# On the Asymptotic Minimum Energy Required to Transport Packets in Wireless Networks

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***Abstract*— In this paper we study the asymptotic minimum energy required to transport (via multiple hops) data packets from a source to a destination. Under the assumptions that nodes are distributed according to a Poisson point process with node density $n$ in a unit-area square and the distance between a source and a distance is at least a constant, we prove, based on percolation theory, the minimum energy required to carry a packet from a source to a destination is $\Theta(n^{(1-\alpha)/2})$ with probability approaching one as the node density goes to infinity, where $\alpha$ is the path loss exponent.**

**We demonstrate how to apply the derived results to obtain the bounds of the capacity of wireless networks equipped with directional antennas, the capacity of wireless networks that operate in UWB, and finally the upper bound on the lifetime of wireless sensor networks. We believe the results and the proof techniques can be applied to derive asymptotic conditions for other parameters in wireless networks, as long as the limiting factor for the parameters of interest is the energy. Finally, we carry out simulations to validate the derived results and to estimate the constant factor associated with the bounds on the minimum energy. The simulation results indicate that the constant associated with the minimum energy converges to the source-destination distance.**

## I. Introduction

A wireless ad hoc network is a collection of wireless mobile hosts which communicate with each other without the support of fixed infrastructures or centralized administrations. It has gained tremendous attention in recent years because of its capability of providing wireless connectivity without the need for pre-existing infrastructures. Since most wireless devices use batteries as their power sources, it is important that as little energy as possible is consumed to execute proper operations of wireless networks.

It is very often that "proper operations" of wireless networks refer to maintaining connectivity or $k$-connectivity of the network. As such, the problem is often formulated as minimizing the total energy consumption while maintaining connectivity of the network. Researches on this problem are performed roughly along two thrusts.

In the first thrust, researchers aim to devise efficient, distributed algorithms to determine the transmission power of each node in order to minimize the total transmission power of all wireless nodes, while maintaining ($k$-)connectivity. This problem is, in general, NP hard in the Euclidean plane [6], and many researchers have developed localized heuristics [21], [14], [16], [15], or efficient algorithms with bounded approximation ratios [13], [5], [12], [4].

In the second thrust, researchers aim to determine the asymptotic minimum common transmission range or the asymptotic minimum total power required for maintaining connectivity or $k$-connectivity [11], [18], [22], [19], [23], [3], [9], [20]. Of particular interest is how the transmission range or the total power scales as the number of wireless devices increases.

We approach the problem from a different perspective. Since the primary function of a wireless network is to transport data packets, rather than merely maintaining connectivity, we define the "proper operations" of the network as transporting packets, and consider the problem of minimizing energy consumption *while transporting a packet from a source to a destination in a wireless network*. Algorithmically, the problem can be solved using any shortest-path algorithm such as Dijkstra's algorithm. However, it has been left unattended *what is the asymptotic minimum energy required to carry a packet from a source to a destination in a wireless network*, and especially, how this quantity scales as the network size goes to infinity. In this paper we address the problem under the assumption that nodes are distributed as a Poisson point process in a unit square area, and the source and the destination are separated by at least a constant distance.

We solve the formulated problem by deriving, based on percolation theory, an upper bound and a lower bound on the asymptotic minimum energy required to transport a packet from a source to a destination in a random network where nodes are deployed as Poisson point process with density $n$ in a unit-square region. We show that if the source-destination distance is of order $\Theta(1)$, both the upper bound and lower bound are of order $\Theta(n^{(1-\alpha)/2})$ with probability approaching one as the network density goes to infinity, where $\alpha$ is the path-loss exponent.

After the bounds are derived, we discuss how to extend the results to accommodate the cases (i) that the network density is kept as constant but the network size goes to infinity, and (ii) that both the transmitting and receiving operations consume power. In particular, we obtain the following more general result by rescaling: in a network where nodes are distributed according to a Poisson point process with density $n$ on an infinite plane, the minimum energy required to carry a packet from a source to a destination with distance $l$ is $\Theta(n^{(1-\alpha)/2}l)$ with high probability if $nl^2 \to \infty$.

We also demonstrate how to apply the derived results to solve other related theoretical problems. For example, the derived results can be used to determine the capacity bound in the case of communications with the use of directional antennas, and the capacity bound in the case of ultra wide band (UWB) communications. This is because in both cases the limiting factor for the capacity bound is the system energy. As another example, the derived result can be used to determine an upper bound on the network lifetime of wireless sensor networks.

We also carry out simulations to both validate the theoretical derivation and estimate the associated constant in the derived asymptotic function. The simulation results suggest that minimum energy required for transporting a data packet converges to $n^{(1-\alpha)/2}l$ with high probability, where $l$ is the source-destination distance and $n$ is the node density.

The rest of the paper is organized as follows. In Section II, we discuss the system model and assumptions we have made for the rest of the paper. We prove the lower bound on the minimum energy required to transport a packet in Section III and the upper bound in Section IV. Following that, we discuss in Section V the extensions, as well as the applications, of the derived results. Finally, we present the simulation results in Section VI, and conclude the paper in Section VII.

## II. Model Assumptions

The network is composed of a set of wireless nodes, which are distributed according to a Poisson point process of density $n$ in a two-dimensional unit square region $[0,1]^2$. We assume the energy required to directly transmit a packet from a sender to a receiver with distance $d$ is $d^\alpha$,[1] where $\alpha \geq 1$ is the path loss exponent. Let $X_i$ denote the $i$th node location. Let $R$ denote the minimum energy route for a given source-destination pair, i.e, $R = [X_0, X_1, \cdots, X_k]$. The minimum energy of the route (which is also termed as transporting energy in this paper) is thus

$$Q \triangleq \sum_{i=0}^{k-1} |X_i - X_{i+1}|^\alpha, \tag{1}$$

where $|X_i - X_{i+1}|$ denote the Euclidean distance between $X_i$ and $X_{i+1}$. The objective of this paper is to determine the asymptotic bounds of $Q$. The results are shown to hold with high probability (*w.h.p.*), which means probability approaching 1 as the density $n \to \infty$. We assume the source destination distance is $\Theta(1)$.

## III. Lower bound on the energy requirement

In this section, we establish a lower bound on $Q$. The key to the derivation is that, if it is possible for the route $R$ to be composed of mostly short hops, then potentially the minimum energy ($Q$) of a route can be very small. Thus, our major task is to show that there are a sufficiently large number of long hops. The proof is based on the site percolation model.

[1]Clearly, the energy also depends on the packet size and the transmission rate. We assume these factors contribute to a constant factor, and hence are ignored in the derivation.

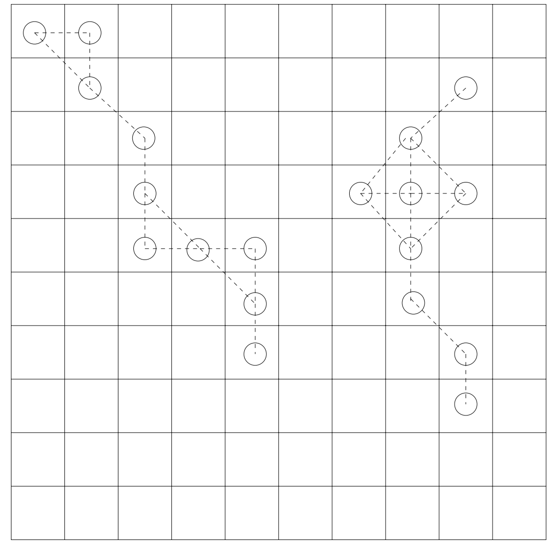


Fig. 1. Construction of the site percolation model. We divide the region into grids of edge length $c_0/\sqrt{n}$. A grid is said to be open if there is at least one Poisson point inside it; and closed otherwise. Two grids are said to be adjacent if two grids share an edge or a vertex, i.e., grid $(i,i)$ is adjacent to $(i-1,i-1), (i-1,i), (i-1,i+1), (i,i-1), (i,i+1), (i+1,i-1), (i+1,i), (i+1,i+1)$. An open grid is denoted with a circle inside it. The dashed lines show all the possible open links.

### A. Construction of the Site Percolation Model

We divide the square region into grids of edge length $c_0/\sqrt{n}$ as depicted in Fig. 1. By adjusting the constant $c_0$, we can adjust the probability that a grid contains at least one node:

$$P(\text{a grid contains at least one node}) = 1 - e^{-c_0^2} \triangleq p. \tag{2}$$

A grid is said to be *open* if it contains at least one node, and *closed* otherwise. Two grids are said to be *adjacent* if they share an edge or a vertex. Any grid is thus adjacent to eight other grids. For notational convenience, we use (i) a *path* to refer to a list of grids such that any two neighboring grids in the list are adjacent; and (ii) a *route* to refer to a list of wireless nodes that are actually used to transport packets from the source to the destination. By the convention in graph theory, we assume a path does not include any grid twice, except that its first grid may be the same as the last grid. A path is said to be *open* (*closed*) if all the grids on the path are open (closed).

*Important Properties of the Site Percolation Model:* As a first step, we observe that if there is an open path in the percolation model from the grid where the source is located to the grid where the destination is located, then we can form a route from the source to the destination by picking one node from each grid on the path. Every hop on this route is bounded from above by $2\sqrt{2}c_0/\sqrt{n}$. On the other hand, if there is no such open path in the percolation model, then on any route (including the minimum energy route) from the source to the destination, at least one hop is of length at least $c_0/\sqrt{n}$. Indeed, if $c_0$ and consequently $p$ are sufficiently small, and the distance, $d$, between the source and the destination is sufficiently large, there exists no open path between them in the percolation model *w.h.p.*. We formally state and prove the above property in the lemma below.

**Lemma 1** *Let $p$ be the probability that a grid is open in the site percolation model we have defined (Eq. (2)). Then the probability that there exists an open path of length $m$ starting from a source is upper bounded by*

$$P(N(m) \geq 1) \leq \frac{8}{7}(7p)^m, \quad (3)$$

*where $N(m)$ is the number of open paths of length $m$ starting from a given source.*

*Proof.* The total number of paths of length $m$ are upper bounded by $8 \cdot 7^{m-1}$, because in the first hop there are at most 8 choices, and in each subsequent hop there are at most 7 choices. Each path of length $m$ is open with a probability of $p^m$. Thus, the expected number of open paths of length $m$ starting from a given source is $E[N(m)] = 8 \cdot 7^{m-1} \cdot p^m$. It then follows by the Markov inequality that

$$P(N(m) \geq 1) \leq E[N(m)] = \frac{8}{7}(7p)^m. \quad (4)$$

□

If we choose $p < 1/7$ and the distance (in terms of grids) between the source and the destination goes to infinity, then *w.h.p.* there is no open path between them.

The next result is patterned on the results derived in [10] (Eq. (2.49)) in which the bond percolation model is used. Note, however, that we consider the *site* percolation model, and hence the proof is not exactly the same.

**Lemma 2** *Let $A$ be the event that there exists an open path of length $m$ starting from a given source and $F_A$ the minimum number of grids that need to be turned open from closed in order for the event $A$ to take place. Then we have*

$$P_p(A) \geq \left(\frac{p-p'}{1-p'}\right)^r P_{p'}(F_A \leq r) \quad (5)$$

*for any $0 < p' < p < 1$, where $P_p$ ($P_{p'}$) denote the probability measure with the site-open probability $p$ ($p'$), which is the probability that a grid is open .*

*Proof.* See Appendix I.

### B. Derivation of Lower Bound

We are now in a position to prove the following major result.

**Theorem 1** *Assume that nodes are distributed in a unit square area according to a Poisson point process with density $n$. If the distance between a source-destination pair is $d \geq \epsilon > 0$, the energy $Q$ (defined in Eq. (1)) of the minimum energy route between them is at least $c_1 n^{(1-\alpha)/2}$* w.h.p. *for some constant $c_1 > 0$. Specifically,*

$$P(Q > c_1 n^{(1-\alpha)/2}) \geq 1 - \frac{8}{7} \cdot \exp(-c_2\sqrt{n}), \quad (6)$$

*as $n \to \infty$, for some constant $c_1, c_2 > 0$.*

*Proof.* For any route between the source and the destination, we can construct a walk (which may include some grids more than once) in the site percolation model, by including all the grids that intersect with the route. The walk can be further trimmed into a path which contains the minimum number of closed grids by removing unnecessary grids. The trimming process is illustrated in Fig. 2. We denote $T^*$ as an optimally trimmed path that contains the minimum number of closed grids. In what follows, we derive a bound on the probability that the optimally trimmed path $T^*$ contains at most $c_3\sqrt{n}$ closed grids, where $c_3$ is a constant yet to be determined.

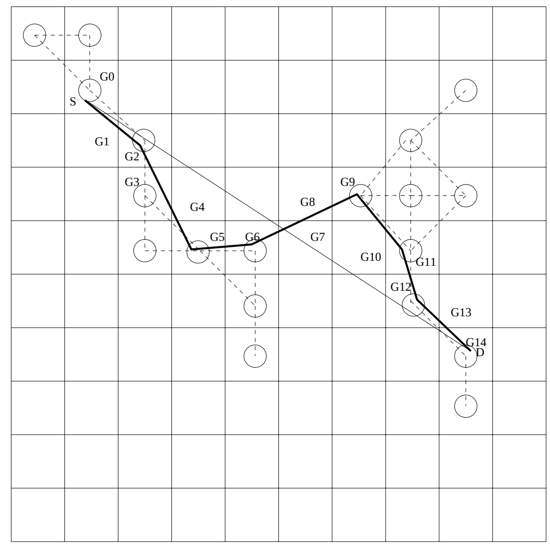



Fig. 2. The bold lines show a route from source $S$ to destination $D$. We can construct a walk (which is also a path) that is composed of grids that intersect with the route: $[G_0, G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9, G_{10}, G_{11}, G_{12}, G_{13}, G_{14}]$. Some of the grids can be removed from the path. For example, $G_1$ can be removed because $G_0$ and $G_2$ are connected (in our percolation model). Similarly, $G_4, G_8, G_{10}, G_{13}$ can all be removed. There are multiple ways of trimming the path. For example, we can also remove $G_3, G_5$ but keep $G_4$. Among all the trimmed paths, we pick as $T^*$ the one that contains the minimum number of closed grids. Ties are broken arbitrarily. In the above example, the path $[G_0, G_2, G_3, G_5, G_6, G_7, G_9, G_{11}, G_{12}, G_{14}]$ contains minimum number (which is 1 in this case) of closed grids.

Note that the distance between the source-destination pair in terms of grids is at least $m \triangleq d/(\sqrt{2}c_0/\sqrt{n}) = d\sqrt{n}/(\sqrt{2}c_0)$. This implies the path length of $T^*$ is at least $m$.

If $T^*$ contains at most $c_3\sqrt{n}$ closed grids, then we can construct an open path from the source to the destination, by turning at most $c_3\sqrt{n}$ closed grids into open grids. This further indicates that by turning at most $c_3\sqrt{n}$ closed grids into open ones, we can obtain an open path of length at least $m$ starting from the source. Now we apply Lemma 2. Let $A$ denote the event that there is an open path of length $m$ starting from the source, and $F_A$ the minimum number of closed grids that need to be turned into open in order for event $A$ to take place. We conclude that $F_A \leq r = c_3\sqrt{n}$ if the trimmed path $T^*$ contains at most $c_3\sqrt{n}$ closed grids. By Lemma 2,

$$P_{p'}(F_A \leq c_3\sqrt{n}) \quad \leq \quad P_p(A)\left(\frac{p-p'}{1-p'}\right)^{-c_3\sqrt{n}}. \quad (7)$$

By Lemma 1,

$$P_p(A) = \frac{8}{7} \cdot (7p)^m = \frac{8}{7} \cdot (7p)^{d\sqrt{n}/(\sqrt{2}c_0)}. \quad (8)$$

We can choose $c_0$ such that $p = 1 - e^{-c_0^2} < 1/7$. After fixing $c_0$ and $p$, we can choose $k > 1/p$ and $p' = \frac{kp-1}{k-1} < p$. Now plugging the expression of $p'$ and Eq. (8) into Eq. (7), we have

$$\begin{aligned} & P_{p'}(F_A \le c_3\sqrt{n}) \\ \le \quad & \frac{8}{7} \cdot (7p)^{d\sqrt{n}/(\sqrt{2}c_0)} \cdot k^{c_3\sqrt{n}} \\ = \quad & \frac{8}{7} \cdot \exp\left(\sqrt{n}\left(\frac{d\log(7p)}{\sqrt{2}c_0} + c_3 \log k\right)\right). \end{aligned} \tag{9}$$

If we choose $0 < c_3 < -\frac{\epsilon \log(7p)}{\sqrt{2}c_0 \log k} < -\frac{d \log(7p)}{\sqrt{2}c_0 \log k}$, we obtain

$$P_{p'}(F_A \le c_3\sqrt{n}) \le \frac{8}{7} \cdot \exp(-c_2\sqrt{n}) \to 0 \tag{10}$$

as $n \to \infty$, where

$$c_2 = -\frac{\epsilon \log(7p)}{\sqrt{2}c_0} - c_3 \log k > 0. \tag{11}$$

Hence, the optimally trimmed path $T^*$ contains more than $c_3\sqrt{n}$ closed grids with probability at least $p_1 \triangleq 1 - \frac{8}{7} \cdot e^{-c_2\sqrt{n}}$ if we choose the grid size $c_0'/\sqrt{n}$ such that $1 - e^{-c_0'^2} = p'$.

For each closed grid on $T^*$, there is exclusively one line segment with length at least $c_0'/\sqrt{n}$ completely contained in a link on the minimum energy route. (Since for each such closed grid $g$, there must be a link $l$ on the route that crosses the grid either from two parallel sides of the grid $g$ or from two adjacent sides of $g$. In the former case, the line segment on the link and contained inside the grid $g$ has length at least $c_0'/\sqrt{n}$. In the latter case, we consider two neighbor grids of $g$ that also intersect with the link $l$. At least one of them is either not closed or not on the optimally trimmed path $T^*$ (otherwise, we can remove the closed grid $g$ from the path $T^*$). The line segment on the link contained inside the grid $g$ and the neighbor grid that is either not on the path $T^*$ or not closed has length at least $c_0'/\sqrt{n}$. By induction on the number of grids intersecting a given link, we can prove that any part of the above obtained line segment will not be reclaimed by other closed grid on $T^*$. An illustration is given in Fig. 3.

Therefore, we conclude that for each closed grid on $T^*$, there is a link on the minimum energy route that intersects with it. In addition, if a link on the route intersects with $j$ closed grids on $T^*$, the link has length at least $jc_0'/\sqrt{n}$. To derivate the lower bound of the energy of the route, we can assume each link only intersects at most one grid in $T^*$, because if a link intersects with $j$ closed grids on $T^*$, its energy will be greater than the energy of $j$ links each with length $c_0'/\sqrt{n}$ since its length is at least $jc_0'/\sqrt{n}$.[2] Thus the route contains at least $c_3\sqrt{n}$ links each with length at least $c_0'/\sqrt{n}$ with probability at least $p_1$. Hence the total energy of the route is at least $c_3\sqrt{n} \cdot \left(\frac{c_0'}{\sqrt{n}}\right)^\alpha = c_3 c_0'^\alpha n^{(1-\alpha)/2}$ with probability at least $p_1$. Denote $c_1 = c_3 c_0'^\alpha$, and we obtain

$$P(Q > c_1 n^{(1-\alpha)/2}) \ge p_1 = 1 - \frac{8}{7} \cdot \exp(-c_2\sqrt{n}). \tag{12}$$

□

[2] Here, we make use of the assumption $\alpha \ge 1$; otherwise, the statement may not hold.

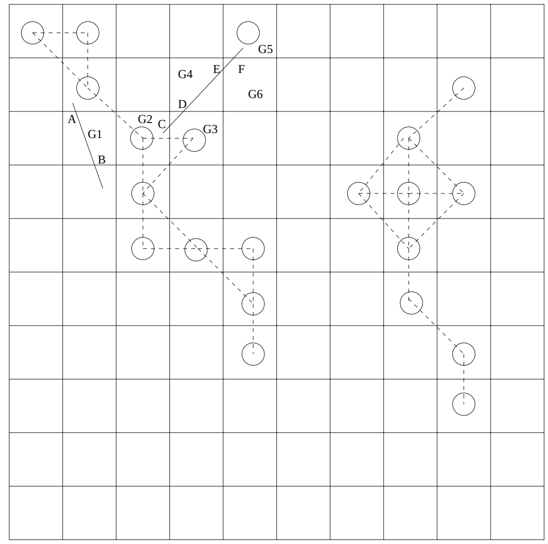



Fig. 3. Illustration that for each closed grid on $T^*$, there exists exclusively one line segment completely contained in a link on the minimum energy route with length at least $c_0'/\sqrt{n}$. If a link crosses a closed grid at the two opposite edges (as in the grid $G_1$), the line segment ($AB$) on the link that is contained by the grid have length at least $c_0'/\sqrt{n}$. Hence without loss of generality, we can assume a link enters a closed grid from its bottom and exits from its right (such as grid $G_4$). If grid $G_3$ is on the path $T^*$, then $G_6$ is either not on the path $T^*$ or open, because otherwise $G_4$ can be removed from the path. In this case, the line segment $DF$ has length at least $c_0'\sqrt{n}$. Similarly if $G_2$ but not $G_3$ is on the path $T^*$, the line segment $CE$ has length at least $c_0'\sqrt{n}$. If a link intersects more than one grid on the path $T^*$, similar analysis can be performed.

## IV. An Upper Bound on the Minimum Energy Consumption

In order to derive an upper bound on the minimum energy required to carry a packet over a wireless network, we leverage the routing scheme devised in [7], [8] and show that there exists a routing scheme that can achieve the energy bound. The energy bound turns out to be of the same order of the lower bound that we have derived in Section III.

### A. Construction of the Backbone Network

The routing scheme lays a wireless backbone network that carries packets across the network at the desired rate. The backbone network is composed of short hops (and hence consumes low energy), and is obtained through the percolation theory.

To construct the backbone network, we divide the area into square grids of edge length $c_5/(\sqrt{2n})$. The new grid system is depicted in Fig. 4 (a). Note that the grid system is constructed differently from that in Section IV: grids in this grid system are not arranged in a vertical-horizontal fashion, but are 45 degree tilted. Then, as depicted in Fig. 4 (b), for each of the one half of grids in the system, we draw a horizontal edge across it and for each of the other half of grids, a vertical edge across it. (Both the horizontal and vertical edges are depicted in thick lines in Fig. 4 (b).) An edge is said to be open if there exists at least one node (from the Poisson point process) in the grid that contains the edge and closed otherwise. In this fashion we obtain a bond percolation model. The probability that an edge is open is independent of all other edges, and can be expressed as

$$p = 1 - e^{-c_5^2/2}. \quad (13)$$

Next we divide the network area into horizontal rectangles, $\bar{R}_n$, of size $1 \times \frac{c_5}{\sqrt{n}} \log \frac{\sqrt{n}}{c_5}$. Each of the rectangles thus has $m \times \log m$ grids in the bond percolation model, with $m = \sqrt{n}/c_5$ (as the edges have length $\frac{1}{m}$). As proved in [7] (Theorem 1), there exist many open paths from left to right inside each such rectangle $\bar{R}_n$.

**Lemma 3** *(Theorem 1 in [7]) If $c_5$ is sufficiently large, there exists a constant $\beta = \beta(c_5) > 0$ such that* w.h.p. *there are $\beta \log m = \beta \log \frac{\sqrt{n}}{c_5}$ disjoint open paths that cross each rectangle $\bar{R}_n$ from left to right.*

This result does not give a bound on the length of the paths. However, we can bound the length of the shortest open path in each rectangle $\bar{R}_n$ as follows.

**Lemma 4** *If $c_5$ is sufficiently large, there exists a constant $\beta = \beta(c_5) > 0$ such that* w.h.p. *the shortest open path crossing each rectangle $\bar{R}_n$ has length not larger than $2m/\beta$.*

*Proof* By Lemma 3, *w.h.p.*, there are $\beta \log m$ *disjoint* open paths in each rectangle $\bar{R}_n$. If every open path in a rectangle has a length greater than $2m/\beta$, the total number of edges held by all the disjoint, open paths in the rectangle is larger than $2m \log m$. However, the total number of edges in each rectangle is equal to the number of original grids (as depicted in Fig. 4(a)) in that rectangle, which is $2m \log m$. By the pigeonhole principle, we reach a contradiction, and hence at least one open path in each rectangle has length not greater than $2m/\beta$ and so does the shortest open path. □

We can also divide the area into vertical rectangles and obtain the same results for paths that cross the area from the bottom to the top. With the use of a simple union bound argument, we conclude that there exist at least one horizontal open path and one vertical open path with length at most $m/\beta$ in each horizontal rectangle and vertical rectangle simultaneously *w.h.p.*. These paths constitute the backbone network we are going to use in the routing scheme.

## B. Routing Scheme in the Wireless Network

Packets are transported from sources to destinations in the above backbone network via three phases: the *draining phase*, the *backbone phase*, and the *delivery phrase*. In the first (draining) phase, the source sends packets directly to a node on a horizontal open path of the backbone network. In the second (backbone) phase, packets are transported along the horizontal open path and reach a vertical open path. In the third (delivery) phase, a node in the vertical open path sends packets directly to the destination. In what follows we discuss the detailed operations in each phase.

*1) Draining phase:* In the draining phase, packets are carried from the source to the backbone network. We first evenly divide the square area into $m/\log m$ horizontal slabs of width $\frac{\log m}{m}$. Now since there are exactly as many slabs as the rectangles, we can enforce that nodes in the $i$th slab send their packets using the shortest open path in the $i$th rectangle. More precisely, an entry point in the $i$th horizontal path can be assigned to each source in the $i$th slab. As shown in Fig. 5, the entry point is chosen to be the node on the shortest open path in the $i$th horizontal rectangle that is closest to the vertical line drawn from the source point. By Lemma 3, the distance between a source and its corresponding entry point is never larger than $(c_5/\sqrt{n})\log(\sqrt{n}/c_5) + c_5/\sqrt{n}$ (since the source and the entry point are in the same rectangle $\bar{R}_n$ their vertical distance is at most $(c_5/\sqrt{n})\log(\sqrt{n}/c_5)$, and their horizontal distance is at most $c_5/\sqrt{n}$ by the choice of the entry point).

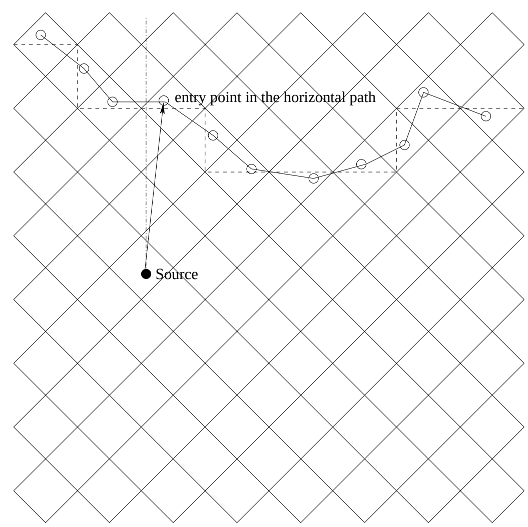


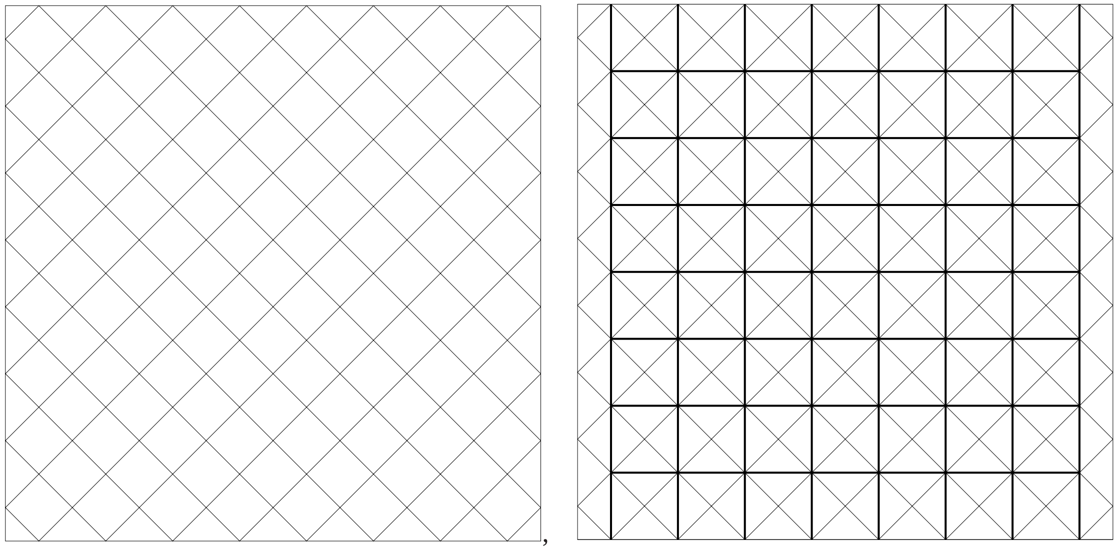

Fig. 5. A source transmits packets directly to the entry point on a horizontal open path.

*2) Backbone phase:* Similarly we can divide the square area into $m/\log m$ vertical slabs. Once a packet is transmitted to the entry point, it is carried along the corresponding horizontal path until it reaches the crossing point with the target vertical open path. The target vertical open path is determined by the vertical slab that contains the destination node, i.e, if the destination is in the $i$th vertical slab, the target vertical open path is the shortest open path in the $i$th vertical rectangle.

*3) Delivery phase:* In the delivery phase, packets are transported from the exit point of the vertical open path to the destination directly. The *exit point* for a given destination is defined as a node in the grid on the vertical open path whose center (i.e., the center of the grid) is closest to the horizontal line drawn from the destination. Again, the destination from the exit point to the destination is at most $(c_5/\sqrt{n})\log(\sqrt{n}/c_5) + c_5/\sqrt{n}$.

## C. Energy consumptions for transporting a packet

We now show that the energy consumed to transport a packet using the routing scheme presented in Section IV-B



(a) New grid system

(b) Edges for the bond percolation model

Fig. 4. Construction of the bond percolation model. We divide the unit square area into square grids of side length $c_5/(\sqrt{2n})$. A grid is said to be *open* if it contains at least one point in the Poisson point process and *closed* otherwise. The edge that crosses an open (closed) grid is said to be *open* (*closed*).

is $O(n^{\frac{1-\alpha}{2}})$. Clearly it is sufficient to show this is true in each phase of the routing scheme.

*1) Draining phase:* Since the distance from each source $X$ to the entry point $X_1$ is never larger than $\frac{c_5}{\sqrt{n}}(\log\frac{\sqrt{n}}{c_5}+1)$, the required energy in this hop is

$$\begin{aligned} q_1 &= |X - X_1|^\alpha \\ &\leq \left(\frac{c_5(1+\log(\sqrt{n}/c_5))}{\sqrt{n}}\right)^\alpha \\ &\leq c_6 n^{(1-\alpha)/2}. \end{aligned} \tag{14}$$

Clearly the last inequality holds if $n$ is sufficiently large for any $c_6 > 0$.

*2) Backbone phase:* By Lemma 4, a message needs to travel at most $2m/\beta$ hops on a horizontal open path and at most $2m/\beta$ hops on a vertical open path before it reaches the exit point. Since each hop length is at most $2c_5/\sqrt{n}$, the total energy consumption on the backbone phase is

$$\begin{aligned} q_2 &\leq 2\cdot 2m/\beta\cdot(2c_5/\sqrt{n})^\alpha \\ &= \frac{4}{\beta}\sqrt{n}/c_5(2c_5/\sqrt{n})^\alpha \\ &= \frac{2^{\alpha+2}}{\beta}c_5^{\alpha-1}\cdot n^{\frac{1-\alpha}{2}}. \end{aligned} \tag{15}$$

*3) Delivery phase:* In the delivery phase, an exit point on the vertical path sends packets to the destination node directly. Again, the distance from the exit point to the destination node is upper bounded by $(c_5/\sqrt{n})(\log(\sqrt{n}/c_5)+1)$. With a similar analysis performed in the draining phase, we can upper bound the energy consumption by

$$q_3 \leq c_6 n^{(1-\alpha)/2}, \tag{16}$$

if $n$ is sufficiently large for any given $c_6 > 0$.

Summing up the energy consumption in all three phases, we obtain an upper bound on the energy required to transport a packet in a network from source to destination.

**Theorem 2** *With the assumptions we have made in Section II, the minimum energy required to transport a packet from a source to a destination in a unit-square area is upper bounded by* $c_7(n^{(1-\alpha)/2})$ w.h.p., *where* $c_7 = 2c_6 + 2^{\alpha+2}c_5^{\alpha-1}/\beta$ *is independent of* $n$.

*Remarks on load balancing:* In the derivation, we do not consider the issue of load balancing, as our focus is on the minimum energy required to transport a packet between a source-destination pair. However, load balancing can be achieved while maintaining the same order of minimum energy consumption. By applying the pigeonhole principle, for any $0 < \gamma < \beta$ ($\beta$ is from Lemma 3), one can show that *w.h.p.* at least $\gamma \log m$ disjoint open paths in each rectangle have a length of at most $2m/(\beta-\gamma)$. Combining all the open paths in all rectangles, we can obtain $\gamma m$ disjoint horizontal open paths and $\gamma m$ disjoint vertical open paths, all of which are of length at most $2m/(\beta-\gamma)$. Thus we can evenly distribute the traffic on the $\gamma m$ open paths to balance the traffic load while incurring minimum energy consumption.

Combining Theorems 1 and 2, we reach the major conclusion in this paper.

**Corollary 1** *Assume that nodes are distributed in a unit square area according to a Poisson point process with density* $n$. *If the distance between a source-destination pair is* $\Theta(1)$, *the minimum energy required to transport a packet from the source to the destination is* $\Theta(n^{(1-\alpha)/2})$ w.h.p..

TABLE I

CORRESPONDING VALUES IN THE RESCALED (LARGE) SQUARE AND THOSE IN THE ORIGINAL UNIT-AREA SQUARE.

| values of interest | in the rescaled square | in the unit-area square |
|---|---|---|
| side length | $l$ | $l' = 1$ |
| density | $D$ | $\lambda' = Dl^2$ |

## V. DISCUSSION

In this section, we discuss how to extend the results to accommodate the case (i) that the network density is kept as constant but the network size goes to infinity, and (ii) that both the transmitting and receiving operations consume power. We also demonstrate how to apply the derived results to solve other related theoretical problems.

### A. Extension to the Case That the Network Size Grows

One may wonder if the derivation holds true in the case of very short distance scale (e.g., of order less than or equal to $n^{-1/2}$), as the energy, $d^\alpha$, incurred in transmission may not be accurate in this case (where $d$ is the distance between a sender and a receiver). Our understanding is that the unit area square (which is widely used) is a miniature of the real world and can be "resized" to obtain parameters of interest. That is, we consider a (rescaled) network with a fixed node density $D$ in a square region with side length $l \to \infty$. If we scale such network back to unit-area with side length 1, the node density in the unit-area network is now $Dl^2$ (Table I shows the corresponding values in the rescaled network and those in the original one). By Corollary 1, the minimum energy required in the unit-area disk is $(Dl^2)^{(1-\alpha)/2}$. As compared with the unit-area network, each edge in the rescaled (large) network is multiplied by $l$, and the energy consumed at each hop (and hence the total energy consumption) is multiplied by $l^\alpha$. Therefore, the energy consumed to carry a packet from a source to a destination with distance $\Theta(l)$ in the rescaled network is

$$\Theta((Dl^2)^{(1-\alpha)/2} \cdot l^\alpha) = \Theta(D^{(1-\alpha)/2} l). \tag{17}$$

It is interesting to observe that although the energy consumed at a single hop scales as $l^\alpha$, the energy consumed through multiple hops scales linearly with $l$ and decreases as the node density increases. We also note that Eq. (17) does not require that $D$ be constant. Instead, the only assumption required (for the asymptotic proof in Section III and IV) is that $Dl^2$ goes to infinity.

More generally, for any given source destination pair with distance $l$, we can construct a square with a side length of $\Theta(l)$ which contains both the source and the destination. Therefore, we can easily obtain the following result.

**Corollary 2** *Assume that nodes are distributed as a Poisson point process with density $D$ in an infinite plane. If the distance between a source-destination pair is $l$, the minimum energy required to transport a packet from the source to the destination is* $\Theta(D^{(1-\alpha)/2} l)$ w.h.p., *as* $Dl^2 \to \infty$.

### B. Extension to the Case That Both Transmitting and Receiving Operations Consume Power

In the derivation in Sections III–IV, we only consider the energy consumption incurred in the transmission activities. It has been suggested (e.g. [2]) that a wireless node also consumes energy when it receives packets. As the amount of energy consumed in receiving a packet is usually a constant,[3] the total energy consumed at the receivers only depends on the number of hops.

To figure in the energy consumed in receiving packets, we consider the rescaled network as above. The number of hops has been shown to be upper bounded by $O(\sqrt{Dl^2})$. Therefore the total energy required to transport a packet from a source to a destination with distance $\Theta(l)$ is upper bounded by $O(D^{1/2}(1 + D^{-\alpha/2})l)$ *w.h.p.*. The lower bound, however, depends on the relation between the receiving energy and the transmitting energy, and is more difficult to obtain. Derivation of such a lower bound is subject to further investigation.

### C. Applications of Derived Results to Other Energy-Related Problems

As mentioned in Section I, the derived results on the minimum energy required to transport a packet in wireless networks can be used to derive the upper bounds on the network capacity and lifetime in certain types of wireless networks, as long as the limiting factor for the parameters of interest (e.g., network capacity and lifetime) is the energy. In the following discussions, we use the results derived in the unit-area square to be consistent with those in the literature (so that comparisons can be made).

*1) Network capacity with the use of directional antennas:* Peraki and Servetto [1] studied the problem of network capacity in wireless networks with directional antennas. As directional antennas can generate arbitrarily narrow beams, wireless interference can be ignored and the major constraint for limiting the network capacity is the energy consumption! One of the major results obtained in [1] is that the network capacity with the use of directional antennas is upper bounded by $\Theta(\sqrt{n \log n})$ if all nodes choose a common power to maintain connectivity. As given in [11], the common transmission radius $r$ required to maintain connectivity satisfies

$$\pi r^2 = \frac{\log n + \eta_n}{n}, \tag{18}$$

for some $\eta_n \to \infty$ as $n \to \infty$.

Now we show how the results derived in this paper can be used to obtain the same capacity bound. Without loss of generality, we assume the transmission power for a transmission radius $r$ is $r^\alpha$. Based on Eq. (18), the total available transmission power (which is the total available energy in one unit of time) in the network is $\Theta(n \cdot (\frac{\log n}{n})^{\alpha/2})$. As we have proved in Sections III–IV, for each source-destination with distance at least $\epsilon > 0$, the minimum energy required to transport a packet from the source to the destination is

[3] The energy consumed in transmitting a packet also contains a constant term (in addition to the $d^\alpha$ term), and can be figured in together with the receiving energy.

$\Theta(n^{(1-\alpha)/2})$. Therefore, the total network capacity is upper bounded by

$$\frac{\Theta(n \cdot (\frac{\log n}{n})^{\alpha/2})}{\Theta(n^{(1-\alpha)/2})} = \Theta(\sqrt{n(\log n)^{\alpha}}). \tag{19}$$

At the first glance, this bound is higher than the upper bound $\Theta(\sqrt{n \log n})$ given in [1] since usually $2 \leq \alpha \leq 4$ in practice. However, throughout the derivation in Sections III–IV we only assume $\alpha \geq 1$. Taking $\alpha = 1$, we obtain the same upper bound as that in [1].

*2) Network capacity in the case of UWB:* Negi and Rajeswaran [17] derived the capacity bounds of power constrained ad-hoc networks, and showed that under the assumption that arbitrarily large bandwidth can be used, the per node capacity in a wireless network in a unit square is upper bounded by $O((n \log n)^{(\alpha-1)/2}$ and lower bounded by $\Omega(\frac{n^{(\alpha-1)/2}}{(\log n)^{(\alpha+1)/2}})$. We now derive tighter bounds using the the bounds of the transporting energy we have obtained in Section III–IV. First, as shown in [17], the total power used on a minimum power route $R$ is

$$P(R) = rN_0Q, \tag{20}$$

where $r$ is the transmission rate (throughput) on the route, $N_0/2$ is the power spectrum density of noise, and $Q$ is defined in Eq. (1). Since the number of minimum power route is the same as the number of source nodes (one source has one route), and each node can use at most some constant power, in average, each minimum power route can be assigned with at most some constant power, i.e., $P(R)$ in Eq. (20) is a constant. Under the assumption that there are $n/2$ randomly chosen source-destination pairs, one can prove that at least a fraction of these source-destination pairs have distance at least $\epsilon$ for $0 < \epsilon < 1$. We have shown that $Q$ (i.e., the transporting energy in this paper) is lower bounded by $\Theta(n^{(1-\alpha)/2})$ for those source-destination pairs with distance at least $\epsilon$. Therefore, by Eq. (20) the throughput ($r$) on each route is upper bounded by $O(n^{(\alpha-1)/2})$. Note that this upper bound is sharper than that presented in [17]!

A lower bound on the network capacity in the case of UWB communications can be obtained in a similar way to the analysis that we have made to derive the upper bound on the minimum energy required to transport a packet between a source-destination pair. Specifically, a packet is still delivered in three phases. However, in the first phase, we divide the region into $\beta\sqrt{n}/c_5$ (which is equal to the number of disjoint horizontal paths) horizontal slabs, and let a source node in the $i$th slab directly send its message to the node (called entry point) with the shortest horizontal distance in the $i$th horizontal path. And similarly divide the region into $\beta\sqrt{n}/c_5$ vertical slabs. In the second phase, once a packet gets on a horizontal path, it is carried along the path until it reaches the target vertical path, which is determined by the vertical slab that contains the destination node (i.e., if a destination is in the $j$th vertical slab, its target vertical path is the $j$th vertical path). In the last phase, the packet is delivered to the destination directly from the exit point (which is determined in a way similar to the entry point) on the vertical path.

With the above routing strategy, we now show that any source-destination pair can achieve a throughput of order $n^{(\alpha-1)/2}$. As showed in [17], the link capacity in the case of arbitrarily large bandwidth is (by Shannon's information theory)

$$c_l = B \log(1 + \frac{W_0 d^{-\alpha}}{N_0 B}) \rightarrow \frac{W_0}{N_0 d^{\alpha}} \tag{21}$$

as $B \rightarrow \infty$, where $W_0$ is the transmission power, $d$ is the distance between the transmitter and the receiver, $N_0/2$ is the power spectrum density of noise, and $B$ is the bandwidth. Therefore, the link capacity (under UWB) is proportional to $d^{-\alpha}$. In the first phase, it is not hard to see that the distance of the first hop (i.e., from the source to the entry point in the horizontal path) is at most $\Theta(\frac{\log\sqrt{n}}{\sqrt{n}})$ (since the source and the entry point are in the same rectangle $\bar{R}_n$, their vertical distance is at most $\Theta(\frac{\log\sqrt{n}}{\sqrt{n}})$, and their horizontal distance is at most $\Theta(1/\sqrt{n})$ by the choice the entry point). Therefore, the capacity in the first hop is at least

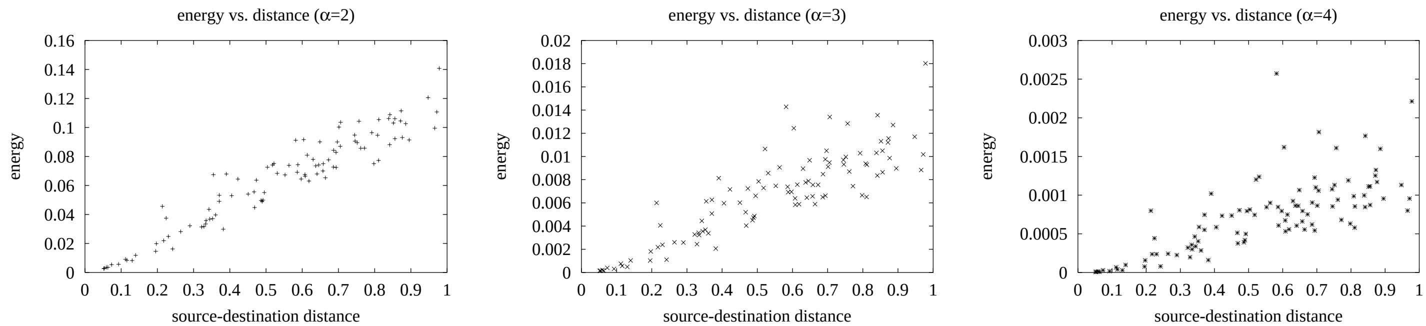
$$\Theta(\frac{\sqrt{n}}{\log\sqrt{n}})^{\alpha} \geq \Theta(n^{(\alpha-1)/2}) \tag{22}$$

In the second phase, the distance of each hop is $\Theta(1/\sqrt{n})$ and so each horizontal (and vertical) path can support a rate of $\Theta(n^{\alpha/2})$. Since at most $\Theta(\sqrt{n})$ source-destination pairs use one horizontal (and vertical) path (proved in [7]), each source-destination can obtain a transmission rate of $\Theta(n^{(\alpha-1)/2})$. The analysis of the third phase is identical to that of the first phase. Therefore, we conclude that the above routing strategy can support a per-node throughput of $\Theta(n^{(\alpha-1)/2})$.

*3) Upper bound on the lifetime of wireless sensor networks:* If a power management scheme can be properly deployed in a wireless sensor network to determine when sensor nodes should go to sleep (in the lack of communications/sensing activities) and when they should wake up (to perform their sensing/communications tasks), we can assume that energy is only consumed when a sensor transmits/receives data packets. (Even in the case that a power management scheme is not used and all the sensor nodes are kept awake consuming energy in their idle states, the derived upper bound on the network lifetime still serves as an upper bound.) In power-managed sensor networks, the results derived in Sections III–IV can be used to obtain an upper bound on the network lifetime. For example, if we assume each sensor node has a constant initial energy and each node transmits to a random destination with a constant rate, then the network lifetime is upper bounded by $\Theta(\frac{1}{Q}) = \Theta(n^{(\alpha-1)/2})$.

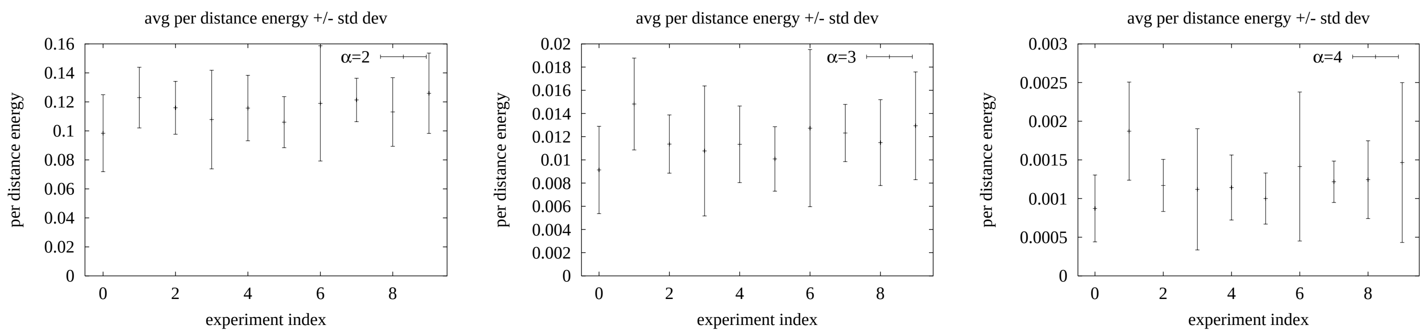
## VI. Simulation Results

We have conducted a simulation study to validate the derived results and to estimate the associated constant in the energy equation $\Theta(n^{(1-\alpha)/2})$ in Corollary 1. The reason for validating the derived results is because the network behavior/property is analyzed in the asymptotic sense (e.g., as the network size grows to infinity). It is not clear whether or not the results hold in a finite region (or, alternatively, to what extent after the network size grows the network exhibits the asymptotic properties).




(a) $\alpha$ = 2 (b) $\alpha$ = 3 (c) $\alpha$ = 4

Fig. 6. The relationship between the minimum energy incurred on a multiple-hop path and the source-destination distance.




(a) $\alpha$ = 2 (b) $\alpha$ = 3 (c) $\alpha$ = 4

Fig. 7. The average, energy incurred per unit of distance on a multi-hop, minimum-energy path and its standard deviation. Note that the size of the error bar is twice the standard deviation.

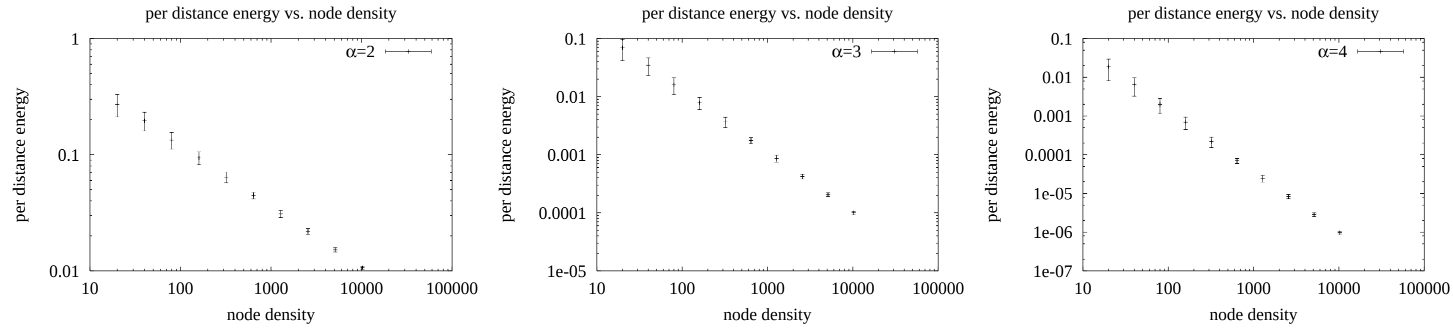
*Energy vs distance:* A total of 10 simulation runs are carried out in a unit-area square with a node density 100 nodes per unit-area. The node positions are uniformly and randomly distributed. In each simulation run we randomly choose 10 source-destination pairs and calculate the minimum energy required to transport packets between each source-destination pair using Dijkstra's algorithm. Figure 6 shows the minimum energy versus the source-destination distance under different values of path loss exponent $\alpha$. As shown in Figure 6, there exists a clear linear relationship between the minimum energy and the source-destination distance. Such a relationship has been predicted in Eq. (17).

To remove the effect of the source-destination distance, we consider the *energy consumed per unit of distance*, defined as the minimum energy divided by the source-destination distance. For each of the 10 experiments, we evaluate the average energy consumed per unit of distance among the 10 source-destination pairs as well as its standard deviation. As shown in Figure 7, for a fixed node density, the energy consumed per unit of distance is some constant value.

*Energy consumed per unit of distance vs node density:* Now we study how the energy consumed per unit of distance changes as the node density increases. The second set of simulation runs are similar to the first set, except that the node density is varied from 10 to $10^5$. For each value of node density, we carry out 10 simulation runs, in each of which 10 source-destination pairs are randomly selected. For each value of node density, we calculate the average energy consumed per unit of distance, and its standard deviation, over all source-destination pairs that are apart from each other by a distance of at least 0.5.[4] Figure 8 gives the energy consumed per unit of distance versus the node density. The linear relation in the double log scale graph suggests a power relation between the energy and the node density. Again this has been predicted in Eq. (17) as well as Corollary 1.

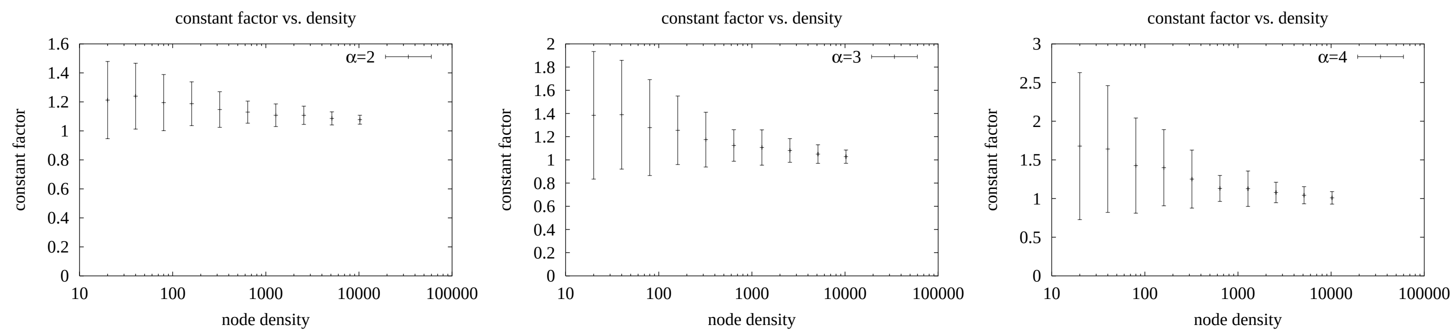
Finally we would like to quantify the constant associated with Eq. (17) and study whether or not the constant converges. Figure 9 gives $\frac{E}{n^{(1-\alpha)/2}l}$ vs. the node density, where $E$ is the minimum energy required to transport packets, $n$ the node density, and $l$ the source-destination distance. The value, $c_n \triangleq \frac{E}{n^{(1-\alpha)/2}l}$, shown in the $y$-axis is the *constant factor* in Eq. (17). As shown in Figure 9, the constant factor converges to 1 with high probability as the node density goes to infinity (because the standard deviation becomes smaller and smaller

[4]This complies with the assumption in our theoretical analysis that the distance between a source and a destination is non-diminishing as the node density increases. This assumption is necessary because if the source-destination distance is extremely small such that they are one-hop away on the the minimum energy route, the minimum energy required to transport packets between them is $l^{\alpha}$, and is independent of $n$.




(a) $\alpha$ = 2 (b) $\alpha$ = 3 (c) $\alpha$ = 4

Fig. 8. The relationship between the energy consumed per unit of distance on a multi-hop, minimum energy path and the node density.




(a) $\alpha$ = 2 (b) $\alpha$ = 3 (c) $\alpha$ = 4

Fig. 9. The relationship between the constant factor and the node density.

as the density increases). Based on the observation, we make the following conjecture.

**Conjecture 1** *Assume that nodes are distributed in a unit square area according to a Poisson point process with density $n$. Given a fixed source-destination pair and their distance $l \geq \epsilon > 0$, the minimum energy required to transport a packet from the source to the destination is $n^{(1-\alpha)/2}l$* w.h.p., *as the density $n \rightarrow \infty$.*

## VII. Conclusion

In this paper, we have derived both the lower and upper bounds of the asymptotic minimum energy required to transport packets from a source to a destination in a random wireless network. Under the assumption that nodes are deployed as a Poisson point process with the node density $n$ in a unit square area and the source-destination distance is non-diminishing, we prove, based on the percolation theory, that the minimum energy required to transport a data packet from a source to a destination is $\Theta(n^{(1-\alpha)})$ *w.h.p.*, where $\alpha$ is the path loss exponent. We have discussed how to extend the results to accommodate the cases (i) that the network density is kept as constant but the network size goes to infinity, and (ii) that both the transmitting and receiving operations consume power.

We have demonstrated how to apply the derived results to derive the network capacity of wireless networks equipped with directional antennas, the network capacity of wireless networks that operate in UWB, and finally the upper bound on the lifetime of wireless sensor networks. We believe the results and the proof techniques can be applied to derive the asymptotic conditions on other parameters in wireless networks, as long as the limiting factor for the parameters of interest is the energy.

We have also carried out simulation to validate the derived results and to estimate the constant factor associated with the bounds on the minimum energy. Based on the simulation results, we conjecture that the minimum energy required to transport packets between a source-destination pair that is separated by the distance $l$ converges to $n^{(1-\alpha)/2}l$ *w.h.p.*. This is subject to further theoretical investigation.

## References

[1] C. Peraki and. S. Servetto. On the maximum stable throughput problem in random wireless networks with directional antennas. In *ACM Mobihoc*, 2003.

[2] M. Bhardwaj and A. P. Chandrakasan. Bounding the lifetime of sensor network via optimal role assignments. In *Proc. of IEEE Infocom*, 2002.

[3] D.M. Blough, M. Leoncini, G. Resta, and P. Santi. On the symmetric range assignment problem in wireless ad hoc networks. In *Proc. of the 2nd IFIP International Conference on Theoretical Computer Science*, pages 71–82, Montreal, Aug. 2002.

[4] G. Calinescu and P-J. Wan. Range assignment for high connectivity in wireless ad hoc networks. In *Adhoc-Now*, 2003.

[5] G. Călinescu, I. L. Măndoiu, and A. Zelikovsky. Symmetric connectivity with minimum power consumption in radio networks. In *Proc. of 17th IFIP World Computer Congress*, pages 119–130, 2002.

[6] A.E.F. Clementi, P. Penna, and R. Silvestri. Hardness results for the power range assignment problem in packet radio networks. In *Proceedings of the 2nd International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)*, pages 197–208, 1999.

[7] M. Franceschetti, O. Dousse, D. Tse, and P. Thiran. Closing the gap in the capacity of random wireless networks. *Preprint under submission*, 2004.

[8] M. Franceschetti, O. Dousse, D. Tse, and P. Thiran. Closing the gap in the capacity of random wireless networks. In *Proc. of IEEE International Symposium on Information Theory (ISIT'04)*, June 2004.

[9] J. Gomez and A. Campbell. A case for variable-range transmission power control in wireless multihop networks. In *Proc. of IEEE Infocom 2004*.

[10] Geoffrey Grimmett. *Percolation*. Springer, 1998.

[11] P. Gupta and P.R. Kumar. Critical power for asymptotic connectivity in wireless networks. *Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W.H. Fleming, 1998*.

[12] M. Hajiaghayi, N. Immorlica, and V. S. MIrrokni. Power optimization in fault-tolerant topology control algorithms for wireless multi-hop networks. In *ACM Mobicom 2003*, September 2003.

[13] L. M. Kirousis, E. Kranakis, D. Krizanc, and A. Pelc. Power consumption in packet radio networks. *Theoretical Computer Science*, 243(1-2):289–305, 2000.

[14] L. Li, J. Halpern, V. Bahl, Y.M. Wang, and R. Wattenhofer. Analysis of a cone-based distributed topology control algorithm for wireless multi-hop networks. In *Proceedings of ACM Symposium on Principle of Distributed Computing (PODC)*, pages 264–273, 2001.

[15] N. Li and J. Hou. Topology control in heterogeneous wireless networks: problems and solutions. In *Proc. of IEEE Infocom 2004*, March 2004.

[16] N. Li, J. Hou, and L. Sha. Design and analysis of a mst-based topology control algorithm. In *Proc. of IEEE Infocom 2003*, April 2003.

[17] R. Negi and A. Rajeswaran. Capacity of power constrained ad-hoc networks. In *Proc. of IEEE Infocom 2004*, March 2004.

[18] M.D. Penrose. The longest edge of the random minimal spanning tree. *Annals of Applied Probability*, 7:340–361, 1997.

[19] M.D. Penrose. On k-connectivity for a geometric random graph. *Random Structures & Algorithms*, 15:145–164, 1999.

[20] B. Rengarajan, J. Chen, S. Shakkottai, and T. S. Rappaport. Connectivity of sensor networks with power control. In *Proc. of 37th Asilomar Conference on Signals, Systems and Computers*, November 2003.

[21] V. Rodoplu and T. H. Meng. Minimum energy mobile wireless networks. *IEEE Journal of Selected Areas in Communications*, 17(8):1633–1639, 1999.

[22] P. Santi and D. M. Blough. The critical transmitting range for connectivity in sparse wireless ad hoc networks. *IEEE transactions on Mobile Computing*, 2(1):25–39, 2003.

[23] P-J. Wan and C. Yi. Asymptotic critical transmission radius and critical neighbor number for k-connectivity in wireless ad hoc networks. In *ACM Mobihoc 2004*.


## APPENDIX I
## PROOF OF LEMMA 2

We prove a generalized version of Lemma 2 in the context of the site percolation model. Let $\Omega = \Pi_{s\in\mathbb{Z}^d}\{0,1\}$ be the sample space in the underline probability space, where $\mathbb{Z} = \{\cdots,-1,0,1,\cdots\}$. Points in $\Omega$ are represented as $\omega = (\omega(s) : s \in \mathbb{Z}^d)$ and called *configurations*. The value $\omega(s) = 0$ corresponds to the site (grid) $s$ being closed and $\omega(s) = 1$ corresponds to the site $s$ being open. An event $A$ is called *increasing* if $I_A(\omega) \le I_A(\omega')$ whenever $\omega \le \omega'$, where $I_A$ is the indicator function of the event $A$. (Interested readers should refer to [10] for more details of the definitions.) Let $A$ be an increasing event. For $\omega \in \Omega$, let $F_A(\omega)$ denote the "distance" of $\omega$ from $A$, i.e.,

$$F_A(\omega) = \inf\left\{\sum_s(\omega'(s)-\omega(s)) : \omega' \ge \omega, \omega' \in A\right\}. \quad (23)$$

Note that $F_A(\omega) = 0$ if $\omega \in A$. The generalized version of Lemma 2 is

$$P_{p_2}(A) \ge \left(\frac{p_2-p_1}{1-p_1}\right)^r P_{p_1}(F_A \le r) \quad (24)$$

for any $0 < p_1 < p_2 < 1$. With Eq. (24), Lemma 2 is obvious since the event that there is an open path of length $m$ starting from a given source is an increasing event.

*Proof.* Suppose that $X(s) : s \in \mathbb{Z}^d$ is a family of independent random variables indexed by the grid (site) set $\mathbb{Z}^d$, where each $X(s)$ is uniformly distributed on [0,1]. We may couple together all the site percolation processes on $\mathbb{Z}^d$ in the following way. Let $0 \le p \le 1$ and define $\eta_p \in \Omega$ by

$$\eta_p(s) = \begin{cases} 1 & \text{if } X(s) \le p, \\ 0 & \text{otherwise.} \end{cases} \quad (25)$$

We may think of $\eta_p$ as the random outcome of the site percolation process on $\mathbb{Z}^d$ with the site-open probability $p$. It is clear that $\eta_{p_1} \le \eta_{p_2}$ whenever $p_1 < p_2$. Thus we may couple two percolation processes with site-open probability $p_1$ and $p_2$ in such a way that the set of open sites of the first process is a subset of the set of the open sites of the second.

Suppose that $0 \le p_1 \le p_2 \le 1$ and $A$ is an increasing event. Denote $I_r(A) = \{\omega : F_A(\omega) \le r\}$. If $\eta_{p_1} \in I_r(A)$, there exists a (random) collection $C = C(\eta_{p_1})$ of sites such that

(a) $|C| \le r$;
(b) $\eta_{p_1}(s) = 0$ for all $s \in C$; and
(c) the configuration $\eta$ obtained from $\eta_{p_1}$ by declaring all sites in $C$ to be open, satisfies $\eta \in A$.

Suppose now that every $s$ in the set $C$ satisfies $p_1 \le X(s) \le p_2$. It follows from (c) above that $\eta_{p_2} \in A$. Conditioning on (b) above, the probability of $p_1 \le X(s) \le p_2$ is $((p_2-p_1)/(1-p_1))^{|C|}$. Therefore,

$$P(\eta_{p_2} \in A | \eta_{p_1} \in I_r(A)) \ge \left(\frac{p_2-p_1}{1-p_1}\right)^r, \quad (26)$$

since $|C| \le r$. Eq. (24) follows easily.