PROJECTIONS, SLICINGS AND FOURIER TRANSFORMS IN THE HEISENBERG GROUP

BY

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DISSERTATION

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ABSTRACT

This thesis starts by giving an expository introduction to the study of projection and slicing problems in the Heisenberg group from the point of view of Hausdorff dimension distortion. It then gives a short summary of research done in this area up to date before introducing and explaining my own contributions to it. Finally, the group Fourier transform in the Heisenberg group is introduced and a novel connection with Hausdorff dimension is discussed.

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CHAPTER 1

INTRODUCTION

This thesis is mainly based on the following three papers:

- Intersection of projections and slicing theorems for the isotropic Grassmannian and the Heisenberg group, [95]
- Dimension distortion by right coset projections in the Heisenberg group [61]
- A Fourier coefficient approach to Hausdorff dimension in the Heisenberg group, [94]

All three papers focus on geometry and analysis in the Heisenberg group, which is considered to be the simplest example of a subRiemannian space.

The first paper studies the Hausdorff dimension of the intersection of images of sets by homogeneous projection in the Heisenberg group. It is shown that if two sets are large enough, the intersection of their image under certain homogeneous projections will have positive measure for a large collection of such projections. This paper also studies the Hausdorff dimension of the intersection of sets with fibers of the aforementioned homogeneous projections. It is shown that if a set has dimension larger than the co-dimension of the fiber, then the co-dimension of the intersection is the sum of the co-dimensions of the set and the fiber.

The second paper, which was done in collaboration with Terrance L.J. Harris and Chi N.Y. Huynh, studies Hausdorff dimension distortion by a related family of homogeneous projections. The Grushin plane, another important subRiemannian space, arises as a quotient of the Heisenberg group. In fact, the Heisenberg group contains a one parameter family of Grushin planes and with it, projection mappings arise naturally. Bounds for the almost sure dimension distortion by these projections are proven. Additionally, these results allow us to also improve the best known dimension distortion bounds for the well-studied problem of the standard family of vertical projections.

The third paper establishes a connection between the non-commutative Fourier theory of the Heisenberg group and Hausdorff dimension of sets. It is shown that integrability of the Fourier transform of a measure implies its absolute continuity. It is also shown that the

Hausdorff dimension of a set can be computed via energy integrals over an appropriately defined frequency space.

In addition to these papers, this thesis will also cover an adaptation of the potential theoretic approach to box and packing dimensions developed Euclidian space, to the Heisenberg group.

My hope is not only to convey the ideas on these papers, but to also provide a clear exposition of the ideas and techniques used in the study of projection mappings in the context of the geometric measure theory of the Heisenberg group. As well as to have a self-contained exposition of its Fourier theory. The goal is to make this exposition accessible to early graduate students with interest in this area of research. The Heisenberg group has been extensively studied for several decades and in many different context. Thus, the amount of related literature is vast and, although much of it lies outside the scope of my work, I will make references to a some of it while, at least, hinting at the research directions treated in each respective source.

The thesis is structured as follows, Chapters 2, 3 and 4 are expository in nature. Chapter 2 contains and short introduction to Hausdorff measure and dimensions as well as a quick mention of box-counting and packing dimensions. It also contains a short exposition in the connection of the Fourier transform and Hausdorff dimension. Chapter 3 is somewhat historical in nature, introducing and explaining developments in the area of fractal projections in Euclidean space. Chapter 4 is a gentle introduction to the Heisenberg group and how it arises in quantum mechanics.

In Chapter 5 starts off with an exposition of the state of the research in fractal projections in the Heisenberg group. Sections 5.8 and 5.3 include my work in this area. Chapter 6 is also novel, it adapts the potential theoretic approach to box-counting and packing dimensions from Euclidean space to the Heisenberg group. Finally, Chapter 7 is based on my work establishing a connection between the non-commutative Fourier theory of the Heisenberg group and Hausdorff dimension.

CHAPTER 2

HAUSDORFF MEASURE AND DIMENSION

One of the central notions in geometric measure theory is that of Hausdorff measure and dimension. These are extensively treated in the literature and many expository books and surveys give in-depth detailed introduction to these notions. Some of these are [46], [77] and [35], among others. In the name of simplicity, I will state many of the standard theorems without proof. For all of them, proofs can be found in the aforementioned references.

In the technical language of measure theory, the Hausdorff measure is really an outer measure defined on the power set of any metric space, (X, d), as follows,

Definition 2.1 (Hausdorff measure)**.** *For $A \subset X$ and $\delta > 0$, the (σ, δ)-Hausdorff pre-measure is*

$$\mathcal{H}^\sigma_\delta(A) := \inf\{\sum_{j=1}^{\infty}(diamE_j)^\sigma : \ A \subset \cup_j E_j \ and \ , diamE_j < \delta\}.$$

Where $diamE = \sup\{d(x, y) : x, y \in E\}$. The σ-Hausdorff measure is then defined as,

$$\mathcal{H}^\sigma(A) = \lim_{\delta \to 0} \mathcal{H}^\sigma_\delta(A).$$

For any set $A \in X$, there is a real number $\sigma' \geq 0$ such that

$$\mathcal{H}^\sigma(A) = \begin{cases} \infty & \text{if } \sigma < \sigma' \\ 0 & \text{if } \sigma > \sigma'. \end{cases} \tag{2.1}$$

This number is taken to be the Hausdorff dimension of the set A.

Definition 2.2 (Hausdorff dimension)**.** *The Hausdorff dimension of a set $A \subset X$ is defined as,*

$$\dim A := \sup\{\sigma : \mathcal{H}^\sigma(A) = \infty\} = \inf\{\sigma : \mathcal{H}^\sigma(A) = 0\}.$$

Note that the value of $\mathcal{H}^\sigma(A)$ is left out of (2.1). This is because if $\dim A = \sigma'$, $\mathcal{H}^{\sigma'}(A)$ can take on any value in $[0, \infty]$. Aside from the defining properties of outer measures, the

Hausdorff measure has many other important properties. Many of these will be discussed later as they become relevant, but here are some of the more immediate ones, all of them can be found with proofs in [77, Chapter 4].

Theorem 2.1 (Some properties of Hausdorff measures and dimension)**.**

1. *$\mathcal{H}^\sigma_\delta$ is non-increasing in δ. In particular, $\mathcal{H}^\sigma(A) = \sup_\delta \mathcal{H}^\sigma_\delta(A)$.*

2. *$\mathcal{H}^\sigma(A) = 0$ if, and only if, $\mathcal{H}^\sigma_\infty(A) = 0$. Therefore if* $\dim A < \sigma$, $\forall \epsilon > 0$, $\exists \{E_j\}_{j\in\mathbb{N}}$ *s.t.* $A \subset \cup_j E_j$, *and* $\sum_j (diam E_j)^\sigma < \epsilon$.

3. *Borel sets are $\mathcal{H}^\sigma$ measurable. In particular, $\mathcal{H}^\sigma$ is countably additive on Borel sets.*

4. *$\mathcal{H}^\sigma$ is Borel regular, inner regular, and outer regular.*

Another central notion is geometric measure theory is Lipchitz maps.

Definition 2.3 (Lipschitz map)**.** *Given metric spaces (X, d_X) and (Y, d_Y), a map $f : X \to Y$ is Lipschitz if there is a constant $C \geq 0$ such that*

$$d_Y(f(x), f(y)) \leq C d_X(x, y).$$

The least such constant is known as the Lipschitz constant of f. The space of all Lipchitz maps between X and Y is denoted $\mathrm{Lip}(X, Y)$ and can be endowed with the semi-norm

$$\mathrm{Lip}(f) = \inf\{C : d_Y(f(x), f(y)) \leq C d_X(x, y) \text{ for all } x, y \in X\}.$$

If f has Lipschitz constant L, we say f is $L-Lipchitz$. An injection $f : X \to Y$ is said to be bi-Lipschitz if $f^{-1}\lfloor_{f(X)}$: $f(X) \to Y$ is also Lipschitz. If there is a bi-Lipchitz bijection between X and Y, the metric spaces are said to be bi-Lipschitz equivalent.

Needless to say, Lipschitz maps respect much of the metric structure. In particular they interact nicely with Hausdorff measure and dimension. If $f : X \to Y$ is L-Lipschitz, then for $A \subset X$,

$$\mathcal{H}^\sigma(f(A)) \leq L^\sigma \mathcal{H}^\sigma(A), \text{ and} \tag{2.2}$$

$$\dim f(A) \leq \dim A. \tag{2.3}$$

In particular, if $f : X \to Y$ is bi-Lipschitz, then $\dim f(A) = \dim A$ for every $A \subset X$. That is to say that Hausdorff dimension is a bi-Lipschitz invariant. It is a central problem in the field of analysis on metric spaces, to find quantities that characterize metric spaces up to bi-Lipchitz equivalence. An important notion in relation of Lipschitz maps is that of rectifiable sets, these are the metric analogue of manifolds.

Definition 2.4 (Rectifiable/Unrectifiable sets). *A Borel set $A \in X$ is,*

k-rectifiable *if there exist a countable family of Lipschitz maps $\{f_j : \mathbb{R}^k \to X\}_{j\in\mathbb{N}}$ such that*

$$\mathcal{H}^k(A \setminus \cup_j f_j(\mathbb{R}^k)) = 0.$$

purely k-unrectifiable *if for any Lipschitz map $f : \mathbb{R}^k \to X$,*

$$\mathcal{H}^k(A \cap f(\mathbb{R}^k)) = 0.$$

Borel set can be neither rectifiable nor unrectifiable. However, the following decomposition theorem holds.

Theorem 2.2. *If $A \subset X$ is a Borel set with $\mathcal{H}^k(A) < \infty$ then there are two Borel sets A_R and A_U such that*

(i) $A = A_R \cup A_U$,

(ii) A_R *is k-rectifiable, and*

(iii) A_U *is purely k-unrectifiable.*

This decomposition is unique up to null sets.

Being that Lipschitz maps play such central role in geometric measure theory, and Hausdorff dimension is a bi-Lipschitz invariant, it makes sense to study Hausdorff measure and dimension in great detail, including how it interact with different classes of functions. A recurring question in this thesis will be the following; given a family of maps $\{f_\lambda : X \to Y\}_{\lambda\in\Lambda}$ and a set $A \subset X$ of known Hausdorff dimension (resp. measure), what can be said about the generic Hausdorff dimensions (resp. measure) of the sets $f_\lambda(A)$. To answer such questions it will be helpful to have several tools at hand to compute and/or estimate Hausdorff dimension of sets.

One such powerful tool is Frostman lemma.

Theorem 2.3 (Frostman Lemma). *Let (X, d) be a complete, separable metric space and A a Borel subset of X. Then the following are equivalent,*

1. $\mathcal{H}^\sigma(A) > 0$.

2. *There is a Radon measure μ compactly supported on A such that $0 < \mu(A) < \infty$, and*

$$\mu(B(x,r)) \lesssim r^\sigma \text{ for all } x \in X \text{ and } r > 0.$$

Here by $\mu(B(x,r)) \lesssim r^\sigma$ is meant that there is a constant $C > 0$ such that $\mu(B(x,r)) \leq Cr^\sigma$. Similarly, the notation $A \simeq B$ will be used to denote that the quantities A and B are comparable, that is, there are constant $c, C > 0$ such that $cA \leq B \leq CA$. A Borel measure is said to be a Radon measure if it is locally finite and Borel regular. Given a subset A of a metric space we denote by $\mathcal{M}(A)$ the collection of all compactly supported, Radon measure whose support is contained in A and such that $0 < \mu(A) < \infty$.

Frostman lemma was proven in $\mathbb{R}^n$ by Otto Frostman as part of his dissertation. A constructive proof of Frostman's lemma for compact subsets of $\mathbb{R}^n$ can be found in [77, Theorem 8.8]. The more general case of Suslin subsets of $\mathbb{R}^n$ is treated by Carleson in [21]. A non-constructive proof, due to Howroyd ([65]), using Hahn-Banach theorem gives the result in the generality stated above. Frostman's lemma can also be used to obtain another powerful tool which allows us to compute the Hausdorff dimension of a set in terms of energies of measures supported on that set. This is sometimes referred to as the "energy version" of Frostman's lemma.

Corollary 2.1. *If A is a Borel subset of (X,d) and we denote by $\mathcal{M}(A)$ the set of all compactly supported Radon measures on A, then*

$$\dim A = \sup\{\sigma \geq 0 : \exists \mu \in \mathcal{M}(A), \ s.t. \ I_\sigma(\mu) < \infty\}.$$

Here

$$I_\sigma(\mu) := \iint d(x,y)^{-\sigma} d\mu(y) d\mu(x), \tag{2.4}$$

denotes the σ-energy of the measure μ

As we will soon see, this approach is extremely powerful when estimating the effect that certain classes of maps have on the Huasdorff dimension of sets. Motivated by (2.4), the more general notion of "mutual energy" is also introduced.

Definition 2.5. *For $\mu, \nu \in \mathcal{M}(X)$ their mutual σ-energy is defined as*

$$I_\sigma(\mu,\nu) = \iint d(x,y)^{-\sigma} d\nu(y) d\mu(x).$$

Note since both μ and ν are positive measures, $I_\sigma(\mu,\nu) > 0$. Mutual energies will be useful when estimating the size of the intersection of images of sets by projection maps.

Aside from Hausdorff measure and dimension, other metrically defined notions of measure and dimension will be used later, so I introduce them here.

Definition 2.6 (Box-counting dimension)**.** *Given a set* $A \subset X$ *denote*

$$\mathcal{N}_\delta(A) := \inf\{N \in \mathbb{N} : \exists \{E_j\}_{j\in\mathbb{N}} \ s.t \ \ \forall\ j \in \mathbb{N},\ \ diam E_j = \delta \ and \ A \subset \cup_j E_j\}.$$

Then, the upper and lower box-counting dimension of A *are respectively defined as*

$$\overline{\dim}_B A = \limsup_{\delta\to 0} \frac{\log \mathcal{N}_\delta(A)}{-\log\delta}, \ \ and$$

$$\underline{\dim}_B A = \liminf_{\delta\to 0} \frac{\log \mathcal{N}_\delta(A)}{-\log\delta}.$$

In analogy to the second statement on Theorem 2.1, the lower box-counting dimensions can also be computed by

$$\underline{\dim}_B(A) = \inf\{\sigma \geq 0 : \forall\epsilon > 0, \exists\ \{E_j\}_{j=1}^N,\ \text{s.t.}$$
$$\text{diam}E_j = \text{diam}E_i, A \subset \cup_j E_j,\ \text{and} \sum_j^N \text{diam}E_j^\sigma < \epsilon\}. \quad (2.5)$$

Box-counting dimension is, in a sense, a restricted version of Hausdorff dimension. To compute Hausdorff dimension one covers the set A by sets of any diameter less than δ before letting $\delta \to 0$, whereas to compute box-counting dimension one restricts further to covers by sets whose diameters are exactly δ. It is therefore not hard to see that for a any set $A \subset X$

$$\dim A \leq \underline{\dim}_B(A) \leq \overline{\dim}_B(A).$$

Another notion of dimension that will come up later is packing dimension. This was introduced in [105] as a dual to Hausdorff dimension in the sense that instead considering coverings by sets, one considers packing small, disjoint, open balls inside the set in question.

Definition 2.7 (Packing measure)**.** *The* (σ, δ)*-packing pre-measure of* A *is*

$$\mathcal{P}_\delta^\sigma(A) := \sup\{\sum_j (diam B(x_j, r_j))^\sigma : \{x_j\}_{j\in\mathbb{N}} \subset A,$$
$$B(x_j, r_j) \cap B(x_i, r_i) = \varnothing,\ \ and\ diam B(x_j, r_j) \leq \delta\}.$$

The s-packing pre-measure is then given by

$$\mathcal{P}_0^\sigma(A) = \lim_{\delta\to 0} \mathcal{P}_\delta^\sigma(A).$$

In contrast with the Hausdorff measure, $\mathcal{P}_0^\sigma$ does not define a measure. Instead, the s-packing measure is

$$\mathcal{P}^\sigma(A) = \inf\{\sum_{j\in\mathbb{N}} \mathcal{P}_0^\sigma(E_j) : A \subset \cup_j E_j\}$$

The packing dimension is then defined in a similar way to the Hausdorff dimension

Definition 2.8 (Packing dimension)**.** *For $A \subset X$,*

$$\dim_{\mathcal{P}} A := \sup\{\sigma : \mathcal{P}^\sigma(A) = \infty\} = \inf\{\sigma : \mathcal{P}^\sigma(A) = 0\}.$$

The relation between these different notions of dimension is

$$\dim A \leq \dim_{\mathcal{P}} A \leq \underline{\dim}_B A \leq \overline{\dim}_B A.$$

Moreover, upper box-counting dimension and packing dimension are more closely related by the following.

Theorem 2.4. *Let $A \subset \mathbb{R}^n$ be a compact set,*

$$\dim_{\mathcal{P}} A = \inf\{\sup_j \overline{\dim}_B E_j : E_j \text{ is compact for all } j, \text{ and } A \subset \cup_j E_j\},$$

Finally, one concept that will also be of great importance in the following sections is that of push-forward measures.

Definition 2.9. *Let $f : X \to Y$ be a map between metric spaces. Let μ be a measure on X, then the push-forward of μ by f is the measure $f_\#\mu$ on Y defined by*

$$f_\#\mu(A) = \mu(f^{-1}A), \text{ for all } A \subset Y.$$

Turns out that continuous functions map $\mathcal{M}(X)$ into $\mathcal{M}(Y)$.

Theorem 2.5. *If $f : X \to Y$ is a continuous map and μ is in $\mathcal{M}(X)$ then $f_\#\mu$ is in $\mathcal{M}(Y)$.*

A proof of this theorem can be found in [77, pp. 16].

2.1 Fourier Transform Approach

Perhaps unsurprisingly, when $X = \mathbb{R}^n$, Fourier analysis has a great deal of applications to geometric measure theory. Few other areas in mathematics are as influential as Fourier analysis. For this reason, there are countless references, introductory and otherwise, on Fourier and Harmonic analysis, for instance [99], [70], [55], and [87], among many others. In this section I will introduce the Fourier transform focusing mostly on the properties that will be relevant later in this thesis. Results are stated without proof but many of the proofs are classical and readily available in all of the above mentioned references. Hereafter, for $p \in [0, \infty]$, $L^p(\mathbb{R}^n)$ denotes p-integrable complex-valued functions.

For $f \in L^1(\mathbb{R}^n)$, its Fourier transform is the continuous, bounded function

$$\widehat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)dx. \tag{2.6}$$

The theory of Fourier analysis is concerned with studying the interplay between $\mathcal{F}$ and various other operators. One of the most important on those interplays is with convolutions. For f and g in $L^1(\mathbb{R}^n)$, their convolution is the function

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy. \tag{2.7}$$

A quick application of Fubini's theorem shows that

$$\mathcal{F}(f * g)(\xi) = \widehat{f}(\xi)\widehat{g}(\xi). \tag{2.8}$$

Two other operators that will come back later on are the dilation and translation operator. For $r \in \mathbb{R}$, and $a \in \mathbb{R}^n$, define $\delta_r : L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $\tau_a : L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ by

$$\delta_r f(x) = f(rx), \text{ and } \tau_a f(x) = f(x+a).$$

By change of variables one checks,

$$\mathcal{F}(\delta_r f) = r^{-n}\delta_{r^{-1}}\widehat{f}, \text{ and } \mathcal{F}(\tau_a f) = e(a)\widehat{f}, \tag{2.9}$$

where $e(y)f(\xi) = e^{iy\cdot\xi}f(\xi)$. Although (2.6) defines $\mathcal{F}$ as an operator in $L^1(\mathbb{R}^n)$, since $L^1 \cap L^p(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, the Fourier transform can be extended to an operator in $L^p(\mathbb{R}^n)$ by approximation. As mentioned in the definition, $\mathcal{F}$ maps $L^1(\mathbb{R}^n)$ continuously into $L^\infty(\mathbb{R}^n)$ but more generally we have Hausdorff-Young's theorem.

Theorem 2.6 (Hausdorff-Young)**.** *For* $1 < p \leq 2$*, let* $p' = \frac{p}{p-1}$*. Then,*

$$\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$$

is a bounded linear operator.

This theorem was originally proven for some values of p by W. H. Young and then generalized to all $p \in [1, 2]$ by Hausdorff. Modern proofs can be found in the references mentioned at the beginning of the section.

In the particular case of $p = 2$, a stronger statement can be made.

Theorem 2.7 (Plancherel's Theorem)**.** *For* $f \in L^2(\mathbb{R}^n)$

$$||f||_2 = ||\widehat{f}||_2.$$

As will be seen later, it turns out that $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isometric isomorphism of Hilbert spaces. Now, if the goal is to apply Fourier theory to measures, it is convenient to have a Fourier theory for distributions. In order to discuss this, one must first introduce the Schwartz class.

Definition 2.10 (Schwartz class)**.** *A function* $\psi : \mathbb{R}^n \to \mathbb{R}$ *is said to be in the Schwartz class, denoted* $\mathscr{S}(\mathbb{R}^n)$*, if*

1. $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$,

2. *For every pair of multi-indices* $\alpha, \beta \in \mathbb{N}^n$

$$||\psi||_{\mathscr{S},\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \psi(x)| < \infty.$$

The definition uses the usual multi-index notation $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $D^\beta \psi = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n} \psi$.

One of the reason why the Schwartz class is so convenient in Fourier analysis is that $\mathcal{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is a bi-continuous bijection. This, in part, follows from the fact that $\mathcal{F}$ turns smoothness into decay and vice-versa.

Theorem 2.8. *For any multi-index* $\alpha \in \mathbb{N}^n$ *and* $\psi \in \mathscr{S}(\mathbb{R}^n)$*,*

1. $\widehat{D^\alpha \psi}(\xi) = (-2\pi i \xi)^\alpha \widehat{\psi}(\xi)$.

2. $D^\alpha \widehat{\psi}(\xi) = [\widehat{(-2\pi i (\cdot))^\alpha \psi}](\xi)$

An important corollary follows from this theorem. Denoting by $\Delta : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ the Laplacian operator

$$\Delta\psi = \sum_{j=1}^{n} \partial_j^2 \psi,$$

one has,

Corollary 2.2. *For $\psi \in \mathscr{S}(\mathbb{R}^n)$,*

$$\widehat{\Delta\psi}(\xi) = -(2\pi|\xi|)^2 \widehat{\psi}(\xi).$$

Explicitly, the inverse of $\mathcal{F}$, $\mathcal{F}^{-1}$ is given by the well known inversion formula,

Theorem 2.9. *The map $\mathcal{F}^{-1} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ given by*

$$\mathcal{F}^{-1}\psi(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \psi(\xi) d\xi,$$

is the continuous inverse of $\mathcal{F}$. That is, for $\psi \in \mathscr{S}(\mathbb{R}^n)$

$$\psi(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \widehat{\psi}(\xi) d\xi.$$

The Schwartz class is dense in $L^p(\mathbb{R}^n)$ for every $p \in [1, \infty]$, so in the particular case of $L^2(\mathbb{R}^n)$, the inversion formula combined with Plancherel's theorem show, as mentioned earlier, that $\mathcal{F}$ is a Hilbert space isomorphism. Another reason why $\mathscr{S}(\mathbb{R}^n)$ is so convenient is that it helps extend the Fourier transform beyond functions. The space $\mathscr{S}(\mathbb{R}^n)$ endowed with the family of semi-norms $\{|| \cdot ||_{\mathscr{S},\alpha,\beta}\}$ is a Frechét space, so it makes sense to talk about its topological dual $\mathscr{S}'(\mathbb{R}^n)$. Given the decay properties of Schwartz functions, it is not hard to see that $\mathscr{S}'(\mathbb{R}^n)$ includes $L^p_{loc}(\mathbb{R}^n)$ for any $p \in [0, \infty]$, but moreover, $\mathscr{S}'(\mathbb{R}^n)$ has elements which are not functions. For instance the evaluation map $\delta_x\psi = \psi(x)$, which corresponds to the Dirac distribution at $x \in \mathbb{R}^n$, is in $\mathscr{S}'(\mathbb{R}^n)$. The space $\mathscr{S}'$ is referred to as the space of tempered distributions. Throughout this thesis, for $T \in \mathscr{S}'(\mathbb{R}^n)$ and $\psi \in \mathscr{S}(\mathbb{R}^n)$, the dual action will be denoted by $\langle T|\psi\rangle_{\mathscr{S}'(\mathbb{R}^n)\times\mathscr{S}(\mathbb{R}^n)}$ but I will often omit the subscript if the context leads to no confusion.

Definition 2.11 (Distributional Fourier transform)**.** *For $T \in \mathscr{S}'(\mathbb{R}^n)$, its Fourier transform $\widehat{T}$ is another tempered distribution given by*

$$\langle \widehat{T}|\psi\rangle = \langle T|\widehat{\psi}\rangle. \tag{2.10}$$

This readily brings us to measures. If μ is a finite Radon measure on $\mathbb{R}^n$, it is clear that $\mu \in \mathscr{S}'(\mathbb{R}^n)$. Thus, the Fourier transform of μ is defined via (2.10). However, distributions often have point-wise representations, this turns out to be the case for the distribution $\widehat{\mu}$. Indeed, let $\mathcal{F}(\mu)$ be the bounded Lipchitz function

$$\mathcal{F}(\mu)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} d\mu(x), \tag{2.11}$$

then for $\psi \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle \widehat{\mu} | \psi \rangle = \int_{\mathbb{R}^n} \psi(\xi)\mathcal{F}(\mu)(\xi) d\xi.$$

This is easy to see by applying Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n} \psi(\xi)\mathcal{F}(\mu)(\xi) d\xi &= \int \psi(\xi) e^{-2\pi i x\cdot\xi} d\mu(x) d\xi \\
&= \int\int \psi(\xi) e^{-2\pi i x\cdot\xi} d\xi d\mu(x) \\
&= \int \widehat{\psi}(x) d\mu(x) = \langle \mu | \psi \rangle \\
&= \langle \widehat{\mu} | \psi \rangle.
\end{aligned}$$

It follows that distribution $\widehat{\mu}$ is given by integration against the function $\mathcal{F}(\mu)$. In view of this, I abuse notation and use $\widehat{\mu}$ (and/or $\mathcal{F}(\mu)$) to mean both the function given by (2.11) and the corresponding distribution. A nice expository reference for theory of Fourier transform of measures in Euclidean space is [80] where the connection with Hausdorff dimension is also treated.

Theorem 2.10 (Some properties of $\mathcal{F}$). *Let μ be a finite Radon measures, f and g be functions in L^1, ψ and φ be functions in $\mathscr{S}(\mathbb{R}^n)$, and define $f * \mu(x) = \int f(x-y)d\mu(y)$. Then*

1. $\int \widehat{f} g = \int f \widehat{g}$,
2. $\int \widehat{\mu} f = \int \widehat{f} d\mu$,
3. $\int \widehat{\mu} d\nu = \int \widehat{\nu} d\mu$,
4. $\widehat{f * \mu} = \widehat{f}\widehat{\mu}$,
5. $\int \psi d\mu = \int \psi \overline{\widehat{\mu}}$,
6. $\int \widehat{f}\widehat{\overline{g}} d\mu = \int (\widehat{\mu} * \overline{g}) f$.

In addition, there is a direct connection between integrability of $\widehat{\mu}$ and regularity of the measure μ itself.

Theorem 2.11. *Let μ be a finite Radon measure. Then*

1. *If $\widehat{\mu} \in L^2(\mathbb{R}^n)$, then $d\mu = f dx$ where $f \in L^2(\mathbb{R}^n)$.*

2. *If $\widehat{\mu} \in L^1(\mathbb{R}^n)$, then $d\mu = g dx$ where $g \in \mathcal{C}(\mathbb{R}^n)$.*

The main connection between Fourier analysis and Hausdorff dimension comes from the fact that one can compute energies of measures as integrals over the frequency space. To make this more precise, for $0 < \sigma \leq n$ let R_σ denote the σ-Riesz kernel

$$R_\sigma(x) = |x|^{-\sigma}.$$

For a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$, its energy integral can be rewritten as

$$I_\sigma(\mu) = \int R_\sigma * \mu(x) d\mu(x). \tag{2.12}$$

The distributional Fourier transform of R_σ is $cR_{n-\sigma}$ for some constant c, sow properties 4,5 and 6 in Theorem 2.10 can be formally applied to obtain

Theorem 2.12. *There is a constant, $c(n, \sigma)$, depending only on n and σ such that*

$$I_\sigma(\mu) = c(n, \sigma) \int |\xi|^{\sigma-n} |\widehat{\mu}(\xi)|^2 d\xi,$$

In a similar vain, mutual energies can also be written as energies on the frequency side. Whenever $s + t > 2\sigma$ we have that if $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$ then

$$I_\sigma(\mu, \nu) = c(n, \sigma) \int |\xi|^{\sigma-n} \widehat{\mu}(\xi) \overline{\widehat{\nu}}(\xi) d\xi \lesssim I_s(\mu)^{1/2} I_t(\nu)^{1/2}. \tag{2.13}$$

A priori there is no reason why Theorem 2.10 can be applied to (2.12), R_σ is not in L^p for any $p \in [1, \infty]$. Nevertheless one may use the distributional Fourier transform and approximation arguments to show that Theorem 2.12 does hold. A proof of this can be found in many places, for instance [80, Section 3.5]. This theorem has proven extremely useful over time having been successfully applied to solve problems like dimension distortion by orthogonal projections, dimension of planar slicing of sets, dimension of intersections of general sets and measures and the distance set problem among many others. The usefulness of this approach is also the biggest motivation behind my work presented on Chapter 7

CHAPTER 3

PROJECTION AND SLICING THEOREMS IN EUCLIDEAN SPACE

Before I will be discuss the theory of fractal projections and planar slices in the Heisenberg group, it is very helpful to have a basic understanding of the theory in Euclidean space. Many of the techniques used in Euclidean space are also valid, after minor adaptations, in the Heisenberg group. As expected, most of the notions discussed in the previous chapter were initiated in Euclidean space. The exceeding amount of structure in $\mathbb{R}^n$ makes for incredibly rich theory of fractal projections. This theory stems from the following, loosely stated, question.

Question 3.1. *Given a probability space* $(\Lambda, \mathbb{P})$*, a family of projection* $\{P_\lambda : \mathbb{R}^n \to \mathbb{R}^m\}_{\lambda\in\Lambda}$*, and a set* $A \subset \mathbb{R}^n$ *What can be said about the* $\mathbb{P}$*-almost sure size of the set* $P_\lambda A$ *in terms of the size of* A*?*

Along similar lines, and using the same set up, the related notion of "planar slices of sets" is driven by the following question.

Question 3.2. *What can be said about the* $\mathbb{P}\times\mathcal{H}^m$*-generic size of the set* $A\cap P_\lambda^{-1}(y)$*,* $(y \in \mathbb{R}^m)$*, in terms of the size of* A*?*

As might be evident, these two kinds of problems are closely related. An exhaustive and rigorous exposition of the theory of fractal projections in $\mathbb{R}^n$ would likely span the length of a book. As such, this chapter merely aims to put my own work into historical context by providing a quick summary of existing results in Euclidean space. I will only focus on the more classical results concerning the standard family of orthogonal projections as well the more recent work on the family of projections onto isotropic subspaces of even dimensional Euclidean space. I will mention, without discussion, many related results including some that have seen recent progress. No proofs are provided but reference to original and expository works are given. Many of the results require the sets in questions to have certain amount measurability, most of the results that will be stated have pathological counterexamples if one allows arbitrary sets. Therefore, to not be repetitive, all sets are assumed to be Borel, although some of the results hold with slightly weaker measurability conditions.

3.1 The Standard Family of Orthogonal Projections

Perhaps the most intuitive example of a family of projections, and the one that kick-started the theory, is the family of orthogonal projections onto all m-dimensional linear subspaces of $\mathbb{R}^n$. Given any m-dimensional linear subspace, V, of $\mathbb{R}^n$, the orthogonal projection $P_V : \mathbb{R}^n \to V$ is a 1-Lipschitz map, therefore one has that for any set $A \subset \mathbb{R}^n$ $\dim P_V A \leq \dim A$. The goal is then to understand to what extend the lower bound also holds. Before we can do that, we must understand the parametrizing space (i.e. the space of all m-dimensional subspaces).

The space of all m-dimensional linear subspaces of $\mathbb{R}^n$ is a compact smooth manifold known as the Grassmannian, and denoted $G(n, m)$. These manifolds are ubiquitous in many areas of mathematics and a lot has been written about them. I will only go over properties that will be relevant for us.

The orthonormal group $O(n) \subset GL(n)$ acts smoothly and transitively on $G(n, m)$, so by fixing $V_0 \in G(n, m)$, the Grassmannian can be written as the homogeneous space $O(n)/\mathcal{O}_{V_0}$, where $\mathcal{O}_{V_0}$ is the stabilizer of V_0. It is not hard to see that $\mathcal{O}_{V_0} = O(m) \times O(n-m)$ so that $G(n, m) = O(n)/(O(m) \times O(n-m))$. It follows that $\dim G(n, m) = m(n-m)$, and that $G(n, m)$ inherits a unique $O(n)$-invariant probability measure, denoted $\gamma_{n,m}$, form the Haar measure of $O(n)$, denoted θ_n. The measure $\gamma_{n,m}$ is given, for $D \subset G(n, m)$, by

$$\gamma_{n,m}(D) = \theta_n(g \in O(n) : gV_0 \in D).$$

In turn, the measure θ_n is given in terms of the surface measure of the $(n-1)$-sphere , σ^{n-1} as

$$\theta_n(\Omega) = \sigma^{n-1}\left(\mathbb{S}^{n-1} \cap (\cup_{g\in\Omega} g u_0)\right),$$

where u_0 is any fixed unit vector and $\Omega \subset O(n)$. The measure $\gamma_{n,m}$, however, is independent of the choice of V_0, and turns $G(n, m)$ into a probability space. Question 3.1 can be rephrased in this specific context, and it is answered by the following theorem.

Theorem 3.1 (Marstrand-Kaufman-Mattila projection theorem). *Let $A \subset \mathbb{R}^n$,*

1. *If $\dim A \leq m$, then $\dim P_V A = \dim A$ for $\gamma_{n,m}$-almost every $V \in G(n, m)$.*
2. *If $\dim A > m$, then $\mathcal{H}^m(P_V A) > 0$ for $\gamma_{n,m}$-almost every $V \in G(n, m)$.*
3. *If $\dim A > 2m$, then $Int(P_V A) \neq \varnothing$ for $\gamma_{n,m}$-almost every $V \in G(n, m)$.*

The first two statements where originally proven my J. Marstrand in $\mathbb{R}^2$ ([75]). Later in [71], R. Kauffmann introduced the potential theoretic approach which streamlined

Marstrand's proof. The new approach was now more suitable to extensions and generalizations, and in [76] P. Mattila used it to generalize Marstrand's theorem to all dimensions. The proof of this theorem is quite elegant, the idea is to use Theorem 2.1 to pick a measure μ in $\mathcal{M}(A)$ with finite σ-energy for $\sigma < \dim A$. Then, by Theorem 2.5, the measure $P_{V\#}\mu$ is in $\mathcal{M}(P_V A)$, so by studying properties of $P_{V\#}\mu$ one can obtain information on the dimension and measure of $P_V A$. For instance, to get the first part, one bounds the integral $\int I_\sigma(P_{V\#}\mu) d\gamma_{n,m}$ by $I_\sigma(\mu)$. This has become a standard approach in the theory of fractal projections.

The Marstrand-Kaufman-Mattila theorem establishes that the set of hyperplanes for which each of the statements fail is $\gamma_{n,m}$-null. One could ask how big these sets of exceptions can be. The following theorem is, once again, a summary of separate results obtained by Marstrand, Kaufmann, and Mattila.

Theorem 3.2. *Let* $A \subset \mathbb{R}^n$ *with* $\dim A = \alpha$.

If $0 < \sigma < \alpha < m$

$$\dim(V \in G(n,m) : \dim P_V A < \sigma) \leq m(n-m) - (m-\sigma).$$

If $\alpha \geq m > \sigma$

$$\dim(V \in G(n,m) : \dim P_V A < \sigma) \leq m(n-m) - (\alpha-\sigma).$$

If $\alpha > m$

$$\dim(V \in G(n,m) : \mathcal{H}^m(P_V A) = 0) \leq m(n-m) - (\alpha-m).$$

If $\alpha > 2m$

$$\dim(V \in G(n,m) : Int(P_V A) = \varnothing) \leq m(n-m) - (\alpha-2m).$$

This result is strictly stronger than Theorem 3.1. A proof of this theorem can be found in [80, Section 5.3].

Other than looking at the projection of a single set, one could also ask for how often projections of sets intersect, and how big that intersection is. This was recently studied by P. Mattila and T. Orponen in [81]. Their motivation in studying this problem is an application to radial projections and visibility. The main result is

Theorem 3.3. *Let* $A, B \subset \mathbb{R}^n$ *be such that* $\dim A = \alpha,\ \dim B = \beta$.

1. *If* $\alpha, \beta > m$ *then,*

$$\gamma_{n,m}(V \in G(n,m) : \mathcal{H}^m(P_V A \cap P_V B) > 0) > 0.$$

2. *If* $\alpha, \beta > 2m$ *then,*

$$\gamma_{n,m}(V \in G(n,m) : Int(P_V A \cap P_V B) \neq \varnothing) > 0.$$

3. *If* $\alpha > m,\ \beta \leq m,$ *but* $\alpha + \beta > 2m$ *then for all* $\epsilon > 0$*,*

$$\gamma_{n,m}(V \in G(n,m) : \dim(P_V A \cap P_V B) > \beta - \epsilon) > 0.$$

The proof of this theorem makes use of the mutual energy of measures and, once again, by applying Fourier transform methods, the proof becomes quite elegant.

Aside form dimension, measure and non-empty interior, the family of projections $\{P_V : V \in G(n,m)\}$ can also be used to uncover other metric-geometric properties of set. For example, in the case of set with integer dimension, this family of projections can uncover information about the set's (un)rectifiability.

In Euclidean space, rectifiable sets are a direct (metric) analogue of smooth manifolds. A celebrated theorem of Radamacher (see for instance [77, Theorem 7.3]) states that every Lipschitz function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is differentiable $\mathcal{L}^n$-almost everywhere. As a countable union of Lipschitz graphs, rectifiable sets can therefore be endowed with a differential structure defined at almost every point. This highlights the importance of characterizing (un)rectifiability in $\mathbb{R}^n$. Projections provide one such characterization via Besicovitch-Federer projection theorem.

Theorem 3.4 (Besicovitch-Federer projection theorem)**.** *Let* $A \subset \mathbb{R}^n$ *be a Borel set with* $0 < \mathcal{H}^m(A) < \infty$ *for some integer* m*. Then the following are equivalent*

1. A *is purely* m*-unrectifiable.*

2. $\mathcal{H}^m(P_V A) = 0$ *for* $\gamma_{n,m}$*-almost every* $V \in G(n,m)$*.*

This theorem was first proven in $\mathbb{R}^2$ by A. Besicovitch in [14, 15, 16], and then in higher dimensions by H. Federer in [45]. There has also been a significant effort put into constructing sets with prescribed projections. The following theorem of K. Falconer and R. Davies ([36, 32]) holds,

Theorem 3.5. *For each $V \in G(n,m)$, Let $A_V \subset V$ be sets such that $\cup_{V\in G(n,m)} A_V$ is $\mathcal{H}^n$-measurable. Then, there exists a Borel set $A \subset \mathbb{R}^n$ such that for $\gamma_{n,m}$-almost every $V \in G(n,m)$*

$$\mathcal{H}^m(A_V \setminus P_V A) = \mathcal{H}^m(P_V A \setminus A_V) = 0.$$

This theorem is sometimes referred to as the digital sundial theorem since in the case $n = 3,\ m = 2$ it implies the existence of a set that when the sun hits it, the shadow casted shows the digits of the time of day at that instance.

3.2 Projections Onto Isotropic Subspaces

Instead of considering the family of all orthogonal projections, sometimes is useful to consider a specific subfamily of projections. However, if one considers a subfamily $S \subset G(n,m)$ such that $\gamma_{n,m}(S) = 0$ one should not expect, a priori, an analogue of Theorem 3.1 to hold as S could fall entirely within the set of exceptional directions of a set A. However, for certain choices of S some positive results can be obtained.

A well studied subfamily is the family of linear subspaces that are isotropic with respect to the standard symplectic form. The standard symplectic form in $\mathbb{R}^{2n}$ is the map $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ given by $\omega((x,y),(x',y')) = x \cdot y' - y \cdot x'$. A plane $V \in G(n,m)$ is said to be isotropic (with respect to ω) if $\omega\lfloor_{V\times V} \equiv 0$. The collection of all m-dimensional isotropic subspaces of $\mathbb{R}^{2n}$ is denoted by $G_h(2n,m)$. The spaces $G_h(2n,m)$ are very heavily studied in several areas of mathematics. The space $G_h(2n,n)$ in particular is known as the Lagrangian manifold, and it is central in symplectic geometry and Hamiltonian mechanics. Here I will state, without proof, the properties of $G_h(2n,m)$ that will be relevant later on. For proof of the properties stated here refer to [7, Section 2].

The space $G_h(2n,m)$ is a smooth submanifold of $G(2n,m)$ whose dimension is strictly less than that of $G(2n,m)$. Just as $G(n,m)$, $G_h(2n,m)$ can be endowed with the structure of a homogeneous space. The unitary group $U(n) \subset O(2n)$ acts smoothly and transitively on $G_h(2n,m)$. For a fix isotropic plane $V_0 \in G_h(2n,m)$, the stabilizer $\mathcal{U}_{V_0}$ is isomorphic to $O(m) \times U(n-m)$, so that $G_h(2n,m) = U(n)/(O(m) \times U(n-m))$. It follows that $\dim G_h(2n,m) = n^2 - \left(\frac{m(m-1)}{2} + (n-m)^2\right) =: \eta^n_m$. The maximal dimension of an isotropic linear subspace is n (i.e. half the dimension of the space) so the smallest of the isotropic Grassmannian's, relative to the size of the standard Grassmannian, is the Lagrangian manifold $G_h(2n,n)$ whose dimension is $\frac{1}{2}n(n+1)$. The largest of them is

$G_h(2n,1)$ which actually coincides with $G(2n,1)$ (i.e. All lines through the origin in $\mathbb{R}^{2n}$ are isotropic). The space $G_h(2n,m)$ inherits a unique, $U(n)$-invariant probability measure here denoted by $\mu_{2n,m}$, from the probability Haar measure, ϑ_n, on $U(n)$. The measure $\mu_{2n,m}$ is given, for $D \subset G_h(2n,m)$, by

$$\mu_{2n,m}(D) = \theta_n(g \in U(n) : gV_0 \in D).$$

In turn, the measure ϑ_n is given in terms of the surface measure of the $(n-1)$-sphere , σ^{n-1} as

$$\mu_{2n,m}(\Omega) = \sigma^{n-1}\left(\mathbb{S}^{n-1} \cap (\cup_{g\in\Omega} g u_0)\right),$$

where u_0 is any fixed unit vector and $\Omega \subset G_h(2n,m)$. Just as the measure $\gamma_{n,m}$, $\mu_{2n,m}$ is independent of the choice of V_0. The following property of $\mu_{2n,m}$ roughly says that isotropic planes are uniformly distributed.

Lemma 3.1. *For $V \in G_h(2n,m)$,*

1. $\mu_{2n,m}(V \in G_h(2n,m) : |P_V x| \leq \delta) \lesssim \delta^m |x|^{-m}$.
2. $\mu_{2n,m}(V \in G_h(2n,m) : |P_{V^\perp} x| \leq \delta) \lesssim \delta^{n-m} |x|^{m-n}$.

By now it should be clear that the probability space $(G_h(2n,m), \mu_{2n,m})$ shares many of the properties of its parent space $(G(2n,m), \gamma_{2n,m})$ so despite the fact that $\mu_{2n,m}$ is singular with respect to $\gamma_{2n,m}$, it is not surprising that the analogues of all the results discussed in the previous section also hold in this case. The analogue of Theorem 3.1 was first proven by the authors in [7], while the corresponding exceptional set result (analogue of Theorem 3.2) was proven by R. Hovila in [64] where it was also shown that isotropic projections characterize unrectifiability in the same way as in Theorem 3.4. Perhaps more importantly for later applications, is the following,

Lemma 3.2. *Let $\mu \in \mathcal{M}(\mathbb{R}^n)$,*

(i) If $I_m(\mu) < \infty$ then for $\mu_{2n,m}$-almost every $V \in G_h(2n,m)$, $P_{V\#}\mu \ll \mathcal{H}^m$ with density in L^2.

(ii) If $I_{2m}(\mu) < \infty$ the same is true but with continuous density.

These facts will are used in [95] to prove the analogue of Theorem 3.3 which will be discussed in Chapter 5.8.

3.3 Other Projection Results

Other, more general subfamilies of projections have also been considered. In particular the following question has received much attention:

Let $\Lambda \subset \mathbb{R}^k$ be an open set and $S = \{V(\lambda) : \lambda \in \Lambda\}$ a smoothly parametrized subfamily of linear subspaces. Then, for a given set A, what can be said about the $\mathcal{H}^k$-almost sure dimension of $P_\lambda(A) \subset V(\lambda)$.

For example, if S is a smoothly parametrized curve then it follows trivially from Theorem 3.2, that for $\mathcal{H}^1$-almost every λ, $\dim P_\lambda A \geq \min\{\dim A - 1, 1\}$. This was made more general By the authors in [67, 66]. In the case of R^3, if a curvature condition is added to the map $\lambda \to V(\lambda)$, even stronger conclusions can be made. This was studied by K. Fässler and T. Orponen in [43]. Later in [88] the authors used Fourier restriction techniques to give partial improvements in this direction. Fourier restriction methods continued to play a role in studying dimension distortion by restricted families of projections. In [90], T. Orponen and L. Venieri applied this method to the particular family of projections given by the directions in $W \cap \mathbb{S}^2$ where W is an affine plane with $0 \notin W$. Their result improved the results of [88] for this particular case. More recently, T. Harris improved it even further in [60].

Yet another area that has been explored, is figuring out for which types of sets one can obtain information about projections in particular directions. The results discussed so far concern the size of the projection onto a generic subspace on a given family of subspaces, however, they provide no information about the size in any given particular direction. However, the set being projected has enough structure some positive result can be obtained. In this context self-similar and self-affine sets have been studied [42, 93, 63, 97, 47].

More general notions of "projections" have also been studied. For instance the study of radial projections ([81, 89]), in connection with visibility ([29]), is still an active area of research ([74]). But perhaps one of the most influential notions is that of transversal families of maps introduced by Y. Peres and W. Schlag in [92]. Their result encompasses all the results discussed in Sections 3.1 and 3.2, as well as radial projections and many other families such as Bernoulli convolutions which was their original motivation. There are many more related problems that have been studied and many that are currently active research areas. For more exhaustive surveys in the subject see [41, 78, 79].

3.4 Planar Slices of Sets

Planar slices of sets are closely related to projections. For a set $A \subset \mathbb{R}^n$ and an affine plane $W \subset \mathbb{R}^n$ the "slice" of A by W is the set $A \cap W$. If $\dim A > m$ then for $\gamma_{n,m}$-almost every plane $V \in G(n,m)$ the set $P_V A$ has positive $\mathcal{H}^m$ measure. This tells us that for $\mathcal{H}^m$- positively many $v \in V$, the $(n-m)$-dimensional affine plane $P_V^{-1}(v) = V^\perp + v$ has non-empty intersection with A. It is therefore natural to ask just how big this intersection is. As is a common theme in fractal geometry, this problem (and its solution) where introduce for the plane in Marstran's paper [75] and later generalized to higher dimensions by Mattila in [76]. Since then, several variations of the problem have been studied, for instance [81, 30]. In this section I will discuss some of the main results.

The planar slicing result proven by Marstrand and Mattila reads as follows,

Theorem 3.6. *Let $s > m$, and $A \subset \mathbb{R}^n$ be a set such that $0 < \mathcal{H}^s(A) < \infty$. Then for $\gamma_{n,m}$-almost every plane $V \in G(n,m)$,*

$$\mathcal{H}^m(\{v \in V : \dim[A \cap (V^\perp + v)] = \dim A - m\}) > 0$$

More recently Mattila and Orponen showed the following,

Theorem 3.7. *Let $s > m$, and $A \subset \mathbb{R}^n$ be a set such that $0 < \mathcal{H}^s(A) < \infty$. Then there is a Borel set $B \in \mathbb{R}^n$ such that $\dim B \leq m$ and for all $x \in \mathbb{R}^n \setminus B$,*

$$\gamma_{n,m}(\{V \in G(n,m) : \dim[A \cap (V^\perp + x)] = \dim A - m\}) > 0$$

Moreover, as will be seen in Chapter 5.8, these results also hold for the isotropic Grassmannian. In Theorem 3.6 if one replaces the equality for "$\leq$", a stronger result holds

Lemma 3.3 (Eilenberg's Inequality)**.** *Let $s > m$ and $A \subset \mathbb{R}^n$. For every plane $V \in G(n,m)$*

$$\int_V \mathcal{H}^{s-m}[A \cap (V^\perp + v)] d\mathcal{H}^m(v) \lesssim \mathcal{H}^s(A).$$

A more general version of this theorem is stated and proven in [19, Theorem 13.3.1]. Note that this implies, if $\mathcal{H}^s(A) < \infty$, that

$$\mathcal{H}^{s-m}(A \cap (V^\perp + v)) < \infty, \text{ for } \mathcal{H}^m - a.e\ v \in V,$$

which is stronger than to say that for all V,

$$\dim[A \cap (V^\perp + v)] \leq s - m, \text{ for } \mathcal{H}^m - a.e\ v \in V.$$

Hence, the bulk of the work goes into proving the lower bound in Theorem 3.6. This requires the introduction of sliced measures. Since they will be used again in later chapters I will briefly introduce them here. A very detailed construction and overview of these measures can be found in [77, Section 10.1].

Given a fixed m-dimensional plane, V, a measure $\mu \in \mathcal{M}(\mathbb{R}^{2n})$ and function $\varphi \in \mathcal{C}_c^+(\mathbb{R}^{2n})$ one can define a new measure by $\mu_\varphi(A) = \int_A \varphi d\mu$. The measure $P_{V\#}\mu_\varphi$ is a measure in $\mathcal{M}(V)$ so by the differentiation theorem, the derivative

$$\mu_{V^\perp+v}(\varphi) := \lim_{\delta\to 0}(2\delta)^{-m} P_{V\#}\mu_\varphi(B(v,\delta)) = \lim_{\delta\to 0}(2\delta)^{-m}\int_{\mathcal{N}(V^\perp+v,\delta)} \varphi d\mu,$$

exists and is finite for $\mathcal{H}^m$-a.e. $v \in V$. Here $\mathcal{N}(V^\perp + v, \delta)$ is the δ neighborhood around the plane $V^\perp + v$. One can check, after some work, that $\mu_{V^\perp+v}$ defines a positive linear functional on $\mathcal{C}_c^+(\mathbb{R}^{2n})$. Therefore, by Riesz representation theorem we can associate a positive Radon measure to $\mu_{V^\perp+v}$ that I denote in the same way. This measure is now supported on $(V^\perp + v) \cap spt\mu$.

Given two Radon measures, λ and γ, the Radon-Nikodym derivative $\frac{d\lambda}{d\gamma}$ satisfies

$$\int_A \frac{d\lambda}{d\gamma}(x)d\gamma(x) \leq \lambda(A),$$

for all Borel sets A, and with equality whenever $\lambda \ll \gamma$ (Theorem 2.12, [77]). Therefore, for any Borel set $B \subset V$ the measure $\mu_{V^\perp+v}$ satisfies,

$$\int_B \int g d\mu_{V^\perp+v} d\mathcal{H}^m(v) \leq \int_{P_V^{-1}(B)} g d\mu, \tag{3.1}$$

for any Borel function g, and with equality whenever $P_{V\#}\mu \ll \mathcal{H}^m$. In this case, taking $g = \mathbb{1}_{P_V^{-1}(B)}$,

$$\mu(P_V^{-1}(B)) = \int_B \mu_{V^\perp+v}(\mathbb{R}^{2n}) d\mathcal{H}^m(v). \tag{3.2}$$

However, denoting by μ_V the density of $P_{V\#}\mu$, the definition of the push-forward measure tells us that

$$\mu(P_V^{-1}(B)) = \int_B \mu_V(v) d\mathcal{H}^m(v). \tag{3.3}$$

Since this is true for any Borel set $B \subset V$, for $\mathcal{H}^m$-almost every $v \in V$

$$\mu_{V^\perp+v}(\mathbb{R}^{2n}) = \mu_V(v) \tag{3.4}$$

In particular, if $P_{V\#}\mu \ll \mathcal{H}^m$ then,

$$\mu(\mathbb{R}^{2n}) = \int \mu_{V^\perp+v}(\mathbb{R}^{2n}) d\mathcal{H}^m(v). \tag{3.5}$$

As we will see later, the way of proving slicing results like the ones mentioned above, is to use Theorem 2.1 to pick a measure $\mu \in A$ with appropriate finite energy, and slice it to obtain a family of measures $\{\mu_{V^\perp+v} \in \mathcal{M}(A \cap (V^\perp + v))\}$ with appropriate finite energies.

3.5 Box and Packing Dimsenion

As now customary with all problems studied in the context of Hausdorff dimension, dimension distortion by projections has also been studied in the context of box-counting and packing dimensions. Just as with Hausdorff dimension, the fact that projections are Lipschitz means that the the bound $\dim_B P_A \leq \dim_B A$ holds for all V. However, the story gets more complicated when one looks at lower bounds and one realizes the dimension is not necessarily almost-surely preserved. The following bounds were proven by K. Falconer and J. Howroyd in [39].

Theorem 3.8. *Let $A \subset \mathbb{R}^n$ be a compact set,*

$$\frac{\underline{\dim}_B A}{1 + (\frac{1}{m} - \frac{1}{n})\underline{\dim}_B A} \leq \underline{\dim}_B P_V A. \tag{3.6}$$

There these bounds were proven sharp for lower box-counting dimension by exhibiting a set for which the maximal dimension drop allowed by (3.6) was attained almost surely. The same bound was shown (also sharply) for upper box-counting dimension and packing dimensions in [68]. However, it remained unanswered wether $\dim_B P_V A$ was $\gamma_{n,m}-$almost surely constant. This was later addressed by Falconer and Howroyd with their introduction of dimension profiles accompanied by a potential theoretic approach to box-counting and packing dimensions [40, 37].

For $r > 0, \sigma \geq 0$, consider the kernel

$$\varphi_r^\sigma(x) = \min\left\{1, \frac{r^\sigma}{|x|^\sigma}\right\}.$$

One can define potentials for $\mu \in \mathcal{M}(\mathbb{R}^n)$ in terms of this kernel, as

$$\mathcal{F}_r^\sigma(x,\mu) := \int_{\mathbb{R}^n} \varphi_r^\sigma(x-y)d\mu(y),$$

and energies

$$\mathcal{E}_r^\sigma(\mu) = \int_{\mathbb{R}^n} \mathcal{F}_r^\sigma(x,\mu)d\mu(x).$$

The (r,σ)-capacity of a set $A \in \mathbb{R}^n$ is then defined by

$$C_r^\sigma(A) = \frac{1}{\inf \mathcal{E}_r^\sigma(\mu)},$$

where the infimum is taken over $\mu \in \mathscr{P}(A) := \{\mu \in \mathcal{M}(A) : \mu(A) = 1\}$. The connection with box-counting dimension comes from the following theorem.

Theorem 3.9. *Let*

$$\underline{\dim}_B^\sigma(A) := \liminf_{r\to 0} \frac{\log C_r^\sigma(A)}{\log 1/r}, \;\; \textit{and} \; \overline{\dim}_B^\sigma(A) := \limsup_{r\to 0} \frac{\log C_r^\sigma(A)}{\log 1/r}.$$

Then,

$$\underline{\dim}_B^\sigma(A) = \underline{\dim}_B(A), \;\; \textit{and} \; \overline{\dim}_B^\sigma(A) = \overline{\dim}_B(A),$$

whenever $\sigma \geq n$.

This gives a way to study box-counting dimension by using many powerful results from potential theory. also, By Theorem 2.4, defining

$$\dim_{\mathcal{P}}^\sigma A = \inf\{\sup_j \overline{\dim}_B^\sigma E_j : E_j \text{ is compact for all } j, \text{ and } A \subset \cup_j E_j\},$$

we see that the same approach is valid for packing dimension. This also motivates the study of the quantities $\dim_B^\sigma A$ and $\dim_{\mathcal{P}}^\sigma A$, for $\sigma < n$. Turns out, this are exactly the correct quantities to consider for dimension distortion by projections.

Theorem 3.10. *Let $A \subset \mathbb{R}^n$ be a compact set. Then*

$$\underline{\dim}_B P_V A = \underline{\dim}_B^m A, \;\; \textit{and} \; \overline{\dim}_B P_V A = \overline{\dim}_B^m A,$$

for $\gamma_{n,m}$-almost every $V \in G(n,m)$. The same holds for packing dimension.

Besides projection theorems, it is useful in general to have a potential theoretic approach to dimension. In Chapter 6 I will be showing that an analogous potential theoretic approach is also valid for box-counting and packing dimensions in the Heisenberg group.

CHAPTER 4

THE HEISENBERG GROUP

In this chapter I will introduce the Heisenberg group in a geometric way which will motivate much of the intuition that will be useful later. Then I will discuss how the Heisenberg group arises naturally in quantum mechanics. This Quantum-mechanical construction will reveal much of the representation theory of the Heisenberg group, which in turn, will be useful in the later chapter dedicated to Fourier theory. The goal of this Chapter is to be a clear but concise introduction to the Heisenberg group, suitable for early graduate student getting started on the topic. Everything discussed here is more or less contain in [20], [56], and [103].

4.1 The Heisenberg Group as a Manifold

The first Heisenberg group is the set $\mathbb{H} = \mathbb{C} \times \mathbb{R}$, which is also identified with $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ in the canonical way. I denote a typical point by $p = (z, t) \in \mathbb{C} \times \mathbb{R}$ or $p = (x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ where the two notations are related by $z = x + iy \in \mathbb{C}$. This set is endowed with the Euclidean topology, and differential structure. This turns $\mathbb{H}$ into a manifold with a global chart given b the identity map. $\mathbb{H}$ is also endowed with the group law

$$(z, t)(z', t') = (z + z', t + t + \frac{1}{2}\text{Im}(z\bar{z}')), \tag{4.1}$$

or alternatively,

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(yx' - xy')). \tag{4.2}$$

It is worth pointing out that $\text{Im}(z\bar{z}') = -\omega(z, z')$ where $\omega : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ is the standard symplectic form in $\mathbb{C}$. It is not hard to check that this law turns $\mathbb{H}$ into a group with identity $(0, 0)$ and inverse $(z, t)^{-1} = (-z, -t)$. Since both the group product and inversion are polynomials in the coordinates, it follows that $\mathbb{H}$ is in fact a Lie group. The left invariant frame is computed by considering infinitesimal left translations in each coordinate,

$$\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0}(\epsilon,0,0)(x,y,t) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0}(x+\epsilon,y,t-\frac{1}{2}y\epsilon) = (1,0,-\frac{1}{2}y),\\
\frac{d}{d\epsilon}\Big|_{\epsilon=0}(0,\epsilon,0)(x,y,t) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0}(x,y+\epsilon,t+\frac{1}{2}x\epsilon) = (0,1,\frac{1}{2}x),\\
\frac{d}{d\epsilon}\Big|_{\epsilon=0}(0,0,\epsilon)(x,y,t) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0}(x,y,t+\epsilon) = (0,0,1).
\end{aligned}$$

From here we see that a frame for left invariant vector fields is given by

$$\begin{aligned}
X &= \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial t},\\
Y &= \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial t},\\
T &= \frac{\partial}{\partial t}.
\end{aligned} \tag{4.3}$$

The Lie algebra of $\mathbb{H}$, denoted by $\mathfrak{h}$, is given by $\mathfrak{h} = \operatorname{span}\{X, Y, T\}$. Perhaps the one of the most important properties of the $\mathbb{H}$ is that it satisfies the bracket relation $[X, Y] = T$, while all other brackets are zero. Setting $\mathfrak{h}_0 = \operatorname{span}\{X, Y\}$, and $\mathfrak{h}_1 = \operatorname{span}\{T\}$, we see that $\mathfrak{h}$ satisfies $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ with $\dim \mathfrak{h}_0 = 2$, $\dim \mathfrak{h}_1 = 1$, and $[\mathfrak{h}_0, \mathfrak{h}_0] = \mathfrak{h}_1$, $[\mathfrak{h}_0, \mathfrak{h}_1] = 0$, and $[\mathfrak{h}_1, \mathfrak{h}_1] = 0$. Loosely speaking, this tells us that the distribution $\mathfrak{h}_0$ encodes enough information to recover all of $\mathfrak{h}$, so that restricting movement in $\mathbb{H}$ to the distribution $\mathfrak{h}_0$ at every point still allows to connect any two points in $\mathbb{H}$. This is formalized as follows, a curve $\gamma : I \subset \mathbb{R} \to \mathbb{H}$ is said to be *horizontal* if it is absolutely continuous, and $\dot{\gamma}(s) \in \mathfrak{h}_0\big|_{\gamma(s)}$ for almost every $s \in I$. That is to say, there are bounded functions, $a, b : I \to \mathbb{R}$, such that for almost every $s \in I$

$$\dot{\gamma}(s) = a(s)X\big|_{\gamma(s)} + b(s)Y\big|_{\gamma(s)}. \tag{4.4}$$

By declaring X and Y to be orthogonal, we obtain a metric on $\mathfrak{h}_0$ which induces the horizontal path length

$$|\gamma|_{\mathfrak{h}_0} = \int_I \sqrt{a(s)^2 + b(s)^2}\, ds. \tag{4.5}$$

If we write γ in coordinates, $\gamma(s) = (x(s), y(s), t(s))$, then γ is horizontal if, and only if $\dot{\gamma} = \dot{x}(\frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial t}) + \dot{y}(\frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial t})$. At the same time $\dot{\gamma} = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{t}\frac{\partial}{\partial t}$. Therefore the horizontality condition (4.4) says that γ is horizontal if, and only if,

$$2\dot{t}(s) = x(s)\dot{y}(s) - y(s)\dot{x}(s). \tag{4.6}$$

This, in turn, induces a horizontal path-distance on $\mathbb{H}$ given for $p, q \in \mathbb{H}$ by

$$d_{cc}(p, p') = \inf\{|\gamma|_{\mathfrak{h}_0} : \gamma : [0,1] \to \mathbb{H} \text{ is horizontal and } \gamma(0) = p, \gamma(1) = p'\}. \tag{4.7}$$

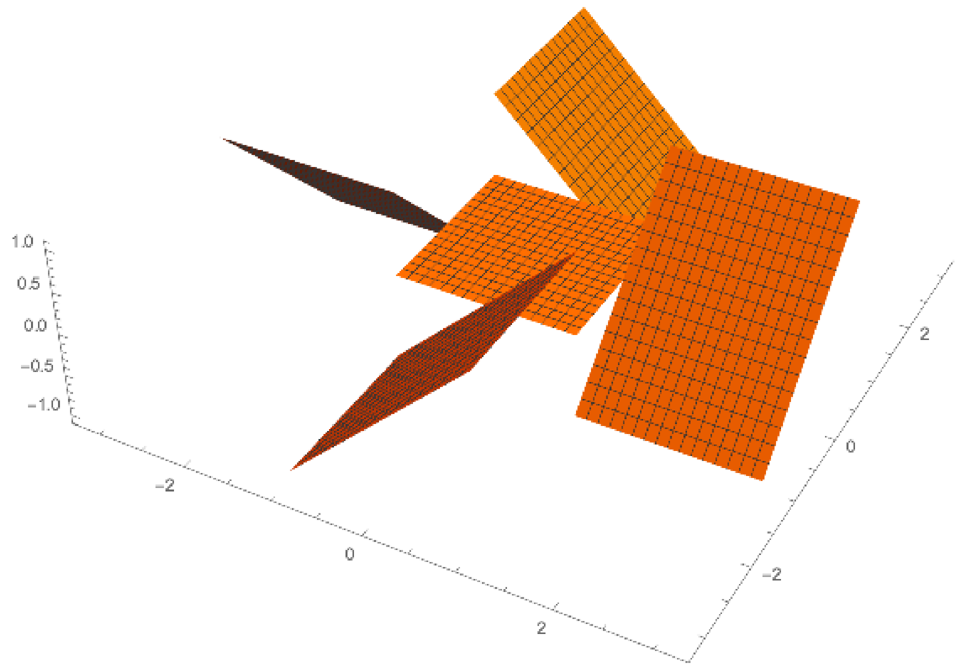

Figure 4.1: The horizontal distribution along the x and y axes

A rather remarkable fact is that, despite having positive co-dimension at every point, the distribution $\mathfrak{h}_0$ induces a finite distance. That is to say, $\mathbb{H}$ is horizontally path connected. This is a spacial case of a more general result of Chow and Rashevskii that I will be covering later. For the sake of completeness in the exposition, i present a proof of this special case.

Theorem 4.1. *Given any two points $p, p' \in \mathbb{H}$, there exists a horizontal path γ such that $\gamma(0) = p$ and $\gamma(1) = p'$*

Proof. We will first see that it is enough to show that any point in the t-axis is horizontally connected to the origin.

First note that since the frame $\{X, Y\}$ is left invariant, so is the horizontality condition. That is to say, denoting $L_q : \mathbb{H} \to \mathbb{H}$ the left translation map $L_q(p) = qp$, if γ is a horizontal path from p to p' then $L_q\gamma$ is a horizontal path from qp to qp'. So, without loss of generality, we may assume $p' = 0$.

Next we show that straight lines contained in $\mathbb{C} \times \{0\}$ passing through the origin, are horizontal. Indeed, any such line is given by $\ell_\theta(s) = (s\cos\theta, s\sin\theta, 0)$ for some $\theta \in [0, \pi]$ The horizontality condition is trivially checked as $s\cos\theta \frac{d}{ds}(s\sin\theta) - s\sin\theta \frac{d}{ds}(s\cos\theta) = 0$. It follows that, denoting $\pi : \mathbb{H} \to \mathbb{C}$ the map $\pi(z,t) = z$, for any point $p \in \mathbb{H}$, $(\pi(p), 0)$ is horizontally connected to the origin.

It is left to show that $(\pi(p), 0)$ and p are horizontally connected. The theorem will then follow by concatenation. Once again, appealing to left invariance, we may assume $\pi(p) = 0$ and $p \in \{(0,0,t) : t > 0\}$. In order to connect $(0,0,0)$ to $(0,0,t)$ consider the paths

$$\gamma_1(s) = \begin{cases} (2s\sqrt{t}, 0, 0), \ 0 \le s \le 1 \\ (0,0,0), \ \text{otherwise} \end{cases}$$

$$\gamma_2(s) = \begin{cases} (2\sqrt{t}, 2(s-1)\sqrt{t}, (s-1)t), \ 1 \le s \le 2 \\ (0,0,0), \ \text{otherwise} \end{cases}$$

$$\gamma_3(s) = \begin{cases} (2(3-s)\sqrt{t}, 2(s-3)\sqrt{t}, t), \ 2 \le s \le 3 \\ (0,0,0), \ \text{otherwise.} \end{cases}$$

It is not hard to check that each of them satisfy the horizontality condition (4.6), so that the path $\gamma : [0,1] \to \mathbb{H}$ given by $\gamma(s) = \gamma_1(3s) + \gamma_2(3s) + \gamma_3(3s)$ is horizontal, and satisfies $\gamma(0) = 0$ and $\gamma(1) = (0,0,t)$. This completes the proof. □

While this theorem shows that any two points can be connected by a horizontal curve, it gives no information about length minimizing curves, i.e. geodesics. It should be clear at this point that left translates of length minimizing curves are themselves length minimizing. So to understand geodesics in $\mathbb{H}$, it is enough to understand geodesics from the origin. Let $\gamma : [0,1] \to \mathbb{H}$ be a horizontal path from $(0,0,0)$ to (x_0, y_0, t_0). Write γ in coordinates as $\gamma(s) = (x(s), y(s), t(s))$. The horizontal length of γ agrees with the Euclidean length of the path $\pi \circ \gamma : \mathbb{H} \to \mathbb{R}^2$ from $(0,0)$ to $\pi(p) = (x_0, y_0)$. Hence minimizing $|\gamma|_{\mathbb{H}}$ is equivalent to minimizing $|\pi \circ \gamma|_{\mathbb{R}^2}$. Since γ is horizontal,

$$t(s) = \frac{1}{2}\int_0^s (x(\xi)\dot{y}(\xi) - y(\xi)\dot{x}(\xi))d\xi. \tag{4.8}$$

Therefore, the problem of finding horizontal length minimizing curves from $(0,0,0)$ to p in $\mathbb{H}$ can be restated as finding length minimizing paths $\Gamma = (x,y) : [0,1] \to \mathbb{R}^2$ with

$\Gamma(0) = (0, 0)$, $\Gamma(1) = (x_0, y_0)$ subjected to the constrain that

$$t_0 = \frac{1}{2} \int_0^1 (x(\xi)\dot{y}(\xi) - y(\xi)\dot{x}(\xi))d\xi.$$

If $t_0 = 0$, finding geodesics becomes trivial as straight lines satisfy the constrain. Assume $t_0 \neq 0$, and either $x_0 \neq 0$ or $y_0 \neq 0$. Let us take a closer look at this constrain. First, let $\Gamma' : I \to \mathbb{R}^2$ be the straight line between $(0, 0)$ and (x_0, y_0) and let $\widetilde{\Gamma}$ be the concatenation of Γ and Γ'. Note that $\widetilde{\Gamma}$ is a close loop bounding a compact region $D \subset \mathbb{R}^2$.

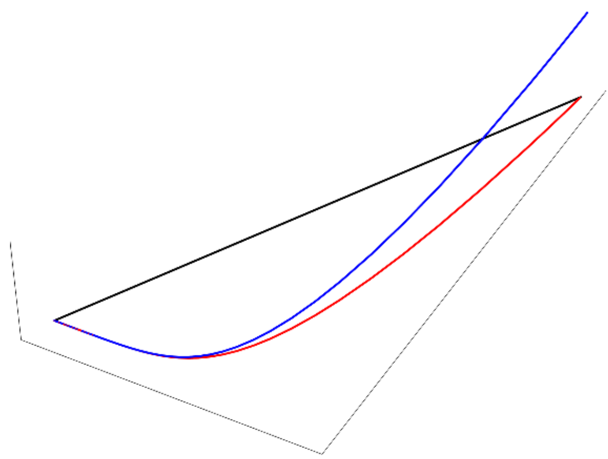

Figure 4.2: Horizontal curve shown in blue with its projection shown in red together with the straight line and bounded region in $\mathbb{R}^2$.

Let $P(x, y) = -y$ and $Q(x, y) = x$, so by Green's theorem

$$\int_{\widetilde{\Gamma}} Pdx + Qdy = \iint D\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \tag{4.9}$$

The right side of (4.9) becomes 2Area(D). Meanwhile, since Γ' is a straight line it can be written of the from $\Gamma'(x, y) = (x, mx)$, so that $P(\Gamma'(s))dx'(s) - Q(\Gamma'(s)dy'(s) = 0$. The left side of (4.9) becomes exactly

$$\int_0^1 (x(\xi)\dot{y}(\xi) - y(\xi)\dot{x}(\xi)d\xi).$$

The problem of finding horizontal geodesics can now be interpreted as minimizing the perimeter of the region D while keeping a fix area equal to t_0. The iso-perimetric inequality of $\mathbb{R}^2$ reveals that Γ must be an arc of a circle. Lastly, if $x_0 = y_0 = 0$, then $\pi \circ \gamma$ is a closed loop itself and using Γ' is not needed. Condition (4.8) still fixes the area bounded by $\pi \circ \gamma$ so the iso-perimetric problem still applies. In this case the length is minimized when $\pi \circ \gamma$ is a circle. Note, however, that there are infinitely many circles, containing the origin, with any given area. So in this case, geodesics are not unique. It follows that geodesics from the origin in $\mathbb{H}$ are either straight lines in $\mathbb{C} \times \{0\}$, lifts of arcs of circle, or lifts of full circles, lifted by defining $t(s)$ via (4.8). In the case of geodesics between the origin and a point $(0, 0, t_0)$ in the t- axis, although geodesics are not unique, they all have the same curvature. Indeed, since the circles $\pi \circ \gamma$ must bound a fix area of exactly t_0, they all must have the same radius, and therefore the same curvature. This curvature is explicitly given by $c = \left(\frac{t_0}{\pi}\right)^{-\frac{1}{2}}$. The structure of geodesics, as explained here, has also been explain in more details in several expository books in the subject, for instance [20], [83] and [10].

The structure of the Heisenberg group does not stop here, as it also admits a homogeneous structure. For each $r > 0$ consider the map $\delta_r : \mathbb{H} \to \mathbb{H}$, given by

$$\delta_r(z, t) = (rz, r^2 t). \tag{4.10}$$

It is not hard to see that this maps are group homomorphism, but moreover for any function $f \in \mathbb{C}^1(\mathbb{H})$, $X(f \circ \delta_r) = r(Xf) \circ \delta_r$, and $Y(f \circ \delta_r) = r(Yf) \circ \delta_r$. therefore, if γ is a horizontal path, so is $\delta_r \circ \gamma$ and, moreover, $|\delta_r \circ \gamma|_{\mathbb{H}} = r|\gamma|_{\mathbb{H}}$. It follows that the dilations $\{\delta_r : r > 0\}$ are homogeneous of degree 1 with respect to the Carnot-Caratheodory distance. That is, for $p, q \in \mathbb{H}$,

$$d_{cc}(\delta_r p, \delta_r q) = r d_{cc}(p, q). \tag{4.11}$$

The Heisenber group also admits a gauge norm which induces another left-invariant metric. The Kornáyi gauge norm $|| \cdot ||_{\mathbb{H}} : \mathbb{H} \to \mathbb{R}$ is given by

$$||(z, t)||_{\mathbb{H}}^4 = |z|^2 + 16t^2. \tag{4.12}$$

The Kornayi distance is then given by

$$d_{\mathbb{H}}(p, q) = ||q^{-1}p||_{\mathbb{H}}. \tag{4.13}$$

It is quite easy to see that $d_{\mathbb{H}}(p, q) = 0 \iff p = q$, and that $d_{\mathbb{H}}(p, q) = d_{\mathbb{H}}(q, p)$. In order to show that $d_{\mathbb{H}}$ is in fact a metric, one must show that it satisfies the triangle inequality. It is enough to show that the Korányi norm satisfies $||pq||_{\mathbb{H}} \leq ||p||_{\mathbb{H}} + ||q||_{\mathbb{H}}$. Put $p = (z, t)$, and $q = (w, s)$ and compute,

$$\begin{aligned}||pq||_{\mathbb{H}}^4 &= |z+w|^4 + 16(t+s+\frac{1}{2}\text{Im}(z\bar{w}))^2 \\ &= ||z+w|^2 + 4i(t+s+\frac{1}{2}\text{Im}(z\bar{w}))|^2 \\ &= ||z|^2 + 4it + 2z\bar{w} + 4is + |w|^2|^2 \\ &\leq (||p||_{\mathbb{H}}^2 + 2|z||w| + ||q||_{\mathbb{H}}^2)^2 \\ &= (||p||_{\mathbb{H}} + ||q||_{\mathbb{H}})^4.\end{aligned}$$

The left invariance of $d_{\mathbb{H}}$ follows directly from the definition. More can be said; it is quite clear form the explicit formula for $|| \cdot ||_{\mathbb{H}}$, that $||\delta_r p||_{\mathbb{H}} = r||p||_{\mathbb{H}}$. Thus, homogeneous dilations are also homogeneous of degree 1 with respect to the metric $d_{\mathbb{H}}$.

Theorem 4.2. *There exist a constant $C > 0$ such that for all $p, q \in \mathbb{H}$,*

$$\frac{1}{C} d_{cc}(p,q) \leq d_{\mathbb{H}}(p,q) \leq C d_{cc}(p,q).$$

Proof. Since both distances are left invariant, we may assume $q = 0$. Moreover, δ_r is homogeneous of degree 1 with respect to both of these distances, we may dilate by $d_{cc}(0,p)^{-1}$. Therefore, it is enough to show that there is a constant $C > 0$ such that for all $p \in \{p' \in \mathbb{H} : d_{cc}(p,0) = 1\} =: \mathbb{S}^2_{cc}$,

$$\frac{1}{C} \leq d_{\mathbb{H}}(0,p) = ||p||_{\mathbb{H}} \leq C.$$

Now, since $\mathbb{S}^2_{cc}$ is separated form the origin, both $||\cdot||_{\mathbb{H}}$ and $\frac{1}{||\cdot||_{\mathbb{H}}||}$ are continuous. Moreover, since $\mathbb{S}^2_{cc}$ is compact, it follows that both functions are bounded on it. Letting $C = \max\{\sup_{p \in \mathbb{S}^2_{cc}} ||p||_{\mathbb{H}}, \sup_{p \in \mathbb{S}^2_{cc}} \frac{1}{||p||_{\mathbb{H}}}\}$ the claim follows. ∎

This says that from a Lipschitz point of view, these two metrics are equivalent. When studying questions that are bi-Lipschitz invariant, like those regarding Hausdorff dimension, either metric can be used without altering the results. The Koranyi metric makes it quite easy to see the fractal nature of the Heisenberg group. If $d_{\mathbb{H}}$ is restricted to the t-axis, it coincides (up to a constant) with the square root of the Euclidean metric. Therefore the Hausdorff dimension of the t-axis, with respect to the metric $d_{\mathbb{H}}$ (or d_{cc}), is 2. In fact, more is true, the Hausdorff dimension of any $\mathcal{C}^1$ curve which is not horizontal, is 2 and the overall Hausdorff dimension of $\mathbb{H}$ is 4. This peculiarities, among others, make it a central object of study in analysis on metric spaces.

To summarize, the Heisenberg group is a non-commutative Lie group, with a nilpotent and graded Lie Algebra. It admits a Carnot-Caratheodory metric which makes is into a subRiemannian manifold. It also admits a homogeneous structure by the non isotropic dilations δ_r, and a gauge norm which induces a left invariant distance equivalent to the Carnot-caratheodory metric. Moreover, In addition to being an interesting space with extremely rich structure, it also arises very naturally in different contexts and areas of mathematics. In Section 4.3, I will discuss in detail one of these appearances, but before then, in the next section, I will talk about various types of spaces that generalize some of the properties of the Heisenberg group.

4.2 Generalizations of the Heisenber group

In this section I will go over several generalizations of the Heisenberg group. As noted before, $\mathbb{H}$ is a special case of some, more general, types of spaces. I will discuss some of these types spaces and explain how the Heisenberg group fits into a larger context regarding them. Some of the standard, expository, references in this regard are [20], and [56] among others. The first section of [72] also contains a short introduction to the specific topic of Carnot groups which I will be talking about in this section.

The first obvious generalization, and the one that will be most relevant throughout this thesis, is higher dimensional Heisenberg groups. The n$^{\text{th}}$ Heisenberg group is the smooth manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, also identified with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ in the usual way. I denote a typical point by $p = (z, t) \in \mathbb{C}^n \times \mathbb{R}$ or alternatively $p = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with $z = x + iy$. The group law now becomes

$$(z, t)(z', t') = (z + z', t + t' + \frac{1}{2} \sum_{j=1}^{n} \mathrm{Im}(z_j \bar{z}'_j)). \tag{4.14}$$

As with the group law in $\mathbb{H}$, the group law in $\mathbb{H}^n$ can be written as $(z, t)(z', t') = (z + z', t + t' - \frac{1}{2}\omega(z, z'))$. Written in real coordinates (4.14) rewrites as $(x, y, t)(x', y', t') = (x + x', y + y', t + t' + \frac{1}{2} \sum_{j=1}^{n}(y_j x'_j - x_j y'_j))$. There are now $2n + 1$ left invariant vector fields,

$$
\begin{aligned}
X_j &= \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial t} \\
Y_j &= \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial t} \\
T &= \frac{\partial}{\partial t},
\end{aligned}
\tag{4.15}
$$

with $j = 1, \ldots, n$. In this case, n of the brackets are non-trivial. indeed, one can check that for $j = 1, \ldots, n$ $[X_j, Y_j] = T$, while all other brackets are zero. Now, the horizontal distribution is defined as $\mathfrak{h}_0^n = \text{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\} \subset \mathfrak{h}^n = T\mathbb{H}^n$. The rest of the structure is defined in the same way as before. That is, the frame $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ is declared orthonormal, inducing a metric on $\mathfrak{h}_0$ which in turns induces a horizontal path length on $\mathbb{H}^n$. This path length is then used to define a left invariant, Carnoth-Caratheodory path distance which I will also denote by d_{cc}. In this higher dimensional case, the structure of geodesics is more complicated. Lines through the origin in, contained in $\mathbb{C}^n \times \{0\}$ are still geodesics but geodesics from zero to points with non-zero t coordinate are not determined by a simple application of the iso-perimetric inequality. A horizontal curve, γ, between the origin and (z, t) satisfies that the sum of the areas bounded by the projections of γ onto each of the $x_j y_j$-planes, and the straight line from zero to $\pi_{x_j y_j}(z)$, is equal to t. However, even for distance minimizing curves, these projections might have self intersections so this signed area has to account for this. I will skip over the details of this, however the structure of geodesics in $\mathbb{H}^n$ has been treated in many places, for instance [84], [13], and [1]. In addition, in [58], the authors give a proof using different, perhaps simpler, techniques to the ones used in the other 3 references.

$\mathbb{H}^n$ also admits a homogeneous structure induced by the same dilations as before, where now the $\mathbb{C}^n$ coordinates are scaled by r and the t coordinate is scaled by r^2. I keep the same notation, $\delta_r(z, t) = (rz, r^2 t)$. These dilations are group homomorphisms and homogeneous of degree one with respect to d_{cc}. The Korányi gauge takes the same form, $||(z, t)||_{\mathbb{H}^n}^4 = |z|^4 + 16t^2$. and induces a left invariant metric, $d_{\mathbb{H}^n}$, in the same way. As before, d_{cc} and $d_{\mathbb{H}^n}$ are bi-Lipschitz equivalent.

Throughout the rest of this thesis I will be working on $\mathbb{H}^n$ given that results proven here obviously apply to the first Heisenberg group. However, it should be noted that, in some situations, there are some differences between $\mathbb{H}$ and $\mathbb{H}^n$ for $n > 1$. For example subgroups which are closed under dilations and contain the t-axis are never horizontally path connected in $\mathbb{H}$ but they might be in $\mathbb{H}^n$.

Another way to generalize the structure of the Heisenberg group, in a way that keeps most of its properties, is with the notion of Carnot groups. These where first introduces by G. Folland in [49] under the name of “stratified groups”. The name “Carnot groups” was coined by P. Pansú in his doctoral thesis who later studied them extensively, for instance in [91].

Let G be a connected, simply connected Lie group. We say its Lie algebra, $\mathrm{Lie}(G) = \mathfrak{g}$, admits a stratification if there exists subspaces, $V_1, \dots, V_s \subset \mathfrak{g}$, such that

1. $\mathfrak{g} = V_1 \oplus + \cdots + \oplus V_s$
2. $V_{j+1} = [V_j, V_1]$, for $j = 1, \dots s-1$ and all other brackets are trivial.

We call this, a stratification of step s. A group whose Lie algebra admits a stratification is called *stratified group*. The second condition is known as “bracket generating condition” or “Horomander's condition” and, as we will see later, it plays a pivotal role in the theory of Carnot groups, and more generally subRiemannian spaces. If G is a stratified group one may define a distribution H (i.e. a subbundle of the tangent bundle) in a left invariant way by denoting by $L_p : G \to G$ the left translation by $p \in M$ and setting $H_p := (L_p)_* V_1$. The distribution H is referred to as the horizontal distribution. Choosing an inner product, g_{V_1}, in V_1 induces a left invariant norm in all of H via

$$||v||_{g_{V_1}} = g_{V_1}(L_p^{-1}v, L_p^{-1}v)^{1/2} \ (v \in H_p). \tag{4.16}$$

An absolutely continuous curve, $\gamma : [0,1] \to G$, is said to be horizontal if $\dot{\gamma}(t) \in H_{\gamma(s)}$ for almost every s. The choice of g_{V_1} induces a length on horizontal curves by

$$|\gamma|_{g_{V_1}} = \int_0^1 ||\dot{\gamma}(s)||_{g_{V_1}} ds. \tag{4.17}$$

In turn, this horizontal curve length induces a left invariant path distance on G, given by

$$d_{g_{V_1}}(p,q) = \inf\{|\gamma|_{g_{V_1}} : \gamma \in \mathcal{C}^1([0,1], G), \dot{\gamma} \in H \text{ and}, \gamma(0) = p, \gamma(1) = q\} \tag{4.18}$$

The metric space $(G, d_{g_{V_1}})$ is called a Carnot group, sometimes referred to in the literature as a subRiemannian Carnot group. If instead of choosing an inner product on V_1 one simply chooses a norm, $|| \cdot ||_{V_1}$, the rest of construction follows through in the same way, also giving raise to a left invariant, constrained, path distance distance on G. The resulting metric space is referred to in some of the literature as a subFinsler Carnot group. Much of the literature however, uses the blanket term "Carnot group" to refer to stratified groups, even before a choice of norm in the first layer of the stratification.

The second condition in the definition of stratified Lie algebra, implies that stratified Lie algebras are nilpotent. This in turn implies that the Baker-Campbell-Hausdorff formula is a polynomial, and not an infinite series. As a consequence, for Carnot groups, the exponential map $exp : \mathfrak{g} \to G$ is a global diffeomorphism. This allows us to endow G with homogeneous dilations, very much in the same way as done in $\mathbb{H}^n$. Specifically, for $r > 0$ one defines $\tilde{\delta}_r : \mathfrak{g} \to \mathfrak{g}$, by defining $\tilde{\delta}_r X = r^j X$ for $X \in V_j$ and extending it linearly to all of $\mathfrak{g}$. Then the dilations $\delta_r : G \to G$ are given by $\delta_r(p) = exp \circ \tilde{\delta}_r \circ exp^{-1}(p)$. By exponentiating each layer of the stratification, elements of G can be written in coordinates $(v_1, \ldots, v_s)$ such that $exp^{-1}(0, \ldots, v_j, \ldots, 0) \in V_j$. Dilations can be express explicitly in these coordinates as $\delta_r(v_1, \ldots, v_s) = (rv_1, r^2 v_2, \ldots, r^s v_s)$. By definition, for $X \in V_1$ and $f \in \mathbb{C}^1(G)$, $X(f \circ \delta_r) = r(Xf) \circ \delta_r$. It follows that the group homomorphisms δ_r, are homogeneous of degree 1 with respect to the metric $d_{g_{V_1}}$.

It should be clear at this point that $\mathbb{H}^n$ is a specific example of a Carnot group where $\mathfrak{h}^n$ admits a step 2 stratification with $V_1 = \text{span}\{X_j, Y_j : j = 1, \ldots, n\}$, and $V_2 = \text{span}\{T\}$. Heisenberg groups are considered to be the simplest non-trivial examples of Carnot group and many of the research questions considered in Carnot groups can be solved in $\mathbb{H}^n$ long before general solutions, applicable to all Carnot groups, are found. Examples of such problems are, smoothness of geodesics (e.g. [107], [85]), classification of monotone sets (e.g. [31], [86]), bi-Lipschitz embeddings into Banach spaces (e.g. [22], [23]), among many others. The theory of Carnot groups continues to be a very active area of research with many problems in many different areas including geometric analysis, differential equations, conformal theory, mapping theory, control theory, etc. Their rich structure makes them very useful and interesting objects of study.

The last generalization that I will go into some details on, is that of subRiemannian manifolds. subRiemannian manifolds are similar (as manifolds) to Carnot groups, but lack a group structure. Therefore, some other differences arise. A subRiemannian manifold is a triple (M, H, g) where M is a smooth manifold, H is a bracket generating distribution (i.e. subbundle) of TM and g is a choice of smoothly varying inner product on H. A distribution, $H \subset TM$ is said to be bracket generating if there are independent vector fields $V_1, \ldots, V_d \in H$ such that at each $p \in M$,

1. $H_p = \text{span}\{V_{j_1}, V_{j_2}, \ldots, V_{j_{k_p}}\}$
2. $T_pM = \text{span}\{V_{j_1}, [V_{j_2}, V_{j_1}], [V_{j_3}, [V_{j_2}, V_{j_1}]], \ldots\}$,

where $1 \leq j_1, \ldots, j_{k_p} \leq d$, and k_p is an integer depending on p

In contrast with Carnot groups, the horizontal distribution does not need to have constant rank. That is to say, $\dim H_p$ may vary with p. If $\dim H_p$ is constant, (M, H, g) is said to be equiregular. On Carnot groups, since $HpM = (L_p)_* V_1$, the horizontal distribution has constant rank. Just as in the case of Carnot groups, subRiemannian manifolds inherit a "constrained" path metric induced by g. An absolutely continuous path $\gamma : [0, 1] \to M$ is horizontal if $\dot{\gamma}(\xi) \in H_{\gamma(\xi)}$ for almost every $\xi \in [0, 1]$. The horizontal length of γ is defined as

$$|\gamma|_{M,g} = \int_0^1 ||\dot{\gamma}(\xi)||_{g_{\gamma(\xi)}} d\xi.$$

Then, the distance between any two points in M is by the minimal horizontal path length between them. As mentioned earlier, it is remarkable that despite H_p potentially having positive co-dimension at every point, any two points in the manifold can be connected by horizontal path. This is exactly the contents of Chow–Rashevskii theorem ([26]) which states that if H is a bracket generating distribution on a connected smooth manifold then any two points are connected by a H-horizontal path. This theorem justifies the definition of the distance on Carnot groups via (4.18) but applies more generally to any subRiemannian manifold.

These spaces are heavily studied as they have direct connections to many of the problems talked about before, like the problem of smoothness of geodesics and Lipschitz embeddability. These spaces also have a great deal of applications in control theory.

There are many other types of spaces that generalize the properties of $\mathbb{H}^n$. For instance, subFinsler manifolds (see [27]), and scalable groups (see [73]), among others. These further generalization will not play any role in this thesis so no further details are included.

4.3 The Heisenberg Group in Quantum Mechanics

In this section I discuss how the Heisenberg group arises naturally in quantum physics. This will lay the ground work for discussing the representation theory of $\mathbb{H}^n$. Many of the details are skipped as the quantum mechanical technicalities will not play a role in my work. There are, however, many places in the literature that discuss the important connection between quantum theory and the Heisenberg group. For instance [103], [104], [50], as well as [109] and [102].

In physics, an "observable" of a physical system is a physical quantity that can be measured. For example, if the system is a free particle moving in n dimensions, position and momentum are two of many observables. In the classical approach, an observable is a real valued function defined on the space of all possible states of the system. However, in quantum mechanics the possible states of such a particle are encoded in its wave function. This function belongs to a Hilbert space representing the state space, i.e. the space of all possible states for the system. In this approach, observables are linear operators acting on this Hilbert space. For example, our spin-less, free particle in n dimensions has a time-dependent wave function, $\psi(\cdot,t) \in L^2(\mathbb{R}^n)$, normalized so that

$$||\psi(\cdot,t)||_{L^2}^2 = \int_{\mathbb{R}^n} |\psi(\xi,t)|^2 d\xi = 1 \text{ for all } t \in \mathbb{R}.$$

The probability of finding the particle in a region $D \subset \mathbb{R}^n$ at a time t, is given by

$$\int_D |\psi(\xi,t)|^2 d\xi.$$

The expected value of the position of the particle at time t is then given by,

$$\int_{\mathbb{R}^n} \xi|\psi(\xi,t)|^2 d\xi.$$

Since

$$\int_{\mathbb{R}^n} \xi|\psi(\xi,t)|^2 d\xi = \int_{\mathbb{R}^n} \xi\psi(\xi,t)\overline{\psi(\xi,t)} d\xi = \int_{\mathbb{R}^n} \overline{\psi(\xi,t)} \sum_{j=1}^{n} \xi_j \psi(\xi,t)\mathbf{e}_j d\xi,$$

where $\mathbf{e}_j$ is the jth coordinate unit vector, this leads to the consideration of the jth coordinate position operator

$$Q_j\psi(\xi,t) = \xi_j\psi(\xi,t). \tag{4.19}$$

Of similar interest is the momentum operator. For this, we look at the plane-wave solution to Schödinger's equation, which is

$$\varphi(\xi,t) = e^{\frac{i}{\hbar}(\xi\cdot p - Et)},$$

where $\xi \in \mathbb{R}^n$ is the position of the particle, p is the momentum vector, E is the energy, and $\hbar$ is the reduced Planck constant which we, unashamedly, set equal to 1. These functions are not in $L^2(\mathbb{R}^n)$, however they form the "so-called" momentum basis. The momentum basis is not a basis in the usual sense but, via the Fourier transform, one can still express any $L^2(\mathbb{R}^n)$ wave function in terms of these plane waves. We skip these details and work with these plane waves formally. It is clear that

$$\frac{\partial}{\partial \xi_j}\varphi(\xi,t) = ip_j\varphi(\xi,t),$$

which leads us to consider the jth coordinate momentum operator

$$D_j\varphi(\xi,t) = -i\frac{\partial}{\partial \xi_j}\varphi(\xi,t).$$

These two observables will play an important role in the subsequent discussion, so for the sake of simplicity, and to formalize the discussion, consider the Hilbert space $L^2(\mathbb{R}^n)$ and think of the unbounded operators Q_j, and D_j as defined on suitable subspaces of $L^2(\mathbb{R}^n)$, say for instance the Schwartz space, although one could be more general. Also consider the operators,

$$Q = \sum_{j=1}^{n} Q_j\mathbf{e}_j, \text{ and}$$
$$D = \sum_{j=1}^{n} D_j\mathbf{e}_j.$$

For $x \in \mathbb{R}^n$ one can check that,

$$\exp(ix\cdot Q)\psi(\xi) = \sum_{k=1}^{\infty}\frac{(ix\cdot\xi)^k}{k!}\psi(\xi) = e^{ix\cdot\xi}\psi(\xi) =: [e(x)\psi](\xi).$$

Similarly, it is possible to compute $\exp(iy\cdot D)$ explicitly. First note that $D\psi = -i\nabla\psi$, denote by $\tau(y)$ the operator of translation by y, $[\tau(y)\psi](\xi) = \psi(\xi+y)$, then from the definition of the gradient,

$$\lim_{y\to 0}\frac{|[\tau(y)\psi](\xi) - \psi(\xi) + [iy\cdot D\psi](\xi)|}{||y||}. \tag{4.20}$$

Since for any positive integer N, $\tau(y) = \tau(\frac{y}{N})^N$, it follows from (4.20) that

$$\tau(y) = \lim_{N\to\infty}\tau(\frac{y}{N})^N = \lim_{N\to\infty}(1 - \frac{iy\cdot D}{N})^N = \exp(iy\cdot D).$$

So, the position and momentum operators each generate a one parameter group of unitary operators, $\{e(x) : x \in \mathbb{R}^n\}$, and $\{\tau(y) : y \in \mathbb{R}^n\}$ respectively. These families of operators do not commute with one another. Indeed,

$$[e(x)\tau(y)\psi](\xi) = e^{ix\cdot\xi}\psi(\xi + y),$$

whereas,

$$[\tau(y)e(x)\psi](\xi) = e^{ix\cdot(\xi+y)}\psi(\xi + y) = e^{ix\cdot y}e^{ix\cdot\xi}\psi(\xi + y),$$

so that

$$\tau(y)e(x) = e^{ix\cdot y}e(x)\tau(y). \tag{4.21}$$

This means that the family of operators $\{e(x)\tau(y) : x, y \in \mathbb{R}^n\}$ does not form a group. This family, however, can be extended to a group by adding the family $\{\chi(t) : t \in \mathbb{R}\}$ given by

$$\chi(t)\psi(\xi) = e^{it}\psi(\xi).$$

This is to say, $\{\chi(t)e(x)\tau(y) : x, y \in \mathbb{R}^n,\ t \in \mathbb{R}\}$ is a group of unitary operators. Indeed,

$$\begin{aligned}
&\chi(t)e(x)\tau(y)\chi(t')e(x')\tau(y') \\
&= \chi(t)\chi(t')e(x)e(x')\tau(y)\tau(y')e^{ix'\cdot y} \\
&= \chi(t + t' + x' \cdot y)e(x + x')\tau(y + y').
\end{aligned}$$

Thus, this last family of unitary operators forms the group $\mathbb{R}^{2n+1}$ with law

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + x' \cdot y). \tag{4.22}$$

This non-commutative group is known as the polarized Heisenberg group, and denoted $\mathbb{H}^n_{\text{pol}}$. Now, the group law in (4.22) is not symmetric with respect to the variables so instead, one may consider the joint exponential $\exp(ix\cdot Q+iy\cdot D)$. In a way similar to the preceding computation, one may check that $\exp(ix\cdot Q+iy\cdot D) = e^{i\frac{x\cdot y}{2}}e(x)\tau(y) =: \rho(x, y)$. Considering the family $\{\rho(x, y, t) = \chi(t)\pi(x, y) : x, y \in \mathbb{R}^n,\ t \in \mathbb{R}\}$, gives

$$\rho(x, y, t)\rho(x', y', t') = \rho(x + x', y + y', t + t' - \frac{1}{2}(xy' - yx')),$$

which, recalling (4.14), tells us that this group of unitary operators is isomorphic to $\mathbb{H}^n$. In fact, the groups $\mathbb{H}^n_{\text{pol}}$ and $\mathbb{H}^n$ are isomorphic as Carnot groups. This way, the Heisenberg group arises as the phase space of a quantum system where position and momentum are the observables of interest. This also leads to the consideration of the maps $\rho_\lambda : \mathbb{H}^n \to \mathcal{B}(L^2(\mathbb{R}^n))$ given by,

$$\rho_\lambda(x, y, t)\psi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + \frac{x\cdot y}{2})}\psi(\xi + y). \tag{4.23}$$

By the previous discussion, it follows that for each $\lambda \in \mathbb{R}^* : \mathbb{R}\backslash\{0\}$, ρ_λ is a homomorphism, in fact $\{\rho_\lambda\}_\lambda$ give a complete description of the representations of $\mathbb{H}^n$. This will be covered in more detail in a later section.

In view of the commutation relation between $e(x)$ and $\tau(y)$, it is relevant at this point to state Stone-VonNeumann's theorem.

Theorem 4.3. *Let $\{U(x) : x \in \mathbb{R}^n\}$ and $\{V(y) : y \in \mathbb{R}^n\}$ be two one-parameter groups of unitary operators. If for every $x, y \in \mathbb{R}^n$,*

$$V(y)U(x) = e^{ix\cdot y}U(x)V(y),$$

then $V(y)$ and $U(x)$ are unitarily equivalent to $\tau(y)$ and $e(x)$. That is to say, $\exists$ unitary operator W on $L^2(\mathbb{R}^n)$ such that

$$\begin{aligned} W^*V(y)W &= \tau(y) \\ W^*U(x)W &= e(x). \end{aligned}$$

There are many equivalent versions, and thus many proofs, of this influential result. For statements and proofs most related to the structure of $\mathbb{H}^n$ as discussed here refer to [103] and [50]. There is a lot more that can be said about the role of the Heisenberg group in quantum mechanics and I will come back to it in a later section when discussing its Fourier theory. For further details refer to the references mentioned at the beginning of the section.

CHAPTER 5

PROJECTION AND SLICING THEOREMS IN THE HEISENBERG GROUP

In $\mathbb{H}^n$, the homogeneous structure induced by dilations, allows the definition of "subspace-like" subgroups and thus projections. In this chapter I will review some of the work done regarding these projections, as well as slicing theorems related to them. As discussed earlier, Hausdorff dimension defined with respect to the Heisenberg distance is different to that define with respect to the Euclidean distance in $\mathbb{R}^{2n+1}$. However, as we will see, there is a close interplay between these and they are both used in most contexts involving dimension distortion. In order to avoid confusion I will be using the notation $\dim_E$ and $\dim_{\mathbb{H}}$ to denote Euclidean and Heisenberg Hausdorff dimensions respectively. Perhaps the most important relation between the two is the following dimension comparison ([8, 9]).

Theorem 5.1 (Dimension Comparison Principle)**.** *Let $A \subset \mathbb{H}^n$ and set $\alpha := \dim_E A$, $\beta := \dim_{\mathbb{H}} A$. Then*

$$\max\{\alpha, 2\alpha - 2n\} \leq \beta \leq \min\{2\alpha, \alpha + 1\}. \tag{5.1}$$

Moreover, for any pair of numbers (α, β) satisfying (5.1)*, there is a bunded Borel set E such that $\dim_E E = \alpha$ and $\dim_{\mathbb{H}} E = \beta$.*

This principle can be applied to great effect when studying dimension distortion by homogeneous projections.

5.1 Homogeneous projections and vertical plane sections

The parabolic dilations defined as in (4.10), give $\mathbb{H}^n$ a homogeneous structure, so it makes sense to talk about homogeneous subgroups.

Definition 5.1. *A subgroup $G \subset \mathbb{H}^n$ is homogeneous if*

$$\delta_r(G) \subset G \text{ for all } r > 0.$$

These subgroups are analogous to vector subspaces of $\mathbb{R}^n$. Homogeneous subgroups come in 2 kinds, those that are completely contained in $\mathbb{C}^n \times \{0\}$ and those that contain all of the t-axis. First I discuss the first kind.

Let $G \subset \mathbb{C}^n \times \{0\}$ be a homogeneous subgroup, and $(z,0),(w,0) \in G$. Then one must have that

$$(z,0)(w,0) = (z+w, \frac{1}{2}(z,w)) \in G.$$

From here we see 2 things. Firstly, $z+w \in \pi(G)$ so $\pi(G) \subset \mathbb{C}^n$ must be closed under addition. Moreover, since $\delta\lfloor_{\mathbb{C}^n \times \{0\}}$ is the classical scalar multiplication by r, it follows that $\pi(G)$ must be a linear subspace of $\mathbb{C}^n$. Secondly, we must have $\omega(z,w) = 0$, so $\pi(G)$ must be an isotropic linear subspace of $\mathbb{C}^n = \mathbb{R}^{2n}$. It is not hard to check that the opposite is true. If $V \in G_h(2n,m)$ then $\mathbb{V} = V \times \{0\} \subset \mathbb{H}^n$ is a homogeneous subgroup. For this reason, hereafter, I use the notation $\mathbb{V}$ to refer to the homogeneous subgroup corresponding to $V \in G_h(2n,m)$. These types of homogeneous subgroups are called "horizontal" subgroups.

Now suppose the homogeneous subgroup G is not contained in $\mathbb{C}^n \times \{0\}$, so there is a point $p = (x,y,t) \in G$ with $t \neq 0$. Then

$$\delta_{\frac{1}{2}}[(x,y,t)(x,y,t)](-x,-y,-t) = (0,0,\frac{1}{2}t).$$

So, by dilation and inversion, G must contain all points of the form $(0,0,t)$ with $t \in \mathbb{R}$. The t-axis itself is a homogeneous subgroup. Any other homogeneous subgroup of this kind, must also be a linear subspace of of $\mathbb{R}^{2n+1}$. This is simply because for $(x,y,t)(u,v,s) \in G$,

$$(x+u, y+v, t+s) = (0,0,\frac{1}{2}(x \cdot v - y \cdot u))(x,y,t)(u,v,s) \in G.$$

These homogeneous subgroups are known as "vertical" subgroups. vertical subgroups are normal in $\mathbb{H}^n$, in fact the t-axis is the center of $\mathbb{H}^n$.

In studying the metric geometry of $\mathbb{H}^n$, greater importance is given to vertical subgroups of the form $V^\perp \times \mathbb{R}$ where $V \in G_h(2n,m)$, in other words, $\mathbb{V}^\perp$ where $\mathbb{V}$ is a horizontal subgroup. This is because for any given $V \in G_h(2n,m)$, $\mathbb{H}^n$ admits semi-direct splittings

$$\mathbb{H}^n = \mathbb{V} \ltimes \mathbb{V}^\perp, \text{ and } \mathbb{H}^n = \mathbb{V}^\perp \rtimes \mathbb{V}.$$

That is to say, every point $p \in \mathbb{H}^n$ can be written, in a unique way, as a product $p_{\mathbb{V}} p_{\mathbb{V}^\perp}$, or $p'_{\mathbb{V}^\perp} p_{\mathbb{V}}$ depending on the order of the splitting. This defines three different projection maps, $P_{\mathbb{V}} : \mathbb{H}^n \to \mathbb{V}$ and $P^L_{\mathbb{V}^\perp}, P^R_{\mathbb{V}^\perp} : \mathbb{H}^n \to \mathbb{V}^\perp$, given by "reading off" the corresponding component. The horizontal projection map coincides with the Euclidean orthogonal projection map $P_{V\times\{0\}} : \mathbb{R}^{2n+1} \to V \times \{0\}$, while the vertical projections are given by

$$P^L_{\mathbb{V}^\perp}(p) = pP_{\mathbb{V}}(p)^{-1}, \text{ and } P^R_{\mathbb{V}^\perp}(p) = P_{\mathbb{V}}(p)^{-1}p.$$

Due to the non-commutativity of $\mathbb{H}^n$, the two vertical projection maps are actually different, and behave differently with respect to the metric $d_{\mathbb{H}^n}$. The maps $P^L_{\mathbb{V}^\perp}$ are, in a sense, more natural in the structure of the Heisenberg group. This is simply because the geometric structure of $\mathbb{H}^n$ is constructed to be left invariant and the fibers if $P^L_{\mathbb{V}^\perp}$ are left cosets of the horizontal subgroup $\mathbb{V}$. In particular, the fibers of $P^L_{\mathbb{V}^\perp}$ are horizontal curves and therefore are metrically identical with the same Hausdorff dimension. This is not the case with the fibers of the map $P^R_{\mathbb{V}^\perp}$ which are right cosets of $\mathbb{V}$. For this reason, the maps $P^L_{\mathbb{V}^\perp}$ have a prominent presence in the literature having been studied in the context of dimension distortion ([7, 6]), Sobolev mappings ([5]), uniform measures ([25]), and intrinsic rectifiability ([82, 24]) among others. In contrast, up until recently ([61]), the maps $P^R_{\mathbb{V}^\perp}$ were nearly absent from the literature, appearing only in the context of iso-perimetric problems in the Grushin plane, where a connection with $\mathbb{H}^n$ appears by way of these maps ([2]).

The specific problem which is most relevant to my work is that of Hausdorff dimension distortion by homogeneous projections. Let us first discuss the case of the family of horizontal projections $\{P_{\mathbb{V}} : V \in G_h(2n,m)\}$. Since $\omega\lfloor_{V\times V}\equiv 0$ for all $V \in G_h(2n,m)$, it follows that $d_{\mathbb{H}^n}\lfloor_{\mathbb{V}}\equiv d_E\lfloor_V$. Therefore, for any set $A \subset \mathbb{H}^n$, $\dim_{\mathbb{H}} P_{\mathbb{V}}(A) = \dim_E P_V(A)$. Moreover, $P_{\mathbb{V}} = P_V \circ \pi$ and since we already know the dimension distortion (or lack thereof) caused by the family $\{P_V : V \in G_h(2n,m)\}$, the question of dimension distortion by $\{P_{\mathbb{V}} : V \in G_h(2n,m)\}$ can be answer by studying the distortion caused by the map $\pi : (\mathbb{H}^n, d_{\mathbb{H}^n}) \to (\mathbb{R}^{2n}, d_E)$.

Theorem 5.2. *For $A \subset \mathbb{H}^n$,*

$$\dim_{\mathbb{H}} A - 2 \le \dim_E \pi(A) \le \dim_{\mathbb{H}} A.$$

Proof. The map π is 1-Lipchitz. Indeed,

$$d_E(\pi(z,t),\pi(w,s)) = |z-w| \le [|z-w|^4 + 16(t-s-\frac{1}{2}\omega(z,w))^2]^{\frac{1}{4}} = d_{\mathbb{H}^n}((z,t),(w,s)).$$

Therefore, by (2.3) the upper bound follows.
The original proof of the lower bound, in [7], uses a covering argument. Here i prove it using energies. If $\dim_{\mathbb{H}} A \leq 2$ then the lower bound is trivial, so assume $\dim_{\mathbb{H}} A > 2$ and pick $2 < s < \dim_{\mathbb{H}} A$. By Frostman lemma there is a measure $\mu \in \mathcal{M}(A)$ such that $\mu(B_{\mathbb{H}^n}(p, r)) \lesssim r^s \ \forall \ p \in \mathbb{H}^n$, and $r > 0$. Since $\pi_\# \mu \in \mathcal{M}(\pi(A))$, the aim is to show that for any $0 < \sigma < s - 2$, $I_\sigma(\pi_\# \mu) < \infty$. By definition of the push-forward measure,

$$I_\sigma(\pi_\# \mu) = \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} |\pi(q) - \pi(p)|^{-\sigma} \, d\mu(q) \, d\mu(p)$$

Since $\operatorname{spt}(\mu)$ is compact, we can fix $R > 0$ such that $\operatorname{spt}(\mu) \subset B_E(0, R)$. For $z \in \mathbb{R}^{2n}$ the set $\{q \in \mathbb{H}^n : |\pi(q) - z| \leq r\}$ is a cylinder with radius r, so $\{q \in \mathbb{H}^n : |\pi(q) - z| \leq r\} \cap \operatorname{spt}(\mu) \subset B_E^{2n}(z, r) \times [-R, R]$. This cylinder can be covered by at most $\lceil Cr^{-2} \rceil$ balls of radius r, where $C = C(n, R)$ is independent of z and r ([7, Lemma 6.5]). It follows that $\mu(\{q \in \mathbb{H}^n : |\pi(q) - z| \leq r\}) \lesssim r^{s-2}$. Therefore,

$$\begin{aligned}
\int_{\mathbb{H}^n} |\pi(q) - z|^{-\sigma} d\mu(q) &= \int_0^\infty \mu\left(\left\{q \in \mathbb{H}^n : |\pi(q) - z| \leq r^{-1/\sigma}\right\}\right) \, dr \\
&= \int_0^1 \mu\left(\left\{q \in \mathbb{H}^n : |\pi(q) - z| \leq r^{-1/\sigma}\right\}\right) \, dr \\
&\quad + \int_1^\infty \mu\left(\left\{q \in \mathbb{H}^n : |\pi(q) - z| \leq r^{-1/\sigma}\right\}\right) \, dr \\
&\lesssim \mu(\mathbb{H}^n) + \int_1^\infty r^{\frac{2-s}{\sigma}} \, dr.
\end{aligned}$$

Since $\sigma < s - 2$ it follows that $\int_1^\infty r^{\frac{2-s}{\sigma}} \, dr < \infty$. This tells us that

$$\int_{\mathbb{H}^n} \int_{\mathbb{H}^n} |\pi(q) - \pi(p)|^{-\sigma} \, d\mu(q) \, d\mu(p) \lesssim \mu(\mathbb{H}^n) \left(\mu(\mathbb{H}^n) + \int_1^\infty r^{\frac{2-s}{\sigma}} \, dr \right) < \infty.$$

Since σ and s are chosen arbitrarily close to $\dim_{\mathbb{H}} A$ the lower bound follows. □

From here, understanding dimension distortion by horizontal projections is quite simple.

Corollary 5.1. *Let $A \subset \mathbb{H}^n$,*

1. *If $\dim_{\mathbb{H}} A \leq m + 2$ then $\dim_{\mathbb{H}} A - 2 \leq \dim_E P_{\mathbb{V}}(A) \leq \dim_{\mathbb{H}} A$ for $\mu_{2n,m}$-almost every $V \in G_h(2n, m)$.*

2. *If $\dim_{\mathbb{H}} A > m + 2$ then $\mathcal{H}^m(P_{\mathbb{V}} A) > 0$ for $\mu_{2n,m}$-almost every $V \in G_h(2n, m)$.*

3. *If $\dim_{\mathbb{H}} A > 2m + 2$ then $Int(P_{\mathbb{V}} A) \neq \varnothing$ for $\mu_{2n,m}$-almost every $V \in G_h(2n, m)$.*

The proof follows from applying the analogous theorem for isotropic planes to the set $\pi(A)$. Moreover, projecting the t-axis shows that this theorem is sharp.

The family of vertical projections $\{P^L_{\mathbb{V}^\perp} : V \in G_h(2n, m)\}$ were first considered in the context of dimension distortion in [6, 7]. This problem is much more complicated and continues to be an active area of research. For each $V \in G_h(2n, m)$, the map $P^L_{\mathbb{V}^\perp}$ is not a group homomorphism, nor linear, and it is not Lipschitz. They are locally $\frac{1}{2}$-Hölder, so in principle these map could not only decrease Hausdorff dimension in some directions, but they could also increase it. The following example shows that this is in fact the case.

Example 5.1. *In $\mathbb{H}$, let $\mathbb{V}_0 = \{(x, 0, 0) : x \in \mathbb{R}\}$, so that $\mathbb{V}_0^\perp = \{(0, y, t) : y, t \in \mathbb{R}\}$. Let S be the segment of the horizontal line $S = \{(0, s, 0) : s \in [0, 1]\}$ and A be the left translate of S by $(1, 0, 0)$; that is $A = \{(1, s, -\frac{s}{2}) : s \in [0, 1]\}$. Since A is a segment of a horizontal curve, $\dim_{\mathbb{H}} A = 1$. However, it is not hard to see that*

$$P^L_{\mathbb{V}_0^\perp} A = \{(0, s, -s) : s \in [0, 1]\}$$

is not contained in any horizontal curve, therefore $\dim_{\mathbb{H}} P_{\mathbb{V}_0^\perp} A = 2$. In fact, one can check that $\dim_{\mathbb{H}} P_{\mathbb{V}^\perp} A = 2$ for all but one choice of $\mathbb{V}^\perp$.

Despite the difficulties that this family of maps presents, significant progress has been made towards bounding the almost sure dimension of $P^L_{\mathbb{V}^\perp} A$ in terms of the dimension of A. The following theorem shows the best known almost sure bounds, it is a combination of the results obtained in [7] and the improvements made recently in [61]

Theorem 5.3. *Let $A \subset \mathbb{H}^n$ then*

$$\dim_{\mathbb{H}^n} P^L_{V^\perp}(A) \leq \begin{cases} 2\dim_{\mathbb{H}} A & \text{if } \dim_{\mathbb{H}} A \in [0, 1] \\ \dim_{\mathbb{H}} A + 1 & \text{if } \dim_{\mathbb{H}} A \in [1, 2n - m] \\ \frac{\dim_{\mathbb{H}} A - m}{2} + n + 1 & \text{if } \dim_{\mathbb{H}} A \in [2n - m, 2n + 2 - m] \\ 2n + 2 - m & \text{if } \dim_{\mathbb{H}} A \in [2n + 2 - m, 2n + 2]. \end{cases}$$

and,

$$\dim_{\mathbb{H}^n} P^L_{V^\perp}(A) \geq \begin{cases} \dim_{\mathbb{H}^n} A & \textit{if } \dim_{\mathbb{H}^n} A \in [0,1] \\ 1 & \textit{if } \dim_{\mathbb{H}^n} A \in [1,2] \\ \dim_{\mathbb{H}^n} A - 1 & \textit{if } \dim_{\mathbb{H}^n} A \in [2, 2n-m+1] \\ 2n-m & \textit{if } \dim_{\mathbb{H}^n} A \in [2n-m+1, 2n+1] \\ 2(\dim_{\mathbb{H}^n} A - n - 1) - m & \textit{if } \dim_{\mathbb{H}^n} A \in [2n+1, 2n+2], \end{cases}$$

for $\mu_{2n,m}$*-almost every* $V \in G_h(2n,m)$*.*

Better almost sure lower bounds exist in the first Heisenberg group [44, 59]. It is also conjectured that $\dim_{\mathbb{H}} P^L_{\mathbb{V}^\perp} A \geq \dim_{\mathbb{H}} A$ for $\mu_{2n,m}$-almost every $V \in G_h(2n,m)$, but this remains an open problem. Dimension distortion by the family $\{P^R_{\mathbb{V}^\perp} : V \in G_h(2n,m)\}$ will be discussed in Section 5.3.

The problem of planar slices of sets has also been studied in the Heisenberg group. Because the dimension distortion by horizontal projections is well understood, It is possible to figure out the dimension of slices of sets by the fibers of these maps (which are translates of vertical planes). In [7] the authors showed,

Theorem 5.4. *If* $s > m+2$ *and* $A \subset \mathbb{H}^n$ *is a set such that* $0 < \mathcal{H}^s(A) < \infty$*, then for* $\mu_{2n,m}$*-almost every plane* $V \in G_h(2n,m)$

$$\mathcal{H}^m(u \in \mathbb{V} : \dim_{\mathbb{H}}[A \cap (V^\perp * u)] = s - m) > 0.$$

Here $V^\perp * u$ denotes the right translation by u of the vertical subgroup $V^\perp$. It is worth noting that since $V^\perp$ is a normal subgroup $u * V^\perp = V^\perp * u$. Other slicing theorems, including the analogue of Theorem 3.7 were obtained in [95] and will be discussed in the following chapter.

5.2 Intersection of Projections, and Planar Slices in the Heisenberg group

This section is based on the paper *Intersection of Projections and Slicing Theorems for the Isotropic Grassmannian and the Heisenberg Group* [95]. There the main goal was to further analyze dimensional properties of vertical-planar slices of sets in $\mathbb{H}^n$, in particular, to obtain a dimension estimate on the set of exceptions to the slicing theorem. Analogues to Theorem 3.3 and 3.7 where studied and the main results are the following.

Theorem 5.5. *Let $A, B \subset \mathbb{H}^n$ be Borel sets, let $\mathbb{V}$ denote the horizontal subgroup corresponding to the isotropic plane $V \in G_h(2n, m)$, and let $P_{\mathbb{V}}$ denote the horizontal projection onto $\mathbb{V}$.*

1. *If $\dim_{\mathbb{H}} A, \dim_{\mathbb{H}} B > m + 2$ then,*

$$\mu_{2n,m}(V \in G_h(2n, m) : \mathcal{H}^m(P_{\mathbb{V}}A \cap P_{\mathbb{V}}B) > 0) > 0.$$

2. *If $\dim_{\mathbb{H}} A, \dim_{\mathbb{H}} B > 2m + 2$ then,*

$$\mu_{2n,m}(V \in G_h(2n, m) : Int(P_{\mathbb{V}}A \cap P_{\mathbb{V}}B) \neq \varnothing) > 0.$$

3. *If $\dim_{\mathbb{H}} A > m + 2,\ \dim_{\mathbb{H}} B \leq m + 2$ but $\dim_{\mathbb{H}} A + \dim_{\mathbb{H}} B > 2m + 4$, then*

$$\mu_{2n,m}(V \in G_h(2n, m) : \dim_{\mathbb{H}}[P_{\mathbb{V}}A \cap P_{\mathbb{V}}B] > \dim_{\mathbb{H}} B - 2 - \epsilon) > 0.$$

Theorem 5.6. *Let $m + 2 < s \leq 2n$. If $A \subset \mathbb{H}^n$ is a Borel set such that $0 < \mathcal{H}^s(A) < \infty$, then for almost every $p \in \mathbb{H}^n$*

$$\mathcal{H}^{s-m}[A \cap (\mathbb{V}^{\perp} * p)] \leq s - m \text{ for } \mu_{2n,m} - a.e.\ V \in G_h(2n, m).$$

And,

Theorem 5.7. *Let $s \in \mathbb{R}$ be such that $m + 2 < s \leq 2n + 2$, and $A \subset \mathbb{H}^n$ be a Borel set with $0 < \mathcal{H}^s(A) < \infty$. Then, there is a Borel set $B \subset \mathbb{H}^n$ with $\dim B \leq m + 2$, such that for all $p \in \mathbb{H}^n \setminus B$,*

$$\mu_{2n,m}(V \in G_h(2n, m) : \dim[A \cap (\mathbb{V}^{\perp} * p)] = s - m) > 0. \tag{5.2}$$

In order to prove Theorem 5.5, and analogous result is first proven about the family of isotropic planes in $\mathbb{R}^{2n}$.

Theorem 5.8. *Let $A, B \subset \mathbb{R}^{2n}$ be Borel sets, and for $V \in G_h(2n, m)$ let P_V be the orthogonal projection onto V.*

1. *If $\dim A > m$ and $\dim B > m$, then*

$$\mu_{2n,m}(V \in G_h(2n, m) : \mathcal{H}^m(P_V A \cap P_V B) > 0) > 0.$$

2. *If* $\dim A > 2m$ *and* $\dim B > 2m$, *then*

$$\mu_{2n,m}(V \in G_h(2n,m) : Int(P_V A \cap P_V B) \neq \varnothing) > 0.$$

3. *If* $\dim A > m$, $\dim B \leq m$, *and* $\dim A + \dim B > 2m$, *then for all* $\epsilon > 0$,

$$\mu_{2n,m}(V \in G_h(2n,m) : \dim[P_V A \cap P_V B] > \dim B - \epsilon) > 0.$$

Once this theorem is proven, Theorem 5.5 follows by simply applying this theorem to the sets $\pi(A)$ and $\pi(B)$.

Similarly, analogues of Theorems 5.5 and 5.7 for $G_h(2n,m)$ were proven in [95] but these statements do not imply their Heisenberg analogues. In both cases however, the proofs follows the same techniques so I will present the proofs of Theorems 5.5 and 5.7 later in this section. Now I focus on the proof of Theorem 5.8, this proof is similar to the arguments used by the authors in [81], and it highlights some of the standard techniques in the theory of fractal projections.

However, before I can proceed with the proof of these theorems, I must introduce the following integral-geometric formula,

Lemma 5.1 (Isotropic disintegration formula)**.** *There exists a positive constant* $c = c(n,m)$ *such that, for all* $f \in L^1(\mathbb{R}^{2n})$,

$$\int_{\mathbb{R}^{2n}} f(x)dx = c\int_{G_h(2n,m)} \int_V |u|^{2n-m} f(u) d\mathcal{H}^m(u) d\mu_{2n,m}(V).$$

Proof. Using spherical coordinates on V we can compute the integral on the left as follows,

$$\begin{aligned}&\int_{G_h(2n,m)} \int_0^\infty \int_{V\cap\mathbb{S}^{2n-1}} r^{2n-m} f(rv) d\sigma^{m-1}(v) r^{m-1} dr d\mu_{2n,m}(V)\\ &= \int_0^\infty r^{2n-1} \int_{G_h(2n,m)} \int_{V\cap\mathbb{S}^{2n-1}} f(rv) d\sigma^{m-1}(v) d\mu_{2n,m}(V) dr.\end{aligned}$$

Now we take a closer look at the inner double-integral. We know that $U(n)$ acts transitively on $\mathbb{S}^{2n-1}$, therefore, up to multiplication by a constant, there is a unique $U(n)$-invariant measure on $\mathbb{S}^{2n-1}$. Since σ^{2n-1} is $O(n)$ invariant, it is in particular $U(n)$ invariant. Hence any $U(n)$-invariant measure on $\mathbb{S}^{2n-1}$ must be a constant multiple of the surface measure σ^{2n-1}. Now, for a function φ on $\mathbb{S}^{2n-1}$, one can check that the measure on $\mathbb{S}^{2n-1}$ given by, $\int_{G_h(2n,m)} \int_{V\cap\mathbb{S}^{2n-1}} \varphi(v) d\sigma^{m-1}(v) d\mu_{2n,m}(V)$, is $U(n)$-invariant. So as noted before, there exist a constant $c = c(n,m)$ such that

$$\int_{G_h(2n,m)} \int_{V\cap\mathbb{S}^{2n-1}} \varphi(v) d\sigma^{m-1}(v) d\mu_{2n,m}(V) = c \int_{\mathbb{S}^{2n-1}} \varphi(v) d\sigma^{2n-1}(v).$$

Going back to our overall integral, we now have,

$$\begin{aligned} I &= \int_0^\infty r^{2n-1} \int_{G_h(2n,m)} \int_{V\cap\mathbb{S}^{2n-1}} f(rv) d\sigma^{m-1}(v) d\mu_{2n,m}(V) dr \\ &= c \int_0^\infty r^{2n-1} \int_{\mathbb{S}^{2n-1}} f(rv) d\sigma^{2n-1}(v) dr = c \int_{\mathbb{R}^{2n}} f(x) dx. \end{aligned}$$

□

Now all is set to proceed with the proofs.

Proof of Theorem 5.8.

1. By the energy version of Frostman's theorem, there is $\mu \in \mathcal{M}(A)$ and $\nu \in \mathcal{M}(B)$ such that $I_m(\mu) < \infty$ and $I_m(\nu) < \infty$. By Lemma 3.2, this implies that for $\mu_{2n,m}$-almost all $V \in G(n,m)$, $P_{V\#}\mu$ and $P_{V\#}\nu$ are absolutely continuous with respect to $\mathcal{H}^m$ with densities $\mu_V, \nu_V \in L^2(V, \mathcal{H}^m)$ respectively. By Hölder's inequality, the function $\mu_V\nu_V$ is in $L^1(V, \mathcal{H}^m)$ so the measure $\mu_V\nu_V d\mathcal{H}^m$, which is compactly supported on $P_VA \cap P_VB$, is finite. The claim is proven by showing that $\mu_V\nu_V d\mathcal{H}^m$ is a positive measure for a $\mu_{2n,m}$-positive measure set of planes. Since μ_V and ν_V are in L^2, by Plancherel's theorem, so are $\widehat{\mu}_V$ and $\widehat{\nu}_V$, so their product is in L^1. This allows the application of Theorem 5.1 to compute,

$$\begin{aligned} &\int_{G(n,m)} \int_V \mu_V(v)\nu_V(v) d\mathcal{H}^m(v) d\mu_{2n,m}(V) \\ &\qquad = \int_{G(n,m)} \int_V \widehat{\mu}_V(v)\overline{\widehat{\nu}}_V(v) d\mathcal{H}^m(v) d\mu_{2n,m}(V) \\ &\qquad = \int_{G(n,m)} \int_V \widehat{\mu}(v)\overline{\widehat{\nu}}(v) d\mathcal{H}^m(v) d\mu_{2n,m}(V) \\ &\qquad \simeq \int_{\mathbb{R}^n} \widehat{\mu}(x)\overline{\widehat{\nu}(x)} |x|^{m-n} dx \\ &\qquad \simeq I_m(\mu,\nu) > 0. \end{aligned}$$

   It follows that for a $\mu_{2n,m}$-positive measure subset of $G(n,m)$, $\int_V \mu_V(v)\nu_V(v) d\mathcal{H}^m(v) > 0$. This completes the prove of the first statement.

2. Since $\alpha, \beta > 2m$, one can choose σ and τ such that $2m < \sigma < \alpha$, $2m < \tau < \beta$ and $\mu \in \mathcal{M}(A), \nu \in \mathcal{M}(B)$ with $I_\sigma(\mu) < \infty$, and $I_\tau(\nu) < \infty$. As seen in the proof of Theorem 3.1, the densities, μ_V and ν_V, of $P_{V\#}\mu$ and $P_{V\#}\nu$ are now continuous, and so is their product. Since for V in a $\mu_{2n,m}$-positive measure subset fo $G(n,m)$, $\int_V \mu_V \nu_V d\mathcal{H}^m > 0$ it follows that for such V $\mu_V\nu_V$ remains positive in some open set. Since $spt(\mu_V\nu_V) \subset P_V A \cap P_V B$, the claim follows.

3. Begin in the same way, by choosing $s, t \in \mathbb{R}$ such that $m < s < \dim A$, $0 < t < \dim B$ and $s + t > 2m$, and $\mu \in \mathcal{M}(A)$ and $\nu \in \mathcal{M}(B)$ with finite s and t energies respectively. Since $s > m$ we have that for $\mu_{2n,m} - a.e.\ V$, $P_{V\#}\mu \ll d\mathcal{H}^m$ with density $\mu_V \in L^2(V)$. Following the same lines as the proof of part 1, we aim to finds a family of measures $\rho_V \in \mathcal{M}(P_V(A) \cap P_V(B))$, but this time we also require that $I_t(\rho_V) < \infty$. In principle, we would like to use the family of measures $\rho_V = \mu_V dP_{V\#}\nu$. However, a priori, we do not know that $\mu_V \in L^1(P_{V\#}\nu)$. Instead, let $\mu_\delta = \mu * \psi_\delta$ be the standard convolution approximation to μ, where ψ is smooth and compactly supported in $B(0,1)$. Using Plancherel's theorem and Lemma 5.1 we have,

$$\begin{aligned} &\int_{G_h(2n,m)} \int_V P_{V\#}\mu_\delta dP_{V\#}\nu(v) d\mu_{2n,m} \\ &= \kappa(n,m) \int |x|^{m-2n} \widehat{\mu_\delta}(x)\widehat{\nu}(x)dx. \end{aligned} \tag{5.3}$$

Now, as $\delta \to 0$, $\widehat{\mu_\delta} = \widehat{\mu\psi_\delta} \to \widehat{\mu}\widehat{\psi}(0) = \widehat{\mu}$. Hence the right hand side of (5.3) goes to $\kappa'(n,m)I_m(\mu,\nu)$. By the choice of μ and ν, and since $s+t > 2m$, (2.13) tells us that $0 < I_m(\mu,\nu) < \infty$. Hence, $\exists\ c, C > 0$ such that $\forall\ \delta > 0$,

$$c < \int \int P_{V\#}\mu_\delta(v) dP_{V\#}\nu(v) d\mu_{2n,m} < C. \tag{5.4}$$

The aim is now to show that $\mu_V \in L^1(P_{V\#}\nu)$ and that

$$\iint \mu_V dP_{V\#}\nu d\mu_{2n,m} = \lim_{\delta\to 0} \iint P_{V\#}\mu_\delta dP_{V\#}\nu d\mu_{2n,m}. \tag{5.5}$$

This, together with (5.4) would show that

$$\mu_V dP_{V\#}\nu \in \mathcal{M}(P_V A \cap P_V B).$$

To prove (5.5) we follow a similar argument to the one used in [81] for the analogous statement.

First not that since $m - 2\frac{s-m}{2} = 2m - s < t$, and $I_{2m-s}(P_{V\#}\nu) \lesssim I_t(P_{V\#}\nu)$, using Lemma 5.1

$$\begin{aligned}\int I_{2m-s}(P_{V\#}\nu)d\mu_{2n,m} &\lesssim \int I_t(P_{V\#}\nu)d\mu_{2n,m} \\ &\lesssim \iint |v|^{m-t}|\widehat{P_{V\#}\nu}|^2 d\mathcal{H}^m d\mu_{2n,m} \\ &\lesssim \int_{\mathbb{R}^{2n}} |x|^{2n-t}|\widehat{\nu}(x)|^2 dx \lesssim I_t(\nu) < \infty.\end{aligned} \tag{5.6}$$

Next, note that we can write $P_{V\#}\mu_\delta(v) = \psi_\delta^V * P_{V\#}\mu$, where $\psi^V(v) = \int_{V^\perp}\psi(v + w)d\mathcal{H}^{2n-m}(w)$ for $v \in V$.

Once again by Lemma 5.1,

$$\iint |v|^{s-m}|\widehat{\mu_V}(v)|^2 d\mathcal{H}^m(v)d\mu_{2n,m}(V) = \kappa(n,m)\int |x|^{s-2n}|\widehat{\mu}(x)|^2 dx \lesssim I_s(\mu) < \infty$$

which tells us that for $\mu_{2n,m}$ almost every V, μ_V is in the fractional Sobolev space $H^{\frac{s-m}{2}}(V)$, with

$$\int_{G_h(2n,m)} ||\mu_V||^2_{H^{\frac{s-m}{2}}(V)} d\mu_{2n,m}(V) < \infty. \tag{5.7}$$

Therefore, taking $\alpha = \frac{s-m}{2}$ in [77, Theorem 17.3], we conclude that for $\mu_{2n,m}-$almost every V, the maximal function $M_V P_{V\#}\mu(v) = \sup_{\delta>0}|\psi_\delta^V * P_{V\#}\mu(v)|$ is in the space $L^1(P_{V\#}\nu)$ with

$$\int_V M_V P_{V\#}\mu(v)dP_{V\#}\nu(v) \lesssim [I_{s-2m}(P_{V\#}\nu)]^{1/2}||\mu_V||^2_{H^{\frac{s-m}{2}}(V)}.$$

By the dominated convergence theorem it follows that for $\mu_{2n,m}-$almost every $V \in G_h(2n,m)$ the sequence $P_{V\#}\mu_\delta$ converges to $f_V := \mu_V\lfloor_{spt(P_{V\#}\nu)}$ in $L^1(P_{V\#}\nu)$. That is to say, the sequence of function

$$F_\delta(V) = \int_V P_{V\#}\mu_\delta(v)dP_{V\#}\nu(v)$$

converges to the function

$$F(V) = \int_V f_V(v)dP_{V\#}\nu(v),$$

for $\mu_{2n,m}-$almost every $V \in G_h(2n,m)$. But Moreover, for such V

$$|F_\delta(V)| \lesssim [I_{s-2m}(P_{V\#}\nu)]^{1/2}||\mu_V||^2_{H^{\frac{s-m}{2}}(V)}.$$

So by Cauchy-Schwarz inequality, (5.6), and (5.7),

$$\int_{G_h(2n,m)} [I_{s-2m}(P_{V\#}\nu)]^{1/2}||\mu_V||^2_{H^{\frac{s-m}{2}}(V)} d\mu_{2n,m}(V)$$
$$\leq \left(\int I_{s-2m}(P_{V\#}\nu) d\mu_{2n,m}\right)^{1/2} \left(\int ||\mu_V||^2_{H^{\frac{s-m}{2}}(V)} d\mu_{2n,m}\right)^{1/2} < \infty.$$

Once more by dominated convergence this gives us (5.5), and by (5.4) we get

$$c < \iint f_V(v) dP_{V\#}\nu(v) d\mu_{2n,m}(V) = \iint P_{V\#}\mu_\delta(v) dP_{V\#}\nu(v) d\mu_{2n,m} < C. \tag{5.8}$$

With this, consider the measure $f_V P_{V\#}\nu$. By (5.8) we know that for $\mu_{2n,m}$-positively many V, f_V is positive and finite on a set of positive $P_{V\#}\nu$ measure. Therefore $f_V P_{V\#}\nu$ is a non-trivial measure supported on $spt(\mu_V) \cap spt(\nu_V) \subset P_V(A) \cap P_V(B)$. That is, $f_V P_{V\#}\nu \in \mathcal{M}(P_V(A) \cap P_V(B))$. For each such V, pick a large enough constant C_V so that the measure $\mathbb{1}_{\{f_V \leq C_V\}} f_V P_{V\#}\nu$ is still non-trivial, and so that $\mathbb{1}_{\{f_V \leq C_V\}} f_V P_{V\#}\nu$ has finite t-energy. Since t can be chosen arbitrarily close to $\dim B$ the claim follows.

□

The proof of Theorem 5.6 takes advantage of Eilenberg's inequality and the properties of push-forward measures.

Proof of Theorem 5.6. First, we show that for any $R > 0$, for $\mathcal{L}^{2n+1}$-almost every $p \in \mathbb{H}^n$

$$\mathcal{H}^{s-m}[A \cap (\mathbb{V}^\perp * p)] \mathbb{1}_{N_E(\mathbb{V},R)}(p) < \infty, \tag{5.9}$$

for $\mu_{2n,m}$-almost every $V \in G_h(2n,m)$. Here, as before, $N_E(\mathbb{V},R)$ is the Euclidean tubular neighborhood of $\mathbb{V}$ of radius R and $\mathcal{L}^{2n+1}$ is the $(2n+1)$-dimensional Lebesgue measure. I think it is important to remark that there is a constant $d = d(n)$, depending only on n, such that $\mathcal{L}^{2n+1} = d\mathcal{H}^{2n+2}$. So using $\mathcal{L}^{2n+1}$ is equivalent to using the, arguably more intrinsic, measure $\mathcal{H}^{2n+2}$.

To show (5.9), let $\lambda = \mathcal{L}^{2n+1}\lfloor_{N_E(\mathbb{V},R)}$. Since $N_E(\mathbb{V},R) = B_{V^\perp,E}(0,R) \times \mathbb{V}$, for any measurable set $S \subset \mathbb{V}$ we have

$$\begin{aligned} P_{\mathbb{V}\#}\lambda(S) &= \lambda(P_{\mathbb{V}}^{-1}(S)) \\ &= \mathcal{L}^{2n+1}(P_{\mathbb{V}}^{-1}(S) \cap N_E(\mathbb{V},R)) \\ &= \mathcal{L}^{2n+1-m}(B_{V^\perp,E}(0,R))\mathcal{L}^m(S) \\ &= C(R,n,m)\mathcal{H}^m(S). \end{aligned}$$

Here $C(R,n,m)$ is a constant, depending only on R, n, and m, which involves the $\mathcal{L}^{2n+1-m}$-volume of the ball of radius R in $\mathbb{R}^{2n+1-m}$, as well as the universal constant c such that $c\mathcal{H}^m = \mathcal{L}^m$ in $\mathbb{R}^m$.

For each $\mathbb{V}$, the map $P_{\mathbb{V}}$ is 1-Lipschitz, so we can apply Elienberg's lemma to get

$$\begin{aligned} &\int_{\mathbb{H}^n}\int_{G_h(2n,m)} \mathcal{H}^{s-m}[A \cap (\mathbb{V}^\perp * p)]\mathbb{1}_{N_E(\mathbb{V},R)}(p)d\mu_{2n,m}(V)d\mathcal{L}^{2n+1}(p) \\ &= \int_{G_h(2n,m)}\int_{\mathbb{H}^n} \mathcal{H}^{s-m}[A \cap (\mathbb{V}^\perp * P_{\mathbb{V}}(p))]d\lambda(p)d\mu_{2n,m}(V) \\ &= C(R,m,n)\int_{G_h(2n,m)}\int_{\mathbb{V}} \mathcal{H}^{s-m}[A \cap (\mathbb{V}^\perp * v)]d\mathcal{H}^m(v)d\mu_{2n,m}(V) \\ &\lesssim \mathcal{H}^s(A) < \infty, \end{aligned}$$

where the last line follows by integrating Elinebger's inequality with respect to $\mu_{2n,m}$.

Now supppose there is $E \subset \mathbb{H}^n$ is such that $\mathcal{L}^{2n+1}(E) > 0$ and for every $p \in E$,

$$\mathcal{H}^{s-m}[A \cap (\mathbb{V}^\perp * p)] = \infty,$$

for a $\mu_{2n,m}$-positive measure subset of $G_h(2n,m)$. Without lost of generality we may assume E is compact, so there is $R_0 > 0$ such that $E \subset B_E(0,R_0) \subset N_E(\mathbb{V},R_0)$ for every $\mathbb{V}$. So we have that for every $p \in E$

$$\mathcal{H}^{s-m}[A \cap (\mathbb{V}^\perp * p)]\mathbb{1}_{N_E(\mathbb{V},R_0)}(p) = \mathcal{H}^{s-m}[A \cap (\mathbb{V}^\perp * P_{\mathbb{V}}(p))] = \infty,$$

for a $\mu_{2n,m}$-positive measure subset of $G_h(2n,m)$ which is a contradiction. It follows that for $\mathcal{L}^{2n+1}$-almost every $p \in \mathbb{H}^n$

$$\mathcal{H}^{s-m}[A \cap (\mathbb{V}^\perp * p)] < \infty \text{ for } \mu_{2n,m} - a.e. V \in G_h(2n,m).$$

□

Note that this implies that for almost every $p \in \mathbb{H}^n$

$$\dim_{\mathbb{H}}[A \cap (\mathbb{V}^{\perp} * p)] \leq s - m, \text{ for } \mu_{2n,m} - a.e.\ V \in G_h(2n, m), \tag{5.10}$$

however, it gives no information about the dimension of the set of Heisenberg points for which this dimensional bound fails. Turns out, the dimensional bound (5.10) holds for all $p \in \mathbb{H}^n$.

Lemma 5.2. *Let $A \subset \mathbb{H}^n$ be a Borel set with $0 < \mathcal{H}^s(A) < \infty$ for some $m+2 < s \leq 2n+2$. Then, for all $p \in \mathbb{H}^n$*

$$\dim[A \cap (\mathbb{V}^{\perp} * p)] \leq s - m \textit{ for } \mu_{2n,m} - a.e.\ V \in G_h(2n, m).$$

The proof of this lemma follows from the proof of Theorem 6.8 in [7], details can be found in [95]. this allows us to focus only on the dimensional lower bound in order to prove Theorem 5.7

Proof of Theorem 5.7. Without loss of generality assume A is compact and set

$$B := \{p \in \mathbb{H}^n : \mu_{2n,m}(V : \dim[A \cap (\mathbb{V}^{\perp} * p)] \geq s - m) = 0\}.$$

Since A is compact, $(p, \mathbb{V}) \to \dim[A \cap (\mathbb{V}^{\perp} * p)]$ is a Borel function, so B is a Borel set. Suppose $\dim B > \sigma > m + 2$ and pick $\nu \in \mathcal{M}(B)$ so that $\nu(B_{\mathbb{H}^n}(p, r)) \lesssim r^{\sigma}$ so that by [7, Proposition 6.1], $P_{\mathbb{V}\#}\nu \ll \mathcal{H}^m$ for $\mu_{2n,m}$-almost every $V \in G_h(2n, m)$. Similarly let $\mu = \mathcal{H}^s\lfloor_A \in \mathcal{M}(A)$, possibly restricted to a further subset, so that $\mu(B_{\mathbb{H}^n}(p, r)) \lesssim r^s$. It is also true that $P_{\mathbb{V}\#}\mu \ll \mathcal{H}^m$ for $\mu_{2n,m}$-almost every $V \in G_h(2n, m)$. By [7, Theorem 6.8], for μ-almost all $p \in \mathbb{H}^n$

$$\dim[A \cap (\mathbb{V}^{\perp} * p)] \geq s - m, \text{ for } \mu_{2n,m} - a.e.\ V \in G_h(2n, m). \tag{5.11}$$

By assumption, for ν-almost every $q \in \mathbb{H}^n$

$$\dim[A \cap (\mathbb{V}^{\perp} * q)] < s - m, \text{ for } \mu_{2n,m} - a.e.\ V \in G_h(2n, m). \tag{5.12}$$

Applying Tonelli's theorem we may switch the order of the measures so that for $\mu_{2n,m}$-almost every $V \in G_h(2n, m)$

$$\dim[A \cap (\mathbb{V}^{\perp} * p)] \geq s - m, \text{ for } \mu - a.e.\ p \in \mathbb{H}^n, \tag{5.13}$$

and,

$$\dim[A \cap (\mathbb{V}^\perp * q)] < s - m, \text{ for } \nu - a.e.\ q \in \mathbb{H}^n. \tag{5.14}$$

A contradiction is found by finding $p \in spt(\mu)$ and $q \in spt(\nu)$ satisfying (5.13) and (5.14) respectively, and such that $P_{\mathbb{V}}(p) = P_{\mathbb{V}}(q)$. For $\mu_{2n,m}$-almost every $V \in G_h(2n, m)$, $P_{\mathbb{V}\#}\mu$ and $P_{\mathbb{V}\#}\nu$ have densities μ_V and ν_V respectively. For such V define,

$$\begin{aligned}
A_{\mathbb{V}} &:= \{p \in \mathbb{H}^n : \dim[A \cap (\mathbb{V}^\perp * p)] \geq s - m\} \\
B_{\mathbb{V}} &:= \{q \in \mathbb{H}^n : \dim[A \cap (\mathbb{V}^\perp * q)] < s - m\} \\
C_{\mathbb{V}} &:= \{v \in \mathbb{V} : \mu_V(v)\nu_V(v) > 0\}.
\end{aligned}$$

For $\mu_{2n,m}$-almost every $\mathbb{V}$, $\mu(\mathbb{H}^n \setminus A_{\mathbb{V}}) = 0$ and $\nu(\mathbb{H}^n \setminus B_{\mathbb{V}}) = 0$. By Theorem 5.5, for a $\mu_{2n,m}$-positive measure subset of $G_h(2n, m)$, $\mathcal{H}^m(C_{\mathbb{V}}) > 0$. We can pick a horizontal subgroup where all 3 things are satisfied simultaneously. By (3.2) (or rather it's $\mathbb{H}^n$ analogue),

$$\int_{\mathbb{H}^n} \mu_{\mathbb{V}^\perp * v}(\mathbb{H}^n \setminus A_{\mathbb{V}}) d\mathcal{H}^m(v) = \mu(\mathbb{H}^n \setminus A_{\mathbb{V}}) = 0,$$

and

$$\int_{\mathbb{H}^n} \nu_{\mathbb{V}^\perp * v}(\mathbb{H}^n \setminus B_{\mathbb{V}}) d\mathcal{H}^m(v) = \nu(\mathbb{H}^n \setminus B_{\mathbb{V}}) = 0.$$

By (the $\mathbb{H}^n$ analogue of) (3.4), $0 < \mu_V(v) = \mu_{\mathbb{V}^\perp * v}(\mathbb{H}^n)$, and $0 < \nu_V(v) = \nu_{\mathbb{V}^\perp * v}(\mathbb{H}^n)$. Hence, $\mu_{\mathbb{V}^\perp * v}(A_{\mathbb{V}}) > 0$ and $\nu_{\mathbb{V}^\perp * v}(B_{\mathbb{V}}) > 0$ for almost every $v \in \mathbb{V}$. In particular, there is $v \in C_{\mathbb{V}}$ for which both $\mu_{\mathbb{V}^\perp * v}$ and $\nu_{\mathbb{V}^\perp * v}$ are positive, so we find $p \in A_{\mathbb{V}}$, $q \in B_{\mathbb{V}}$ such that $P_{\mathbb{V}}(p) = P_{\mathbb{V}}(q) = v$. This completes the proof. □

5.3 Right Coset Projections in the Heisenberg group

This section is based on work done in collaboration with Terence L.J. Harris and Chi N.Y. Hunynh ([61]).

As mentioned before, the vertical projection maps are related by $P^R_{\mathbb{V}^\perp}(A) = -P^L_{\mathbb{V}^\perp}(-A)$ so they image of a set under these is different. Moreover, the maps have different behaviors with respect to the metric. This can be seen in Example 5.1 as one can check that

$$P^R_{\mathbb{V}_0^\perp} A = \{(0, s, 0) : s \in [0, 1]\}$$

is horizontal. Thus, $2 = \dim_{\mathbb{H}} P^L_{\mathbb{V}_0^\perp} A \neq \dim_{\mathbb{H}} P^R_{\mathbb{V}} A = 1$. As mentioned before, this is because the maps $P^R_{\mathbb{V}^\perp}$, in a sense, are not natural maps in the metric structure of $\mathbb{H}^n$. However, they do arise naturally in a different context accompanied by a more natural metric on the target space $\mathbb{V}^\perp$. In order to motivate this section, I will first describe the problem in the setting of the first Heisenberg group, where a connection with the Grushin plane arises.

5.3.1 The Grushin plane

In order to study Hausdorff dimension distortion by right coset projections, the plane $\mathbb{V}^\perp$ must be endowed with a metric. When studying left coset projections it is standard to consider the ambient distance restricted to $\mathbb{V}^\perp$. However, it will be more natural to consider a different metric which is closely related to the metric of the Grushin plane. The Grushin plane is the manifold $\mathbb{G} = \mathbb{R}^2$ with vector fields

$$\begin{cases} T = -v\frac{\partial}{\partial \tau} \\ V = \frac{\partial}{\partial v}, \end{cases} \tag{5.15}$$

where $(v, \tau) \in \mathbb{R}^2$. These vector fields span the whole tangent space at every point outside of the singular set $\{v = 0\}$, and by taking them to be orthonormal there, one gets a line form

$$ds^2 = dv^2 + \frac{d\tau^2}{v^2}$$

on $\mathbb{R}^2 \setminus \{(0, \tau) : \tau \in \mathbb{R}\}$. One can check that $[T, V] = \frac{\partial}{\partial \tau}$, which allows to extend this metric to a Carnot-Carathéodory path distance in all of $\mathbb{R}^2$. The resulting metric, denoted by $d_{\mathbb{G}}$, turns $\mathbb{G}$ into a subRiemannian manifold whose horizontal curves are curves that have horizontal tangent at every point where they cross the critical line. That is to say, $\gamma : [0, 1] \to \mathbb{G}$ is horizontal if there exist absolutely continuous functions a and b such that

$$\dot{\gamma}(s) = a(s)T + b(s)V,$$

for a.e. $s \in [0, 1]$. The length of γ is then given by

$$\int_0^1 \left[a(s)^2 + b(s)^2\right]^{1/2} ds.$$

If we write $\gamma(s) = (v(s), \tau(s))$, a more explicit formula for the length is

$$\Lambda_{\mathbb{G}} = \int_0^1 \left[\dot{v}(s)^2 + \frac{\dot{\tau}(s)^2}{v(s)^2} \right]^{1/2} ds. \tag{5.16}$$

For the purposes of computing Hausdorff dimension of sets, it is useful to have a more explicitly computable distance formula. This is exactly the content of the following theorem (See for instance Section 2.3 in [11]).

Theorem 5.9. *Let*

$$d'_{\mathbb{G}}((x,y),(u,v)) = \max\left\{ |x-u|, \min\left\{ |y-v|^{1/2}, \frac{|y-v|}{\max\{|x|,|u|\}} \right\} \right\}. \tag{5.17}$$

Then there exists a constant $C > 0$ such that $\frac{1}{C} d_{\mathbb{G}}(z,w) \leq d'_{\mathbb{G}}(z,w) \leq C d_{\mathbb{G}} z,w)$ for all $z, w \in \mathbb{G}$.

The space $\mathbb{G}$ has a homogeneous structure provided by the dilations $\delta_r : \mathbb{G} \to \mathbb{G}$, $(v,\tau) \to (rv, r^2\tau)$. Indeed, it is not hard to see from (5.16)) (resp. (5.17)), that δ_r is homogeneous of degree 1 with respect to the metric $d_{\mathbb{G}}$ (resp. $d'_{\mathbb{G}}$). In addition, notice that for each $\tau_0 \in \mathbb{R}$ the map $L_{\tau_0}(v,\tau) = (v, \tau+\tau_0)$ is an isometry of $(\mathbb{G}, d_{\mathbb{G}})$. Indeed, if γ is a path between (v,τ) and (x,y), it is clear that $\gamma_{\tau_0} = L_{\tau_0}\gamma$ is also a path between $(v, \tau+\tau_0)$ and $(x, y+\tau_0)$. Moreover, If γ is horizontal, we have that

$$\dot{\gamma}(s) = a(s)T + b(s)V.$$

It is clear that the push-forward of L_{τ_0} is the identity matrix, so it follows that

$$\dot{\gamma}_{\tau_0}(s) = a(s)T + b(s)V.$$

This tells us that γ_{τ_0} is also horizontal and $\Lambda_{\mathbb{G}}(\gamma_{\tau_0}) = \Lambda_{\mathbb{G}}(\gamma)$. The claim then follows by taking the infimum over all such paths.

5.3.2 Right coset quotient space in $\mathbb{H}^1$

Hereafter we only consider the projections $P^R_{\mathbb{V}^\perp}$ which we will simply denote by $P_{\mathbb{V}^\perp}$. For each fixed $\mathbb{V}$ we can consider the quotient space of right cosets of $\mathbb{V}$, denoted $\mathbb{V}\backslash\mathbb{H}$, with quotient distance

$$d_{\mathbb{V}\backslash\mathcal{H}}(\mathbb{V}p, \mathbb{V}q) = \inf\{d_{cc}(gp,q) : g \in \mathbb{V}\}. \tag{5.18}$$

Since $0 \in \mathbb{V}$ for every $\mathbb{V}$ it follows that $d_{\mathbb{V}\backslash\mathbb{H}}(\mathbb{V}p, \mathbb{V}q) \leq d_{cc}(p,q)$. Each element of $\mathbb{V}\backslash\mathbb{H}$ can be written as $\mathbb{V}p$ for exactly one $p \in \mathbb{V}^\perp$; this allows us to identify $\mathbb{V}\backslash\mathbb{H}$ with the plane $\mathbb{V}^\perp$ via the map $\mathbb{V}p \mapsto p$. Turns out, the image of $\mathbb{V}p$ under this map coincides with $P_{\mathbb{V}^\perp}q$ for any $q \in \mathbb{V}p$. That is to say, as a map of sets: $P_{\mathbb{V}^\perp}(\mathbb{V}p) = \{p\}$. This gives an identification of $\mathbb{V}\backslash\mathbb{H}$ with $\mathbb{R}^2$. In what follows, we will study the metric on $\mathbb{R}^2$ which turns this set bijection into an isometry. This exposition follows the arguments in [2].

The unitary group $U(1)$ acts smoothly and transitively on $\mathbb{G}_h(1,1) \approx \mathbb{S}^1$. Given any two horizontal subgroups $V, V' \in \mathbb{G}_h(1,1)$, there is a unitary matrix $R \in U(1)$ such that $\mathcal{R}\mathbb{V} = \mathbb{V}'$ and $\mathcal{R}\mathbb{V}^\perp = \mathbb{V}'^\perp$. Since unitary rotations are isometric automorphisms of $\mathbb{H}^n$, we only need to consider the problem for a specific choice of $\mathbb{V}$. To simplify our computations we fix

$$\mathbb{V}_0 = \{(x,0,0):\ x \in \mathbb{R}\},$$

so that

$$\mathbb{V}_0^\perp = \{(0,y,t):\ y,t \in \mathbb{R}\}.$$

In this case, the vertical projection map turns into

$$P_{\mathbb{V}_0^\perp}(x,y,t) = \left(0, y, t - \frac{xy}{2}\right),$$

so the map $\mathbb{H}^n \to \mathbb{R}^{2n}$, given by $(x,y,t) \to \left(y, t - \frac{xy}{2}\right)$ induces the aforementioned identification of $\mathbb{V}_0\backslash\mathbb{H}$ with $\mathbb{R}^2$ via the quotient map. Abusing notation we will also denote this identification map by $P_{\mathbb{V}_0^\perp}$, and think of it as the same map.

For $\zeta \in \mathbb{R}^2$ with $\zeta = (u,v)$, consider the analytic change of variables in $\mathbb{H}$,

$$\Psi(\zeta,\tau) = [\zeta,\tau] = \left(\zeta, \tau + \frac{uv}{2}\right).$$

Under this change of variables we have that $\Phi(\zeta,\tau) = (0,v,\tau)$, where $\Phi := P_{\mathbb{V}_0^\perp} \circ \Psi$ is the projection in the new coordinates. The horizontal vector fields in the new variables become

$$\begin{cases} \widetilde{X} = \frac{\partial}{\partial u} - v\frac{\partial}{\partial \tau}, \\ \widetilde{Y} = \frac{\partial}{\partial v}, \end{cases} \tag{5.19}$$

where $\widetilde{X} := \Psi_*^{-1}X$ and $\widetilde{Y} := \Psi_*^{-1}Y$. Under the new coordinates, the pushforward of Φ

can be represented by the constant matrix

$$(\Phi_*)_{(u,v,\tau)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $T = \Phi_* \widetilde{X}$, and $V = \Phi_* \widetilde{Y}$ can be easily computed to be given by

$$\begin{cases} T = -v\frac{\partial}{\partial \tau}, \\ V = \frac{\partial}{\partial v}. \end{cases} \tag{5.20}$$

Here the pushforward $\Phi_* W$ of a vector field W on $\mathbb{H}$ is defined to be the unique vector field Z on the Grushin plane satisfying $Z_{\Phi(u,v,\tau)} = (\Phi_*)_{(u,v,\tau)} W_{(u,v,\tau)}$ for all (u, v, τ). The pushforward under $P_{\mathbb{V}_0^\perp}$ is defined similarly, and satisfies $P_{\mathbb{V}_0^\perp *} = \Phi_* \Psi_*^{-1}$. One might notice that the vector fields in (5.20) are exactly the vector fields from (5.15) that give $\mathbb{R}^2$ the Grushin plane structure.

Theorem 5.10. *The space* $(\mathbb{V}_0 \backslash \mathbb{H}, d_{\mathbb{V}_0 \backslash \mathbb{H}})$ *is isometrically isomorphic to* $(\mathbb{G}, d_{\mathbb{G}})$.

Proof. If $\Gamma : [0,1] \to \mathbb{H}$ is a horizontal path in $\mathbb{H}$ then there exist absolutely continuous functions $a, b : [0,1] \to \mathbb{R}$ such that

$$\dot{\Gamma} = aX + bY.$$

It follows that

$$\dot{P}_{\mathbb{V}_0^\perp}(\Gamma) = P_{\mathbb{V}_0^\perp} * \dot{\Gamma} = \Phi_* \Psi_*^{-1} \dot{\Gamma} = aT + bV,$$

so that $\gamma := P_{\mathbb{V}_0^\perp}(\Gamma) : [0,1] \to \mathbb{G}$ is horizontal and satisfies

$$\Lambda_{\mathbb{G}}(\gamma) = \int_0^1 \left(a^2(s) + b^2(s)\right)^{1/2} ds = \Lambda_{\mathbb{H}}(\Gamma). \tag{5.21}$$

This tells us that given $p, p' \in \mathbb{V}_0^\perp$, every $\mathbb{H}$-horizontal path between the fibers $\mathbb{V}_0 p$ and $\mathbb{V}_0 p'$ induces a $\mathbb{G}$-horizontal path between p and p' of the same length. Therefore,

$$d_{\mathbb{G}}(p, p') \leq \inf\{d_{cc}(qp, p') : q \in \mathbb{V}_0\} = d_{\mathbb{V}_0 \backslash \mathbb{H}}(p, p').$$

Now we aim to show that every $\mathbb{G}$-horizontal path between $p, p' \in \mathbb{V}_0^\perp$ has a $\mathbb{H}$-horizontal lift between $\mathbb{V}_0 p$ and $\mathbb{V}_0 p'$. This would imply $d_{\mathbb{V}_0 \backslash \mathbb{H}}(p, p') \leq d_{\mathbb{G}}(p, p')$ and finish the proof.

Suppose $\gamma : [0,1] \to \mathbb{G}$ is $\mathbb{G}$-horizontal, given by $\gamma(s) = (v(s), \tau(s))$, so that

$$\dot{\gamma} = \frac{-\dot{\tau}}{v} T + \dot{v} V.$$

Put

$$u(s) = -\int_0^s \frac{\dot{\tau}(\eta)}{v(\eta)}\, d\eta, \tag{5.22}$$

and set $\Gamma : [0,1] \to \mathbb{H}$ to be $\Gamma(s) = (u(s), v(s), \tau(s))$. The integrand in the definition of u is in $L^1[0,1]$ since γ has finite length in $\mathbb{G}$. The Fundamental Theorem of Calculus for the Lebesgue integral therefore implies that Γ is absolutely continuous, and satisfies

$$\begin{aligned}
\dot{\Gamma} &= \dot{u}\frac{\partial}{\partial u} + \dot{\gamma} \\
&= \frac{-\dot{\tau}}{v}\frac{\partial}{\partial u} + \frac{\dot{\tau}}{v} v \frac{\partial}{\partial \tau} + \dot{v}\frac{\partial}{\partial v} \\
&= \frac{-\dot{\tau}}{v}\left(\frac{\partial}{\partial u} - v\frac{\partial}{\partial \tau}\right) + \dot{v}\frac{\partial}{\partial v},
\end{aligned}$$

almost everywhere. Comparing to (5.19) we see that $\Psi \circ \Gamma$ is a horizontal lift of γ in $\mathbb{H}$, and by (5.21),

$$\Lambda_{\mathbb{H}}\left(\Psi \circ \Gamma\right) = \int_0^1 \left[\dot{v}(s)^2 + \left(\frac{\dot{\tau}(s)}{v(s)}\right)^2\right]^{1/2} ds = \Lambda_{\mathbb{G}}(\gamma).$$

This completes the proof. □

As mentioned before, everything shown for $\mathbb{V}_0$ carries over to any horizontal line $\mathbb{V}$, so we have a family of mappings $\left\{P_{\mathbb{V}^\perp} : (\mathbb{H}, d_{cc}) \to (\mathbb{V}^\perp, d_{\mathbb{G}})\right\}$. This last theorem has several applications, for instance, in [2] the authors used this fact to solve a certain iso-perimetric problem in $\mathbb{G}$ by projecting geodesics in $\mathbb{H}^n$ via the map $P_{\mathbb{V}_0^\perp}$. It is therefore natural to ask about the generic effect of this map on Hausdorff dimension. Moreover, it motivates exploring the problem in higher dimensions.

5.3.3 The right coset quotient space in $\mathbb{H}^n$

This section generalizes the right coset quotient space to higher dimensional Heisenberg groups. In $\mathbb{H}^n$, given $V \in G_h(2n, m)$, consider the quotient space of right cosets of $\mathbb{V}$,

$$\mathbb{V}\backslash\mathbb{H}^n := \left\{\mathbb{V}p : p \in \mathbb{H}^n\right\},$$

endowed with the quotient distance

$$d_{\mathbb{V}^\perp\backslash\mathbb{H}^n}(\mathbb{V}p, \mathbb{V}p') = \inf\left\{d_{cc}(qp, p') : \ q \in \mathbb{V}\right\}.$$

Just as in $\mathbb{H}$, there is a unique way to write elements of $\mathbb{V}^\perp\backslash\mathbb{H}^n$ as $\mathbb{V}q$ with $q \in \mathbb{V}^\perp$. Therefore $\mathbb{V}^\perp\backslash\mathbb{H}^n$ is identified with $\mathbb{V}^\perp$ by the map $\mathbb{V}q \mapsto q$. This map coincides with the map on $\mathbb{V}^\perp\backslash\mathbb{H}^n$ induced by $P_{\mathbb{V}^\perp}$, that is $P_{\mathbb{V}^\perp}(\mathbb{V}p) = \{P_{\mathbb{V}^\perp}(p)\}$. For each fixed $\mathbb{V}$, the map

$$P_{\mathbb{V}^\perp} : (\mathbb{H}^n, d_{cc}) \to \left(\mathbb{V}^\perp, d_{\mathbb{V}^\perp\backslash\mathbb{H}^n}\right)$$

is 1-Lipschitz. Indeed, if $p, p' \in \mathbb{H}^n$,

$$d_{\mathbb{V}^\perp\backslash\mathbb{H}^n}(P_{\mathbb{V}^\perp}(p), P_{\mathbb{V}^\perp}(p')) = \inf_{q\in\mathbb{V}} d_{cc}(qP_{\mathbb{V}^\perp}(p), P_{\mathbb{V}^\perp}(p')).$$

An upper bound is found by choosing a specific $q \in \mathbb{V}$. In particular, choosing $q = P_{\mathbb{V}}(p')^{-1}P_{\mathbb{V}}(p)$, and appealing to left invariance of d_{cc} we see that,

$$d_{\mathbb{V}^\perp\backslash\mathbb{H}^n}(P_{\mathbb{V}^\perp}(p), P_{\mathbb{V}^\perp}(p')) \leq d_{cc}(p, p').$$

Denoting by π_W the Euclidean orthogonal projection onto W, an explicit formula for the projection is given by

$$P_{\mathbb{V}^\perp}(z, t) = \left(\pi_{V^\perp}(z), t - \frac{1}{2}\omega(\pi_V(z), \pi_{V^\perp}(z))\right). \tag{5.23}$$

Unlike the case of $\mathbb{H}$, the space $\mathbb{V}^\perp\backslash\mathbb{H}^n$ for $n \geq 2$ does not resemble, at least not immediately, any well understood subRiemannian space. Because of this, we do not have an explicit formula to compute distances as we do in $\mathbb{G}$ with (5.17). Nevertheless, $\mathbb{V}^\perp\backslash\mathbb{H}^n$ inherits a rich structure from $\mathbb{H}^n$ which allows us to have a more intuitive understanding of the space.

Since $U(n)$ acts smoothly and transitively on $G_h(2n, m)$, Understanding the metric properties of $\mathbb{V}_0^\perp\backslash\mathbb{H}^n$ for a fixed $\mathbb{V}_0$ will get us the same properties for $\mathbb{V}^\perp\backslash\mathbb{H}^n$ in general. As we did in the previous section, we fix the horizontal subgroup

$$\mathbb{V} = \mathbb{V}_0 := \{(x_1, \ldots, x_m, 0, \ldots, 0) : x_j \in \mathbb{R}\},$$

for the rest of this section. This gives us

$$\mathbb{V}_o^\perp = \{(0, \ldots, 0, x_{m+1}, \ldots, x_n, y_1, \ldots, y_n, t) : \ x_j, y_j, t \in \mathbb{R}\}.$$

We discuss some of the symmetries of the space $\mathbb{V}^\perp\backslash\mathbb{H}^n$.

Homogeneous dilations

For each $r > 0$ the map $\delta_r : \mathbb{V}^\perp\backslash\mathbb{H}^n \to \mathbb{V}^\perp\backslash\mathbb{H}^n$ given by

$$\delta_r(0, \dots, 0, x_{m+1}, \dots, y_n, t) = \left(0, \dots, 0, rx_{m+1}, \dots, ry_n, r^2 t\right),$$

is homogeneous of degree 1 with respect to $d_{\mathbb{V}^\perp\backslash\mathbb{H}^n}$. Indeed:

$$d_{\mathbb{V}^\perp\backslash\mathbb{H}^n}(\delta_r(p), \delta_r(p')) = \inf_{q\in\mathbb{V}} d_{cc}(q\delta_r(p), \delta_r(p')) = r \inf_{q\in\mathbb{V}} d_{cc}(\delta_{1/r}(q)p, p') = r d_{\mathbb{V}^\perp\backslash\mathbb{H}^n}(p, q).$$

The last equality follows from the fact that $\mathbb{V}$ is homogeneous (so that $\delta_{1/r}(q) \in \mathbb{V}$).

Group action by $\mathbb{H}^{n-m}$

We embed $\mathbb{H}^{n-m}$ in $\mathbb{H}^n$ by the map $\xi \mapsto \widehat{\xi}$ given by,

$$(u_1, \dots, u_{n-m}, v_1, \dots, v_{n-m}, \tau) \mapsto (0, \dots, 0, u_1, \dots, u_{n-m}, 0, \dots, 0, v_1, \dots, v_{n-m}, \tau),$$

where in the right hand side the first m coordinates and coordinates $n+1$ through $n+m$ are all zero. With this notation we can see that $\mathbb{H}^{n-m}$ acts on $\mathbb{H}^n$ by "left translation" via the map

$$L_\xi p = \widehat{\xi} p.$$

To see that this action is isometric, note that for each $\xi \in \mathbb{H}^{n-m}$, $\widehat{\xi}$ commutes with elements of $\mathbb{V}$. Indeed, writing $q = (z, 0) \in \mathbb{V}$ and $\widehat{\xi} = (\widehat{w}, \tau)$, it is not hard to see that $\omega(\widehat{w}, z) = 0$. Because of this,

$$\begin{aligned}
d_{\mathbb{V}^\perp\backslash\mathbb{H}^n}(L_\xi p, L_\xi p') &= \inf_{q\in\mathbb{V}} d_{cc}\left(q\widehat{\xi}p, \widehat{\xi}p'\right) \\
&= \inf_{q\in\mathbb{V}} d_{cc}\left(\widehat{\xi}qp, \widehat{\xi}p'\right) \\
&= \inf_{q\in\mathbb{V}} d_{cc}(qp, p') = d_{\mathbb{V}^\perp\backslash\mathbb{H}^n}(p, p').
\end{aligned}$$

This action is smooth with respect to the quotient topology but it is not transitive. For a point $(0, \dots, 0, x_{m+1}, \dots, x_n, y_1, \dots, y_n, t) \in V^\perp$ its orbit consist exactly of all other points of the form $(0, \dots, 0, x'_{m+1}, \dots, x'_n, y_1, \dots y_m, y'_{m+1}, y'_n, t')$. Therefore, the orbit space is parametrized by $\mathbb{R}^m$.

Group action by $U(n-m)$

Similarly, we embed $U(n-m)$ into $U(n)$ via the map $R \mapsto \widetilde{R}$ given for each $z = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$ by

$$\widetilde{R}z = \widetilde{z}.$$

Here

$$\widetilde{z} = (x_1 \ldots, x_m, \widetilde{x}_{m+1}, \ldots, \widetilde{x}_n, y_1 \ldots, y_m, \widetilde{y}_{m+1}, \ldots, \widetilde{y}_n)$$

with

$$(\widetilde{x}_{m+1}, \ldots, \widetilde{x}_n, \widetilde{y}_{m+1}, \ldots, \widetilde{y}_n) = R(x_{m+1}, \ldots, x_n, y_{m+1}, \ldots, y_n).$$

In this way $U(n-m)$ acts on $\mathbb{V}^\perp \backslash \mathbb{H}^n$ via $p \mapsto \widehat{\mathcal{R}}p := \left(\widetilde{R}z, t\right)$ where $p = (z,t) \in V^\perp \simeq \mathbb{V}^\perp \backslash \mathbb{H}^n$. Once again, it is not hard to check that this action, as an action naturally extended to all of $\mathbb{H}^n$, fixes $\mathbb{V}$ pointwise. Therefore $\widehat{\mathcal{R}}(qp) = q\widehat{\mathcal{R}}p$ for each $q \in \mathbb{V}$ and $p \in V^\perp$. Since $U(n)$ acts isometrically on $\mathbb{H}^n$, it follows that

$$\begin{aligned} d_{\mathbb{V}^\perp \backslash \mathbb{H}^n}\left(\widehat{\mathcal{R}}p, \widehat{\mathcal{R}}p'\right) &= \inf_{q \in \mathbb{V}} d_{cc}\left(q\widehat{\mathcal{R}}p, \widehat{\mathcal{R}}p'\right) \\ &= \inf_{q \in \mathbb{V}} d_{cc}\left(\widehat{\mathcal{R}}(qp), \widehat{\mathcal{R}}p'\right) \\ &= \inf_{q \in \mathbb{V}} d_{cc}(qp, p') = d_{\mathbb{V}^\perp \backslash \mathbb{H}^n}(p, p'). \end{aligned}$$

Like the $\mathbb{H}^{n-m}$ action, the action by $U(n-m)$ is smooth but not transitive. The orbit of a point $(0, \ldots, 0, x_{m+1}, \ldots, x_n, y_1, \ldots, y_n, t) \in V^\perp$ consists of all other points of the form $(0, \ldots, 0, x'_{m+1}, \ldots, x'_n, y_1, \ldots y_m, y'_{m+1}, y'_n, t)$. Therefore, the orbit space is parametrized by $\mathbb{R}^{m+1}$.

The group action by $\mathbb{H}^{n-m}$ reveals that there are "$\mathbb{R}^m$ many" copies of the set $\mathbb{H}^{n-m}$ embedded in $V^\perp$ in a natural way. More precisely, using the notation $p = (x_1, x_2, y_1, y_2, t) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R} = \mathbb{H}^n$, for a fixed $\tilde{y} \in \mathbb{R}^m$ we denote by $U_{\tilde{y}}$ the orbit $U_{\tilde{y}} = \{L_\xi(0, \tilde{y}, 0, 0) \in \mathbb{H}^n : \xi \in \mathbb{H}^{n-m}\}$. The map $\mathbb{H}^n \to U_{\tilde{y}}$ given by $(x, y, t) \to (0, x, \tilde{y}, y, t)$ gives a natural embedding of the set $\mathbb{H}^n$ into $V^\perp$.

Proposition 5.1. *The restrictions of $d_{\mathbb{V}_0^\perp \backslash \mathbb{H}^n}$ and d_{cc} to $U_{\tilde{0}}$ coincide.*

Proof. The proof begins with the following claim: For any $x_1 \in \mathbb{R}^m, x_2, y_2 \in \mathbb{R}^{n-m}$, and

$t \in \mathbb{R}$

$$d_{cc}((x_1, x_2, 0, y_2, t), 0) \geq d_{cc}((0, x_2, 0, y_2, t), 0).$$

To prove the claim assume γ is a horizontal path such that $\gamma(0) = 0$ and $\gamma(1) = (x_1, x_2, 0, y_2, t)$. The horizontality condition tells us that

$$\dot{t}(s) = \frac{1}{2}(\dot{y}_1(s)x(s) + \dot{y}_2(s)x_2(s) - \dot{x}_1(s)y_1(s) - \dot{x}_2(s)y_2(s)),$$

but since $y_1(s) = 0$ it follows that γ satisfies

$$\dot{t}(s) = \frac{1}{2}(\dot{y}_2(s)x_2(s) - \dot{x}_2(s)y_2(s)).$$

Therefore, the curve $\tilde{\gamma}(s) = (0, x_2(s), 0, y_2(s), t(s))$ is horizontal with $\tilde{\gamma}(0) = 0$ and $\tilde{\gamma}(1) = (0, x_2, 0, y_2, t)$. The claim follows by taking infimums.

Now, as mentioned earlier, it is easy to check that $\omega(\mathbb{V}, U_{\tilde{0}}) = 0$ so that $\mathbb{V}$ and $U_{\tilde{0}}$ commute, and moreover, for $q \in \mathbb{V}$ and $p \in U_{\tilde{0}}$, $qp = q + p$. In particular, if $p, p' \in U_{\tilde{0}}$ it follows that

$$\begin{aligned} d_{\mathbb{V}_0^\perp \backslash \mathbb{H}^n}(p', p) &= \inf_{q \in \mathbb{V}} d_{cc}(qp', p) \\ &= \inf_{q \in \mathbb{V}} d_{cc}(p^{-1}qp', 0) \\ &= \inf_{q \in \mathbb{V}} d_{cc}(q + p^{-1}p', 0) \\ &= d_{cc}(p^{-1}p', 0) = d_{cc}(p', p), \end{aligned}$$

where the first equality in the last line follows from the claim. This completes the proof of the proposition. □

Corollary 5.2. *The map* $\iota : (\mathbb{H}^{n-m}, d_{cc,\mathbb{H}^{n-m}}) \to (V^\perp, d_{\mathbb{V}_0^\perp \backslash \mathbb{H}^n})$ *given by* $\iota(x, y, t) = (0, x, 0, y, t)$ *is an isometric embedding.*

Proof. It is clear that $\iota : \mathbb{H}^{n-m} \to U_{\tilde{0}} \subset V^\perp$ is bijective. By Proposition 5.1,

$$\begin{aligned} &d_{\mathbb{V}_0^\perp \backslash \mathbb{H}^n}(\iota(x, y, t), \iota(u, v, s)) \\ &\quad = d_{\mathbb{V}_0^\perp \backslash \mathbb{H}^n}((0, x, 0, y, t), (0, u, 0, v, s)) \\ &\quad = d_{cc}((0, x, 0, y, t), (0, u, 0, v, s)) \\ &\quad = d_{cc,\mathbb{H}^{n-m}}((x, y, t), (u, v, s)). \end{aligned}$$

□

Proposition 5.1 and its corollary, do not hold for $\tilde{y} \neq 0$. In particular, for $\tilde{y} \neq 0$, the natural embedding of the orbit $U_{\tilde{y}}$ is not an isometric, nor bi-Lipschitz, embedding of $(\mathbb{H}^{n-m}, d_{cc})$. Indeed, if $\tilde{y} \neq 0$ and $p = (0, x, \tilde{y}, y, 0), q = (0, u, \tilde{y}, v, 0) \in U_{\tilde{y}}$, we have

$$d_{\mathbb{H}^n}(p, q) = [(|x-u|^2 + |y-v|^2)^2 + 4(u \cdot y - x \cdot v)^2]^{1/4}, \tag{5.24}$$

whereas,

$$\begin{aligned} d_{\mathbb{V}_0^\perp \backslash \mathbb{H}^n}(p, q) &\simeq \inf_{p' \in \mathbb{V}} d_{\mathbb{H}^n}(p'p, q) \\ &= \inf_{\tilde{x} \in \mathbb{R}^m} ||(\tilde{x}, x-u, 0, y-v, -\tilde{x} \cdot \tilde{y} - \frac{1}{2}(x \cdot v - y \cdot u))||_{\mathbb{H}^n}. \end{aligned}$$

In particular, choosing $\tilde{x} = -\frac{1}{2}(x \cdot v - y \cdot u)\frac{\tilde{y}}{|\tilde{y}|^2}$ gives the upper bound

$$d_{\mathbb{V}_0^\perp \backslash \mathbb{H}^n}(p, q) \lesssim [\frac{1}{4}(x \cdot v - y \cdot u)^2 + |x-u|^2 + |y-v|^2]^{1/2}.$$

Comparing with (5.24) one sees that $d_{\mathbb{V}_0^\perp \backslash \mathbb{H}^n}\lfloor_{U_{\tilde{y}}}$ cannot be bi-Lipschitz equivalent to $d_{\mathbb{H}^n}\lfloor_{U_{\tilde{y}}}$, and therefore to $d_{cc}\lfloor_{U_{\tilde{y}}}$. This is analogous to the situation in $\mathbb{H}$. Indeed, following the standard notation for the n$^{\text{th}}$ Heisenberg group, $\mathbb{H}^0 = \mathbb{R}^{2(0)} \times \mathbb{R}$ is not $\{0\}$ but rather the $t-$axis. In other words $(\mathbb{H}^0, d_{cc})$ is simply the snowflaked real line $(\mathbb{R}, d_E^{1/2})$. In particular, in $\mathbb{H}^1$, Corollary 5.2 makes reference to the fact that the critical line of $\mathbb{G}$ is an isometric copy of $(\mathbb{R}, d_E^{1/2})$ whereas all other vertical lines are Riemannian copies of the real line.

We expect the space $\mathbb{V}^\perp \backslash \mathbb{H}^n$, in the case $n > 1$, to behave in an analogous way to $\mathbb{G}$, in that the metric should be Riemannian away from the critical subspace $U_{\tilde{0}}$ and extend as a Carnot-Caratheodory metric to $U_{\tilde{0}}$. We were unable to prove this, so it remains an interesting problem to check if $(V^\perp, d_{\mathbb{V}^\perp \backslash \mathbb{H}^n})$ is isometrically equivalent (or at least bi-Lipschitz equivalent) to a non equi-regular Carnot-Caratheodory space. In other words, can one find bracket generating vector fields in $\mathbb{R}^{2n-m+1}$ such that $\mathbb{R}^{2n-m+1}$ with the induced Carnot-Caratheodory distance is isometrically (or even bi-Lipschitz) equivalent to $(V^\perp, d_{\mathbb{V}^\perp \backslash \mathbb{H}^n})$?

Despite not having an explicitly computable formula for the distance, it is still possible to obtain dimension distortion bounds for the family of right-coset projections. The main results in [61] are,

Theorem 5.11. *For $1 \le m \le n$ and any Borel set $A \subseteq \mathbb{H}^n$,*

$$\min\{2\dim_E A, \dim_E A + 1\}$$
$$\ge \dim_E P^R_{\mathbb{V}^\perp}(A) \ge \begin{cases} \dim_E A & \text{if } \dim_E A \in [0, 2n-m] \\ 2n-m & \text{if } \dim_E A \in [2n-m, 2n] \\ \dim_E A - m & \text{if } \dim_E A \in [2n, 2n+1] \end{cases} \tag{5.25}$$

for $\mu_{n,m}$-a.e. $V \in G_h(n,m)$, and

$$\dim_{\mathbb{V}\backslash\mathbb{H}^n} P^R_{\mathbb{V}^\perp}(A) \ge \begin{cases} \frac{\dim_{\mathbb{H}^n} A}{2} & \text{if } \dim_{\mathbb{H}^n} \in [0,2] \\ \dim_{\mathbb{H}^n} A - 1 & \text{if } \dim_{\mathbb{H}^n} \in [2, 2n-m+1] \\ 2n-m & \text{if } \dim_{\mathbb{H}^n} \in [2n-m+1, 2n+1] \\ \dim_{\mathbb{H}^n} A - m - 1 & \text{if } \dim_{\mathbb{H}^n} \in [2n+1, 2n+2] \end{cases} \tag{5.26}$$

for $\mu_{n,m}$-a.e. $V \in G_h(n,m)$. If $\dim_E A \le 2n-m$ then (5.25) is sharp, and if $\dim_{\mathbb{H}^n} A \le 2n+1-m$ then (5.26) is sharp.

In addition, this results allowed us to obtain the following bound for the case of standard (left coset) projections in $\mathbb{H}^n$ with respect to the ambient metric.

Theorem 5.12. *For $1 \le m \le n$ and any Borel set $A \subseteq \mathbb{H}^n$,*

$$\dim_{\mathbb{H}^n} P^L_{\mathbb{V}^\perp}(A) \ge \begin{cases} \frac{\dim_{\mathbb{H}^n} A}{2} & \text{if } \dim_{\mathbb{H}^n} \in [0,2] \\ \dim_{\mathbb{H}^n} A - 1 & \text{if } \dim_{\mathbb{H}^n} \in [2, 2n-m+1] \\ 2n-m & \text{if } \dim_{\mathbb{H}^n} \in [2n-m+1, 2n+1] \\ \dim_{\mathbb{H}^n} A - m - 1 & \text{if } \dim_{\mathbb{H}^n} \in [2n+1, 2n+2] \end{cases} \tag{5.27}$$

for $\mu_{n,m}$-a.e. $V \in G_h(n,m)$.

Theorem 5.12 improves this almost sure lower bound in the range $\dim_{\mathbb{H}^n} A \in [2, 2n+1]$. The new lower bound reads

$$\dim_{\mathbb{H}^n} P^L_{V^\perp}(A) \ge \begin{cases} \dim_{\mathbb{H}^n} A & \text{if } \dim_{\mathbb{H}^n} A \in [0,1] \\ 1 & \text{if } \dim_{\mathbb{H}^n} A \in [1,2] \\ \dim_{\mathbb{H}^n} A - 1 & \text{if } \dim_{\mathbb{H}^n} A \in [2, 2n-m+1] \\ 2n-m & \text{if } \dim_{\mathbb{H}^n} A \in [2n-m+1, 2n+1] \\ 2(\dim_{\mathbb{H}^n} A - n - 1) - m & \text{if } \dim_{\mathbb{H}^n} A \in [2n+1, 2n+2], \end{cases}$$

for $\mu_{2n,m}$-almost every $V \in G_h(2n,m)$.

In proving these theorems, the bulk of the work goes into proving (5.25). Both (5.26) and (5.27) follow directly from (5.25) by applying the Dimension Comparison Principle. In turn, the proof of 5.25 follows the same potential-theoretic technique used in the proof of the standard projection theorems in Euclidean space. By way of Theorem 2.1, one picks a measure $\mu \in \mathcal{M}(A)$ with finite σ-energy for some $\sigma < \dim_{\mathbb{H}} A$ and then tries to bound the integral $\int I_s(P^R_{\mathbb{V}^\perp \#}\mu, d_{\mathbb{V}^\perp \backslash \mathbb{H}^n}) d\mu_{2n,m}$ by $I_\sigma(\mu)$ for an appropriate choice of s. The proof is technical and uses different ad-hoc strategies in different parts of the range of dimensions for A, so it is omitted from this thesis but interested readers are referred to [61, Section 4].

CHAPTER 6

A POTENTIAL THEORETIC APPROACH TO BOX AND PACKING DIMENSIONS IN THE HEISENBERG GROUP

The potential theoretic approach to Hausdorff dimension discussed in Chapter 2, is known to be valid in quite general metric spaces. So motivated by Section 3.5 one is led to consider the more general energies

$$\mathcal{E}_r^\sigma(\mu) = \int_X \int_X \min\left\{1, \frac{r^\sigma}{d(x,y)^\sigma}\right\} d\mu(y)d\mu(x),$$

and ask what are the most general metric spaces for which Theorem 3.9 holds. Using Assuad's Embedding theorem, one can show without much effort, that a version of Theorem 3.9 holds for doubling metric measure spaces but the value of σ after which $\dim_B^\sigma(A) = \dim_B(A)$, depends on the snowflaking constant as well as the dimension of the co-domain of the Assuad embedding, and can be much larger than the doubling dimension of the metric space X. I am interested in the case $X = \mathbb{H}^n$ and $d = d_{\mathbb{H}^n}$. In that case we get,

Theorem 6.1. *Let A be a compact subset of $\mathbb{H}^n$, then*

$$\overline{\dim}_{\mathbb{H},B}^\sigma(A) = \overline{\dim}_{\mathbb{H},B}(A), \text{ and } \underline{\dim}_{\mathbb{H},B}^\sigma(A) = \underline{\dim}_{\mathbb{H},B}(A),$$

for all $\sigma \geq Q = 2n+2$.

Here $\dim_{\mathbb{H},B}$ denotes the box counting dimension with respect to the metric $d_{\mathbb{H}}$, and $\dim_{\mathbb{H},B}^\sigma$ denotes the σ-dimensional profile defined with respect to the kernel $\Phi_r^\sigma(p) = \min\{1, \frac{r^\sigma}{||p||_{\mathbb{H}}^\sigma}\}$. Moreover, this also holds for packing dimension. In the proof of Theorem 6.1 I make use of Strichartz's self similar tilings of $\mathbb{H}$. These tilings are the Heisenberg analogue of tilings by cubes in Euclidean space and are ofter refer to as "Heisenberg cubes". For a complete description of this tiles the reader is referred to [100], [101] and [106]. Here we simply introduce the tilings for higher dimensional Heisenberg groups, $\mathbb{H}^n$, and the properties that will be relevant in the proof.

Fix an integer $b \geq 2$, write $\mathbb{H}_{\mathbb{Z}}^n := \{p = (z,t) \in \mathbb{H}^n : z \in \mathbb{Z} + i\mathbb{Z}, t \in \mathbb{Z}\}$, and let

$D := \{p = (z,t) \in \mathbb{H}^n_{\mathbb{Z}} : |z|_\infty \leq \frac{b-1}{2}, |t| \leq \frac{b^2-1}{2}\}$. Then, there exist a non-empty, compact, self similar set $T_0 \in \mathbb{H}^n$ such that

$$\delta_b(T_0) = \bigcup_{p \in D} p * T_0.$$

By induction, one has a tiling of $\mathbb{H}^n$ by self similar sets:

$$\mathbb{H}^n = \bigcup_{p \in \mathbb{H}^n_{\mathbb{Z}}} p * T_0.$$

More generally, for $m \in \mathbb{Z}$, and denoting by T^m_p the tile $p * \delta_{b^m}(T_0)$, we have

$$\mathbb{H}^n = \bigcup_{p \in \delta_{b^m}(\mathbb{H}^n_{\mathbb{Z}})} T^m_p.$$

Now we state without proof some of the properties that we will need. Write $\mathcal{T}_m := \{T^m_p : p \in \delta_{b^m}(\mathbb{H}^n_{\mathbb{Z}}\}$, then

(i) If $T, T' \in \mathcal{T}_m$ are such that $T \neq T'$, then $Int(T) \cap Int(T') = \varnothing$.

(ii) $\mathcal{T}_m$ refines $\mathcal{T}_{m+1}$ in the sense that every tile in $\mathcal{T}_m$ is completely contained inside a unique tile in $\mathcal{T}_{m+1}$.

(iii) For every $m \in \mathbb{Z}$ and $p \in \delta_{b^m}(\mathbb{H}^n_{\mathbb{Z}})$, we have $B_{\mathbb{H}}(p, r_{in} b^m) \subset T^m_p \subset B_{\mathbb{H}}(p, r_{out} b^m)$, where $r_{in} := \sqrt{\frac{1}{4n} - \frac{1}{2b+2}}$, and $r_{out} := \begin{cases} \sqrt{\frac{1}{4n} + \frac{1}{2b+2}} & n = 1 \\ \frac{1}{2} & n \geq 2 \end{cases}$.

(iv) If we call the tiles T^m_p and T^m_q adjacent, with $p = (z,t), q = (w,s)$, whenever $|z - w|_\infty \leq b^m$, and $|t - s| \leq 18nb^{2n}$, then each tile in $\mathcal{T}_m$ is adjacent to $3^{2n}(36n+1)$ other tiles in $\mathcal{T}_m$. Note that this notion of adjacent does not mean the tiles are adjacent in the literal sense.

(v) The distance between non-adjacent tiles is at least $2(1 - r_{out})b^m \geq b^m/2$

Lemma 6.1. *Suppose $diam(A) = \frac{b^m}{3}$, then A can intersect at most $3^{2n}(36n+1)$ tiles in $\mathcal{T}_m$*

Proof. Let $T \in \mathcal{T}_m$ be such that $A \cap T \neq \varnothing$, then A can only intersect tiles adjacent to T. Indeed, if A intersects a non-adjacent tile $T' \in \mathcal{T}_m$ we have that

$$diam(A) \geq dist(T, T') \geq \frac{b^m}{2} > \frac{b^m}{3}$$

□

We may also prescribe the diameters of the tiles by dilating $\mathcal{T}_m$ by the appropriate scalar. Set $d_0 =$diam(T_0) so that diam$(T_p^m) = b^m d_0$. We might dilate each tile in $\mathcal{T}_m$ by $\delta_{rb^{-m}d_0^{-1}}$ to obtain a tiling of $\mathbb{H}^n$ by self-similar tiles of diameter r. Denote by $\mathcal{T}_m^r$ the collection of such tiles. It is clear from Lemma 6.1 that a set A with diam$(A) = \frac{r}{3d_0}$ can intersect at most $3^{2n}(36n+1)$ tiles in $\mathcal{T}_m^r$.

From here, the proof of Theorem 6.1 follows the same lines as the Euclidean proof, by showing that the logarithmic behaviors of $\mathcal{N}_r(A)$ and $C_r(A)$ are comparable as $r \to 0$.

Lemma 6.2. *Let $A \in \mathbb{H}^n$ be a compact set. Suppose there exist $\mu \in \mathcal{M}(A)$ such that for some $\gamma > 0$*

$$(\mu \times \mu)(\{(p,q) : ||q^{-1}p|| \le r) \le \gamma, \tag{6.1}$$

then $\mathcal{N}_{\frac{r}{3b_0}}(A) \ge \frac{c_n}{\gamma}$, where c_n is a constant depending only on n. This is true, in particular, if for some $\sigma > 0$, $\mathcal{E}_r^\sigma(\mu) \le \gamma$.

Proof. Denote by $\mathcal{T}_m^r(A)$ the set of tiles in $\mathcal{T}_m^r$ that intersect A, and suppose there are $T'(A)$ of them. Then

$$\begin{aligned}
1 = \mu^2(A) &= \left(\sum_{\mathcal{T}_m^r(A)} \mu(T)\right)^2 \\
&= T'(A) \sum_{\mathcal{T}_m^r(A)} \mu(T)^2 \\
&= T'(A) \sum_{\mathcal{T}_m^r(A)} (\mu \times \mu)(\{(p,q) \in T \times T\}) \\
&\le T'(A)(\mu \times \mu)(\{(p,q) : ||q^{-1}p|| \le r\}) \\
&\le T'(A)\gamma \le 3^{2n}(36n+1)\mathcal{N}_{\frac{r}{3b_0}}(A)\gamma.
\end{aligned}$$

The last inequality follows from the the fact that each set accounted in $\mathcal{N}_{\frac{r}{3b_0}}$ intersects at most $3^{2n}(36n+1)$ tiles in $\mathcal{T}_m^r$. It is clear that

$$\mathbb{1}_{\mathbb{B}_{\mathbb{H}}}(p,r) \le \Phi_r^\sigma(p),$$

Therefore if there is $\mu \in \mathcal{M}(A)$ such that $\mathcal{E}_r^\sigma(\mu) \le \gamma$ then (6.1) holds.

□

Lemma 6.3. *Let $A \subset \mathbb{H}^n$ be a non-empty compact set and let $\mu \in \mathcal{M}(A)$ be such that* *for* $\sigma \geq Q := 2n+2$ *and some* $\gamma > 0$, $\mathcal{F}_r^\sigma \geq \gamma$ *for every* $p \in A$. *Then*

$$\mathcal{N}_r(A) \leq \begin{cases} \frac{c_n \lceil \log_b \frac{diam A}{r} + 1 \rceil}{\gamma} & \text{if } \sigma = Q \\ \frac{c_{\sigma,n}}{\gamma} & \text{if } \sigma > Q \end{cases} \tag{6.2}$$

Proof. To simplify notation, let $M = \lceil \log_b \frac{\text{diam} A}{r} \rceil$. For every $p \in A$ we have:

$$\begin{aligned} \mathcal{F}_r^\sigma(p,\mu) &= \int_{\mathbb{H}^n} \Phi_r^\sigma(q^{-1}p) d\mu(q) \\ &\leq \mu(\mathbb{B}_{\mathbb{H}^n}(p,r)) + \sum_{m=0}^{M-1} \int_{\mathbb{B}_{\mathbb{H}^n}(p,b^{m+1}r) \setminus \mathbb{B}_{\mathbb{H}^n}(p,b^m r)} b^{-m\sigma} d\mu(q) \\ &\leq \mu(\mathbb{B}_{\mathbb{H}^n}(p,r)) + \sum_{m=0}^{M-1} b^{-m\sigma} \mu(\mathbb{B}_{\mathbb{H}^n}(p,b^{m+1}r)) \\ &\leq b^\sigma \sum_{m=0}^{M} b^{-m\sigma} \mu(\mathbb{B}_{\mathbb{H}^n}(p,b^m r)). \end{aligned} \tag{6.3}$$

Now, let $\{\mathbb{B}_{\mathbb{H}^n}(p_i,r)\}_{i=1}^{B'_r(A)}$ be a packing of A by balls where $B'_r(A)$ is the maximal number of disjoint balls of radius r that can be chosen with $p_i \in A$. Equation (6.3) tells us that for each i,

$$\gamma \leq \int \Phi_r^{\sigma(q^{-1}p)} d\mu(q) \leq \sum_{m=0}^{M} b^{\sigma(1-m)} \mu(\mathbb{B}_{\mathbb{H}^n}(p_i, b^m r)).$$

Therefore summing over i gives,

$$B'_r(A)\gamma \leq \sum_{m=0}^{M} b^{\sigma(1-m)} \sum_{i=1}^{B'_r(A)} \mu(\mathbb{B}_{\mathbb{H}^n}(p_i, b^m r)). \tag{6.4}$$

In addition, for each $p \in \mathbb{H}^n$ and $R > 0$, $Vol(\mathbb{B}_{\mathbb{H}^n}(p,R)) \sim R^Q$. It is clear then that, by volume comparison, for $p \in A$ at most cb^{mQ} of the $p_i's$ lie inside $\mathbb{B}_{\mathbb{H}^n}(p, b^m r)$. Therefore p belongs to at most $c'B^{mQ}$ of the balls $\mathbb{B}_{\mathbb{H}^n}(p_i, b^m r)$. From here we get that for $\sigma \geq Q$,

$$\sum_{i=0}^{B'_r(A)} \mu(\mathbb{B}_{\mathbb{H}^n}(p_i, b^m r)) \leq b^{mQ}\mu(A) = b^{Q+\sigma} b^{-\sigma(1-m)} b^{m(Q-\sigma)} \leq b^{Q+\sigma} b^{-\sigma(1-m)}. \tag{6.5}$$

The computation now splits into 2 cases.

Case 1: $\sigma = Q$

In this case, from (6.4) we obtain that for some m,

$$\frac{B'_r(A)\gamma}{M+1} \leq b^{\sigma(1-m)} \sum_{i=1}^{B'_r(A)\mu} (\mathbb{B}_{\mathbb{H}^n}(p_i, b^m r)).$$

Combining this with (6.5), we obtain the bound

$$B'_r(A) \leq \frac{b^2 Q(M+1)}{\gamma}.$$

Finally using the fact that $\mathcal{N}_r(A) \leq c'_n B'_r(A)$ the claim for $\sigma = Q$ follows.

Case 2: $\sigma > Q$.
We claim that for some m,

$$B'_r(A)\gamma b^{m(Q-\sigma)}(1 - b^{Q-\sigma}) \leq b^{\sigma(1-m)} \sum_{i=0}^{B'_r(A)} \mu(\mathbb{B}_{\mathbb{H}^n}(p_i, b^m r)).$$

Indeed if this failed for every m then we have an upper bound by a geometric sum,

$$\begin{aligned} \sum_{m=0}^{M} b^{\sigma(1-m)} \sum_{i=0}^{B'_r(A)} \mu(\mathbb{B}_{\mathbb{H}^n}(p_i, b^m r)) &\leq B'_r(A)\gamma(1 - b^{Q-\sigma}) \sum_{m=0^M} b^{m(Q-\sigma)} \\ &\leq B'_r(A)\gamma(1 - b^{Q-\sigma}) \frac{1 - b^{(Q-\sigma)M}}{1 - b^{Q-\sigma}} \\ &\leq B'_r(A)\gamma(1 - (M/r)^{Q-\sigma}) \leq B'_r(A)\gamma, \end{aligned}$$

which contradicts (6.4). So once again, using (6.5), we get,

$$\mathcal{N}_r(A) \leq c'_n B'_r(A) \leq c'_n \frac{b^{Q+\sigma} b^{-\sigma(1-m)}}{\gamma b^{m(Q-\sigma)}(1 - b^{Q-\sigma})} \leq \frac{C_{n,\sigma}}{\gamma},$$

as claimed. □

Proof of Theorem 6.1. For $\sigma \geq Q$, Lemma 6.2 gives us that

$$\liminf_{r\to 0} \frac{\log(\mathcal{N}_r(A))}{\log(1/r)} \leq \liminf_{r\to 0} \frac{\log(C^\sigma_r(A))}{\log(1/r)}.$$

On the other hand, Lemma 6.3 gives,

$$\liminf_{r\to 0}\frac{\log(C_r^\sigma(A))}{\log(1/r)} \le \liminf_{r\to 0}\frac{\log(\mathcal{N}_{\frac{r}{3b_0}}(A))}{\log(1/r)} = \liminf_{r\to 0}\frac{\log(\mathcal{N}_{\frac{r}{3b_0}}(A))}{\log(3b_0/r)} = \liminf_{r\to 0}\frac{\log(\mathcal{N}_r(A))}{\log(1/r)}.$$

It follows that

$$\underline{\dim}^\sigma_{\mathbb{H},B}(A) = \underline{\dim}_{\mathbb{H},B}(A).$$

Similarly,

$$\overline{\dim}^\sigma_{\mathbb{H},B}(A) = \overline{\dim}_{\mathbb{H},B}(A).$$

□

Note that the equivalent statement for packing dimension follows directly from Theorem 2.4.

A box-counting (and packing) dimension distortion result for horizontal projections in $\mathbb{H}^n$, follows trivially from the work of Falconer and Howroyd. In [37] the following is proven

Theorem 6.2 (Theorem 2.7, [38]). *Let $A \subset \mathbb{R}^n$ $(\Omega, \mathbb{P})$ be a probability space and $\{f_\omega : A \to \mathbb{R}^m\}$ be a family of maps such that there exist constants $c, \gamma, \sigma > 0$ with*

$$\mathbb{P}(|f_\omega(x) - f_\omega(y)| \le r) \le c\varphi^\sigma_{r^\gamma}(x-y).$$

Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$dim_{E,B}(A) \ge \gamma dim^m_{E,B}(A)$$

Since Euclidean orthogonal projections are 1-Lipschitz, we trivially have

$$\dim_{E,B} P_V(A) = \dim^m_{E,B} P_V(A) \le \dim^m_{E,B} A$$

But moreover, by Lemma 3.1, isotropic projections satisfy,

$$\varphi^m_r(x) \le \mu_{n,m}\{V \in G_h(2n,m) : |P_V(x)| \le r\} \le c_{n,m}\varphi^m_r(x), \tag{6.6}$$

so the family of maps $\{P_V : V \in G_h(2n,m)\}$ with the probability measure $\mu_{2n,m}$ on $G_h(2n,m)$, satisfies Theorem 6.2 with $\gamma = 1, \sigma = m$. We therefore obtain,

Corollary 6.1. *Let $A \subset \mathbb{R}^{2n}$ be a compact set. Then*

$$\dim_{E,B} P_V A = \dim^m_{E,B} A, \text{ for } \mu_{n,m} - a.e. V \in G_h(2n,m),$$

This gives a direct relation between the box counting dimension of $P_V(A)$, for $A \subset \mathbb{H}^n$, and the m box counting profile of the set $\pi(A)$.

Corollary 6.2. *Let $A \subset \mathbb{H}^n$ be a compact set. Then,*

$$\dim_{\mathbb{H}^n,B} P_{\mathbb{V}}A = \dim_{E,B}^m \pi(A), \text{ for } \mu_{n,m} - a.e. V \in G_h(2n,m),$$

This shows that $\dim_{\mathbb{H}^n,B} P_{\mathbb{V}}A$ is $\mu_{2n,m}$-almost surely constant. This however, makes no use of the potentials with respect to the intrinsic metric on $\mathbb{H}^n$. It would be interesting to see if this potential theoretic approach can be used to estimate box-counting and packing dimension distortion by vertical projections.

CHAPTER 7

A FOURIER THEORETIC APPROACH TO HAUSDORFF DIMENSION IN THE HEISENBERG GROUP

This chapter is based on the pre-print [94]. There, the main goal is to use the group Fourier transform on $\mathbb{H}^n$ to establish results analogous to Theorem 2.11, and most importantly, Theorem 2.12. In order to do this, I must first review the Fourier theory of the Heisenberg group. Moreover, in order to get an analogous result to Theorem 2.12, I used an alternative approach to Fourier transform developed recently by H. Bahouri, J.Y. Chemin, and R. Danchin in [3, 4]. My hope is that this chapter, aside from discussing my own work, will serve as a short introduction to both approaches, suitable for a graduate student getting started in the subject. Many of the computations are quite technical and many others are merely algebraic manipulations, as such, most of them are not presented here. The intention is to convey the general ideas of the theory while discussing my contributions to it. All the details and computations can be found in the original paper. For the sake of simplifying certain computations, it is convenient to work with a re-normalized (yet equivalent) definition of the Euclidean Fourier transform. So throughout this chapter, for $f \in L^1(\mathbb{R}^n)$,

$$\widehat{f}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

7.1 Group Fourier Transform in $\mathbb{H}^n$

In the general theory of Fourier analysis on groups, in order to introduce the Fourier transform, one first needs a complete description of the representations of the group. The representation theory of $\mathbb{H}^n$ is not too complicated, it arises naturally from the quantum construction discussed in Section 4.3. Before discussing it further, I need to introduce some notation and recall some definitions. Let H be a Hilbert space and $\mathcal{B}(H)$ the set of bounded operators on H. An operator $U \in \mathcal{B}(H)$ is *unitary* if $U^*U = UU^* = I$, where U^* denotes the adjoint of U. Denote by $\mathcal{U}(H) \subset \mathcal{B}(H)$ the group of unitary operators on H.

For a group G, a map $\rho : G \to \mathcal{U}(H)$ is said to be a *strongly continuous representation* if it is a group homomorphism which is continuous with respect to the strong operator topology on $\mathcal{U}(H)$. That is, for each $\varphi \in H$, the map $\rho^\varphi : G \to H$ given by $\rho^\varphi(g) = \rho(g)\varphi$ is continuous.

A strongly continuous representation, ρ, of a group G, is said to be *irreducible* if there is no non-trivial subspace $W \subset H$ such that $\rho(g)W \subset W$ for all $g \in G$.

In the case of the $\mathbb{H}^n$, for each $\lambda \in \mathbb{R}^*$, the map $\rho_\lambda : \mathbb{H}^n \to \mathcal{U}(L^2(\mathbb{R}^n))$ defined by (4.23), is a strongly continuous, unitary representation of $\mathbb{H}^n$. The fact that it is a homomorphism follows form the discussion in Section 4.3. Strong continuity and irreducibility are not too hard to check, see for instance [103, pp. 8]. In addition, as a consequence of the Stone-Von Neumann theorem, the maps $\{\rho_\lambda : \lambda \in \mathbb{R}^*\}$ give a complete description of the representations of $\mathbb{H}^n$.

Theorem 7.1. *If $\vartheta : \mathbb{H}^n \to \mathcal{U}(L^2(\mathbb{R}^n))$ is a strongly continuous, irreducible representation which is non-trivial at the center, then ϑ is unitarily equivalent to ρ_λ for some value of λ.*

Proof. Since ϑ is non-trivial at the center (i.e. the t-axis) by Schur's lemma, it must map it holomorphically to the center of $\mathcal{U}(L^2(\mathbb{R}^n))$. Moreover, since $\vartheta(0,0,t)$ must be unitary, we must have $c(t) = e^{i\lambda t}$ for some λ. Similarly, the group law requires that

$$\vartheta(x,0,0)\vartheta(0,y,0) = \vartheta(x,y,0)e^{-i\lambda\frac{x\cdot y}{2}},$$

and that

$$\vartheta(0,y,0)\vartheta(x,0,0) = \vartheta(x,y,0)e^{i\lambda\frac{x\cdot y}{2}}.$$

Therefore

$$\vartheta(x,0,0)\vartheta(0,y,0) = e^{i\lambda x\cdot y}\vartheta(0,y,0)\vartheta(x,0,0).$$

By Stone-Von Neumann's theorem, $\vartheta(x,0,0)$ and $\vartheta(0,y,0)$ must be unitarily equivalent to $e(\lambda x)$ and $\tau(y)$. This completes the proof. □

This allows us to introduce the Fourier transform of integrable functions in $\mathbb{H}^n$. For $f \in L^1(\mathbb{H}^n)$ the Fourier transform of f is the operator-valued function

$$\mathcal{F}(f)(\lambda) = \widehat{f}(\lambda) : \mathbb{R}^* \to \mathcal{B}(L^2(\mathbb{R}^n)),$$

given by

$$\widehat{f}(\lambda)\varphi = \int_{\mathbb{H}^n} f(p)\rho_\lambda(p)\varphi dp. \tag{7.1}$$

The integral is in the Bochner sense, that is to say $\widehat{f}(\lambda)\varphi$ is the function such that for every $\psi \in L^2(\mathbb{R}^n)$,

$$\langle \widehat{f}(\lambda)\varphi, \psi\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{H}^n} f(p)\langle \rho_\lambda(p)\varphi, \psi\rangle_{L^2(\mathbb{R}^n)} dp.$$

It is not hard to see that for each λ, $\widehat{f}(\lambda)$ is indeed bounded. In fact, if we denote by $||\cdot||_{op}$ the operator norm in $\mathcal{B}(L^2(\mathbb{R}^n))$, we have $||\widehat{f}(\lambda)||_{op} \leq ||f||_1$. As expected, the Fourier transform takes convolutions to "products", where now the product is composition of operators. Since neither the group law, nor composition of operators are commutative one has to be consistent with the order in which the functions are placed in the convolution. Here,

$$f * g(p) = \int_{\mathbb{H}^n} f(pq^{-1})g(q)dq,$$

so that

$$\widehat{f * g}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda).$$

As mentioned before, the group Fourier transform shares many properties with the classical Fourier transform in $\mathbb{R}^n$. In what follows some of this properties will be discussed, many of them follow from the more general theory of Fourier theory on groups but for the sake of completeness Heisenberg-specific proofs are included.

As one expects from the Fourier transform, depending on the regularity of f, one can obtain more than just boundedness of the operators $\widehat{f}(\lambda)$. To state this formally, I first need to introduce more notation.

A compact operator T is said to be in the Schatten p-class, denoted $S_p(L^2(\mathbb{R}^n))$, if its singular values are in ℓ_p. That is

$$tr[(TT^*)^{p/2}] < \infty.$$

The Schatten p-norm of $T \in S_p(L^2(\mathbb{R}^n))$ is $||T||_{S_p} = Tr[(TT^*)^{p/2}]^{1/p}$. Note that S_1 is the trace class and S_2 is the class of Hilbert-Schmidt operators.

Now, let ϱ be the measure on $\mathbb{R}^*$ given by

$$d\varrho(\lambda) = \frac{|\lambda|^n}{(2\pi)^{n+1}},$$

and denote by $L^p(\mathbb{R}^*, S_p(L^2(\mathbb{R}^n)); d\varrho)$ the space of $S_p(L^2(\mathbb{R}^n))$-valued functions on $\mathbb{R}^*$ which are ϱ measurable and $\int_{\mathbb{R}*} ||T(\lambda)||^p_{S_p} d\varrho(\lambda) < \infty$.

The spaces $L^p(\mathbb{R}^*, S_p(L^2(\mathbb{R}^n)); d\varrho)$ are examples of non-commutative L^p spaces. In particular, the space $L^2(\mathbb{R}^*, S_2(L^2(\mathbb{R}^n)); d\varrho)$ is a non-commutative Hilbert space with inner product

$$\langle T, S\rangle_{L^2(S_2)} = \int_{\mathbb{R}^*} tr(T(\lambda)S(\lambda)^*)d\varrho(\lambda).$$

The simplified notation $L^p(S_p)$ is used to denote these spaces. With these definitions in place, we can continue our discussion by drawing parallels between the Fourier transform in $\mathbb{H}^n$ and the classical Euclidean theory. For instance, one very important property of the classical Fourier transform is Plancherel's theorem. Turns out, there is a non-commutative analogue that holds in $\mathbb{H}^n$.

Theorem 7.2 (Plancherel Theorem)**.** *If $f \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$ then $\widehat{f} \in L^2(\mathcal{S}_2)$ with*

$$||f||_2 = ||\widehat{f}||_{L^2(\mathcal{S}_2)}. \tag{7.2}$$

Moreover, if f and g are in $L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$ then

$$\int_{\mathbb{H}^n} f(p)g(p)dp = \int_{\mathbb{R}^*} tr(\widehat{g}(\lambda)^*\widehat{f}(\lambda)d\varrho(\lambda) \tag{7.3}$$

Since $L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$ is dense in $L^2(\mathbb{H}^n)$, Plancharel theorem extends to all of $L^2(\mathbb{H}^n)$ making the Fourier transform a Hilbert space isomorphism between $L^2(\mathbb{H}^n)$ and $L^2(\mathcal{S}_2)$.

Another important aspect of the Euclidean Fourier transform is its interaction with the Schwartz class. The space $\mathscr{S}(\mathbb{H}^n)$ coincides with $\mathscr{S}(\mathbb{R}^{2n+1})$. As expected, the regularity of functions in $\mathscr{S}(\mathbb{H}^n)$ translate as regularity of their Fourier transform (this will be made much more precise later) and allows the following inversion formula analogous to the Euclidean one.

Theorem 7.3. *If $f \in \mathscr{S}(\mathbb{H}^n)$, then*

$$f(p) = \int_{\mathbb{R}^*} tr(\rho_\lambda(p)^*\widehat{f}(\lambda))d\varrho(\lambda). \tag{7.4}$$

In analogy with Euclidean space, if $\mu \in \mathcal{M}(\mathbb{H}^n)$ one defines its Fourier transform as the $\mathcal{B}(L^2(\mathbb{R}^n))$-valued map $\widehat{\mu} : \mathbb{R}^* \to \mathcal{B}(L^2(\mathbb{R}^n))$ given by

$$\widehat{\mu}(\lambda)\varphi = \int_{\mathbb{H}^n} \rho_\lambda(p)\varphi d\mu(p). \tag{7.5}$$

This is a specific case of the more general theory of characteristic functions of measures on locally compact groups (see for instance H. Heyer [62] and E. Siebert [98]). It is not hard to see that $\widehat{\mu}(\lambda)$ is indeed bounded with $||\widehat{\mu}(\lambda)||_{op} \leq \mu(\mathbb{H}^n)$. Many properties of the Fourier transform of functions also extend to measures. For instance, group convolutions are defined as usual by

$$f * \mu(p) = \int_{\mathbb{H}^n} f(pq^{-1})d\mu(q),$$

and their Fourier transform is a composition of operators

$$\widehat{f * \mu} = \widehat{f}(\lambda)\widehat{\mu}(\lambda).$$

In studying the group Fourier transform of measures, it is extremely useful to use convolution approximations. Approximations to the identity in the convolution algebra $L^1(\mathbb{H}^n)$ coincide with classical approximations to the identity in $L^1(\mathbb{R}^{2n+1})$. The following properties are easy to check

Lemma 7.1. *Let $\{\psi_\epsilon\}_{\epsilon>0}$ be an approximation to the identity in $L^1(\mathbb{H}^n)$.*

1. *$\psi_\epsilon \to \delta_0$ in the weak sense.*
2. *If $f \in L^1(\mathbb{H}^n)$, $\psi * f \to f$ in $L^1(\mathbb{H}^n)$.*
3. *If $g \in \mathcal{C}(\mathbb{H}^n)$ is bounded, $\psi * g \to g$ point-wise.*
4. *If $\mu \in \mathcal{M}(\mathbb{H}^n)$, $\psi_\epsilon * \mu \to \mu$ in the weak sense.*

Here, δ_0 denotes the Dirac distribution. It is possible, by way of (7.5), to explicitly compute $\widehat{\delta_0}$. For $\varphi \in L^2(\mathbb{R}^n)$,

$$\widehat{\delta_0}(\lambda)\varphi = \int_{\mathbb{H}^n} \rho_\lambda(p)\varphi d\delta_0(p) = \rho_\lambda(0)\varphi = \varphi.$$

That is, $\widehat{\delta_0} = I$, the identity operator. One expects that the weak convergence of approximations to the identity, ψ_ϵ, implies some form of convergence of $\widehat{\psi_\epsilon}$ to I. This is indeed the case, for each λ, $\widehat{\psi_\epsilon}(\lambda) \to I$, in the strong operator topology. This is a common theme on Fourier analysis on groups, weak convergence on the group side, translates to convergence on the strong operator topology on the Fourier side.

Indeed, a corollary of this proposition, and the convolution theorem for measures, is that if $\mu \in \mathcal{M}(\mathbb{H}^n)$, the function $\mu_\epsilon = \psi_\epsilon * \mu$ satisfies $\widehat{\mu_\epsilon}(\lambda) \to \widehat{\mu}(\lambda)$ in the strong operator topology.

As mentioned in the introduction, a quick consequence of the extension of $\mathcal{F}_E$ to distributions is that if $f \in \mathscr{S}(\mathbb{R}^n)$ and $\mu \in \mathcal{M}(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} f d\mu = \int_{\mathbb{R}^n} \mathcal{F}_E(f)\overline{\mathcal{F}_E(\mu)} d\xi.$$

Since, as of now, there is no satisfactory extension of the group Fourier transform to the space of distributions, the analogous results in the Heisenberg case does not follow as easily. Nevertheless, with some additional work one can prove the following

Theorem 7.4. *Let $f \in \mathscr{S}(\mathbb{H}^n)$ and $\mu \in \mathcal{M}(\mathbb{H}^n)$, then*

$$\int_{\mathbb{H}^n} f(p) d\mu(p) = \int_{\mathbb{R}^*} tr(\widehat{f}(\lambda)\widehat{\mu}(\lambda)^*) d\varrho(\lambda) \tag{7.6}$$

This extension of Plancherel's formula to measures is what allows to establish my result connecting the integrability of $\widehat{\mu}$ to the density of μ.

Proposition 7.1. *Let $\mu \in \mathcal{M}(\mathbb{H}^n)$.*

(i) If $\widehat{\mu} \in L^2(\mathcal{S}_2)$, then $d\mu = f dp$ for $f \in L^2(\mathbb{H}^n)$.

(ii) If $\widehat{\mu} \in L^1(\mathcal{S}_1)$, then $d\mu = g dp$ for $g \in \mathcal{C}(\mathbb{H}^n)$.

Proof. (i) Since the Fourier transform is an isometric isomorphism between $L^2(\mathbb{H}^n)$ and $L^2(\mathcal{S}_2)$, it follows that $\exists\ f \in L^2(\mathbb{H}^n)$ such that $\widehat{f}(\lambda) = \widehat{\mu}(\lambda)$ for $\varrho - a.e.\ \lambda$. The aim is to show that this f is the density of μ. As before, let $\{\psi_\epsilon\}_{\epsilon>0} \subset \mathcal{C}_c^\infty$ be an approximation to the identity and put $\mu_\epsilon = \psi_\epsilon * \mu$, and $f_\epsilon = \psi_\epsilon * f$. By the convolution theorem,

$$\widehat{\mu_\epsilon} = \widehat{\psi_\epsilon}\widehat{\mu} = \widehat{\psi_\epsilon}\widehat{f} = \widehat{f_\epsilon},\ \ \varrho - \text{almost everywhere },$$

and since $\mu_\epsilon, f_\epsilon \in L^2(\mathbb{H}^n)$, one has that $\mu_\epsilon = f_\epsilon$ almost everywhere. Let ψ be a continuous bounded function on $\mathbb{H}^n$, then

$$\begin{aligned}\int_{\mathbb{H}^n} \psi(p) d\mu(p) &= \lim_{\epsilon\to 0} \int_{\mathbb{H}^n} \psi(p)\mu_\epsilon(p) dp \\ &= \lim_{\epsilon\to 0} \int_{\mathbb{H}^n} \psi(p) f_\epsilon(p) dp \\ &= \int_{\mathbb{H}^n} \psi(p) f(p) dp,\end{aligned}$$

which proves the claim.

(ii) If $\widehat{\mu} \in L^1(\mathcal{S}_1)$ then, $|tr(\rho_\lambda(p)^*\widehat{\mu}(\lambda))| \leq tr(|\rho_\lambda(p)^*\widehat{\mu}(\lambda)|) \leq tr(|\widehat{\mu}(\lambda)|)$. It follows that

$$g(p) := \int_{\mathbb{R}^*} tr(\rho_\lambda(p)^*\widehat{\mu}(\lambda))d\varrho(\lambda)$$

is a well defined continuous function on $\mathbb{H}^n$. As before, the aim is to show that g is the density of μ. To see this, let μ_ϵ be as before. Since $\mu_\epsilon \in \mathscr{S}(\mathbb{H}^n)$, $\widehat{\mu}_\epsilon(\lambda) \in S_1$ for each λ. Also, $\widehat{\mu}_\epsilon(\lambda) \to \widehat{\mu}(\lambda)$ in the strong operator topology. Now, it is possible to extract a subsequence $\{\mu_k\}$ such that

- $\mu_k \in \mathscr{S}(\mathbb{H}^n)$,
- $\mu_k \to \mu$ weakly,
- $\widehat{\mu}_k(\lambda) \to \widehat{\mu}(\lambda)$ in the strong operator topology,
- and $tr(\rho_\lambda(p)^*\widehat{\mu}_k(\lambda)) \to tr(\rho_\lambda(p)^*\widehat{\mu}(\lambda))$.

By Theorem 7.4,

$$\mu_k(p) = \int_{\mathbb{R}^*} tr(\rho_\lambda(p)^*\widehat{\mu}_k(\lambda))d\varrho(\lambda),$$

but also,

$$|tr(\rho_\lambda(p)^*\widehat{\mu}_k(\lambda))| \leq ||\rho_\lambda(p)^*||_{op}||\widehat{\psi}_k(\lambda)||_{op}tr(|\widehat{\mu}(\lambda)|) \lesssim tr(|\widehat{\mu}(\lambda)|) \in L^1(S_1).$$

By the dominated convergence theorem, we have a point-wise limit

$$\begin{aligned}\lim_{k\to\infty} \mu_k(p) &= \lim_{k\to\infty} \int_{\mathbb{R}^*} tr(\rho_\lambda(p)^*\widehat{\mu}_k(\lambda))d\varrho(\lambda) \\ &= \int_{\mathbb{R}^*} tr(\rho_\lambda(p)^*\widehat{\mu}(\lambda))d\varrho(\lambda) = g(p).\end{aligned}$$

Since, for all $k \in \mathbb{N}$, $||\psi_k||_{L^1(\mathbb{H}^n)} \leq 1$, it follows that pointwise convergence, implies weak convergence. That is, for any continuous bounded function ψ,

$$\int_{\mathbb{H}^n} \psi(p)d\mu(p) = \lim_{k\to\infty} \int_{\mathbb{H}^n} \psi(p)\mu_k(p)dp = \int_{\mathbb{H}^n} \psi(p)g(p)dp.$$

This completes the proof.

□

7.2 Fourier Coefficients in $\mathbb{H}^n$

In the Heisenberg group, energy integrals of measures take the form

$$I_\sigma(\mu) = \iint ||q^{-1}p||_{\mathbb{H}^n}^{-\sigma} d\mu(q)d\mu(p).$$

The inner integral is of the form

$$\mu * K_\sigma(p), \tag{7.7}$$

where "$*$" denotes group convolution, and K_σ denotes the Riesz kernel for the Koranyi norm, $K_\sigma(p) = ||p||_{\mathbb{H}^n}^{-\sigma}$ for $0 \leq \sigma < 2n+2$. One would like to use the convolution theorem to compute energy integrals in the frequency domain. However, K_σ is not in L^p for any p, therefore first step in such a computation would be to extend the group Fourier transform to tempered distributions. This has been recently done by the authors in [3] and [4] where they introduced a frequency space for $\mathbb{H}^n$ and, with it, an alternative definition for the Fourier transform in the Heisenberg group. This section will present this approach and all results that will be relevant to the proof of the energy formula.
The main idea behind these Fourier coefficients is, instead of studying the operator $\widehat{f}$, to study its matrix coefficients. The matrix coefficients of an operator, are dependent on the choice of an orthonormal basis, however in this case, it turns out to be convenient to choose a basis comprise of eigenfunctions for the Euclidean Fourier transform.

Definition 7.1 (Hermite functions and re-scaled Hermite functions.)**.** *For $x \in \mathbb{R}$ and $k \in \mathbb{N}$, the k-th Hermite polynomial is the polynomial*

$$\mathscr{H}_k(x) := (-1)^k \left[\frac{d^k}{dx^k} \{e^{-x^2}\} e^{x^2} \right].$$

The (normalized) k-Hermite function is

$$\phi_k(x) := (2^k \sqrt{\pi} k!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} \mathscr{H}_k(x).$$

In higher dimensions, for $x \in \mathbb{R}^n$ and multi-index $\alpha \in \mathbb{N}$,

$$\phi_\alpha(x) := \Pi_{j=1}^n \phi_{\alpha_j}(x_j).$$

Finally, the re-scaled Hermite functions are given for $\alpha \in \mathbb{N}$, $\lambda \in \mathbb{R}^*$, and $x \in \mathbb{R}^n$, by

$$\phi_{\alpha,\lambda}(x) := |\lambda|^{n/4} \phi_\alpha(|\lambda|^{1/2} x).$$

Let $\widetilde{\mathbb{H}}^n = \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{R}^*$, and denote typical points by $\zeta = (\alpha, \beta, \lambda)$. For $f \in L^1(\mathbb{H}^n)$ let $\mathcal{F}_{\mathbb{H}^n}(f) : \widetilde{\mathbb{H}}^n \to \mathbb{C}$ be the map

$$\mathcal{F}_{\mathbb{H}^n}(f)(\zeta) = \mathcal{F}_{\mathbb{H}^n}(f)(\alpha, \beta, \lambda) := \langle \widehat{f}(\lambda)\phi_{\alpha,\lambda}, \phi_{\beta,\lambda}\rangle_{L^2(\mathbb{R}^n)}. \tag{7.8}$$

For simplicity, $\mathcal{F}_{\mathbb{H}^n}(f)$ is interchangeably denoted by $\widehat{f}_{\mathbb{H}^n}$. This transform shares many important properties, or at least analogues thereof, with the classical Fourier transform. Some of these properties are what allowed the authors to extend it to tempered distribution by characterizing the image of $\mathscr{S}(\mathbb{H}^n)$. Many of these properties will be of great relevance later in this work when computing energies of measures via integrals on the frequency domain. Because of this, a brief introduction to this Fourier coefficient approach is given next. Many of the proofs are skipped, but readers are referred to the aforementioned references where all the proofs are presented and more details are given.

Perhaps the most clear advantage of this approach is that the function $\widehat{f}_{\mathbb{H}^n}$ can be written as an integral of f with respect to an appropriate kernel. Specifically, letting $\Phi : \mathbb{H}^n \times \widetilde{\mathbb{H}}^n \to \mathbb{C}$ be given by

$$\Phi(p, \zeta) = \langle \rho_\lambda(p)\phi_{\alpha,\lambda}, \phi_{\beta,\lambda}\rangle_{L^2(\mathbb{R}^n)} = e^{i\lambda t}\int_{\mathbb{R}^n} e^{i\lambda x\cdot\xi}\phi_{\alpha,\lambda}(\xi + \frac{y}{2})\phi_{\beta,\lambda}(\xi - \frac{y}{2})d\xi, \tag{7.9}$$

a quick computation using Fubini's theorem shows that

$$\widehat{f}_{\mathbb{H}^n}(\zeta) = \int_{\mathbb{H}^n} \Phi(p, \zeta) f(p)dp. \tag{7.10}$$

It is worth mentioning that the integral $\int_{\mathbb{R}^n} e^{i\lambda x\cdot\xi}\phi_{\alpha,\lambda}(\xi+\frac{y}{2})\phi_{\beta,\lambda}(\xi-\frac{y}{2})d\xi$ is what is known as the Fourier-Wigner transform of $\phi_{\alpha,\lambda}$ and $\phi_{\beta,\lambda}$. In general, the Fourier-Wigner transform of two functions $\varphi, \psi \in L^2(\mathbb{R}^n)$ is the function on $\mathbb{R}^{2n}$ given by

$$V_\lambda(f, g)(z) := \langle \rho_\lambda(z, 0)f, g\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} e^{i\lambda x\cdot\xi} f(\xi + \frac{y}{2})\overline{g(\xi - \frac{y}{2})}d\xi.$$

This integral formulation of $\mathcal{F}_{\mathbb{H}^n}$ simplifies many computations. For instance, the inversion formula for $\mathscr{S}(\mathbb{H}^n)$ becomes much simpler. Before I state it, the space $\widetilde{\mathbb{H}}^n$ must be endowed with a measure. For $\eta : \widetilde{\mathbb{H}}^n \to \mathbb{C}$, let $d\zeta$ be the measure given by

$$\int_{\widetilde{\mathbb{H}}^n} \eta(\zeta)d\zeta = \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\int_{\mathbb{R}^*} \eta(\alpha, \beta, \lambda)d\rho(\lambda). \tag{7.11}$$

One has, for $f \in \mathscr{S}(\mathbb{H}^n)$,

$$f(p) = \mathcal{F}_{\mathbb{H}^n}^{-1}(\widehat{f}_{\mathbb{H}^n})(p) = \int_{\widetilde{\mathbb{H}}^n} \overline{\Phi(p,z)} \widehat{f}_{\mathbb{H}^n}(\zeta) d\zeta.$$

One of the most important properties of the Fourier transform is its effect on convolutions. With this approach the convolution theorem recasts as follows,

$$\mathcal{F}_{\mathbb{H}^n}(f * g)(\alpha, \beta, \lambda) = \sum_{\gamma \in \mathbb{N}^n} \widehat{f}_{\mathbb{H}^n}(\gamma, \beta, \lambda) \widehat{g}_{\mathbb{H}^n}(\alpha, \gamma, \lambda) =: \widehat{f}_{\mathbb{H}^n} \cdot \widehat{g}_{\mathbb{H}^n}(\alpha, \beta, \lambda). \tag{7.12}$$

Yet another property of the map $\mathcal{F}_{\mathbb{H}^n}$ is that it turns smoothness into decay.

Lemma 7.2. *For any $q \in \mathbb{N}$, there exist $N_q \in \mathbb{N}$ and constant $C_q > 0$ such that*

$$[1 + |\lambda|(|\alpha| + |\beta| + n) + |\alpha - \beta|]^q \, |\widehat{f}_{\mathbb{H}^n}(\alpha, \beta, \lambda)| \leq C_q ||f||_{N_q, \mathscr{S}}, \tag{7.13}$$

where $||f||_{N_q,\mathscr{S}}$ is the classical family of Schwartz semi-norms,

$$||f||_{N, \mathscr{S}(\mathbb{H}^n)} := \sup_{|\alpha| \leq N} ||(1 + |z|^2 + t^2)^{N/2} \partial_{z,t}^{\alpha} f||_{L^\infty(\mathbb{H}^n)}.$$

This decay result motivates the definition of the metric $d_{\widetilde{\mathbb{H}}^n}$ on $\widetilde{\mathbb{H}}^n$ defined by

$$d_{\widetilde{\mathbb{H}}^n}(\zeta, \zeta') = |\lambda(\alpha + \beta) - \lambda'(\alpha' + \beta')|_{\ell^1} + |(\alpha - \beta) - (\alpha' - \beta')|_{\ell^1} + |\lambda - \lambda'|. \tag{7.14}$$

The metric $d_{\widetilde{\mathbb{H}}^n}$ will be relevant later. Now we focus on the characterization of $\mathcal{F}_{\mathbb{H}^n}(\mathscr{S}(\mathbb{H}^n))$, which requires several definitions and results first.

Definition 7.2. *Let $\eta : \widetilde{\mathbb{H}}^n \to \mathbb{C}$ be a function differentiable with respect to λ, and f be a function in $\mathscr{S}(\mathbb{H}^n)$. For $j \in \mathbb{N}$, denote by $\delta_j \in \mathbb{N}^n$ the point with a "1" in the j^{th} coordinate and zeros elsewhere, and set $\zeta_j^{\pm} = (\alpha \pm \delta_j, \beta \pm \delta_j, \lambda)$. Define the following six operators,*

$$\widehat{\Delta}\eta(\zeta) := \frac{1}{|\lambda|}(|\alpha + \beta| + n)\eta(\zeta) + \frac{1}{|\lambda|}\sum_{j=1}^{n}(\sqrt{\alpha_j \beta_j}\eta(\zeta_j^-) + \sqrt{(\alpha_j + 1)(\beta_j + 1)}\eta(\zeta_j^+)), \tag{7.15}$$

$$\widehat{D}_\lambda \eta(\zeta) := \frac{d\eta}{d\lambda}(\zeta) + \frac{n}{\lambda}\eta(\zeta) + \frac{1}{|\lambda|}\sum_{j=1}^{n}(\sqrt{\alpha_j \beta_j}\eta(\zeta_j^-) - \sqrt{(\alpha_j + 1)(\beta_j + 1)}\eta(\zeta_j^+)), \tag{7.16}$$

$$\mathcal{P}f(p) := \int_{-\infty}^{t} (f(z,s) - f(z,-s))ds \tag{7.17}$$

$$\widehat{\Sigma}_0(\eta) := \frac{\eta(\alpha,\beta,\lambda) - (-1)^{|\alpha+\beta|}\eta(\alpha,\beta,-\lambda)}{\lambda} \tag{7.18}$$

$$(M^2 f)(z,t) := |z|^2 f(z,t) \tag{7.19}$$

$$(M_0 f)(z,t) := itf(z,t). \tag{7.20}$$

Then, the following holds

Theorem 7.5. *For $f \in \mathscr{S}(\mathbb{H}^n)$,*

$$\mathcal{F}_{\mathbb{H}^n}(M^2 f) = -\widehat{\Delta}(\mathcal{F}_{\mathbb{H}^n} f) \tag{7.21}$$

$$\mathcal{F}_{\mathbb{H}^n}(M_0 f) = \widehat{D}_\lambda(\mathcal{F}_{\mathbb{H}^n} f) \tag{7.22}$$

$$\mathcal{F}_{\mathbb{H}^n}(\mathcal{P} f) = -i\widehat{\Sigma}_0(\mathcal{F}_{\mathbb{H}^n} f) \tag{7.23}$$

In a sense, this is analogous to the fact that the Fourier transform turns decay into smoothness. With this at hand one may define the Schwartz space on $\widetilde{\mathbb{H}}^n$ as follows,

Definition 7.3. $\eta \in \mathscr{S}(\widetilde{\mathbb{H}}^n)$ *if*

1. *For any given $(\alpha,\beta) \in \mathbb{N}^n$, $\eta(\alpha,\beta,\cdot) : \mathbb{R}^* \to \mathbb{C}$ is smooth.*
2. *For any $N \in \mathbb{N}$ the functions $\widehat{\Delta}^N\eta$, $\widehat{D}_\lambda^N\eta$, and $\widehat{\Sigma}_0\widehat{D}_\lambda^N\eta$, all decay faster than any power of $d_0 = |\lambda|(|\alpha+\beta|+n) + |\alpha-\beta|$.*

The space $\mathscr{S}(\widetilde{\mathbb{H}}^n)$ is equipped with the family of semi-norms

$$||\eta||_{N,N',\mathscr{S}(\widetilde{\mathbb{H}}^n)} := \sup_{\zeta\in\widetilde{\mathbb{H}}^n} (1 + d_0(\zeta))^N [\widehat{\Delta}^{N'}\eta + \widehat{D}_\lambda^{N'}\eta + \widehat{\Sigma}_0\widehat{D}_\lambda^{N'}\eta] \tag{7.24}$$

Theorem 7.6. *The map $\mathcal{F}_{\mathbb{H}^n} : \mathscr{S}(\mathbb{H}^n) \to \mathscr{S}(\widetilde{\mathbb{H}}^n)$ is a a continuous isomorphism with continuous inverse given by*

$$\mathcal{F}_{\mathbb{H}^n}^{-1}\eta(p) = \int_{\widetilde{\mathbb{H}}^n} \overline{\Phi(p,\zeta)}\eta(\zeta)d\zeta.$$

Perhaps the biggest inconvenience of the metric space $(\widetilde{\mathbb{H}}^n, d_{\widetilde{\mathbb{H}}^n})$ is that it is not complete, however, for any $f \in L^1(\mathbb{H}^n)$, $\widehat{f}_{\mathbb{H}^n}$ is uniformly continuous on $(\widetilde{\mathbb{H}}^n, d_{\widetilde{\mathbb{H}}^n})$. It is therefore natural to extend $\widehat{f}_{\mathbb{H}^n}$ to the metric completion of $(\widetilde{\mathbb{H}}^n, d_{\widetilde{\mathbb{H}}^n})$ which is denoted by $(\widehat{\mathbb{H}}^n, d_{\widehat{\mathbb{H}}^n})$. The metric space $(\widehat{\mathbb{H}}^n, d_{\widehat{\mathbb{H}}^n})$ is explicitly given by $\widehat{\mathbb{H}}^n = \widetilde{\mathbb{H}}^n \cup \widehat{\mathbb{H}}^n_0$, where $\widehat{\mathbb{H}}^n_0 = \mathbb{R}^n_{\mp} \times \mathbb{Z}^n$, with $\mathbb{R}^n_{\mp} = (\mathbb{R}_-)^n \cup (\mathbb{R}_+)$. For $\zeta \in \widehat{\mathbb{H}}^n$, if $\zeta \in \widehat{\mathbb{H}}^n_0$ we denote it by $\zeta = (\dot{x}, k)$. The metric $d_{\widehat{\mathbb{H}}^n}$ is given by

$$\begin{aligned}
&d_{\widehat{\mathbb{H}}^n}(\zeta, \zeta') = d_{\widetilde{\mathbb{H}}^n}(\zeta, \zeta'), && \text{if } \ \zeta, \zeta' \in \widetilde{\mathbb{H}}^n \\
&d_{\widehat{\mathbb{H}}^n}((\alpha, \beta, \lambda), (\dot{x}, k)) = |\lambda(\alpha+\beta) - \dot{x}|_{\ell^1} + |\alpha - \beta - k|_{\ell^1} + |\lambda|, && \text{if } \ (\alpha, \beta, \lambda) \in \widetilde{\mathbb{H}}^n, \ (\dot{x}, k) \in \widehat{\mathbb{H}}^n_0 \\
&d_{\widehat{\mathbb{H}}^n}((\dot{x}, k), (\dot{x}', k')) = |\dot{x} - \dot{x}'|_{\ell^1} + |k - k'|_{\ell^1}, && \text{if } \ (\dot{x}, k), (\dot{x}', k') \in \widehat{\mathbb{H}}^n_0.
\end{aligned}$$

Abusing notation, the extension of $\widehat{f}_{\mathbb{H}^n}$ is also denoted by $\widehat{f}_{\mathbb{H}^n}$. The space $\mathscr{S}(\widehat{\mathbb{H}}^n)$ is defined to be the space of all continuous functions, η, in $\widehat{\mathbb{H}}^n$ such that $\eta\lfloor_{\widetilde{\mathbb{H}}^n} \in \mathscr{S}(\widetilde{\mathbb{H}}^n)$. It is not hard to see that

$$\int_{\widetilde{\mathbb{H}}^n} \mathbb{1}_{\{|\lambda||\alpha+\beta|+|\alpha-\beta|\leq R\}}(\zeta)\mathbb{1}_{\{|\lambda|\leq\epsilon\}}(\zeta)d\zeta \lesssim R^{2n}\epsilon.$$

Therefore, it is natural to extend $d\zeta$ to $\widehat{\mathbb{H}}^n$ by defining, for all $\psi \in \mathcal{C}_c(\widehat{\mathbb{H}}^n)$,

$$\int_{\widehat{\mathbb{H}}^n} \psi(\zeta)d\zeta := \int_{\widetilde{\mathbb{H}}^n} \psi\lfloor_{\widetilde{\mathbb{H}}^n}(\zeta)d\zeta.$$

The convolution theorem extends as follows,

$$\mathcal{F}_{\mathbb{H}^n}(f * g)(\dot{x}, k) = \sum_{k' \in \mathbb{Z}^n} \widehat{f}_{\mathbb{H}^n}(\dot{x}, k - k')\widehat{g}_{\mathbb{H}^n}(\dot{x}, k').$$

The space $\mathscr{S}(\widehat{\mathbb{H}}^n)$ with the semi-norms (7.24) is a Frechét space, so It makes sense to talk about its topological dual. This will be the class of tempered distributions on $\widetilde{\mathbb{H}}^n$ and will be denoted $\mathscr{S}'(\widehat{\mathbb{H}}^n)$. This allows the following definition.

Definition 7.4. *For $T \in \mathscr{S}'(\mathbb{H}^n)$, $\mathcal{F}_{\mathbb{H}^n}(T) \in \mathscr{S}'(\widehat{\mathbb{H}}^n)$ is given by*

$$\langle \mathcal{F}_{\mathbb{H}^n} T | \eta \rangle_{\mathscr{S}'(\widehat{\mathbb{H}}^n) \times \mathscr{S}(\widehat{\mathbb{H}}^n)} = \langle T | \mathcal{F}^{\tau}_{\mathbb{H}^n} \eta \rangle_{\mathscr{S}'(\mathbb{H}^n) \times \mathscr{S}(\mathbb{H}^n)}.$$

Here $\mathcal{F}^{\tau}_{\mathbb{H}^n} : \mathscr{S}(\widehat{\mathbb{H}}^n) \to \mathscr{S}(\mathbb{H}^n)$ is the map

$$\mathcal{F}^{\tau}_{\mathbb{H}^n}\eta(p) := \int_{\widehat{\mathbb{H}}^n} \Phi(p, \zeta)\eta(\zeta)d\zeta.$$

Turns out $\mathcal{F}_{\mathbb{H}^n} : \mathscr{S}'(\mathbb{H}^n) \to \mathscr{S}'(\widehat{\mathbb{H}}^n)$ is a continuous injection.

7.2.1 Fourier coefficients of measures

This distributional definition of $\mathcal{F}_{\mathbb{H}^n}$ clearly applies to measures in $\mathcal{M}(\mathbb{H}^n)$. The goal of this section is to explore properties of this distributional transform keeping in mind the goal of computing the Fourier transform of K_σ. This is precisely a first step in obtaining a frequency formulation of energy integrals. In addition to the distributional definition of $\mathcal{F}_{\mathbb{H}^n}$, the definition of $\mathcal{F}_{\mathbb{H}^n}$ on functions can be extended to $\mathcal{M}(\mathbb{H}^n)$ by using either (7.8) or (7.10). A quick use of Fubini's theorem shows that, just as with functions, these give equivalent definitions. This gives us a point-wise definition of $\widehat{\mu}_{\mathbb{H}^n}$ for $\mu \in \mathcal{M}(\mathbb{H}^n)$,

$$\widehat{\mu}_{\mathbb{H}^n}(\zeta) = \int_{\mathbb{H}^n} \Phi(p,\zeta) d\mu(p). \tag{7.25}$$

Lemma 7.3. *Let $\widehat{\mu}_{\mathbb{H}^n}$ be as in* (7.25)*, and let $\mathcal{F}_{\mathbb{H}^n}\mu$ denote the distributional Fourier transform μ as in Definition* 7.4*. Then, $\mathcal{F}_{\mathbb{H}^n}\mu = \widehat{\mu}_{\mathbb{H}^n}$ as distributions. That is to say, $\mathcal{F}_{\mathbb{H}^n}\mu$ is given by integration against the function $\widehat{\mu}_{\mathbb{H}^n}$.*

Proof. The proof is a straight forward computation using Fubini's theorem. Let $\eta \in \mathscr{S}(\widehat{\mathbb{H}}^n)$, then

$$\begin{aligned}
\langle \mathcal{F}_{\mathbb{H}^n}\mu | \eta \rangle &= \langle \mu | \mathcal{F}^{\tau}_{\mathbb{H}^n} \eta \rangle \\
&= \int_{\mathbb{H}^n} \int_{\widetilde{\mathbb{H}}^n} \Phi(p,\zeta)\eta(\zeta) d\zeta d\mu(p) \\
&= \int_{\widetilde{\mathbb{H}}^n} \eta(\zeta) \int_{\mathbb{H}^n} \Phi(p,\zeta) d\mu(p) d\zeta = \int_{\widetilde{\mathbb{H}}^n} \eta(\zeta) \widehat{\mu}_{\mathbb{H}^n}(\zeta) d\zeta.
\end{aligned}$$

□

The following lemma, which before took some work to prove, is now immediate.

Lemma 7.4. *For $f \in \mathscr{S}(\mathbb{H}^n)$, $\mu \in \mathcal{M}(\mathbb{H}^n)$*

$$\int_{\mathbb{H}^n} f(p) d\mu(p) = \int_{\widetilde{\mathbb{H}}^n} \widehat{f}_{\mathbb{H}^n}(\zeta) \overline{\widehat{\mu}_{\mathbb{H}^n}(\zeta)} d\zeta.$$

Proof. Once again the proof is a simple computation involving Fubini's theorem and

Theorem 7.6,

$$\begin{aligned}\int_{\mathbb{H}^n} f(p)d\mu(p) &= \int_{\mathbb{H}^n} [\mathcal{F}^{-1}_{\mathbb{H}^n}\widehat{f}_{\mathbb{H}^n}](p)d\mu(p)\\ &= \int_{\mathbb{H}^n}\int_{\widetilde{\mathbb{H}}^n} \overline{\Phi(p,\zeta)}\widehat{f}_{\mathbb{H}^n}(\zeta)d\zeta d\mu(p)\\ &= \int_{\widetilde{\mathbb{H}}^n} \widehat{f}_{\mathbb{H}^n}(\zeta)\int_{\mathbb{H}^n}\overline{\Phi(p,\zeta)}d\mu(p) = \int_{\widetilde{\mathbb{H}}^n}\widehat{f}_{\mathbb{H}^n}(\zeta)\overline{\widehat{\mu}_{\mathbb{H}^n}(\zeta)}d\zeta.\end{aligned}$$

□

The hope is to extend this Lemma to the integral in (7.7). However, while this applies to $f \in \mathscr{S}(\mathbb{H}^n)$, $K_\sigma * \mu$ is not in the Schwarz class and, a priori, may not even be a tempered distribution. Moreover, as of yet, there is no suitable extension of the convolution theorem to distributions. If $f, g, h \in \mathscr{S}(\mathbb{H}^n)$ a quick computation yields

$$\int_{\mathbb{H}^n} f * g(p)h(p)dp = \int_{\mathbb{H}^n} g(q)\tilde{f} * h(q)dq,$$

where $\tilde{f}(p) = f(-p)$. This motivates the definition of the convolution of distributions with Schwartz functions. We recall the definition here,

Definition 7.5. *For $g \in \mathscr{S}(\mathbb{H}^n)$, $T \in \mathscr{S}'(\mathbb{H}^n)$, $g * T$ is the distribution given, for all $f \in \mathscr{S}(\mathbb{H}^n)$, by*

$$\langle g * T|f\rangle = \langle T|\tilde{g} * f\rangle.$$

In a similar way, for $\eta, \theta, \psi \in \mathscr{S}(\widehat{\mathbb{H}}^n)$ we compute

$$\begin{aligned}\int_{\widehat{\mathbb{H}}^n} \eta\cdot\theta(\zeta)\psi(\zeta)d\zeta &= \int_{\widetilde{\mathbb{H}}^n} \eta\cdot\theta(\zeta)\psi(\zeta)d\zeta\\ &= \sum_{\alpha,\beta}\int_{\mathbb{R}^*}\sum_{\gamma}\eta(\gamma,\beta,\lambda)\theta(\alpha,\gamma,\lambda)\psi(\alpha,\beta,\lambda)d\rho(\lambda)\\ &= \sum_{\gamma,\alpha}\int_{\mathbb{R}^*}\theta(\alpha,\gamma,\lambda)\sum_{\beta}\eta(\gamma,\beta,\lambda)\psi(\alpha,\beta,\lambda)d\rho(\lambda)\\ &= \int_{\widetilde{\mathbb{H}}^n}\theta(\zeta)(\eta_\tau\cdot\psi)(\zeta)d\zeta = \int_{\widehat{\mathbb{H}}^n}\theta(\zeta)(\eta_\tau\cdot\psi)(\zeta)d\zeta,\end{aligned}$$

where $\eta_\tau(\alpha,\beta,\lambda) = \eta(\beta,\alpha,\lambda)$, and $\eta_\tau(\dot{x},k) = \eta(\dot{x},-k)$. This motivates the following,

Definition 7.6. *For $\eta \in \mathscr{S}(\widehat{\mathbb{H}}^n)$ and $\Psi \in \mathscr{S}'(\widehat{\mathbb{H}}^n)$, $\eta\cdot\Psi$ is defined as the distribution given, for all $\theta \in \mathscr{S}(\widehat{\mathbb{H}}^n)$, by,*

$$\langle \eta\cdot\Psi|\theta\rangle = \langle\Psi|\eta_\tau\cdot\theta\rangle$$

Along side the transform $\mathcal{F}^{\tau}_{\mathbb{H}^n}$ the following transform is also introduced: $\mathcal{F}^{-\tau}_{\mathbb{H}^n} : \mathscr{S}(\mathbb{H}^n) \to \mathscr{S}(\widehat{\mathbb{H}}^n)$, given by

$$\mathcal{F}^{-\tau}_{\mathbb{H}^n} f(\zeta) = \int_{\mathbb{H}^n} \overline{\Phi(p,\zeta)} f(p) dp.$$

With this, we have a total of 4 "Fourier-like" transforms $\mathcal{F}_{\mathbb{H}^n}, \mathcal{F}^{-\tau}_{\mathbb{H}^n} : \mathscr{S}(\mathbb{H}^n) \to \mathscr{S}(\widehat{\mathbb{H}}^n)$ and $\mathcal{F}^{-1}_{\mathbb{H}^n}, \mathcal{F}^{\tau}_{\mathbb{H}^n} : \mathscr{S}(\widehat{\mathbb{H}}^n) \to \mathscr{S}(\mathbb{H}^n)$. The following relations between these are easily verified.

Lemma 7.5. *Let $f \in \mathscr{S}(\mathbb{H}^n)$, $\eta \in \mathscr{S}(\widehat{\mathbb{H}}^n)$ and for any function $\vartheta : \widehat{\mathbb{H}}^n \to \mathbb{C}$ denote $\vartheta-(\alpha,\beta,\lambda) = \vartheta(\alpha,\beta,-\lambda)$.*

1. $(\mathcal{F}_{\mathbb{H}^n})^{-1} = \mathcal{F}^{-1}_{\mathbb{H}^n}$,
2. $(\mathcal{F}^{\tau}_{\mathbb{H}^n})^{-1} = \mathcal{F}^{-\tau}_{\mathbb{H}^n}$
3. $\mathcal{F}^{-\tau}_{\mathbb{H}^n} f = [\mathcal{F}_{\mathbb{H}^n} f]_-$
4. $\mathcal{F}^{-1}_{\mathbb{H}^n} \eta = \mathcal{F}^{\tau}_{\mathbb{H}^n}(\eta_-)$
5. $\mathcal{F}^{-\tau}_{\mathbb{H}^n} \tilde{f}(\zeta) = [\mathcal{F}_{\mathbb{H}^n} f]_{\tau}(\zeta)$

Lemma 7.6. *For $f \in \mathscr{S}(\mathbb{H}^n)$, and $T \in \mathscr{S}'(\mathbb{H}^n)$, $\mathcal{F}_{\mathbb{H}^n}(f * T) = \widehat{f}_{\mathbb{H}^n} \cdot \widehat{T}_{\mathbb{H}^n}$ in the sense of distributions.*

Proof. Let $f \in \mathscr{S}(\mathbb{H}^n)$, $T \in \mathscr{S}'(\mathbb{H}^n)$ and $\eta \in \mathscr{S}(\widehat{\mathbb{H}}^n)$. Then, using (5) on Lemma 7.5,

$$\begin{aligned}
\langle \mathcal{F}_{\mathbb{H}^n}(f * T) | \eta \rangle &= \langle f * T | \mathcal{F}^{\tau}_{\mathbb{H}^n} \eta \rangle \\
&= \langle T | \tilde{f} * \mathcal{F}^{\tau}_{\mathbb{H}^n} \eta \rangle \\
&= \langle T | \mathcal{F}^{\tau}_{\mathbb{H}^n}(\mathcal{F}^{-\tau}_{\mathbb{H}^n} f \cdot \eta) \rangle \\
&= \langle \mathcal{F}_{\mathbb{H}^n} T | \mathcal{F}^{-\tau}_{\mathbb{H}^n} f \cdot \eta \rangle \\
&= \langle \mathcal{F}_{\mathbb{H}^n} T | \mathcal{F}^{-\tau}_{\mathbb{H}^n} \tilde{f} \cdot \eta \rangle \\
&= \langle \mathcal{F}_{\mathbb{H}^n} T | [\mathcal{F}_{\mathbb{H}^n} f]_{\tau} \cdot \eta \rangle \\
&= \langle \mathcal{F}_{\mathbb{H}^n} f \cdot \mathcal{F}_{\mathbb{H}^n} T | \eta \rangle.
\end{aligned}$$

This completes the proof. □

Before finishing this section, let's put the distributional Fourier transform to work by computing $\mathcal{F}_{\mathbb{H}^n} \delta_0$ in the sense of distributions, where δ_0 is the Dirac distribution at zero. This is done by the authors in [4] but given that this explicit computation will be of great relevance when computing the Fourier transform of K_σ, I present the (rather simple) computation here. For $\eta \in \mathscr{S}(\mathbb{H}^n)$,

$$\begin{aligned}\langle \mathcal{F}_{\mathbb{H}^n}\delta_0|\eta\rangle &= \langle \delta_0|\mathcal{F}^\tau_{\mathbb{H}^n}\eta\rangle \\ &= \mathcal{F}^\tau_{\mathbb{H}^n}\eta(0) = \int_{\widetilde{\mathbb{H}}^n}\Phi(0,\zeta)\eta(\zeta)d\zeta \\ &= \int_{\widetilde{\mathbb{H}}^n}\int_{\mathbb{R}^n}\phi_{\alpha,\lambda}(\xi)\phi_{\beta,\lambda}(\xi)d\xi\eta(\zeta)d\zeta.\end{aligned}$$

Since $\{\phi_{\alpha,\lambda}\}_{\alpha\in\mathbb{N}^n}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$ it follows that the inner integral is $\mathbb{1}_{\{\alpha=\beta\}}$. Therefore,

$$\langle \mathcal{F}_{\mathbb{H}^n}\delta_0|\eta\rangle = \int_{\widetilde{\mathbb{H}}^n}\mathbb{1}_{\{\alpha=\beta\}}(\zeta)\eta(\zeta)d\zeta.$$

It follows that $\mathcal{F}_{\mathbb{H}^n}\delta_0 = \mathbb{1}_{\{\alpha=\beta\}}$ in the sense of distributions.

7.2.2 The Fourier transform of the Koranyi-Riesz kernel

Another inprotant property of the classical Fourier transform is its interaction with differential operators. In $\mathbb{H}^n$ one should expect any reasonable definition of the Fourier transform to interact nicely with the subLaplacian operator

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^n X_j^2 + Y_j^2.$$

Turns out that $\Delta_{\mathbb{H}^n}$ is a $\mathcal{F}_{\mathbb{H}^n}$ multiplier with $\mathcal{F}_{\mathbb{H}^n}\Delta_{\mathbb{H}^n}f = |\lambda|(2|\alpha|+n)\mathcal{F}_{\mathbb{H}^n}f,\ f\in\mathscr{S}(\mathbb{H}^n)$. In [48], G.B. Folland showed that there is a constant $d = d(n)$ depending only on n such that the fundamental solution of $\Delta_{\mathbb{H}^n}$ is $d(n)^{-1}K_{Q-2}$. This allows to, quite easily, compute $\mathcal{F}_{\mathbb{H}^n}K_{Q-2}$ in the sense of distributions. This computation will be done later in greater generality (i.e. for $K_\sigma,\ 0<\sigma<Q$) but it turns out that

$$\mathcal{F}_{\mathbb{H}^n}K_{Q-2}(\alpha,\beta,\lambda) = d(n)\frac{1}{|\lambda|(2|\alpha|+n)}\mathbb{1}_{\alpha=\beta}(\alpha,\beta,\lambda).$$

This gives an explicit formula for $\mathcal{F}_{\mathbb{H}^n}K_\sigma$ for a particular choice of σ. To obtain the more general case of arbitrary $0\le\sigma\le Q$, one might be tempted to consider a fundamental solution of the fractional subLaplacian $\Delta_{\mathbb{H}^n}^{\sigma/2}$, and use a similar computation. Although one can show that if R_σ is a fundamental solution of $\Delta_{\mathbb{H}^n}^{\sigma/2}$ then $\mathcal{F}_{\mathbb{H}^n}R_\sigma = \tilde{d}(n,\sigma)\frac{\mathbb{1}_{\alpha=\beta}}{[|\lambda|(2|\alpha|+n)]^{\sigma/2}}$ in the sense of distributions, however it is not true that R_σ is a constant multiple of K_σ (unless, of course, $\sigma=2$). In fact, there is no known explicit formula for R_σ. There has been work done in the direction of realizing R_σ as explicitly as possible, see for instance [12], and more recently [108] where a specific case is treated. Instead, it proves useful to consider conformally invariant fractional powers of the subLaplacian. This operator arises naturally in CR geometry as studied for instance in [18], [17], [52] and [69]. It also arises in studying the extension problem in $\mathbb{H}^n$, [51], and has been studied in the context of Hardy inequalities in the Heisenberg group, [96]. In what follows, this operator will be introduced and studied in the context of $\mathcal{F}_{\mathbb{H}^n}$.

The function Φ introduced before, is of the form $e^{i\lambda t}|\lambda|^{-\frac{n}{2}}(2\pi)^{\frac{n}{2}}\Theta$ where

$$\Theta(z,\zeta) = \left(\frac{|\lambda|}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\lambda x\cdot\xi}\phi_{\alpha,\lambda}(\xi+\frac{y}{2})\phi_{\beta,\lambda}(\xi-\frac{y}{2})d\xi = \left(\frac{|\lambda|}{2\pi}\right)^{\frac{n}{2}} \langle \rho_\lambda(z,0)\phi_{\alpha,\lambda}|\phi_{\beta,\lambda}\rangle,$$

are the rescaled special Hermite functions. The set $\{\Theta(\cdot,\zeta)\}_{\alpha,\beta\in\mathbb{N}^n}$ forms a complete orthonormal basis for $L^2(\mathbb{C}^n)$. In particular, for $f \in \mathscr{S}(\mathbb{H}^n)$

$$f^\lambda(z) = \sum_\alpha \sum_\beta \langle f^\lambda|\Theta(\cdot,\zeta)\rangle\Theta(z,\zeta). \tag{7.26}$$

This expansion has a more compact form in terms of Laguerre functions. For $k \in \mathbb{N}$, the kth Laguerre polynomial of type $\delta > -1$ is

$$L_k^\delta(t)e^{-t}t^\delta = \frac{1}{k!}(\frac{d}{dt})^k(e^{-t}t^{k+\delta}).$$

The Laguerre functions are given in terms of L_k^δ by

$$\ell_k^\delta(z) := L_k^\delta(\frac{1}{2}|z|^2)e^{-\frac{1}{2}|z|^2}.$$

Just as with the Hermite functions, ℓ_k^δ can be adapted to the representation theory of $\mathbb{H}^n$ by a simple rescaling. The rescaled Laguerre function is given by $\ell_{k,\lambda}^\delta(z) = \ell_k^\delta(\sqrt{|\lambda|}z)$. These functions are closely related to the functions $\Theta(z,\zeta)$. In fact, (7.26) rewrites as

$$f^\lambda(p) = \frac{|\lambda|^n}{(2\pi)^n}\sum_{k=0}^\infty (f^\lambda *_{-\lambda} \ell_{k,\lambda}^{n-1})(z). \tag{7.27}$$

This relation between the special Hermite expansion and the Laguerre expansion will be instrumental in computing $\widehat{K}_\sigma$, but first, conformally invariant fractional powers of the subLaplacian are introduced.

Definition 7.7. *For $0 \leq \sigma \leq Q$ the conformally invariant σ-fractional subLaplacian is the operator $\mathcal{L}_\sigma : \mathscr{S}(\mathbb{H}^n) \to \mathscr{S}(\mathbb{H}^n)$ given by*

$$\mathcal{L}_\sigma f(p) = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} (2|\lambda|)^{\sigma/2} \frac{\Gamma(k + \frac{Q+\sigma}{4})}{\Gamma(k + \frac{Q-\sigma}{4})} f^\lambda *_\lambda \ell_{k,\lambda}^{n-1}(z) e^{-i\lambda t} d\rho(\lambda). \tag{7.28}$$

This 'modified' subLaplacian is of great importance here because its fundamental solution is precisely $d(n,\sigma)^{-1} K_{Q-\sigma}$ where $d(n,\sigma)$ is a constant depending only on n and σ. A proof of this can be found in [96, Section 3] (see, specifically, equation (3.10)). One can see that taking $\sigma = 2$

$$(2|\lambda|)^{\sigma/2} \frac{\Gamma(k + \frac{Q+\sigma}{4})}{\Gamma(k + \frac{Q-\sigma}{4})} = 2|\lambda| \frac{\Gamma(k + \frac{n}{2} + 1)}{\Gamma(k + \frac{n}{2})} = 2|\lambda|(k + \frac{n}{2}) = |\lambda|(2k+n).$$

This highlights the relation between $\mathcal{L}_\sigma$ and $\Delta_{\mathbb{H}^n}^{\sigma/2}$. Indeed, $\Delta_{\mathbb{H}^n}^{\sigma/2} = U(\sigma)\mathcal{L}_\sigma$ where $U(\sigma)$ is a bounded operator depending only on σ. In particular, $U(2)$ is the identity operator. Moreover, due to the relation between Laguerre and Hermite expansions, one can check that $\mathcal{L}_\sigma$ is a multiplier for $\mathcal{F}_{\mathbb{H}^n}$.

Theorem 7.7. *For $f \in \mathscr{S}(\mathbb{H}^n)$, $\mathcal{L}_\sigma$ satisfies*

$$\mathcal{L}_\sigma f = \mathcal{F}_{\mathbb{H}^n}^{-1}[(2|\lambda|)^{\sigma/2} \frac{\Gamma(|\alpha| + \frac{Q+\sigma}{4})}{\Gamma(|\alpha| + \frac{Q-\sigma}{4})} \mathcal{F}_{\mathbb{H}^n} f].$$

Using the interplay between $\mathcal{F}_{\mathbb{H}^n}$ and $\mathcal{F}_{\mathbb{H}^n}^\tau$ from Lemma 7.5, on can verify the following identity on the the frequency side.

Corollary 7.1. *For $\eta \in \mathscr{S}(\widehat{\mathbb{H}}^n)$,*

$$\mathcal{L}_\sigma \mathcal{F}_{\mathbb{H}^n}^\tau \eta = \mathcal{F}_{\mathbb{H}^n}^\tau \left[(2|\lambda|)^{\sigma/2} \frac{\Gamma(|\alpha| + \frac{Q+\sigma}{4})}{\Gamma(|\alpha| + \frac{Q-\sigma}{4})} \eta \right]$$

In order to compute $\mathcal{F}_{\mathbb{H}^n} K_\sigma$ explicitly, Theorem 7.7 needs to be extended to distributions. A quick computation using Lemma 7.5 shows that for $f, g \in \mathscr{S}(\mathbb{H}^n)$

$$\int_{\mathbb{H}^n} \mathcal{L}_\sigma f(p) g(p) dp = \int_{\mathbb{H}^n} f(p) \mathcal{L}_\sigma g(p) dp. \tag{7.29}$$

Therefore, $\mathcal{L}_\sigma$ is extended to distributions in the usual way. For $T \in \mathscr{S}'(\mathbb{H}^n)$, $\mathcal{L}_\sigma T \in$

$\mathscr{S}'(\mathbb{H}^n)$ is given, for $g \in \mathscr{S}(\mathbb{H}^n)$, by

$$\langle \mathcal{L}_\sigma T | g \rangle = \langle T | \mathcal{L}_\sigma g \rangle.$$

From here one can check that if $T \in \mathscr{S}'(\mathbb{H}^n)$,

$$\mathcal{F}_{\mathbb{H}^n} \mathcal{L}_\sigma T = (2|\lambda|)^{\sigma/2} \frac{\Gamma(|\alpha| + \frac{Q+\sigma}{4})}{\Gamma(|\alpha| + \frac{Q-\sigma}{4})} \mathcal{F}_{\mathbb{H}^n} T, \tag{7.30}$$

In the $\mathscr{S}'(\widehat{\mathbb{H}}^n)$ sense. Indeed,

$$\begin{aligned}
\langle \mathcal{F}_{\mathbb{H}^n} \mathcal{L}_\sigma T | \eta \rangle &= \langle \mathcal{L}_\sigma T | \mathcal{F}^\tau_{\mathbb{H}^n} \eta \rangle \\
&= \langle T | \mathcal{L}_\sigma \mathcal{F}^\tau_{\mathbb{H}^n} \eta \rangle \\
&= \langle T | \mathcal{F}^\tau_{\mathbb{H}^n} [(2|\lambda|)^{\sigma/2} \frac{\Gamma(|\alpha| + \frac{Q+\sigma}{4})}{\Gamma(|\alpha| + \frac{Q-\sigma}{4})} \eta] \rangle \\
&= \langle (2|\lambda|)^{\sigma/2} \frac{\Gamma(|\alpha| + \frac{Q+\sigma}{4})}{\Gamma(|\alpha| + \frac{Q-\sigma}{4})} \mathcal{F}_{\mathbb{H}^n} T | \eta \rangle.
\end{aligned}$$

From here one obtains the desired result.

Proposition 7.2.

$$\mathcal{F}_{\mathbb{H}^n} K_{Q-\sigma} = d(n, Q-\sigma)(2|\lambda|)^{-\sigma/2} \frac{\Gamma(|\alpha| + \frac{Q-\sigma}{4})}{\Gamma(|\alpha| + \frac{Q+\sigma}{4})} \mathbb{1}_{\{\alpha=\beta\}},$$

in the sense of distributions.

Proof. Recall that $d(\sigma, Q)^{-1} K_{Q-\sigma}$ is the fundamental solution of $\mathcal{L}_\sigma$. Therefore, since

$$\mathcal{F}_{\mathbb{H}^n} \mathcal{L}_\sigma K_{Q-\sigma} = (2|\lambda|)^{\sigma/2} \frac{\Gamma(|\alpha| + \frac{Q+\sigma}{4})}{\Gamma(|\alpha| + \frac{Q-\sigma}{4})} \mathcal{F}_{\mathbb{H}^n} K_{Q-\sigma},$$

it follows that

$$d(n, Q-\sigma) \mathcal{F}_{\mathbb{H}^n} \delta_0 = (2|\lambda|)^{\sigma/2} \frac{\Gamma(|\alpha| + \frac{Q+\sigma}{4})}{\Gamma(|\alpha| + \frac{Q-\sigma}{4})} \mathcal{F}_{\mathbb{H}^n} K_{Q-\sigma}.$$

Recalling the transform of δ_0 we obtain

$$(2|\lambda|)^{\sigma/2} \frac{\Gamma(|\alpha| + \frac{Q+\sigma}{4})}{\Gamma(|\alpha| + \frac{Q-\sigma}{4})} \mathcal{F}_{\mathbb{H}^n} K_{Q-\sigma} = d(n, Q-\sigma) \mathbb{1}_{\{\alpha=\beta\}},$$

in the sense of $\mathscr{S}'(\widehat{\mathbb{H}}^n)$, as claimed. $\square$

Hereafter, to simplify notation, I denote

$$\mathfrak{K}_\sigma(\alpha,\lambda) = (2|\lambda|)^{-\sigma/2}\frac{\Gamma(|\alpha| + \frac{Q-\sigma}{4})}{\Gamma(|\alpha| + \frac{Q+\sigma}{4})}.$$

The group Fourier transform of the Koranyi-Riez kernel, has been known for quite some time. It was originally computed by Cowling and Haagerup in [28]. Their computation yields

$$\widehat{K}_{Q-\sigma}(\lambda) = \tilde{d}_n|\lambda|^{-\sigma/2}\frac{\Gamma(\frac{\sigma}{2})}{\Gamma(\frac{Q-\sigma}{4})}\sum_{k=1}^{\infty}\frac{\Gamma(k + \frac{Q-\sigma}{4})}{\Gamma(k + \frac{Q+\sigma}{4})}P_k(\lambda), \tag{7.31}$$

where $\tilde{d}_n$ is a constant, and $P_k(\lambda)$ is the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the eigenspace $\{\phi_{\alpha,\lambda} : |\alpha| = k\}$. A quick, computation renders

$$\langle P_k(\lambda)\phi_{\alpha,\lambda}, \phi_{\beta,\lambda}\rangle_{L^2(\mathbb{R}^n)} = \mathbb{1}_{\{\alpha=\beta \text{ and } |\alpha|=k\}},$$

which, at least formally, further supports Proposition 7.2.

7.2.3 Application to energy integrals

The explicit expression for $\mathcal{F}_{\mathbb{H}^n}K_\sigma$ brings us a step closer to the coveted analogue of (2.12). In fact, it is now possible to compute energies of Schwartz functions as an integral over the frequency space, $\widehat{\mathbb{H}}^n$. For $g \in \mathscr{S}(\mathbb{H}^n)$, the σ- energy of g is defined as the integral

$$I_\sigma(g) := \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} ||q^{-1}p||^{-\sigma} g(q)g(p)dqdp.$$

Lemma 7.7. *For $g \in \mathscr{S}(\mathbb{H}^n)$*

$$I_\sigma(g) = d(n,\sigma)\int_{\widetilde{\mathbb{H}}^n} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda)|\widehat{g}_{\mathbb{H}^n}(\zeta)|^2 d\zeta.$$

Proof.

$$\begin{aligned}
I_\sigma(g) &= \int_{\mathbb{H}^n} g * K_\sigma(p) g(p) dp = \langle g * K_\sigma | g \rangle \\
&= \langle \mathcal{F}_{\mathbb{H}^n}(g * K_\sigma) | \overline{\mathcal{F}_{\mathbb{H}^n} g} \rangle = \langle \mathcal{F}_{\mathbb{H}^n} K_\sigma | (\mathcal{F}_{\mathbb{H}^n} g)_\tau \cdot \overline{\mathcal{F}_{\mathbb{H}^n}}(g) \rangle \\
&= d(n,\sigma) \int_{\widetilde{\mathbb{H}}^n} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda) \mathbb{1}_{\{\alpha=\beta\}} (\widehat{g}_{\mathbb{H}^n})_\tau \cdot \overline{\widehat{g}_{\mathbb{H}^n}}(\zeta) d\zeta \\
&= d(n,\sigma) \sum_\alpha \int_{\mathbb{R}^*} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda) (\widehat{g}_{\mathbb{H}^n})_\tau \cdot \overline{\widehat{g}_{\mathbb{H}^n}}(\alpha,\alpha,\lambda) d\rho(\lambda) \\
&= d(n,\sigma) \sum_\alpha \int_{\mathbb{R}^*} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda) \sum_{\gamma \in \mathbb{N}^n} \widehat{g}_{\mathbb{H}^n}(\alpha,\gamma,\lambda) \overline{\widehat{g}}_{\mathbb{H}^n}(\alpha,\gamma,\lambda) d\rho(\lambda) \\
&= d(n,\sigma) \sum_{\alpha,\gamma} \int_{\mathbb{R}^*} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda) |\widehat{g}(\alpha,\gamma,\lambda)|^2 d\rho(\lambda) \\
&= d(n,\sigma) \int_{\widetilde{\mathbb{H}}^n} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda) |\widehat{g}_{\mathbb{H}^n}(\zeta)|^2 d\zeta.
\end{aligned}$$

□

The goal is to replace $g \in \mathscr{S}(\mathbb{H}^n)$ by $\mu \in \mathcal{M}(\mathbb{H}^n)$. This can be done with the standard convolution approximation.

Proposition 7.3. *Let* $\mu \in \mathcal{M}(\mathbb{H}^n)$

$$I_\sigma(\mu) = d(n,\sigma) \int_{\widehat{\mathbb{H}}^n} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda) |\widehat{\mu}_{\mathbb{H}^n}(\zeta)|^2 d\zeta$$

Proof. Let ψ be a compactly supported, smooth function so that defining $\psi_\epsilon(p) = \epsilon^{-Q}\psi(\delta_{1/\epsilon}p)$ makes $\{\psi_\epsilon\}_{\epsilon>0}$ a compactly supported, smooth approximation to the identity. Set $\mu_\epsilon = \mu * \psi_\epsilon$ so that $\mu \in \mathcal{C}_c^\infty(\mathbb{H}^n)$ as well. By Lemma 7.7,

$$I_\sigma(\mu_\epsilon) = d(n,\sigma) \int_{\widehat{\mathbb{H}}^n} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda) |\widehat{\mu}_{\epsilon,\mathbb{H}^n}(\zeta)|^2 d\zeta.$$

The idea is to show that $I_\sigma(\mu) = \lim_{\epsilon\to 0} I_\sigma(\mu_\epsilon)$ and also

$$\lim_{\epsilon\to 0} \int_{\widehat{\mathbb{H}}^n} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda) |\widehat{\mu}_{\epsilon,\mathbb{H}^n}(\zeta)|^2 d\zeta = \int_{\widehat{\mathbb{H}}^n} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda) |\widehat{\mu}(\zeta)|^2 d\zeta.$$

I will first deal with the frequency side, and show the convergence by splitting the proof in 2 cases.

First assume that

$$\int_{\widehat{\mathbb{H}}^n} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda) |\widehat{\mu}_{\mathbb{H}^n}(\zeta)|^2 d\zeta = \infty.$$

Since Φ is continuous and bounded, it follows from weak convergence that $\widehat{\mu}_{\epsilon,\mathbb{H}^n} \to \widehat{\mu}_{\mathbb{H}^n}$ point-wise. Therefore, by Fatou's Lemma,

$$\infty \leq \liminf_{\epsilon\to 0} \int_{\widehat{\mathbb{H}}^n} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda)|\widehat{\mu}_{\epsilon,\mathbb{H}^n}(\zeta)|^2 d\zeta,$$

so the equality is trivial.

Now assume

$$\int_{\widehat{\mathbb{H}}^n} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda)|\widehat{\mu}_{\mathbb{H}^n}(\zeta)|^2 d\zeta \leq \infty.$$

The following identity will be used: Let H be a Hilbert space and $\{e_j\}_{j\in\mathbb{N}}$ and orthonormal basis. For an operator $A \in \mathcal{B}(H)$,

$$\sum_j |\langle Ae_i|e_j\rangle_H|^2 = ||Ae_i||_H^2.$$

Since $\mathfrak{K}_{Q-\sigma}$ is independent of β, it follows that $\sum_\beta |\widehat{\mu}_{\mathbb{H}^n}(\alpha,\beta,\lambda)|^2$ is finite for each fixed $\beta \in \mathbb{N}^n$ and ϱ-almost all $\lambda \in \mathbb{R}^*$. Moreover it is integrable with respect to the product measure $\# \times \varrho$ where $\#$ stands for the counting measure on $\mathbb{N}^n$. Recalling (7.8),

$$\begin{aligned}\sum_\beta |\widehat{\mu}_{\epsilon,\mathbb{H}^n}(\alpha,\beta,\lambda) - \widehat{\mu}_{\mathbb{H}^n}(\alpha,\beta,\lambda)|^2 \\ &= \sum_\beta |\langle(\widehat{\mu}_\epsilon(\lambda) - \widehat{\mu}(\lambda))\phi_{\alpha,\lambda}, \phi_{\beta,\lambda}\rangle_{L^2(\mathbb{R}^n)}|^2 \\ &= ||(\widehat{\mu}_\epsilon(\lambda) - \widehat{\mu}(\lambda))\phi_{\alpha,\lambda}||^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Since $\widehat{\mu}_\epsilon(\lambda) \to \widehat{\mu}(\lambda)$ in the strong operator topology, it follows that for each fixed $\lambda \in \mathbb{R}^*$ and $\alpha \in \mathbb{N}^n$,

$$\lim_{\epsilon\to 0} \sum_\beta |\widehat{\mu}_{\epsilon,\mathbb{H}^n}(\alpha,\beta,\lambda) - \widehat{\mu}_{\mathbb{H}^n}(\alpha,\beta,\lambda)|^2 = 0.$$

In particular,

$$\lim_{\epsilon\to 0} \sum_\beta |\widehat{\mu}_{\epsilon,\mathbb{H}^n}(\alpha,\beta,\lambda)|^2 = \sum_\beta |\widehat{\mu}_{\mathbb{H}^n}(\alpha,\beta,\lambda)|^2.$$

Furthermore, $\sum_\beta |\widehat{\mu}_{\epsilon,\mathbb{H}^n}(\alpha,\beta,\lambda)|^2$ can be dominated as follows,

$$\begin{aligned}\sum_\beta |\widehat{\mu}_{\epsilon,\mathbb{H}^n}(\alpha,\beta,\lambda)|^2 &= \sum_\beta |\langle \widehat{\psi}_\epsilon(\lambda)\widehat{\mu}(\lambda)\phi_{\alpha,\lambda}, \phi_{\beta,\lambda}\rangle_{L^2(\mathbb{R}^n)}|^2 \\ &= ||\widehat{\psi}_\epsilon(\lambda)\widehat{\mu}(\lambda)\phi_{\alpha,\lambda}||^2_{L^2(\mathbb{R}^n)} \\ &\leq ||\widehat{\psi}_\epsilon(\lambda)||^2_{op}||\widehat{\mu}(\lambda)\phi_{\alpha,\lambda}||^2_{L^2(\mathbb{R}^n)} \\ &= ||\psi_\epsilon||^2_{L^1(\mathbb{H}^n)}\sum_\beta |\widehat{\mu}_{\mathbb{H}^n}(\alpha,\beta,\lambda)|^2 = \sum_\beta |\widehat{\mu}_{\mathbb{H}^n}(\alpha,\beta,\lambda)|^2.\end{aligned}$$

So, by the dominated convergence theorem,

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{\widehat{\mathbb{H}}^n} \mathfrak{K}_{Q-\sigma}(\alpha,\lambda)|\widehat{\mu}_{\epsilon,\mathbb{H}^n}(\zeta)|^2 d\zeta \\ &\qquad = \sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}^*}\mathfrak{K}_{Q-\sigma}(\alpha,\lambda)\lim_{\epsilon\to 0}\sum_\beta |\widehat{\mu}_{\epsilon,\mathbb{H}^n}(\zeta)|^2 d\zeta = \int_{\widehat{\mathbb{H}}^n}\mathfrak{K}_{Q-\sigma}(\alpha,\lambda)|\widehat{\mu}_{\mathbb{H}^n}(\zeta)|^2 d\zeta.\end{aligned}$$

Now the energy on the spatial domain is studied.

Once again, if $I_\sigma(\mu) = \infty$, Fatou's Lemma gives,

$$\infty \leq \liminf_{\epsilon\to 0} I_\sigma(\mu_\epsilon),$$

so the equality is trivial.

Assume $I_\sigma(\mu) < \infty$. By expanding the convolution $\psi_\epsilon * \mu$, and applying Fubini's theorem,

$$I_\sigma(\mu_\epsilon) = \iint \left(\iint ||q^{-1}p||^{-\sigma}_{\mathbb{H}^n}\psi_\epsilon(a^{-1}p)\psi_\epsilon(b^{-1}q)dpdq\right) d\mu(a)d\mu(b).$$

Using $W = \delta_{1/\epsilon}(a^{-1}p)$ and $Z = \delta_{1/\epsilon}(b^{-1}q)$, the inner integral becomes

$$\iint ||\delta_\epsilon(Z)b^{-1}a\delta_\epsilon(W)||^{-\sigma}_{\mathbb{H}^n}\psi(Z)\psi(W)dZdW.$$

As $\epsilon \to 0$, this integral goes to $||b^{-1}a||^{-\sigma}_{\mathbb{H}^n}$ as long as $a \neq b$. Moreover, it shows that

$$\iint ||q^{-1}p||^{-\sigma}_{\mathbb{H}^n}\psi_\epsilon(a^{-1}p)\psi_\epsilon(b^{-1}q)dpdq \lesssim ||b^{-1}a||^{-\sigma}_{\mathbb{H}^n}.$$

By assumption, $||b^{-1}a||_{\mathbb{H}^n}^{-\sigma}$ is integrable, so by the Dominated Convergence Theorem

$$\begin{aligned}\lim_{\epsilon\to 0} I_\sigma(\mu_\epsilon) &= \lim_{\epsilon\to 0} \iint \left(\iint ||q^{-1}p||_{\mathbb{H}^n}^{-\sigma} \psi_\epsilon(a^{-1}p)\psi_\epsilon(b^{-1}q)dpdq \right) d\mu(a)d\mu(b) \\ &= \iint \left(\lim_{\epsilon\to 0} \iint ||q^{-1}p||_{\mathbb{H}^n}^{-\sigma} \psi_\epsilon(a^{-1}p)\psi_\epsilon(b^{-1}q)dpdq \right) d\mu(a)d\mu(b) \\ &= \iint ||a^{-1}b||_{\mathbb{H}^n}^{-\sigma} d\mu(a)d\mu(b) = I_\sigma(\mu).\end{aligned}$$

This completes the proof. □

There is a potential application of Proposition 7.3 to the distance set problem in the Heisenberg group. In Euclidean space, it is conjectured that if $A \subset \mathbb{R}^n$ is a Borel set with $\dim A > \frac{n}{2}$, then $\mathcal{H}^1(D(A)) > 0$ where $D(A) = \{|x-y| : x, y \in A\}$. The current best known sufficient lower bounds for this problem are as follows

$$\dim_E A > \begin{cases} \frac{5}{4} & \text{if } n = 2 \text{ ([57])} \\ \frac{9}{5} & \text{if } n = 3 \text{ ([33])} \\ \frac{n}{2} + \frac{1}{4} + o(1) & \text{if } n \geq 4 \text{ ([34])} \end{cases}.$$

The analogous problem for the Heisenberg distance has not been studied, however I suspect this Fourier approach can be used to study this problem. Let $A \subset \mathbb{H}^n$ be a Borel set and $\mu \in \mathcal{M}(A)$. Put $\mu_d := d_{\mathbb{H}^n \#}(\mu \times \mu)$, that is for $h \in \mathcal{C}_c(\mathbb{R})$,

$$\int_{\mathbb{R}} h(r)d\mu_d(r) = \iint h(||q^{-1}p||_{\mathbb{H}^n})d\mu(q)d\mu(p).$$

It is not hard to see that $\mu_d \in \mathcal{M}(D_{\mathbb{H}^n}(A))$, where $D_{\mathbb{H}^n}(A) = \{||q^{-1}p||_{\mathbb{H}^n} : p, q \in A\}$. Moreover, if μ has smooth density $g \in \mathcal{C}_c^\infty$, then $\mu_d \ll \mathcal{H}^1$ with density

$$g_d(r) = \int (g * \sigma_r)(p)g(p)dp,$$

where σ_r is the surface measure of the Koranyi sphere of radius r. Indeed, using Fubini's theorem and change of variables we can check,

$$\begin{aligned}
\int h(r)d\mu_d(r) &= \iint h(||q^{-1}p||_{\mathbb{H}^n} g(q)g(p)dp \\
&= \int \left(\int h(||q||)g(pq^{-1})dq \right) g(p)dp \\
&= \int \left(\int_{\mathbb{R}^n} \int_{\mathbb{S}^{Q-1}_{\mathbb{H}^n,r}} h(||q||)g(pq^{-1})d\sigma_r(q)dr \right) g(p)dp \\
&= \int \left(\int_{\mathbb{R}^n} h(r)(g*\sigma_r)(p)dr \right) g(p)dp \\
&= \int_{\mathbb{R}^n} h(r) \left(\int_{\mathbb{H}^n} (g*\sigma_r)(p)g(p)dp \right) dr.
\end{aligned}$$

If μ is a general measure, one can take $\mu_\epsilon = \psi_\epsilon * \mu$ as in the proof of Theorem 7.3. On one hand, by weak convergence one can check that $\mu_{\epsilon,d} \to \mu_d$ as $\epsilon \to 0$. On the other hand, $\mu_{\epsilon,d}$ is a function,

$$\mu_{\epsilon,d}(r) = \int_{\mathbb{H}^n} (\mu_{\epsilon,d} * \sigma_r)(p)\mu_{\epsilon,d}(p)dp.$$

Arguing as in the proof of Lemma 7.7, we can obtain

$$\mu_{\epsilon,d}(r) = \int_{\widehat{\mathbb{H}}^n} R_r(\alpha,\lambda)|\widehat{\mu}_{\mathbb{H}^n,\epsilon}(\alpha,\beta,\lambda)|^2 d\zeta, \tag{7.32}$$

where $R_r(\alpha,\lambda)$ is related to $\widehat{\sigma}_{\mathbb{H}^n}$ in that

$$\widehat{\sigma}_{\mathbb{H}^n}(\alpha,\beta,\gamma) = R_r(\alpha,\lambda)\mathbb{1}_{\alpha=\beta}$$

in the sense of distributions ([54]). If there is a value $s_0 > 0$ such that $R_r(\alpha,\lambda) \lesssim_r \mathfrak{K}_{Q-s_0}(\alpha,\lambda)$, then picking μ such that $I_{s_0}(\mu) < \infty$ would allows us, by way of by (7.32), to bound $|\mu_{\epsilon,d}(r)|$ uniformly. By dominated convergence this would imply that μ_d is also given by a function. In turn, we could conclude that $\dim A > s_0 \implies \mathcal{H}^1(D_{\mathbb{H}^n}(A)) > 0$.

It is worth noting that in $\mathbb{R}^n$, the analogue is true with $s_0 = \frac{n+1}{2}$, since the surface measure of the Euclidean sphere, ς_r^{n-1}, satisfies

$$\widehat{\varsigma}_r^{n-1}(\xi) \lesssim_r |\xi|^{\frac{1-n}{2}}.$$

However, the best dimension bounds discussed before are obtained using slightly different, albeit still Fourier theoretic, approaches.

The coefficients $R_r(\alpha, \lambda)$ have been computed in terms of integrals of Laguerre functions (see for instance [53]). Specifically, there is a constant, $d'(n)$, depending only on n such that

$$R_r(\alpha, \lambda) = d'(n) \frac{k!}{(k+n-1)!} \int_{-\pi/2}^{\pi/2} \ell_{k,\lambda}^{n-1}(r\sqrt{\cos\theta}) e^{i\frac{\lambda}{4}r^2 \sin\theta} (\cos\theta)^{n-1} d\theta. \tag{7.33}$$

Further studying these coefficients, and in particular the joint asymptotic behavior for large values of α and λ, might yield some results in the direction of the Heisenberg distance set problem. Beyond the distance set problem, I hope this approach will prove useful in tackling other problems relating to Hausdorff dimension in $\mathbb{H}^n$.

REFERENCES

[1] L. Ambrosio and S. Rigot. Optimal mass transportation in the Heisenberg group. *J. Funct. Anal.*, 208(2):261–301, 2004.

[2] Nicola Arcozzi and Annalisa Baldi. From Grushin to Heisenberg via an isoperimetric problem. *J. Math. Anal. Appl.*, 340(1):165–174, 2008.

[3] Hajer Bahouri, Jean-Yves Chemin, and Raphael Danchin. A Frequency Space for the Heisenberg Group. *arXiv e-prints*, page arXiv:1609.03850, Sep 2016.

[4] Hajer Bahouri, Jean-Yves Chemin, and Raphael Danchin. Tempered distributions and Fourier transform on the Heisenberg group. *arXiv e-prints*, page arXiv:1705.02195, May 2017.

[5] Zoltán Balogh, Jeremy T. Tyson, and Kevin Wildrick. Frequency of Sobolev dimension distortion of horizontal subgroups in Heisenberg groups. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 17(2):655–683, 2017.

[6] Zoltán M. Balogh, Estibalitz Durand-Cartagena, Katrin Fässler, Pertti Mattila, and Jeremy T. Tyson. The effect of projections on dimension in the Heisenberg group. *Rev. Mat. Iberoam.*, 29(2):381–432, 2013.

[7] Zoltán M. Balogh, Katrin Fässler, Pertti Mattila, and Jeremy T. Tyson. Projection and slicing theorems in Heisenberg groups. *Adv. Math.*, 231(2):569–604, 2012.

[8] Zoltán M. Balogh, Matthieu Rickly, and Francesco Serra Cassano. Comparison of Hausdorff measures with respect to the Euclidean and the Heisenberg metric. *Publ. Mat.*, 47(1):237–259, 2003.

[9] Zoltán M. Balogh, Jeremy T. Tyson, and Ben Warhurst. Sub-Riemannian vs. Euclidean dimension comparison and fractal geometry on Carnot groups. *Adv. Math.*, 220(2):560–619, 2009.

[10] André Bellaïche. The tangent space in sub-Riemannian geometry. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 1–78. Birkhäuser, Basel, 1996.

[11] André Bellaïche and Jean-Jacques Risler, editors. *Sub-Riemannian geometry*, volume 144 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1996.

[12] Chal Benson, A. H. Dooley, and Gail Ratcliff. Fundamental solutions for powers of the Heisenberg sub-Laplacian. *Illinois J. Math.*, 37(3):455–476, 1993.

[13] V. N. Berestovskiĭ. Geodesics of nonholonomic left-invariant inner metrics on the Heisenberg group and isoperimetrics of the Minkowski plane. *Sibirsk. Mat. Zh.*, 35(1):3–11, i, 1994.

[14] A. S. Besicovitch. On the fundamental geometrical properties of linearly measurable plane sets of points. *Math. Ann.*, 98(1):422–464, 1928.

[15] A. S. Besicovitch. On the fundamental geometrical properties of linearly measurable plane sets of points (II). *Math. Ann.*, 115(1):296–329, 1938.

[16] A. S. Besicovitch. On the fundamental geometrical properties of linearly measurable plane sets of points (III). *Math. Ann.*, 116(1):349–357, 1939.

[17] Thomas Branson, Gestur Ólafsson, and Bent Ø rsted. Spectrum generating operators and intertwining operators for representations induced from a maximal parabolic subgroup. *J. Funct. Anal.*, 135(1):163–205, 1996.

[18] Thomas P. Branson, Luigi Fontana, and Carlo Morpurgo. Moser-Trudinger and Beckner-Onofri's inequalities on the CR sphere. *Ann. of Math. (2)*, 177(1):1–52, 2013.

[19] Yu. D. Burago and V. A. Zalgaller. *Geometric inequalities*, volume 285 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1988. Translated from the Russian by A. B. Sosinskiĭ, Springer Series in Soviet Mathematics.

[20] Luca Capogna, Donatella Danielli, Scott D. Pauls, and Jeremy T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, volume 259 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

[21] Lennart Carleson. *Selected problems on exceptional sets*. Van Nostrand Mathematical Studies, No. 13. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[22] Jeff Cheeger and Bruce Kleiner. Differentiating maps into L^1, and the geometry of BV functions. *Ann. of Math. (2)*, 171(2):1347–1385, 2010.

[23] Jeff Cheeger and Bruce Kleiner. Metric differentiation, monotonicity and maps to L^1. *Invent. Math.*, 182(2):335–370, 2010.

[24] Vasileios Chousionis, Katrin Fässler, and Tuomas Orponen. Intrinsic Lipschitz graphs and vertical β-numbers in the Heisenberg group. *Amer. J. Math.*, 141(4):1087–1147, 2019.

[25] Vasilis Chousionis, Valentino Magnani, and Jeremy T. Tyson. On uniform measures in the Heisenberg group. *Adv. Math.*, 363:106980, 2020.

[26] Wei-Liang Chow. Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung. *Math. Ann.*, 117:98–105, 1939.

[27] Jeanne N. Clelland and Christopher G. Moseley. Sub-Finsler geometry in dimension three. *Differential Geom. Appl.*, 24(6):628–651, 2006.

[28] Michael Cowling and Uffe Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Invent. Math.*, 96(3):507–549, 1989.

[29] Marianna Csörnyei. On the visibility of invisible sets. *Ann. Acad. Sci. Fenn. Math.*, 25(2):417–421, 2000.

[30] Marianna Csörnyei. On the visibility of invisible sets. *Ann. Acad. Sci. Fenn. Math.*, 25(2):417–421, 2000.

[31] Donatella Danielli, Nicola Garofalo, and Duy-Minh Nhieu. Notions of convexity in Carnot groups. *Comm. Anal. Geom.*, 11(2):263–341, 2003.

[32] R. O. Davies. On accessibility of plane sets and differentiation of functions of two real variables. *Proc. Cambridge Philos. Soc.*, 48:215–232, 1952.

[33] Xiumin Du, Larry Guth, Yumeng Ou, Hong Wang, Bobby Wilson, and Ruixiang Zhang. Weighted restriction estimates and application to Falconer distance set problem. *arXiv e-prints*, page arXiv:1802.10186, February 2018.

[34] Xiumin Du and Ruixiang Zhang. Sharp L^2 estimate of Schrödinger maximal function in higher dimensions. *arXiv e-prints*, page arXiv:1805.02775, May 2018.

[35] K. J. Falconer. *The geometry of fractal sets*, volume 85 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1986.

[36] K. J. Falconer. Sets with prescribed projections and Nikodým sets. *Proc. London Math. Soc. (3)*, 53(1):48–64, 1986.

[37] K. J. Falconer. A capacity approach to box and packing dimensions of projections and other images. *arXiv e-prints*, page arXiv:1711.05316, Nov 2017.

[38] K. J. Falconer. A capacity approach to box and packing dimensions of projections and other images. *arXiv e-prints*, page arXiv:1711.05316, Nov 2017.

[39] K. J. Falconer and J. D. Howroyd. Projection theorems for box and packing dimensions. *Math. Proc. Cambridge Philos. Soc.*, 119(2):287–295, 1996.

[40] K. J. Falconer and J. D. Howroyd. Packing dimensions of projections and dimension profiles. *Math. Proc. Cambridge Philos. Soc.*, 121(2):269–286, 1997.

[41] Kenneth Falconer, Jonathan Fraser, and Xiong Jin. Sixty years of fractal projections. In *Fractal geometry and stochastics V*, volume 70 of *Progr. Probab.*, pages 3–25. Birkhäuser/Springer, Cham, 2015.

[42] Ábel Farkas. Projections of self-similar sets with no separation condition. *Israel J. Math.*, 214(1):67–107, 2016.

[43] Katrin Fässler and Tuomas Orponen. On restricted families of projections in $\mathbb{R}^3$. *Proc. Lond. Math. Soc. (3)*, 109(2):353–381, 2014.

[44] Fässler, Katrin and Hovila, Risto. Improved Hausdorff dimension estimate for vertical projections in the Heisenberg group. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 15(3495435):459–483, 2016.

[45] Herbert Federer. The (φ, k) rectifiable subsets of n-space. *Trans. Amer. Soc.*, 62:114–192, 1947.

[46] Herbert Federer. Colloquium lectures on geometric measure theory. *Bull. Amer. Math. Soc.*, 84(3):291–338, 1978.

[47] Andrew Ferguson, Thomas Jordan, and Pablo Shmerkin. The Hausdorff dimension of the projections of self-affine carpets. *Fund. Math.*, 209(3):193–213, 2010.

[48] G. B. Folland. A fundamental solution for a subelliptic operator. *Bull. Amer. Math. Soc.*, 79:373–376, 1973.

[49] G. B. Folland. Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.*, 13(2):161–207, 1975.

[50] Gerald B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[51] Rupert L. Frank, María del Mar González, Dario D. Monticelli, and Jinggang Tan. An extension problem for the CR fractional Laplacian. *Adv. Math.*, 270:97–137, 2015.

[52] Rupert L. Frank and Elliott H. Lieb. Sharp constants in several inequalities on the Heisenberg group. *Ann. of Math. (2)*, 176(1):349–381, 2012.

[53] Pritam Ganguly and Sundaram Thangavelu. On the lacunary spherical maximal function on the Heisenberg group. *arXiv e-prints*, page arXiv:1912.11302, December 2019.

[54] Daryl Geller. Fourier analysis on the Heisenberg group. *Proc. Nat. Acad. Sci. U.S.A.*, 74(4):1328–1331, 1977.

[55] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[56] Mikhael Gromov. Carnot-Carathéodory spaces seen from within. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 79–323. Birkhäuser, Basel, 1996.

[57] Larry Guth, Alex Iosevich, Yumeng Ou, and Hong Wang. On Falconer's distance set problem in the plane. *Invent. Math.*, 219(3):779–830, 2020.

[58] Piotr Hajł asz and Scott Zimmerman. Geodesics in the Heisenberg group. *Anal. Geom. Metr. Spaces*, 3(1):325–337, 2015.

[59] Terence L. J. Harris. An a.e. lower bound for Hausdorff dimension under vertical projections in the Heisenberg group. *arXiv e-prints*, page arXiv:1811.12559, Nov 2018.

[60] Terence L. J. Harris. Improved bounds for restricted projection families via weighted Fourier restriction. *arXiv e-prints*, page arXiv:1911.00615, Nov 2019.

[61] Terence L. J. Harris, Chi N. Y. Huynh, and Fernand o Roman-Garcia. Dimension Distortion by Right Coset Projections in the Heisenberg Group. *arXiv e-prints*, page arXiv:2002.04789, Feb 2020.

[62] Herbert Heyer. L'analyse de Fourier non-commutative et applications à la théorie des probabilités. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 4:143–164, 1968.

[63] Michael Hochman and Pablo Shmerkin. Local entropy averages and projections of fractal measures. *Ann. of Math. (2)*, 175(3):1001–1059, 2012.

[64] Risto Hovila. Transversality of isotropic projections, unrectifiability, and Heisenberg groups. *Rev. Mat. Iberoam.*, 30(2):463–476, 2014.

[65] J. D. Howroyd. On dimension and on the existence of sets of finite positive Hausdorff measure. *Proc. London Math. Soc. (3)*, 70(3):581–604, 1995.

[66] E. Järvenpää, M. Järvenpää, F. Ledrappier, and M. Leikas. One-dimensional families of projections. *Nonlinearity*, 21(3):453–463, 2008.

[67] Esa Järvenpää, Maarit Järvenpää, and Tamás Keleti. Hausdorff dimension and non-degenerate families of projections. *J. Geom. Anal.*, 24(4):2020–2034, 2014.

[68] Maarit Järvenpää. On the upper Minkowski dimension, the packing dimension, and orthogonal projections. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (99):34, 1994.

[69] Kenneth D. Johnson and Nolan R. Wallach. Composition series and intertwining operators for the spherical principal series. I. *Trans. Amer. Math. Soc.*, 229:137–173, 1977.

[70] Yitzhak Katznelson. *An introduction to harmonic analysis*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition, 2004.

[71] Robert Kaufman. On Hausdorff dimension of projections. *Mathematika*, 15:153–155, 1968.

[72] Enrico Le Donne. A metric characterization of Carnot groups. *Proc. Amer. Math. Soc.*, 143(2):845–849, 2015.

[73] Enrico Le Donne, Sean Li, and Terhi Moisala. Gâteaux differentiability on infinite-dimensional Carnot groups. *arXiv e-prints*, page arXiv:1812.07375, Dec 2018.

[74] Bochen Liu. On Hausdorff dimension of radial projections. *arXiv e-prints*, page arXiv:1903.12093, Mar 2019.

[75] J. M. Marstrand. Some fundamental geometrical properties of plane sets of fractional dimensions. *Proc. London Math. Soc. (3)*, 4:257–302, 1954.

[76] Pertti Mattila. Hausdorff dimension, orthogonal projections and intersections with planes. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 1(2):227–244, 1975.

[77] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[78] Pertti Mattila. Hausdorff dimension, projections, and the Fourier transform. *Publ. Mat.*, 48(1):3–48, 2004.

[79] Pertti Mattila. Recent progress on dimensions of projections. In *Geometry and analysis of fractals*, volume 88 of *Springer Proc. Math. Stat.*, pages 283–301. Springer, Heidelberg, 2014.

[80] Pertti Mattila. *Fourier analysis and Hausdorff dimension*, volume 150 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2015.

[81] Pertti Mattila and Tuomas Orponen. Hausdorff dimension, intersections of projections and exceptional plane sections. *Proc. Amer. Math. Soc.*, 144(8):3419–3430, 2016.

[82] Pertti Mattila, Raul Serapioni, and Francesco Serra Cassano. Characterizations of intrinsic rectifiability in Heisenberg groups. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 9(4):687–723, 2010.

[83] Richard Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[84] Roberto Monti. Some properties of Carnot-Carathéodory balls in the Heisenberg group. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 11(3):155–167 (2001), 2000.

[85] Roberto Monti. The regularity problem for sub-Riemannian geodesics. In *Geometric control theory and sub-Riemannian geometry*, volume 5 of *Springer INdAM Ser.*, pages 313–332. Springer, Cham, 2014.

[86] Roberto Monti and Matthieu Rickly. Geodetically convex sets in the Heisenberg group. *J. Convex Anal.*, 12(1):187–196, 2005.

[87] Camil Muscalu and Wilhelm Schlag. *Classical and multilinear harmonic analysis. Vol. I*, volume 137 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013.

[88] Daniel Oberlin and Richard Oberlin. Application of a Fourier restriction theorem to certain families of projections in $\mathbb{R}^3$. *J. Geom. Anal.*, 25(3):1476–1491, 2015.

[89] Tuomas Orponen. On the dimension and smoothness of radial projections. *Anal. PDE*, 12(5):1273–1294, 2019.

[90] Tuomas Orponen and Laura Venieri. Improved bounds for restricted families of projections to planes in $\mathbb{R}^3$. *arXiv e-prints*, page arXiv:1711.08934, Nov 2017.

[91] Pierre Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)*, 129(1):1–60, 1989.

[92] Yuval Peres and Wilhelm Schlag. Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions. *Duke Math. J.*, 102(2):193–251, 2000.

[93] Yuval Peres and Pablo Shmerkin. Resonance between Cantor sets. *Ergodic Theory Dynam. Systems*, 29(1):201–221, 2009.

[94] Fernando Roman-Garcia. A Fourier Coefficients Approach to Hausdorff Dimension in the Heisenberg Group. *arXiv e-prints*, page arXiv:1801.09022, January 2018.

[95] Fernando Román-García. Intersections of Projections and Slicing Theorems for the Isotropic Grassmannian and the Heisenberg group. *Anal. Geom. Metr. Spaces*, 8(1):15–35, 2020.

[96] Luz Roncal and Sundaram Thangavelu. Hardy's inequality for fractional powers of the sublaplacian on the Heisenberg group. *Adv. Math.*, 302:106–158, 2016.

[97] Pablo Shmerkin and Boris Solomyak. Absolute continuity of self-similar measures, their projections and convolutions. *Trans. Amer. Math. Soc.*, 368(7):5125–5151, 2016.

[98] Eberhard Siebert. Fourier analysis and limit theorems for convolution semigroups on a locally compact group. *Adv. in Math.*, 39(2):111–154, 1981.

[99] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[100] Robert S. Strichartz. Self-similarity on nilpotent Lie groups. In *Geometric analysis (Philadelphia, PA, 1991)*, volume 140 of *Contemp. Math.*, pages 123–157. Amer. Math. Soc., Providence, RI, 1992.

[101] Robert S. Strichartz. Self-similarity in harmonic analysis. *J. Fourier Anal. Appl.*, 1(1):1–37, 1994.

[102] Leon A. Takhtajan. *Quantum mechanics for mathematicians*, volume 95 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[103] Sundaram Thangavelu. *Harmonic analysis on the Heisenberg group*, volume 159 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1998.

[104] Sundaram Thangavelu. *An introduction to the uncertainty principle*, volume 217 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2004. Hardy's theorem on Lie groups, With a foreword by Gerald B. Folland.

[105] Claude Tricot, Jr. Two definitions of fractional dimension. *Math. Proc. Cambridge Philos. Soc.*, 91(1):57–74, 1982.

[106] Jeremy T. Tyson. Global conformal Assouad dimension in the Heisenberg group. *Conform. Geom. Dyn.*, 12:32–57, 2008.

[107] Davide Vittone. The regularity problem for sub-Riemannian geodesics. In *Geometric measure theory and real analysis*, volume 17 of *CRM Series*, pages 193–226. Ed. Norm., Pisa, 2014.

[108] Hai-meng Wang and Qing-yan Wu. On fundamental solution for powers of the sub-Laplacian on the Heisenberg group. *Appl. Math. J. Chinese Univ. Ser. B*, 32(3):365–378, 2017.

[109] Peter Woit. *Quantum Mechanics for Mathematicians : The Heisenberg group and the Schrödinger Representation*. 2012.