-ring spectrum, with $(\mathbb{Z}/N)^\times$ acting on it by $\mathbb{E}_\infty$-ring automorphisms as described in Proposition 5.3.*

Proof. This proposition contains three parts:

1. $J(N)$ is an $\mathbb{E}_\infty$-ring spectrum since it is the homotopy pullback of $\mathbb{E}_\infty$-ring maps between $\mathbb{E}_\infty$-ring spectra.
2. $J(N)$ is K-local since $J(N)^\wedge_p \simeq S^0_{K/p}\left(p^{v_p(N)}\right)$ is K/p-local for all primes p by Corollary 5.3.
3. The action of $\left(\mathbb{Z}/p^{v_p(N)}\right)^\times$ on $J(N)^\wedge_p \simeq S^0_{K/p}\left(p^{v_p(N)}\right)$ is $\mathbb{E}_\infty$ by the Goerss-Hopkins theorem. Thus the action of $(\mathbb{Z}/N)^\times \simeq \prod_{p|N}\left(\mathbb{Z}/p^{v_p(N)}\right)^\times$ is $\mathbb{E}_\infty$ on the upper right corner of (5.3). This implies the induced $(\mathbb{Z}/N)^\times$-action on lower right corner is also $\mathbb{E}_\infty$. The trivial $(\mathbb{Z}/N)^\times$-action on $S^0_{\mathbb{Q}}$ is $\mathbb{E}_\infty$. We conclude $(\mathbb{Z}/N)^\times$ acts by $\mathbb{E}_\infty$-ring maps on $J(N)$ in Proposition 5.3, since the action is assembled from $\mathbb{E}_\infty$-actions on the other three corners of (5.3).

□

Remark 5.4. The homotopy fixed points $J(N)^{h(\mathbb{Z}/N)^\times}$ is in general not equivalent to J. As a result $J(N)$ is in general not a $(\mathbb{Z}/N)^\times$-Galois extension of J. One example is when $N = 3$, we have

$$\left(J(3)^{h(\mathbb{Z}/3)^\times}\right)^\wedge_2 \simeq \left(S^0_{K/2}\right)^{h(\mathbb{Z}/3)^\times} \simeq \left(S^0_{K/2}\right)_{h(\mathbb{Z}/3)^\times} \simeq (B\Sigma_2)_{K/2} \not\simeq S^0_{K/2} \simeq J^\wedge_2.$$

Here we use the following facts:

- Homotopy fixed points commute with p-completion.
- $J(3)^\wedge_2 \simeq S^0_{K/2}$ by Corollary 5.3.
- Homotopy fixed points of finite group actions in the $K(1)$-local category are equivalent to homotopy orbits.
- $(\mathbb{Z}/3)^\times$ acts on $S^0_{K/2}$ trivially and $(\mathbb{Z}/3)^\times \simeq C_2 \simeq \Sigma_2$.
- $(B\Sigma_p)_+ \simeq S^0_{K/p} \times S^0_{K/p}$ in the K/p-local category by [Hop14, Lemma 3.1].

In general, $J(N)^{h(\mathbb{Z}/N)^\times}$ is equivalent to J after inverting $\prod_{p|N}(p-1)$.

Analogous to (5.2), we now compute $\pi_*(J(N))$.

Proposition 5.5. *The computation of* $\pi_*(J(N))$ *has two cases:* $4 \mid N$ *and* N *is odd (since* $J(N) \simeq J(2N)$ *for odd* N*). Define* $D_{2k,N}$ *by*

$$D_{2k,N} = \begin{cases} ND_{2k}/(2\Pi), & \text{if } 4 \mid N; \\ ND_{2k}/\Pi, & \text{if } 2 \nmid N, \end{cases} \quad \text{where } \Pi = \prod_{p \mid N, (p-1) \mid (2k)} p.$$

When $4 \mid N$*, we get*

$$\pi_i(J(N)) = \begin{cases} \mathbb{Z}, & i = 0; \\ \mathbb{Q}/\mathbb{Z}, & i = -2; \\ \mathbb{Z}/D_{|2k|,N}, & i = 4k-1 \neq -1; \\ \mathbb{Z}/N, & i \equiv 1 \mod 4; \\ 0, & \text{otherwise.} \end{cases} \tag{5.4}$$

When N *is odd, we get*

$$\pi_i(J(N)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2, & i = 0; \\ \mathbb{Q}/\mathbb{Z}, & i = -2; \\ \mathbb{Z}/D_{|2k|,N}, & i = 4k-1 \neq -1; \\ \mathbb{Z}/N \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 1 \mod 8; \\ \mathbb{Z}/N, & i \equiv 5 \mod 8; \\ \mathbb{Z}/2, & i \equiv 0, 2 \mod 8 \text{ and } i \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 5.5. One can check from (5.4) that

$$\mathrm{Hom}(\pi_i(J(4N)), \mathbb{Q}/\mathbb{Z}) \simeq (\pi_{-2-i}(J(4N)))^\wedge$$

holds for all N and i, where $(-)^\wedge$ is the profinite completion of a group. The formula is true up to summands of $\mathbb{Z}/2$ for $J(N)$ when N is odd. This isomorphism suggests a possible Brown-Comenetz duality $I_{\mathbb{Q}/\mathbb{Z}}(J(4N)) \simeq \Sigma^2 J(4N)$. In particular, $\pi_{4k-1}(J(4)) \simeq \pi_{4k-1}(J) = \mathbb{Z}/D_{|2k|}$, whose order is equal to the denominator of $\zeta(1-2k)$ (expressed as a fraction in lowest terms). The suggested Brown-Comenetz duality for $J(4)$ is similar to the functional equation of the Riemann ζ-function:

$$\zeta(2k) = \frac{(2\pi i)^{2k}}{2(2k-1)!} \cdot \zeta(1-2k).$$

5.3 Constructing Moore spectra with group actions

Another ingredient needed to construct the Dirichlet J-spectra and $K(1)$-local spheres is a Moore spectrum with a $(\mathbb{Z}/N)^\times$-action induced by a (p-adic) Dirichlet character $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ (or $\mathbb{C}_p^\times$). The first observation is following:

Lemma 5.2. *There is a unique number n such that χ factorizes as*

$$\begin{array}{ll} \chi : (\mathbb{Z}/N)^\times \longrightarrow\!\!\!\!\!\rightarrow C_n \lhook\joinrel\longrightarrow (\mathbb{Z}[\zeta_n])^\times \lhook\joinrel\longrightarrow \mathbb{C}^\times, & \qquad \textit{when } \chi \textit{ is } \mathbb{C}\textit{-valued;} \\ \chi : (\mathbb{Z}/N)^\times \longrightarrow\!\!\!\!\!\rightarrow C_n \lhook\joinrel\longrightarrow (\mathbb{Z}_p[\zeta_n])^\times \lhook\joinrel\longrightarrow \mathbb{C}_p^\times, & \qquad \textit{when } \chi \textit{ is } \mathbb{C}_p\textit{-valued,} \end{array}$$

where C_n is the cyclic group of order n and the second maps send a generator $g \in C_n$ to a primitive n-th root of unity ζ_n.

Then it suffices to construct the Moore spectra $M(\mathbb{Z}[\zeta_n])$ and $M(\mathbb{Z}_p[\zeta_n])$ with C_n-actions such that the induced C_n-action on H_0 (equivalently π_0) is equivalent to that on $\mathbb{Z}[\zeta_n]$ and $\mathbb{Z}_p[\zeta_n]$. The latter is called the integral/p-adic cyclotomic representation of C_n. Properties of such representations needed in this section are summarized in Appendix B.

We can further reduce to cases $n = p^v$ by noting from Lemma B.1:

$$\begin{array}{rcc} & \mathbb{Z}[\zeta_n] \simeq \bigotimes\limits_{p\mid n} \mathbb{Z}[\zeta_{p^{v_p(n)}}] & \mathbb{Z}_p[\zeta_n] \simeq \bigotimes\limits_{q\mid n} \mathbb{Z}_p[\zeta_{q^{v_q(n)}}], \\ \xRightarrow{\text{non-equivariantly}} & M(\mathbb{Z}[\zeta_n]) \simeq \bigwedge\limits_{p\mid n} M(\mathbb{Z}[\zeta_{p^{v_p(n)}}]) & M(\mathbb{Z}_p[\zeta_n]) \simeq \bigwedge\limits_{q\mid n} M(\mathbb{Z}_p[\zeta_{q^{v_q(n)}}]). \end{array}$$

The constructions now split into three cases:

1. In the integral case, we give an explicit construction suggested by Charles Rezk.
2. The p-adic case where $n = p^v$ is the p-completion of the corresponding integral case.
3. The p-adic case where $(n, p) = 1$ uses Cooke's obstruction theory [Coo78] to lift group actions on homotopy groups to the homotopy category of spectra. The comparison of this case with the integral case uses the obstruction theory to uniqueness of the lifting.

The integral case.

Construction 5.2 (Charles Rezk)**.** From the short exact sequence of C_{p^v}-representations in Lemma B.2:

$$0 \longrightarrow \mathbb{Z}[\zeta_{p^v}] \longrightarrow \mathbb{Z}[C_{p^v}] \longrightarrow \mathbb{Z}[C_{p^{v-1}}] \longrightarrow 0, \tag{5.5}$$

we define $M(\mathbb{Z}[\zeta_{p^v}])$ as the de-suspension of the cofiber of the quotient map $C_{p^v} \twoheadrightarrow C_{p^{v-1}}$. That is, there is a cofiber sequence:

$$S^0 \wedge (C_{p^v})_+ \longrightarrow S^0 \wedge (C_{p^{v-1}})_+ \longrightarrow \Sigma M(\mathbb{Z}[\zeta_{p^v}]). \tag{5.6}$$

$M(\mathbb{Z}[\zeta_{p^v}])$ inherits a natural $(\mathbb{Z}/p^v)^\times$-action from its suspension as the cofiber of a C_{p^v}-equivariant map.

Proposition 5.6. *$M(\mathbb{Z}[\zeta_{p^v}])$ constructed above is a Moore spectrum for $\mathbb{Z}[\zeta_{p^v}]$. The induced $(\mathbb{Z}/p^v)^\times$-action on $H_0(M(\mathbb{Z}[\zeta_{p^v}]);\mathbb{Z})$ is equivalent to the cyclotomic action of C_{p^v} on $\mathbb{Z}[\zeta_{p^v}]$.*

Proof. Applying $H_*(-;\mathbb{Z})$ to the cofiber sequence (5.6), we can show that $M(\mathbb{Z}[\zeta_{p^v}])$ is a Moore spectrum. The rest follows from (5.5). ☐

Below are some examples of the C_{p^v}-equivariant cell structures of $\Sigma M(\mathbb{Z}[\zeta_{p^v}])$:

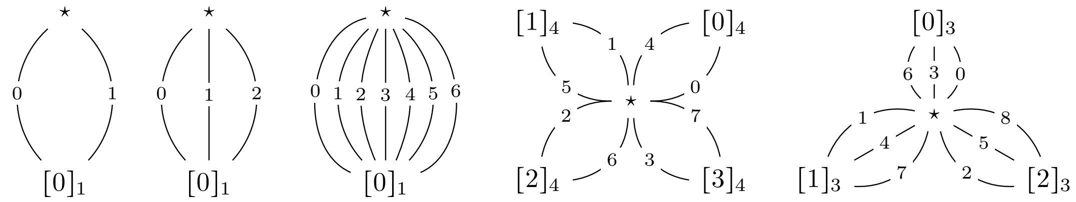

Figure 1: C_{p^v}-cell structures of $\Sigma M(\mathbb{Z}[\zeta_{p^v}])$ for $p^v = 2,3,7,8,9$

- $\star$ is the base point and is fixed by the C_n-action.
- $[a]_b := (a \mod b)$ is the label of (non-equivariant) 0-cells.
- $a := (a \mod n)$ is the label of (non-equivariant) 1-cells.
- $g \in C_n \simeq \mathbb{Z}/n$ acts on the labels by mapping $(a \mod b)$ to $(a + g \mod b)$.

Here is another description of this construction:

1. $M(\mathbb{Z}[\zeta_2]) \simeq S^{\sigma-1}$, where σ is the sign representation of C_2.

2. C_n acts on $\mathbb{C}$ by multiplication by n-th roots of unity. Denote the associated C_n-representation by ρ_{cyclo} and the representation sphere by $S^{\rho_{\text{cyclo}}}$. When $n = p$, the C_p-cell structure of $\Sigma M(\mathbb{Z}[\zeta_p])$ above shows

$$S^{\rho_{\text{cyclo}}} \simeq \Sigma M(\mathbb{Z}[\zeta_p]) \cup (C_p \times D^2).$$

   As a result, $M(\mathbb{Z}[\zeta_p])$ is the 1-skeleton in this equivariant cell structure of the representation sphere $S^{\rho_{\text{cyclo}}}$.

3. Foling Zou has observed and proved the following relation between $M(\mathbb{Z}[\zeta_{p^v}])$ and $M(\mathbb{Z}[\zeta_p])$ via private conversations with the author:

Proposition 5.7 (Foling Zou). *There is a C_{p^v}-equivariant equivalence:*

$$M(\mathbb{Z}[\zeta_{p^v}]) \simeq (C_{p^v})_+ \bigwedge_{C_p} M(\mathbb{Z}[\zeta_p]),$$

where $a \in \mathbb{Z}/p \simeq C_p$ acts on $\mathbb{Z}/p^v \simeq C_{p^v}$ by sending $(b \mod p^v)$ to $(b + ap^{v-1} \mod p^v)$.

Proof. Notice that $C_{p^{v-1}} \simeq C_{p^v}/C_p$, we can rewrite this quotient as pointed sets by

$$(C_{p^{v-1}})_+ \simeq S^0 \bigwedge_{C_p} (C_{p^v})_+,$$

where C_p acts on C_{p^v} as described in the proposition. From this we get:

$$\begin{aligned}
\Sigma M(\mathbb{Z}[\zeta_{p^v}]) :=& \mathrm{Cofib}\left(S^0 \bigwedge (C_{p^v})_+ \longrightarrow S^0 \bigwedge (C_{p^{v-1}})_+\right) \\
\simeq& \mathrm{Cofib}\left(S^0 \bigwedge (C_p)_+ \bigwedge_{C_p} (C_{p^v})_+ \longrightarrow S^0 \bigwedge S^0 \bigwedge_{C_p} (C_{p^v})_+\right) \\
\simeq& \mathrm{Cofib}\left(S^0 \bigwedge (C_p)_+ \longrightarrow S^0 \bigwedge S^0\right) \bigwedge_{C_p} (C_{p^v})_+ \\
\simeq& \Sigma M(\mathbb{Z}[\zeta_p]) \bigwedge_{C_p} (C_{p^v})_+.
\end{aligned}$$

□

Taking external smash product of $M(\mathbb{Z}[\zeta_{p^v}])$ with the prescribed C_{p^v}-actions over all $p \mid n$, we have constructed a Moore spectrum $M(\mathbb{Z}[\zeta_n])$ with a C_n-action such that the induced action on $H^0(-;\mathbb{Z})$ is equivalent to the cyclotomic action of C_n. We now give an explicit description of the C_n-equivariant simplicial structure of $M(\mathbb{Z}[\zeta_n])$.

Write $n = p_1^{v_1} \cdots p_m^{v_m}$. $X_n := \Sigma^m M(\mathbb{Z}[\zeta_n])$ is constructed as follows:

1. Set the 0-th skeleton by $\mathrm{sk}_0 X_n := \star \coprod C_n/C_{p_1 \cdots p_m}$, where $\star$ is the base point fixed by the $(\mathbb{Z}/N)^\times$-action.
2. Assuming we have defined $\mathrm{sk}_{k-1} X_n$ for $k < m$, then define the k-th skeleton to be:

$$\mathrm{sk}_k X_n := \mathrm{sk}_{k-1} X_n \bigcup \left(\coprod_{i_1 < \cdots < i_{m-k}} C_n/C_{p_{i_1} \cdots p_{i_{m-k}}} \right) \times \Delta^k.$$

The attaching map of an equivariant k-simplex $C_n/C_{p_{i_1} \cdots p_{i_{m-k}}} \times \Delta^k$ is described by the following:

- The 0-th face $C_n/C_{p_{i_1} \cdots p_{i_{m-k}}} \times \Delta^k_{[0]}$ is attached to the base point $\star$.

- Let $\{j_1 < \cdots < j_k\}$ be the complement of $\{i_1, \cdots i_{m-k}\} \subseteq \{1, \cdots, m\}$. Then the l-th face $C_n/C_{p_{i_1}\cdots p_{i_{m-k}}} \times \Delta^k_{[l]}$ for $1 \leq l \leq k$ is attached to the equivariant $(k-1)$-complex

$$C_n/C_{p_{i_1}\cdots p_{i_{m-k}}\cdot p_{j_l}} \times \Delta^{k-1}$$

via the quotient map of orbits.

3. The top simplex is $C_n \times \Delta^m$. The 0-th face $C_n \times \Delta^m_{[0]}$ is attached to the base point $\star$. The l-th face $C_n \times \Delta^m_{[l]}$ for $1 \leq l \leq m$ is attached to the $(m-1)$-equivariant simplex $C_n/C_{p_l} \times \Delta^{m-1}$ via the quotient map $C_n \twoheadrightarrow C_n/C_{p_l}$.

Remark 5.6. The non-equivariant Euler number of $X_n = \Sigma^m M(\mathbb{Z}[\zeta_n])$ is equal to $1 + (-1)^m \phi(n)$ since it is non-equivariantly a wedge sum of $\phi(n)$ many copies of S^m. On the other hand, by counting the number of non-equivariant simplices in each dimension from the above construction, we get

$$\begin{aligned} 1 + (-1)^m \phi(n) = e(X_n) &= 1 + \sum_{k=0}^{m-1} \left((-1)^k \sum_{i_1 < \cdots < i_{m-k}} \frac{n}{p_{i_1} \cdots p_{i_{m-k}}} \right) + (-1)^m n \\ \Longrightarrow \quad \phi(n) &= n + \sum_{k=1}^{m} \left((-1)^k \sum_{i_1 < \cdots < i_k} \frac{n}{p_{i_1} \cdots p_{i_k}} \right). \end{aligned}$$

This is precisely the formula of $\phi(n) := |\{a \in \mathbb{N} \mid 1 \leq a \leq n, (a,n) = 1\}|$ via the Inclusion and Exclusion Principle.

Remark 5.7. The construction above is not unique. For example when $n = 2$, $M(\mathbb{Z}[\zeta_2])$ is by definition S^0 with a C_2-action such that the induced action of C_2 on $\pi_*(S^0)$ is the sign representation in all degrees. Figure 1 shows our model for $M(\mathbb{Z}[\zeta_2])$ is $S^{\sigma-1}$. But one can check $S^{(2k-1)(\sigma-1)}$ also satisfies the assumptions for all $k \in \mathbb{Z}$ and these are non-equivalent C_2-actions on S^0.

The p-adic case with $n = p^v$. By Corollary B.1, $(\mathbb{Z}[\zeta_{p^v}])^\wedge_p \simeq \mathbb{Z}_p[\zeta_{p^v}]$. From this we can simply define the Moore spectrum with a C_{p^v}-action by setting

$$M(\mathbb{Z}_p[\zeta_{p^v}]) := M(\mathbb{Z}[\zeta_{p^v}])^\wedge_p.$$

The p-adic case with $p \nmid n$. In this case, Proposition B.2 implies that $(\mathbb{Z}[\zeta_n])^\wedge_p \ncong \mathbb{Z}_p[\zeta_n]$, since the two sides have different ranks as $\mathbb{Z}_p$-modules. As a result, the construction in the $n = p^v$ case does not apply. Instead, we use Cooke's obstruction theory in [Coo78] to lift the C_n-action on $\mathbb{Z}_p[\zeta_n] = \pi_0(M(\mathbb{Z}_p[\zeta_n]))$ to the Moore spectrum $M(\mathbb{Z}[\zeta_n])$.

Let X be a spectrum and $h\mathrm{Aut}(X)$ be the group of self-homotopy equivalences of X. $h\mathrm{Aut}(X)$ is an associative H-space. Then $\pi_0(h\mathrm{Aut}(X))$ is the group of homotopy classes of homotopy equivalences of X. Denote the identity component of $h\mathrm{Aut}(X)$ by $h\mathrm{Aut}_1(X)$. There is an short exact sequence of H-spaces:

$$\begin{tikzcd} 1 \arrow[r] & h\mathrm{Aut}_1(X) \arrow[r] & h\mathrm{Aut}(X) \arrow[r] & \pi_0(h\mathrm{Aut}(X)) \arrow[r] & 1. \end{tikzcd}$$

This induces a fiber sequence by taking classifying spaces:

$$\begin{tikzcd} Bh\mathrm{Aut}_1(X) \arrow[r] & Bh\mathrm{Aut}(X) \arrow[r] & B\pi_0(h\mathrm{Aut}(X)). \end{tikzcd}$$

An action of a group G on $\pi_0(X)$ is then a group homomorphism $\alpha : G \to \pi_0(h\mathrm{Aut}(X))$.

Theorem 5.3. [Coo78, Theorem 1.1] *There is an obstruction theory to lift α to an action on X:*

$$\begin{tikzcd} & Bh\mathrm{Aut}(X) \arrow[d] \\ BG \arrow[ur, dashed] \arrow[r, "B\alpha"'] & B\pi_0(h\mathrm{Aut}(X)). \end{tikzcd}$$

The obstruction classes to the existence of such liftings live in

$$H^n(G; \{\pi_{n-2}(h\mathrm{Aut}_1(X))\}), \quad n \geq 3.$$

In particular, one can always lift a G-action on $\pi_*(X)$ to X if G is finite and $|G|$ is invertible in $\pi_n(h\mathrm{Aut}_1(X))$ for all $n \geq 1$.

Corollary 5.4. *When $p \nmid n$, any of C_n-action on π_* of a p-complete spectrum can be lifted to an action on the spectrum itself.*

Proof. As n is invertible in $\mathbb{Z}_p$, group cohomology of C_n with coefficients in $\mathbb{Z}_p$-modules vanishes in positive degrees. As a result, the obstruction classes in Theorem 5.3 all vanish. ☐

As a result, there exists a C_n-action on the p-adic Moore spectrum $M(\mathbb{Z}_p[\zeta_n])$ such that the induced action on π_0 agrees with p-adic cyclotomic representation of C_n.

One last thing to check is the compatibility of the constructions in the integral and p-adic cases when $p \nmid n$. Fix an embedding $\iota : \mathbb{Z}[\zeta_n] \hookrightarrow \mathbb{Z}_p[\zeta_n]$. ι induces a map of Galois groups:

$$\begin{tikzcd} \iota^* : \mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p) \arrow[r, hook] & \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}). \end{tikzcd}$$

By Proposition B.4, there is an equivalence of p-adic C_n-representations:

$$\mathbb{Z}[\zeta_n] \otimes \mathbb{Z}_p \simeq \bigoplus_{[\sigma] \in \operatorname{Coker} \iota^*} (\mathbb{Z}_p[\zeta_n])_{\iota \circ \sigma}, \tag{5.7}$$

where C_n acts on the summand $(\mathbb{Z}_p[\zeta_n])_{\iota\circ\sigma}$ by

$$C_n \hookrightarrow (\mathbb{Z}[\zeta_n])^\times \xrightarrow{\sigma} (\mathbb{Z}[\zeta_n])^\times \xrightarrow{\iota} (\mathbb{Z}_p[\zeta_n])^\times.$$

By Corollary 5.4, there is a C_n-action on $M(\mathbb{Z}_p[\zeta_n])^{\vee|\operatorname{Coker}\iota^*|}$ such that the induced C_n-action on π_0 agrees with the right hand side of (5.7). On the other hand, the C_n-action $M(\mathbb{Z}[\zeta_n])_p^\wedge$ induces an equivalent C_n-representation on π_0. To check the two C_n-actions on the p-adic Moore spectrum are equivalent, we use the uniqueness part of Cooke's obstruction theory.

Proposition 5.8. *In Theorem 5.3, the obstruction classes to the uniqueness of the liftings live in*

$$H^n(G; \{\pi_{n-1}(h\mathrm{Aut}_1(X))\}), \quad n \geq 2.$$

Corollary 5.5. *Let X be a p-complete spectrum. When $p \nmid n$, any two lifts of a C_n-action from $\pi_*(X)$ to X are C_n-equivariantly equivalent.*

As a result, there is a C_n-equivalence:

$$M(\mathbb{Z}[\zeta_n])_p^\wedge \simeq \bigvee_{[\sigma]\in\operatorname{Coker}\iota^*} (M(\mathbb{Z}_p[\zeta_n]))_{\iota\circ\sigma}.$$

Remark 5.8. When $n = p^v$, there could be non-equivalent C_{p^v}-actions on $M(\mathbb{Z}_p[\zeta_{p^v}])$ inducing the same action on π_0. One counterexample in the integral case is C_2-equivariant spheres $S^{2\sigma-2}$ and S^0 – both induce trivial action on the homotopy groups.

Pre-composing with the map $(\mathbb{Z}/N)^\times \twoheadrightarrow C_n$ in Lemma 5.2, we have shown in this section:

Theorem 5.4. *Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ or $\mathbb{C}_p^\times$ be a Dirichlet character.*

1. *There is a Moore spectrum $M(\mathbb{Z}[\chi])$ or $M(\mathbb{Z}_p[\chi])$ with a $(\mathbb{Z}/N)^\times$-action such that the induced action on π_0 is equivalent to that induced by χ.*
2. *Let $\iota : \mathbb{Z}[\chi] \hookrightarrow \mathbb{Z}_p[\chi]$ be an embedding. There is a $(\mathbb{Z}/N)^\times$-equivariant equivalence:*

$$M(\mathbb{Z}[\chi])_p^\wedge \simeq \bigvee_{[\sigma]\in\operatorname{Coker}\iota^*} M(\mathbb{Z}_p[\iota\circ\sigma\circ\chi]). \tag{5.8}$$

5.4 The homotopy eigen-spectra

Now we are ready to twist the J-spectrum and the $K(1)$-local spheres with a Dirichlet character. Analogous to Proposition 1.5, the twisting is realized as the "homotopy χ-eigen-spectrum".

Construction 5.3. Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character of conductor N. We define the **Dirichlet J-spectrum** by:

$$J(N)^{h\chi} := \mathrm{Map}(M(\mathbb{Z}[\chi]), J(N))^{h(\mathbb{Z}/N)^\times}, \tag{5.9}$$

Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ be a primitive p-adic Dirichlet character of conductor N and set $v = v_p(N)$. We define the **Dirichlet $K(1)$-local sphere** to be

$$S^0_{K(1)}(p^v)^{h\chi} := \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/N)^\times}. \tag{5.10}$$

The $(\mathbb{Z}/N)^\times$-actions on the Moore spectrum and $J(N)$ are described in Theorem 5.4 and Proposition 5.3, respectively. $(\mathbb{Z}/N)^\times$ acts on $S^0_{K(1)}(p^v)$ through the Galois action of its quotient group $(\mathbb{Z}/p^v)^\times$.

Remark 5.9. The spectra $J(N)^{h\chi}$ and $S^0_{K(1)}(p^v)^{h\chi}$ depend on the constructions of the $(\mathbb{Z}/N)^\times$-actions on $M(\mathbb{Z}[\chi])$ and $M(\mathbb{Z}_p[\chi])$, which is not unique in general as illustrated in Remark 5.7. When $N = 4, p = 2$ and $\chi : (\mathbb{Z}/4)^\times \simeq C_2 \to \mathbb{C}_2^\times$, different models of $M(\mathbb{Z}_2[\chi])$ lead to different $S^0_{K(1)}(4)^{h\chi}$. We will explain the differences in more detail in Remark 6.5.

One immediate consequence of this construction is

Proposition 5.9. *If χ_1 and χ_2 are Dirichlet characters of conductor N with isomorphic induced representations, then $J(N)^{h\chi_1} \simeq J(N)^{h\chi_2}$. In particular, $J(N)^{h\chi} \simeq J(N)^{h(\sigma\circ\chi)}$ for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$.*

Remark 5.10. As $S^0_{K(1)}(p^v)$ is $K(1)$-local, we have

$$S^0_{K(1)}(p^v)^{h\chi} \simeq \mathrm{Map}_{K(1)\text{-loc}}\left(M(\mathbb{Z}_p[\chi])_{K(1)}, S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/N)^\times}$$

is also $K(1)$-local.

Proposition 5.10. *The E_2-pages of the HFPSS to compute $\pi_*\left((J(N))^{h\chi}\right)$ and $\pi_*\left(S^0_{K(1)}(p^v)^{h\chi}\right)$ are identified with*

$$E_2^{s,t} \simeq \mathrm{Ext}^s_{\mathbb{Z}[(\mathbb{Z}/N)^\times]}\left(\mathbb{Z}[\chi], \pi_t(J(N))\right) \Longrightarrow \pi_{t-s}\left(J(N)^{h\chi}\right) \tag{5.11}$$

$$E_2^{s,t} \simeq \mathrm{Ext}^s_{\mathbb{Z}_p[(\mathbb{Z}/N)^\times]}\left(\mathbb{Z}_p[\chi], \pi_t\left(S^0_{K(1)}(p^v)\right)\right) \Longrightarrow \pi_{s-t}\left(S^0_{K(1)}(p^v)^{h\chi}\right) \tag{5.12}$$

where $a \in (\mathbb{Z}/N)^\times$ *acts on* $\mathbb{Z}[\chi]$ *and* $\mathbb{Z}_p[\chi]$ *by multiplication by* $\chi(a)$*.*

Proof. We give a proof of (5.11). The proof of (5.12) is similar. By construction, the E_2-page of the HFPSS for (5.9) is

$$E_2^{s,t} = H^s((\mathbb{Z}/N)^\times; \pi_t(\mathrm{Map}(M(\mathbb{Z}[\chi]), J(N)))).$$

Denote the rank of $\mathbb{Z}[\chi]$ as a free $\mathbb{Z}$-module by r. Then $M(\mathbb{Z}[\chi])$ is non-equivariantly equivalent to $\left(S^0\right)^{\vee r}$. The Atiyah-Hirzebruch spectral sequence:

$$E_2^{s,t} = H^s(M(\mathbb{Z}[\chi]); \pi_t(J(N))) \Longrightarrow \pi_{s+t}(\mathrm{Map}(M(\mathbb{Z}[\chi]), J(N)))$$

collapses on the E_2-page since $H^*(M(\mathbb{Z}[\chi]); -)$ is concentrated in degree 0. Together with the universal coefficient theorem, this implies:

$$\begin{aligned}\pi_t(\mathrm{Map}(M(\mathbb{Z}[\chi]), J(N))) \simeq & H^0(M(\mathbb{Z}[\chi]); \pi_t(J(N))) \\ \simeq & \mathrm{Hom}_{\mathbb{Z}}(H^0(M(\mathbb{Z}[\chi]); \mathbb{Z}), \pi_t(J(N))) \\ \simeq & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\chi], \pi_t(J(N))).\end{aligned}$$

By Theorem 5.4, $(\mathbb{Z}/N)^\times$ acts on $\mathbb{Z}[\chi] \simeq H^0(M(\mathbb{Z}[\chi]); \mathbb{Z})$ by χ. Since $\mathbb{Z}[\chi]$ is a finite free $\mathbb{Z}$-module, the Grothendieck spectral sequence

$$E_2^{s,t} = H^s((\mathbb{Z}/N)^\times; \mathrm{Ext}^t_{\mathbb{Z}}(\mathbb{Z}[\chi], \pi_t(J(N)))) \Longrightarrow \mathrm{Ext}^{s+t}_{\mathbb{Z}[(\mathbb{Z}/N)^\times]}(\mathbb{Z}[\chi], \pi_t(J(N)))$$

collapses on the E_2-page, yielding

$$H^s((\mathbb{Z}/N)^\times; \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\chi], \pi_t(J(N)))) \simeq \mathrm{Ext}^s_{\mathbb{Z}[(\mathbb{Z}/N)^\times]}(\mathbb{Z}[\chi], \pi_t(J(N)))\,.$$

□

Remark 5.11. The E_2-page of (5.11) consists of the derived χ-eigenspaces of $\pi_*(J(N))$. Moreover, $J(N)^{h\chi}$ is defined as the **homotopy χ-eigen-spectrum** of $J(N)$. In this sense, we will call (5.11) the **homotopy eigen spectral sequence** (HESS). [1]

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[1] The alternative name "homotopy eigen-spectrum spectral sequence" would be too redundant.

5.5 Local structures of the Dirichlet J-spectra

While it is not hard to compute the E_2-page of (5.11) directly, the differentials are non-trivial as the cohomological dimension of $(\mathbb{Z}/N)^\times$ with coefficients in $\mathbb{Z}$-modules is infinite. Instead, we will compute $\pi_* \left(J(N)\right)^{h\chi}$ rationally and completed at each prime p.

Over $\mathbb{Q}$, the spectral sequence is concentrated in the 0-th line, since $(\mathbb{Z}/N)^\times$ is a finite group. By Corollary 5.3, $J(N)_\mathbb{Q} \simeq S^0_\mathbb{Q}$ and $(\mathbb{Z}/N)^\times$ acts on it trivially. We conclude from these facts:

Proposition 5.11. *The homotopy groups of $\left(J(N)^{h\chi}\right)_\mathbb{Q}$ are given by*

$$\pi_i\left(\left(J(N)^{h\chi}\right)_\mathbb{Q}\right) \simeq \begin{cases} \mathbb{Q}, & i = 0 \text{ and } \chi = \chi^0; \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 5.6. *$\left(J(N)^{h\chi}\right)_\mathbb{Q}$ is contractible unless $\chi = \chi^0$ is trivial. In that case, $N = 0$ and $J(N)^{h\chi}_\mathbb{Q} \simeq J_\mathbb{Q} \simeq S^0_\mathbb{Q}$.*

Proof. By Corollary 5.3, $J(N)_\mathbb{Q} \simeq S^0_\mathbb{Q}$. Then $E_2^{s,t} \otimes \mathbb{Q} = 0$ for all $(s,t) \neq (0,0)$ (5.11). The remaining entry $E_2^{0,0} \simeq \mathbb{Q}(\chi^{-1})^{(\mathbb{Z}/N)^\times}$ is non-zero only when $\chi = \chi^0$ is trivial, implying the claim. ☐

Proposition 5.12. *Fix an embedding $\iota : \mathbb{Q}(\chi) \hookrightarrow \mathbb{C}_p$. The p-completion of the Dirichlet J-spectrum decomposes as*

$$\left(J(N)^{h\chi}\right)^\wedge_p \simeq \bigvee_{[\sigma]\in \operatorname{Coker}\iota^*} S^0_{K(1)}(p^v)^{h(\iota\circ\sigma\circ\chi)},$$

where $\iota^* : \operatorname{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p) \hookrightarrow \operatorname{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is defined in (B.2).

Proof. Since homotopy fixed points and p-completions commute and that the p-completion of $J(N)$ is $S^0_{K(1)}(p^v)$

$$\left(J(N)^{h\chi}\right)^\wedge_p \simeq \operatorname{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}[\chi])^\wedge_p, S^0_{K(1)}(p^v)\right)^{h(\mathbb{Z}/N)^\times}$$

The rest follows from (5.8). ☐

Now we give explicit descriptions of how $\left(J(N)^{h\chi}\right)^\wedge_p$ decomposes when $N = p^v$.

Examples 5.1. Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a Dirichlet character of conductor $N = p^v$. Fix an embedding $\iota : \mathbb{Z}[\chi] \hookrightarrow \mathbb{C}_p$. There are two cases.

- $p = 2$. The $v = 1$ case is trivial. For $v > 1$, $(\mathbb{Z}/2^v)^\times \simeq \{\pm 1\} \times \mathbb{Z}/2^{v-2}$. When $v = 2$, χ is primitive when it is non-trivial, i.e. $\chi(-1) = -1$. When $v > 2$, χ is primitive of conductor 2^v iff $\mathbb{Z}[\chi] \simeq \mathbb{Z}[\zeta_{2^{v-2}}]$. In both cases, we have by Proposition B.2, $(\mathbb{Z}[\zeta_{2^{v-2}}])_2^\wedge \simeq \mathbb{Z}_2[\zeta_{2^{v-2}}]$. As a result,

$$\left(J(2^v)^{h\chi}\right)_2^\wedge \simeq S^0_{K(1)}(2^v)^{h(\iota\circ\chi)}.$$

Notice for any two 2-adic Dirichlet characters χ_1 and χ_2 of conductor 2^v with the same parity, there is a $\sigma \in \mathrm{Gal}(\mathbb{Q}_2(\zeta_{2^{v-2}})/\mathbb{Q}_2)$ such that $\chi_1 = \sigma \circ \chi_2$. By Proposition 5.9, the above isomorphism does not depend on ι, since $\iota \circ \chi(-1)$ is independent of the choice of ι.

- $p > 2$. In this case, $(\mathbb{Z}/p^v)^\times \simeq (\mathbb{Z}/p)^\times \times \mathbb{Z}/p^{v-1}$. When $v = 1$, χ is primitive iff it is non-trivial. When $v > 1$, χ is primitive iff $\zeta_{p^{v-1}} \in \mathbb{Z}[\chi]$, i.e. $\chi|_{\mathbb{Z}/p^{v-1}}$ is injective. By Corollary B.3, there is an isomorphism of p-adic $(\mathbb{Z}/p^v)^\times$-representations:

  $$(\mathbb{Z}[\chi])_p^\wedge \simeq \bigoplus_{\substack{0\le a\le p-2 \\ \ker\omega^a = \ker\chi|_{(\mathbb{Z}/p)^\times}}} \mathbb{Z}_p[\chi_a],$$

  where $\chi_a = \omega^a \cdot (\iota \circ \chi|_{\mathbb{Z}/p^{v-1}})$ and $\omega : (\mathbb{Z}/p)^\times \to \mathbb{Z}_p^\times$ is the **Teichmüller** character. This implies a decomposition of the p-completion of the Dirichlet J-spectrum as in Proposition 5.12:

  $$\left(J(p^v)^{h\chi}\right)_p^\wedge \simeq \bigvee_{\substack{0\le a\le p-2 \\ \ker\omega^a = \ker\chi|_{(\mathbb{Z}/p)^\times}}} S^0_{K(1)}(p^v)^{h\chi_a}. \tag{5.13}$$

Remark 5.12. When $N = p > 2$, we will show in Corollary 6.1 that summands in (5.13) are $K(1)$-local invertible spectra of finite order in the **$K(1)$-local Picard group** $\mathrm{Pic}_{K(1)}$. The $N = 4$ and $p = 2$ case will be discussed in Remark 6.5.

Chapter 6

Computations of the Dirichlet J-spectra

In this chapter, we compute homotopy groups of the Dirichlet J-spectra. By Proposition 5.12, we can recover the p-primary parts of the homotopy groups of Dirichlet J-spectra from the corresponding summands of Dirichlet $K(1)$-local spheres. Let $\chi : (\mathbb{Z}/N)^{\times} \to \mathbb{C}_p^{\times}$ be a p-adic Dirichlet character of conductor N. The computations of $\pi_*\left(S^0_{K(1)}(p^v)^{h\chi}\right)$ break up into four cases:

1. $N = 1$.
2. $N = p^v$ and $p > 2$.
3. $N = 2^v$.
4. N has prime factors other than p.

In the $N = 1$ case, we recover the classical $K(1)$-local sphere, whose homotopy groups are computed in (2.6) when $p > 2$ and in (2.9) when $p = 2$. When N is power of p, we use HFPSS/HESS to compute homotopy groups of the Dirichlet $K(1)$-local spheres. One important technique here is to lift the $(\mathbb{Z}/p^v)^{\times}$-action to a $\mathbb{Z}_p^{\times}$-action. When N has prime factors other than p, the Dirichlet $K(1)$-local spheres are contractible in many cases. Finally we assemble our computations at each prime and compare $\pi_{2k-1}(J(N)^{h\chi})$ with Carlitz's result of arithmetic properties of $B_{k,\chi^{-1}}/k$ in Theorem 1.2.

6.1 The $N = p^v$ and $p > 2$ cases

Let's start with the $N = p > 2$ case. We will compute $\pi_*\left(S^0_{K(1)}(p)\right)^{h\chi}$ for $p > 2$ using the homotopy eigen spectral sequence (HESS) introduced in (5.12). The E_2-page of this spectral sequence is:

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathbb{Z}_p[(\mathbb{Z}/p)^{\times}]}\left((\mathbb{Z}_p)_\chi, \pi_t\left(S^0_{K(1)}(p)\right)\right) \Longrightarrow \pi_{t-s}\left(S^0_{K(1)}(p)^{h\chi}\right), \tag{6.1}$$

where $a \in (\mathbb{Z}/p)^{\times}$ acts on $(\mathbb{Z}_p)_\chi$ by multiplication by $\chi(a)$.

Remark 6.1. When χ is the trivial character χ^0, we recover the HFPSS in (2.3).

Let $g \in (\mathbb{Z}/p)^\times$ be a generator. A projective resolution of $(\mathbb{Z}_p)_\chi$ as a $\mathbb{Z}_p[(\mathbb{Z}/p)^\times]$-module is

$$\cdots \longrightarrow \mathbb{Z}_p[(\mathbb{Z}/p)^\times] \xrightarrow{\times(\sum \chi(g)^{-i}g^i)} \mathbb{Z}_p[(\mathbb{Z}/p)^\times] \xrightarrow{\times(g-\chi(g))} \mathbb{Z}_p[(\mathbb{Z}/p)^\times] \xrightarrow{g\mapsto\chi(g)} (\mathbb{Z}_p)_\chi.$$

By (2.6), the homotopy groups of $S^0(p)$ are

$$\pi_t\left(S^0_{K(1)}(p)\right) = \begin{cases} \mathbb{Z}_p, & t = 0 \text{ or } -1; \\ \mathbb{Z}/p^{v_p(k)+1}, & t = 2k-1 \neq -1; \\ 0, & \text{otherwise.} \end{cases}$$

Descending from the Adams operations on $\left(K_p^\wedge\right)_t$, $(\mathbb{Z}/p)^\times$ acts trivially on π_0 and π_{-1} and by $\chi = \omega^k$ on π_{2k-1} of $S^0_{K(1)}(p)$. A direct computation shows

Proposition 6.1. *When $\chi = \omega^a$, $a \neq 0$, the E_2-page of (6.1) is*

$$E_2^{s,t} = \begin{cases} \mathbb{Z}/p^{v_p(k)+1}, & s = 0, t = 2k-1, \text{ and } (p-1) \mid (k-a); \\ 0, & \text{otherwise.} \end{cases}$$

As the spectral sequence collapses on the E_2-page, we conclude

$$\pi_t\left(S^0_{K(1)}(p)^{h\omega^a}\right) = \begin{cases} \mathbb{Z}/p^{v_p(k)+1}, & t = 2k-1, \text{ and } (p-1) \mid (k-a); \\ 0, & \text{otherwise.} \end{cases} \tag{6.2}$$

There is another way to formulate this computation, which will be useful later. Recall that $\pi_*(S^0(p))$ was computed by HFPSS in (2.3):

$$E_2^{r,s} = H^r\left(1 + p\mathbb{Z}_p; \left(K_p^\wedge\right)_s\right) \Longrightarrow \pi_{r-s}\left(S^0_{K(1)}(p)\right).$$

Combining (2.3) and (6.1), we get a two-step spectral sequence:

$$\mathrm{Ext}^r_{\mathbb{Z}_p[(\mathbb{Z}/p)^\times]}\left((\mathbb{Z}_p)_\chi, H^s\left(1 + p\mathbb{Z}_p; \left(K_p^\wedge\right)_t\right)\right) \Longrightarrow \pi_{r+s-t}\left(S^0_{K(1)}(p)^{h\chi}\right).$$

This two-step spectral sequence can also be computed using the Hochschild-Serre spectral sequence as in (2.10). To do that, first denote by $\widetilde{\chi}: \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ the composition:

$$\widetilde{\chi}: \mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p)^\times \xrightarrow{\chi} \mathbb{Z}_p^\times.$$

Let $(\mathbb{Z}_p)_{\widetilde{\chi}}$ be the $\mathbb{Z}_p^\times$-representation associated to $\widetilde{\chi}$. By precomposing the $(\mathbb{Z}/p)^\times$-action on $M(\mathbb{Z}_p[\chi]) \simeq \left(S^0\right)_p^\wedge$ with the quotient map $\mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p)^\times$, we get a $\mathbb{Z}_p^\times$-action on $S_p^0 := \left(S^0\right)_p^\wedge$. The induced $\mathbb{Z}_p^\times$-action on π_0 is equivalent to $\widetilde{\chi}$. Denote this naïve $\mathbb{Z}_p^\times$-spectrum by $\left(S_p^0\right)_{\widetilde{\chi}}$.

Proposition 6.2. *When $N = p > 2$, the Dirichlet $K(1)$-local sphere defined in Construction 5.3 can be reformulated as*

$$S^0_{K(1)}(p)^{h\chi} \simeq \left(K_p^\wedge\right)^{h\widetilde{\chi}} := \mathrm{Map}_{\mathbb{Z}_p}\left(\left(S_p^0\right)_{\widetilde{\chi}}, K_p^\wedge\right)^{h\mathbb{Z}_p^\times}.$$

Proof. Recall from definition $S^0_{K(1)}(p) := \left(K_p^\wedge\right)^{h(1+p\mathbb{Z}_p)}$, we have

$$\begin{aligned} S^0_{K(1)}(p)^{h\chi} :=& \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), \left(K_p^\wedge\right)^{h(1+p\mathbb{Z}_p)}\right)^{h(\mathbb{Z}/p)^\times} \\ \simeq& \left(\mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), \left(K_p^\wedge\right)\right)^{h(1+p\mathbb{Z}_p)}\right)^{h(\mathbb{Z}/p)^\times} \\ \simeq& \mathrm{Map}_{\mathbb{Z}_p}\left(\left(S_p^0\right)_{\widetilde{\chi}}, K_p^\wedge\right)^{h\mathbb{Z}_p^\times}. \end{aligned}$$

In the above formulas, $1 + p\mathbb{Z}_p$ acts trivially on $M(\mathbb{Z}_p[\chi])$ and we use the fact that $M(\mathbb{Z}_p[\chi])$ is non-equivariantly equivalent to S_p^0 in the second line. ☐

Corollary 6.1. *$S^0_{K(1)}(p)^{h\chi} \simeq \left(K_p^\wedge\right)^{h\widetilde{\chi}}$ is a $K(1)$-local invertible spectrum, corresponding to the character $\widetilde{\chi}^{-1} \in \mathrm{End}(\mathbb{Z}_p^\times) \simeq \mathrm{Pic}^{alg}_{K(1)}$. As a result, $S^0_{K(1)}(p)^{h\chi}$ has finite order in the Picard group.*

Corollary 6.2. *There is another HESS to compute $\pi_*\left(S^0_{K(1)}(p)^{h\chi}\right)$:*

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]}\left((\mathbb{Z}_p)_{\widetilde{\chi}}, \left(K_p^\wedge\right)_t\right) \Longrightarrow \pi_{t-s}\left(S^0_{K(1)}(p)^{h\chi}\right). \tag{6.3}$$

The two approaches to compute $\pi_*\left(S^0_{K(1)}(p)^{h\chi}\right)$ are related by the diagram:

$$\begin{tikzcd}
\mathrm{Ext}^r_{\mathbb{Z}_p[(\mathbb{Z}/p)^\times]}\left((\mathbb{Z}_p)_\chi, H^s\left(1+p\mathbb{Z}_p; \left(K_p^\wedge\right)_t\right)\right) \arrow[r, Rightarrow, "\text{HSSS}"] \arrow[d, Rightarrow, "\text{HFPSS}"'] & \mathrm{Ext}^{r+s}_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]}\left((\mathbb{Z}_p)_{\widetilde{\chi}}, \left(K_p^\wedge\right)_t\right) \arrow[d, Rightarrow, "\text{HESS}"] \\
\mathrm{Ext}^r_{\mathbb{Z}_p[(\mathbb{Z}/p)^\times]}\left((\mathbb{Z}_p)_\chi, \pi_{t-s}\left(S^0_{K(1)}(p)\right)\right) \arrow[r, Rightarrow, "\text{HESS}"'] & \pi_{t-r-s}\left(S^0_{K(1)}(p)^{h\chi}\right)
\end{tikzcd} \tag{6.4}$$

Retrospectively from this diagram, we get when $\chi = \omega^a$, $a \neq 0$:

$$\mathrm{Ext}^s_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]}\left((\mathbb{Z}_p)_{\widetilde{\chi}}, \left(K_p^\wedge\right)_t\right) = \begin{cases} \mathbb{Z}/p^{v_p(k)+1}, & s = 1, t = 2k, (p-1) \mid (k-a); \\ 0, & \text{otherwise.} \end{cases} \tag{6.5}$$

When $N = p^v > p > 2$, we compute the homotopy groups of the Dirichlet $K(1)$-local spheres by lifting the group actions from $(\mathbb{Z}/p^v)^\times$ to $\mathbb{Z}_p^\times$, whose cohomological dimension is 1 with coefficients in $\mathbb{Z}_p$-modules. As in Proposition 6.2, there is an identification:

$$S^0_{K(1)}(p^v)^{h\chi} \simeq \left(K_p^\wedge\right)^{h\widetilde{\chi}} := \mathrm{Map}_{\mathbb{Z}_p}\left(M(\mathbb{Z}_p[\chi]), K_p^\wedge\right)^{h\mathbb{Z}_p^\times},$$

where $\widetilde{\chi}$ is defined by

$$\widetilde{\chi} : \mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p^v)^\times \xrightarrow{\chi} (\mathbb{Z}_p[\chi])^\times. \tag{6.6}$$

Using the resolution in (2.4), we get the E_2-page of the HESS:

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathbb{Z}_p\llbracket\mathbb{Z}_p^\times\rrbracket}\left(\mathbb{Z}_p[\chi], \pi_t\left(K_p^\wedge\right)\right) = \begin{cases} \mathbb{Z}_p[\chi]\big/\left(\chi(g) - g^{t'}\right), & s = 1, t = 2t'; \\ 0, & \text{otherwise}, \end{cases} \tag{6.7}$$

where g is a topological generator of $\mathbb{Z}_p^\times$.

Lemma 6.1. *Let* $\chi|_{(\mathbb{Z}/p)^\times} = \omega^a$. *Then*

$$\mathbb{Z}_p[\chi]\big/\left(\chi(g) - g^t\right) = \begin{cases} \mathbb{Z}/p, & t \equiv a \mod (p-1); \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Since χ is primitive, we have $\chi(g) = \chi|_{(\mathbb{Z}/p)^\times}(g) \cdot \zeta_{p^{v-1}} = \omega^a(g)\zeta_{p^{v-1}}$. Rewrite $\chi(g) - g^t$ as

$$g^t - \chi(g) = g^t - \omega^a(g)\zeta_{p^{v-1}} = \omega^a(g)(1 - \zeta_{p^{v-1}}) + g^t - \omega^a(g).$$

As $1 - \zeta_{p^{v-1}}$ is a uniformizer of $\mathbb{Z}_p[\chi] \simeq \mathbb{Z}_p[\zeta_{p^{v-1}}]$, $g^t - \chi(g)$ is invertible whenever $g^t - \omega^a(g)$ is. This happens when $t \not\equiv a \mod (p-1)$. When $t \equiv a \mod (p-1)$, $v_p(g^t - \omega^a(g)) \geq 1 > v_p(1 - \zeta_{p^{v-1}})$, yielding

$$(g^t - \chi(g)) = (1 - \zeta_p^{v-1}) \implies \mathbb{Z}_p[\chi]\big/\left(\chi(g) - g^t\right) \simeq \mathbb{Z}/p.$$

□

Again let $\chi|_{(\mathbb{Z}/p)^\times} = \omega^a$. The spectral sequence collapses at the E_2-page and we conclude:

$$\pi_i\left(S^0_{K(1)}(p^v)^{h\chi}\right) = \begin{cases} \mathbb{Z}/p, & i = 2(a + k(p-1)) - 1; \\ 0, & \text{otherwise}. \end{cases} \tag{6.8}$$

Question 6.1. *Let* χ *be a* p*-adic Dirichlet character of conductor* $N = p^v > p$ *and* $\chi|_{(\mathbb{Z}/p)^\times}$ *is non-trivial.*

Comparing (6.2) *and* (6.8)*, we have* $\pi_i\left(S^0_{K(1)}(p^v)^{h\chi}\right) \simeq \left(\pi_i\left(S^0_{K(1)}(p)^{h\chi|_{(\mathbb{Z}/p)^\times}}\right)\right)\Big/(p)$ *for all* i*. One might wonder if there is an equivalence of spectra:*

$$S^0_{K(1)}(p^v)^{h\chi} \overset{?}{\simeq} S^0_{K(1)}(p)^{h\chi|_{(\mathbb{Z}/p)^\times}} \bigwedge M(\mathbb{Z}/p).$$

6.2 The $N = 2^v$ case

We start with the $N = 4$ case, when the only non-trivial 2-adic Dirichlet character of conductor 4 is the Teichmüller character $\omega : (\mathbb{Z}/4)^\times \to \mathbb{Z}_2^\times$. Like Proposition 6.2, the Dirichlet $K(1)$-local sphere is identified with

$$S^0_{K(1)}(4)^{h\omega} \simeq (K_2^\wedge)^{h\tilde{\omega}} \simeq \left((K_2^\wedge)^{h\omega}\right)^{h(1+4\mathbb{Z}_2)}. \tag{6.9}$$

Parallel to the computation of the classical $K(1)$-local sphere at $p = 2$ in Section 2.3, we will first identify $(K_2^\wedge)^{h\omega}$ geometrically.

Proposition 6.3. *Let* σ *be the sign representation of* C_2 *on* $\mathbb{Z}$ *Define* $K^{h\sigma}$ *to be the homotopy* σ*-eigenspectrum of the complex* K*-theory. Then we have an identification:*

$$K^{h\sigma} := \mathrm{Map}(M(\mathbb{Z}[\sigma]), K)^{hC_2} \simeq \Sigma^2 KO.$$

Proof. By Figure 1, $M(\mathbb{Z}[\sigma])$ is C_2-equivariantly equivalent to $S^{\sigma-1}$. Complex K-theory together with the C_2-action by complex conjugation is by definition Atiyah's $K\mathbb{R}$-theory in [Ati66]. Now using the $(1+\sigma)$-periodicity of $K\mathbb{R}$, we have a C_2-equivalence

$$\mathrm{Map}(S^{\sigma-1}, K\mathbb{R}) \simeq \Sigma^{1-\sigma} K\mathbb{R} \simeq \Sigma^2 K\mathbb{R}.$$

The claim now follows from the equivalence $K\mathbb{R}^{hC_2} \simeq KO$. □

Remark 6.2. This statement depends on the actual model of $M(\mathbb{Z}[\sigma])$. If we start with $S^{1-\sigma}$, where C_2 also acts by the sign representation on $\pi_*(S^0)$, we will have

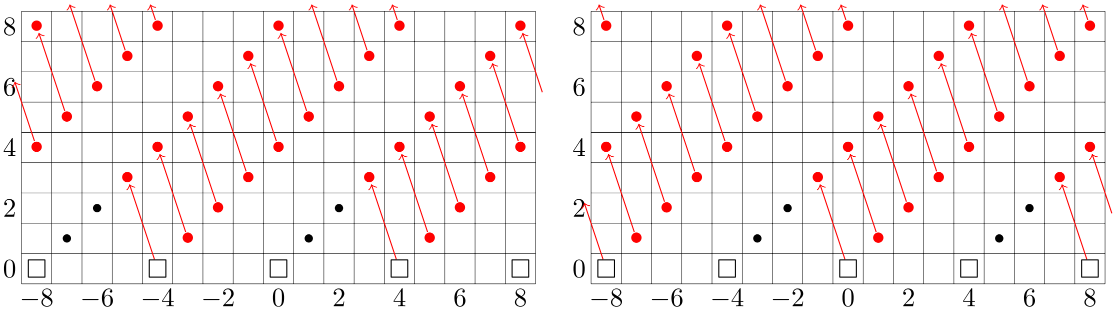
$$\mathrm{Map}(S^{1-\sigma}, K\mathbb{R})^{hC_2} \simeq \Sigma^{-2} KO.$$

In terms of the HFPSS computations, the E_2-pages of $\mathrm{Map}(S^{\sigma-1}, K\mathbb{R})^{hC_2}$ and $\mathrm{Map}(S^{1-\sigma}, K\mathbb{R})^{hC_2}$ are the same. The difference is the d_3-differentials, which are invisible in algebra. Likewise, one can check the

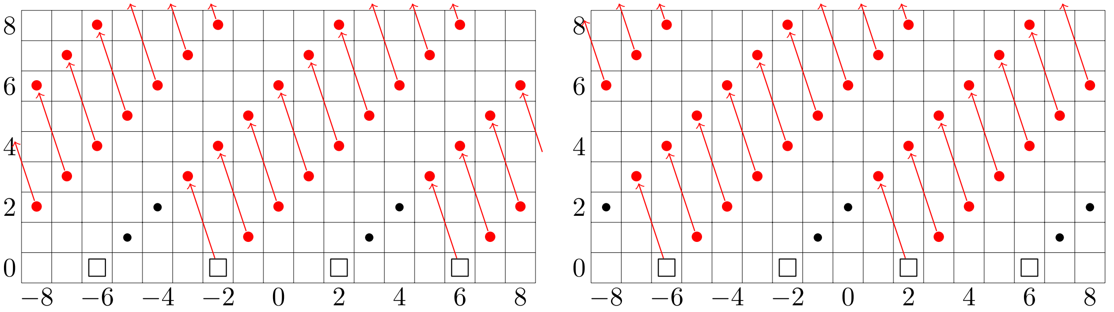
HFPSS for

$$\mathrm{Map}(S^{2\sigma-2}, K\mathbb{R})^{hC_2} \simeq \Sigma^4 KO \simeq KSp$$

has the same E_2-page as that for $K\mathbb{R}^{hC_2} \simeq KO$. Again the difference is the d_3-differentials that are invisible in algebra.



(a) $\left(\Sigma^{4k(1-\sigma)}K\mathbb{R}\right)^{hC_2} \simeq \Sigma^{8k}KO \simeq KO$ (b) $\left(\Sigma^{(4k+2)(1-\sigma)}K\mathbb{R}\right)^{hC_2} \simeq \Sigma^{8k+4}KO \simeq \Sigma^4 KO$



(c) $\left(\Sigma^{(4k+1)(1-\sigma)}K\mathbb{R}\right)^{hC_2} \simeq \Sigma^{8k+2}KO \simeq \Sigma^2 KO$ (d) $\left(\Sigma^{(4k+3)(1-\sigma)}K\mathbb{R}\right)^{hC_2} \simeq \Sigma^{8k+6}KO \simeq \Sigma^6 KO$

Figure 2: d_3-differentials in the HFPSS for different C_2-actions on the K-theory spectrum (Adams grading. $\square = \mathbb{Z}$ and $\bullet = \mathbb{Z}/2$. (A) and (B) are the same as Figures 3 and 6 in [HS14].)

Remark 6.3. A more explicit construction is the following. For any compact space X, $K^{h\sigma}(X)$ consists of virtual complex vector bundles $[E]$ over X such that $\psi^{-1}([E]) = [\overline{E}] = -[E]$. For any such virtual vector bundle, its tensor product with the complexification of a real vector also satisfies this condition. As a result, $K^{h\sigma}$ is a KO-module spectrum.

Let ξ be the tautological complex line bundle over $\mathbb{CP}^1 \simeq S^2$. Then $[\xi] - [\overline{\xi}] \in K^{h\sigma}(S^2)$. The proof above implies the external tensor product with $\xi - \overline{\xi}$ induces an isomorphism:

$$(\xi - \overline{\xi}) \boxtimes (-)_{\mathbb{C}} : KO(X) \xrightarrow{\sim} K^{h\sigma}(S^2 \times X).$$

As elements in $K^{h\sigma}(X)$ satisfy $[\overline{E}] = -[E]$, $K^{h\sigma}$ can be thought of as the *purely imaginary* K-theory, compared to the real K-theory $KO \simeq K^{hC_2}$.

Corollary 6.3. *$(K_2^\wedge)^{h\omega} \simeq \Sigma^2 KO_2^\wedge$ and its homotopy groups are given by:*

$i \mod 8$	0	1	2	3	4	5	6	7
$\pi_i\left((K_2^\wedge)^{h\omega}\right)$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}_2$	0

Remark 6.4. The equivalence $(K_2^\wedge)^{h\omega} \simeq \Sigma^2 KO_2^\wedge$ is NOT $(1+4\mathbb{Z}_2)$-equivariant.

The next step is to compute the HFPSS:

$$E_2^{s,t} = H^s\left(1+4\mathbb{Z}_2; \pi_t\left((K_2^\wedge)^{h\omega}\right)\right) \Longrightarrow \pi_{t-s}\left(S^0_{K(1)}(4)^{h\omega}\right).$$

Let $g \in 1+4\mathbb{Z}_2$ be a topological generator. Descending the Adams operations on $K_2^\wedge$ to $(K_2^\wedge)^{h\omega}$, we get g acts on $\pi_{4t+2}\left((K_2^\wedge)^{h\omega}\right)$ by g^{2t+1}. The actions on the $\mathbb{Z}/2$-terms are trivial since $\mathbb{Z}/2$ has only trivial automorphism. Using the continuous resolution (2.4), we compute the E_2-page of the HFPSS:

$$E_2^{s,t} = H^s\left(1+4\mathbb{Z}_2; \pi_t\left((K_2^\wedge)^{h\omega}\right)\right) = \begin{cases} \mathbb{Z}/4, & s=1, t\equiv 2 \mod 4; \\ \mathbb{Z}/2, & s=0,1, t\equiv 3,4 \mod 8; \\ 0, & \text{otherwise.} \end{cases} \tag{6.10}$$

Proposition 6.4. *The extension problems of this spectral sequence are trivial.*

Proof. We need solve the extension problems at $t-s\equiv 3 \mod 8$. The argument here is analogous to Proposition 2.4. As $(K_2^\wedge)^{h\omega} \simeq \Sigma^2 KO_2^\wedge$ is a $KO_2^\wedge$-module spectrum, we denote the non-zero element in $\pi_3\left((K_2^\wedge)^{h\omega}\right)$ by $\Sigma^2\eta$. This is an element of order 2 and represents a permanent cycle in $E_2^{0,1}$ of (6.10). As $\Sigma^2\eta$ represents an element of order 2 in $\pi_3\left(S^0_{K(1)}(4)^{h\omega}\right)$, the extension problem is trivial. For general $t-s=8k+3$, replace $\Sigma^2\eta$ by $\beta^t\cdot\Sigma^2\eta$ in the argument above, where $\beta\in\pi_8(KO_2^\wedge)$ is the Bott element. ☐

From this, we conclude:

$$\pi_i\left(S^0_{K(1)}(4)^{h\omega}\right) = \begin{cases} \mathbb{Z}/4, & i\equiv 1 \mod 4; \\ \mathbb{Z}/2, & i\equiv 2,4 \mod 8; \\ \mathbb{Z}/2\oplus\mathbb{Z}/2, & i\equiv 3 \mod 8; \\ 0, & \text{otherwise,} \end{cases} \tag{6.11}$$

We also record the E_2-page of the HESS associated to (6.9):

$$\operatorname{Ext}^s_{\mathbb{Z}_2[\![\mathbb{Z}_2^\times]\!]}\left((\mathbb{Z}_2)_{\varpi}, (K_2^\wedge)_t\right) = \begin{cases} \mathbb{Z}/4, & s = 1, t \equiv 2 \mod 4; \\ \mathbb{Z}/2, & s > 1, t \equiv 2 \mod 4; \\ \mathbb{Z}/2, & s > 0, 4 \mid t; \\ 0, & \text{otherwise.} \end{cases} \tag{6.12}$$

Remark 6.5. As explained in Remark 5.7, we could have chosen $M(\mathbb{Z}[\zeta_2]) = S^{1-\sigma}$ when defining the Dirichlet J-spectra and $K(1)$-local spheres. Denote the resulting homotopy eigen-spectra by

$$X^{h'\omega} := \operatorname{Map}_{\mathbb{Z}_2}(S^{1-\sigma}, X)^{hC_2},$$

where $\omega : C_2 \simeq (\mathbb{Z}/4)^\times \to \mathbb{Z}_2^\times$ is the 2-adic Teichmüller character. Then by Remark 6.2, $(K_2^\wedge)^{h'\omega} \simeq \Sigma^{-2}KO_2^\wedge$. A similar computation as above yields:

$$\pi_i\left(S^0_{K(1)}(4)^{h'\omega}\right) = \begin{cases} \mathbb{Z}/4, & i \equiv 1 \mod 4; \\ \mathbb{Z}/2, & i \equiv -2, 0 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv -1 \mod 8; \\ 0, & \text{otherwise,} \end{cases}$$

Note that $\pi_{2k-1}\left(S^0_{K(1)}(4)^{h\chi}\right) = \pi_{2k-1}\left(S^0_{K(1)}(4)^{h'\chi}\right)$ when $(-1)^k = \chi(-1)$.

Both $S^0_{K(1)}(4)^{h\omega}$ and $S^0_{K(1)}(4)^{h'\omega}$ are elements of order 4 in the $K(1)$-local Picard group $\operatorname{Pic}_{K(1)}$ at prime 2. Their difference in $\operatorname{Pic}_{K(1)}$ is the **exotic element**, an element whose HFPSS has the same E_2-page as that for the $K(1)$-local sphere. A construction of this element is given in [HS14, Section 9]. By identifying the exotic element with $(KSp_2^\wedge)^{h(1+4\mathbb{Z}_2)}$, we can compute its homotopy groups as in (2.9):

$$\pi_i\left((KSp_2^\wedge)^{h(1+4\mathbb{Z}_2)}\right) = \begin{cases} \mathbb{Z}_2, & i = 0, -1; \\ \mathbb{Z}/2, & i \equiv 4, 6 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 5 \mod 8; \\ \mathbb{Z}/2^{v_2(t')+3}, & i = 4t' - 1 \neq -1; \\ 0, & \text{otherwise.} \end{cases}$$

When $p = 2$ and $N = 2^v > 4$, we first lift the character to $\mathbb{Z}_2^\times$ as before:

$$S^0_{K(1)}(2^v)^{h\chi} \simeq (K_2^\wedge)^{h\widetilde{\chi}} := \operatorname{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi]), K_2^\wedge\right)^{h\mathbb{Z}_2^\times}.$$

Lemma 6.2. $S^0_{K(1)}(2^v)^{h\chi} \simeq \mathrm{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi]), (K_2^\wedge)^{h\chi|_{(\mathbb{Z}/4)^\times}}\right)^{h(1+4\mathbb{Z}_2)}$.

Proof. We prove the claim by breaking the $\mathbb{Z}_2^\times$-homotopy fixed points into two steps.

$$\begin{aligned}
S^0_{K(1)}(2^v)^{h\chi} \simeq & \mathrm{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi]), K_2^\wedge\right)^{h\mathbb{Z}_2^\times} \\
\simeq & \mathrm{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi\cdot\chi|_{(\mathbb{Z}/4)^\times}]), \mathrm{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi|_{(\mathbb{Z}/4)^\times}]), K_2^\wedge\right)\right)^{h\mathbb{Z}_2^\times} \\
\simeq & \mathrm{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi]), \mathrm{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi|_{(\mathbb{Z}/4)^\times}]), K_2^\wedge\right)^{h(\mathbb{Z}/4)^\times}\right)^{h(1+4\mathbb{Z}_2)} \\
\simeq & \mathrm{Map}_{\mathbb{Z}_2}\left(M(\mathbb{Z}_2[\chi]), (K_2^\wedge)^{h\chi|_{(\mathbb{Z}/4)^\times}}\right)^{h(1+4\mathbb{Z}_2)}.
\end{aligned}$$

In the third line, we used the fact $\chi\cdot\chi|_{(\mathbb{Z}/4)^\times}$ is trivial when restricted to $(\mathbb{Z}/4)^\times$ and is equal to $\widetilde{\chi}$ when restricted to $1+4\mathbb{Z}_2$. □

Let g be the topological generator of $1+4\mathbb{Z}_2$. Denote by $\mathrm{Ann}\left(\widetilde{\chi}(g)-1\right)$ the ideal of annihilators of $\widetilde{\chi}(g)-1$ in $\mathbb{Z}_2[\chi]/(2)$. The computation now splits into two subcases depending on the parity of χ:

- When $\chi(-1)=1$, $(K_2^\wedge)^{h\chi|_{(\mathbb{Z}/4)^\times}} \simeq KO_2^\wedge$. By (2.4) and (2.7), E_2-page of the HESS is:

$$\begin{aligned}
E_2^{s,t} =& \mathrm{Ext}^s_{\mathbb{Z}_2\llbracket 1+4\mathbb{Z}_2\rrbracket}\left(\mathbb{Z}_2[\chi], \pi_t\left(KO_2^\wedge\right)\right) \\
=& \begin{cases} \mathbb{Z}_2[\chi]\Big/\left(\widetilde{\chi}(g)-g^{2t'}\right), & s=1, t=4t'; \\ \mathrm{Ann}\left(\widetilde{\chi}(g)-1\right), & s=0, t\equiv 1,2 \mod 8; \\ \mathbb{Z}_2[\chi]/\left(2,\widetilde{\chi}(g)-1\right), & s=1, t\equiv 1,2 \mod 8; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

- When $\chi(-1)=-1$, $(K_2^\wedge)^{h\chi|_{(\mathbb{Z}/4)^\times}} \simeq \Sigma^2 KO_2^\wedge$ by Proposition 6.3. The E_2-page of the HESS is:

$$\begin{aligned}
E_2^{s,t} =& \mathrm{Ext}^s_{\mathbb{Z}_2\llbracket 1+4\mathbb{Z}_2\rrbracket}\left(\mathbb{Z}_2[\chi], \pi_t\left(\Sigma^2 KO_2^\wedge\right)\right) \\
=& \begin{cases} \mathbb{Z}_2[\chi]\Big/\left(\widetilde{\chi}(g)-g^{2t'+1}\right), & s=1, t=4t'+2; \\ \mathrm{Ann}\left(\widetilde{\chi}(g)-1\right), & s=0, t\equiv 3,4 \mod 8; \\ \mathbb{Z}_2[\chi]/\left(2,\widetilde{\chi}(g)-1\right), & s=1, t\equiv 3,4 \mod 8; \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

In both cases, the spectral sequences collapse at the E_2-pages. Analogous to Proposition 2.4 (Proposition 6.4), the extension problems at $t-s\equiv 1 \mod 8$ ($t-s\equiv 3 \mod 8$, resp.) are trivial. We further simplify the formulas using the following facts about $\mathbb{Z}_2[\chi]$ from Proposition B.2.

Lemma 6.3. *Let χ be a primitive 2-adic Dirichlet character of conductor $2^v \geq 8$. Let g be a topological generator of $1 + 4\mathbb{Z}_2$.*

1. *$\mathbb{Z}_2[\chi]$ is a totally ramified extension of $\mathbb{Z}_2$ of ramification index 2^{v-3}.*
2. *$1 - \widetilde{\chi}(g)$ is a uniformizer of $\mathbb{Z}_2[\chi]$ and $\mathbb{Z}_2[\chi]/(1 - \widetilde{\chi}(g)) \simeq \mathbb{Z}/2$.*
3. *The ideal of annihilators of $\widetilde{\chi}(g) - 1 \in \mathbb{Z}_2[\chi]/(2)$ is isomorphic to $\mathbb{Z}/2$.*
4. *$\mathbb{Z}_2[\chi]/(\widetilde{\chi}(g) - g^k) = \mathbb{Z}/2$ for any k.*

Proof. Only (4) needs a proof. $\widetilde{\chi}(g) = \zeta_{2^{v-2}}$ since χ is primitive. Write $\widetilde{\chi}(g) - g^k = \widetilde{\chi}(g) - 1 + 1 - g^k$. By (2), $\widetilde{\chi}(g) - 1$ is a uniformizer. On the other hand $v_2(1 - g^k) \geq 2 > v_2(\widetilde{\chi}(g) - 1)$, since $g \equiv 1 \mod 4$. This implies:

$$(\widetilde{\chi}(g) - g^k) = (\widetilde{\chi}(g) - 1) \implies \mathbb{Z}_2[\chi]/(\widetilde{\chi}(g) - g^k) = \mathbb{Z}/2.$$

$\square$

Proposition 6.5. *When $\chi(-1) = 1$, we have*

$$\pi_i\left(S^0_{K(1)}(2^v)^{h\chi}\right) = \begin{cases} \mathbb{Z}/2, & i \equiv 0, 2, 3, 7 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 1 \mod 8; \\ 0, & \text{otherwise.} \end{cases} \tag{6.13}$$

When $\chi(-1) = -1$, we have

$$\pi_i\left(S^0_{K(1)}(2^v)^{h\chi}\right) = \begin{cases} \mathbb{Z}/2, & i \equiv 1, 2, 4, 5 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 3 \mod 8; \\ 0, & \text{otherwise.} \end{cases} \tag{6.14}$$

Remark 6.6. The computations above depend on the actual model of the C_2-actions on the Moore spectra:

- When $\chi(-1) = 1$, if we choose $S^{2-2\sigma}$ as a model for the C_2-action on S^0 with trivial induced action on π_*, (6.13) becomes:

$$\pi_i\left(S^0_{K(1)}(2^v)^{h'\chi}\right) = \begin{cases} \mathbb{Z}/2, & i \equiv 3, 4, 6, 7 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 5 \mod 8; \\ 0, & \text{otherwise.} \end{cases}$$

- When $\chi(-1) = -1$, if we choose $S^{1-\sigma}$ as a model for the C_2-action on S^0 that induces sign representations on π_*, (6.14) becomes:

$$\pi_i\left(S^0_{K(1)}(2^v)^{h'\chi}\right) = \begin{cases} \mathbb{Z}/2, & i \equiv 0,1,5,6 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 7 \mod 8; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\pi_{2k-1}\left(S^0_{K(1)}(2^v)^{h\chi}\right) = \pi_{2k-1}\left(S^0_{K(1)}(2^v)^{h'\chi}\right)$ when $(-1)^k = \chi(-1)$.

Question 6.2. *Like the odd prime case, we have when* $\chi(-1) = -1$,

$$\pi_i\left(S^0_{K(1)}(2^v)^{h\chi}\right) = \pi_i\left(S^0_{K(1)}(4)^{h\chi|_{(\mathbb{Z}/4)^\times}}\right)/2.$$

So one might wonder in this case if there is an equivalence:

$$S^0_{K(1)}(2^v)^{h\chi} \overset{?}{\simeq} S^0_{K(1)}(4)^{h\chi|_{(\mathbb{Z}/4)^\times}} \bigwedge M(\mathbb{Z}/2).$$

6.3 The $N = p^v N'$ with $p \nmid N' > 1$ case

In this case, a Dirichlet character $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ factorizes into a product $\chi = \chi_p \cdot \chi'$, where χ_p has conductor p^v and χ' has conductor N'. The subgroup $(\mathbb{Z}/N')^\times$ of $(\mathbb{Z}/N)^\times$ acts trivially on $S^0_{K(1)}(p^v)$.

Proposition 6.6. *Write* $(\mathbb{Z}/N')^\times = G'_p \times G'$, *where* G'_p *is the Sylow* p*-subgroup of* $(\mathbb{Z}/N')^\times$. *If* $\chi'|_{G'}$ *is non-trivial, then the Dirichlet* $K(1)$*-local sphere is contractible.*

Proof. We have identifications:

$$\begin{aligned} S^0_{K(1)}(p^v)^{h\chi} &\simeq \left(S^0_{K(1)}(p^v)^{h\chi_p}\right)^{h\chi'} \simeq \left(\left(K_p^\wedge\right)^{h\widetilde{\chi_p}}\right)^{h\chi'} \simeq \left(\left(K_p^\wedge\right)^{h\chi'}\right)^{h\widetilde{\chi_p}}, \\ \left(K_p^\wedge\right)^{h\chi'} &\simeq \left(\left(K_p^\wedge\right)^{h\chi'|_{G'}}\right)^{h\chi'|_{G'_p}}. \end{aligned}$$

Thus it suffices to show $\left(K_p^\wedge\right)^{h\chi'|_{G'}}$ is contractible. As the order of the group G' is coprime to p, its group cohomology is concentrated in degree 0. In degree 0, the action of G' on $\mathrm{Hom}_{\mathbb{Z}_p}\left(\mathbb{Z}_p[\chi'|_{G'}], \left(K_p^\wedge\right)_{2t}\right) \simeq \mathbb{Z}_p[(\chi')^{-1}|_{G'}]$ has no fixed points. This implies all entries vanish in the HFPSS (or HESS) to compute $\pi_*\left(\left(K_p^\wedge\right)^{h\chi'|_{G'}}\right)$, from which we conclude $\left(K_p^\wedge\right)^{h\chi'|_{G'}}$, and hence $S^0_{K(1)}(p^v)^{h\chi}$ are contractible. ☐

Corollary 6.4. *$S^0_{K(1)}(p^v)^{h\chi}$ is contractible when $(p, \phi(N')) = 1$ and χ is primitive of conductor $N = p^v N'$ with $p \nmid N'$. In particular, we have*

1. *When $N = q \neq p$ is a prime with $p \nmid (q-1)$, $\left(S^0_{K/p}\right)^{h\chi}$ is contractible.*
2. *When $N = q^v > 2q$ for any prime not equal to p, $\left(S^0_{K/p}\right)^{h\chi}$ is contractible.*

Proof. In (1), the assumption implies the order of the group $|(\mathbb{Z}/q)^\times| = q-1$ is coprime to q and χ is non-trivial. In (2), write $(\mathbb{Z}/q^v)^\times \simeq (\mathbb{Z}/q)^\times \times \mathbb{Z}/q^{v-1}$ ($(\mathbb{Z}/2^v)^\times \simeq (\mathbb{Z}/4)^\times \times \mathbb{Z}/q^{v-2}$ when $q = 2$). $\chi|_{\mathbb{Z}/q^{v-1}}$ ($\chi|_{\mathbb{Z}/2^{v-2}}$ when $q = 2$) is non-trivial since χ is primitive of conductor $N = q^v > 2q$. The claim now follows from Proposition 6.6. ☐

When $\chi|_{G'}$ is trivial, we have

$$\left(K_p^\wedge\right)^{h\chi'} \simeq \left(K_p^\wedge\right)^{h\chi'|_{G_p}}$$

The entries on the E_2-page of the HESS to compute $\pi_*\left(\left(K_p^\wedge\right)^{h\chi'|_{G_p}}\right)$ are group cohomology of G_p', whose cohomological dimension with coefficients in $\mathbb{Z}_p$-modules is infinite. The spectral sequence collapses at E_2-page because of parity, but the author does not know how to solve the extension problem in that case.

Example 6.1. Let $N = 3$, $p = 2$ and $\chi = \sigma$ be the non-trivial 2-adic Dirichlet character of conductor 3. By definition $\left(S^0_{K(1)}\right)^{h\sigma}$ is the homotopy fixed points of $S^0_{K(1)}$ under the reflection action of C_2. As C_2 is a finite group, this homotopy fixed points in the $K(1)$-category are equivalent to the homotopy orbits:

$$\left(S^0_{K(1)}\right)^{h\sigma} \simeq \left(S^0_{K(1)}\right)_{h\sigma}.$$

One can show this homotopy orbit is not contractible as in Remark 5.4.

We record the 0-th line of (5.12) in this case

Proposition 6.7. *Write $N = p^v N'$ with $p \nmid N' > 1$. When $(-1)^t = \chi(-1)$, we have*

$$\mathrm{Hom}_{\mathbb{Z}_p[(\mathbb{Z}/N)^\times]}\left(\mathbb{Z}_p[\chi], \pi_{2t-1}\left(S^0_{K(1)}(p^v)\right)\right) = 0.$$

Proof. Recall that $\pi_{2t-1}\left(S^0_{K(1)}(p^v)\right) \simeq H^1\left(1 + p^v\mathbb{Z}_p; \left(K_p^\wedge\right)_{2t}\right)$ when $(-1)^t = \chi(-1)$ from the computations in Section 2.3. Again, write $\chi = \chi_p \cdot \chi'$, where χ_p has conductor p^v and χ' has conductor N' coprime to p. We have

$$\mathrm{Hom}_{\mathbb{Z}_p[(\mathbb{Z}/N')^\times]}\left(\mathbb{Z}_p[\chi'], \left(K_p^\wedge\right)_{2t}\right) = 0.$$

since the $(\mathbb{Z}/N)^{\times}$-action induced by χ' has no fixed points on $\mathbb{Z}_p[\chi']$ and $(\mathbb{Z}/N')^{\times}$ acts on the torsion free module $\left(K_p^{\wedge}\right)_{2t}$ trivially. By exchanging Ext-groups repeatedly, we get:

$$\begin{aligned}
&\operatorname{Hom}_{\mathbb{Z}_p[(\mathbb{Z}/N)^{\times}]}\left(\mathbb{Z}_p[\chi], \pi_{2t-1}\left(S^0_{K(1)}(p^v)\right)\right)\\
\simeq&\operatorname{Hom}_{\mathbb{Z}_p[(\mathbb{Z}/N')^{\times}]}\left(\mathbb{Z}_p[\chi'], \operatorname{Hom}_{\mathbb{Z}_p[(\mathbb{Z}/p^v)^{\times}]}\left(\mathbb{Z}_p[\chi_p], \pi_{2t-1}\left(S^0_{K(1)}(p^v)\right)\right)\right)\\
\simeq&\operatorname{Hom}_{\mathbb{Z}_p[(\mathbb{Z}/N')^{\times}]}\left(\mathbb{Z}_p[\chi'], \operatorname{Ext}^1_{\mathbb{Z}_p[\![\mathbb{Z}_p^{\times}]\!]}\left(\mathbb{Z}_p[\chi_p], \left(K_p^{\wedge}\right)_{2t}\right)\right)\\
\simeq&\operatorname{Ext}^1_{\mathbb{Z}_p[\![\mathbb{Z}_p^{\times}]\!]}\left(\mathbb{Z}_p[\chi_p], \operatorname{Hom}_{\mathbb{Z}_p[(\mathbb{Z}/N')^{\times}]}\left(\mathbb{Z}_p[\chi'], \left(K_p^{\wedge}\right)_{2t}\right)\right)\\
\simeq&0.
\end{aligned}$$

□

6.4 Dirichlet J-spectra and L-functions

In this section, we assemble homotopy groups of $J(N)^{h\chi}$ from the computations in the previous section and observe their similarities with the Dirichlet L-functions.

Theorem 6.1. *Let χ be a primitive Dirichlet character $(\mathbb{Z}/N)^{\times} \to \mathbb{C}^{\times}$ of conductor N.*

1. *When $N = p > 2$, we have*

$$\pi_i\left(J(p)^{h\chi}\left[\frac{1}{p-1}\right]\right) = \begin{cases} \mathbb{Z}/p^{v_p(k)+1}, & i = 2k-1 \text{ and } \ker\omega^k = \ker\chi;\\ 0, & \text{otherwise.}\end{cases}$$

2. *When $N = p^v$, $v > 1$ and $p > 2$, we have:*

$$\pi_i\left(J(p^v)^{h\chi}\right) = \begin{cases} \mathbb{Z}/p, & i = 2k-1 \text{ and } \ker\omega^k = \ker\chi|_{(\mathbb{Z}/p)^{\times}};\\ 0, & \text{otherwise.}\end{cases}$$

3. *When $N = 4$, the only non-trivial character satisfies $\chi(-1) = -1$. We have:*

$$\pi_i\left(J(4)^{h\chi}\right) = \begin{cases} \mathbb{Z}/4, & i = 4k+1;\\ \mathbb{Z}/2, & i \equiv 2,4 \mod 8;\\ \mathbb{Z}/2\oplus\mathbb{Z}/2, & i \equiv 3 \mod 8;\\ 0, & \text{otherwise.}\end{cases}$$

4. *When* $N = 2^v > 4$ *and* $\chi(-1) = 1$, *we have:*

$$\pi_i\left(J(2^v)^{h\chi}\right) = \begin{cases} \mathbb{Z}/2, & i \equiv 0,2,3,7 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 1 \mod 8; \\ 0, & \text{otherwise.} \end{cases}$$

5. *When* $N = 2^v > 4$ *and* $\chi(-1) = -1$, *we have:*

$$\pi_i\left(J(2^v)^{h\chi}\right) = \begin{cases} \mathbb{Z}/2, & i \equiv 1,2,4,5 \mod 8; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & i \equiv 3 \mod 8; \\ 0, & \text{otherwise.} \end{cases}$$

6. *When* N *is a square-free composite number,* $J(N)^{h\chi}$ *is contractible after inverting* $\prod_{p|N}(p-1)$. *If* N *is composite number with a non-trivial square factor, then* $J(N)^{h\chi}$ *is contractible.*

Theorem 6.2. *Let* $\mathcal{D}_{k,\chi}$ *be the ideal of* $\mathbb{Z}[\chi]$ *generated by the denominator of* $\frac{B_{k,\chi}}{2k} \in \mathbb{Q}(\chi)$. *When* $N = p^v$ *and* $(-1)^k = \chi(-1)$, *we have*

$$\pi_{2k-1}\left(J(N)^{h\chi}\left[\frac{1}{p-1}\right]\right) \simeq \mathbb{Z}[\chi]/\mathcal{D}_{|k|,\chi^{-1}}.$$

Remark 6.7. By Remark 6.5 and Remark 6.6, the statements above are independent of the models of $M(\mathbb{Z}[\chi])$ when $(-1)^k = \chi(-1)$.

Proof. In the first fives cases in Theorem 6.1, the Dirichlet J-spectra are equivalent to their p-completions by Corollary 5.6, Proposition 5.12 and Corollary 6.4. (6) also follows from the three statements. The only thing remains to check is π_{2k-1} where $(-1)^k = \chi(-1)$ and $N = p^v > 1$. For that, it suffices to compare the arithmetic properties of $B_{k,\chi}$ in Theorem 1.2 with computations in Section 6.1.

1. $N = p > 2$. Comparing the decomposition in Examples 5.1 and computation in (6.2) with Theorem 1.2, we need to check the following:
    - Let g be a primitive $(p-1)$-st root of unity mod p. The ideal $\mathfrak{p} := (p, 1-\chi(g)g^k)$ of $\mathbb{Z}[\chi]$ is not equal to (1) iff $\ker\chi = \ker\omega^{-k}$. To see this, notice by Corollary B.2, there is an isomorphism of $(\mathbb{Z}/p)^\times$-representations:

$$\mathbb{Z}[\chi]/p \simeq \bigoplus_{\substack{0\le a\le p-2 \\ \ker\omega^a = \ker\chi}} (\mathbb{Z}/p)_{\omega^a} \simeq \bigoplus_{\substack{0\le a\le p-2 \\ \ker\omega^a = \ker\chi}} (\mathbb{Z}/p)^{\otimes a}.$$

Then $1-\chi(g)g^k$ is invertible in $\mathbb{Z}[\chi]/p$ iff $1 \equiv g^a \cdot g^k \mod p$ for some a satisfying $0 \le a \le p-2$ and $\ker\chi = \ker\omega^a$. Since g is a primitive $(p-1)$-st root of unity mod p, this condition is further equivalent to saying $(p-1) \mid (a+k)$ for such an a. From this we conclude $\ker\chi = \ker\omega^{-k}$.

- When $\mathfrak{p} \neq (1)$, the congruence (1.2) $pB_{k,\chi} \equiv p-1 \mod \mathfrak{p}^{v_p(k)+1}$ implies $\mathbb{Z}[\chi]/\mathcal{D}_{k,\chi} \simeq \mathbb{Z}/p^{v_p(k)+1}$.

  It suffices to check this formula holds p-adically and 2-adically since the denominator ideal of $\frac{B_{k,\chi}}{k}$ is p-primary by Theorem 1.2. As $2 \mid (p-1)$, $\mathcal{D}_{k,\chi}$ has no 2-primary factors by (1.2). p-adically, $\mathfrak{p}$ is the same as (p) when it is not (1). Now (1.2) becomes

$$pB_{k,\omega^a} \equiv p-1 \mod p^{v_p(k)+1} \implies \frac{B_{k,\omega^a}}{2k} \equiv \frac{p-1}{2pk} \mod \mathbb{Z}_p,$$

  where a satisfies $\ker\omega^a = \ker\chi$ and $(p-1) \mid (k+a)$. This implies

$$\mathbb{Z}[\chi]/\mathcal{D}_{k,\chi^{-1}} \simeq \mathbb{Z}/p^{v_p(k)+1} \simeq \pi_{2k-1}\left(J(p)^{h\chi}\left[\frac{1}{p-1}\right]\right).$$

2. $N = p^v$, $v > 1$ and $p > 2$. By Lemma 6.1, $\mathfrak{p} = (p, 1-\chi(g)g^k) \neq (1)$ when $\ker\chi|_{(\mathbb{Z}/p)^\times} = \ker\omega^{-k}$. In that case, $\mathfrak{p} = (1-\zeta_{p^{v-1}}, p) = (1-\zeta_{p^{v-1}})$. On the other hand, since $1+p$ is a generator of the subgroup $\mathbb{Z}/p^{v-1} \subseteq (\mathbb{Z}/p^v)^\times$ and χ is primitive, $\chi(1+p)$ is also a primitive p^{v-1}-th root of unity. As a result, (1.2) translates into

$$(1-\chi(p+1))\frac{B_{k,\chi}}{k} \equiv 1 \mod \mathfrak{p} \implies \frac{B_{k,\chi}}{k} \equiv \frac{1}{1-\zeta_{p^{v-1}}} \mod \mathbb{Z}_p[\zeta_{p^{v-1}}].$$

   Thus $\mathcal{D}_{k,\chi}$ is either $(1-\zeta_{p^{v-1}})$ or $(2(1-\zeta_{p^{v-1}}))$. Whereas by Theorem 6.1, $\pi_{2k-1}\left(J(p^v)^{h\chi}\right) \simeq \mathbb{Z}/p \simeq \mathbb{Z}[\chi]/(1-\zeta^{p_{v-1}})$.

3. $N = 4$. In this case $\chi = \chi^{-1}$ since $(\mathbb{Z}/4)^\times \simeq C_2$. By (1.4), we have when k is odd:

$$\frac{B_{k,\chi}}{k} \equiv \frac{1}{2} \mod 1 \implies \frac{B_{k,\chi}}{2k} \equiv \pm\frac{1}{4} \mod 1.$$

   Thus $\mathcal{D}_{k,\chi} = \mathcal{D}_{k,\chi^{-1}}$ is equal to the ideal (4) of $\mathbb{Z}[\chi] \simeq \mathbb{Z}$. This matches the computation in (6.11) that $\pi_{2k-1}\left(S^0_{K(1)}(4)^{h\omega}\right) \simeq \mathbb{Z}/4$ when k is odd.

4. $N = 2^v > 4$. Theorem 1.2 says $\frac{B_{k,\chi}}{k}$ is an algebraic integer. It follows from Theorem 3.3 and Case IV in Theorem 4.1 that $\mathcal{D}_{k,\chi} = (1-\zeta_{2^{v-2}})$. This matches with the computation in Theorem 6.1 that $\pi_{2k-1}\left(J(2^v)^{h\chi}\right) \simeq \mathbb{Z}/2 \simeq \mathbb{Z}[\chi]/(1-\zeta_{2^{v-2}})$.

□

We summarize another the computation of HESS used in Section 6.1.

Proposition 6.8. *Let χ be a p-adic Dirichlet character of conductor $N = p^v > 1$. There is a spectral sequence*

$$E_2^{s,t} = H^s(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes t}[\chi^{-1}]) \Longrightarrow \pi_{2t-s}\left(S^0_{K(1)}(p^v)^{h\chi}\right),$$

where $a \in \mathbb{Z}_p^\times$ acts on $\mathbb{Z}_p^{\otimes t}[\chi^{-1}]$ by multiplication by $a^t \cdot \chi^{-1}(a)$. This spectral sequence collapses at the E_2-page when $p > 2$. In particular, when $(-1)^k = \chi(-1)$, the following holds for all primes p:

$$H^1(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes k}[\chi^{-1}]) \simeq \pi_{2k-1}\left(S^0_{K(1)}(p^v)^{h\chi}\right).$$

Proof. Applying derived adjunction on (6.3),(6.12), and (6.7), we have

$$H^s(\mathbb{Z}_p^\times; \mathbb{Z}_p^{\otimes t}[\chi^{-1}]) \simeq \mathrm{Ext}^s_{\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]}(\mathbb{Z}_p[\chi], \mathbb{Z}_p^{\otimes t}).$$

The result follows from the computations in Section 6.1. ☐

Remark 6.8. Like in Remark 5.5, we observe the duality phenomena in the homotopy groups of $J(N)^{h\chi}$ and $S^0_{K(1)}(p^v)^{h\chi}$ from the computations in this chapter.

When p is odd and χ is a p-adic Dirichlet character of conductor p^v, we observe from (6.2) and (6.8) that

$$\mathrm{Hom}_{\mathbb{Z}_p}\left(\pi_i\left(S^0_{K(1)}(p^v)^{h\chi}\right), \mathbb{Q}_p/\mathbb{Z}_p\right) \simeq \pi_{-2-i}\left(S^0_{K(1)}(p^v)^{h\chi^{-1}}\right).$$

Also, when p is odd and χ is a complex-valued Dirichlet character of conductor p^v, we observe from Theorem 6.1 that

$$\mathrm{Hom}_{\mathbb{Z}}\left(\pi_i\left(J(p^v)^{h\chi}\left[\frac{1}{p-1}\right]\right), \mathbb{Q}/\mathbb{Z}\right) \simeq \pi_{-2-i}\left(J(p^v)^{h\chi^{-1}}\left[\frac{1}{p-1}\right]\right).$$

When $p = 2$, the formulas above hold up to summands of $\mathbb{Z}/2$. These formulas suggest a possible Brown-Comenetz duality:

$$I_{K(1)}\left(S^0_{K(1)}(p^v)^{h\chi}\right) \overset{?}{\simeq} \Sigma^2 S^0_{K(1)}(p^v)^{h\chi^{-1}} \quad \text{and} \quad I_{\mathbb{Q}/\mathbb{Z}}\left(J(p^v)^{h\chi}\left[\frac{1}{p-1}\right]\right) \overset{?}{\simeq} \Sigma^2\left(J(p^v)^{h\chi^{-1}}\left[\frac{1}{p-1}\right]\right).$$

In the view of Theorem 6.2, this possible duality resembles the functional equations of the Dirichlet L-functions. Let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$ be a primitive Dirichlet character of conductor N and k is a positive integer such that $(-1)^k = \chi(-1)$. Then we have the following functional equation of $L(k;\chi)$:

$$L(k;\chi) = \frac{\tau(\chi)}{2(k-1)!} \cdot \left(\frac{2\pi i}{N}\right)^k \cdot L(1-k;\chi^{-1}), \text{ where } \tau(\chi) = \sum_{a=1}^{N} \chi(a) e^{\frac{2\pi i a}{N}}.$$

Appendices

Appendix A

Dieudonné modules of formal groups

A.1 Formal groups and p-divisible groups

We start by reviewing basic concepts of formal groups and formal A-modules.

Definitions A.1. A **formal group** $\widehat{G}$ over a ring R is a formal abelian variety over R that is locally isomorphic to the affine formal scheme $\widehat{\mathbb{A}}^d$. This d is called the **dimension** of $\widehat{G}$.

When $\widehat{G} \simeq \operatorname{Spf} A[\![x_1, \cdots, x_d]\!]$, the comuliplication map

$$\Delta : A[\![x_1, \cdots, x_d]\!] \to A[\![x'_1, \cdots, x'_d, x''_1, \cdots, x''_d]\!]$$

is determined by the images of the x_i's. Denote the variables $(x'_1, \cdots, x'_d)$ and $(x''_1, \cdots, x''_d)$ by x' and x'' respectively. $F_i(x', x'') := \Delta(x_i)$ is a formal power series of $2d$-variables. Denote $(F_1, \cdots, F_d)$ by F. Then F satisfies the following identities:

1. $F(x', 0) = x'$ and $F(0, x'') = x''$,
2. $F(x', F(x'', x''')) = F(F(x', x''), x''')$,
3. $F(x', x'') = F(x'', x')$.

In general, formal power series satisfying the above identities is called a **formal group law**. It is conventional to write $F(x', x'') = x' +_F x''$. The inverse map in the formal group corresponds to a power series ι_F such that $x +_F \iota_F(x) = \iota_F(x) +_F x = 0$.

Examples A.1. One way to produce formal groups is to complete abelian varieties at their origins. Over an algebraically closed field, a one dimensional abelian variety is either the additive group, the multiplicative group or an elliptic curve. Completing them at the origin, we get

1. The additive formal group $\widehat{G}_a$, whose associated formal group law is $F(x, y) = x + y$.
2. The multiplicative formal group $\widehat{G}_m$, whose associated formal group law is $F(x, y) = x + y + xy$.

3. An elliptic formal group $\widehat{G}_{\text{ell}}$, see [Sil09, IV.1] for the formula of the associated formal group law.

A morphism of formal groups is a morphism of the underlying formal schemes that is compatible with the group scheme structure. In terms of local coordinates, we have

Definition A.1. Let F_1 and F_2 be formal group laws over a ring R. A *morphism of formal group laws* $f : F_1 \to F_2$ is a power series $f(x) \in R[\![x]\!]$ satisfying

1. $f(0) = 0$,
2. $f(F_1(x,y)) = F_2(f(x), f(y))$.

Lemma A.1. *A formal power series $f(t) \in R[\![t]\!]$ has a composition inverse f^{-1} iff $f'(0)$ is invertible.*

The lemma shows that a morphism of formal group laws $f : F_1 \to F_2$ is an isomorphism iff $f'(0)$ is invertible. In addition, f is called a **strict isomorphism** if $f'(0) = 1$.

The pullback of a formal group by a map of schemes admits a natural formal group structure. In terms of formal group laws, we have

Definition A.2. If $f : R \to S$ is a ring homomorphism and F is a formal group law over R, $F(x,y) = \sum a_{ij}x^iy^j$, we define a formal group law f^*F over S by setting $f^*F(x,y) = \sum f(a_{ij})x^iy^j$.

The classification of formal groups depend on the characteristic base ring.

Proposition A.1. *Let F be a 1-dimnsional formal group law over $\mathbb{Q}$-algebra, then there is a unique strict isomorphism $f : F \xrightarrow{\sim} \widehat{G}_a$. This strict isomorphism is called the **logarithm** of F.*

Proof. Let F be a formal group law over $\mathbb{Q}$. Let's construct the strict isomorphism $f : F \to \widehat{G}_a$. From the definition, $f(t) \in \mathbb{Q}[\![t]\!]$ satisfies

$$f(F(x,y)) = f(x) + f(y) \tag{A.1}$$

and $f'(0) = 1$. Now we can solve for f from these two equations. Take $\left.\frac{\partial}{\partial y}\right|_{y=0}$ on both sides of (A.1). We get

$$F_y(x,0)f'(x) = 1 \qquad \Longrightarrow \qquad f(x) = \int_0^x \frac{dt}{F_y(t,0)}. \tag{A.2}$$

Here, we can do integration because all the nonzero integers are invertible in $\mathbb{Q}$. Now it can be shown that such f is a strict isomorphism from F to $\widehat{G}_a$. ☐

Over a ring of characteristic p, formal groups admit a invariant called the height. The height is defined using the p-series of the formal group laws.

Definition A.3. The n-series of a formal group law F is defined inductively by

- $[0]_F(t) = 0$
- $[n+1]_F(t) = [n]_F(t) +_F t$ when $n > 0$.
- $[-n]_F(t) = \iota_F([n]_F(t))$ when $n > 0$.

Lemma A.2. *Let R be a ring of characteristic p and F be a formal group law over R, then either $[p]_F t = 0$, or there is a unique integer h and a power series $g(t)$ with $g(0) = 0$ and $g'(0) \neq 0$ such that $[p]_F(t) = g\left(t^{p^h}\right)$.*

Definitions A.2. This integer h is called the **height** of F. If in addition $g'(0)$ is invertible, we say F is of **pure height** h. In this paper, we will omit "pure" unless otherwise specified. If $[p]_F t = 0$, we say the height of F is ∞. A formal group law is called p**-divisible** if $h < \infty$.

Examples A.2. Over a finite field κ of characteristic p, the heights of the formal groups in Examples A.1 are

1. $\widehat{G}_a$ has height ∞ at all primes p, since its p-series is $[p]_a(t) = pt \equiv 0 \mod p$.
2. $\widehat{G}_m$ has height 1 at all primes p, since its p-series is $[p]_m(t) = (1+t)^p - 1 \equiv t^p \mod p$.
3. $\widehat{G}_{\text{ell}}$ has height 1 if the elliptic curve is p-ordinary and has height 2 if the elliptic curve is p-supersingular.

In terms of formal groups, height means the following. Let $\widehat{G}_0$ be a formal group over a ring R of characteristic p and $\varphi_0 : R \to R$ be the Frobenius endomorphism (p-th power map). The p-series of $\widehat{G}_0$ factorizes through $\varphi_0^*\widehat{G}_0$ as follows:

$$\begin{tikzcd}
\widehat{G}_0 \arrow[rr, "{[p]}"] \arrow[dr, "\Upsilon_0"'] & & \widehat{G}_0 \\
& \varphi_0^*\widehat{G}_0 \arrow[ur, "\Phi_0"'] &
\end{tikzcd} . \tag{A.3}$$

The map $\varphi^*\widehat{G}_0 \to \widehat{G}_0$ can be further factorized if the induced map on the Lie algebras is zero [Goe08, Lemma 5.1]. $\widehat{G}_0$ has height h, if the map $\left(\varphi_0^{(h)}\right)^* \widehat{G}_0 \to \widehat{G}_0$ below cannot be further factorized:

$$\begin{tikzcd}
\widehat{G}_0 \arrow[rr, "{[p]}"] \arrow[dr] & & \widehat{G}_0 \\
& \left(\varphi_0^{(h)}\right)^* \widehat{G}_0 \arrow[ur] &
\end{tikzcd} . \tag{A.4}$$

In addition, $\widehat{G}_0$ has pure height h iff the map $\left(\varphi_0^{(h)}\right)^* \widehat{G}_0 \to \widehat{G}_0$ is an isomorphism.

Now let $\widehat{G}_0$ be a formal group of height h over R. By the definition above, $[p]_{\widehat{G}_0}$ is an isogeny of degree p^h. So the closed subscheme $\widehat{G}_0[p]$ of $\widehat{G}_0$ defined by the power series $[p]_{\widehat{G}_0}$ is a finite subscheme of rank p^h over R. It is actually a sub-group scheme of $\widehat{G}_0$. Similarly, one can show in general $\widehat{G}_0[p^m]$ is a finite group scheme flat over R of rank p^{hm}. Assembling all the p^m-torsions of a formal group, we get its associated p-divisible group.

Definition A.4. A **p-divisble group** over a ring R (or more generally a scheme) of height h is a sequence of finite flat group schemes $\{G_v\}$ with maps $i_v : G_v \to G_{v+1}$ for $i \geq 1$ such that

- G_v is finite flat of rank p^{hv} over $\operatorname{Spec} R$.
- There is a short exact sequence of finite group schemes:

$$0 \longrightarrow G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{[p^v]} G_{v+1} \longrightarrow 0.$$

Examples A.3. Starting with a formal group or an abelian variety, we can produce a p-divisible group by taking G_v to be its p^v-torsion

1. Let A be an abelian variety of dimension d, then its associated p-divisible group $A[p^\infty]$ has height $2d$.
2. Let $\widehat{G}$ be a 1-dimensional formal group of height h, then its associated p-divisible group $\widehat{G}[p^\infty]$ has height h.

One can recover the information of a p-divisible formal group from its associated p-divisible group.

Definition A.5. The **connected-étale decomposition** of a p-divisible group is a short exact sequence of p-divisible groups

$$0 \longrightarrow G^0 \longrightarrow G \longrightarrow G^{ét} \longrightarrow 0,$$

where G^0 and $G^{ét}$ are p-divisible groups with $(G^0)_v$ connected and $(G^{ét})_v$ étale over R for all v. A p-divisible group G is called **connected** if $G^{ét} = 0$.

Theorem A.1. *[Tat67, Proposition 1] Let R be a complete noetherian local ring. Then the functor $\widehat{G} \to \widehat{G}[p^\infty]$ is an equivalence of categories from p-divisible formal groups over R to connected p-divisible groups over R.*

In particular, we can study p-divisible formal groups from their p^m-torsion subgroups.

A.2 Dieudonné modules associated to formal groups

In this section, we introduce the Dieudonné modules of formal groups following [Kat81]. Formal groups over a finite field κ are not necessarily isomorphic to the additive formal group, so the logarithm cannot always be defined. Still, we want to use logarithm to study formal groups over finite fields. To do that, the first step is to lift a formal group $\widehat{G}_0$ from k to the Witt vectors $\mathbb{W}\kappa$. This is possible by the Lazard's theorem:

Theorem A.2 (Lazard)**.** *The functor from commutative rings to sets that assigns to a ring R the set of formal group laws over R is represented by a polynomial algebra of infinite generators $L \simeq \mathbb{Z}[\![b_1, b_2, \cdots]\!]$.*

Corollary A.1. *Let $R_1 \to R_2$ be a surjective ring homomorphism, then any formal group law over R_2 can be lifted to R_1.*

Let $\widehat{G}$ be a formal group over $\mathbb{W}\kappa$ lifting $\widehat{G}_0$. We may want to compute the logarithm of $\widehat{G}$ as in Proposition A.1. Everything works except for the very last step. Monomials like t^{pk-1} are not integrable over $\mathbb{W}\kappa$: p is not invertible in $\mathbb{W}\kappa$. But still, we can define the integrand in (A.2), since $F_y(t,0)$ is a formal power series with constant term 1. This integrand $\omega_{\widehat{G}}$ is called an **invariant differential** of the lift $\widehat{G}$. Again as in Proposition A.1, we can integrate $\omega_{\widehat{G}}$ if p is inverted. Let $f_{\widehat{G}}$ be the integration of $\omega_{\widehat{G}}$ in $\mathbb{Q}_p \otimes \mathbb{W}\kappa$ with zero constant term. $f_{\widehat{G}}$ is the logarithm of $\widehat{G}$. We have an identification:

$$\Omega^1(\mathbb{W}\kappa[\![t]\!]/\mathbb{W}\kappa) \simeq \{f(t) \in \mathbb{Q}_p \otimes \mathbb{W}\kappa[\![t]\!] \mid f(0) = 0, f'(t) \in \mathbb{W}\kappa[\![t]\!]\}. \tag{A.5}$$

We now use the integration of integral 1-form to represent the 1-form. By construction $f_{\widehat{G}}$ is the kernel of the operator

$$\partial(f) := f(x) - f(x +_{\widehat{G}} y) + f(y).$$

What if we start with another lift $\widehat{G}'$ of $\widehat{G}_0$, then we'll get a different operator

$$\partial'(f) := f(x) - f(x +_{\widehat{G}'} y) + f(y).$$

Since $\widehat{G}$ and $\widehat{G}'$ are both lifts of $\widehat{G}_0$, we have $\widehat{G}'(x,y) - \widehat{G}(x,y) = pg(x,y)$ for some $g(x,y) \in \mathbb{W}\kappa[\![x,y]\!]$. The difference of ∂ and ∂' is then

$$\begin{aligned}\partial(f) - \partial'(f) =& f(x +_{\widehat{G}'} y) - f(x +_{\widehat{G}} y)\\ =& f(x +_{\widehat{G}} y + pg(x,y)) - f(x +_{\widehat{G}} y)\\ =& \sum_{n\geq 1} \frac{p^n}{n!} f^{(n)}(x +_{\widehat{G}} y) \cdot (g(x,y))^n.\end{aligned} \tag{A.6}$$

Notice $\partial(f) - \partial'(f)$ is integral since $p^n/n! \in \mathbb{W}\kappa$ and $f'(t) \in \mathbb{W}\kappa[\![t]\!]$. So to get "logarithms" of $\widehat{G}_0$, we need to find the kernel of ∂ up to an integral power series. Thus it makes sense to consider the kernel of ∂ in $H^1_{\mathrm{dR}}(\widehat{G}/\mathbb{W}\kappa)$. Here, we have an identification like in (A.5):

$$H^1_{\mathrm{dR}}(\widehat{G}/\mathbb{W}\kappa) \simeq \frac{\{f(t) \in \mathbb{Q}_p \otimes \mathbb{W}\kappa[\![t]\!] \mid f(0) = 0, f'(t) \in \mathbb{W}\kappa[\![t]\!]\}}{\{f(t) \in \mathbb{W}\kappa[\![t]\!] \mid f(0) = 0\}}. \tag{A.7}$$

The kernel of ∂ in H^1_{dR} is called the **primitives** and we have an identification:

$$PH^1_{\mathrm{dR}}(\widehat{G}/\mathbb{W}\kappa) \simeq \frac{\{f(t) \in \mathbb{Q}_p \otimes \mathbb{W}\kappa[\![t]\!] \mid f(0) = 0, df \text{ and } \partial f \text{ are integral}\}}{\{f(t) \in \mathbb{W}\kappa[\![t]\!] \mid f(0) = 0\}}. \tag{A.8}$$

The analysis above shows $PH^1_{\mathrm{dR}}(\widehat{G}/\mathbb{W}\kappa)$ is independent of the lift $\widehat{G}$. Thus it makes sense to define $\mathbb{D}(\widehat{G}_0/k) := PH^1_{\mathrm{dR}}(\widehat{G}/\mathbb{W}\kappa)$.

Proposition A.2. *$\mathbb{D}$ is a contravariant functor from formal groups over κ to $\mathbb{W}\kappa$-modules and is compatible with pullback.*

The factorization in (A.3) induces a factorization of multiplication by p map on $PH^1_{\mathrm{dR}}(\widehat{G}/\mathbb{W}\kappa)$:

$$\begin{tikzcd}
PH^1_{\mathrm{dR}}(\widehat{G}/\mathbb{W}\kappa) \arrow[rr, "p"] \arrow[dr, "V"'] & & PH^1_{\mathrm{dR}}(\widehat{G}/\mathbb{W}\kappa) \\
& \varphi^* PH^1_{\mathrm{dR}}(\widehat{G}/\mathbb{W}\kappa) \arrow[ur, "F"'] &
\end{tikzcd},$$

where φ is a lift of Frobenius to $\mathbb{W}\kappa$. The maps F and V satisfies $FV = p = VF$ on their respective domains. The triple $(\mathbb{D}(\widehat{G}_0/k), F, V)$ is called the **Dieudonné module** associated to $\widehat{G}_0$. This construction also carries to formal groups over a ring R/p, where R is a smooth $\mathbb{W}\kappa$-algebra with a lift of Frobenius φ. The crucial points that this works are:

- The ideal $(p) \trianglelefteq R$ has a **divided power (P.D.)** structure. By (A.6), $PH^1_{\mathrm{dR}}(\widehat{G}/R)$ depends only on $\widehat{G}_0$.
- R is flat over $\mathbb{W}\kappa$. This implies $R \to p^{-1}R$ is a injection and we have a similar identification for $PH^1_{\mathrm{dR}}(\widehat{G}/R)$ as in (A.8).

Definition A.6. A Dieudonné module over R is a triple (M, F, V), where M is a projective R-module of finite rank, $F : \varphi^* M \to M$ and $V : M \to \varphi^* M$ are two maps such that $FV = p = VF$ on the respective domains. We further require M is **reduced** and **uniform** with respect to V. Here, reduced means M is V-complete and uniform means $M/VM \simeq V^l M/V^{l+1} M$ for all l. We say a Dieudonné module is **nilpotent** if V is p-adically nilpotent.

Theorem A.3. *[Blo, Cororally 7.2] Let R/p be smooth over a perfect field κ of characteristic p. The contravariant functor $\mathbb{D} : \widehat{G}_0 \mapsto (M, F, V)$ from p-divisible formal groups over R/p to nilpotent Dieudonné modules is an equivalence of categories.*

Proposition A.3. *Let $\mathbb{D}(\widehat{G}_0) = (M, F, V)$. Then $M/VM \simeq T^*(\widehat{G}_0)$ as R/p-modules.*

There is also a construction of Dieudonné modules for finite flat commutative group schemes. While it is possible to give a direct construction as above, here we follow [Oor74] and define them using the Dieudonné modules associated to formal groups.

Construction A.1. Let G be a finite flat commutative group scheme over R/p, then

1. G admits an embedding into an abelian variety X. If G is connected, then G can be thought of as a subgroup of $\widehat{X}$, the formal completion of X at origin. [dJon93, Remarks 1.4 (3)].
2. $\widehat{X}$ admits a categorical quotient by G. Equivalently, there is a surjective map of formal groups $f : \widehat{X} \twoheadrightarrow \widehat{Y}$ whose kernel G. This map is called a smooth resolution of G.
3. f induces an injective map on the Dieudonné modules $\mathbb{D}(f) : \mathbb{D}(\widehat{Y}) \hookrightarrow \mathbb{D}(\widehat{X})$. The cokernel of $\mathbb{D}(f)$ does not depend on the smooth resolution of G. Now define $\mathbb{D}(G) = \mathrm{Coker}(\mathbb{D}(f))$ with the induced F and V.
4. Equivalence between finite group schemes and Dieudonné modules.

A.3 Some explicit computations

Let's now compute the Dieudonné modules associated to certain formal groups. Before that, we need to find a suitable coordinate for the formal group.

Proposition A.4. *Let $\widehat{G}$ be a formal group over a p-local algebra R. Then $\widehat{G}$ has a coordinate t such that*

$$[p]_{\widehat{G}}(t) = pt +_{\widehat{G}} \sum_{i\geq 1}^{\widehat{G}} v_i t^{p^i}.$$

*This is called the p-**typical** coordinate of $\widehat{G}$. When t is p-typical, (A.8) can be reduced to:*

$$PH^1_{\mathrm{dR}}(\widehat{G}/\mathbb{W}\kappa) \simeq \left\{ f(t) = \sum_{i\geq 0} \frac{m_i}{p^i} t^{p^i} \mid m_i \in \mathbb{W}\kappa/p^i, f([p]_{\widehat{G}} t) - pf(t) \in \mathbb{W}\kappa[\![t]\!] \right\}. \tag{A.9}$$

Example A.1. The additive formal group $\widehat{G}_a$. In $\mathbb{D}(\widehat{G}_a) = (M, F, V)$, M is an infinite dimensional module over R/p with basis $\{\gamma, V\gamma, \cdots\}$ and $F = 0$.

Example A.2. With this identification, we can compute the Dieudonné module associated to the Honda formal groups Γ_h over k, whose p-series is $[p](t) = t^{p^h}$. By definition, Γ_h has height h. Let the p-series for a lift of Γ_h be $[p](t) = pt +_{\widehat{G}} t^{p^h}$. As t^{p^h} differs from $[p](t)$ by a multiple of p, we actually just need to find the $f(t)$'s such that $f(t^{p^h}) - pf(t)$ is integral. Solving for the coefficients of f, we get a basis for $\mathbb{D}(\Gamma_h/k)$:

$$f_0(t) = \sum_{i\geq 0} \frac{t^{p^{ih}}}{p^i}, \quad f_1(t) = \sum_{i\geq 0} \frac{t^{p^{ih+1}}}{p^i}, \quad \cdots, \quad f_{h-1}(t) = \sum_{i\geq 0} \frac{t^{p^{ih+h-1}}}{p^i}.$$

It is straight forward to check that both $f_j'(t)$ and $f_j\left(t^{p^h}\right) - pf(t)$ are integral for all j, and that the f_j's are linearly independent. Since V acts by $(Vf)(t) = f(t^p)$, we get the matrix representations of F and V with respect to this basis $\{f_0, \cdots, f_{h-1}\}$:

$$F = \begin{pmatrix} & pI_{h-1} \\ 1 & \end{pmatrix}, \qquad\qquad V = \begin{pmatrix} & p \\ I_{h-1} & \end{pmatrix}.$$

In particular, we have $V^h(f_0) = f_0\left(t^{p^h}\right) = pf_0(t)$. f_0 is called an **Artin-Hasse exponential**. When $h = 1$, $\Gamma_1 = \widehat{G}_m$. In this case, we have $F(f_0) = f_0$ and $V(f_0) = pf_0$.

Example A.3. If $\widehat{G}_0$ is one dimensional formal group of pure height h, then (A.4) implies in $\mathbb{D}(\widehat{G}_0) = (M, F, V)$, there is a γ, whose reduction mod V is a generator of M/VM and such that

$$F\gamma = V^{h-1}(a_0\gamma + a_1 V\gamma + \cdots). \tag{A.10}$$

Here, we should think of this equation as expressing $F\gamma$ in terms of powers of V. As $V^m M/V^{m+1}M \simeq M/VM$ is locally free rank 1 over R/p, the coefficients a_i's are in R/p and $a_0 \in (R/p)^\times$. When $h = 1$, F is an isomorphism. For general h, this implies $p\gamma = VF\gamma = V^h(a_0\gamma + a_1V\gamma + \cdots)$.

Proposition A.5. *M is projective of rank h over R.*

Proof. Since M is reduced and uniform with respect to V, Equation (A.10) implies $M/p \simeq M/V^h M$ is locally free of rank h over R/p. The claim now follows from Nakayama's lemma. □

Examples A.4. Following Construction A.1, we can compute $\mathbb{D}(G)$ for a finite subgroup G of a formal group scheme $\widehat{G}_0$ of height h.

1. $G \simeq \widehat{G}_0[p^m]$. The induced map of the p^m-isogeny of $\widehat{G}_0$ on $\mathbb{D}(\widehat{G}_0)$ is just multiplication by p^m. This implies in $\mathbb{D}(G) = (M, F, V) \simeq \mathbb{D}(\widehat{G}_0)/p^m$, M is locally free of rank h over R/p^m with F and V satisfy the same identity as in Example A.3.

2. $G \simeq \widehat{G}_0[\Upsilon_0^m]$. A finer filtration of subgroups of $\widehat{G}_0$ is given by the kernels of the iterated powers of Υ_0 in (A.3). Here $\Upsilon_0^m := \Upsilon_0 \circ \varphi_0^* \Upsilon_0 \circ \cdots \circ (\varphi_0^{(m)})^* \Upsilon_0$. The induced map of Υ_0^m on $\mathbb{D}(\widehat{G}_0)$ is V^m. By assumption, we have $\widehat{G}[\Upsilon_0^h] \simeq \widehat{G}_0[p]$. Let $m' = \lceil m/h \rceil$ be the ceiling of m/h. Then in $\mathbb{D}(G) = (M, F, V) \simeq \mathbb{D}(\widehat{G}_0)/V^m$, M is locally free of rank h over $R/p^{m'}$ with F and V satisfying the same identity as in Example A.3 plus $V^m = 0$.

A.4 Galois descent for formal groups

Following Theorem A.3, we can study the endo-(iso-)morphisms of formal groups via their Dieudonné modules. Let $(M, F, V) = \mathbb{D}(\Gamma_h)$ be the Dieudonné module defined in Example A.3 and let $f : (M, F, V) \to (M, F, V)$ be an endomorphism. f is completely determined by the image of γ:

$$f(\gamma) = a_0\gamma + a_1 V\gamma + \cdots + a_{h-1}V^{h-1}\gamma.$$

Since f is a homomorphism, it commutes with V and F, in particular with $p = V^h$. This would imply all the coefficients above satisfy $a_i^{\varphi^h} = a_i$.

Lemma A.3. *If R is flat over $\mathbb{W}\kappa$ and admits an endomorphism φ that lifts the Frobenius (p-th power map) on R/p, then $\{a \in R \mid a^{\varphi^h} = a\} = R \cap \mathbb{W}\mathbb{F}_{p^h}$.*

Proof. As R/p is an integral domain over κ, $[a] \in R/p$ satisfies $[a]^{p^h} = a$ iff $[a] \in \kappa \cap \mathbb{F}_{p^h} = R/p \cap \mathbb{F}_{p^h}$. Now let $a_0 \in \mathbb{W}(\kappa \cap \mathbb{F}_{p^h})$ be the Teichüller lift of $[a]$. Then $a - a_0 = pb_1$ for some $b_1 \in R$. Then $b_1^{\varphi^h} = b_1$ since φ^h is a $\mathbb{W}\mathbb{F}_{p^h}$-algebra map. Then $[b_1] \in \kappa \cap \mathbb{F}_{p^h}$ for the same reason. Let a_1 be the Teichmüller lift of $[b_1]$, we have $a = a_0 + a_1 p + b_2 p^2$ for some $b_2 \in R$. Now by induction, $a = \sum_{i \geq 0} a_i p^i$ where $a_i \in \mathbb{W}(\kappa \cap \mathbb{F}_{p^h})$ is Teichmüller lift of some elements in $\kappa \cap \mathbb{F}_{p^h}$. This implies $a \in R \cap \mathbb{W}\mathbb{F}_{p^h}$. ☐

Assuming $\kappa \supseteq \mathbb{F}_{p^h}$, we just showed

Proposition A.6. *The endomorphism ring of Γ_h over R/p is an associative algebra $\mathcal{O}_h = \mathbb{W}\mathbb{F}_{p^h}\langle S \rangle / (S^h = p, aS = Sa^{\varphi})$. The automorphism group of Γ_h is $\mathcal{O}_h^\times$. From this, we get* $\mathrm{End}(\Gamma_h[\Upsilon_0^m]) \simeq \mathcal{O}_h/V^m$. *In particular,* $\mathrm{End}(\Gamma_h[p^m]) \simeq \mathcal{O}_h/p^m$.

Remark A.1. $\mathcal{O}_h$ is non-commutative when $h \geq 2$. $\mathcal{O}_h \otimes \mathbb{Q}_p$ is a central division algebra over $\mathbb{Q}_p$ with invariant $1/h$.

Another application of Theorem A.3 is the Galois descent theory for formal groups. Assumptions on A as before, we will show

Theorem A.4 (Lazard)**.** *Let $\widehat{G}_0$ and $\widehat{G}'_0$ be two one dimensional formal groups of height h over R/p. For each $m > 0$, there is a finite étale extension of R/p over which $\widehat{G}_0[\Upsilon_0^m]$ and $\widehat{G}'_0[\Upsilon_0^m]$ are isomorphic.*

Corollary A.2. *There is an (ind)-étale extension of R/p over which $\widehat{G}_0$ and $\widehat{G}'_0$ are isomorphic.*

Proof. The proof we give here is essentially the same as [Goe08, 5.29]. It suffices to prove the statement for $\widehat{G}'_0 \simeq \Gamma_h$. Let $\mathbb{D}(\widehat{G}_0) = (M, F, V)$. By Example A.3, there is a $\gamma \in M$, whose reduction mod V is a generator of M/VM and such that $F\gamma = V^{h-1}(a_0\gamma + a_1V\gamma + \cdots)$, where $a_0 \in (R/p)^\times$ and $a_i \in R/p$ for $i > 0$. We now need to find a $\gamma' = t_0\gamma + t_1V\gamma + \cdots$ such that $F\gamma' = V^{h-1}\gamma$, where the t_i's are in some finite étale extension of R/p and t_0 is invertible. We will construction γ' modulo powers of V.

The analysis in Examples A.4 shows $\mathbb{D}(\widehat{G}_0[\Upsilon_0^m])$ and $\mathbb{D}(\Gamma_h[\Upsilon_0^m])$ are identical when $m < h$ (F and V are both zero mod V^m for $m < h$). The first non-trivial case is $m = h$. Let $\gamma' = t_0\gamma \mod V$, we have

$$F\gamma' = V^{h-1}\gamma' \mod V^h \implies V^{h-1}t_0^{p^h}a_0\gamma = V^{h-1}t_0\gamma$$

So it suffices for $t_0^{p^h-1} = 1/a_0$, which exists in a finite étale extension of R/p since it is integrally closed. Now assume we have find a γ with $F\gamma = V^{h-1}\gamma \mod V^{h+m-1}$. This implies $p\gamma = VF\gamma = V^h\gamma \mod V^{h+m}$. Now let $\gamma' = \gamma + t_mV^m\gamma \mod V^{m+1}$ where $t_m \in R/p$. Then $F\gamma' = V^{h-1}\gamma' \mod V^{h+m}$ implies

$$\begin{aligned} V^{h-1}\gamma + V^{h-1}a_mV^m\gamma + t_m^pV^{h+m-1}\gamma =& V^{h-1}\gamma + V^{h-1}t_mV^m\gamma \\ V^{h-1}(a_m + t_m^{p^h})V^m\gamma =& V^{h-1}t_mV^m\gamma. \end{aligned}$$

So it suffices to solve the equation $t_m^{p^h} - t_m + a_m = 0$. Again, the solution exists in some finite étale extension of R/p since it is an integrally closed domain. ☐

Note that $(R/p)^{\mathrm{sep}} \simeq R^{\mathrm{sep}}/p$ by Hensel's lemma. This theorem implies all formal groups of height h are isomorphic to the Honda formal group Γ_h over $(R/p)^{\mathrm{sep}}$. Together with Proposition A.6, we get

Corollary A.3. *The stack $B\mathrm{Aut}(\Gamma_h) = \mathrm{Spec}\,\mathbb{F}_p^{\mathrm{sep}}//\mathcal{O}_h^\times$ represents the sheaf of one dimensional formal groups of height h in the étale topology.*

Using the general theory of Galois descent, we have

Proposition A.7. *Isomorphism classes of height h formal groups over R/p are classified by the non-abelian Galois cohomology $H_c^1(\pi_1^{\acute{e}t}(R/p), \mathcal{O}_h^\times)$, with the zero object corresponding to the Honda formal group Γ_h.*

When $R/p \supseteq \mathbb{F}_{p^h}$, $H_c^1(\pi_1^{\acute{e}t}(R/p), \mathcal{O}_h^\times)$ classifies isomorphism classes of $\pi_1^{\acute{e}t}(R/p)$-representations in free $\mathcal{O}_h$-modules of rank 1. In particular, when $h = 1$, $\mathcal{O}_h^\times \simeq \mathbb{Z}_p^\times$ is abelian and $H_c^1(\pi_1^{\acute{e}t}(R/p), \mathbb{Z}_p^\times) \simeq \mathrm{Hom}(\pi_1^{\acute{e}t}(R/p), \mathbb{Z}_p^\times)$.

Proof. This a special case of faithfully flat descent for formal groups [Goe08, Lemma 1.35]. Let $\widehat{G}_0$ be a formal group of height h over R/p. Let $\pi : R/p \to (R/p)^{\mathrm{sep}}$ be the inclusion of R/p into its separable closure. To simplify notation, denote $\operatorname{Spec} R/p$ by S and $\operatorname{Spec}(R/p)^{\mathrm{sep}}$ by S^{sep}. Then $\pi : S^{\mathrm{sep}} \to S$ is faithfully flat. Now consider the simplicial diagram:

$$\begin{tikzcd}
S & S^{\mathrm{sep}} \arrow[l, "\pi"'] & S^{\mathrm{sep}} \times_S S^{\mathrm{sep}} \arrow[l, shift right, "p_0"'] \arrow[l, shift left, "p_1"] & S^{\mathrm{sep}} \times_S S^{\mathrm{sep}} \times_S S^{\mathrm{sep}} \arrow[l, shift right=1.5] \arrow[l] \arrow[l, shift left=1.5]
\end{tikzcd} \tag{A.11}$$

Faithfully flat descent for formal groups says given a formal group $\widehat{G}'_0$ over S^{sep} and an isomorphism $f : p_0^*\widehat{G}'_0 \xrightarrow{\sim} p_1^*\widehat{G}'_0$ that satisfies the cocycle condition, there is a unique formal group $\widehat{G}_0$ over S such that there is an isomorphism of formal groups $\eta : \pi^*\widehat{G}_0 \xrightarrow{\sim} \widehat{G}'_0$ over S^{sep} and the diagram commutes:

$$\begin{tikzcd}
p_0^*\pi^*\widehat{G}_0 \arrow[r, equal] \arrow[d, "p_0^*\eta"'] & p_1^*\pi^*\widehat{G}_0 \arrow[d, "p_1^*\eta"] \\
p_0^*\widehat{G}'_0 \arrow[r, "f"'] & p_1^*\widehat{G}'_0
\end{tikzcd} \tag{A.12}$$

Here "=" is the canonical isomorphism arising from the fact $p_0 \circ \pi = p_1 \circ \pi$. Now it suffices to show the descent data $(\widehat{G}'_0, f)$ is equivalent to elements in $H^1_c(\pi_1^{\acute{e}t}(R/p), \mathcal{O}_h^\times)$. First, notice $\widehat{G}'_0 \simeq \Gamma_h$ by Theorem A.4. So the descent data is $f : p_0^*\Gamma_h \xrightarrow{\sim} p_1^*\Gamma_h$. As $S^{\mathrm{sep}} \to S$ is a $\pi_1^{\acute{e}t}(S)$-torsor, (A.11) is equivalent to the simplicial diagram:

$$\begin{tikzcd}
S & S^{\mathrm{sep}} \arrow[l, "\pi"'] & \pi_1^{\acute{e}t}(S) \times S^{\mathrm{sep}} \arrow[l, shift right, "\text{proj}"'] \arrow[l, shift left, "\text{action}"] & \pi_1^{\acute{e}t}(S) \times \pi_1^{\acute{e}t}(S) \times S^{\mathrm{sep}} \arrow[l, shift right=1.5] \arrow[l] \arrow[l, shift left=1.5]
\end{tikzcd} \tag{A.13}$$

For each $\sigma \in \pi_1^{\acute{e}t}(S)$, (A.12) becomes the following when restricting to $\{\sigma\} \times S^{\mathrm{sep}}$:

$$\begin{tikzcd}
\pi^*\widehat{G}_0 \arrow[r, "\sigma\otimes 1"] \arrow[d, "\eta"'] & \sigma^*\pi^*\widehat{G}_0 \arrow[r, equal] \arrow[d, "\sigma^*\eta"] & \pi^*\widehat{G}_0 \arrow[d, "\sigma^*\eta"] \\
\Gamma_h \arrow[r, "f(\sigma)"'] & \sigma^*\Gamma_h \arrow[r, equal] & \Gamma_h
\end{tikzcd} \tag{A.14}$$

So we get a map $f : \pi_1^{\acute{e}t}(S) \to \operatorname{Aut}(\Gamma_h) \simeq \mathcal{O}_h^\times$. The cocycle condition says when composing the actions of σ_1, $\sigma_2 \in \pi_1^{\acute{e}t}(S)$, the following diagram commutes:

$$\begin{tikzcd}
\pi^*\widehat{G}_0 \arrow[r, "\sigma_2\otimes 1"] \arrow[d, "\eta"'] & \pi^*\widehat{G}_0 \arrow[r, "\sigma_1\otimes 1"] \arrow[d, "\sigma_2^*\eta"] & \pi^*\widehat{G}_0 \arrow[d, "\sigma_2^*\sigma_1^*\eta"] \\
\Gamma_h \arrow[r, "f(\sigma_2)"] \arrow[rr, bend right, "f(\sigma_1\sigma_2)"] & \Gamma_h \arrow[r, "\sigma_2^* f(\sigma_1)"] & \Gamma_h
\end{tikzcd}$$

That is $f(\sigma_1\sigma_2) = \sigma_2^* f(\sigma_1) \circ f(\sigma_2)$. On the other hand, using a different isomorphism $\eta' : \pi^*\widehat{G}_0 \xrightarrow{\sim} \Gamma_h$, we get another map $f' : \pi_1^{ét}(S) \to \mathcal{O}_h^\times$. By $(\sigma^*\eta')^{-1} \circ f'(\sigma) \circ \eta' = \sigma \otimes 1 = (\sigma^*\eta)^{-1} \circ f'(\sigma) \circ \eta$, f and f' are Galois-twisted conjugates: $f'(\sigma) = \sigma^*(\eta \circ \eta'^{-1})^{-1} \circ f'(\sigma) \circ (\eta \circ \eta'^{-1})$. Here $\eta \circ \eta'^{-1} \in \mathrm{Aut}\Gamma_h \simeq \mathcal{O}_h^\times$. This shows $f \sim f'$ as descent data if there is an $\alpha \in \mathcal{O}_h^\times$ such that $f(\sigma) = \sigma^*(\alpha)^{-1} f'(\sigma)\alpha$. We thus conclude the equivalence classes of descent data for a height h formal group over R/p correspond to elements in $H_c^1(\pi_1^{ét}(S), \mathcal{O}_h^\times)$.

If $R/p \supseteq \mathbb{F}_{p^h}$, then the action of $\pi_1^{ét}(S)$ on $\mathcal{O}_h^\times$ is trivial. The cocycle condition now becomes $f(\sigma_1\sigma_2) = f(\sigma_1) \circ f(\sigma_2)$ and $f \sim f'$ if $f(\sigma) = \alpha^{-1} f'(\sigma)\alpha$. This implies the Galois cohomology $H_c^1(\pi_1^{ét}(S), \mathcal{O}_h^\times)$ is isomorphic to the conjugacy classes in $\mathrm{Hom}(\pi_1^{ét}(S), \mathcal{O}_h^\times)$ as pointed sets. The conjugacy classes of continuous group homomorphisms further correspond to continuous $\pi_1^{ét}(S)$-representations in rank 1 free $\mathcal{O}_h$-modules.

In the $h = 1$ case, $\mathcal{O}_h^\times \simeq \mathbb{Z}_p^\times$ is abelian and the assumption $R/p \supseteq \mathbb{F}_p$ is always true. In this case, we have isomorphism classes of height 1 formal groups over R/p is given by $\mathrm{Hom}(\pi_1^{ét}(S), \mathbb{Z}_p^\times)$. ☐

Proposition A.8. *When $h = 1$, $\mathcal{O}_h^\times \simeq \mathbb{Z}_p^\times$ is abelian and $H_c^1(\pi_1^{ét}(R/p); \mathbb{Z}_p^\times)$ is indeed a group (actually abelian). The addition induces a monoidal structure in the category of one dimensional formal groups of height 1 over R/p. In addition, the functor $\widehat{G}_0 \mapsto \mathrm{Lie}(\widehat{G}_0)$ from height 1 formal groups to line bundles (invertible modules) is mononoidal.*

Recall that the Picard group can also be computed by a Galois cohomology $H_c^1(\pi_1^{ét}(R/p), (R/p)^\times)$. As a result, taking Lie algebra of a height 1 formal group is mononoidal. This corresponds to a homomorphism of Galois cohomologies induced by $\mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p)^\times \hookrightarrow (R/p)^\times$.

Proposition A.9. *Let $\widehat{G}_1$ and $\widehat{G}_2$ be two 1-dimensional formal groups of height 1 over R/p and $\widehat{G}_1 \otimes \widehat{G}_2$ be the monoidal product described above. If $\mathbb{D}(\widehat{G}_i) = (M_i, F_i, V_i)$, then*

$$\mathbb{D}(\widehat{G}_1 \otimes \widehat{G}_2) \simeq (M_1 \otimes_R M_2, F_1 \otimes F_2, V_1 \otimes V_2),$$

where $F_1 \otimes F_2$ and $V_1 \otimes V_2$ are defined by

$$F_1 \otimes F_2 : \varphi^*(M_1 \otimes_R M_2) \simeq \varphi^* M_1 \otimes_{\varphi^* R} \varphi^* M_2 \mathrel{\substack{\xrightarrow{V_1 \otimes V_2} \\ \xleftarrow[F_1 \otimes F_2]{}}} M_1 \otimes_R M_2 : V_1 \otimes V_2.$$

Example A.4 (Galois cohomology and the Hasse invariant)**.** We will give an explicit correspondence between Dieudonné modules associated to height h formal groups over $\mathbb{F}_{p^h}$ and conjugate classes of elements in $\mathrm{Hom}(\widehat{\mathbb{Z}}, \mathcal{O}_h^\times) \simeq \mathcal{O}_h^\times$. To treat all heights at once, define $\mathbb{D}_h := \mathbb{D}(\widehat{G}_a/\mathbb{F}_{p^h}) = \mathbb{F}_{p^h}\langle V\rangle_V^\wedge/\{Va^p = aV\}$ and $\mathbb{D}^{\mathrm{sep}} := \mathbb{D}(\widehat{G}_a/\mathbb{F}_p^{\mathrm{sep}})$. $\mathbb{D}_h$ admits a relative Frobenius, if $\alpha = \sum a_i V^i$, then $\alpha^\varphi = \sum a_i^p V^i$.

By Example A.3, for a height h formal group $\widehat{G}_0$, its Dieudonné module $\mathbb{D}(\widehat{G}_0) = (M, F, V)$ consists of a rank h locally free module over $\mathbb{W}\mathbb{F}_{p^h}$ and the F and V operators. Since the Picard group of $\mathbb{W}\mathbb{F}_{p^h}$ is trivial, $M \simeq (\mathbb{W}\mathbb{F}_{p^h})^{\oplus h}$ as a $\mathbb{W}\mathbb{F}_{p^h}$-module. Let $\gamma \in M$ be an element whose reduction mod V is a generator of $M/VM \simeq \mathbb{F}_{p^h}$. We can now abbreviate (A.10) by $F\gamma = V^{h-1}\alpha\gamma$ for some $\alpha \in \mathbb{D}_h^\times$.

Proposition A.10. *The conjugacy classes of α is independent of the choice of γ. Furthermore if $\widehat{G}_0'$ is another height h formal group over $\mathbb{F}_{p^h}$ with $F\gamma' = \alpha'\gamma'$, then $\widehat{G}_0 \simeq \widehat{G}_0'$ over $\mathbb{F}_p$ iff $\alpha = \beta^{-1}\alpha'\beta$ for some $\beta \in \mathbb{D}_h^\times$.*

Proof. Let $t\gamma$ be another generator of M, where $t \in \mathbb{D}_h^\times$. Then notice $t^{\varphi^h} = t$ and we have

$$F(t\gamma) = t^\varphi F\gamma = t^\varphi V^{h-1}\alpha\gamma = V^{h-1}t\alpha\gamma = V^{h-1}t\alpha t^{-1}(t\gamma).$$

Suppose $\widehat{G} \simeq \widehat{G}'$ over $\mathbb{F}_{p^h}$ then there is a $t \in \mathbb{D}_h^\times$ such that $\gamma' = t\gamma$. Then we have

$$F\gamma' = V^{h-1}\alpha'\gamma' \implies t^\varphi V^{h-1}\alpha\gamma = V^{h-1}\alpha' t\gamma \implies t\alpha = t^{\varphi^h}\alpha = \alpha' t. \tag{A.15}$$

So α and α' are conjugates in $\mathbb{D}_h^\times$. ☐

As a result, the conjugacy class of $\alpha \in \mathbb{D}_h^\times$ uniquely determines the isomorphism class of a height h formal group over $\mathbb{F}_{p^h}$. When $h = 1$, its reduction mod p is the classic Hasse invariant of the formal group associated to a p-ordinary elliptic curve. We will now call the conjugacy class of α the **Hasse invariant** of $\widehat{G}_0$.

Proposition A.11. *The Hasse invariant of $\widehat{G}_0$ is equal to its descent data, i.e. the conjugacy class of $f(\sigma^h) \in \mathcal{O}_h^\times$, where $\sigma^h \in \mathrm{Gal}(\mathbb{F}_p^{\mathrm{sep}}/\mathbb{F}_{p^h})$ is the generator.*

Proof. Let γ' be a generator in $\mathbb{D}(\widehat{G}_0)$ and $F\gamma' = V^{h-1}\alpha\gamma'$. By Example A.3, $F\gamma = V^{h-1}\gamma$ in $\mathbb{D}(\Gamma_h)$. Let $\mathbb{D}(\eta) : \gamma' \mapsto t\gamma$, where $t \in (\mathbb{D}^{\mathrm{sep}})^\times$. Then $\mathbb{D}\left(\left(\sigma^h\right)^* \eta\right)$ maps γ' to $t^{\varphi^h}\gamma$. Now we have

$$F(\eta(\gamma')) = \alpha V^{h-1}\eta(\gamma') \implies t^{\varphi^h} = \alpha t.$$

On the other hand, by (A.14) we have

$$\left(\left(\sigma^h\right)^* \eta\right)^{-1} \circ f(\sigma^h) \circ \eta = \sigma^h \otimes 1 \implies \left(t^{\varphi^h}\right)^{-1} f(\sigma^h) t = 1$$

Comparing the two equations, we conclude $\alpha = f(\sigma^h)$. ☐

Example A.5 (Height h formal groups over $\mathbb{F}_{p^{h+n}}$)**.** The Hasse invariant only works for height h formal groups over $\mathbb{F}_{p^h}$. Over $\mathbb{F}_{p^{2h}}$, we have

Proposition A.12. *Let $\widehat{G}_0$ and $\widehat{G}'_0$ be two height h formal groups over $\mathbb{F}_{p^h}$ with Hasse invariants α and α'. Then $\widehat{G}_0 \simeq \widehat{G}'_0$ over $\mathbb{F}_{p^{2h}}$ iff $\alpha = \pm\alpha'$.*

Proof. For the "only if" part, by (A.15), we have $t^{\varphi^h}\alpha = \alpha' t$ for some $t \in (\mathbb{W}\kappa_{2h})^\times$. Then α, α' and t satisfy $\alpha = \alpha^{\varphi^h}$, $\alpha' = \alpha'^{\varphi^h}$ and $t^{(\varphi^{2h})} = t$. We have

$$t^{\varphi^h} \cdot t^{-1} = \alpha' \cdot \alpha^{-1} \implies t \cdot \left(t^{\varphi^h}\right)^{-1} = (\alpha' \cdot \alpha^{-1})^{\varphi^h} = \alpha' \cdot \alpha^{-1} \implies (\alpha' \cdot \alpha^{-1})^2 = 1.$$

The "if" part follows from the existence of solutions to $t^{\varphi^h} = -t$ in $\mathbb{F}_{p^{2h}}$. ☐

In general, for height h formal groups $\widehat{G}_0$ over k_{h+n}, $\mathbb{D}(\widehat{G}_0)$ and the Galois descent data are related by

Corollary A.4. *If $F\gamma = V^{h-1}\alpha\gamma$ in $\mathbb{D}(\widehat{G}_0)$, then there is a $t \in (\mathbb{D}^{\mathrm{sep}})^\times$ such that $\alpha = t^{\varphi^h} \cdot t^{-1}$ and the descent data for $\widehat{G}_0$ is given by $f(\sigma^{h+n}) = t^{\varphi^{h+n}} \cdot t^{-1}$. It follows then mod V, α is a $(p^{h+n}-1)/(p^h-1)$ root of $f(\sigma^{h+n})$*

Proof. This is essentially the same as Proposition A.11. The only difference is the equation for Galois descent data. Since $\widehat{G}_0$ is defined over $\mathbb{F}_{p^{h+n}}$, (A.14) becomes

$$\left(\left(\sigma^{h+n}\right)^* \eta\right)^{-1} \circ f(\sigma^{h+n}) \circ \eta = \sigma^h \otimes 1 \implies \left(t^{\varphi^{h+n}}\right)^{-1} f(\sigma^{h+n}) t = 1.$$

Since φ is the p-th power map mod V, it now follows α is a $(p^{h+n}-1)/(p^h-1)$ root of $f(\sigma^{h+n})$ mod V. ☐

Appendix B

Cyclotomic representations of cyclic groups

In this appendix, we study the integral and p-adic cyclotomic representations of the cyclic group C_n.

B.1 Integral cyclotomic representations

Let $\Phi_n(t)$ be the n-th cyclotomic polynomial, i.e. the minimal polynomial of a primitive n-th root of unity ζ_n over $\mathbb{Q}$. The integral cyclotomic representation of C_n has underlying abelian group $\mathbb{Z}[\zeta_n] \simeq \mathbb{Z}[t]/\Phi(t)$ and $g \in C_n$ acts by multiplication by a primitive n-th root of unity (or $t \in \mathbb{Z}[t]/\Phi(t)$). The rank of $\mathbb{Z}[\zeta_n]$ as a free abelian group is equal to $\deg \Phi_n(t) = \phi(n)$.

Examples B.1. We consider the following examples:

1. When $n = 5$, $\mathbb{Z}[\zeta_5]$ is a free $\mathbb{Z}$-module of rank 4 as $\phi(5) = 4$. $\{1, \zeta_5, \zeta_5^2, \zeta_5^3\}$ form a basis of $\mathbb{Z}[\zeta_5]$. The minimal polynomial of ζ_5 is $\Phi_5(t) = t^4 + t^3 + t^2 + t + 1$. Let $g \in C_5$ be a generator that acts on $\mathbb{Z}[\zeta_5]$ by multiplication by ζ_n. Then the matrix representation of $g \in C_5$ with respect the basis $\{1, \zeta_5, \zeta_5^2, \zeta_5^3\}$ of $\mathbb{Z}[\zeta_5]$ is
$$g = \begin{pmatrix} & & & -1 \\ 1 & & & -1 \\ & 1 & & -1 \\ & & 1 & -1 \end{pmatrix}.$$

2. When $n = 6$, $\mathbb{Z}[\zeta_6]$ is a free $\mathbb{Z}$-module of rank 2 as $\phi(6) = 2$. $\{1, \zeta_6\}$ form a basis of $\mathbb{Z}[\zeta_6]$. The minimal polynomial of ζ_6 is $\Phi_6(t) = t^2 - t + 1$. Let $g \in C_6$ be a generator that acts on $\mathbb{Z}[\zeta_6]$ by multiplication by ζ_n. Then the matrix representation of $g \in C_6$ with respect the basis $\{1, \zeta_6\}$ of $\mathbb{Z}[\zeta_6]$ is
$$g = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Lemma B.1. *The cyclotomic representation of C_n is equivalent to the external tensor product of the cyclotomic representations of $C_{p^{v_p(n)}}$, i.e. there is an equivalence of C_n-representations:*

$$\mathbb{Z}[\zeta_n] \simeq \bigotimes_{p|n} \mathbb{Z}\left[\zeta_{p^{v_p(n)}}\right]$$

Lemma B.2. *There is a short exact sequence of C_{p^v}-representations:*

$$0 \longrightarrow \mathbb{Z}[\zeta_{p^v}] \longrightarrow \mathbb{Z}[C_{p^v}] \longrightarrow \mathbb{Z}[C_{p^{v-1}}] \longrightarrow 0 \tag{B.1}$$

where C_{p^v} acts on $\mathbb{Z}[C_{p^{v-1}}]$ via the quotient map $C_{p^v} \twoheadrightarrow C_{p^{v-1}}$.

Proof. This follows from the observations that $\Phi_{p^v}(t) = \frac{t^{p^v}-1}{t^{p^{v-1}}-1}$ and $\mathbb{Z}[C_n] \simeq \mathbb{Z}[t]/(t^n - 1)$. ☐

B.2 p-adic cyclotomic representations

From now on, let $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}_p^\times$ be a p-adic Dirichlet character of conductor N and $\mathbb{Z}_p[\chi]$ be the $\mathbb{Z}_p$-subalgebra of $\mathbb{C}_p$ generated by the image of χ. Again, $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\zeta_n]$ for some n. Write $n = p^v \cdot n'$ with $p \nmid n'$, we have $\mathbb{Z}_p[\zeta_n] \simeq \mathbb{Z}_p[\zeta_{p^v}] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_{n'}]$. Now it suffices to analyze C_n-actions on $\mathbb{Z}_p[\zeta_n]$ in the cases when $n = p^v$ or $p \nmid n$. Let's first recall some basic facts of cyclotomic extensions of $\mathbb{Q}$:

Lemma B.3. [Was97, Theorem 2.5, 2.6] *We recall the following basic facts of the cyclotomic extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$.*

1. *$\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is a Galois extension of degree $\phi(n)$ and $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \simeq (\mathbb{Z}/n)^\times$, with $a \in (\mathbb{Z}/n)^\times$ acts by $\zeta_n \mapsto \zeta_n^a$.*
2. *The ring of integers of $\mathbb{Q}(\zeta_n)$ is $\mathbb{Z}[\zeta_n]$. Consequently, for any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, $\sigma(\mathbb{Z}[\zeta_n]) = \mathbb{Z}[\zeta_n]$.*

As a result of this lemma, we can extract the action of $(\mathbb{Z}/N)^\times$ on $\mathbb{Z}[\zeta_n]$ from that on $\mathbb{Q}(\zeta_n)$.

Proposition B.1. *For any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, the $(\mathbb{Z}/N)^\times$-representation induced by the Dirichlet character $\sigma \circ \chi$ is isomorphic to that induced by χ.*

Proof. Let $\mathbb{Z}[\chi] = \mathbb{Z}[\zeta_n]$, where ζ_n is a primitive n-th root of unity. For any $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$, $\sigma(\zeta_n)$ is also a primitive n-th root of unity. As a result, the minimal polynomials of ζ_n and $\sigma(\zeta_n)$ are both $\Phi_n(t)$. It follows that the matrix representations of χ and $\sigma \circ \chi$ are differed by a change of basis induced by σ. Thus, the integral representations induced by χ and $\sigma \circ \chi$ are isomorphic. ☐

Proposition B.2. *Write $n = p^v \cdot n'$, where $p \nmid n'$ and let m be the multiplicative order of p mod n', i.e.*

$$m = \min\{k > 0 \mid p^k \equiv 1 \mod n'\}.$$

Then $\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p$ is a Galois extension of local fields of residue index m and ramification index $\phi(p^v)$. Moreover,

$$\mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p) \simeq \mathrm{Gal}(\mathbb{Q}_p(\zeta_{n'})/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^v})/\mathbb{Q}_p) \simeq (\mathbb{Z}/m) \times (\mathbb{Z}/p^v)^\times,$$

where a generator $\varphi \in \mathbb{Z}/m$ acts on $\mathbb{Q}_p(\zeta_{n'})$ by the lift of the Frobenius (p-th power map) from $\mathbb{Z}_p[\zeta_{n'}]/(p) \simeq \mathbb{F}_{p^m}$ to $\mathbb{Q}_p(\zeta_{n'}) \simeq \mathbb{W}(\mathbb{F}_{p^m})$. In particular, $\varphi(\zeta_{n'}) = \zeta_{n'}^p$.

B.3 p-completions of integral cyclotomic representations

We conclude this appendix with a discussion on how $\mathbb{Z}[\chi]$ decomposes upon p-completion. The simplest case is

Corollary B.1. $\mathbb{Z}_p[\zeta_{p^v}] \simeq \mathbb{Z}[\zeta_{p^v}] \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq (\mathbb{Z}[\zeta_{p^v}])_p^\wedge$.

Proof. By Proposition B.2, $\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p$ is a totally ramified extension of local fields of rank $\phi(p^v)$. This means $\mathbb{Z}_p[\zeta_{p^v}]$ is a free $\mathbb{Z}_p$-module of rank $\phi(p^v)$, which is equal to the rank of $\mathbb{Z}[\zeta_{p^v}]$ as a free $\mathbb{Z}$-module. This implies $\mathbb{Z}[\zeta_{p^v}]$ does not split upon p-completion. □

Comparing Lemma B.3 and Proposition B.2, we have shown:

Proposition B.3. *Fix an embedding $\iota : \mathbb{Q}[\zeta_n] \hookrightarrow \mathbb{C}_p$. For any $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p)$, $\sigma \circ \iota(\mathbb{Q}(\zeta_n)) = \iota(\mathbb{Q}(\zeta_n))$. In addition, the restriction map on the Galois group induced by ι*

$$\iota^* : \mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p) \longrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \tag{B.2}$$

is injective. More precisely, rewrite $\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_{p^v}) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_{n'})$ and $\iota = \iota_p \otimes \iota_{n'}$, where

$$\iota_p : \mathbb{Q}(\zeta_{p^v}) \hookrightarrow \mathbb{C}_p, \quad \iota_{n'} : \mathbb{Q}(\zeta_{n'}) \hookrightarrow \mathbb{C}_p.$$

Then we have

- $\iota_p^* : \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^v})/\mathbb{Q}_p) \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}(\zeta_{p^v})/\mathbb{Q})$ *is an isomorphism.*

- $\iota_{n'}^{*}$: $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{n'})/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_{n'})/\mathbb{Q})$ *is the inclusion of the subgroup of* $(\mathbb{Z}/n')^{\times}$ *generated by the element* $p \in (\mathbb{Z}/n')^{\times}$.

Proposition B.4. *Pick a representative* $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ *for each coset in*

$$\mathrm{Coker}\,\iota^* = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})/\mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p).$$

$\mathbb{Z}[\zeta_n] \otimes \mathbb{Z}_p$ *decomposes as a* $\mathbb{Z}_p$*-algebra by*

$$\mathbb{Z}[\zeta_n] \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow[\sim]{\prod(\iota\circ\sigma)\otimes 1} \prod_{[\sigma]\in\mathrm{Coker}\,\iota^*} \mathbb{Z}_p[\zeta_n] \simeq \bigoplus_{[\sigma]\in\mathrm{Coker}\,\iota^*} \mathbb{Z}_p[\zeta_n].$$

Proof. The minimal polynomial of ζ_n over $\mathbb{Z}$ is

$$\Phi_n(t) = \prod_{\sigma\in\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})} (t - \sigma(\zeta_n)).$$

We have an isomorphism $\mathbb{Z}[\zeta_n] \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathbb{Z}_p[t]/(\Phi_n(t))$. Over $\mathbb{Z}_p$, $\Phi_n(t)$ factorizes as

$$\Phi_n(t) = \prod_{[\sigma]\in\mathrm{Coker}\,\iota^*} \Phi_{n,\sigma}(t), \quad \text{where } \Phi_{n,\sigma}(t) \coloneqq \prod_{\tau\in\mathrm{Gal}(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p)} (t - \tau \circ \iota \circ \sigma(\zeta_n)).$$

For each $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, $\Phi_{n,\sigma}(t)$ is the minimal polynomial of $\iota \circ \sigma(\zeta_n)$ over $\mathbb{Z}_p$. As $\Phi_{n,\sigma}(t)$ are coprime to each other for different cosets $[\sigma] \in \mathrm{Coker}\,\iota^*$ and $\mathbb{Z}_p[t]/(\Phi_{n,\sigma}(t)) \simeq \mathbb{Z}_p[\zeta_n]$ for all σ, the claim now follows from the Chinese Reminder Theorem. ☐

Corollary B.2. *Let* $\chi : (\mathbb{Z}/N)^{\times} \to \mathbb{C}^{\times}$ *be a Dirichlet character with* $\mathbb{Z}[\chi] = \mathbb{Z}[\zeta_n]$. $\mathbb{Z}[\chi] \otimes_{\mathbb{Z}} \mathbb{Z}_p$ *decomposes as a p-adic* $(\mathbb{Z}/N)^{\times}$*-representation by*

$$\mathbb{Z}[\chi] \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \bigoplus_{[\sigma]\in\mathrm{Coker}\,\iota^*} \mathbb{Z}_p[\iota \circ \sigma \circ \chi],$$

where $\iota \circ \sigma \circ \chi$ *is the p-adic Dirichlet character defined by*

$$(\mathbb{Z}/N)^{\times} \xrightarrow{\chi} (\mathbb{Z}[\chi])^{\times} \xrightarrow{\sigma} (\mathbb{Z}[\chi])^{\times} \xhookrightarrow{\iota} \mathbb{C}_p^{\times}.$$

Proof. This is done by forcing the isomorphism in Proposition B.4 to be $(\mathbb{Z}/N)^{\times}$-equivariant. ☐

Corollary B.3. *When* $\chi : (\mathbb{Z}/N)^{\times} \to \mathbb{C}^{\times}$ *is a primitive Dirichlet character of conductor* $N = p^v$ *and* $p > 2$, *there is an equivalence of* $(\mathbb{Z}/p^v)^\times$*-representations:*

$$\mathbb{Z}[\chi]_p^\wedge \simeq \bigoplus_{\substack{0\le a\le p-2\\ \ker\omega^a=\ker\chi|_{(\mathbb{Z}/p)^\times}}} \mathbb{Z}_p[\chi_a],$$

where $\chi_a = \omega^a \cdot (\iota\circ\chi|_{\mathbb{Z}/p^{v-1}})$ *and* $\omega : (\mathbb{Z}/p)^\times \to \mathbb{Z}_p^\times$ *is the Teichmüller character.*

Proof. By Corollary B.2, we need show the following two sets of characters are the same:

$$\{\iota\circ\sigma\circ\chi \mid [\sigma]\in \operatorname{Coker}\iota^*\} = \{\omega^a\cdot(\iota\circ\chi|_{\mathbb{Z}/p^{v-1}}) \mid 0\le a\le p-2, \ker\omega^a = \ker\chi|_{(\mathbb{Z}/p)^\times}\}. \tag{B.3}$$

We first prove the $v = 1$ case. A p-adic character of conductor p is necessarily of the form ω^a for some a, since $\mathbb{Z}_p$ contains all $(p-1)$-st roots of unity. As ι and σ are injections, $\ker\iota\circ\sigma\circ\chi = \ker\chi$. Now it suffices to check the two sets have the same size. Since $\mathbb{Z}_p[\iota\circ\chi] = \mathbb{Z}_p$, we have $|\operatorname{Coker}\iota^*| = |\operatorname{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})| = \operatorname{rank}_{\mathbb{Z}}(\mathbb{Z}[\chi])$. χ factorizes as $(\mathbb{Z}/p)^\times \twoheadrightarrow C_{n'} \hookrightarrow (\mathbb{Z}[\zeta_{n'}])^\times$ for some $n'|(p-1)$. Then $\mathbb{Z}[\chi]$ has rank $\phi(n')$. Let $g\in(\mathbb{Z}/p)^\times$ be a generator, then $\ker\chi$ is the subgroup of $(\mathbb{Z}/p)^\times$ generated by $g^{n'}$. We have

$$\{a \mid 0\le a\le p-2, \ker\omega^a = \ker\chi = \langle g^{n'}\rangle \subseteq (\mathbb{Z}/p)^\times\} = \{a \mid 0\le a\le p-2,\ \text{the order of } a\in(\mathbb{Z}/p)^\times \text{ is } (p-1)/n'\}.$$

The size of this set is $\phi(n')$, which is equal to $|\operatorname{Coker}\iota^*|$, from which we conclude the two sets of characters in (B.3) are the same when $v = 1$.

When $v > 1$, write $\mathbb{Z}[\chi] = \mathbb{Z}[\chi|_{(\mathbb{Z}/p)^\times}] \otimes \mathbb{Z}[\chi|_{\mathbb{Z}/p^{v-1}}]$. χ being primitive implies $\chi|_{\mathbb{Z}/p^{v-1}}$ is injective and $\mathbb{Z}[\chi|_{\mathbb{Z}/p^{v-1}}] = \mathbb{Z}[\zeta_{p^{v-1}}]$. By Corollary B.1, $\mathbb{Z}[\chi|_{\mathbb{Z}/p^{v-1}}]_p^\wedge = \mathbb{Z}_p[\iota\circ\chi|_{\mathbb{Z}/p^{v-1}}]$. On the other hand , write $\iota = \iota_{n'}\cdot\iota_p$ as in Proposition B.3, where $\iota_p : \mathbb{Q}(\zeta_{p^{v-1}}) \hookrightarrow \mathbb{C}_p$ is a field extension. Proposition B.3 says ι_p^* is an isomorphism, which implies $\operatorname{Coker}\iota^* = \operatorname{Coker}\iota_{n'}^*$. The analysis above shows:

$$\begin{aligned}\mathbb{Z}[\chi]_p^\wedge &\simeq \mathbb{Z}[\chi|_{(\mathbb{Z}/p)^\times}]_p^\wedge \bigotimes_{\mathbb{Z}_p} \mathbb{Z}_p[\iota_p\circ\chi|_{\mathbb{Z}/p^{v-1}}]\\ \bigoplus_{[\sigma]\in\operatorname{Coker}\iota^*} \mathbb{Z}_p[\iota\circ\sigma\circ\chi] &\simeq \left(\bigoplus_{[\sigma]\in\operatorname{Coker}\iota_{n'}^*} \mathbb{Z}_p[\iota_{n'}\circ\sigma\circ\chi|_{(\mathbb{Z}/p)^\times}]\right) \bigotimes_{\mathbb{Z}_p} \mathbb{Z}_p[\iota_p\circ\chi|_{\mathbb{Z}/p^{v-1}}]\end{aligned}$$

Now we have reduced this case to the $v = 1$ situation for the character $\chi|_{(\mathbb{Z}/p)^\times}$. □

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