GENERALIZATIONS OF QUASICONVEXITY FOR FINITELY GENERATED GROUPS

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DISSERTATION

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Abstract

For a word-hyperbolic group G, the notion of quasiconvexity of a finitely generated subgroup H of G is independent of the choices of finite generating sets for G and H, and is equivalent to H being quasi-isometrically embedded in G. However, beyond word-hyperbolic groups, the notion of quasiconvexity is not as useful. For a finitely generated group, there are two recent generalizations of the notion of a quasiconvex subgroup of a word-hyperbolic group, a "stable" subgroup and a "Morse" subgroup. Durham and Taylor [33] defined stability and proved stability is equivalent to convex cocompactness in mapping class groups. Another natural generalization of quasiconvexity is given by the notion of a Morse or strongly quasiconvex subgroup of a finitely generated group, studied by Tran [92] and Genevois [37].

For an arbitrary finitely generated group, an infinite subgroup is stable if and only if the subgroup is Morse and hyperbolic. We prove that two properties of being Morse of infinite index and stable coincide for a subgroup of infinite index in the mapping class group of an oriented, connected, finite type surface with negative Euler characteristic [67].

Finding algorithms for detection and decidability of various properties of groups is a fundamental theme in geometric group theory. For a word-hyperbolic group G, Kapovich [55] provided a partial algorithm which, on input a finite set S of G, halts if S generates a quasiconvex subgroup of G and runs forever otherwise. In this thesis, we give various detection and decidability algorithms for stability and Morseness of mapping class groups, right-angled Artin groups, toral relatively hyperbolic groups which contains finitely generated groups discriminated by a locally quasiconvex torsion-free hyperbolic group (for example, ordinary limit groups) [68]. Also, we provide a partial algorithm which, for a finite subset S of a toral relatively hyperbolic group, terminates if S generates a relatively quasiconvex subgroup of G, equivalently, the subgroup generated by S is undistorted in G.

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Chapter 1

Introduction

Geometric group theory studies infinite, finitely generated groups by looking at how they act on various metric spaces, and relating geometric properties of these spaces to algebraic properties of the groups themselves. Dehn in 1912 introduced several fundamental algorithmic problems, including the word, the conjugacy, and the subgroup membership problems, for finitely presented groups. He solved these problems for the fundamental groups of compact surfaces by using the hyperbolic plane where these groups act nicely. Magnus, Stallings, Serre, Rips, and Gromov generalized the work of Dehn and introduced various geometric and combinatorial techniques to geometric group theory, 3-manifold theory, and logic. In the last 30 years, there has been considerable progress in the study of finitely generated groups acting isometrically on hyperbolic spaces, especially *word-hyperbolic groups*, based on the properties of hyperbolic geometry.

We recall that a metric space (X, d) is said to be geodesic if any two points $x, y \in X$ can be joined be a geodesic segment $[x, y]$ that is a naturally parameterized path from x to y whose length is equal to $d(x, y)$.

Definition 1.1 (Hyperbolic space)**.** Let (X, d) be a geodesic metric space. We say that X is δ-*hyperbolic* if there exists $\delta \geq 0$ such that for any geodesic triangle T in X each side of T is contained in the δ-neighborhood of the union of two other sides.

Given a finitely generated group G with a finite generating set S, we have a natural geodesic metric space, namely the *Cayley graph* of G with respect to S, whose vertices are group elements and there exists an edge of length 1 between two vertices $g, h \in G$ if $gh^{-1} \in S \cup S^{-1}$. Note that the length of the shortest path in the Cayley graph between two elements g, h of G corresponds to $d_S(g, h)$ where d_S is the word metric.

Definition 1.2 (Hyperbolic group)**.** Let G be a finitely generated group with a finite generating set S. We say that G is δ-*hyperbolic* if the Cayley graph of G with respect to S is δ -hyperbolic for some $\delta \geq 0$. If G is δ-hyperbolic for some $\delta \geq 0$, we call that G is *word-hyperbolic*.

Note that hyperbolicity is independent of a finite generating set of G. A word-hyperbolic group has laid the foundations of geometric group theory and there have been lots of studies on generalizations of the theory of word-hyperbolic groups. In particular, the notion of a quasiconvex subgroup plays an important role in the theory of word-hyperbolic groups and in its various generalizations.

Definition 1.3 (Quasiconvex subgroup)**.** Let G be a finitely generated group with a finite generating set S. A subgroup $H \leq G$ is *quasiconvex* in G with respect to S if there exists $N \geq 0$ such that every geodesic in the Cayley graph $\Gamma(G, S)$ of G with respect to S that connects a pair of points in H is contained in the N-neighborhood of H.

For a word-hyperbolic group G, quasiconvexity is independent on a finite generating set of G, and for a subgroup being quasiconvex has several equivalent characterizations: being finitely generated and undistorted; being Gromov boundary quasiconvex-cocompact; being rational; and so on.

Outside word-hyperbolic groups, quasiconvexity is not as useful since the notion depends on the choice of a generating set of an arbitrary finitely generated group. There have been two recent generalizations of the notion of quasiconvexity to the context of subgroups of ambient finitely generated groups. Durham and Taylor [33] introduced a strong notion of quasiconvexity for a subgroup of a finitely generated group, namely a *stable* subgroup.

Definition 1.4 (Stable subgroup [33])**.** Let G be a finitely generated group and let H be a finitely generated subgroup of G. We say that H is *stable* in G if H is undistorted in G and if for every (equivalently, some) finite generating set S of G and for every $k \geq 1$ and $c \geq 0$ there is some $L = L(S, k, c)$ such that for every pair of $(k, c)-$quasigeodesics in G with the same endpoints on H, each of these two qausigeodesics is contained in the L-neighborhood of the other.

Tran [92] and Genevois [37] defined another generalization of the notion of quasiconvexity as follows.

Definition 1.5 (Morse subgroup [92, 37])**.** Let G be a finitely generated group and let H be a subgroup of G. We say that H is a *Morse* subgroup of G if for every (equivalently, some) finite generating set S of G, for every $k \geq 1$ and $c \geq 0$ there is some $M = M(k, c)$ such that every $(k, c)-$quasigeodesic in G with endpoints on H is contained in the M-neighborhood of H.

Note that Tran used "strongly quasiconvex subgroups" instead of "Morse groups" in [92]. For a word-hyperbolic group, both stability and Morseness are equivalent to quasiconvexity since every quasigeodesic stays in uniformly bounded distance to a geodesic by the Morse Lemma. However, in general, for a subgroup of a finitely generated group, being stable is not equivalent to being Morse. For example, $\mathbb{Z} \times \mathbb{Z}$ is a non-stable Morse subgroup of itself. Tran [92] showed that an infinite subgroup H of a finitely generated group G is stable if and only if H is hyperbolic and Morse in G.

1.1 Stable and Morse subgroups of mapping class groups

Tran [92] and Genevois [37] showed stability and Morseness are equivalent for subgroups of infinite index in the right-angled Artin group A_Γ of a finite simplicial, connected graph Γ which does not decompose as a nontrivial join. Their result served as a motivation for our first main theorem which states that for a subgroup of a mapping class group being stable is equivalent to being Morse of infinite index.

Theorem A. *Let S be an oriented, connected, finite type surface with $\chi(S) < 0$ which is neither the 1-punctured torus nor the 4-punctured sphere. Let $\mathrm{Mod}(S)$ be the mapping class group of S, and let G be a finitely generated subgroup of $\mathrm{Mod}(S)$. Then the following are equivalent:*

(1) G is convex cocompact.

(2) G is Morse of infinite index in $\mathrm{Mod}(S)$.

For the implication "(1) ⇒ (2)", we use the characterization of convex cocompactness in terms of stability proved in [33]. For the reverse implication "(2) ⇒ (1)", we use the characterization of convex cocompactness in terms of pseudo-Anosov elements proved in [8]. Explicitly, we have the following corollary by combining their results and our main theorem. Furthermore, the next corollary says that stability and Morseness are equivalent notions in $\mathrm{Mod}(S)$.

Corollary B. *Let S be an oriented, connected, finite type surface with $\chi(S) < 0$ which is neither the 1-punctured torus nor the 4-punctured sphere. Let $\mathrm{Mod}(S)$ be the mapping class group of S, and let G be a finitely generated subgroup of $\mathrm{Mod}(S)$. Then the following are equivalent:*

(1) G is convex cocompact.

(2) An orbit map of G into the curve complex $\mathcal{C}(S)$ is a quasi-isometric embedding.

(3) G is finitely generated, undistorted, and purely pseudo-Anosov.

(4) G is stable.

(5) G is Morse of infinite index in $\mathrm{Mod}(S)$.

The equivalence between (1) and (2) was shown in Kent and Leininger [61] and Hamenstädt in [50] independently. The equivalence between (1) and (3) was proved in [8] including the 1-punctured torus $S_{1,1}$ and the 4-punctured sphere $S_{0,4}$. On the other hand, the equivalence between (1) and (4) shown in [33] excludes those two surfaces.

Remark 1.6. The mapping class groups $\mathrm{Mod}(S_{1,1})$ and $\mathrm{Mod}(S_{0,4})$ are commensurable with $GL_2(\mathbb{Z})$ and thus are virtually free (see Chapter 2 in [36]). Both these groups are locally quasiconvex and all of their finitely generated subgroups are stable and Morse. However, $\mathrm{Mod}(S_{1,1})$ and $\mathrm{Mod}(S_{0,4})$ contain Dehn twists and they have finitely generated subgroups that are not convex cocompact. Therefore, the conclusion of Theorem A does not hold for these groups.

The main result of [33] and Corollary 1.2 in [8] are stated slightly incorrectly since they should have omitted $\mathrm{Mod}(S_{1,1})$ and $\mathrm{Mod}(S_{0,4})$ for similar reasons.

Nevertheless, the equivalence between (3) and (4) includes the two surfaces $S_{1,1}$ and $S_{0,4}$ since stability is equivalent to being Morse in a hyperbolic group.

1.2 Algorithms detecting stability and Morseness for finitely generated groups

One of the central Themes in geometric group theory is to find algorithms for detection and decidability of varied properties of groups. For example, for a word-hyperbolic group G and a finite subset S in G, it is interesting to ask whether or not there is an algorithm that, for a finite subset S of G, detects whether the subgroup $H = \langle S \rangle$ generated by S is quasiconvex in G. Kapovich [55] provided a partial algorithm detecting quasiconvexity of a finitely generated subgroup of a word-hyperbolic group. See also [63].

Proposition 1.7 (Proposition 4 in [55])**.** *Let G be a word-hyperbolic group given by a finite presentation $G = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle$ and let $S = \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$. Then there is a uniform algorithm which, given a finite set of words $v_1, .., v_t$ over S, will*

(i) eventually stop and produce the quasiconvexity constant N and the distortion constant C of the subgroup $H = gp(\bar{v}_1, \ldots, \bar{v}_t)$ of G if H is quasiconvex in G, where $\bar{v}_i$ denotes the element of G represented by the word v_i;

(ii) run forever if H is not quasiconvex in G.

Since the above partial algorithm does not detect non-quasiconvex subgroups, one might ask the following question.

Question 1.8. *Given a word-hyperbolic group G, is there an algorithm that, for a finite subset S of G, decides whether or not $H = \langle S \rangle$ is quasiconvex in G?*

In general, the answer is no, see [19]. Motivated by Proposition 2.29 and Question 1.8, we are interested in the following questions for stable and Morse subgroups of a finitely generated group.

Question 1.9. *Let G be a finitely generated group from a particular class of groups. Is there a partial algorithm which, on input a finite subset S of G, halts at least on those inputs such that $H = \langle S \rangle$ is stable and decides whether H is indeed stable or not, but may run forever on inputs such that H is not stable? Is there a complete algorithm that decides whether or not a finitely generated subgroup H of G is stable?*

Question 1.10. *What if we replace stable by Morse in Question 1.9?*

The second main theorem answers to the above questions for mapping class groups as follows.

Theorem C. *Let S be an oriented, connected, finite type surface with $\chi(S) < 0$ which is neither the 1-punctured torus nor the 4-punctured sphere and let $\mathrm{Mod}(S)$ be the mapping class group of S.*

(1) There is a partial algorithm which, for a subgroup H of $\mathrm{Mod}(S)$ given by a finite generating set, will terminate if H is stable in $\mathrm{Mod}(S)$ and run forever if H is not stable in $\mathrm{Mod}(S)$.

(2) There is a complete algorithm which, for an undistorted subgroup H of $\mathrm{Mod}(S)$, decides whether or not H is stable.

(3) There is a partial algorithm which, for a finitely generated subgroup H of $\mathrm{Mod}(S)$ given by a finite generating set, will terminate if H is Morse in $\mathrm{Mod}(S)$ and run forever if H is not Morse in $\mathrm{Mod}(S)$.

A finitely generated subgroup H of $\mathrm{Mod}(S)$ is stable if and only if its orbit into the curve graph $\mathcal{C}(S)$ is a quasi-isometrically embedding. It is known that the distances in $\mathcal{C}(S)$ can be computed algorithmically [14]. For part (1) of Theorem C, we use the "local-to-global" principle to detect subgroups H of $\mathrm{Mod}(S)$ with quasi-isometrically embedded orbits in $\mathcal{C}(S)$. For an undistorted subgroup H of $\mathrm{Mod}(S)$, it is known that H is stable in $\mathrm{Mod}(S)$ if and only if H is purely loxodromic [33, 8]. Therefore, for the proof of (2), we run a partial algorithm in (1) and in parallel, look for a non-loxodromic element in H. It is known that a finitely generated subgroup H of $\mathrm{Mod}(S)$ is Morse if and only if H is stable or has finite index in $\mathrm{Mod}(S)$ [67]. For part (3), we run a partial algorithm in (1) and in parallel, run the Todd-Coxeter algorithm to detect the finite index of a finitely generated subgroup.

Note that since mapping class groups are known to biautomatic by Mosher in [78], every finitely generated Morse subgroup of $\mathrm{Mod}(S)$ is therefore [39] rational with respect to any automatic structure on $\mathrm{Mod}(S)$. That implies, for example, that for a given finitely generated Morse subgroup G of $\mathrm{Mod}(S)$, the membership problem for G in $\mathrm{Mod}(S)$ is solvable in quadratic time.

The theorem answers to Question 1.9 and Question 1.10 for right-angled Artin groups.

Theorem D. *Let Γ be a finite connected and anti-connected graph with at least two vertices and let A_Γ be the corresponding right-angled Artin group.*

(1) There is a complete algorithm which, for a subgroup H of A_Γ given by a finite generating set, will terminate and determine whether or not H is stable in A_Γ.

(2) There is a partial algorithm which, for a subgroup H of A_Γ given by a finite generating set, will terminate if H is Morse in A_Γ and run forever if H is not Morse in A_Γ.

We provide two different algorithms for Theorem D(1). The first algorithm uses the *extension graph* Γ^e [69, 70] and "star metric" on A_Γ which is quasi-isometric to Γ^e and comparable with the standard normal form. A finitely generated subgroup H of A_Γ is stable if and only if its orbit into Γ^e with the graph metric is quasi-isometrically embedding if and only if H is purely loxodromic. Thus, the first algorithm for Theorem D(1) proceeds by running a partial algorithm detecting quasiconvexity of an orbit H in Γ^e and in parallel, iterating elements to check that there is a non-loxodromic element to detect non-stability of H. For the second algorithm for Theorem D(1), we look for a cube complex in [74] encoding all infinite order elements in H and check whether or not a closed loop is labeled by a join word, i.e., non-loxodromic element. Note that under the assumptions of Theorem D, a subgroup H is Morse if and only if either H is stable or H has finite index in G [92, 37]. For part (2), we run a partial algorithm in (1) and in parallel, run the Todd-Coxeter algorithm to detect the finite index of a finitely generated subgroup.

We now consider toral relatively hyperbolic groups for Question 1.9 and Question 1.10. A finitely generated group G is called a *toral relatively hyperbolic group* if G is torsion-free and hyperbolic relative to a (possibly empty) finite collection $\mathbb{P}$ of finitely generated free abelian non-cyclic subgroups of G.

Theorem E. *Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group.*

(1) There is a partial algorithm which, for a subgroup H of G given by a finite generating set, will terminate if H is stable in G and run forever if H is not stable in G.

(2) There is a complete algorithm which, for an undistorted subgroup H of G, decides whether or not H is stable.

(3) There is a partial algorithm which, for a subgroup H of G given by a finite generating set, will terminate if H is Morse in G and run forever if H is not Morse in G.

(4) There is a complete algorithm which, for an undistorted finitely generated subgroup H of G, decides whether or not H is Morse.

Tran [92] gave complete characterizations of stability and Morseness of an undistorted subgroup H of a relatively hyperbolic group $(G, \mathbb{P})$. These characterizations involve checking the properties of all infinite intersections $H \cap P^g$, where $g \in G$ and $P \in \mathbb{P}$. For part (1) of Theorem E, we combine these results of Tran with recent algorithmic results of Kharlampovich, Myasnikov and Weil [63] about toral relatively hyperbolic groups. When an undistorted subgroup is given as (2), we run the algorithm in (1) and in parallel, detect non-stability by Tran's characterization of stability. The approach to (3) is similar. For part (4), we run the algorithm in part (3) to detect Morseness and in parallel, run a partial algorithm detecting non-Morseness by using relatively hyperbolic Dehn fillings [80, 47, 48]. Specifically, we use the results of Groves and Manning [48] on the behavior of relatively quasiconvex subgroups under Dehn fillings. Producing an algorithm for detecting non-Morseness of undistorted finitely generated subgroups in G is the most involved portion of the proof of Theorem E(4) since it requires iteratively applying the above procedures to groups obtained from G by hyperbolic Dehn fillings. Note that for a finitely generated subgroup H of G the algorithm from [63] terminates if H is relatively quasiconvex and satisfies the "peripherally finite index" condition. Hence, the difficulty of proving Theorem E(4) lies in recognizing relatively quasiconvex subgroups that do not have peripherally finite index.

Note that a new result of Kharlampovich and Weil (Theorem 2 in [66]) implies that if $(G, \mathbb{P})$ is a toral relatively hyperbolic group then there is a partial algorithm which, for given $g, h_1, \ldots, h_n \in G$, decides whether or not $g \in H = \langle h_1, \ldots, h_n \rangle$, assuming that H is relatively quasiconvex. This result is related to, but does not imply, our Theorem E(4). The proof of Theorem 2 in [66] utilizes an element-wise separability result of Manning and Martinez-Perdoza [75]. That theorem implies that if H is relatively quasiconvex in a toral relatively hyperbolic group G and if $g \in G \setminus H$, then there exists a Morse subgroup $H_1 \leq G$ such that $H \leq H_1$ and $g \notin H_1$.

Theorem E(4) also has the following useful corollary:

Corollary F. Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group. Then there exists an algorithm that, given an undistorted finitely generated subgroup H of G, decides whether or not H has finite index in G.

Let G be a finitely generated group discriminated by a locally quasiconvex torsion-free hyperbolic group. Then G is a toral relatively hyperbolic group and every finitely generated subgroup H of G is undistorted (see Lemma 4.6 below). Thus, Theorem E(2) and Theorem E(4) imply the following corollary.

Corollary G. *Let G be a finitely generated group discriminated by a locally quasiconvex torsion-free hyperbolic group.*

(1) There is a complete algorithm which, for a subgroup H of G given by a finite generating set, decides

whether or not H is stable.

(2) There is a complete algorithm which, for a subgroup H of G given by a finite generating set, decides whether or not H is Morse.

Recall that a limit group is a finitely generated group G that is discriminated by the free group F_2. Since F_2 is locally quasiconvex torsion-free hyperbolic, there exist such algorithms for G as in Corollary G.

Furthermore, we provide a partial algorithm which, for a finite set S of a toral relatively hyperbolic group, detects if S generated a relatively quasiconvex subgroup.

Theorem H. Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group. Then there exists a partial algorithm which, for a finite set S of G, terminates the subgroup $H := \langle S \rangle$ is relatively quasiconvex but runs forever if H is not relatively quasiconvex.

Martínez-Pedroza [76] showed that for a relatively hyperbolic G, if a subgroup H is relatively quasiconvex in G then the particular amalgamated free product of H and parabolic subgroup along their intersection is again a relatively quasiconvex subgroup of G. We show that the converse is true for a toral relatively hyperbolic group and use the characteristic to find such an algorithm in Theorem H. The proof of Theorem H gives an alternative algorithm for Theorem E(4) without using Dehn fillings.

Theorem H can be promoted to obtain a corollary for the limited version of the uniform membership problem.

Corollary I. Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group. Then there exists a partial algorithm which, for given $u, v_1, \ldots, v_k \in G$, detects if $H = \langle v_1, \ldots, v_k \rangle$ is relatively quasiconvex in G, and if H is relatively quasiconvex, decides whether or not $u \in H$.

1.3 Outline

In Chapter 2, we give some preliminaries about hyperbolic groups and their quasiconvex subgroups. We then discuss stability and Morseness in Chapter 3 for more details. In Chapter 4, we provide some backgrounds about mapping class groups and the proof of Theorem A. We describe the algorithms in Theorem C, Theorem D, Theorem E, Corollary F, Corollary G, Theorem H, and Corollary I in Chapter 5. Motivated by our main theorems, there are some natural questions and we discuss the related open problems in Chapter 6.

Chapter 2

Quasiconvex subgroups of word-hyperbolic groups

In this chapter, we review properties of word-hyperbolic groups and their quasiconvex subgroups.

2.1 Word-hyperbolic groups

Definition 2.1 (Hyperbolic space)**.** Let (X, d) be a geodesic metric space and let $\delta \geq 0$. We say that (X, d) is *δ-hyperbolic* if for all $x, y, z \in X$ and for every geodesic path α from x to y, β from y to z, and γ from z to x, we have $\alpha \subseteq \mathcal{N}_\delta(\beta \cup \gamma)$, $\beta \subseteq \mathcal{N}_\delta(\alpha \cup \gamma)$, and $\gamma \subseteq \mathcal{N}_\delta(\alpha \cup \beta)$. A geometric metric space (X, d) is called *hyperbolic* if there exists $\delta \geq 0$ such that X is δ-hyperbolic.

For other equivalent definitions of hyperbolicity, see [2, 18] for more details. Let G be a finitely generated group with a finite generating set S. The Cayley graph $\Gamma(G, S)$ is a geodesic metric space with the associated word metric d_S. By using the Cayley graph, we define word-hyperbolic groups as follows:

Definition 2.2 (Hyperbolic group)**.** A finitely generated group G is called *word-hyperbolic* if there exists a finite generating set S such that the Cayley graph $\Gamma(G, S)$ is hyperbolic.

It is known that for a word-hyperbolic group G, the Cayley graph is hyperbolic for every finite generating set of G, that is, hyperbolicity of G is independent of a generating set of G. We recall the notion of a quasi-isometry between two metric spaces.

Definition 2.3 (Quasi-isometry)**.** Let (X, d_X) and (Y, d_Y) be two metric spaces and let $k \geq 1$ and $c \geq 0$. For a map $f : X \to Y$ we say that f is a *(k, c)-quasi-isometric embedding* if for every $x, y \in X$ we have

$$\tfrac{d_X(x,y)}{k} - c \leq d_Y(f(x), f(y)) \leq k d_X(x, y) + c.$$

A (k,c)-quasi-isometric embedding f is a *(k, c)-quasi-isometry* if for every $y \in Y$ there is some $x \in X$ with $d_Y(f(x), y) \leq c$. A map $f : X \to Y$ is called a *quasi-isometric embedding (respectively, quasi-isometry)* if f is a (k,c)-quasi-isometric embedding (respectively, quasi-isometry) for some $k \geq 1$ and $c \geq 0$. Two metric spaces X and Y are said to be *quasi-isometric* if there exists a quasi-isometry $f : X \to Y$.

Definition 2.4 (Quasigeodesic)**.** Let (X, d) be a metric space, let $k \geq 1$ and $c \geq 0$. A (k, c)-*quasigeodesic* in X is a map $f : I \to X$, where I is an interval in $\mathbb{R}$, such that for all $t_1, t_2 \in I$,

$$\frac{|t_1 - t_2|}{k} - c \leq d_Y(f(t_1), f(t_2)) \leq k|t_1 - t_2| + c.$$

That is, a (k, c)-quasigeodesic is a (k, c)-quasi-isometric embedding of some interval in $\mathbb{R}$ into X.

It is known that quasigeodesics in a hyperbolic space have the Morse property (see Theorem 1.7 in Chapter III.H in [18]):

Proposition 2.5 (Morse property)**.** *Let X be a hyperbolic space, that is, X is δ-hyperbolic for some $\delta \geq 0$. Let $k \geq 1$ and $c \geq 0$. Then there exists a constant $N = N(\delta, k, c)$ such that If a (k, c)-quasigeodesic γ in X and $[p, q]$ is a geodesic segment joining the endpoints of γ then the Hausdorff distance between $[p, q]$ and γ is less than N.*

Proposition 2.6. *Let (X, d_X) and (Y, d_Y) be geodesic metric spaces which are quasi-isometric. Then X is hyperbolic if and only if Y is hyperbolic.*

For the proof of the above proposition, see Theorem 1.9 in Chapter III. H in [18]. There is another way to verify hyperbolicity of a group by using a group action on a hyperbolic metric space. We recall that a metric space X is *proper* if all closed metric balls in X are compact.

Definition 2.7 (Geometric group action)**.** Let (X, d) be a proper geodesic metric space and let G be a group. A group action of G on X is called *geometric* if it satisfies the following conditions:

(i) Every element of G acts isometrically on X.

(ii) The action is cocompact, i.e., X/G is compact.

(iii) The action is properly discontinuous action, i.e., for any compact $K \subset X$ the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite.

Theorem 2.8 (Švarc–Milnor Lemma)**.** *Let X be a proper geodesic metric space and let G be a group with a geometric action on X. Then*

(i) G is finitely generated; and

(ii) for some (equivalently, any) finite generated set S of G and for every $x_0 \in X$ the orbit map $f : (G, d_s) \to X$, $g \mapsto gx_0$ is a quasi-isometry.

For the proof of Švarc–Milnor Lemma, for example, see Proposition 8.19 in Chapter I.8 in [18]. Since hyperbolicity is a quasi-isometric invariant of geodesic metric spaces, we have the following corollary from Švarc–Milnor Lemma.

Corollary 2.9. *Let X be a proper hyperbolic space and let G be a group with a geometric action on X. Then G is a word-hyperbolic group.*

It is known that every word-hyperbolic group is finitely presented and for a finitely presented group being hyperbolic is equivalent to satisfying linear isoperimetric inequality and also equivalent to possessing a finite Dehn presentation (see Section 2 in [2]).

Here are some examples of word-hyperbolic groups (see [44, 2, 40] for more details): finite groups, finitely generated free groups, finitely presented small cancellation groups satisfying $C(7), C(5)-T(4), C(4)-T(5)$ or $C(3)-T(7)$, free products of word-hyperbolic groups, free factors of a word-hyperbolic group, and fundamental groups of closed Reimannian manifolds all of whose sectional curvatures are negative.

We list some properties of word-hyperbolic groups. We use [44, 2] as background references.

Theorem 2.10. *Let G be a word-hyperbolic group with a finite generating set S.*

(1) The group G is finitely presented.

(2) The group G has solvable word problem, that is, there exists algorithm such that for a word w over S, the algorithm determines whether or not $w =_G 1$.

(3) The group G has solvable conjugacy problem, that is, there exists an algorithm that determines whether or not w and u represent conjugate elements of the group G.

(4) The isomorphism problem is algorithmically solvable in the class of torsion-free word hyperbolic groups, that is, there exists an algorithm which accepts as input two group presentations and determines whether or not they represent isomorphic groups [86].

(5) The group G is automatic [34].

(6) Any subgroup of G is either virtually cyclic or contains a free subgroup of rank 2 [41].

(7) The group G does not contain a free abelian subgroup of rank 2 [39].

(8) For a short exact sequence $1 \to N \to G \to Q \to 1$ where N is finite, then G is word-hyperbolic if and only if Q is word-hyperbolic.

A word-hyperbolic group G can be equipped with a well-defined boundary, which is a compact metrizable space.See [57] for details on boundaries of hyperbolic groups.

Definition 2.11. Let (X, d) be a proper hyperbolic space. We say that two geodesic rays $\gamma_1 : [0, \infty) \to X$ and $\gamma_2 : [0, \infty) \to X$ are *equivalent* and write $\gamma_1 \sim \gamma_2$ if there is $K > 0$ such that for any $t \geq 0$, $d(\gamma_1(t), \gamma_2(t)) \leq K$. Then $\sim$ is an equivalence relation on the set of geodesic rays in X and write $[\gamma]$ as the equivalence class of a geodesic ray γ in X.

Definition 2.12 (Geodesic boundary of a proper hyperbolic space)**.** Let (X, d) be a proper hyperbolic metric space and let $x \in X$ be a base-point. We define the *geodesic boundary* of X as

$$\partial X := \{[\gamma] | \gamma : [0, \infty) \to X \text{ is a geodesic ray in } X\}.$$

We also define the *geodesic boundary of X with respect to the base-point x* as

$$\partial_x X := \{[\gamma] | \gamma : [0, \infty) \to X \text{ is a geodesic ray with } \gamma(0) = x \text{ in } X\}.$$

Remark 2.13. In general, we can define the boundary ∂X of a hyperbolic metric space X that is not necessarily proper. For a non-proper hyperbolic space X, one needs to use the definition of boundary points via equivalence classes of sequences rather than of geodesic rays (see Section 2 [57]).

For a proper hyperbolic space X and any two points $x, y \in X$, there exists a canonical identification $\mathcal{J}_{x,y} : \partial_x X \to \partial_y X$. Specifically, for any geodesic ray γ_1 starting from $x \in X$ and other point $y \in X$, there exists a geodesic ray γ_2 starting from y such that $[\gamma_1] = [\gamma_2]$ (see Section 2 in [57]) and we put $\mathcal{J}_{x,y}[\gamma_1] = [\gamma_2]$. This fact also implies that for every $x \in X$ the natural inclusion map $\mathcal{J}_x : \partial_x X \to \partial X$ is a bijection.

Definition 2.14. Let X be a metric space and x, y, and z be points on X. Then the *Gromov product* of y and z with respect to x is

$$(y, z)_x := \frac{1}{2}(d(x, y) + d(x, z) - d(y, x)).$$

Note that for a δ-hyperbolic metric space X and $x, y, z \in X$, $|\, d(x, [y, z]) - (y, z)_x| \leq 2\delta$, where $[y, z]$ is a geodesic between y and z.

Definition 2.15 (Topology on the geodesic boundary)**.** Let (X, d) be a proper hyperbolic space and let $x \in X$ be a base-point. For any $p \in \partial_x X$ and $r \geq 0$ we define the set

$$V_x(p, r) := \{q \in \partial_x X | \text{ for some geodesic rays } \gamma_1, \gamma_2 \text{ starting at } x \text{ and with } [\gamma_1] = p, [\gamma_2] = q \\ \text{we have } \lim\inf_{t\to\infty} (\gamma_1(t), \gamma_2(t))_x \geq r\}.$$

For any $p \in \partial_x X$, the basis of neighborhoods is the collection $\{V_x(p,r) | r \geq 0\}$. Then we topologize $\partial_x X$ by using the basis of neighborhoods $\{V_x(p,r) | r \geq 0\}$ for any $p \in \partial_x X$.

The resulting topology on $\partial_x X$ is independent of the choice of a base-point $x \in X$ in the sense that for any $x, y \in X$ the identification $\mathcal{J}_{x,y} : \partial_x X \to \partial_y X$ is homeomorphism. Then for a proper hyperbolic space X, we topologize ∂X via the bijection $\mathcal{J}_x : \partial_x X \to \partial X$, where $x \in X$ is any base-point. Note that this topology is independent of x.

Furthermore, we can endow $X \cup \partial X$ with a natural topology. For $y \in X$ we choose the same basis of neighborhoods in $X \cup \partial X$ as in X. For a point $p \in \partial X$ and $x \in X$ let γ is a geodesic starting from x and $[\gamma] = p$. Then we use the basis of neighborhoods of p in $X \cup \partial X$ to be the collection

$$V_x(p,r) \cup \{y \in X \mid \text{ for a geodesic } \beta \text{ from } x \text{ to } y \ \liminf_{t\to\infty} (\gamma(t), \beta(t))_x \geq r\}.$$

In fact, for a proper hyperbolic metric space X, both X and ∂X are compact and we can consider $X \cup \partial X$ as a compactification of X. It is known that for a proper hyperbolic space X, this topology on $X \cup \partial X$ does not depend on $x \in X$ (see Section 2 in [57]).

Proposition 2.16. *Let X and Y be proper hyperbolic spaces and let $f : X \to Y$ be a quasi-isometry. Then f induces a homeomorphism between ∂X and ∂Y.*

See Theorem 3.9 in Chapter III.H in [18] for a proof of Proposition 2.16. Note that for a finitely generated group G and finite generating sets S_1 and S_2 of G, the identity map $id_G : G \to G$ induces a quasi-isometry from Cayley graph $\Gamma(G, S_1)$ to the Cayley graph $\Gamma(G, S_2)$.

Definition 2.17 (Gromov boundary of a hyperbolic group)**.** Let G be a word-hyperbolic group so that for some (equivalently, every) finite generating set S of G the Cayley graph $\Gamma(G, S)$ is hyperbolic. We define the *Gromov boundary* ∂G of G as the geodesic boundary $\partial\Gamma(G, S)$ of $\Gamma(G, S)$ for some (equivalently, every) finite generating set S of G.

For a hyperbolic group G changing a finite generating set is a quasi-isometry between Cayley graphs, and we have an induced homeomorphism from the Gromov boundary of one Cayley graph to the Gromov boundary of the other. Thus, both G and $G \cup \partial G$ are well-defined topological spaces.

Note that a word-hyperbolic group G acts by isometries on its Cayley graph $\Gamma(G, S)$ and this action extends to the action of G on ∂G by homeomorphisms. Any infinite order element $g \in G$ acts as a *loxodromic* isometry of $\Gamma(G, S)$, that is, there are exactly two points $g^+ = \lim_{n\to\infty} g^n$ and $g^- = \lim_{n\to\infty} g^{-n}$ in ∂G fixed by the element g. The Gromov boundary ∂G has been an useful tool to study the hyperbolic group G because of the topological, dynamical, metric, and quasiconformal structure of ∂G (see [57] for more details).

2.2 Quasiconvex subgroups

In the theory of word-hyperbolic groups, quasiconvex subgroups play a significant role.

Definition 2.18 (Quasiconvex subset)**.** Let X be a geodesic metric space. A subset $Y \subseteq X$ is *quasiconvex* in X if there exists $N \geq 0$ such that every geodesic in X that connects a pair of points in Y is contained in the N-neighborhood of Y.

In particular, for a hyperbolic space X every geodesic or quasi-geodesic ray in X is quasiconvex. In this case, it is also known that every quasigeodesic in X stays in uniformly bounded distance to a geodesic by the Morse Lemma.

Definition 2.19 (Quasiconvex subgroup)**.** Let G be a finitely generated group with a finite generating set S. A subgroup $H \leq G$ is *quasiconvex in G with respect to S* if the subset H of the Cayley graph $\Gamma(G, S)$ is quasiconvex in $\Gamma(G, S)$. A subgroup H of G is *quasiconvex* in G if H is quasiconvex in G with respect to some generating set of G.

Quasiconvex subgroups with respect to some generating set of an ambient finitely generated group are themselves finitely generated. In general, for a subgroup H of a finitely generated group G, quasiconvexity of H depends on the choice of finite generating sets for G.

Example 2.20. *Let a and b be free generators of a free abelian group $\mathbb{Z}^2$ of rank 2. Then $\langle ab \rangle$ is not quasiconvex in $\mathbb{Z}^2$ with respect to $\{a, b\}$, while the subgroup $\langle ab \rangle$ is quasiconvex with respect to $\{ab, b\}$.*

However, in the case of a word-hyperbolic group G, quasiconvexity of subgroups does not depend upon the choice of a finite generating set for G.

Proposition 2.21. *Let G be a word-hyperbolic group and let H be a subgroup of G. If H is quasiconvex in G with respect to some finite generating set of G then H is quasiconvex with respect to every finite generating set of G.*

Definition 2.22 (Undistorted subgroup)**.** Let G be a finitely generated group let H be a finitely generated subgroup of G. Then H is called *undistorted* if for every (equivalently, some) finite generating set S of G and every (equivalently, some) finite generating set T of H, the inclusion map $(H, d_T) \to (G, d_S)$ is a quasi-isometric embedding.

We recall that for a word-hyperbolic group G with a finite generating set S and a base-point x in the Cayley graph (G, S), the *limit set* $\Lambda H \subseteq \partial G$ of H as the union of all limits in ∂G of sequences of elements from an H-orbit of x. We define the *convex hull* $C_S(H)$ with respect to S as the union of all bi-infinite geodesics in the Cayley graph of G with both endpoints in ΛH.

Now we list some characteristics of quasiconvex subgroups of a word-hyperbolic group.

Theorem 2.23. *Let G be a word hyperbolic group with a finite generating set S.*

(1) A subgroup H is quasiconvex in G if and only if H is finitely generated and undistorted in G [2].

(2) Let (L, S) be an automatic structure on G. A subgroup H of G is quasiconvex in G if and only if H is L-rational with respect to L, that is the pre-image L_H of H in L is a regular language [91].

(3) An infinite subgroup H is quasiconvex in G if and only if $C_S(H)/H$ has finite diameter [60].

We recall properties of quasiconvex subgroups of a word-hyperbolic group. See [2, 60] as references.

Theorem 2.24. *Let G be a word hyperbolic group with a finite generating set S.*

(1) If H is a quasiconvex subgroup of G then H is finitely generated, finitely presented, and word-hyperbolic.

(2) If $H_1, \ldots, H_k$ are quasiconvex subgroups of G then $H = H_1 \cap \cdots \cap H_k$ is quasiconvex in G.

(3) If C is a virtually cyclic subgroup of G then C is quasiconvex in G.

(4) If H is quasiconvex in G and K is a subgroup of H, that is, $K \leqslant H \leqslant G$, then K is quasiconvex in G if and only if K is quasiconvex in H.

(5) If H is an infinite quasiconvex subgroup of G then H has finite index in its commensurator

$$Comm_G(H) = \{g \in G \mid |H : H \cap gHg^{-1}| < \infty, \, |gHg^{-1} : H \cap gHg^{-1}| < \infty\}.$$

(6) If H is an infinite quasiconvex subgroup of G then there are only finitely many subgroups $K \leqslant G$ such that K contains H as a subgroup of finite index.

To give another property, we recall the notions of *finite height* and *finite width.*

Definition 2.25. Let G be a group and let H be a subgroup of G.

(1) Conjugates $g_1 H g_1^{-1}, \ldots, g_k H g_k^{-1}$ are *essentially distinct* if the cosets $g_1 H, \ldots, g_k H$ are distinct.

(2) H has *finite height* n in G if the intersection of every $(n+1)$ essentially distinct conjugates is finite and n is minimal possible.

(3) H has *finite width* n in G if n is the maximal cardinality of the set $\{g_iH : |g_iHg_i^{-1} \cap g_jHg_j^{-1}| = \infty\}$, where $\{g_iH\}$ ranges over all collections of distinct cosets.

We note that every finite subgroup and every subgroup of finite index have finite height and width, and every infinite normal subgroup of infinite index has infinite height and width. The following theorem states that quasiconvex subgroups of a word-hyperbolic group G are far from being normal in G.

Theorem 2.26 ([43]). *A quasiconvex subgroup of a word-hyperbolic group has finite height and finite width.*

In general, not all finitely generated subgroups for a word-hyperbolic group are quasiconvex.

Example 2.27. *By Rips' construction [83], for an arbitrary finitely generated group Q there exists a short exact sequence*

$$1 \to K \to G \to Q \to 1$$

such that G is torsion-free, non-elementary (infinite and not virtually cyclic), word-hyperbolic and such that K is two-generated. For instance, if $Q = \mathbb{Z}\times\mathbb{Z}$ then G is a word-hyperbolic and K is an infinite two-generated normal subgroup of infinite index in G. Hence K is not quasiconvex in G.

Another example is the fundamental group $G = \pi_1(M)$ of a closed hyperbolic 3–manifold M fibering over a circle. Then G splits as a semi-direct product $G = H \ltimes \mathbb{Z}$, where H is the fundamental group of a fiber, and $1 \to H \to G \to \mathbb{Z} \to 1$. Therefore, H is not quasiconvex in G.

Quasiconvex subgroups of a word-hyperbolic group are word-hyperbolic themselves and have nice algorithmic aspects. For a finitely presented group $G = \langle S \,|\, R \rangle$ and a subgroup H of G, we say that G has *solvable membership problem* (or *solvable generalized word problem*) for G with respect to H with respect to H if there exists an algorithm which, for any word w over S, decides whether w belongs to H.

Proposition 2.28 (Proposition 7.5 in [59]). *Let G be a word-hyperbolic group and let H be a quasiconvex subgroup of G. Then G has solvable membership problem with respect to H in linear time.*

Kapovich in [55] gave a partial algorithm which, for a finite subset S of a word-hyperbolic group G, detects if S generates a quasiconvex subgroup of G.

Proposition 2.29 (Proposition 4 in [55]). *Let G be a word-hyperbolic group given by a finite presentation $G = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_m\rangle$ and let $S = \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$. Then there is a uniform algorithm which, given a finite set of words $v_1, .., v_t$ over S, will*

(i) eventually stop and produce the quasiconvexity constant N and the distortion constant C of the subgroup $H = gp(\bar{v}_1, \ldots, \bar{v}_t)$ of G if H is quasiconvex in G, where $\bar{v}_i$ denotes the element of G represented by the word v_i;

(ii) run forever if H is not quasiconvex in G.

Note that for a word-hyperbolic group quasiconvex subgroups have played a key role in amalgamated free products and HNN-extensions of hyperbolic groups (see [44, 9, 10, 64, 5, 42]). For example, Bestvina and Feighn [9, 10] showed that for the fundamental group G of a finite graph of hyperbolic groups, if all the edge-monomorphisms are quasi-isometric embeddings then G is word-hyperbolic.

Chapter 3

Stable and Morse subgroups of finitely generated groups

There have been several important recent generalizations of the notion of quasiconvexity to the context of subgroups of arbitrary finitely generated groups, namely, stability and Morseness.

3.1 Stable groups of finitely generated groups

Durham and Taylor [33] introduced stable subgroups of a finitely generated group, where they proved that the stable subgroups of mapping class groups are exactly the convex cocompact subgroups.

Definition 3.1. Let A and X be two geodesic metric spaces and let $f : A \to X$ be a quasi-isometric embedding. We say A is *stable* in X if for every $k \geq 1$ and $c \geq 0$ there is some $L = L(k, c)$ such that for every pair of $(k, c)-$quasigeodesic in X with the same endpoints in $f(A)$, each quasigeodesic is contained in the L-neighborhood of the other.

Definition 3.2 (Stable subgroup)**.** Let G be a finitely generated group with a finite generating set S. Let H be a finite generated subgroup of G. We say H is *stable* in G if H is undistorted in G and $H \subseteq \Gamma(G, S)$ is stable for some (equivalently, any) choice of a finite generating set of H.

Stability has been characterized in many ways. For instance, in the recent studies of a generalization of the Gromov boundary, called the Morse boundary, Cordes and Durham in [27] characterized stability in terms of "boundary convex cocompactness". See Definition 3.4 and Section 4 in Cordes' survey paper [26] for details.

Abbott, Behrstock, Durham [1] provided a complete classification of stable subgroups of a hierarchically hyperbolic group by the hierarchically hyperbolic structure and projections. Many interesting families of groups are hierarchically hyperbolic groups, such as mapping class groups, right-angled Coxeter groups, right-angled Artin groups,and most 3-manifold groups.

Note that a stable subgroup of an arbitrary group is necessarily hyperbolic by the definition. Hence the geometry of the subgroup might not reflect the geometry of the whole group.

3.2 Morse subgroups of finitely generated groups

Another recent generalization of quasiconvexity is given by the notion of a Morse subset of a geodesic metric space. See [22, 25, 4, 28, 27, 26] for more details and additional related results.

Definition 3.3 (Morse subset)**.** Let X be a geodesic metric space. A subset $Y \subseteq X$ is *Morse* if every $k \geq 1$ and $c \geq 0$ there is some $M = M(k, c)$ such that every $(k, c)-$quasigeodesic in X with endpoints on Y is contained in the M-neighborhood of Y.

This notion has been of particular interest in the case where Y is a geodesic or quasigeodesic. Note that Arzhantseva, Cashen, Gruber, and Hume in [4] characterized Morse quasigeodesics in terms of superlinear divergence and sublinear contraction with characterizations of Morse sets.

Morse geodesics have led to the notion of a Morse boundary introduced by Cordes in [25]. In the case of a proper CAT(0) space the Morse boundary is the contracting boundary of Charney and Sultan [22], and in the case of a proper Gromov hyperbolic space this boundary is the Gromov boundary.

Definition 3.4 (Morse boundary)**.** Let X be a proper metric space. We say that the *Morse boundary* of a space X, $\partial_M X$, is the set of all Morse geodesic rays in X where two geodesic rays $\gamma, \gamma' : [0, \infty) \to X$ are identified if there is a constant K such that $d(\gamma(t), \gamma'(t)) < K$ for all $t > 0$.

Contracting boundaries in [22] and Morse boundaries in [25] are equipped with a direct limit topology on the Morse boundary. As a result, the Morse boundary is not first countable. Therefore, Cashen and Mackay [20] used a metrizable topology on the Morse boundary. We also note that Qing and Rafi [82] recently introduced another boundary for a CAT(0) space, namely the *sublinearly Morse boundary*.

For the case where X is the Cayley graph $\Gamma(G, S)$ of a finitely generated group G with a finite generating set S, and $Y \subseteq X$ is a subgroup of G, the notion of a Morse subset naturally leads to the following:

Definition 3.5 (Morse or strongly quasiconvex subgroup [92, 37])**.** Let G be a finitely generated group and let H be a subgroup of G. We say that H is a *Morse* or *strongly quasiconvex* subgroup of G if for every (equivalently, some) finite generating set S of G, the subgroup $H \subseteq \Gamma(G, S)$ is Morse.

If H is a Morse subset in $\Gamma(G, S)$ for some finite generating set S of G, then the same is true for every finite generating set of G. Moreover, H is finitely generated and undistorted in G in this case.

For a finitely generated group G, an element $g \in G$ of infinite order is called a *Morse element* if the orbit of $\langle g \rangle$ in any Cayley graph is a Morse quasigeodesic. The notion of a Morse element has been studied for some time now. This case corresponds to cyclic Morse subgroups of finitely generated groups, and the cyclic Morse subgroups are indeed stable.

However, the actual notion of a Morse subgroup has not been defined until recently. This notion was originally explicitly defined under the name "strongly quasiconvex subgroup" by Tran in July 2017 [92] with its characterization via lower relative divergence. Genevois independently introduced the term "Morse subgroup" for the notion from Definition 3.5 above in September 2017 [92], with its characterizations in a cubulable group. We think that the term "Morse subgroup" may be preferable to "strongly quasiconvex subgroup" because of how the notion fits into the theory of Morse subsets and Morse boundary.

For a hyperbolic group, every quasigeodesic stays in uniformly bounded distance to a geodesic (see Proposition 2.5), which implies the following:

Proposition 3.6. *Let G be a word-hyperbolic group. For a finitely generated subgroup of G, being quasiconvex in G is equivalent to being stable in G and also equivalent to being Morse in G.*

Beyond hyperbolic groups, we have many examples to distinguish these two definitions.

Example 3.7. The group $\mathbb{Z}^2$ is a non-stable Morse subgroup of $\mathbb{Z}^2$. Consider two $(3,0)$-quasigeodesics with endpoints $(0,0)$ and $(n,0)$ where $n \in \mathbb{N}$; one consists of three geodesics $[(0,0),(0,n)]$, $[(0,n),(n,n)]$, and $[(n,n),(n,0)]$, and the other consists of three geodesics $[(0,0),(0,-n)]$, $[(0,-n),(n,-n)]$, and $[(n,-n),(n,0)]$. Then if one contains the other in its L-neighborhood then $L > 2n$. This means that there is no uniform bound $L = L(3,0)$ as described in Definition 3.2. Therefore, the group $\mathbb{Z}^2$ is not stable but obviously Morse in $\mathbb{Z}^2$. However, $\mathbb{Z}$ is not Morse in $\mathbb{Z}^2$.

Now we discuss some properties of Morse subgroups. For the details, see Section 4 in [92], where Morse subgroups are studied under the name strongly quasiconvex subgroups. The following proposition tells us what the relation between Morse subgroups and stable subgroups is.

Proposition 3.8 (Proposition 4.8 of [92])**.** *Let G be a finitely generated group and H an infinite subgroup of G. Then H is stable in G if and only if H is hyperbolic and Morse in G.*

Tran also gave another characterization of stable subgroups in terms of the relatively divergence of G with respect to H in Theorem 4.11 in [92]. The following proposition gives us a way to get another Morse subgroup from a Morse subgroup.

Proposition 3.9 (Theorem 4.11 in [92])**.** *Let G be a finitely generated group and let A be an undistorted subgroup of G. If H is a Morse subgroup of G, then $A \cap H$ is Morse in A.*

The above proposition directly implies that for two Morse subgroups H_1 and H_2 of G the intersection $H_1 \cap H_2$ is Morse in G because a Morse subgroup is undistorted in G. The next proposition states that Morse subgroups are far from being normal.

Proposition 3.10 (Theorem 1.2 in [92]). *Let G be a finitely generated group and let H be a Morse subgroup. Then H has finite height and finite width.*

For a finitely generated group G, if H is an infinite Morse subgroup of G then H has finite index in its commensurator $Comm_G(H) = \{g \in G \,|\, |H : H \cap gHg^{-1}| < \infty, \, |gHg^{-1} : H \cap gHg^{-1}| < \infty\}$.

Motivated by Example 3.7, we obtain the following lemma which is easily derived from Proposition 3.10.

Lemma 3.11. *Let G be a finitely generated group and let H be a Morse subgroup of G. For $g \in G$ and $h \in H$, if g and h commute and $\langle h, g \rangle \cong \mathbb{Z}^2$ is undistorted in G, then there is an integer $m > 0$ such that g^m is contained in H.*

Proof. Suppose that g^m is not contained in H for every $m > 0$. Then we have $g^i H \neq g^j H$ if $i \neq j$. By Proposition 3.10, the subgroup H has finite height n in G for some n. Since $h \in H$ and g commute, we have $\langle h \rangle \in g^i H g^{-i}$ for $i = 1, \ldots, n+1$, and the intersection $\cap_{i=1}^{n+1} g^i H g^{-i}$ is infinite. This contradicts that H has finite height n in G. □

Chapter 4

Stable and Morse subgroups of mapping class groups

For subgroups of a certain right-angled Artin group, being stable is equivalent to being Morse of infinite index. The precise statement is as follows.

Theorem 4.1 (Theorem 1.16 in [92], Theorem B.1 in [37])**.** *For a non-trivial, infinite index subgroup of the right-angled Artin group A_Γ of a simplicial, finite, connected graph Γ which does not decompose as a nontrivial join, a Morse subgroup of A_Γ is stable in A_Γ.*

Inspired by the above theorem, we compare stability and Morseness for mapping class groups in this chapter.

4.1 Mapping class groups

For the material in this section, we use [36, 61, 35, 33, 8] as background references.

Let S be an oriented, connected, finite type surface with negative Euler characteristic $\chi(S)$. We note that a surface S is of finite type if and only if the fundamental group of S is finitely generated. The *(extended) mapping class group* $\mathrm{Mod}(S)$ of S is the group of isotopy classes of homeomorphisms of S. From the Nielsen-Thurston classification (see Section 13 in [35]), for an element f of $\mathrm{Mod}(S)$, we have three possible cases; we say f is *periodic* if some power of f is the identity; f is *reducible* if it permutes some finite collection of pairwise disjoint simple closed curves in S; and f is *Pseudo-Anosov* if it is neither periodic nor reducible. Also, if f is reducible then some power of f preserves a simple closed curve on S up to isotopy (see [35]).

Farb and Mosher [36] introduced and developed the notion of a convex cocompact subgroup of the mapping class group of a closed, connected, and oriented surface by its action on Teichmüller space $\mathcal{T}(S)$. In [61], Kent and Leininger extended the definition for the case of an oriented, connected, finite area hyperbolic surface, which is equivalent to an oriented, connected, finite type surface with $\chi(S) < 0$.

Definition 4.2. Let S be an oriented, connected, finite type surface with $\chi(S) < 0$, and let $\mathrm{Mod}(S)$ be its mapping class group. Then a subgroup $G < \mathrm{Mod}(S)$ is *convex cocompact* if for some $x \in \mathcal{T}(S)$ the orbit $G \cdot x$ is quasiconvex with respect to the Teichmüller metric on $\mathcal{T}(S)$.

We remark that from the definition, if G is convex cocompact, then G is finitely generated, and every infinite order element in G is pseudo-Anosov, i.e., G is purely pseudo-Anosov. There have been many equivalent characterizations of convex cocompactness for subgroups of $\mathrm{Mod}(S)$.

We first recall that there is a natural simplicial complex, associated to a surface S, called the *curve complex* $\mathcal{C}(S)$ on which $\mathrm{Mod}(S)$ acts by simplicial automorphisms. We utilize the *curve graph* $\mathcal{C}(S)$, one-skeleton of the curve complex, whose vertices are isotopy classes of essential simple closed curves on S, and two distinct isotopy classes are joined by an edge if they are disjointly realizable.

Remark 4.3. For a surface S which has either genus at least 2 or at least 5 punctures, the curve graph $\mathcal{C}(S)$ is connected. The only surfaces making non empty curve graphs disconnected with $\chi(S) < 0$ are the 1-punctured torus $S_{1,1}$ and the 4-punctured sphere $S_{0,4}$. In those two cases, the curve graph$\mathcal{C}(S)$ is a countable disjoint union of points. Hence, in the case of $S = S_{1,1}$ or $S = S_{0,4}$, we alter the definition of $\mathcal{C}(S)$ by joining a pair of distinct vertices if they realize the minimal possible geometric intersection in S, which makes $\mathcal{C}(S)$ connected.

In [61], Kent and Leininger show that a finitely generated subgroup G of $\mathrm{Mod}(S)$ is convex cocompact if and only if an orbit map from G to the curve graph $\mathcal{C}(S)$ is undistorted. This fact is independently proved by Hamenstädt in [50]. Furthermore, Durham and Taylor in [33] characterize convex cocompactness in $\mathrm{Mod}(S)$ by using only the geometry of $\mathrm{Mod}(S)$ itself.

Proposition 4.4 (Theorem 1.1 in [33])**.** *Let S be a connected and oriented surface which is neither the 1-punctured torus nor the 4-punctured sphere, and let $\mathrm{Mod}(S)$ be its mapping class group. Then the subgroup G of $\mathrm{Mod}(S)$ is convex cocompact if and only if it is stable.*

In [8], we have another characterization for the convex cocompactness as follows.

Proposition 4.5 (Main Theorem in [8])**.** *Let S be an oriented, connected, finite type surface with $\chi(S) < 0$ and let $\mathrm{Mod}(S)$ be its mapping class group. A subgroup G of $\mathrm{Mod}(S)$ is convex cocompact if and only if it is finitely generated, undistorted, and purely pseudo-Anosov.*

4.2 Proof of Theorem A

From now on, we assume that S is an oriented, connected, finite type surface with negative Euler characteristic which is neither the 1-punctured torus nor the 4-punctured sphere, and $\mathrm{Mod}(S)$ is its mapping class group. For an essential simple closed curve α, we denote by D_α the Dehn twist about α, and denote by $C(D_\alpha)$ the centralizer.

Theorem A. *Let S be an oriented, connected, finite type surface with $\chi(S) < 0$ which is neither the 1-punctured torus nor the 4-punctured sphere. Let $\mathrm{Mod}(S)$ be the mapping class group of S, and let G be a finitely generated subgroup of $\mathrm{Mod}(S)$. Then the following are equivalent:*

(1) G is convex cocompact.

(2) G is Morse of infinite index in $\mathrm{Mod}(S)$.

The implication "(1) $\Rightarrow$ (2)" in Theorem A is straight forward from Proposition 4.4 and Proposition 3.8. In order to prove the reverse implication "(2) $\Rightarrow$ (1)", we need the following two lemmas.

Lemma 4.6. *Let α be an essential simple closed curve on S. Then if K is a Morse subgroup of $C(D_\alpha)$ then K is either finite or has finite index in $C(D_\alpha)$.*

Proof. Suppose that K is infinite and has infinite index in $C(D_\alpha)$. Since every element in K commutes with D_α, and K has finite height in $C(D_\alpha)$ by Proposition 3.10, there exists $m > 0$ such that $D_\alpha^m \in K$. Also, we have $[\,C(D_\alpha) : K\,] = \infty$ by assumption. Therefore, there exists an infinite sequence (g_n) of distinct elements in $C(D_\alpha)$ such that $g_iK \neq g_jK$ for $i \neq j$. Then from $D_\alpha^m \in K$ and $g_i \in C(D_\alpha)$, we have $\langle D_\alpha^m \rangle \subseteq g_iKg_i^{-1}$ for all i, and then the intersection $\cap_{i=1}^{\infty} g_iKg_i^{-1}$ is infinite. This contradicts that K has finite height in $C(D_\alpha)$. □

Lemma 4.7. *Let α be an essential simple closed curve on S. Then for each g_0 and g in $\mathrm{Mod}(S)$, there exists a sequence of subgroups $g_0C(D_\alpha)g_0^{-1} = Q_0, Q_1 \ldots, Q_m = gC(D_\alpha)g^{-1}$ of $\mathrm{Mod}(S)$ such that $Q_i \cap Q_{i+1}$ is infinite for each $i \in \{0, 1, \ldots, m-1\}$.*

Proof. Pick a path between $g_0 \cdot \alpha$ to $g \cdot \alpha$ in the curve graph $\mathcal{C}(S)$ of S. For every vertex on the path, we pick a representative curve in the corresponding isotopy class of essential simple closed curves. Then there is a finite sequence of curves of $g_0 \cdot \alpha = \gamma_0, \gamma_1, \ldots, \gamma_m = g \cdot \alpha$ such that γ_{i-1} is disjoint from γ_i for each i. Now we consider the sequence of subgroups $g_0C(D_\alpha)g_0^{-1}$, $C(D_{\gamma_1})$, $\ldots$, $C(D_{\gamma_{m-1}})$, $gC(D_\alpha)g^{-1}$. We have $\langle D_{\gamma_i} \rangle \subseteq C(D_{\gamma_i}) \cap C(D_{\gamma_{i+1}})$ for $i \in \{1, 2, \ldots, m-2\}$. Since $g_0D_\alpha g_0^{-1} = D_{g_0\cdot\alpha}$ and $gD_\alpha g^{-1} = D_{g\cdot\alpha}$, we have $\langle D_{\gamma_0} \rangle \subseteq g_0C(D_\alpha)g_0^{-1} \cap C(D_{\gamma_1})$ and $\langle D_{\gamma_m} \rangle \subseteq C(D_{\gamma_{m-1}}) \cap gC(D_\alpha)g^{-1}$. This means the intersection of every two consecutive subgroups on the sequence is infinite. □

Remark 4.8. The centralizer of the Dehn twist about an essential simple closed curve is undistorted in $\mathrm{Mod}(S)$ followed by Masur-Minsky distance formula in the marking graph (see [77, 7]). We note that the proof of Theorem A is similar to the proof of Proposition 8.18 in [92]. In case of a right-angled Artin group A_Γ, Tran used a star subgroup to prove that stability is equivalent to being Morse. On the other hand, in $\mathrm{Mod}(S)$, we can use the centralizer of a curve which plays the same role of the cone-off vertex in a star subgroup of A_Γ.

***Proof of* "(2) ⇒ (1)" *in Theorem* A.** Suppose that G is Morse of infinite index but not convex cocompact. By Proposition 4.5, G is not purely pseudo-Anosov. Without loss of generality, we may assume that there exists a reducible element $h \in G$ of infinite order such that h fixes an essential simple closed curve α on S, i.e., $h(\alpha) = \alpha$. Let D_α be the Dehn twist about α, and let $C(D_\alpha)$ be the centralizer of D_α. Since $\langle h \rangle \subseteq G \cap C(D_\alpha)$, it is sufficient to show that $G \cap C(D_\alpha)$ is finite to derive a contradiction. In fact, we will show that for every $g \in \mathrm{Mod}(S)$, $g^{-1}Gg \cap C(D_\alpha)$ is finite. Then in particular, if g is the identity then $G \cap C(D_\alpha)$ is finite.

For a contradiction, suppose that there exists $g_0 \in \mathrm{Mod}(S)$ such that $H_0 = g_0^{-1}Gg_0 \cap C(D_\alpha)$ is infinite. Since $g_0^{-1}Gg_0$ is Morse and $C(D_\alpha)$ is undistorted in $\mathrm{Mod}(S)$, H_0 is Morse in $C(D_\alpha)$ by Proposition 3.9. Take $K = H_0$ in Lemma 4.6, and then we have $[C(D_\alpha) : H_0]$ is finite.

Now we claim that for every $g \in \mathrm{Mod}(S)$, $g^{-1}Gg \cap C(D_\alpha)$ has finite index of $C(D_\alpha)$. By Lemma 4.7, there is a sequence of subgroups $g_0C(D_\alpha)g_0^{-1} = Q_0, Q_1, \ldots, Q_m = gC(D_\alpha)g^{-1}$ such that $|Q_i \cap Q_{i+1}| = \infty$ for each $i \in \{0, 1, \ldots, m-1\}$. Then since $[\, g_0C(D_\alpha)g_0^{-1} : G \cap g_0C(D_\alpha)g_0^{-1}\,] = [C(D_\alpha) : H_0] < \infty$ and $|g_0C(D_\alpha)g_0^{-1} \cap Q_1| = \infty$, we have $G \cap Q_1$ is not finite. Indeed, Q_1 is the centralizer of an essential simple closed curve in the proof of Lemma 4.7. Therefore, Q_1 is undistorted in $\mathrm{Mod}(S)$ and $G \cap Q_1$ is Morse in Q_1. Then by Lemma 4.6, $G \cap Q_1$ has finite index in Q_1. By repeating this process, we end up with getting $[C(D_\alpha) : g^{-1}Gg \cap C(D_\alpha)] = [gC(D_\alpha)g^{-1} : G \cap gC(D_\alpha)g^{-1}] = [Q_m : G \cap Q_m] < \infty$.

By Proposition 3.10, G has finite height k for some k in $\mathrm{Mod}(S)$. Since $[\mathrm{Mod}(S){:}\ G] = \infty$, there exist $k+1$ distinct elements $g_1, \ldots, g_{k+1}$ of $\mathrm{Mod}(S)$ such that $g_iG \neq g_jG$ for $i \neq j$. Then we have $[C(D_\alpha) : {g_i}^{-1}Gg_i \cap C(D_\alpha)] < \infty$ for all g_i where $i = 1, \ldots, k+1$. It follows that $[C(D_\alpha) : (\cap_{i=1}^{k+1} {g_i}^{-1}Gg_i) \cap C(D_\alpha)] < \infty$. However, this means the intersection $\cap_{i=1}^{k+1} {g_i}^{-1}Gg_i$ is infinite, which contradicts that G has finite height k in $\mathrm{Mod}(S)$. Therefore, for any $g \in \mathrm{Mod}(S)$, $g^{-1}Gg \cap C(D_\alpha)$ is finite. This completes the proof. □

Then we can derive the following corollary.

Corollary B. *Let S be an oriented, connected, finite type surface with $\chi(S) < 0$ which is neither the 1-punctured torus nor the 4-punctured sphere. Let $\mathrm{Mod}(S)$ be the mapping class group of S, and let G be a finitely generated subgroup of $\mathrm{Mod}(S)$. Then the following are equivalent:*

(1) G is convex cocompact.

(2) An orbit map of G into the curve complex $\mathcal{C}(S)$ is a quasi-isometric embedding.

(3) G is finitely generated, undistorted, and purely pseudo-Anosov.

(4) G is stable.

(5) G is Morse of infinite index in $\mathrm{Mod}(S)$.

We remark that Russel, Spriano, and Tran [84] provided a characterization of Morse subgroups of hierarchically hyperbolic groups, which include mapping class groups. In particular, Corollary F in [84] implies our Theorem A. Their result, motivated in part by Theorem A, is proved by a rather different argument than our proof of Theorem A.

We also note that Genevois [38] recently characterized all right-angled Coxeter groups whose Morse subgroups are stable or coarsely cover the entire group by using defining graphs.

Chapter 5

Algorithms detecting stability and Morseness for finitely generated groups

In this chapter, we present various algorithms which, for a given finitely generated subgroup H of a particular group G, either detect (or completely decide) whether H is stable or Morse in G. We consider when G is a mapping class group, right-angled Artin group, and toral relatively hyperbolic group in this consequence.

5.1 Mapping class groups

For preliminaries about mapping class groups, see Section 4.1. In this section, we provide algorithms detecting stability and Morseness for mapping class groups by the curve graph of a mapping class group. Furthermore, for a closed hyperbolic surface, we also give an alternative proof by using a short exact sequence.

5.1.1 Curve graph

Let S be a surface as in Definition 4.2. We recall that curve graph $\mathcal{C}(S)$, one-skeleton of the curve complex, has vertices which are isotopy classes of essential simple closed curves on S, and two distinct isotopy classes are joined by an edge if they are disjointly realizable. It is known that the curve graph $\mathcal{C}(S)$ is a δ-hyperbolic space for some δ which can be chosen independent of S, see [14, 51, 3, 16, 23, 52].

We recall the following well-known fact about δ-hyperbolic spaces.

Lemma 5.1 (Local to global principle)**.** *Let X be a δ-hyperbolic geodesic space. For all integers $k \geq 1$, and $c \geq 0$ there exists K, k', and c' such that, if every length K segment of γ is a (k, c)-quasigeodesic, then γ is a (k', c')-quasigeodesic. Moreover, the constants K, k', and c' can be computed algorithmically from δ, k, and c.*

For the proof of Lemma 5.1, see Theorem 1.4 in Chapter 3 in [24]. Note that Theorem 1.4 in Chapter 3 in [24] does not say that k' and c' are algorithmically computed from δ, k, and c, but the proof provides the explicit bounds of k' and c' which are computable in terms of δ, k, and c.

Note that Leasure [72] found an algorithm to compute the distance between two vertices of $\mathcal{C}(S)$, and after that other algorithms were produced by Shackleton [90], Webb [94], Watanabe [93], and Birman, Margalit, and Menasco [13]. Bowditch [14] provided an algorithm to compute geodesics in $\mathcal{C}(S)$.

Theorem C. *Let S be an oriented, connected, finite type surface with $\chi(S) < 0$ which is neither the 1-punctured torus nor the 4-punctured sphere and let $\mathrm{Mod}(S)$ be the mapping class group of S.*

(1) There is a partial algorithm which, for a subgroup H of $\mathrm{Mod}(S)$ given by a finite generating set, will terminate if H is stable in $\mathrm{Mod}(S)$ and run forever if H is not stable in $\mathrm{Mod}(S)$.

(2) There is a complete algorithm which, for an undistorted subgroup H of $\mathrm{Mod}(S)$, decides whether or not H is stable.

(3) There is a partial algorithm which, for a finitely generated subgroup H of $\mathrm{Mod}(S)$ given by a finite generating set, will terminate if H is Morse in $\mathrm{Mod}(S)$ and run forever if H is not Morse in $\mathrm{Mod}(S)$.

First proof of Theorem C(1). Pick a base vertex x in the curve graph $\mathcal{C}(S)$. Suppose that H is a subgroup of $\mathrm{Mod}(S)$ generated by a finite set A in $\mathrm{Mod}(S)$ such that $A = A^{-1}$. For each $a \in A$, by using Bowditch's algorithm, compute a geodesic starting from x and terminating at $a \cdot x$ in the curve graph $\mathcal{C}(S)$ and denote it as u_a. If $a \cdot x = x$, we choose u_a to be a closed edge-path of length 2 from x to x in $\mathcal{C}(S)$. Let $f : \mathrm{Mod}(S) \to \mathcal{C}(S)$ be an orbit map sending ϕ to $\phi \cdot x$ for every $\phi \in \mathrm{Mod}(S)$. We can extend the orbit map f to the Cayley graph $\Gamma_{\mathrm{Mod}(S)}$ of $\mathrm{Mod}(S)$ by sending an edge $(\phi, \phi a)$ to the path $\phi \cdot u_a$ from $\phi \cdot x$ to $(\phi a) \cdot x$, where $a \in A$. Note that, by construction, f sends an edge path of length n in $\Gamma_{\mathrm{Mod}(S)}$ to an edge-path of length at least n in $\mathcal{C}(S)$. By Corollary B, the subgroup H is stable in $\mathrm{Mod}(S)$ if and only if there exist integers $k \geq 1, c \geq 0$ such that for every geodesic w in $\Gamma_{\mathrm{Mod}(S)}$ the image $f(w)$ is a (k, c)-quasigeodesic in $\mathcal{C}(S)$. This is also equivalent to the existence of k and c such that for every geodesic w starting from 1 in $\Gamma_{\mathrm{Mod}(S)}$ the image $f(w)$ is a (k, c)-quasigeodesic in $\mathcal{C}(S)$. The algorithm for detecting stability proceeds as follows.

Start enumerating all pairs of integers $k \geq 1, c \geq 0$. For every such pair (k, c), compute the constants K, $k' \geq 1$ and $c' \geq 0$ from Lemma 5.1. Using the solution of the word problem in $\mathrm{Mod}(S)$, list all the geodesic edge-paths w in $\Gamma_{\mathrm{Mod}(S)}$ of length at most K. For every such w, compute a path $f(w)$ and check whether or not $f(w)$ is a (k', c')-quasigeodesic in $\mathcal{C}(S)$. If the answer is 'yes' for all such w, we terminate the algorithm. Otherwise we proceed to the next pair of k and c.

This algorithm terminates if and only if H is stable. □

Proof of Theorem C(2). Recall that an element $\phi \in Mod(S)$ acts loxodromically on $\mathcal{C}(S)$ if and only if ϕ is pseudo-Anosov. Recall also that there exists an algorithm that, given $\phi \in Mod(S)$, decides whether or not ϕ is pseudo-Anosov [11].

For a given undistorted finitely generated subgroup H of $\mathrm{Mod}(S)$, we run the partial algorithm in Theorem C(1) and, in parallel, we start enumerating all elements of H and look for a non-loxodromic element of H. By Corollary B, either H is stable or else H contains a non-loxodromic element. If the algorithm in Theorem C(1) terminates on H, we declare that H is stable in $\mathrm{Mod}(S)$. If we find a non-loxodromic element in H, we declare that H is not stable in $\mathrm{Mod}(S)$.

This procedure produces a complete algorithm for deciding whether or not H is stable in $\mathrm{Mod}(S)$. □

Proof of Theorem C(3). Let H be a finitely generated subgroup of $\mathrm{Mod}(S)$ given by a finite generating set. By Corollary B, the subgroup H is Morse in $\mathrm{Mod}(S)$ if and only if either H is stable in $\mathrm{Mod}(S)$ or H has finite index in $\mathrm{Mod}(S)$.

We now run the partial algorithm for detecting Morseness in Theorem C(1), and, in parallel, run the Todd-Coxeter coset enumeration algorithm (see Chapter III, Section 12 in [73]) for detecting finiteness of the index of H in $\mathrm{Mod}(S)$. If the algorithm in Theorem C(1) terminates with the conclusion that H is stable, we declare that H is Morse in $\mathrm{Mod}(S)$. If H has finite index in $\mathrm{Mod}(S)$, the Todd-Coxeter algorithm eventually terminates and discovers this fact. In that case, we declare that H is Morse in $\mathrm{Mod}(S)$. Otherwise we continue running both the algorithm in Theorem C(1) and the Todd-Coxeter algorithm run forever.

Taken together, this procedure provides a partial algorithm for detecting Morseness of H in $\mathrm{Mod}(S)$, as required. □

Remark 5.2. At the moment, it is not known if there exists a complete algorithm that, given an undistorted finitely generated subgroup H of $\mathrm{Mod}(S)$, decides whether or not H has finite index in $\mathrm{Mod}(S)$. If such an algorithm is found, we can promote Theorem C(3) to a complete algorithm for deciding whether or not an undistorted subgroup of $\mathrm{Mod}(S)$ is Morse.

In Section 5.1.2, we provide another algorithm for Theorem C(1) when S is a closed hyperbolic surface.

5.1.2 Short exact sequence

In this section, we assume that S is a closed hyperbolic surface. For a finitely generated subgroup H of $\mathrm{Mod}(S)$ the *extension group* E_H is obtained from the short exact sequence $1 \to \pi_1(S) \to E_H \to H \to 1$ induced by the Birman exact sequence $1 \to \pi_1(S) \to \mathrm{Mod}(S \setminus \{p\}) \to \mathrm{Mod}(S) \to 1$ for a point $p \in S$. For a closed hyperbolic surface S, it is known that E_H is hyperbolic if and only if H is convex cocompact in $\mathrm{Mod}(S)$ [36, 50]. Suppose that we know a finite presentation of H. Then we can provide a presentation for E_H algorithmically as follows.

Lemma 5.3. *Let H be a finitely presented subgroup of $\mathrm{Mod}(S)$ for a closed hyperbolic surface S. Then the extension group E_H is finitely presented. Moreover, we can algorithmically find a finite presentation for E_H, given a finite generating set $Y \subseteq \mathrm{Mod}(S)$ for H and a finite presentation $H = \langle Y \mid Z \rangle$ for H.*

Proof. Let $\pi_1(S) = \langle X \mid R \rangle$ be a finite presentation of $\pi_1(S)$ and let $H = \langle Y \mid Z \rangle$ be a finitely presented subgroup of $\mathrm{Mod}(S)$. By the exactness of the sequence $1 \to \pi_1(S) \to E_H \to H \to 1$, the fundamental group $\pi_1(S)$ of S can be identified with a normal subgroup of E_H and the group homomorphism $\pi : E_H \to H$ is surjective. Pick a lift $\phi : Y \to E_H$ of the quotient map $\pi : E_H \to H$. For each $y \in Y$ denote $y' = \phi(y)$, and for every $z = y_1^{\varepsilon_1} \dots y_n^{\varepsilon_n} \in Z$ denote $z' = (y_1')^{\varepsilon_1} \dots (y_n')^{\varepsilon_n}$. Let $A := \{y', x \mid y \in Y, x \in X\}$. Since $\pi_1(S) = Ker(\pi)$ is normal in E_H, for every $x \in X, y \in Y, \varepsilon \in \{\pm 1\}$, there exists a word $u = u_{x,y,\varepsilon} \in F(X)$ such that $(y')^{\varepsilon} x (y')^{-\varepsilon} = u$ in E_H. Moreover, such u can be found algorithmically as follows. Start enumerating all words from $F(X)$ and, using the solution of the word problem in $\mathrm{Mod}(S)$, for each of these words check whether it is equal to $(y')^{\varepsilon} x (y')^{-\varepsilon}$ in E_H. Eventually this process stops and produces a desired word $u_{x,y,\varepsilon}$. For every $z \in Z$ there exists some word $v = v_z \in F(X)$ such that $z' = v$ in E_H and we can find such v algorithmically in a similar manner. By this procedure, we can compute the following finite set U algorithmically:

$$U := \{r,\, (y')^{\varepsilon} x (y')^{-\varepsilon},\, z' v_z^{-1} \mid r \in R,\, x \in X, y \in Y, \varepsilon \in \{\pm 1\}, z \in Z\}.$$

Note that the set U is contained in $F(A)$.

(i) Claim 1 : The group E_H is generated by the set A.

Let w be an element of E_H. Then $\pi(w) \in H$ and $\pi(w) = y_1^{e_1} y_2^{e_2} \dots y_n^{e_n}$ for some $y_i \in Y$ and $e_i \in \mathbb{Z}$. For each i, we have $\phi(y_i) = y_i' \in E_H$ and $\pi(w) = \pi(y_1')^{e_1} \pi(y_2')^{e_2} \dots \pi(y_n')^{e_n} = \pi((y_1')^{e_1} (y_2')^{e_2} \dots (y_n')^{e_n})$. Then we have $w((y_1')^{e_1} (y_2')^{e_2} \dots (y_n')^{e_n})^{-1} \in \ker(\pi) = \pi_1(S)$. Therefore, $w((y_1')^{e_1} (y_2')^{e_2} \dots (y_n')^{e_n})^{-1} =_{E_H} x_1^{f_1} x_2^{f_2} \dots x_m^{f_m}$ for some $x_j \in X$ and $f_i \in \mathbb{Z}$. Hence, $w =_{E_H} x_1^{f_1} x_2^{f_2} \dots x_m^{f_m} (y_1')^{e_1} (y_2')^{e_2} \dots (y_n')^{e_n}$ and therefore Claim 1 holds since w is arbitrary.

(ii) Claim 2 : The set U is a set of defining relations for E_H on A, that is, E_H has the presentation $E_H = \langle A \mid U \rangle$.

By construction, every element of U is a relation in E_H. Suppose now $w \in F(A)$ is such that $w = 1$ in E_H. Using the conjugation relations from U and pushing the letters from Y to the right, we can rewrite w in the form $w =_{E_H} u (y_1')^{e_1} \dots (y_n')^{e_n}$ for some word $u \in F(X)$, $y_i \in Y$, and $e_i \in \mathbb{Z}$. Since $\pi(w) = 1$ in H, we have $y_1^{e_1} \dots y_n^{e_n} = 1$ in H. Then we can reduce the word $y_1^{e_1} \dots y_n^{e_n}$ to the empty word in H using the relations from Z. Each application of a relation z from Z consists of replacing a subword of this relation by its complementary portion in z. Starting from the word $(y_1')^{e_1} \dots (y_n')^{e_n}$, for each such move we perform the corresponding move in $F(A)$ and replace a subword of z' by the complementary portion of $z'v_z^{-1}$ there, and then pushing all newly created letters from X to the left using the conjugation relations from U. Iterating this process, using the relations from U we can rewrite w as $w =_{E_H} uv$ where $v \in F(X)$. Then $uv =_{\pi_1(S)} 1$, and we can rewrite uv to the empty word using the relations from R. Therefore, $w \in \langle\langle U \rangle\rangle_{F(A)}$, as claimed.

It follows that $\langle A \,|\, U \rangle$ is indeed a finite presentation for E_H. □

For a closed surface S, possibly with a finite set of punctures, Mosher [78] showed that the mapping class group $\mathrm{Mod}(S)$ of S is automatic. For an automatic group G with an automatic structure (L, A) where L is a regular language in the free monoid A^* on a finite set A of semigroup generators for G, a subgroup H of G is called *L-rational* if its full preimage in L, $L \cap \pi^{-1}(H)$ where $\pi : A^* \to G$ is the natural monoid homomorphism, is a regular language. See [34] for further details. Gersten and Short [39] proved that a rational subgroup H of an automatic group G is finitely presented. The algorithm given by Kapovich [55], applied to such H, produces both a rationality constant for H in G and a finite presentation for H. Gersten and Short [39] show that the subgroup H is L-rational if and only if H is L-quasiconvex in G. Since Morseness implies quasiconvexity, a Morse subgroup H of an automatic group G is rational, and therefore the algorithm in [55] applied to H eventually terminates.

We are ready to give the second proof of Theorem C(1).

Theorem C. *Let S be an oriented, connected, finite type surface with $\chi(S) < 0$ which is neither the 1-punctured torus nor the 4-punctured sphere and let $\mathrm{Mod}(S)$ be the mapping class group of S.*

(1) There is a partial algorithm which, for a subgroup H of $\mathrm{Mod}(S)$ given by a finite generating set, will terminate if H is stable in $\mathrm{Mod}(S)$ and run forever if H is not stable in $\mathrm{Mod}(S)$.

***Second proof of Theorem C(1)** when S is a closed hyperbolic surface.* For a closed hyperbolic surface S, let $\langle X|R \rangle$ be the standard presentation for $\pi_1(S)$. Suppose that H is a finitely generated subgroup of $\mathrm{Mod}(S)$. We run the [55] procedure for detecting rationality of H in $\mathrm{Mod}(S)$. If it terminates, take a presentation for H obtained from this procedure, compute a presentation for E_H by using Lemma 5.3 and then run Papasoglu's algorithm [81] for detecting hyperbolicity on E_H. If Papasoglu's algorithm discovers that E_H is hyperbolic, terminate the entire algorithm and declare that H is stable in $\mathrm{Mod}(S)$.

It is known that E_H is hyperbolic if and only if H is convex cocompact [36, 50]. Hence, by Corollary B, the above algorithm terminates if and only if H is stable in $\mathrm{Mod}(S)$. □

5.2 Right-angled Artin groups

For the material in this section, see [21, 6] as background references.

Definition 5.4 (Right-angled Artin groups)**.** Let Γ be a finite simplicial graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma) \subset V \times V$. The *right-angled Artin group* on Γ has the finite group presentation:

$$A_\Gamma := \langle\ V(\Gamma) \mid vu = uv \text{ whenever } (v,u) \in E(\Gamma)\ \rangle.$$

For an induced subgraph Λ of Γ it is known that the subgroup of A_Γ generated by $V(\Lambda)$ is the right-angled Artin group on Λ, and we denote this subgroup $A_\Lambda \leqslant A_\Gamma$.

Definition 5.5. For two graphs Γ_1 and Γ_2, the *join* of Γ_1 and Γ_2 is a graph obtained by connecting every vertex of Γ_1 to every vertex of Γ_2 by an edge. If $|V(\Gamma_1)| = 1$ or $|V(\Gamma_2)| = 1$ then the join of Γ_1 and Γ_2 is called a *trivial join.* A graph Γ is *anti-connected* if Γ does not decompose as a nontrivial join. For an induced subgraph Λ of Γ which decomposes as a nontrivial join, the subgroup A_Λ is called a *join subgroup* of A_Γ.

Definition 5.6. Let Γ be a finite simplicial, connected, and anti-connected graph. An element g of A_Γ is called *loxodromic* if the element g is not conjugate into a join subgroup of A_Γ and *elliptic* otherwise.

Definition 5.7. Let Γ be a finite simplicial, connected, and anti-connected graph. A word w over the alphabet $V(\Gamma)^{\pm 1}$ is said to be in a *normal form* for A_Γ if w is freely reduced and does not contain any subwords of the form $x^\varepsilon u x^{-\varepsilon}$, where $x \in V(\Gamma), \varepsilon = \pm 1$, and where x commutes with every letter from u. A word w over the alphabet $V(\Gamma)^{\pm 1}$ is said to be in a *cyclically reduced normal form* for A_Γ if w is in a normal form for A_Γ and every cyclic permutation of w is in a normal form.

For a finite connected graph Γ, Kim and Koberda [69, 70] introduced the *extension graph* Γ^e of Γ. The vertex set of Γ^e is $\{v^g = g^{-1}vg \mid v \in V(\Gamma),\, g \in A_\Gamma\}$ and two distinct vertices u^g and v^h are adjacent if and only if they commute in A_Γ. For a finite and connected graph Γ, the extension graph Γ^e is a quasi-tree and thus a hyperbolic metric space [69]. We have the following characterizations of loxodromic elements in A_Γ, see [89, 6, 69, 70].

Theorem 5.8. *Let Γ be a finite simplicial, connected, and anti-connected graph with at least two vertices, and let g be an element of A_Γ. Then the following are equivalent:*

(1) The element g is loxodromic.

(2) The element g acts as a rank 1 isometry on the universal cover $\widetilde{S_\Gamma}$ of the Salvetti complex S_Γ.

(3) The centralizer $C_{A_\Gamma}(g)$ of g is infinite cyclic.

(4) The element g acts as a loxodromic isometry of the extension graph Γ^e.

(5) Some (equivalently, any) cyclically reduced normal form g is not in a join subgroup of A_Γ.

Note that Theorem 5.8(5) provides an algorithm that, given a word in the generators of A_Γ, decides whether or not this word represents a loxodromic element. Koberda, Mangahas, and Taylor [71] gave a characterization of stable subgroups of a right-angled Artin group as follows.

Theorem 5.9 (Theorem 1.1 in [71])**.** *Let Γ be a finite simplicial, connected, and anti-connected graph and let H be a finitely generated subgroup of A_Γ. Then the following are equivalent:*

(1) Some (any) orbit map from H into Γ^e is a quasi-isometric embedding.

(2) The subgroup H is stable in A_Γ.

(3) The subgroup H is purely loxodromic, i.e., every nontrivial element of H is loxodromic in A_Γ.

Tran [92] and Genevois [37] showed, independently, that stability and Morseness are equivalent for an infinite index subgroup of the right-angled Artin group A_Γ.

Theorem 5.10 (Theorem 1.16 in [92] or Theorem B.1 in [37])**.** *Let Γ be a finite simplicial, connected, and anti-connected graph. Let H be a finitely generated infinite index subgroup of A_Γ. Then H is stable in A_Γ if and only if H is Morse in A_Γ.*

We give algorithms detecting stability of right-angled Artin groups in Section 5.2.1 and Section 5.2.2 by star length and a cube complex. And then algorithms detecting Morseness of right-angled Artin groups are given at the end of Section 5.2.2.

5.2.1 Star Length

Recall that for a graph Γ, the *link* of a vertex v in Γ, denoted by $\mathrm{Lk}(v)$, is the set of the vertices in Γ which are adjacent to v and the *star* of v, denoted by $\mathrm{St}(v)$, is the union of $\mathrm{Lk}(v)$ and $\{v\}$. Kim and Koberda [70] defined the *star metric* on A_Γ and showed that the extension graph Γ^e with standard graph metric is quasi-isometric to the Cayley graph of A_Γ with the star metric.

Definition 5.11 (Star length)**.** For a right-angled Artin group A_Γ and an element $g \in A_\Gamma$ the *star length* of an element g is the minimum l such that g can be written as the product of l elements in $\bigcup_{v\in V(\Gamma)}\langle \mathrm{St}(v)\rangle$.

We denote the star length g by $|g|_*$. The star length induces a metric d_* on A_Γ by left invariance: $d_*(g,h) := |g^{-1}h|_*$.

Theorem 5.12 (Theorem 15 in [70]). *Let Γ be a finite connected graph. The metric spaces (A_Γ, d_*) and (Γ^e, d_{Γ^e}) are quasi-isometric, and the orbit map $A_\Gamma \to \Gamma^e$, $g \mapsto v^g = g^{-1}vg$ (where v is a base-vertex of Γ^e) is a quasi-isometry.*

Definition 5.13. Let Γ be a simplicial, finite, connected, and anti-connected graph. A product $g = g_1 \dots g_k$ is called a *star-geodesic* if $g_i \in \mathrm{St}(v_i)$ for $i = 1, \dots, k$ where $v_i \in V(\Gamma)$ and $k = |g|_*$. A word w over $V(\Gamma)^{\pm 1}$ is called a *star-block* if there is some $v \in V(\Gamma)$ such that every letter of w belongs to $\mathrm{St}(v)$.

The following lemma allows us to compute the star length for an element in A_Γ by using its normal form. Note that this fact can also be derived from Lemma 20 in [70].

Lemma 5.14. *Let Γ be a finite simplicial, connected, and anti-connected graph. For a nontrivial element $g \in A_\Gamma$ the star length $|g|_*$ is equal to the smallest k such that there exists a word $w = w_1 \dots w_k$ representing g such that each w_i is a nontrivial star-block and that w is in a normal form for A_Γ.*

Proof. Take an element $1 \neq g \in A_\Gamma$, and put $k = |g|_*$. Now look at all representations of g as $g = w_1 \dots w_k$ where each w_i is a star-block word and among them choose a representation $w = w_1 \dots w_k$ with $|w| = \sum_{i=1}^{k} |w_i|$ minimal. Note that this choice of w implies that for all $1 \leq i < j \leq k$, $|w_i \dots w_j|_* = j - i + 1$.

We claim that this representation $g = w_1 \dots w_k$ is in a normal form for A_Γ. Note that the minimality assumption on $|w|$ implies that w is a freely reduced word. By contradiction, suppose that w is not in a normal form for A_Γ. Then w contains a subword of the form xux^{-1} where x or x^{-1} is a vertex of Γ and u is a nontrivial word and where x commutes with every letter from u. For simplicity, say $x \in V(\Gamma)$. Since all letters from u commute with x, we have $u \in \langle \mathrm{Star}(x) \rangle$. We take an innermost xux^{-1} of this type so that u is in normal form. Since we chose w with $\sum_{i=1}^{k} |w_i|$ minimal, if x occurs in w_i, then x^{-1} comes from w_j with $i < j$. There are several cases to consider:

(i) The letter x occurs in w_i and x^{-1} occurs in w_j where $j \geq i + 3$.

In this case, w_{i+1} and w_{i+2} are both subwords of u in xux^{-1}. Since u commutes with x letterwise, the words w_{i+1} and w_{i+2} can be viewed as words in the generators of $\mathrm{Star}(x)$. Then g can be expressed as a product of strictly less than k star-block words, which contradicts $|g|_* = k$. Hence, this case does not happen.

(ii) The letter x occurs in w_i and x^{-1} occurs in w_{i+2}.

The entire w_{i+1} is a subword of u, and we can view w_{i+1} as a word in the generators of $\mathrm{Star}(x)$. We then rewrite $w_i w_{i+1} w_{i+2} = w'_i w'_{i+1} w'_{i+2}$, where $w_i = w'_i x$, $w'_{i+1} = u$, $w_{i+2} = x^{-1} w'_{i+2}$. We thus get a star-decomposition of g as $w' = w_1 \dots w'_i w'_{i+1} w'_{i+2} \dots w_k$ with a smaller $|w'|$ than $|w|$, which contradicts the choice of w.

(iii) The letter x occurs in w_i and x^{-1} occurs in w_{i+1}.

In this case, $w_i = zxu_1$, and $w_{i+1} = u_2 x^{-1} y$ for some reduced words u_1 and u_2. Since $u_1, u_2 \in \langle \mathrm{Star}(x) \rangle$, $w_i =_{A_\Gamma} zu_1 x$ and $w_{i+1} =_{A_\Gamma} x^{-1} u_2 y$. Then g is equal to the word $w' = w_1 \dots w_{i-1}(zu_1)(u_2 y) w_{i+2} \dots w_k$ in A_Γ. Note that zu_1 and $u_2 y$ are star-block words. Hence, $|w'| < |w|$ which contradicts the minimal choice of w.

The claim now directly implies the statement of the lemma. □

Lemma 5.14 allows us to have the following algorithm.

Corollary 5.15. *Let Γ be a finite simplicial, connected, and anti-connected graph. There exists an algorithm that, given a word w in the generators of A_Γ representing an element g, computes $|g|_*$ and produces a star-geodesic representative of g.*

Proof. Given a word w in the generators of A_Γ, compute all the normal forms of the element g represented by w. Note that these normal forms can be computed algorithmically [31] and that they all are related to each other by a process of shuffling the letters using the commutativity relations in A_Γ [53]. Among these normal forms, find the one which decomposes as a product of the smallest number of star-block words. Lemma 5.14 implies that this numbers equals $|g|_*$. □

Theorem D. *Let Γ be a finite connected and anti-connected graph with at least two vertices and let A_Γ be the corresponding right-angled Artin group.*

(1) There is a complete algorithm which, for a subgroup H of A_Γ given by a finite generating set, will terminate and determine whether or not H is stable in A_Γ.

First proof of Theorem D(1). Pick a base-vertex $v \in V(\Gamma)$. We also regard v as a base-vertex of the extension graph Γ^e. Suppose that H is a finitely generated subgroup of A_Γ given by a finite generating set S in A_Γ such that $S = S^{-1}$. Let $f : A_\Gamma \to \Gamma^e$ be an orbit map sending g to v^g for every $g \in A_\Gamma$. We can extend f to the Cayley graph Γ_{A_Γ} of A_Γ sending an edge (g, gs) to (v^g, v^{gs}). By Theorem 5.12, $f : (A_\Gamma, d_*) \to \Gamma^e$ is a quasi-isometry. Let $g : (\Gamma_{A_\Gamma}, d) \to (\Gamma_{A_\Gamma}, d_*)$ be the identity map, where d is the word metric on Γ_{A_Γ}. Note that g sends a geodesic of length n in (Γ_{A_Γ}, d) to a path of d_*-distance between its endpoints at most n

in (Γ_{A_Γ}, d_*). By Theorem 5.9, the subgroup H is stable in A_Γ if and only if there exist integers $k \geq 1, c \geq 0$ such that for every geodesic w in (Γ_{A_Γ}, d) the image $f \cdot g(w)$ is (k, c)-quasigeodesic in Γ^e. By Theorem 5.12, this condition is equivalent to the existence of $k' \geq 1, c' \geq 0$ such that for every geodesic w starting from 1 in (Γ_{A_Γ}, d) the image $g(w) = w$ is (k', c')-quasigeodesic in (Γ_{A_Γ}, d_*). The algorithm for deciding stability is as follows.

Start enumerating all pairs of integers $k \geq 1, c \geq 0$. For every such pair k, c, compute the constants K, $k' \geq 1$ and $c' \geq 0$ from Lemma 5.1. Using the solution of the word problem in A_Γ, list all the geodesic edge-paths w in (Γ_{A_Γ}, d) of length at most K. For every such w, check whether or not w is a (k', c')-quasigeodesic in (Γ_{A_Γ}, d_*) by Lemma 5.15. If the answer is 'yes' for all such w, we terminate the algorithm and declare that H is stable in A_Γ. Otherwise we proceed to the next pair of k and c. This algorithm terminates if and only if H is stable. In parallel, we keep enumerating elements of H, find their cyclically reduced forms and, using Theorem 5.8 check whether the element of H under consideration is loxodromic. If we find a non-loxodromic element of H, we terminate the algorithm and declare that H is not stable by Theorem 5.9.

Taken together, this procedure provides a complete algorithm for detecting stability of H in A_Γ. □

5.2.2 Cube complex

In this section, we use [49, 74] as background references. Let Γ be a finite simplicial graph. Recall the construction of the Salvetti complex S_Γ for A_Γ. Begin with a wedge of circles attached to a vertex x_0 and labeled by the elements of $V(\Gamma)$. For each edge, say from v_i to v_j in Γ, attach a 2-torus with boundary labeled by the relator $v_i^{-1}v_j^{-1}v_iv_j$. For each triangle in Γ connecting three vertices v_i, v_j, v_k, attach a 3-torus with faces corresponding to the tori for the three edges of the triangle. Continue this process, attaching a k-torus for each set of k mutually commuting generators. The resulting space is called the *Salvetti complex* S_Γ for A_Γ. Note that $\pi_1(S_\Gamma, x_0) = A_\Gamma$ and that the universal cover $\widetilde{S_\Gamma}$ of S_Γ is a CAT(0) cube complex.

Lemma 5.16 (Lemma 3.5 in [74]). *Let Γ be a finite simplicial graph and let H be a quasiconvex subgroup of A_Γ with respect to the standard generators of A_Γ. Then there exists a pointed finite connected cube complex (C, x) and a cubical local isometry $\phi : (C, x) \to (S_\Gamma, x_0)$ with $H = \phi_*(\pi_1(C, x))$.*

Lemma 5.17 (Lemma 3.6 in [74]). *Let Γ be a finite simplicial graph and let H be a quasiconvex subgroup of A_Γ with respect to the standard generators of A_Γ. Let (C, x) be the pointed finite connected cube complex as in Lemma 5.16. Then oriented edge-loops at x which are combinatorial local geodesics are in bijective correspondence with minimal length words in H with respect to the standard generators of A_Γ.*

Lemma 5.18. *Let Γ be a finite simplicial graph and let H be a quasiconvex subgroup of A_Γ with respect to* *the standard generators of A_Γ. Let (C, x) be the pointed finite connected cube complex as in Lemma 5.16. Then*

(i) for an infinite order element $h \in H$, there exists a cyclically reduced normal form w of h such that w corresponds to a nontrivial closed loop based at some vertex (not necessarily equal to x) of C, and

(ii) the subgroup H contains a non-loxodromic element if and only if there is a nontrivial simple loop in C whose label is a join word.

Proof. Let h be an infinite order element in H. By Lemma 5.17, a minimal length representative w of h corresponds to an edge-loop based at x which is a combinatorial local geodesic in C. If w is cyclically reduced then we are done. Suppose now that w is not cyclically reduced. Then there exists a normal form w' of h which has the form $w' = u(v_i \dots v_j)u^{-1}$ where u is nontrivial and $v_i \dots v_j$ is a cyclically reduced normal form of h, with $v_i \in V(\Gamma)^{\pm}$. Note that, by Lemma 5.16, the map from the complex C to the Salvetti complex S_Γ is an immersion so that the 1-skeleton of C is a folded graph in the free group sense, see [58] for more details. That guarantees that there is at most one path starting at the base point x and labeled by u. The end point of this path is a vertex y of C and the portion of w corresponding to $v_i \dots v_j$ gives a loop in C based at y and labeled by $u_i \dots u_j$. This completes the proof of part (i).

For part (ii), first suppose that there is a simple closed loop based at a vertex y in C whose label is a join word. Let u be the label of the path starting from x terminating at y and let v be the label of the simple closed loop. Since $\pi_1(C, x) = H$, the non-loxodromic element uvu^{-1} is in H. Suppose now that H contains a nontrivial non-loxodromic element h. Then by part (i) there exists a nontrivial loop α in C at some vertex y with the label w of α being a cyclically reduced normal form of h. Then w is a join word, and every subword of w is a join word. We can find a nontrivial subpath β of α such that β is a simple closed loop at some vertex of C. The label of β is a subword of w and thus is a join word. This completes the proof of part (ii). □

Theorem D. *Let Γ be a finite connected and anti-connected graph with at least two vertices and let A_Γ be the corresponding right-angled Artin group.*

(1) There is a complete algorithm which, for a subgroup H of A_Γ given by a finite generating set, will terminate and determine whether or not H is stable in A_Γ.

(2) There is a partial algorithm which, for a subgroup H of A_Γ given by a finite generating set, will terminate if H is Morse in A_Γ and run forever if H is not Morse in A_Γ.

Second proof of Theorem D(1). Let H be a finitely generated subgroup of A_Γ given by a finite generating set S. Start enumerating candidate finite base-pointed connected cube complexes (C, x) admitting cubical locally isometric maps to S_Γ. For each such (C, x), pick a finite generating set Y for the fundamental group $\pi_1(C^{(1)}, x)$ of the 1-skeleton of C. Start enumerating all words in $S^{\pm 1}$ and the labels of all loops in $C^{(1)}$ based at x. Then check, using the word problem for A_Γ, whether elements of S all appear in the second list and whether all elements of Y appear in the first list. Suppose that we find such (C, x) with $\pi_1(C, x) = H$. Now check all finite simple loops in $C^{(1)}$. If all such loops are labeled by not a join word then we declare that H is stable in A_Γ, and otherwise we declare that H is not stable. In parallel, with the above process involving cube complexes (C, x), we keep enumerating elements of H and check them for being non-loxodromic. If there is a non-loxodromic element in H, then we declare that H is not stable.

Note that if the subgroup H is stable, then Lemma 5.16 guarantees that we eventually find such (C, x) where we check the existence of a non-loxodromic element of H by Lemma 5.18(2). Note also that by Theorem 5.9 the subgroup H is stable if and only if H is purely loxodromic. Therefore, this procedure gives a complete algorithm for deciding whether or not H is stable in A_Γ. □

Proof of Theorem D(2). Let H be a finitely generated subgroup of A_Γ given by a finite generating set S. We first run one of the complete algorithms in Theorem D(1) for deciding stability of H in A_Γ. If the algorithm terminates with the conclusion that H is stable, then we declare that H is Morse in A_Γ. Suppose now that the algorithm in Theorem D(1) terminates with the conclusion that H is not stable in A_Γ. We then keep running the Todd-Coxter algorithm on H, for detecting finiteness of the index of H in A_Γ. If the Todd-Coxeter algorithm terminates then we declare that H is Morse in A_Γ. Otherwise we continue running the Todd-Coxeter algorithm forever.

Note that the subgroup H is Morse in A_Γ if and only if either H is stable or H has finite index in A_Γ by Theorem 5.10. Hence, taken together, the above procedure provides a partial algorithm for detecting Morseness of H in A_Γ. □

Remark 5.19. Similar to mapping class groups, there is no known algorithm which, given a finitely generated subgroup H of A_Γ, decides whether H has infinite index in A_Γ. If such an algorithm is found, we can improve Theorem D(2) to a complete algorithm deciding Morseness of H in A_Γ.

5.3 Toral relatively hyperbolic groups

There are various definitions of a relatively hyperbolic group, see [79, 54] for more details. In this paper we use the following definition of relative hyperbolicity, due to Bowditch [15].

Definition 5.20 (Relatively hyperbolic groups)**.** Let G be a finitely generated group and let $\mathbb{P}$ be a (possibly empty) finite collection of finitely generated subgroups of G. Suppose that G acts on a δ-hyperbolic graph K with finite edge stabilizers and finitely many orbits of edges (and hence also of vertices). Suppose that each edge of K is contained in only finitely many circuits of length n for each integer n, and that $\mathbb{P}$ is a set of representatives of the conjugacy classes of infinite vertex stabilizers. Then $(G, \mathbb{P})$ is a *relatively hyperbolic* group with respect to $\mathbb{P}$. An element P of $\mathbb{P}$ is called a *peripheral* subgroup of G.

For a relatively hyperbolic group $(G, \mathbb{P} = \{P_1, \dots, P_n\})$, we allow the case $n = 0$, in which situation the family $\mathbb{P}$ is empty and the group G is word-hyperbolic.

Definition 5.21. Let $(G, \mathbb{P})$ be a relatively hyperbolic group. An element $g \in G$ is *elliptic* if it has finite order, *parabolic* if it has infinite order and is conjugate to an element of some $P \in \mathbb{P}$, and *hyperbolic* or *loxodromic* otherwise. A subgroup H of G is *elliptic* if it is finite, *parabolic* if it is infinite and contained in a conjugate of a peripheral subgroup $P \in \mathbb{P}$, and *hyperbolic* otherwise.

The notion of relatively quasiconvex subgroups plays an important role in the theory of relatively hyperbolic groups. Note that there are several definitions of relative quasiconvexity, and they are equivalent for a finitely generated relatively hyperbolic group [54]. Recall that for a group G with a collection of subgroups $\mathbb{P} = \{P_1, \dots, P_n\}$, a subset S of G is called *relative generating set* for the pair $(G, \mathbb{P})$ if the set $S \cup P_1 \cup \cdots \cup P_n$ generates G.

Definition 5.22 (Relatively quasiconvex subgroups)**.** Let $(G, \mathbb{P})$ be a relatively hyperbolic group. A subgroup H of G is *relatively quasiconvex* if the following holds. Let S be some (any) finite relative generating set for $(G, \mathbb{P})$, and let $\mathcal{P}$ be the union of all $P \in \mathbb{P}$. Consider the Cayley graph $\Gamma = \mathrm{Cayley}(G, S \cup \mathcal{P})$ with all edges of length one. Let d be some (any) proper, left invariant metric on G. Then there is a constant $k = k(S, d)$ such that for each geodesic c in Γ connecting two points of H, every vertex of c lies within a d-distance k of H.

It is known that every undistorted finitely generated subgroup of a relatively hyperbolic group $(G, \mathbb{P})$ is relatively quasiconvex in G [54]. Since stable and Morse subgroups are undistorted in G, stable and Morse subgroups of G are relatively quasiconvex. Also, each peripheral subgroup $P \in \mathbb{P}$ is relatively quasiconvex in G since $P \in \mathbb{P}$ is Morse in G [32]. For an undistorted subgroup H of G, Tran [92] gave the following complete characterizations of stability and Morseness of H. Recall our convention regarding denoting conjugates of elements and of subgroups for a group G: $x^g = g^{-1}xg$, $H^g = g^{-1}Hg$ for $x, g \in G$ and $H \leqslant G$.

Theorem 5.23 (Theorem 1.9 in [92])**.** *Let $(G, \mathbb{P})$ be a finitely generated relatively hyperbolic group and let H be an undistorted finitely generated subgroup of G. Then the following are equivalent:*

(1) The subgroup H is Morse in G.

(2) The subgroup $H \cap P^g$ is Morse in P^g for each conjugate P^g of a peripheral subgroup in $\mathbb{P}$.

(3) The subgroup $H \cap P^g$ is Morse in G for each conjugate P^g of a peripheral subgroup in $\mathbb{P}$.

Corollary 5.24 (Corollary 1.10 in [92])**.** *Let $(G, \mathbb{P})$ be a finitely generated relatively hyperbolic group and let H be an undistorted finitely generated subgroup of G. Then the following are equivalent:*

(1) The subgroup H is stable in G.

(2) The subgroup $H \cap P^g$ is stable in P^g for each conjugate P^g of a peripheral subgroup in $\mathbb{P}$.

(3) The subgroup $H \cap P^g$ is stable in G for each conjugate P^g of a peripheral subgroup in $\mathbb{P}$.

We now concentrate on a particular type of a relatively hyperbolic group, namely a toral relatively hyperbolic group. A relatively hyperbolic group $(G, \mathbb{P})$ is called *toral* if G is torsion-free and the elements of $\mathbb{P}$ are finitely generated free abelian non-cyclic subgroups of G.

Lemma 5.25. *Let $(G, \mathbb{P})$ be a relatively hyperbolic group where every $P \in \mathbb{P}$ is finitely generated abelian. Then every relatively quasiconvex subgroup of G is undistorted.*

Proof. Let H be a relatively quasiconvex subgroup of G. It is known that the distortion of H in G is a combination of the distortions of the infinite subgroups $H^g \cap P$ of $P \in \mathbb{P}$, see Theorem 1.4 in [54]. Since a peripheral subgroup $P \in \mathbb{P}$ is finitely generated abelian, the distortion of any infinite subgroup of P is linear. Hence, the distortion of H in G is linear, that is, H is undistorted in G. ☐

5.3.1 Relatively quasiconvex subgroups with peripherally finite index

For the remainder of this section, except for Proposition 5.48 and Corollary F, we assume that $(G, \mathbb{P})$ is a toral relatively hyperbolic group where the finite family $\mathbb{P}$ of free abelian non-cyclic groups is nonempty. Also note that in the case where $\mathbb{P}$ is empty, G is torsion-free word hyperbolic. In this case, for a finitely generated subgroup H of G, being Morse is equivalent to being stable, which is equivalent to being quasiconvex, and also equivalent to being undistorted. The conclusions of Theorem E then follow from Proposition 2.29.

Note that for a finitely generated free abelian non-cyclic group P, the only stable subgroup of P is trivial, and a Morse subgroup is either trivial or has finite index in P. Hence, Theorem 5.23 and Corollary 5.24 imply the following:

Corollary 5.26. *Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group and let be H an undistorted finitely generated subgroup of G.*

(i) The subgroup H is stable in G if and only if $H \cap P^g$ is trivial for each conjugate P^g of a peripheral subgroup in $\mathbb{P}$.

(ii) The subgroup H is Morse in G if and only if $H \cap P^g$ either is trivial or has finite index in P^g for each conjugate P^g of a peripheral subgroup in $\mathbb{P}$.

Corollary 5.26 says that we only need to check the intersection $H \cap P^g$ for each conjugate P^g of peripheral subgroup in $\mathbb{P}$ to detect stability or Morseness of H in G. Kharlampovich, Miasnikov, and Weil [63] provided a partial algorithm for computing the intersection of two given relatively quasiconvex subgroups with "peripherally finite index" of a toral relatively hyperbolic group $(G, \mathbb{P})$.

Definition 5.27 (Peripherally finite index)**.** A subgroup H of a finitely generated relatively hyperbolic group $(G, \mathbb{P})$ has *peripherally finite index* in G, if, for each peripheral subgroup $P \in \mathbb{P}$ and each $g \in G$, the subgroup $H^g \cap P$ is either finite or has finite index in P.

Definition 5.28. A subgroup P of a group G is called *almost malnormal* if for every $g \in G \setminus P$ the intersection $P \cap P^g$ is finite. A family $\{P_1, \ldots, P_k\}$ of subgroups of G is called almost malnormal if whenever $g \in G, P_i$, and P_j are such that $P_i \cap P_j^g$ is infinite then $i = j$ and $g \in P_i$ (so that $P_i = P_j^g$).

For a relatively hyperbolic group $(G, \mathbb{P})$ it is known that the family $\mathbb{P}$ is almost malnormal in G. Hence, every peripheral subgroup $P \in \mathbb{P}$ has peripherally finite index in G.

Theorem 5.29. *Let $(G, \mathbb{P})$ be relatively hyperbolic where every $P \in \mathbb{P}$ is finitely generated abelian. Let H be a relatively quasiconvex subgroup of G. Then:*

(i) Every infinite parabolic subgroup of H is contained in a unique maximal parabolic subgroup of H.

(ii) There are only finitely many H-conjugacy classes of maximal infinite parabolic subgroups of H.

Theorem 5.29 follows from Proposition 7.19 in [63] because every $P \in \mathbb{P}$ is relatively quasiconvex. Note that Proposition 7.19 in [63] assumes that G is a toral relatively hyperbolic group but the proof also works for a relatively hyperbolic group where all peripheral subgroups are finitely generated abelian. Also note that in a relatively hyperbolic group $(G, \mathbb{P})$, every infinite parabolic subgroup Q of G is contained in a unique conjugate P^g of some $P \in \mathbb{P}$.

Definition 5.30. Let $(G, \mathbb{P})$ be a relatively hyperbolic group where all peripheral subgroups are finitely generated abelian. Let H be a relatively quasiconvex subgroup of G and let $\mathbb{D}$ be a collection of representatives of H-conjugacy classes of maximal infinite parabolic subgroups of H. The collection $\mathbb{D}$ is called the *induced peripheral structure* for H from $(G, \mathbb{P})$.

Note that if $(G, \mathbb{P}), (H, \mathbb{D})$ are as in Definition 5.30, and if $g \in G, P \in \mathbb{P}$ are such that $H \cap P^g$ is infinite then $H \cap P^g$ is conjugate in H to some $D \in \mathbb{D}$. That is, $\mathbb{D}$ is the set of representatives of H-conjugacy classes of infinite subgroups of H of the form $H \cap P^g$ where $P \in \mathbb{P}$ and $g \in G$. Moreover, the collection $\mathbb{D}$ is finite and $(H, \mathbb{D})$ is relatively hyperbolic [48]. Note that the subgroup H has peripherally finite index if and only if for every $D \in \mathbb{D}$, whenever $D \leq P^g$, $P \in \mathbb{P}$ then D has finite index in P^g.

Remark 5.31. Let $(G, \mathbb{P})$ be a relatively hyperbolic group where all peripheral subgroups are finitely generated abelian and let H be a relatively quasiconvex subgroup of G with induced peripheral structure $\mathbb{D}$. Groves and Manning [48] gave a definition of relatively quasiconvexity of H which is equivalent to Definition 5.22, and their work implies that a peripheral structure $\mathbb{D}$ on H compatible with P is unique in the following sense (see Definition 2.9 and the following paragraph in [48]). Suppose that $\mathbb{D}'$ is a finite family of infinite subgroups D' of H such that each D' is infinite parabolic in G and such that $(H, \mathbb{D}')$ is relatively hyperbolic. Then there exists a bijective correspondence between families $\mathbb{D}$ and $\mathbb{D}'$ such that if D is sent to D' under this correspondence then for some $h \in H$ the subgroups $D^h = D'$.

Theorem 5.32. *Let $(G, \mathbb{P})$ be a relatively hyperbolic group such that all $P \in \mathbb{P}$ are finitely generated abelian. Then there is a partial algorithm which, given a finite tuple generating a subgroup H of G,*

- *halts if and only if the subgroup H is relatively quasiconvex with peripherally finite index;*
- *when it halts, computes (by producing their finite generating sets) a family $\mathbb{D}$ as in Definition 5.30.*

Theorem 5.32 follows from Proposition 7.20 in [63] because every peripheral subgroup $P \in \mathbb{P}$ is relatively quasiconvex with peripherally finite index. Note that Proposition 7.20 in [63] assumes that G is a toral relatively hyperbolic group but their proof also works for a relatively hyperbolic group where all peripheral subgroups are finitely generated abelian. Moreover, Corollary 7.9 in [63] implies the following proposition that allows us, in particular, to find a finite generating set for an element of such $\mathbb{D}$ as in Theorem 5.32.

Proposition 5.33. *Let $(G, \mathbb{P})$ be a relatively hyperbolic group such that all $P \in \mathbb{P}$ are finitely generated abelian. Then there is a partial algorithm which, given finite generating sets for relatively quasiconvex subgroups H, K, halts if both H and K have peripherally finite index in G and then computes a finite generating set for $H \cap K$, and runs forever otherwise.*

We are ready to prove (1), (2), and (3) in Theorem E.

Theorem E. *Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group.*

(1) There is a partial algorithm which, for a subgroup H of G given by a finite generating set, will terminate if H is stable in G and run forever if H is not stable in G.

(2) *There is a complete algorithm which, for an undistorted subgroup H of G, decides whether or not H is stable.*

(3) *There is a partial algorithm which, for a subgroup H of G given by a finite generating set, will terminate if H is Morse in G and run forever if H is not Morse in G.*

Proof of Theorem E(1). Let H be a finitely generated subgroup of a toral relatively hyperbolic group $(G, \mathbb{P})$ given by a finite generating set of H. We run the partial algorithm in Theorem 5.32 on the subgroup H. Suppose that the partial algorithm terminates, determines that H is relatively quasiconvex of peripherally finite index in $(G, \mathbb{P})$, and computes such a family $\mathbb{D}$ as in Definition 5.30. If the family $\mathbb{D}$ is empty then we declare that H is stable in G.

Note that if the subgroup H is stable in G, then H is relatively quasiconvex and has peripherally finite index in G by Corollary 5.26. Therefore, if the subgroup H is stable, the partial algorithm in Theorem 5.32 for H eventually terminates. Conversely, by Lemma 5.25 and Corollary 5.26, if this algorithm terminates then H is stable in G. Thus the above procedure does detect stability of H in G as, required. □

Proof of Theorem E(2). Let H be an undistorted subgroup of a toral relatively hyperbolic group $(G, \mathbb{P})$. We run the partial algorithm in Theorem E(1). If the algorithm halts then we declare that H is stable in G. In parallel, we enumerate elements of H in G, enumerate conjugates of elements of P for each $P \in \mathbb{P}$, and look for an infinite order element in some $H \cap P^g$. If we find an infinite order element, then we declare that H is not stable. By Corollary 5.26, this procedure decides whether or not H is stable in G. □

Proof of Theorem E(3). Let H be a finitely generated subgroup of a toral relatively hyperbolic group G. We run the partial algorithm in Theorem 5.32 on H. Suppose that the partial algorithm terminates, determines that H is relatively quasiconvex and peripherally of finite index in $(G, \mathbb{P})$, and computes such a family $\mathbb{D}$ as in Definition 5.30. If the family $\mathbb{D}$ is empty we declare that H is Morse. Otherwise, for each infinite subgroup $U = H \cap P^g$ in the collection $\mathbb{D}$, compute the index of U in the finitely generated free abelian group P^g. If all such subgroups U have finite index in the corresponding P^g, we declare that H is Morse in G and terminate the procedure.

Note that if the subgroup H is Morse in G, then H is relatively quasiconvex, and has peripherally finite index by Corollary 5.26. Therefore, if H is Morse, the partial algorithm in Theorem 5.32 for H eventually terminates. Conversely, by Lemma 5.25 and Corollary 5.26, if the above procedure terminates then H is Morse in G. Thus, this partial algorithm detects Morseness of H in G, as required. □

5.3.2 Dehn fillings of relatively hyperbolic groups

To improve Theorem E(3) to a complete algorithm deciding whether or not an undistorted subgroup H is Morse, that is, to prove Theorem E(4), we need to be able to decide whether or not H has peripherally finite index in G. For solving this problem, we use the algorithms given by Kharlampovich, Miasnikov, and Weil [63] combined with the Groves and Manning's result [48] on relatively hyperbolic Dehn fillings. Before stating Groves and Manning's result on relatively hyperbolic Dehn fillings, we recall some definitions, see [80, 47, 48] for further details.

Definition 5.34 (Dehn fillings)**.** Let $(G, \mathbb{P})$ be a relatively hyperbolic group. A *Dehn filling* of $(G, \mathbb{P})$ is the quotient $G/\langle\langle \bigcup N_P \rangle\rangle$ determined by normal subgroups $N_P \trianglelefteq P \in \mathbb{P}$, together with the quotient map $\pi : G \to \bar{G}$, where we denote the quotient $G/\langle\langle \bigcup N_P \rangle\rangle$ by $\bar{G}$. We say that $G/\langle\langle \bigcup N_P \rangle\rangle$ is the *Dehn filling determined by* $(N_P)_{P\in\mathbb{P}}$.

Definition 5.35. Let G be a relatively hyperbolic group and let $\pi : G \to \bar{G}$ be a Dehn filling of G with $\bar{G} = G/\langle\langle \bigcup N_P \rangle\rangle$. For a finite subset $Z \subset \bigcup_{P\in\mathbb{P}} P \setminus \{1\}$, we say that a Dehn filling is *Z-long* if $N_P \cap Z = \emptyset$ for all $P \in \mathbb{P}$. We say that a statement holds for *all sufficiently long fillings* if there exists a finite set Z such that the statement holds for all Z-long fillings.

Osin [80] and Groves and Manning [47] proved, independently, that for a relatively hyperbolic group G, there exists a finite set $Z = Z(G)$ such that any Z-long Dehn filling $\bar{G}$ with $N_P \cap Z = \emptyset$ for all $P \in \mathbb{P}$ is again a relatively hyperbolic group:

Theorem 5.36. *[80, 47] Let $(G, \mathbb{P})$ be a relatively hyperbolic group. There exists a finite $Z \subset (\bigcup_{P\in\mathbb{P}} P) \setminus \{1\}$ such that for every Z-long filling $\pi : G \to \bar{G} = G/\langle\langle \bigcup N_p \rangle\rangle$ we have*

1. *for each $N_p \trianglelefteq P$, the Dehn filling π induces an embedding of P/N_p in $\bar{G}$ whose image we identify with P/N_p,*
2. *$(\bar{G}, \{\, P/N_p \mid P \in \mathbb{P} \,\})$ is relatively hyperbolic,*

For a relatively hyperbolic group G and a relatively quasiconvex subgroup H of G, Groves and Manning [48] proposed the notion of H-wide fillings, and studied the behavior of H under sufficiently long and H-wide fillings.

Definition 5.37. Let P be a group, let B be a subgroup of P, and let S be a finite set. A normal subgroup N of P is *(B, S)-wide* in P if whenever $b \in B$, $s \in S$ are such that $bs \in N$, then $s \in B$.

Definition 5.38. Let $(G, \mathbb{P})$ be relatively hyperbolic and let $(H, \mathbb{D})$ be a relatively quasiconvex subgroup with the induced peripheral structure $\mathbb{D}$ as in Definition 5.30. Then for any $D \in \mathbb{D}$ there exists $P_D \in \mathbb{P}$ and $c_D \in G$ so that $D \leq P_D^{c_D}$. Let $S \subset (\bigcup_{P \in \mathbb{P}} P) \setminus \{1\}$. A Dehn filling $\pi : G \to \bar{G}$ determined by $(N_{P_D})_{P_D \in \mathbb{P}}$ of G is *(H, S)-wide* if for any $D \in \mathbb{D}$ the normal subgroup N_{P_D} is $(D^{{c_D}^{-1}}, S \cap P_D)$-wide in P_D.

We remark that Definition 5.38 depends on the choice of P_D and c_D for each $D \in \mathbb{D}$. Note that is $S \subseteq S'$ and a filling is (H, S')-wide, then this filling is also (H, S)-wide.

Definition 5.39. Let $(G, \mathbb{P})$ be relatively hyperbolic and let H be a relatively quasiconvex subgroup. We say that a property holds for *all sufficiently long and H-wide fillings* if there is a finite set $S \subset (\bigcup_{P \in \mathbb{P}} P) \setminus \{1\}$ so that the property holds for any (H, S)-wide filling $G \to G/\langle\langle \bigcup N_P \rangle\rangle$ where $N_P \cap S = \emptyset$ for each $N_P \trianglelefteq P$.

Groves and Manning [48] proved the following properties on the images of H under sufficiently long and H-wide fillings.

Proposition 5.40 (Proposition 4.5 in [48])**.** *Let $(G, \mathbb{P})$ be relatively hyperbolic and let H be a relatively quasiconvex subgroup of G. Then for all sufficiently long and H-wide fillings $\pi : G \to \bar{G}$, the subgroup $\pi(H)$ is relatively quasiconvex in $\bar{G}$.*

The following theorem is a special case of Proposition 6.2 in [48] which shows that if all peripheral subgroups of $(G, \mathbb{P})$ are finitely generated free abelian and a subgroup H is relatively quasiconvex in G, then we can find a sufficiently long and H-wide filling $G \to \bar{G}$ such that the image of H is relatively quasiconvex in $\bar{G}$. Specifically, Theorem 5.41 is obtained by applying Proposition 6.2 of [48] to the family $\mathbb{H} = \{H\} \cup \mathbb{P}$, where H is a relatively quasiconvex subgroup of $(G, \mathbb{P})$. Note that in this case, as follows from Definition 5.37 and Definition 5.38, a filling π of $(G, \mathbb{P})$ is (H, S)-wide if and only if π is (H', S)-wide for every $H' \in \mathbb{H}$.

Theorem 5.41. *Suppose that $(G, \mathbb{P})$ is relatively hyperbolic, that each element of $\mathbb{P}$ is finitely generated free abelian, that H is a relatively quasiconvex subgroup of G, and that $S \subset (\bigcup_{P \in \mathbb{P}} P) \setminus \{1\}$ is a finite set. Then there exist finite index subgroups $\{K_p \leqslant P \mid P \in \mathbb{P}\}$ so that, for any subgroups $N_P \leq K_P$, the filling*

$$G \to G/\langle\langle \bigcup N_P \rangle\rangle\rangle = \bar{G}$$

is (H, S)-wide. Moreover, for an element $b \in G$ and $P \in \mathbb{P}$, if $1 \notin PHb$, then there is no element of $\langle\langle \bigcup N_P \rangle\rangle\rangle$ in PHb, that is, $1 \notin \pi(PHb) = \pi(P)\pi(H)\pi(b)$.

Definition 5.42. Let $(G, \mathbb{P})$ be a relatively hyperbolic group where all peripheral subgroups are finitely generated abelian and let $(H, \mathbb{D})$ be a relatively quasiconvex subgroup of G with the induced peripheral structure $\mathbb{D}$ as in Definition 5.30. For every $D \in \mathbb{D}$ there exists $P_D \in \mathbb{P}$ and $c_D \in G$ so that $D \leq P_D^{c_D}$.

For a Dehn filling $\pi : G \to \bar{G} = G/\langle\langle \bigcup N_P \rangle\rangle$ the *induced filling kernels* for $(H, \mathbb{D})$ are the collection $\mathcal{D} = \{D \cap N_{P_D}^{c_D} \mid D \in \mathbb{D}\}$. This defines the *induced filling* $\pi' : H \to \bar{H} = H/\langle\langle \mathcal{D} \rangle\rangle$ of H. We denote $\bar{D} = \pi'(D)$ for $D \in \mathbb{D}$ and denote by $\bar{\mathbb{D}}$ the list of all those $\bar{D}$, where $D \in \mathbb{D}$, such that $\bar{D}$ is infinite.

Proposition 5.43 (Proposition 4.6 in [48]). *Let $(G, \mathbb{P})$ be relatively hyperbolic and let H be a relatively quasiconvex subgroup of G. For all sufficiently long and H-wide fillings $\pi : G \to \bar{G}$, the map from the induced filling $\pi(H)$ of H to $\bar{G}$ is injective.*

We say that two finite lists $\mathbb{A} = A_1, \ldots, A_k$ and $\mathbb{B} = B_1, \ldots, B_s$ of infinite subgroups of a group W are *the same up to conjugation* in W, if $k = s$ and there exists a permutation $\sigma \in S_k$ such that for every $1 \le i \le k$, $B_{\sigma(i)} = A_i^{w_i}$ for some $w_i \in W$.

Proposition 5.44. *Let $(G, \mathbb{P})$ be relatively hyperbolic and let H be a relatively quasiconvex subgroup of G with the induced peripheral structure $\mathbb{D}$ from G. For all sufficiently long and H-wide fillings, the induced peripheral structure on $\pi(H)$ from $\bar{G}$ is the same as the peripheral structure $\bar{\mathbb{D}}$ on $\bar{H}$, up to conjugation in $\bar{H} = \pi(H)$.*

Proof. By Theorem 5.36 and Proposition 5.40, for a sufficiently long and H-wide filling $\pi : G \to \bar{G}$, the image $\pi(H)$ is relatively quasiconvex in the new relatively hyperbolic group $(\bar{G}, \{\pi(P) \mid P \in \mathbb{P}\})$. The induced peripheral structure on $\pi(H)$ from $\bar{G}$ consists of the infinite intersections $\pi(H) \cap \pi(P)^h$ where $h \in \bar{G}$. Remark 5.31 and Proposition 5.43 imply that if for $D \in \mathbb{D}$ the image $\pi(D)$ is infinite then $\pi(D) = \pi(H) \cap \pi(P)^h$ for some $h \in \bar{G}$. Thus, the conclusion of Proposition 5.44 holds as required. ☐

Proposition 5.45. *Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group with $\mathbb{P} = \{P_1, \ldots, P_k\}$. Let H be a relatively quasiconvex subgroup of G. Let $a \in G$ be a fixed element of G. Then for all fillings $\pi : G \to \bar{G}$ that are sufficiently long and H-wide, the following holds:*

If $\pi(H^a \cap P)$ is finite where $P \in \mathbb{P}$ then $\pi(H)^{\pi(a)} \cap \pi(P)$ is also finite.

Proof. By replacing H by H^a, without loss of generality, we can assume that $a = 1$. Suppose that $\pi(H \cap P)$ is finite but $\pi(H) \cap \pi(P)$ is infinite. Take an element $\pi(u)$ of infinite order in $\pi(H) \cap \pi(P)$ where $u \in P$. Since the element $\pi(u)$ is parabolic of infinite order in $\pi(H)$, Remark 5.31 implies that $\pi(u)$ have the form $\pi(v^{bh}) = \pi(h^{-1}b^{-1}vbh)$ where $h \in H$ and where $v \in H^{b^{-1}} \cap P$, and where $H \cap P^b$ is another element of $\mathbb{D}$, different from $H \cap P$. Hence, $\pi(u) = \pi(v^{bh}) = \pi(v)^{\pi(bh)}$. The groups $H \cap P$ and $H \cap P^b$ from $\mathbb{D}$ are in different H-conjugacy classes and therefore $b \notin PH$ (since P is abelian). In $\bar{G} = \pi(G)$, the elements $\pi(u)$ and $\pi(v)$ are infinite order elements of the finitely generated infinite abelian group $\pi(P)$ such that $\pi(u)$ and $\pi(v)$ are conjugate in $\bar{G}$. This implies that $\pi(u) = \pi(v)$ by almost-malnormality of $\pi(P)$ in $\bar{G}$. Then we have $\pi(v) = \pi(u) = \pi(v)^{\pi(bh)}$. Since $\pi(v)$ has infinite order in $\pi(P)$, almost malnormality of $\pi(P)$ implies that $\pi(bh) \in \pi(P)$ and hence $\pi(b) \in \pi(P)\pi(H)$. However, the fact that $b \notin PH$ implies, by the second part of Proposition 5.41, $\pi(b) \notin \pi(H)\pi(P)$. This gives a contradiction. □

We now define a particular class of Dehn fillings of a relatively hyperbolic group G, satisfying certain conditions relative to a relatively quasiconvex subgroup of G.

Definition 5.46 (Benign Dehn fillings)**.** Let $(G, \mathbb{P})$ be a relatively hyperbolic group where all peripheral subgroups of G are finitely generated free abelian, and where $\mathbb{P} = \{P_1, \ldots, P_k\}$. Let H be a relatively quasiconvex subgroup of G. Let $Z \subset (\bigcup_{P \in \mathbb{P}} P) \setminus \{1\}$ be a finite subset provided by Theorem 5.36. A Dehn filling $\bar{G}$ of G, determined by a collection $N_1 \trianglelefteq P_1, \ldots, N_k \trianglelefteq P_k$ of normal subgroups, is called *benign with respect to* H if:

(a) The Dehn filling $\pi : G \to \bar{G}$ is Z-long.

(b) There is some index i such that $H^{g_i} \cap N_i$ is infinite, that $\pi(H^{g_i}) \cap \pi(P_i)$ is finite, and that $\pi(P_i) = P_i/N_i$ is infinite.

(c) The subgroup $\pi(H)$ is relatively quasiconvex with peripherally finite index in $\bar{G}$.

Condition(a) in Definition 5.46 guarantees that the Dehn filling $\bar{G}$ is relatively hyperbolic with respect to $\pi(P_i) = P_1/N_1, \ldots, \pi(P_k) = P_k/N_k$, and if $\pi(P_i)$ is infinite then $\pi(P_i)$ is a maximal parabolic subgroup in $\bar{G}$ by Theorem 5.36. The following proposition says that the existence of such a benign Dehn filling of G with respect to H is equivalent to the existence of $P \in \mathbb{P}$, $g \in G$ such that the intersection $H^g \cap P$ is infinite and has infinite index index in P.

Proposition 5.47. *Let $(G, \mathbb{P})$ be a relatively hyperbolic group where all peripheral subgroups are finitely generated free abelian and let H be a relatively quasiconvex subgroup of G. There exists a benign Dehn filling of G with respect to H if and only if H does not have peripherally finite index in G.*

Proof. Suppose there exists a benign Dehn filling $\pi : G \to \bar{G}$ with respect to H. Let H^{g_i} and P_i be as in part (b) of Definition 5.46, so that $H^{g_i} \cap N_i$ is infinite, $\pi(H^{g_i}) \cap \pi(P_i)$ is finite and $P_i/N_i = \pi(P_i)$ is infinite. Since $\pi(H^{g_i} \cap P_i) \leq \pi(H^{g_i}) \cap \pi(P_i)$, it follows that $\pi(H^{g_i} \cap P_i)$ has infinite index in $\pi(P_i)$. Hence, $H^{g_i} \cap P_i$ is infinite and has infinite index in P_i, and therefore H does not have peripherally finite index in G.

Suppose now that H is relatively quasiconvex but does not have peripherally finite index. List all distinct representatives $\mathbb{D}$ of H-conjugacy classes of infinite parabolic subgroups of the form $H \cap P^g$ of H. Group them according to which P_i they come from. Take a specific P_i and suppose that the subgroups in the above list are $D_1 = H \cap P_i^{g_i}, \ldots, D_m = H \cap P_i^{g_m}$. Put $U_j = D_j^{g_j^{-1}}$ so that U_j is an infinite finitely generated subgroup of the free abelian group P_i. Take a maximal subcollection $U_1, \ldots, U_t$ of U_j's in P_i such that $\bigcup_{i=1}^{t} U_i$ generates a subgroup M_i of P_i of infinite index in P_i. If all U_j have finite index in P_i, take the subcollection as an empty set. Then adding any extra U_y to this collection generates a subgroup of finite index in P_i. Now choose a subgroup N_i of finite index in M_i such that $N_1, \ldots, N_k$ is a sufficiently long and H-wide filling.

We claim that the Dehn filling $\pi : G \to G/\langle\langle \bigcup N_i \rangle\rangle$ is benign with respect to H. By construction, Condition(a) in Definition 5.46 is satisfied. By Proposition 5.40, $\pi(H)$ is relatively quasiconvex in $\bar{G}$. Each of $U_1, \ldots, U_t$ is commensurable with a subgroup of N_i and therefore has finite image in $\bar{G}$. Each $U_y \leq P_i$ not from the list $U_1, \ldots, U_t$ has its image $\pi(U_y)$ having finite index in the infinite group $\pi(P_i) = P_i/N_i$.

By Remark 5.31, the induced peripheral structure on $\pi(H)$ is given exactly by all the infinite groups among $\bar{D} = \pi(D)$ where $D \in \mathbb{D}$. Therefore, $\pi(H)$ has peripherally finite index in $\bar{G}$, and so Condition(c) in Definition 5.46 is satisfied. Since H does not have peripherally finite index in G, there is a P_i such that N_i has infinite index in P_i and that N_i contains a subgroup commensurable with some infinite subgroup $H^a \cap P_i$ of P_i for some $D = H \cap P_i^{a^{-1}} \in \mathbb{D}$. Then $\pi(H^a \cap P_i)$ is a finite subgroup in the infinite group $\pi(P_i) = P_i/N_i$. By Proposition 5.45, the intersection $\pi(H^a) \cap \pi(P_i)$ is finite, so Condition(b) in Definition 5.46 also holds. Hence, the filling $\pi : G \to G/\langle\langle \bigcup N_i \rangle\rangle$ is benign with respect to H, as required. □

Theorem E. *Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group.*

(4) There is a complete algorithm which, for an undistorted finitely generated subgroup H of G, decides whether or not H is Morse.

Proof of Theorem E(4). Fix a finite set $Z = Z(G)$ as in the conclusion of Theorem 5.36.

Let H be an undistorted subgroup of a toral relatively hyperbolic group $(G, \mathbb{P} = \{P_1, \ldots, P_k\})$. Since H is undistorted, H is relatively quasiconvex in G (see Theorem 1.5 in [54]).

We run the algorithm in Theorem E(3), and if the algorithm terminates, then we declare that H is Morse in G.

In parallel, we run the following procedure. We start enumerating all plausible candidates for being a benign Dehn filling of G with respect to H as follows. Start enumerating all nontrivial elements of the form h^g where $h \in H$ and $g \in G$ and enumerating all elements of $P_1, \ldots, P_k$ and checking if h^g is equal to an element of P_i for some $1 \leq i \leq k$. Using this check, we enumerate all nontrivial elements γ such that $\gamma \in H^g \cap P_i$ where $g \in G$, $1 \leq i \leq k$. Using this list, start enumerating all tuples $\tau = (\gamma, Y_1, \ldots, Y_k)$ where $1 \neq \gamma \in H^{g_j} \cap P_j$ for some P_j and some $g_j \in G$, and where $Y_1 \subseteq P_1, \ldots, Y_k \subseteq P_k$ are finite subsets. Given each such tuple τ, do the following. Put $N_i = \langle Y_i \rangle \trianglelefteq P_i$. Then check if $N_i \cap Z = \emptyset$ for each $i = 1, \ldots, k$. If not, we discard this tuple τ and move to the next one. Suppose now that $N_i \cap Z = \emptyset$ for each $i = 1, \ldots, k$, so that the filling π defined by $N_1, \ldots, N_k$ is Z-long. We declare that the Dehn filling π is a candidate for a benign Dehn filling. We then run the partial algorithm in Theorem 5.32 on $\pi(H)$. Suppose the algorithm terminates and discovers that $\pi(H)$ is a relatively quasiconvex with peripherally finite index in $\bar{G}$. Note that each $P_i/N_i = \pi(P_i)$ is parabolic and thus relatively quasiconvex in $\bar{G}$. We use the algorithm in Proposition 5.33 to compute a generating set of $\pi(H)^{\pi(g_j)} \cap \pi(P_i)$. If $\pi(H)^{\pi(g_j)} \cap \pi(P_i)$ is finite but $\pi(P_j) = P_j/N_j$ is infinite, then the filling π is benign with respect to H. Then we terminate the entire procedure and declare that H is not Morse in G.

Recall that the subgroup H is relatively quasiconvex in G. If H is Morse in G, that is, if H has peripherally finite index, then the algorithm in Theorem E(3) eventually discovers this fact and terminates. Suppose that H is not Morse in G, that is H does not have peripherally finite index in G. By Proposition 5.47, this happens if and only if there exists a benign Dehn filling of G with respect H. Our second process above will eventually discover such a benign filling and declare that H is not Morse in G. Therefore the above algorithm correctly decides whether or not H is Morse in G, as required. □

We use [29] as background reference to the Bowditch boundary of a relatively hyperbolic group in the following proposition.

Proposition 5.48. *Let $(G, \mathbb{P})$ be a relatively hyperbolic group where $\mathbb{P}$ is a finite collection of finitely generated subgroups. Let H be a relatively quasiconvex subgroup of G with $[G : H] = \infty$. Then there exists a loxodromic element $g \in G$ such that $H \cap \langle g \rangle = \{1\}$.*

Proof. For a relatively hyperbolic group G and a relatively quasiconvex subgroup H, take the limit set $\Lambda(H)$ in the Bowditch boundary ∂G. Then $\Lambda(H)$ is a closed subset of ∂G, and since H is a relatively quasiconvex subgroup of infinite index in G, the complement $\partial G \setminus \Lambda(H)$ is nonempty (see Proposition 1.8 in [29]). Since the poles of loxodromic elements of G are dense in ∂G, there exists a loxodromic element $g \in G$ such that $g^\infty \in \partial G \setminus \Lambda(H)$. Then $\langle g \rangle \cap H = \{1\}$ since otherwise some positive power g^n of g belongs to H and hence $g^\infty \in \partial H$. □

Corollary F. *Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group. Then there exists an algorithm that, given an undistorted finitely generated subgroup H of G, decides whether or not H has finite index in G.*

Proof of Corollary F. Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group (where $\mathbb{P}$ is possibly empty). Given an undistorted subgroup H of G, we do the following.

First, run the algorithm in Theorem E(4) for deciding whether H is Morse in G. If H is determined to be non-Morse in G, then H has infinite index in G.

Suppose now that H turned out to be Morse, that is peripherally of finite index in G. We then run in parallel the following two processes:

Run the Todd-Coxeter coset enumeration for detecting finiteness of index of H in G. In parallel, start enumerating all loxodromic elements $g \in G$. Note that for a given word g in the generators of G we can decide whether or not g is loxodromic (see Theorem 5.6 in [79]). For each such g the subgroup $\langle g \rangle$ of G is relatively quasiconvex and peripherally of finite index. We then use Proposition 5.33 to compute the subgroup $H \cap \langle g \rangle$. If this subgroup is trivial, we declare that H has infinite index in G. If this subgroup is nontrivial (and thus has finite index in $\langle g \rangle$) then we move to the next loxodromic element $g \in G$.

By Proposition 5.48, the above procedure decides whether or not H has finite index in G. □

Note that, except for Proposition 5.48 and Corollary F, we assume that $(G, \mathbb{P})$ is a toral relatively hyperbolic group where the finite family $\mathbb{P}$ of free abelian non-cyclic groups is nonempty. Also note that in the case where $\mathbb{P}$ is empty, G is torsion-free word hyperbolic. In this case, for a finitely generated subgroup H of G, being Morse is equivalent to being stable, it is equivalent to being quasiconvex, and it is also equivalent to being undistorted. The conclusions of Theorem E then follow from Proposition 2.29.

5.3.3 Groups discriminated by a locally quasiconvex torsion-free hyperbolic group

In this section, we consider a special toral relatively hyperbolic group, that is, a group discriminated by a locally quasiconvex torsion-free hyperbolic group.

Definition 5.49. We say that a group G is *discriminated* by another group Γ if for every finite set $\{g_1, \ldots, g_n\}$ of non-trivial elements of G there exists a homomorphism $f : G \to \Gamma$ such that $f(g_i) \neq 1$ for $i = 1, \ldots, n$.

Note that for a finitely generated group to being discriminated by another group Γ for a relatively hyperbolic group is equivalent to being a limit group over Γ. See [87, 85, 65, 88, 46, 62] as references. Sela started a study of ordinary limit groups over the free group $\mathbb{F}_2$ in [87, 85] and showed that the limit groups have the same elementary theory as $\mathbb{F}_2$ (also see [65]). In [88], Sela considered limit groups over a torsion-free hyperbolic group Γ and characterized them as the same elementary theory as Γ. Groves [46, 45] and Kharlampovich and Myasnikov [62] investigated limit groups over a toral relatively hyperbolic group.

In particular, Kharlampovich and Myasnikov [62] proved that limit groups over a toral relatively hyperbolic group Γ embed into a group obtained from Γ by finitely many extensions of centralizers.

Definition 5.50. Let A be a toral relatively hyperbolic group. An *extension of centralizer* of A is a group presented by

$$B = \langle A, t_1, \ldots, t_r \,|\, [\,C(g), t_i\,] = [\,t_i, t_j\,] = 1,\, 1 \le i,\, j \le r \rangle,$$

where $1 \neq g \in A$ and $C(g)$ is the centralizer of g in A.

Theorem 5.51 (Theorem B in [62]). *Let G be a finitely generated group and let Γ be a toral relatively hyperbolic group. Then the following are equivalent.*

(i) G has the same universal theory as Γ .

(ii) G embeds into the Lyndon's completion $\Gamma^{\mathbb{Z}[t]}$ of the group Γ (equivalently, G embeds into a group obtained from Γ by finitely many extensions of centralizers).

(iii) G is discriminated by Γ.

(iv) G is a limit group over Γ.

We consider a finitely generated group discriminated by a locally quasiconvex torsion-free hyperbolic group. Here, a word-hyperbolic group A is called *locally quasiconvex* if every finitely generated subgroup of A is quasiconvex. Similarly, a finitely generated relatively hyperbolic group A is *locally relatively quasiconvex* if every finitely generated subgroup of A is relatively quasiconvex. In particular, ordinary limit groups are finitely generated groups discriminated by the free group $\mathbb{F}_2$ which is locally quasiconvex torsion-free hyperbolic .

Note that if $(G, \mathbb{P})$ is toral relatively hyperbolic and $1 \neq g \in G$ then the centralizer $C(g)$ of g in G is the maximal abelian subgroup of G containing g. Moreover, if g is loxodromic then $C(g)$ is infinite cyclic. If g is parabolic then $C(g)$ is equal to P^a where P^a is the unique conjugate of $P \in \mathbb{P}$ such that $g \in P^a$.

Proposition 5.52. *Let A be a torsion-free hyperbolic group and let B be an extension of the centralizer of A. Then B is toral relatively hyperbolic. Moreover, if A is locally relatively quasiconvex then so is B.*

For the first part of Proposition 5.52, see combination theorems in [29] and for the second part, see Theorem 3.1 in [12]. Theorem 5.53 follows from Theorem B and Theorem C in [62].

Theorem 5.53. *Let Γ be a torsion-free hyperbolic group and let G be a finitely generated group discriminated by Γ. There exists a sequence of centralizer extensions $\Gamma = G_0 < G_1 < \cdots < G_n$ where G_{i+1} is an extension of a centralizer of G_i and an embedding $G \hookrightarrow G_n$, and where each G_i is toral relatively hyperbolic.*

Lemma 5.54. *Let Γ be a torsion-free locally quasiconvex word-hyperbolic group and let G be a finitely generated group discriminated by Γ. Then G is a locally relatively quasiconvex toral relatively hyperbolic group (and in particular G is finitely presented).*

Proof. Let $\Gamma = G_0 \leq \cdots \leq G_n$ be a sequence of extensions of centralizers provided by Theorem 5.53, where $G \leq G_n$. Note that G_n as in Theorem 5.53 is a locally relatively quasiconvex toral relatively hyperbolic group, by iteratively applying Proposition 5.52. Then the finitely generated group G is relatively quasiconvex in G_n and G is relatively hyperbolic with respect to the induced peripheral structure from G_n. Therefore G is toral relatively hyperbolic.

Note that the induced peripheral structure on G from G_n may contain some infinite cyclic peripheral subgroups. However, since infinite cyclic subgroups are word-hyperbolic, they can be dropped from the list of peripheral subgroups, and G will still be relatively hyperbolic with respect to the remaining free abelian (non-cyclic) groups on the list. Thus G is indeed toral relatively hyperbolic.

Let H be a finitely generated subgroup of G. Since $H \leq G \leq G_n$ and G_n is locally relatively quasiconvex, H and G are relatively quasiconvex in G_n. Then by Lemma 5.25, H and G are undistorted in G_n. Therefore H is undistorted in G, and this implies that H is relatively quasiconvex in G [54]. Thus, G is toral relatively hyperbolic and locally relatively quasiconvex, as required. □

Corollary 5.55. *Let G be a finitely generated group discriminated by a locally quasiconvex torsion-free hyperbolic group. Then every finitely generated subgroup is undistorted in G.*

Proof. Let H be a finitely generated subgroup of G. Lemma 5.54 implies that G is locally relatively quasiconvex toral relatively hyperbolic, and therefore H is relatively quasiconvex in G. Then by Lemma 5.25 the subgroup H is undistorted in G. □

Corollary G. *Let G be a finitely generated group discriminated by a locally quasiconvex torsion-free hyperbolic group.*

(1) *There is a complete algorithm which, for a subgroup H of G given by a finite generating set, decides whether or not H is stable.*

(2) *There is a complete algorithm which, for a subgroup H of G given by a finite generating set, decides whether or not H is Morse.*

Proof of Corollary G. Let G be a finitely generated group discriminated by a locally quasiconvex word-hyperbolic group Γ. For a finitely generated subgroup H of G given by a finite generating set of H, we run the following procedure. Note that G is toral relatively hyperbolic and H is undistorted in G by Corollary

5.55. For Corollary G(1) we run the algorithm in Theorem E(2) on H and decide whether or not H is stable in G. For Corollary G(2), we run the algorithm in Theorem E(4) on H and decide whether or not H is Morse in G. □

5.3.4 Relatively quasiconvex subgroups

Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group. Note that the proof of Theorem E uses the partial algorithm provided by Kharlampovich, Miasnikov, and Weil [63] which detects a relatively quasiconvex subgroup with peripherally finite index in G. The methods in [63] use partial algorithms, provided by Kapovich [55], detecting L-rational subgroups of G, where L is an automatic (or bi-automatic) structure on G. As proved in [63], if a subgroup H of G is relatively quasiconvex and peripherally of finite index then H is L-rational. That is why the partial algorithm [63] terminates on a relatively quasiconvex subgroup with peripherally finite index. It follows from a general result of Birdson [17] that that if $(G, \mathbb{P})$ is a toral relatively hyperbolic where the collection $\mathbb{P}$ is nonempty, and if L is a bi-automatic structure on G, then there exists a relatively quasiconvex (in fact, parabolic) subgroup H of G such that H does not L-rational and that H does not have peripherally finite index in G. To remove the peripherally finite index condition, we can ask the following question.

Question 5.56. *Is there a partial algorithm detecting relatively quasiconvex subgroups of a toral relatively hyperbolic group?*

To answer Question 5.56, we use the following combination theorem on relatively quasiconvex subgroups of relatively hyperbolic groups proved by Martínez-Pedroza in [76]:

Theorem 5.57 (Theorem 1.1 in [76])**.** *Let G be a group generated by a finite set X and hyperbolic relative to a collection $\mathbb{P}$ of subgroups of G. For any relatively quasiconvex subgroup H and any maximal parabolic subgroup $P \in \mathbb{P}$, there exists a constant $C = C(H, P) \geq 0$ with the following property. If R is a subgroup of P such that*

(i) $H \cap P \leqslant R$ and

(ii) $|g|_X \geq C$ for any $g \in R \setminus H$,

then the natural homomorphism

$$H *_{H \cap R} R \to G$$

is injective with an image of a relatively quasiconvex subgroup. Moreover, every parabolic subgroup of $\langle H \cup R\rangle \leqslant G$ is either conjugate to a subgroup of H or a subgroup of R in $\langle H \cup R\rangle \leqslant G$.

By using Theorem 5.57, Manning and Martínez-Pedroza [75] proved the following:

Proposition 5.58. *Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group and let H be a relatively quasiconvex subgroup of G. Let $\mathbb{D}$ be a collection of representatives of maximal infinite parabolic subgroups of H up to conjugacy in H, i.e., for $D \in \mathbb{D}$, $D = H \cap P_D{}^{g_D}$ for some $g_D \in G$ and $P_D \in \mathbb{P}$ (so that $\mathbb{D}$ is the induced peripheral structure on H from $(G, \mathbb{P})$). Then there exists a family of subgroups $R_D \leqslant P_D^{g_D}$ of finite index in $P_D^{g_D}$, where $D \in \mathbb{D}$, such that the following holds:*
Put $H_1 := \langle H \cup_{D \in \mathbb{D}} R_D \rangle$. Then H_1 is relatively quasiconvex with peripherally finite index in G, and H_1 is an amalgamated product of H with parabolic subgroups $H \cap R_D$, that is,

$$H_1 = H *_{\langle H \cap R_D | D \in \mathbb{D} \rangle} \langle R_D | D \in \mathbb{D} \rangle.$$

Moreover, the collection $(R_D)_{D \in \mathbb{D}}$ is the induced peripheral structure on H_1 from $(G, \mathbb{P})$.

Note that Theorem 5.58 is a special case of Theorem 1.7 in [75] when a given group is hyperbolic relative to a collection of free abelian subgroups.

For a finitely generated subgroup of a toral relatively hyperbolic, being relatively quasiconvex is equivalent to being undistorted. Kapovich [56] showed that for a finitely generated which splits as a graph of groups, undistortness of the edge groups implies undistortness of vertex groups.

Lemma 5.59 (Lemma 3.5 in [56])**.** *Let G be a finitely generated group that splits as a finite graph of groups with finite generated vertex and edge groups, and if G_v is a vertex group such that all the adjacent edge groups G_e to G_v are quasi-isometric embedded in G then G_v is also quasi-isometric embedded in G.*

Note that the statement of Lemma 3.5 in [56] assumes that G is hyperbolic but that assumption is not actually used in the proof. Note also that Bigdely and Wise [12] showed that for a relatively hyperbolic group which splits as a graph of groups, relative quasiconvexity of vertex groups is equivalent to relative quasiconvexity of the edge groups (see Lemma 4.9 in [12]). The following lemma provides a sufficient condition for a finitely generated subgroup of a toral relatively hyperbolic group to be relatively quasiconvex.

Lemma 5.60. *Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group and let H be a finitely generated subgroup of G. Suppose that there exist and a subgroup R of P such that*

(i) $H \cap P \leqslant R$ *and*

(ii) the homomorphism

$$H *_{H \cap R} R \to G$$

is injective with image a relatively quasiconvex subgroup, where every parabolic subgroup of $\langle H \cup R\rangle \leqslant G$ *is either conjugate to a subgroup of* H *or a subgroup of* R *in* $\langle H \cup R\rangle \leqslant G$.

Then H *is relatively quasiconvex in* G *(equivalently,* H *is undistorted in* G*).*

Proof. If $R = H$ then $\langle H \cup R\rangle = H$ is relatively quasiconvex by the assumption. Suppose now that $R \neq H$. By the assumption $H_1 = H *_{H\cap R} R$ is relatively hyperbolic with respect to $\{R,\, H \cap P_i^{g_i} \mid P_i \in \mathbb{P},\, g_i \in G\}$. Since $H \cap R \leqslant R$ and R is abelian, the intersection $H \cap R$ is undistorted in R. Since R is a parabolic group of H_1, R is undistorted in H_1. Then, since $H \cap R \subseteq R \subseteq H_1$ and R is abelian, $H \cap R$ is undistorted in H_1. Note that for subgroups of a toral relatively hyperbolic, relative quasiconvexity is equivalent to undistorted. Since the edge group $H \cap R$ is relatively quasiconvex in H_1, H is relatively quasiconvex in H_1 by Lemma 5.59. Then since H_1 is relatively quasiconvex in G, H is relatively quasiconvex in G. □

By combining Theorem 5.57, Proposition 5.58, and Lemma 5.60, we can obtain the following partial algorithm detecting undistorted subgroups of toral relatively hyperbolic group and answer Question 5.56.

Theorem H. *Let* $(G, \mathbb{P})$ *be a toral relatively hyperbolic group. There exists an algorithm which, for a finite set* S *of* G*, will terminate the subgroup* $H := \langle S\rangle$ *is relatively quasiconvex but run forever if* H *is not relatively quasiconvex.*

Proof of Theorem H. We run the algorithm from Proposition 5.33 on H. If the process terminates, then we declare that H is relatively quasiconvex and has peripherally finite index in G, that is, H is Morse.

In parallel, we start enumerating all plausible candidates for being isomorphic to an amalgamated product of H and finite index subgroups of parabolic subgroups $P \in \mathbb{P}$ as follows.

(i) (a) Start enumerating all nontrivial elements of the form h^g where $h \in H$ and $g \in G$ and enumerating all elements of $P_1, \dots, P_k$ and checking if h^g is equal to an element of P_i for some $1 \leq i \leq k$. Using this check, we enumerate all nontrivial elements h^g such that $h^g \in H^g \cap P_i$ where $g \in G$ and $1 \leq i \leq k$.

(b) For each $1 \leq i \leq k$, start enumerating all tuples $g_{i1}, \dots, g_{il} \in G$, tuples $\overrightarrow{h_{in}}$ of element of H, and tuples $\overrightarrow{p_{in}}$ of element of P_i such that $H^{g_{in}} \cap P_i$ has a nontrivial element $\overrightarrow{h_{in}}^{g_{in}} = \overrightarrow{p_{in}}$ in G for $1 \leq n \leq l = l(i)$ and $\langle \overrightarrow{p_{in}} \rangle$ freely generates a subgroup of P_i. For such a list, let I be the index subset of $\{1, \dots, k\}$ such that for every $i \in I$ there is an element on the list contained in $H^g \cap P_i$ for some $g \in G$.

(c) For each $i \in I$ and $1 \leq n \leq l(i)$, start enumerating all tuples $\overrightarrow{q_{in}}$ of elements of P_i freely generates a finite index subgroup of P_i containing $\overrightarrow{p_{in}}$. Let H_1 be the following subgroup of G:

$$H_1 := \langle\, S \cup (\, \cup_{i\in I, 1\le n\le l(i)}\, \overrightarrow{q_{in}}^{g_{in}^{-1}}\,)\,\rangle$$

(d) We then run the partial algorithm from Proposition 5.32 on H_1. Suppose that the partial algorithm terminates and detects that H_1 is relatively quasiconvex with peripherally finite index in G. Then the algorithm also computes (by producing their finite generating sets) a induced peripheral structure $\mathbb{D}_{H_1}$ for H_1 from $(G, \mathbb{P})$ as in Definition 5.30. Then we check that $\mathbb{D}_{H_1}$ is the same (up to H_1 conjugacy) as the collection of free abelian subgroups $U_{in} := \langle \overrightarrow{q_{in}}^{g_{in}^{-1}} \rangle$. If that is the case, we proceed to step (i)(e). If not, we consider the next collection of candidate tuples as in (i)(a)-(i)(c) above.

(e) Since H_1 is relatively quasiconvex with peripherally finite index in G, H_1 is rational with respect to the automatic structure on G (see [63]). Then we can compute a finite presentation $\langle X \,|\, R_X \rangle$ for H_1.

(f) Since H_1 has two different generating sets X and $S \cup (\cup \overrightarrow{q_{in}}^{g_{in}^{-1}})$, start rewriting elements of X as words over $(S \cup (\cup \overrightarrow{q_{in}}^{g_{in}^{-1}}))^{\pm}$ by enumerating elements of $(S \cup (\cup \overrightarrow{q_{in}}^{g_{in}^{-1}}))^{\pm}$ and the solution of the word problem in H_1. Then rewrite the relations of R_X as words over $(S \cup (\cup \overrightarrow{q_{in}}^{g_{in}^{-1}}))^{\pm}$. Let $\langle\, S \cup (\cup \overrightarrow{q_{in}}^{g_{in}^{-1}}) \,|\, R \,\rangle$ be the result of a finite presentation of H_1.

(ii) Start enumerating all candidates of amalgamated products of H and finite index subgroups of parabolic subgroups $P \in \mathbb{P}$ for being isomorphic to H_1 as follows.

(a) Start enumerating non-trivial elements $r \in F(S)$ and list all elements r such that $r =_G 1$. Note that if H is an undistorted subgroup of G, that is, H is relatively hyperbolic, then H has the induced peripheral structure on H from $(G, \mathbb{P})$ as a collection of free abelian subgroups of H [54]. We also note that a relatively hyperbolic group with respect to a collection of finitely presented subgroups is finitely presented itself (see, for example, [79]). By using a list of such elements $r_1, \ldots, r_m$, we start to enumerate all candidate finite presentations $\langle\, S \,|\, r_1, \ldots, r_m \,\rangle$ for H.

(b) For each $i \in I$ and $1 \le n \le l(i)$, let B_{in} and C_{in} be the following subgroups of G from the list from (i)(b):

$$B_{in} := \langle \overrightarrow{h_{in}} \rangle, \quad C_{in} := \langle\, \overrightarrow{p_{in}}^{g_{in}^{-1}} \rangle$$

By the construction, $B_{in} =_G C_{in}$, B_{in} is a subgroup of H, C_{in} is a subgroup of $\langle \overrightarrow{q_{in}}^{g_{in}^{-1}} \rangle$, and $\overrightarrow{h_{in}}$ and $\overrightarrow{p_{in}}^{g_{in}^{-1}}$ are free generating sets of B_{in} and C_{in} respectively. By using a candidate presentation for H obtained from (ii)(a), consider the following finite presentation:

$$\langle S \cup (\cup \overrightarrow{q_{in}}^{g_{in}^{-1}}) | r_1, \ldots, r_m,\ xyx^{-1}y^{-1},\ \overrightarrow{h_{in}}^{-1} \overrightarrow{p_{in}}^{g_{in}^{-1}}\, \forall x, y \in \cup A_{in}, \forall i \in I, 1 \le n \le l(i) \rangle \qquad (\star)$$

Note that since the relations in the above presentation ($\star$) are contained in the normal closure of R, there is a natural quotient map α from the presentation ($\star$) to H_1. We consider such group presentations as candidate presentations for being isomorphic to H_1 and the amalgamated product $H *_{(B_{in}=C_{in})} U_{in}$, where this amalgamation is taken over all $i \in I$ and $1 \leq n \leq l(i)$.

(iii) Start comparing the group presentations $\langle\, S \cup (\cup \overrightarrow{q_{in}}^{g_{in}^{-1}}) \,|\, R \,\rangle$ for H_1 and a candidate presentation ($\star$) from (ii)(b). We need to check that two group presentations are isomorphic and the candidate presentation ($\star$) is indeed an amalgamated product of H and $(B_{in})_{in}$.

(a) Start enumerating the normal closure of the set of relations in ($\star$) and look for the ones which come from R (If we fail to find an element of R, add more words $r =_G 1$ over $F(S)$ into ($\star$) and consider the presentation with the extra relations). Suppose that all relations in R are contained in the normal closure of the set of relations in ($\star$). Then we declare that H_1 is isomorphic to the candidate ($\star$).

(b) Let $B'_{in} := \langle \overrightarrow{h_{in}} \rangle$ be a subgroup of $H' = \langle\, S \,|\, r_1, \ldots, r_m \,\rangle$. Start enumerating the normal closure of $\{r_1, \ldots, r_m\}$ and check that generators of B'_{in} commute in H'. If there are non-commutative generators of B'_{in} in H', we add more and more relations until all generators of B'_{in} commute. Since B_{in} is abelian in H and if H is undistorted in G, we eventually get the finite presentation of H.

(c) By the natural quotient map $\alpha : H' \to H$, we have $\alpha(B'_{in}) = B_{in}$. Note that the tuple $\overrightarrow{p_{in}}^{g_{in}^{-1}}$ is a basis of the free abelian group B_{in} with rank N_{in} which is the number of elements in $\overrightarrow{p_{in}}^{g_{in}^{-1}}$. Then the preimage $\alpha^{-1}(\overrightarrow{p_{in}}^{g_{in}^{-1}})$ freely generates a free abelian subgroup of B'_{in} with rank N_{in}. Note that the abelian group B'_{in} can be represented as a quotient of a free abelian group by relations and the numbers of generators of B'_{in} is also N_{in}. This implies that B'_{in} is free abelian with rank N_{in}. Hence, α is a surjective endomorphism of a free abelian group with rank N_{in}. Since a free abelian group is hopfian, α is isomorphism.

(d) From (iii)(a)-(iii)(c), we have $H \leqslant H_1 = \langle\, S \,|\, r_1, \ldots, r_m \,\rangle *_{(B'_{in}=C_{in})} U_{in}$. We now consider the surjective homomorphism map from $\langle\, S \,|\, r_1, \ldots, r_m \,\rangle$ to H. If a word $r \in F(S)$ is trivial in H, then $r = 1$ in $\langle\, S \,|\, r_1, \ldots, r_m \,\rangle *_{(B'_{in}=C_{in})} U_{in}$. Then by the normal form of r in the amalgamated product, $r = 1$ in $\langle\, S \,|\, r_1, \ldots, r_m \,\rangle$. Hence, H is isomorphic to $\langle\, S \,|\, r_1, \ldots, r_m \,\rangle$ and $H_1 = H *_{(B_{in}=C_{in})} U_{in}$.

The above process terminate if and only if the subgroup H is undistorted in G by Theorem 5.57, Proposition 5.58, and Lemma 5.60. □

Remark 5.61. Suppose that the algorithm from Theorem H terminates on H by computing such $H_1 = H *_{(B_{in}=C_{in})} U_{in}$ as shown in the proof of Theorem H, and we declare that H is relatively quasiconvex in G. Note that in part (i)(d) in the proof, we use the partial algorithm from Proposition 5.32 to choose H_1 to be relatively quasiconvex with peripherally finite index in G and compute the induced peripheral structure $\mathbb{D}_{H_1}$ for H_1 from $(G, \mathbb{P})$, which is indeed the collection $\{U_{in} = \langle \overrightarrow{q_{in}}^{g_{in}^{-1}} \rangle\}$. This implies that $\{B_{in}\}$ is the maximal collection of parabolic subgroups of H and so the induced peripheral structure for H from G is $\{B_{in}\}$. Note that B_{in} is equal to $H \cap U_{in}$.

The proof of Theorem H gives an alternative algorithm for Theorem E(4):

Theorem 5.62 (Theorem E(4))**.** *Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group. There is a complete algorithm which, for an undistorted finitely generated subgroup H of G, decides whether or not H is Morse.*

Proof. Since H is undistorted in G, if H is Morse in G then H has peripherally finite index in G. If the algorithm in Proposition 5.33 terminates, we declare that H is Morse in G. We now suppose that the algorithm in Theorem H gives such an enlarged subgroup H_1 of H shown as the proof. Then for $i \in I$ and $1 \leqslant n \leqslant l(i)$, check whether or not B_{in} has finite index in U_{in}. If all such B_{in} have finite index in U_{in}, then we declare that H is Morse in G. If there exists B_{in} having infinite index in U_{in}, then $H^{g_{in}} \cap P_i$ has infinite index in P_i, i.e., H does not have peripherally finite index in G. In this case, we conclude that H is not Morse in G. □

As another application of Theorem H, we have a limited version of the uniform membership problem for a toral relatively hyperbolic group. Before proving Corollary I, we remark well-known facts about the membership problem.

Remark 5.63. Let G be the fundamental group of a finite graph of groups where all vertex groups and edge groups are finitely generated. If the vertex groups of G have solvable word problem and the membership problem for edge groups in vertex groups is decidable, then the word problem for G is solvable and the membership problem for vertex groups in G is decidable. For HNN extensions, see Corollary 2.2 in Chapter IV in [73]. For amalgamated products and more general finite graphs of groups, the normal forms argument is similar.

Remark 5.64. For relatively hyperbolic groups, the proofs of Theorem 5.1 and Corollary 5.5 in [79] implies that the following version of the membership problem for parabolic subgroups in relatively hyperbolic groups is decidable: there exists an algorithm that, given a finite relative presentation of a group G with peripherally finitely generated free abelian subgroups $P_1, \ldots, P_k$, and given a word w in the generators of G, for each

$i = 1, \ldots, k$ decides whether or not $w = 1$ in G and w belongs to P_i. If in addition, we are given a finite subset of some P_i generating a subgroup $U \leq P_i$, the above algorithm also decides whether or not w belongs to U in G.

Corollary I. *Let $(G, \mathbb{P})$ be a toral relatively hyperbolic group. Then there exists a partial algorithm that, given $u, v_1, \ldots, v_k \in G$, detects if $H = \langle v_1, \ldots, v_k \rangle$ is relatively quasiconvex in G, and if H is relatively quasiconvex, decides whether or not $u \in H$.*

Proof. We fix an automatic structure L on G [63]. First we run the partial algorithm from Theorem H on $H = \langle v_1, \ldots, v_k \rangle$ which detects if H is relatively quasiconvex in G.

Suppose the algorithm terminates on H and produces a subgroup H_1 of G such that H_1 is relatively quasiconvex with peripherally finite index in G, and that H_1 is an amalgamated product of H with several parabolic subgroups, that is, $H_1 = H *_{(B_{in}=C_{in})} U_{in}$ as in part (iii)(d) of the proof of Theorem H. By Remark 5.61, the relatively quasiconvex subgroup H of G has an induced peripheral structure as $\{B_{in}\}$.

By Theorem 7.5 in [63], the subgroup H_1 is L-rational. It follows that we can then algorithmically find the regular language $L_1 \subseteq L$ which is the presage of H_1 in L by Proposition 1 in [55]. We rewrite the word u into a word $u' \in L$ such that $u =_G u'$, and then check whether or not $u' \in L_1$.

(i) If $u' \notin L_1$, then $u \notin H_1$ and hence $u \notin H$.

(ii) Suppose now that $u' \in L_1$. Since H is hyperbolic relative to finitely generated free abelian groups $\{B_{in}\}$, the membership problem for $\{B_{in}\}$ in H is decidable in the uniform (with respect to H and $\{B_{in}\}$) sense described in Remark 5.64. Also, the membership problem for a subgroup of the free abelian group $U_{in} = \langle \overrightarrow{q_{in}}^{g_{in}^{-1}} \rangle$ in U_{in} is decidable. Note that since G has solvable word problem, the subgroups H and U_{in} have solvable word problem. Therefore, in the amalgamated product decomposition of H_1, the membership problem for the vertex group H in H_1 is decidable by Remark 5.63, so we can decide whether or not u belongs to the vertex group H.

Therefore, in both cases, we can decide whether or not $u \in H$. ∎

Chapter 6

Open problems

Our algorithmic results in Chapter 5 raise several interesting questions.

6.1 Mapping class groups and right-angled Artin groups

If we can find a partial algorithm for detecting finitely generated subgroups of $\mathrm{Mod}(S)$ which are distorted, then Theorem C together with Corollary B would imply that there exists a total algorithm for deciding whether or not a finitely generated subgroup of $\mathrm{Mod}(S)$ is stable in $\mathrm{Mod}(S)$. Hence, it is natural to ask the following question.

Question 6.1. *For a surface S as in Theorem C, is there a partial algorithm detecting that a finitely generated subgroup is undistorted in Mod(S)?*

Let G be either a mapping class group as in Theorem C or a right-angled Artin group as in Theorem D. Then we do not yet have a partial algorithm for detecting non-Morseness of a given undistorted subgroup H of G. Non-Morseness of H in this setting is equivalent to the fact that H is not stable and of infinite index in G. Since we already have a partial algorithm for detecting non-stability, the problem of deciding whether or not an undistorted finitely generated subgroup of G is Morse in G reduces to the following:

Question 6.2. *Let G be a mapping class group or a right-angled Artin group. Is there a partial algorithm which, for an undistorted finitely generated subgroup H, decides whether or not H has infinite index in G?*

For a mapping class group $\mathrm{Mod}(S)$, a positive answer would promote Theorem C(3) to a complete algorithm which decides whether or not an undistorted subgroup of $\mathrm{Mod}(S)$ is Morse. In the case of a right-angled Artin group A_Γ, an affirmative answer together with Theorem D(2) would give a complete algorithm deciding if an undistorted subgroup of A_Γ is Morse. For a right-angled Artin group, we can consider the associated cubical CAT(0) complex.

6.2 Right-angled Coxeter groups

Another interesting class of groups to study is the class of right-angled Coxter groups. Let Γ be a finite simplicial graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$. Then the *right-angled Coxeter group* on Γ has the finite group presentation:

$$G_\Gamma := \langle\, V(\Gamma) \mid v^2 = 1 \text{ for all } v \in V(\Gamma), \text{and } \; vu = uv \text{ whenever } (v, u) \in E(\Gamma) \,\rangle.$$

Note that for each right-angled Artin group A_Λ there is a right-angled Coxeter group G_Γ which contains A_Λ as a subgroup of finite index. Genevois [38] recently characterized all right-angled Coxeter groups whose Morse subgroups of infinite index are stable by using defining graphs.

Definition 6.3. Let Γ be a simplicial graph. A subgraph $\Lambda \leq \Gamma$ is *square-complete* if every induced square of Γ containing two opposite vertices in Λ must be entirely included into Λ. A *minsquare subgraph* of Γ is a subgraph which is minimal among all the square-complete subgraphs of Γ containing at least one induced square.

Recall that for two graphs Γ_1 and Γ_2, the *join* of Γ_1 and Γ_2 is a graph obtained by connecting every vertex of Γ_1 to every vertex of Γ_2 by an edge.

Theorem 6.4. *[38] Let Γ be a simplicial graph. Every infinite-index Morse subgroup of the right-angled Coxeter group G_Γ is stable if and only if Γ is square-free or if it decomposes as the join of a minisquare subgraph and a complete graph.*

For a graph Γ which is neither square-free nor the join of a minisquare subgraph and a complete graph, the notions of stability and Morseness of infinite index of a subgroup of G_Γ are not equivalent. Motivated by Chapter 5, we ask the following question:

Question 6.5. *For a simplicial graph Γ is there a partial algorithm that, for a finite subset S of a right-angled Coxeter group G_Γ, decides whether or not a finitely generated subgroup H of G is stable? What if we replace stable by Morse?*

At first, we can try to answer Question 6.5 on the class of graphs Γ as in Theorem 6.4, where a subgroup H of G_Γ is stable if and only if H is Morse of infinite index, and then consider an arbitrary simplicial graph.

Dani and Levcovitz [30] provided finite-time algorithms to solve the following problems: For a quasiconvex subgroup H of a right-angled Coxeter group G_Γ given by a finite generating set of words in G_Γ, determine the index of H in G_Γ (even if infinite); given $g \in G_\Gamma$, determine whether or not a positive power of g belongs to H. Dani and Levcovitz's algorithms might answer Question 6.5 positively.

6.3 Complexity bounds of the algorithms in Chapter 5

Since the algorithms in Chapter 5 have to run in parallel various enumeration procedures, they are impractical and we cannot estimate their complexity bounds. Hence, it would be interesting to find other algorithms at least on some more restricted types of subgroups of groups in Chapter 5 such that the algorithms do not use general enumeration arguments and their complexity bounds are computable.

Question 6.6. *Find some more restricted types of subgroups of groups in Chapter 5 where we can obtain some complexity bounds for the complete algorithms obtained in Theorem C, Theorem D, and Theorem E.*

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