ASYMPTOTIC BEHAVIOR OF STOCHASTIC OPTIMAL VELOCITY DYNAMICAL MODEL

BY

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DISSERTATION

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ABSTRACT

The imminent revolution in autonomous vehicles’ driving technologies has led to a wide range of challenges in modeling and understanding the effect of autonomous vehicles. Our main interest in these notes is the mathematical analysis of effects of perturbation in dynamics of autonomous vehicles, in the form of deviation from the unperturbed solution, collision analysis and propagation of noise in the system.

In the first part, we introduce the optimal velocity dynamical model which was initially considered to explain the congestion in the traffic flow. Then we consider the car-following optimal velocity dynamics, which is an extension of the optimal velocity dynamical model by adding a singularity term, as the dynamics of autonomous vehicles.

In the second part of these notes, using some tools from perturbation theory, we investigate the behavior of the optimal velocity dynamical model in response to both deterministic and stochastic perturbations in the system. In particular, in this part, we are mainly concerned with the deviation of the solution from the unperturbed trajectory. In the deterministic cases, we consider fast perturbations. We show that the solution can be approximated by the trajectory of an averaged dynamics and in addition, we discuss rates of such convergence.

Next, we consider stochastic perturbations. We study perturbations by bounded and pathwise continuous random processes as well as Brownian perturbations. By careful analysis in both cases, we can show that the stochastic solution converges to the trajectory of unperturbed dynamics and we discuss the rates of convergence.

In the third part, we focus on the possibility of collision between the leading and following vehicles in the optimal velocity model. This is of special importance when we study the dynamics of autonomous vehicles. Through rigorous analysis, we show that collision does
not happen in the deterministic case when there is no noise affecting the system. Then, we carefully investigate the probability of collision when the system is perturbed by small Brownian noise. In addition, we find a provable bound for the probability of collision in this case.

Finally, we study the propagation of noise in the system. First, we approximate the optimal velocity dynamical model with another dynamical model. We show that the Markov process associated with the solution of the approximating dynamics has a transition density function and we show the construction of an explicit form of such density function.
To my beloved family.
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CHAPTER 1

INTRODUCTION AND RELATED WORKS

Microscopic models are one of the main traffic flow dynamics in which the behavior of each individual vehicle, as a result of interaction with others, is of the main focus. These dynamical models are suitable frameworks for Intelligent Transportation Systems (ITS) including the Adaptive Cruise Control (ACC) and Advanced Driver Assistance Systems (ADS) (for details about the applications, different microscopic models and their assumptions, we refer the interested readers to [1]). From theoretical point of view, these problems are very complex and have many interesting properties that make them theoretically rich and valuable.

The basic car-following dynamical model was introduced by [2, 3] and subsequently various models have been proposed to improve upon these dynamics by improving the assumptions (interested readers can consult some of the fundamental works [3, 4, 5, 6, 7, 8, 9]). Classification, dynamical modelling, and related analysis can be found in [1]. A generic form of car-following dynamics can be formulated as

\[
\ddot{x}^{(n)} = F(\Delta x^{(n)}, y^{(n-1)}, y^{(n)}),
\]

(1.1)

where \(\Delta x^{(n)} = x^{(n-1)} - x^{(n)}\) denotes the distance between the \(n\)-th vehicle and its leading one. \(y^{(n)}\) denotes the velocity of the \(n\)-th vehicle. The function \(F\) defines the acceleration/deceleration which can be interpreted as a response to the other vehicles’ stimuli.

The imminent promise of autonomous vehicles has led to a resurgence of interest in a number of models of car-following. The classic work of Bando et al. [10] models the forces needed to follow a preceding vehicle at a fixed distance. Although simple and with few number of parameters, the optimal velocity model has been shown to efficiently present the
traffic flow and in particular the evolution of congestion and stop-and-go waves.

Let us start by understanding the forces in the optimal velocity dynamical model. We are concerned with a collection of $N$ vehicles governed by optimal velocity dynamics of the form

$$
\ddot{x}^{(n)}_t = -\alpha \left\{ \dot{x}^{(n)}_t - V \left( \frac{x^{(n-1)}_t - x^{(n)}_t}{d} \right) \right\} - \beta \frac{\dot{x}^{(n)}_t - \dot{x}^{(n-1)}_t}{\left( x^{(n)}_t - x^{(n-1)}_t \right)^2}
$$

(1.2)

for $n \in \{1, 2, \ldots, N\}$, with the lead car satisfying

$$
(x^{(0)}_t, \dot{x}^{(0)}_t) = (x_0, v_0),
$$

We consider

$$
V(x) \overset{\text{def}}{=} \tanh(x - 2) - \tanh(-2),
$$

(1.3)

$x_t^{(n)}$ is the location of the $n$-th vehicle, $\alpha, \beta > 0$ are positive parameters. We can define $\frac{1}{\alpha}$ and $\frac{(x^{(n)}_t - x^{(n-1)}_t)^2}{\beta}$ as relaxation times. The scaling parameter $d$ captures the sensitivity of the reference velocity with respect to the following distance. It is assumed that the $n$-th vehicle is located to the left of the $(n-1)$-st one, and hence $x^{(n-1)}_0 > x^{(n)}_0$. The optimal velocity function $V: \mathbb{R} \to \mathbb{R}$ satisfies some regularity condition such as monotone increment, boundedness and smoothness (see Figure 1.1).

The optimal velocity dynamical model was originally introduced by Bando et al. [10, 11]
in the form of equation (1.2) (with parameter $\beta = 0$) in an attempt to explain the evolution of congestion in traffic flow. In this model, the acceleration and deceleration of any vehicle is calculated as the difference between the *optimal velocity* (also known as desired velocity) and the current velocity of the vehicle. In this model, the optimal velocity is in turn calculated based on the distance to the leading vehicle (the details will be discussed later).

While the optimal velocity model successfully explains the traffic flow [12, 13, 14], it is associated with some issues which are mainly attributed to its independence from the difference of the velocities. In other words, in the dynamical model (1.2) (with $\beta = 0$) the acceleration/deceleration is only decided based on the distance between the two vehicles and without any insight on the difference of the velocities. Such independence makes the model unrealistic on some quantitative level and also results in strong dependence on the parameters [1]. Incorporating the car-following term (pre-multiplied by $\beta$) addresses these issues to some extent. In addition, in this case (when $\beta > 0$) if the distance between the two vehicles is extremely large then the sensitivity to the difference of velocities in the second term vanishes, which is reasonable in real applications. Without the dependency of the model on the distance in the denominator of the second term (pre-multiplied by $\beta > 0$), even if the difference of the velocities is large, i.e. vehicles are far apart, the deceleration might be applied and the following vehicle has to unnecessarily slow down.

The literature on stability analysis of optimal velocity model is extensive. A prior work in stability analysis is due to [15]. String stability analysis of optimal velocity model is studied in various works, e.g. [10, 16, 1]. Jiang [17] discusses the linear stability analysis of full velocity dynamical model. Wilson and Ward [9] survey different criteria for the platoon and string stability of a general car-following model. The traffic flow stability in mixed human and autonomous vehicle systems is studied by Talebpour and Mahmassani [18]. Cui et al. [19] study the stabilization of traffic flow by adding a single autonomous vehicle. More recently, Gunter et al. [20] discuss the string stability of adaptive cruise control equipped vehicles. They consider the optimal velocity model with a relative velocity term which includes time delay. Stern et al. [21] study the dissipation of traffic wave by adding a
controlled autonomous vehicle.

Our main focus in Chapter 3 is analyzing the possibility of collision. As we shall see, the basic optimal velocity model (1.2) (with $\beta = 0$) is not sufficiently proper for this purpose (see Figure 1.2). By adding the car-following term (pre-multiplied by $\beta$) and the fact that the denominator in this term creates a singularity as $x_t^{(n)} - x_t^{(n-1)} \nrightarrow 0$, we may overcome this issue.

To this end, let us look at the way the optimal velocity model (1.2) affects the interaction between the consecutive vehicles. If $\beta \equiv 0$ (Bando’s optimal velocity model) and if in addition

$$x_t^{(n-1)} - x_t^{(n)} \leq d \cdot V^{-1}(\dot{x}_t^{(n)}),$$

i.e. if the distance between the two vehicles exceedingly decreases, then the first term, pre-multiplied by $\alpha$, provides a negative force to the system encouraging the deceleration of the following vehicle. However, this force may not be sufficient for the avoiding the collision. The right plot in Figure 1.2 shows that in this case the collision can happen. Similarly, if the distance between the two consecutive vehicles overly increases, a positive force will be applied that suggests acceleration of the following vehicle\(^1\).

Adding the second term, pre-multiplied by $\beta$, intensifies the acceleration/deceleration force. In fact, when the two vehicles are too close to each other, the singularity in the denominator of the second term imposes a very large negative force (in addition to the force which is imposed by the first term). Such an extra force is the main source of avoiding the collision in optimal velocity model (1.2) (see Figure 1.3). We will rigorously prove this property in the next section.

\(^1\)It should be noted that the behavior of the dynamical system depends on the parameters. The discussion about calibration of these parameters is outside the scope of this paper.
Figure 1.2: Optimal velocity model in 2-dim when $\beta = 0$. The initial value is $(x_0, y_0) = (0.5, -1)$, $d = 1$, $v_o = 0.5$ and the result for time duration of $T = 100$ is plotted. In the left figure $\alpha = 2$ and the trajectory does not cross the origin on the $x$-coordinate (no collision). On the right figure $\alpha = 0.5$ and the collision happens since the deceleration force is not sufficient. This in particular, shows the dependence of this model to the parameters.

Figure 1.3: The left figure demonstrates the trajectory of the regularized optimal velocity model in 2-dim for $\alpha = 0.5$, $\beta = 1$, and the right figure illustrates the trajectory for $\alpha = 2$, $\beta = 1$. The initial value is considered to be $(x_0, y_0) = (0.5, -1)$ and all parameters are as in previous figure. In both cases we observe that the collision does not happen. This is as a result of the force imposed by the singular term in the dynamic.
1.1 Stochasticity in Optimal Velocity Model

In Chapters 3 and 4 we are mainly interested in studying the effects of stochasticity in some particular aspects of traffic flow and its dynamical representation. As mentioned before, in many situations the dynamic of motion will be affected by different sources of uncertainty such as the road and weather condition, mechanical conditions, drivers’ uncertain behavior and so on. There are numerous research articles both from macroscopic and microscopic point of views of the traffic flow that emphasize on importance of incorporating uncertainty into the dynamical models to explain the oscillation and perturbation observed in empirical data and in practice. Jost and Nagel [22] discuss the road capacity in presence of uncertainty. They consider the Kruass model [23] and include a random number \( \eta \in [0, 1] \) with intensity \( \sigma_0 \) added to the so called desired velocity. Through experiments, they discuss how different values of these parameters contribute to different states of the traffic flow. Mahnke et al. [24] show that the behavior of the traffic flow can be described similar to the the phase transition in gas-liquid systems. They explain such behavior by time-disceretized stochastic processes. Laval et al. [25] consider an Uhlenbeck- Orntstein process to explain the driver’s uncertain behavior. They do so by incorporating a Brownian noise with constant diffusion to the Newell [26] car following model. They conclude that the instability and oscillations in the collected data can be attributed to the human’s uncertain behavior and not only as a result of unstable nature of some car-following dynamical models. This model is extended in [27] by considering the dependence of the diffusion term on the velocity and constant optimal velocity function. Similar to [25], they consider the drift term to be constant. Ngoduy et al. [28] propose an extended Cox-Ingersoll-Ross process in which the drift term is not constant. They investigate the linear stability of the linearized stochastic dynamical model.

Motivated by the effectiveness of optimal velocity model in explaining the traffic flow dynamics as explained above, and its adaptability to practical data (see [25, 28] and references therein), we consider the stochastic version of the optimal velocity model of (1.2).
For better interpretation of the result, we start by defining the change of variable

$$x^{(n)}_t \overset{\text{def}}{=} x_t^{(0)} - x_t^{(n)} = v_0 t - x_t^{(n)},$$
$$y^{(n)}_t \overset{\text{def}}{=} \dot{x}_t^{(n)} - \dot{x}_t^{(n)} = v_0 - \dot{x}_t^{(n)},$$

(1.4)

for $n \in \{1, \cdots, N\}$, and where

$$x_t^{(0)} = 0$$
$$y_t^{(0)} = 0.$$

This way we re-scale the variables such that the initial positions of the vehicles become positive. Replacing these variables in (1.2), the presentation of the dynamical model will be

$$\dot{x}^{(n)}_t = y^{(n)}_t$$
$$\dot{y}^{(n)}_t = -\alpha \left\{ V \left( \frac{x^{(n)}_t - x^{(n-1)}_t}{d} \right) + y^{(n)}_t - v_0 \right\} - \beta \frac{y^{(n)}_t - y^{(n-1)}_t}{\left( x^{(n)}_t - x^{(n-1)}_t \right)^2}$$

(1.5)

for $n \in \{1, 2, \cdots, N\}$, and the initial condition \((x^{(n)}_0, y^{(n)}_0) = (-x_n, v_0 - v_n)\).

We also consider the stochastic dynamical model

$$dX^{\varepsilon,(n)}_t = Y^{\varepsilon,(n)}_t \, dt$$
$$dY^{\varepsilon,(n)}_t = \left\{ -\alpha \left\{ V \left( \frac{X^{\varepsilon,(n)}_t - X^{\varepsilon,(n-1)}_t}{d} \right) + Y^{\varepsilon,(n)}_t - v_0 \right\} - \beta \frac{Y^{\varepsilon,(n)}_t - Y^{\varepsilon,(n-1)}_t}{\left( X^{\varepsilon,(n)}_t - X^{\varepsilon,(n-1)}_t \right)^2} \right\} dt + \varepsilon_n dW^{(n)}_t$$

(1.6)

for $n \in \{1, \cdots, N\}$, and $x_n$ is introduced in (1.2). The parameters $\varepsilon_n \geq 0$ are the intensity of the noise between the $n$-th and $(n - 1)$-st vehicles, and $W^{(n)}_t$ are i.i.d family of Brownian motions on an abstract probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\), where \((\mathcal{F}_t)\) is a complete and right continuous filtration.
1.2 Dynamical Models

For better accessibility of the dynamical models for future references, in this section we gather different models that will be frequently used throughout these notes. The dynamics of (1.5) for the first two vehicles will be in the form of

\[
\begin{align*}
\dot{x}_t &= y_t \\
\dot{y}_t &= -\alpha \left\{ V \left( \frac{x_t}{d} \right) + y_t - v_o \right\} - \beta \frac{y_t}{x_t^2}
\end{align*}
\]  

(1.7)

In this case, the solution \(x_t\) will be the distance between the two vehicles and \(y_t\) is the difference of velocities. In the stochastic case, these dynamics become

\[
\begin{align*}
dX^\varepsilon_t &= Y^\varepsilon_t dt \\
dY^\varepsilon_t &= \left\{ -\alpha \left\{ V \left( \frac{X^\varepsilon_t}{d} \right) + Y^\varepsilon_t - v_o \right\} - \beta \frac{Y^\varepsilon_t}{X^\varepsilon_t^2} \right\} dt + \varepsilon dW_t.
\end{align*}
\]  

(1.8)

The initial value for both cases is considered to be \(z_o = (x_o, y_o)^T \overset{\text{def}}{=} (-x_1, v_o - v_1)^T\).

Let

\[
B_o(z) \overset{\text{def}}{=} \begin{pmatrix} y \\ -\alpha \left\{ V \left( \frac{z}{d} \right) + y - v_o \right\} - \beta \frac{y}{x^2} \end{pmatrix}, \quad z = (x, y)^T \in \mathbb{R}^2.
\]  

(1.9)

The optimal velocity model (1.7) can be written in the vector form of

\[
\dot{z}_t = B_o(z_t),
\]  

(1.10)

with \(z_t = (x_t, y_t)^T\). The stochastic counterpart will be

\[
dZ^\varepsilon_t = B_o(Z^\varepsilon_t)dt + \varepsilon dW_t,
\]  

(1.11)

with \(Z^\varepsilon_t = (X^\varepsilon_t, Y^\varepsilon_t)^T\).
The vector form of (1.5) can be written as

\[
\begin{align*}
\dot{x}_t &= y_t \\
\dot{y}_t &= b(x_t, y_t),
\end{align*}
\]  

(1.12)

where, for \( x, y \in \mathbb{R}^N \)

\[
b(x, y) \overset{\text{def}}{=} \begin{pmatrix} b^{(1)}(x, y) \\ \vdots \\ b^{(N)}(x, y) \end{pmatrix}
\]  

(1.13)

and

\[
b^{(n)}(x, y) \overset{\text{def}}{=} -\alpha \left\{ \mathcal{V} \left( \frac{x^{(n)}_t - x^{(n-1)}_t}{d} \right) + y^{(n)}_t - v_o \right\} - \beta \frac{y^{(n)}_t - y^{(n-1)}_t}{(x^{(n)}_t - x^{(n-1)}_t)^2}
\]  

(1.14)

Here,

\[
x_t = \begin{pmatrix} x^{(1)}_t \\ \vdots \\ x^{(N)}_t \end{pmatrix}, \quad y_t = \begin{pmatrix} y^{(1)}_t \\ \vdots \\ y^{(N)}_t \end{pmatrix}.
\]

In the stochastic case, we have

\[
dX_t = Y_t dt \\
dY_t = b(X_t, Y_t) dt + \varepsilon dW_t
\]  

(1.15)

where,

\[
W_t = \begin{pmatrix} W_{1,t} \\ \vdots \\ W_{N,t} \end{pmatrix}
\]

and \( W_{i,t} \)’s are independent standard Wiener processes.

We can also consider the vector form of

\[
\dot{z}_t = B(z_t)
\]  

(1.16)
where, for $z = (x, y)^T \in \mathbb{R}^{2N}$ for $x, y \in \mathbb{R}^N$ and
\[
B(z) = \begin{pmatrix} y \\ b(x, y) \end{pmatrix}
\tag{1.17}
\]
and $b(x, y)$ is defined in (1.13). Similarly, for the stochastic case
\[
dZ_t^\varepsilon = B(Z_t^\varepsilon) + \varepsilon \Xi dW_t,
\tag{1.18}
\]
where,
\[
\Xi = \begin{pmatrix} 0_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & 1_{N \times N} \end{pmatrix} \in \mathbb{R}^{2N \times 2N}, \quad W_t = \begin{pmatrix} U_t \\ W_t \end{pmatrix} \in \mathbb{R}^N \times \mathbb{R}^N \tag{1.19}
\]
$W_t$ and $U_t$ are independent $\mathbb{R}^N$-valued Wiener processes.

### 1.2.1 Notation Convention

- We use $x_t$ and $x(t)$ alternatively, as solution of a differential equation. This provides more flexibility in different sections with different subscripts and superscripts.

- The solution of the deterministic system is denoted by small letter, for example $x_t$.

- For presenting a solution of stochastic differential equations, we use the block letters. For example, $X_t$.

- For the solutions in multi-dimension we usually use bold face letters. For example,
\[
x(t) = \begin{pmatrix} x^{(1)}(t) \\ \vdots \\ x^{(N)}(t) \end{pmatrix}.
\]

- For a random process $(\xi_t)$, we use both $\xi_t(\omega)$ and $\xi(t, \omega)$ equivalently depends on the context.
CHAPTER 2

DETERMINISTIC AND STOCHASTIC PERTURBATION OF OPTIMAL VELOCITY DYNAMICAL MODEL

Motivated by applications in autonomous driving and traffic dynamics, we are interested in understanding the behavior of the dynamical models (1.5) and (1.6). Asymptotic methods and in particular perturbation theory and the averaging principles are ubiquitous tools in analyzing nonlinear systems which provide an approximation to the solution of a nonlinear dynamics. After the seminal work of [29], in deterministic perturbation and averaging [30, 31, 32, 33] and in stochastic perturbation [34, 35, 36, 37], among others, made fundamental contributions to the theory.

In this chapter, we consider the optimal velocity dynamics of (1.5) (resp. (1.6)) perturbed by a fast varying function (resp. a random noise). Using some standard tools from perturbation theory, we study the deviation of the perturbed solution from the trajectory of the averaged (resp. unperturbed) dynamics.

In the present chapter, for simplicity of the analysis and since our main focus is not on the analysis of collision, we consider $\beta = 0$ (ignoring the singularity term).

2.1 Deterministic Fast Perturbation

We start by introducing a deterministic fast perturbation into equations of (1.7) (in two-dimension) with $\beta = 0$ and we discuss the asymptotic behavior of these dynamics [38].
More specifically, we consider

\[
\begin{align*}
\dot{x}^\varepsilon(t) &= y^\varepsilon(t) \\
\dot{y}^\varepsilon(t) &= -\alpha \left\{ V\left(\frac{x^\varepsilon(t)}{d}\right) + y^\varepsilon(t) - v_o \right\} + g(t/\varepsilon) \\
(x^\varepsilon(0), y^\varepsilon(0)) &= (x_o, y_o),
\end{align*}
\] (2.1)

for a smooth and bounded function \( g : \mathbb{R} \to \mathbb{R} \). In particular, let

\[
\sup_{t \in \mathbb{R}} |g(t)| \leq M_g.
\] (2.2)

Such perturbations usually present the periodic and rotational noises. We like to study the deviation of (2.1) from the trajectory of an unperturbed dynamic over the time interval \([0, T]\), for some \( T > 0 \) (see Theorem 2.1.2). We need some results in uniform boundedness of the solution first.

**Theorem 2.1.1.** Let \((x^\varepsilon(t), y^\varepsilon(t))^T\) be the flow of dynamics of (2.1) on interval \([0, T]\). Then there exists a constant \( M > 0 \) such that:

\[
\sup_{\varepsilon > 0} \sup_{t \in [0, T]} |y^\varepsilon(t)| \leq M.
\]

**Proof.** We use an idea similar to dissipation in Hamiltonian systems to bound the solution. Let’s define a function \( H : \mathbb{R}^2 \to \mathbb{R} \):

\[
H(x, y) \overset{\text{def}}{=} \frac{1}{2} y^2 + W(x),
\] (2.3)

where

\[
W(x) \overset{\text{def}}{=} \alpha \int_0^x \left\{ V\left(\frac{u}{d}\right) - v_o \right\} du
\]

\[
= -\frac{\alpha d}{2} \log \left(1 - \tanh^2 \left(\frac{x - x_o}{d} \right)\right) + \alpha x \tanh(2) - \alpha x, \quad x \in \mathbb{R}.
\]

Then we have that

\[
W'(x) = \alpha \left\{ V\left(\frac{x}{d}\right) - v_o \right\}.
\]
Figure 2.1: Function $W(x)$, for $d = 5, v_\circ = 0.5, \alpha = 10$.

We consider $v_\circ \in (0, 1 + \tanh(2))$ so that function $V$ is invertiable. Let

$$x_\infty \overset{\text{def}}{=} d \cdot V^{-1}(v_\circ)$$

Then by the fact that function $V$ is monotone increasing function, $W'(x) < 0$ for $x < x_\infty$, $W'(x_\infty) = 0$ and $W'(x) > 0$ for $x > x_\infty$. On the other hand,

$$W''(x) = \frac{\alpha}{d} (1 - \tanh^2(\frac{x}{d} - 2)) > 0$$

Therefore, putting all together and by the fact that $W$ is continuous on $\mathbb{R}$, we have that

$$W \overset{\text{def}}{=} \inf_{x} W(x) > -\infty.$$  

Figure 2.1 shows the behavior of function $W$.

Considering function $H$ defined in (2.3), we have that:

$$\partial_x H(x, y) = \alpha \left\{ V\left(\frac{x}{d}\right) - v_\circ \right\}$$

$$\partial_y H(x, y) = y.$$
Therefore,

\[
\frac{d}{dt} H(x^\varepsilon(t), y^\varepsilon(t)) = \dot{x}^\varepsilon(t) \partial_x H(x^\varepsilon(t), y^\varepsilon(t)) + \dot{y}^\varepsilon(t) \partial_y H(x^\varepsilon(t), y^\varepsilon(t))
\]

\[
= \alpha y^\varepsilon(t) \left\{ V \left( \frac{x^\varepsilon(t)}{d} \right) - v_o \right\} + y^\varepsilon(t) \left\{ -\alpha \left\{ V \left( \frac{x^\varepsilon(t)}{d} \right) - v_o + y^\varepsilon(t) \right\} + g(t/\varepsilon) \right\}
\]

\[
= -\alpha (y^\varepsilon(t))^2 + y^\varepsilon(t) g(t/\varepsilon)
\]

\[
\leq y^\varepsilon(t) g(t/\varepsilon)
\]

Using Young’s inequality, we have that

\[
H(x^\varepsilon(t), y^\varepsilon(t)) - H(x_o, y_o) \leq \frac{1}{2} \int_0^t (y^\varepsilon(s))^2 \, ds + \frac{1}{2} M_g^2 T.
\]

In addition, by the definition of \( H \) and \( W \):

\[
H(x, y) \geq \frac{1}{2} y^2 + W.
\]

This implies that on \([0, T]\):

\[
\frac{1}{2} (y^\varepsilon(t))^2 \leq H(x_o, y_o) - W + \frac{1}{2} \int_0^t (y^\varepsilon(s))^2 \, ds + \frac{1}{2} M_g^2 T.
\]

Therefore by applying Gronwall’s inequality we have

\[
\sup_{\varepsilon > 0} \sup_{t \in [0, T]} |y^\varepsilon(t)| \leq \left\{ \sqrt{2H(x_o, y_o) - 2W + M_g^2 T} \right\} e^{T/2}
\]

which completes the proof.

For continuous functions \( f_1 : [0, T] \to \mathbb{R}, \ f_2 : [0, T] \to \mathbb{R} \) and \( f = (f_1, f_2) : [0, T] \to \mathbb{R}^2 \), we consider the following norms

\[
\|f_i\|_{C_0,T(\mathbb{R})} \overset{\text{def}}{=} \sup_{t \in [0, T]} |f_i(t)|, \quad i = 1, 2.
\]
\[ \| f \|_{C_0,T(\mathbb{R}^2)} \overset{\text{def}}{=} \max_i \| f_i \|_{C_0,T(\mathbb{R})}. \]

We will explain these norms in more details in the next section, where we generalize the results to higher dimensions.

**Theorem 2.1.2.** Let \( g : \mathbb{R} \to \mathbb{R} \) be a bounded and continuous function such that

\[ \bar{g} \overset{\text{def}}{=} \lim_{T \to \infty} \frac{1}{T} \int_0^T g(t) dt, \quad (2.5) \]

is well-defined. Then the dynamics of (2.1) can be uniformly approximated by

\[
\begin{align*}
\dot{x}^\circ(t) &= y^\circ(t), \\
\dot{y}^\circ(t) &= -\alpha \left\{ V \left( \frac{x^\circ(t)}{d} \right) + y^\circ(t) - v_o \right\} + \bar{g}, \\
(x^\circ(0), y^\circ(0)) &= (x_o, y_o), \\
\end{align*}
\]

over time interval \([0, T]\) for sufficiently small \( \varepsilon \). In other words,

\[ \| z^\varepsilon(t) - z^\circ(t) \|_{C_0,T(\mathbb{R}^2)} \to 0, \quad \text{as} \ \varepsilon \to 0, \]

where

\[
\begin{pmatrix} x^\varepsilon(t) \\ y^\varepsilon(t) \end{pmatrix}, \quad \begin{pmatrix} x^\circ(t) \\ y^\circ(t) \end{pmatrix},
\]

and \( z^\circ(t) \) is the trajectory of the averaged dynamics of (2.6).

**Proof.** Let us define

\[ A(t) \overset{\text{def}}{=} \int_0^t \{ g(s) - \bar{g} \} ds, \quad (2.7) \]

which is well-defined by boundedness and continuity of function \( g \). Thus, for any \( t \in \mathbb{R}_+ \) and for any \( \varepsilon > 0 \) we have

\[ \int_0^t \{ g(s/\varepsilon) - \bar{g} \} ds = \varepsilon \int_0^{t/\varepsilon} \{ g(s) - \bar{g} \} ds = \varepsilon A(t/\varepsilon). \quad (2.8) \]
Let
\[ L(x, y) \overset{\text{def}}{=} -\alpha \left\{ V\left(\frac{x}{d}\right) + y - v_0 \right\}. \]

In the integral form, we have
\[
y^\varepsilon(t) = y_0 + \int_0^t L(x^\varepsilon(s), y^\varepsilon(s))ds + \int_0^t g(s/\varepsilon)ds
\]
\[
= y_0 + \int_0^t L(x^\varepsilon(s), y^\varepsilon(s))ds + \int_0^t (g(s/\varepsilon) - \bar{g})ds + \int_0^t \bar{g}ds
\]
\[
= y_0 + \int_0^t L(x^\varepsilon(s), y^\varepsilon(s))ds + \varepsilon A(t/\varepsilon) + \bar{g}t. \tag{2.9}
\]

This implies that
\[
\sup_{t \in [0,T]} \left| y^\varepsilon(t) - \left( y_0 + \int_0^t L(x^\varepsilon(s), y^\varepsilon(s))ds + \bar{g}t \right) \right| = \sup_{t \in [0,T]} |\varepsilon A(t/\varepsilon)|. \tag{2.10}
\]

First, we show that the right hand side is well-defined. By (2.8) we have that
\[
\sup_{t \in [0,T]} |\varepsilon A(t/\varepsilon)| = \sup_{t \in [0,T]} \left| \varepsilon \int_0^{t/\varepsilon} (g(s) - \bar{g})ds \right|. \]

Therefore, by assumption of theorem 2.1.2 on boundedness of \( g \) and consequently boundedness of \( \bar{g} \) by (2.5), for any fixed \( \varepsilon > 0 \) we have that
\[
\sup_{t \in [0,T]} |\varepsilon A(t/\varepsilon)| \leq \varepsilon \int_0^{T/\varepsilon} |g(s) - \bar{g}|ds \leq 2M_gT < \infty. \tag{2.11}
\]

Therefore, by (2.11), by definition (2.7) of the function \( A \) and continuity of the function \( g \) on \( \mathbb{R} \), for any fixed \( \varepsilon > 0 \)
\[
t \in [0, T] \mapsto \varepsilon A(t/\varepsilon) \in \left( C([0, T], \mathbb{R}), ||\cdot||_{C_{0,T}(\mathbb{R})} \right); \tag{2.12}
\]
the Banach space of continuous functions with uniform norm. In addition, for all \( t \in [0, T] \)
we have that
\[
\lim_{\varepsilon \to 0} \varepsilon A(t/\varepsilon) = t \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{t/\varepsilon} g(s)ds - \bar{g}t = 0,
\] (2.13)
where the last equality is by definition (2.5) of $\bar{g}$. Therefore, by (2.12) and (2.13), we conclude that
\[
\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} |\varepsilon A(t/\varepsilon)| = 0,
\]
and hence by (2.10) we have
\[
\sup_{t \in [0,T]} \left| y^\varepsilon(t) - \left( y_0 + \int_0^t \left( L(x^\varepsilon(s), y^\varepsilon(s)) + \bar{g} \right) ds \right) \right| \to 0,
\] (2.14)
as $\varepsilon \to 0$. We need the following result to show that the limit of $(x^\varepsilon, y^\varepsilon)$ exists in some sense.

**Lemma 2.1.3.** The $(x^\varepsilon, y^\varepsilon)$ has a uniform limit point $(x^*, y^*)$ as $\varepsilon \to 0$. In addition, the limit point $(x^*, y^*)$, solves dynamics of (2.6).

**Proof of Lemma 2.1.3.** We define the collection of functions of the form
\[
\mathcal{F} = \{ y^\varepsilon : \varepsilon > 0 \},
\] (2.15)
such that for each $\varepsilon > 0$, $y^\varepsilon \in C([0,T], \mathbb{R})$ and function $t \mapsto y^\varepsilon(t)$ solves (2.1). Then $\mathcal{F} \subset \left( C([0,T], \mathbb{R}), \mathcal{T}_{|| \cdot ||_{C_0,T}} \right)$ where $\mathcal{T}_{|| \cdot ||_{C_0,T}}$ is the topology of uniform convergence. Equation (2.4) implies that $\mathcal{F}$ is bounded. Using dynamical system (2.1), theorem 2.1.1 and boundedness of $g(t)$, it is straightforward to see that:
\[
\sup_{\varepsilon > 0} |y^\varepsilon(t) - y^\varepsilon(s)| \leq C |t - s|, \quad t, s \in [0,T],
\]
for some constant $C > 0$. This proves the **equicontinuity** of the collection $\mathcal{F}$. Therefore by Arzela-Ascoli theorem we conclude that $\mathcal{F}$ is **totally bounded** (precompact). Hence, any sequence $\{y^{\varepsilon_m}\}_{m \in \mathbb{N}} \subset \mathcal{F}$ has a cauchy subsequence which converges in the Banach space $\left( C([0,T], \mathbb{R}), \mathcal{T}_{|| \cdot ||_{C_0,T}} \right)$. Let us consider any sequence $\{y^{\varepsilon_n}\}_{n \in \mathbb{N}}$ in which $\varepsilon_n \searrow 0$ as $n \nearrow \infty$. 

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In addition, let \( y^* \in C([0, T], \mathbb{R}) \) be the limit point of this sequence. This also implies that

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \left| x_0 + \int_0^t y^{\varepsilon_n}_s ds - x_0 - \int_0^t y^*_s ds \right| = \lim_{n \to \infty} \sup_{t \in [0, T]} \left| \int_0^t (y^{\varepsilon_n} - y^*_s) ds \right|
\]

\[
\leq \lim_{n \to \infty} \sup_{t \in [0, T]} \int_0^t |y^{\varepsilon_n}_s - y^*_s| ds
\]

\[
\leq \lim_{n \to \infty} \int_0^T \sup_{s \in [0, T]} |y^{\varepsilon_n}_s - y^*_s| ds = 0
\]

where the last equality is by dominated convergence theorem which is justified by uniform boundedness of \( y^{\varepsilon} \) of Theorem 2.1.1. In other words,

\[
x^{\varepsilon_n}_t = x_0 + \int_0^t y^{\varepsilon_n}_s ds \to x_0 + \int_0^t y^*_s ds \equiv x^*_t
\]

as \( n \to \infty \) in \( C([0, T], \mathbb{R}), \mathcal{F}_{\| \cdot \|_{C_0, T}}(\mathbb{R}) \). In particular,

\[
x^*(t) = x_0 + \int_0^t y^*(s) ds.
\]

Therefore, from (2.14) and using dominated convergence theorem, we have that

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \left| y^{\varepsilon_n}(t) - \left( y_0 + \int_0^t (\mathbf{L}(x^{\varepsilon_n}(s), y^{\varepsilon_n}(s)) + \bar{g}) ds \right) \right|
\]

\[
= \sup_{t \in [0, T]} \left| y^*(t) - (y_0 + \int_0^t (\mathbf{L}(x^*(s), y^*(s)) + \bar{g}) ds) \right| = 0.
\]

This result suggests that \((x^*(t), y^*(t))\) solves

\[
\dot{x}^*(t) = y^*(t),
\]

\[
\dot{y}^*(t) = \mathbf{L}(x^*(t), y^*(t)) + \bar{g}, \quad t \in [0, T]
\]

with the initial condition \((x_0, y_0)\).

Now let’s continue with the proof of the theorem. The ODE (2.6) has a unique solution. In addition, any limit point satisfies the same ODE. Therefore, the limits of \( x^{\varepsilon} \) and \( y^{\varepsilon} \) exist.
in \( \left( C([0,T],\mathbb{R}), \mathcal{F}_{\|\cdot\|_{C_{0,T}}} \right) \) as \( \varepsilon \to 0 \) (and satisfies the ODE). This completes the proof. \( \square \)

2.1.1 Generalization to Higher Dimensions

Now we consider the generalization of the previous results to higher dimensions. First, we need to extend some of the definitions. In particular, we need to construct a suitable functional space for the solution of the dynamical models in higher dimensions. We start by recalling some topological facts which would helpful in extending the results\(^1\).

Complete Metric Spaces

**Definition 2.1.1.** A metric space \((\mathcal{Y},d)\) is said to be complete if any Cauchy sequence in this space converges.

A More practical way to investigate the completeness of a metric space will be

**Lemma 2.1.4.** A metric space \((\mathcal{Y},d)\) is complete if any Cauchy sequence has a convergent subsequence.

**Proof.** It can be shown that if \((y_n)\) is a Cauchy sequence, and has a convergent subsequence, then \((y_n)\) converges. \( \square \)

**Corollary 2.1.1.** If a metric space \((\mathcal{X},d)\) is compact then it is complete.

**Proof.** In metric spaces, the compactness is equivalent to sequentially compactness. Therefore, if \((x_n)\) is a Cauchy sequence in \((\mathcal{Y},d)\), by compactness, it has a convergent subsequence. The previous Lemma then implies the completeness of the metric space. \( \square \)

Uniform and Sup Metrics

In this section, we will introduce some metrics on the space \( \mathcal{G} \) of functions from \( \mathcal{X} \) to \( \mathcal{Y} \). This will lead to some properties of the space \( C(\mathcal{X},\mathcal{Y}) \subset \mathcal{G} \) of continuous functions from \( \mathcal{X} \)

---

\(^1\)The topological results in this section for metric spaces and Ascoli’s theorem, are mostly from Munkres [39].
For a metric space \((Y, d)\), a standard bounded metric corresponding to metric \(d\) will be

\[
\tilde{d}(a, b) = \min \{d(a, b), 1\}, \quad a, b \in Y.
\]

Let \(X\) be a topological space and \((Y, d)\) be a metric space. Then for the function space \(G\) we define the uniform metric

\[
\rho(f, g) = \sup \{\tilde{d}(f(x), g(s)) : x \in X\}, \quad f, g \in G.
\]

It can be shown that \(\rho\) is a metric on \(G\). If sequence \((f_n) \subset G\) converges to a function \(f\) in the uniform metric topology, then the convergence is uniform (which justifies the name) in the sense that

\[
\lim_{n \to \infty} \sup_{x \in X} \tilde{d}(f_n(x), f(x)) = 0.
\]  \hspace{1cm} (2.16)

Therefore, the space the metric space \((G, \rho)\) is complete.

**Lemma 2.1.5.** The space \(C(X, Y)\) is a closed subspace of \(G\) under the uniform metric and consequently it is complete with respect to this metric.

**Proof.** Let \(f \in G\) be a limit point of the space \(C(X, Y)\). Since this is a metric space, there exists a sequence \((f_n) \subset C(X, Y)\) that converges to \(f\) in uniform metric topology. By the previous remark, such convergence is uniform in the sense of (2.16). Therefore, function \(f\) is also continuous and hence \(C(X, Y)\) contains its limit points and so is closed under the uniform metric.

A similar result can be proven for the space \(B(X, Y)\) of the bounded functions from \(X\) to \(Y\). Furthermore, on this space we can define the sup norm

\[
\rho(f, g) = \sup \{d(f(x), g(x)) : x \in X\}.
\]

In particular, \(\rho(f, g) = \min \{\rho(f, g), 1\}\); i.e. the standard bounded metric corresponding to
metric $\rho$. The Cauchy property of a sequence, the convergence of a sequence and consequently the completeness of metric spaces is the same under both metrics. In addition, if $\mathcal{X}$ is a compact topological space, then the sup norm is well defined for space $C(\mathcal{X}, \mathcal{Y})$ as well.

**Compactness in Metric Spaces**

In this section we recall a definition of compactness which is related to the completeness of a metric space. We will then use this result to find compact subsets of $C(\mathcal{X}, \mathbb{R}^N)$ under uniform norm and eventually showing the Ascoli’s theorem.

**Definition 2.1.2.** A metric space $(\mathcal{Y}, d)$ is said to be **totally bounded** if for every $\varepsilon > 0$ there is a finite collection of points $C(\varepsilon) = \{y_1, \cdots, y_n\}$ where $y_j \in \mathcal{Y}$ for all $j$ such that $\bigcup_{j=1}^n \text{Ball}(y_j, \varepsilon)$ covers $\mathcal{Y}$. As a consequence of this definition, if $(\mathcal{Y}, d)$ is totally bounded then for any $y \in \mathcal{Y}$ there exists at least one $y_j \in C(\varepsilon)$ such that $d(y, y_j) < \varepsilon$.

**Theorem 2.1.6.** A metric space $(\mathcal{Y}, d)$ is compact if and only if it is totally bounded and complete.

*Proof.* The if part is easy. As we have shown before, the compactness implies that $\mathcal{Y}$ is complete. Now consider an open covering with $\varepsilon$-balls at each point. But then compactness implies that there should finite open covering and this means that the metric space $\mathcal{Y}$ is totally bounded.

For the converse, we refer the interested reader to [39, Theorem 45.1].

**Definition 2.1.3.** Let $(\mathcal{Y}, d)$ be a metric space and $\mathcal{X}$ a topological space. Let $F \subset C(\mathcal{X}, \mathcal{Y})$. The set $F$ is said to be **equicontinuous** at $x_o \in \mathcal{X}$ if for any $\varepsilon > 0$ the exists a neighborhood $\mathcal{U}$ of $x_o$ such that for all $x \in \mathcal{U}$ and all $f \in F$,

$$d(f(x), f(x_o)) < \varepsilon.$$ 

If set $F$ is equicontinuous at all points of $\mathcal{X}$, then it is said that $F$ is equicontinuous.

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For the Euclidean space $\mathbb{R}^N$ the metric

$$d(a, b) = \max \{|a_i - b_i| : i = 1, \cdots, N\}, \quad a, b \in \mathbb{R}^N$$

is called square metric.

**Lemma 2.1.7.** Let $\mathcal{X}$ be a topological space, and $(\mathcal{Y}, d)$ a metric space. If $\mathcal{F} \subset C(\mathcal{X}, \mathcal{Y})$ is totally bounded in uniform metric corresponding to metric $d$, then $\mathcal{F}$ is equicontinuous with respect to metric $d$.

**Theorem 2.1.8** (Ascoli’s theorem). Let $\mathcal{X}$ be a compact topological space, and $(\mathbb{R}^N, d)$ be the Euclidean space in square metric. Consider $C(\mathcal{X}, \mathbb{R}^N)$ with the uniform topology (topology induced by the uniform/sup metric) corresponding to metric $d$. A subspace $\mathcal{F}$ of $C(\mathcal{X}, \mathbb{R}^N)$ have a compact closure (precompact) if and only if $\mathcal{F}$ is bounded and equicontinuous with respect to metric $d$.

In the case of our problem, we have $\mathcal{X} = [0, T]$ under the standard topology on $\mathbb{R}$ and the sup metric will be defined as

$$\rho(f, g) = \sup \left\{ \max_{i \in I} |f_i(t) - g_i(t)| : t \in [0, T] \right\},$$

for any $f, g \in C([0, T], \mathbb{R}^N)$, and where $I = \{1, \cdots, N\}$.

The sup metric of (2.17) can be induced by the norm

$$\|f\|_{C_0,T(\mathbb{R}^N)} \overset{\text{def}}{=} \max_{i \in I} \|f_i\|_{C_0,T(\mathbb{R})} = \max_{i \in I} \sup_{t \in [0, T]} |f_i(t)|$$

for $f : I \to C([0, T], \mathbb{R})$. In other words, $f \in C([0, T], \mathbb{R}^N)$ can be written in the form of $f = (f_1, \cdots, f_N)$ such that $f_i \in C([0, T], \mathbb{R})$, for any $i \in I$.

**Remark 2.1.9.** By Lemma 2.1.5 and the fact that the metric space $(\mathbb{R}^N, d)$ is complete, we conclude that $C([0, T], \mathbb{R}^N)$ is complete under sup metric. This implies that $C([0, T], \mathbb{R}^N)$ equipped with the norm $\|\cdot\|_{C_0,T(\mathbb{R}^N)}$ is a Banach space. However, we will prove this directly in the next result.
Lemma 2.1.10. The vector space \( C([0, T], \mathbb{R}^N) \) equipped with the norm \( \| \cdot \|_{C_0,T(\mathbb{R}^N)} \) is a Banach space.

Proof. Consider a Cauchy sequence \( \{ f^{(n)} : n \in \mathbb{N} \} \subset C([0, T], \mathbb{R}^N) \); i.e. for any \( \nu > 0 \), there is an \( M > 0 \) such that

\[
\| f^{(n)} - f^{(m)} \|_{C_0,T(\mathbb{R}^N)} < \nu, \quad \text{for any } n, m \geq M.
\]

Equivalently,

\[
\max_i \| f^{(n)}_i - f^{(m)}_i \|_{C_0,T(\mathbb{R})} < \nu, \quad \text{for any } n, m \geq M.
\]

This means that for each \( i \in I \), sequence \( \{ f^{(n)}_i : n \in \mathbb{N} \} \) is a Cauchy sequence. But since \( C([0, T], \mathbb{R}) \) equipped with \( \| \cdot \|_{C_0,T(\mathbb{R})} \) is a Banach space, for any \( i \in I \) there exists an \( f^*_i \in C([0, T], \mathbb{R}) \) such that

\[
\| f^{(n)}_i - f^*_i \|_{C_0,T(\mathbb{R})} \to 0, \quad \text{as } n \to \infty.
\]

Let \( f^* = (f^*_1, \ldots, f^*_N) \). Then

\[
\| f^{(n)} - f^* \|_{C_0,T(\mathbb{R}^N)} = \max_i \| f^{(n)}_i - f^*_i \|_{C_0,T(\mathbb{R})} \to 0, \quad \text{as } n \to \infty,
\]

which implies that sequence \( \{ f^{(n)} : n \in \mathbb{N} \} \) converges and hence the claim follows. \( \square \)

Let \( f_1, f_2 \in C([0, T], \mathbb{R}^N) \). Then for \( f = (f_1, f_2) \in C([0, T], \mathbb{R}^{2N}) \), we define

\[
\| f \|_{C_0,T(\mathbb{R}^{2N})} \overset{\text{def}}{=} \max_i \| f_i \|_{C_0,T(\mathbb{R}^N)}.
\]

The main result of this section is

Theorem 2.1.11. Consider the dynamics of

\[
\begin{align*}
\dot{x}^\varepsilon_i &= y^\varepsilon_i \\
\dot{y}^\varepsilon_i &= b(x^\varepsilon_i, y^\varepsilon_i) + g(t/\varepsilon) \\
(x_0, y_0) &= (x_0, y_0)
\end{align*}
\] (2.18)
where
\[
\mathbf{b}^{(n)}(x_t^\varepsilon, y_t^\varepsilon) = -\alpha \left\{ V \left( \frac{x_t^\varepsilon(n) - x_t^{\varepsilon(n-1)}}{d} \right) + \frac{y_t^\varepsilon(n) - y_t^{\varepsilon(n-1)}}{d} \right\} - \beta \left( \frac{y_t^\varepsilon(n) - y_t^{\varepsilon(n-1)}}{x_t^\varepsilon(n) - x_t^{\varepsilon(n-1)}} \right)^2,
\]
and \( g = (g_1, \cdots, g_N)^T \in C^1_{\bar{b}}(\mathbb{R}, \mathbb{R}^N) \) with
\[
\|g\|_{C_{0, T}(\mathbb{R}^N)} \leq M_g.
\]
Then for the integral curve \( t \mapsto z_t^\varepsilon = (x_t^\varepsilon, y_t^\varepsilon)^T \in \mathbb{R}^N \times \mathbb{R}^N \) we have that
\[
\lim_{\varepsilon \to 0} \|z_t^\varepsilon - z_t\|_{C_{0, T}(\mathbb{R}^{2N})} = 0,
\]
where \( z_t = (x_t, y_t)^T \) is the flow of
\[
\begin{align*}
\dot{x}_t &= y_t \\
\dot{y}_t &= \mathbf{b}(x_t, y_t) + \bar{g},
\end{align*}
\]
with the initial value \((x_0, y_0)\). The vector \( \bar{g} \in \mathbb{R}^N \) is defined (component-wise) by
\[
\bar{g} \overset{\text{def}}{=} \lim_{T \to \infty} \frac{1}{T} \int_0^T g(t) dt.
\]

Proof. We need the following uniform boundedness result first.

**Lemma 2.1.12.** We have that
\[
\sup_{\varepsilon > 0} \|y_t^\varepsilon\|_{C_{0, T}(\mathbb{R}^N)} < \infty.
\]

**Proof of Lemma 2.1.12.** We define \( \Phi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) by
\[
\Phi(q, y) \overset{\text{def}}{=} \frac{1}{2} |y|^2 + \sum_{n=1}^N \alpha \int_0^{q_n} \left\{ V \left( \frac{r}{d} \right) - v_0 \right\} dr.
\]
where \( \mathbf{q} = (q_1, \cdots, q_N)^T \). Here, we define
\[
|\mathbf{y}| \overset{\text{def}}{=} \left( \sum_{i=1}^{N} |y_i|^2 \right)^{\frac{1}{2}} = \|\mathbf{y}\|_{L^2(\mathbb{R}^N)}.
\]

In addition, for simplicity we define
\[
q_{1,t}^\varepsilon \overset{\text{def}}{=} x_t^{\varepsilon(1)},
q_{n,t}^\varepsilon \overset{\text{def}}{=} x_t^{\varepsilon(n)} - x_{t-1}^{\varepsilon(n-1)}, \quad n \geq 2.
\]

Then we have
\[
\frac{d}{dt} \Phi(\mathbf{q}_t^\varepsilon, \mathbf{y}_t^\varepsilon) = \sum_{n=1}^{N} q_{n,t}^\varepsilon \partial_{y_n} \Phi(\mathbf{q}_t^\varepsilon, \mathbf{y}_t^\varepsilon) + \sum_{n=1}^{N} y_{t}^{\varepsilon(n)} \partial_{q_n} \Phi(\mathbf{q}_t^\varepsilon, \mathbf{y}_t^\varepsilon)
\]
\[
= \sum_{n=1}^{N} \alpha \left( y_t^{\varepsilon(n)} - y_{t-1}^{\varepsilon(n-1)} \right) \left\{ V \left( \frac{q_{n,t}^\varepsilon}{d} \right) - v_o \right\}
\]
\[
+ \sum_{n=1}^{N} \left( -\alpha \left\{ V \left( \frac{q_{n,t}^\varepsilon}{d} \right) - v_o \right\} + y_t^{\varepsilon(n)} \right) y_t^{\varepsilon(n)} + \sum_{n=1}^{N} g_n(t/\varepsilon) y_t^{\varepsilon(n)},
\]

where \( x_t^{\varepsilon(0)} = y_t^{\varepsilon(0)} = 0 \). By applying Young’s inequality, we have that
\[
\frac{d}{dt} \Phi(\mathbf{q}_t^\varepsilon, \mathbf{y}_t^\varepsilon) \leq \alpha \sum_{n=1}^{N} y_t^{\varepsilon(n-1)} \left| V \left( \frac{q_{n,t}^\varepsilon}{d} \right) - v_o \right| + \sum_{n=1}^{N} |g_n(t/\varepsilon)| y_t^{\varepsilon(n)}
\]
\[
\leq \frac{\alpha}{2} N \left( |\mathbf{y}_t^\varepsilon|^2 + K^2 \right) + \frac{1}{2} N \left( M_g^2 + |\mathbf{y}_t^\varepsilon|^2 \right)
\]

By integrating both sides, we have that
\[
\Phi(\mathbf{q}_t^\varepsilon, \mathbf{y}_t^\varepsilon) \leq \Phi(q_o, y_o) + \frac{1}{2} \alpha N K \int_0^t |\mathbf{y}_s^\varepsilon|^2 \, ds + \frac{1}{2} \alpha N K^2 T + \frac{1}{2} \int_0^t |\mathbf{y}_s^\varepsilon|^2 \, ds + \frac{1}{2} M_g^2 T
\]
\[
\leq \Phi(q_o, y_o) + K \int_0^t |\mathbf{y}_s^\varepsilon|^2 \, ds + K, \quad t \in [0, T]
\]
where constant $K$ depends on $N$, $T$, $M_g$ and the bound on function $V$. In addition, as proved in 2-dim case,

$$W \triangleq \inf_{u \in \mathbb{R}} W(u) = \inf_{u \in \mathbb{R}} \alpha \int_0^u \left\{ V \left( \frac{u'}{d} \right) - v_0 \right\} du' > -\infty.$$ 

Putting all together, from definition of function $\Phi$ we have that

$$\frac{1}{2} |y_\varepsilon|^2 \leq \Phi (q_\varepsilon, y_\varepsilon) - NW \leq \Phi (q_\varepsilon, y_\varepsilon) - NW + K + K \int_0^t |y_\varepsilon|^2 ds$$

Then by Gronwall’s inequality we have

$$|y_\varepsilon| \leq \sqrt{2(\Phi (q_\varepsilon, y_\varepsilon) + K)} e^{KT}$$

which in turn implies that

$$\sup_{\varepsilon > 0} \|y_\varepsilon\|_{C_{0,T} (\mathbb{R}^N)} = \sup_{\varepsilon > 0} \left( \max_n \sup_{t \in [0,T]} |y_{\varepsilon,n}| \right) \leq \sqrt{2(\Phi (q_\varepsilon, y_\varepsilon) + K)} e^{KT}$$

This completes the proof of uniform boundedness in higher dimension.

In analogy with the 2-dim case, we define

$$A(t) \triangleq \int_0^t (g(s) - \bar{g}) ds, \quad t \in \mathbb{R}_+,$$

where the integral is defined component-wise. By continuity and boundedness of $g$, function $A$ is well-defined on $\mathbb{R}_+$. For any $t \in \mathbb{R}_+$ and $\varepsilon > 0$, we have

$$\int_0^t (g(s/\varepsilon) - \bar{g}) ds = \varepsilon A(t/\varepsilon).$$
Similar to the 2-dim case we can write

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \left| \varepsilon A^{(n)}(t/\varepsilon) \right| = 0,$$

and hence for \( n \geq 1 \)

$$\sup_{t \in [0,T]} \left| y^{\varepsilon,(n)}_t - \left( y_0^{(n)} + \int_0^t b^{(n)}(x_s^\varepsilon, y_s^\varepsilon) ds + \bar{g}_n t \right) \right| \to 0, \quad \text{as} \ \varepsilon \to 0.$$

This implies that

$$\lim_{\varepsilon \to 0} \left\| y^\varepsilon - \left( y_0 + \int_0^t [b(x_s^\varepsilon, y_s^\varepsilon) + \bar{g}] ds \right) \right\|_{C_0,T(\mathbb{R}^N)} = 0. \quad (2.20)$$

The next step is to show that \((x^\varepsilon, y^\varepsilon)\) converges uniformly. To do so, we define collection of solutions of (2.18) by

$$\mathcal{F} = \{ y^\varepsilon : \varepsilon > 0 \} \subset \left( C([0,T], \mathbb{R}^N), \mathcal{T}_{\|\cdot\|_{C_0,T(\mathbb{R}^N)}} \right)$$

where \( \mathcal{T}_{\|\cdot\|_{C_0,T(\mathbb{R}^N)}} \) denotes the topology induced by the uniform norm.

By the uniform boundedness of Lemma 2.1.12, \( \mathcal{F} \) is bounded with respect to metric \( d \).

In addition, for any \( n \geq 1 \), by equations of (2.18), by uniform boundedness of Lemma 2.1.12 and boundedness of \( g_n \), we have that

$$\sup_{\varepsilon > 0} \left| y_t^{\varepsilon,(n)} - y_s^{\varepsilon,(n)} \right| \leq C |t - s|, \quad \text{for all} \ s, t \in [0,T].$$

Taking \( \max_{n \in I} \) on the left hand side immediately implies the equicontinuity of \( \mathcal{F} \) with respect to metric \( d \). Therefore, by Ascoli’s theorem 2.1.8 we conclude that \( \mathcal{F} \) is precompact.

Hence, any sequence \( \{ y^{\varepsilon_n} \} \subset \mathcal{F} \) has a convergent subsequence in \( \mathcal{F} \) (closure of \( \mathcal{F} \)) and consequently is convergent in \( \left( C([0,T], \mathbb{R}^N), \mathcal{T}_{\|\cdot\|_{C_0,T(\mathbb{R}^N)}} \right) \). Let \( \{ y^{\varepsilon_n} \}_{n \in \mathbb{N}} \subset \mathcal{F} \) be such a sequence with \( \varepsilon_n \to 0 \) as \( n \to \infty \) (with a slight abuse of notation, we denote the convergent subsequence with the same notation). Suppose, \( y^* \in C([0,T], \mathbb{R}^N) \) be the uniform limit
point of this sequence. On the other hand, this implies that for any \( j \in \{1, \cdots, N\} \) we have

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \left| x_o^j + \int_0^t y_n^{\varepsilon_j} ds - x_o^j - \int_0^t y_s^* ds \right| = \lim_{n \to \infty} \sup_{t \in [0, T]} \left| \int_0^t (y_n^{\varepsilon_j} - y_s^*) ds \right|
\]

\[
\leq \lim_{n \to \infty} \sup_{t \in [0, T]} \int_0^t |y_n^{\varepsilon_j} - y_s^*| ds
\]

\[
\leq \lim_{n \to \infty} \int_0^T \sup_{s \in [0, T]} |y_n^{\varepsilon_j} - y_s^*| ds = 0
\]

where the last equality is by dominated convergence theorem which is justified by uniform boundedness of \( y^\varepsilon \). In other words, for each \( j \in \{1, \cdots, N\} \)

\[
x_o^{\varepsilon_n,j} \equiv x_o^j + \int_0^t y_n^{\varepsilon_j} ds \to x_o^j + \int_0^t y_s^* ds \equiv x_s^j
\]

as \( n \to \infty \) in \( C([0, T], \mathbb{R}), \mathcal{F}_{\|\cdot\|_{C_0, T(\mathbb{R})}} \). Therefore, \( x^\varepsilon \to x^* \) in \( C([0, T], \mathbb{R}^N), \mathcal{F}_{\|\cdot\|_{C_0, T(\mathbb{R}^N)}} \)

where

\[
x_t^* = x_o + \int_0^t y_s^* ds. \quad (2.21)
\]

Therefore, from (2.20) we have that

\[
0 = \lim_{n \to \infty} \left\| y_n - \left( y_o + \int_0^t [b(x_s^{\varepsilon_n}, y_s^{\varepsilon_n}) + \bar{g}] ds \right) \right\|_{C_0, T(\mathbb{R}^N)}
\]

\[
\left\| y_t^* - \left( y_o + \int_0^t [b(x_s^*, y_s^*) + \bar{g}] ds \right) \right\|_{C_0, T(\mathbb{R}^N)}.
\]

This result along with (2.21) suggests that \( (x^*, y^*)^T \) satisfies

\[
\dot{x}_t^* = y_t^*
\]

\[
\dot{y}_t^* = b(x_t^*, y_t^*) + \bar{g}, \quad t \in [0, T]
\]

with initial value \((x_o, y_o)^T\). Therefore, by uniqueness of the solution of the averaged ODE and the fact that any uniform limit point of \( x^\varepsilon \) and \( y^\varepsilon \) satisfies the same ODE, the claim follows.
2.2 Convergence Rate of Perturbed Optimal Velocity Dynamical Model

In previous section we showed that the perturbed optimal velocity model converges to the averaged dynamics. In this section, we continue our discussion by showing the rate of such convergence [40].

2.2.1 Notations

Let us fix some notations before we proceed. We will be using all the notations from the previous section.

In addition, for a bounded and continuous function $f$ in $\mathbb{R}^n$, we define

$$\|f\|_{C(\mathbb{R}^n)} \overset{\text{def}}{=} \sup_{x \in \mathbb{R}^n} |f(x)|$$

This norm will be used in several places in this section, including theorem 2.2.2 when we discuss the boundedness of a particular function.

In proving theorem 2.2.2 it is easier to employ the multi-index notation since we will use Taylor series representation of functions. Consider a vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ such that

$$|\alpha| \overset{\text{def}}{=} \alpha_1 + \cdots + \alpha_n = k$$

and a function $f \in C^k(\mathbb{R}^n)$. Then

$$\mathcal{D}^\alpha f(x) \overset{\text{def}}{=} \frac{\partial^\alpha f(x)}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_n} x_n}.$$ 

Interested readers can consult any standard book in analysis for more details on the multi-index notations (e.g. see [41]). The second derivative with respect to $x_1$-coordinate of function $f$ is denoted by $\mathcal{D}^{(2,0,\cdots,0)} f$. If there is no chance of confusion we equivalently use $\partial_1 f \equiv \mathcal{D}^{(1,0,\cdots,0)} f$ (and similarly with respect to other coordinates) for simplicity of
notations.

2.2.2 Preliminaries

We consider the dynamical model (2.1) (optimal velocity dynamical model which is perturbed by a function $g(t/\varepsilon)$), and (2.6) the averaged dynamic. We showed in previous section that

$$z^\varepsilon = (x^\varepsilon, y^\varepsilon) \to z^\circ = (x^\circ, y^\circ), \quad \text{in} \quad \left( C([0,T], \mathbb{R}), \mathcal{H}_s \|_{C_0} \right),$$

as $\varepsilon \to 0$, where $z^\circ$ is the integral curved of the averaged problem.

In this section we study the rate of this convergence. Let’s fix the interval $[0, T]$ for some $T > 0$ throughout this section. Let us start with some definitions and notations.

Recall the definition of $\bar{g}$ in (2.5). This condition is usually satisfied for periodic functions. So, we consider $g : \mathbb{R} \to \mathbb{R}$ to be a continuously differentiable, bounded, and 1-periodic function. We define:

$$\bar{g} \overset{\text{def}}{=} \int_0^1 g(s) ds,$$

$$A(t) \overset{\text{def}}{=} \int_0^t (g(s) - \bar{g}) ds,$$

$$\bar{A} \overset{\text{def}}{=} \int_0^1 A(s) ds,$$

$$A^\dagger(t) \overset{\text{def}}{=} \int_0^t \left( A(s) - \bar{A} \right) ds.$$  \hfill (2.22)

Let’s study the essential properties of these functions that we will need later in this section.

**Lemma 2.2.1.** Let function $g \in C^1_b(\mathbb{R})$ and is 1-periodic. Then the following properties hold:

(a) Function $A$ is bounded.

(b) Function $A$ is 1-periodic.

(c) Function $A^\dagger$ is bounded.
Proof. By the definitions in (2.22), function $A$ is a continuous on $\mathbb{R}$. Considering any integer $n$ and periodicity of function $g$ (which is 1-periodic), we have

$$A(n) = \sum_{n' = 0}^{n-1} \int_{n'}^{n+1} (g(s) - \bar{g}) ds$$
$$= n \int_{0}^{1} (g(s) - \bar{g}) = 0.$$

Thus, continuity of function $A$ implies that in fact $A$ is bounded. To prove the second statement, we consider that

$$A(t + 1) = \int_{0}^{t+1} (g(s) - \bar{g}) ds$$
$$= \int_{0}^{1} (g(s) - \bar{g}) ds + \int_{1}^{t+1} (g(s) - \bar{g}) ds$$
$$= \int_{0}^{t} (g(s) - \bar{g}) ds = A(t).$$

The proof of the last statement is the same as the first part by appropriately replacing the function $g$ by $A$ in the proof.

Next result provides an essential bound which will be used in showing the rate of convergence.

**Theorem 2.2.2.** Let $(x^\varepsilon(t), y^\varepsilon(t))$ be the flow of dynamical system (2.1) and $(x^\circ(t), y^\circ(t))$ be that of the averaged system (2.6). Define

$$\zeta_t^\varepsilon \overset{\text{def}}{=} \frac{y^\varepsilon(t) - y^\circ(t)}{\varepsilon} - A(t/\varepsilon).$$

Then there exists an $\varepsilon_0 > 0$ such that

$$\sup_{0 < \varepsilon < \varepsilon_0} \|\zeta_t^\varepsilon\|_{C_{0,T}} < \infty.$$
Proof. For simplicity, let’s define

\[ \mathbf{L}(x, y) \overset{\text{def}}{=} -\alpha \{ V(x/d) + y - v_0 \}, \quad x, y \in \mathbb{R}. \]

By (2.1) and (2.6)

\[ y^\varepsilon(t) - y^\circ(t) = \int_0^t \left\{ \mathbf{L}(z^\varepsilon(s)) - \mathbf{L}(z^\circ(s)) + (g(s/\varepsilon) - \bar{g}) \right\} ds \]

\[ = \int_0^t \left\{ \mathbf{L}(z^\varepsilon(s)) - \mathbf{L}(z^\circ(s)) \right\} ds + \varepsilon A(t/\varepsilon); \tag{2.24} \]

where the last term is by (2.22).

Function \( \mathbf{L} \in C^\infty(\mathbb{R}^2) \) and for any \( s \in [0, T] \) we write the Taylor series about the trajectory of \( z^\circ(s) \). Denoting the partial derivatives by \( \partial_x \) and \( \partial_y \), we have that

\[ \mathbf{L}(z^\varepsilon(s)) - \mathbf{L}(z^\circ(s)) = \partial_x \mathbf{L}(z^\circ(s))(x^\varepsilon(s) - x^\circ(s)) \]

\[ + \partial_y \mathbf{L}(z^\circ(s))(y^\varepsilon(s) - y^\circ(s)) \]

\[ + \sum_{|\beta|=2} \mathcal{E}_\beta(z^\varepsilon(s); z^\circ(s))(z^\varepsilon(s) - z^\circ(s))^\beta, \tag{2.25} \]

where for any \( z, a \in \mathbb{R}^2 \) the error (the remainder) term is defined by (see section 2.2.1 for the multi-index notations):

\[ \mathcal{E}_\beta(z; a) \overset{\text{def}}{=} \frac{|\beta|}{\beta!} \int_0^1 (1 - \gamma)^{|\beta|-1} \mathcal{D}_\beta \mathbf{L}(a + \gamma(z - a)) d\gamma \]

\[ = \frac{2}{\beta!} \int_0^1 (1 - \gamma) \mathcal{D}_\beta \mathbf{L}(a + \gamma(z - a)) d\gamma. \]

From the definition of function \( \mathbf{L} \) and boundedness of function \( V \) and its second order derivatives, it follows that

\[ |\mathcal{E}_\beta(z^\varepsilon(t); z^\circ(t))| \leq \frac{1}{|\beta|!} \sup_{|\beta|=2} \sup_{z \in \mathbb{R}^2} |\mathcal{D}_\beta \mathbf{L}(z)| < \infty. \]

Replacing the error term in the Taylor series and considering that some of the second
order terms vanish, we can write:

\[
L(z^\varepsilon(s)) - L(z^0(s)) = \partial_x L(z^0(s))(x^\varepsilon(s) - x^0(s)) \\
+ \partial_y L(z^0(s))(y^\varepsilon(s) - y^0(s)) \\
+ (x^\varepsilon(s) - x^0(s))^2 \mathcal{E}_{(2,0)}(z^\varepsilon(s); z^0(s)).
\]

Therefore (2.24) and the fact that \(\partial_y L \equiv -\alpha\), imply that

\[
y^\varepsilon(t) - y^0(t) = \int_0^t \partial_x L(z^0(s)) (x^\varepsilon(s) - x^0(s)) \, ds \\
- \alpha \int_0^t (y^\varepsilon(s) - y^0(s)) \, ds \\
+ \int_0^t \mathcal{E}_{(2,0)}(z^\varepsilon(s); z^0(s))(x^\varepsilon(s) - x^0(s))^2 \, ds \\
+ \varepsilon A(t/\varepsilon).
\]

We can rewrite (2.23) in the form of

\[
y^\varepsilon(t) - y^0(t) = \varepsilon \left( \zeta^\varepsilon_t + A(t/\varepsilon) \right),
\]

and replacing it in (2.26), we have

\[
\zeta^\varepsilon_t = \frac{1}{\varepsilon} \int_0^t \partial_x L(z^0(s)) (x^\varepsilon(s) - x^0(s)) \, ds \\
- \alpha \int_0^t (\zeta^\varepsilon_s + A(s/\varepsilon)) \, ds. \\
+ \frac{1}{\varepsilon} \int_0^t \mathcal{E}_{(2,0)}(z^\varepsilon(s); z^0(s))(x^\varepsilon(s) - x^0(s))^2 \, ds.
\]

In addition, from the equation (2.1)

\[
x^\varepsilon(s) - x^0(s) = \int_0^s (y^\varepsilon(r) - y^0(r)) \, dr \\
= \int_0^s \varepsilon (\zeta^\varepsilon_r + A(r/\varepsilon)) \, dr.
\]
Therefore, from equation (2.27) suggests that

\[
\zeta_t^\varepsilon = \int_0^t \partial_z L(z^\circ(s)) \int_0^s (\zeta_r^\varepsilon + A(r/\varepsilon)) dr \, ds \\
- \alpha \int_0^t (\zeta_r^\varepsilon + A(s/\varepsilon)) \, ds \\
+ \varepsilon \int_0^t \mathcal{E}_{(2,0)}(\zeta_r^\varepsilon(s) ; z^\circ(s)) \left( \int_0^s (\zeta_r^\varepsilon + A(r/\varepsilon)) dr \right)^2 \, ds.
\]

By some algebraic work we get

\[
\zeta_t^\varepsilon = \int_0^t \partial_z L(z^\circ(s)) \int_0^s \zeta_r^\varepsilon dr \, ds \\
+ \int_0^t \partial_z L(z^\circ(s)) \int_0^s A(r/\varepsilon) dr \, ds \\
- \alpha \int_0^t \zeta_r^\varepsilon ds - \alpha \int_0^t A(s/\varepsilon) ds \\
+ \varepsilon \int_0^t \mathcal{E}_{(2,0)}(\zeta_r^\varepsilon(s) ; z^\circ(s)) \left( \int_0^s (\zeta_r^\varepsilon + A(r/\varepsilon)) dr \right)^2 \, ds.
\] (2.29)

Considering the definition of \( A^\dagger \) in (2.22) and a simple change of variables in the integral, we have:

\[
\int_0^s A(r/\varepsilon) dr = \int_0^s (A(r/\varepsilon) - \tilde{A}) dr + \int_0^s \tilde{A} dr \\
= \varepsilon A^\dagger(s/\varepsilon) + \tilde{A}s,
\] (2.30)

and thus, (2.29) and (2.30) imply that:

\[
\zeta_t^\varepsilon = \int_0^t \partial_z L(z^\circ(s)) \int_0^s \zeta_r^\varepsilon dr \, ds \\
+ \varepsilon \int_0^t \partial_z L(z^\circ(s)) A^\dagger(s/\varepsilon) ds + \int_0^t \partial_z L(z^\circ(s)) \tilde{A} s ds \\
- \alpha \int_0^t \zeta_r^\varepsilon ds - \alpha \varepsilon A^\dagger(t/\varepsilon) - \alpha \tilde{A} t \\
+ \varepsilon \int_0^t \mathcal{E}_{(2,0)}(\zeta_r^\varepsilon(s) ; z^\circ(s)) \left( \int_0^s (\zeta_r^\varepsilon + A(r/\varepsilon)) dr \right)^2 \, ds.
\] (2.31)

Looking at the integral equation in (2.31), it is reasonable to expect that \( \lim_{\varepsilon \to 0} \zeta^\varepsilon = \zeta^\circ \) in
in the uniform topology where $\zeta^\circ$ satisfies:
\begin{equation}
\zeta^\circ_t = -\alpha \int_0^t \zeta^\circ_s ds + \int_0^t \partial_x L(z^\circ(s)) \int_0^s \zeta^\circ_r dr ds
+ \int_0^t \partial_x L(z^\circ(s)) \bar{A} s ds - \alpha \bar{A} t.
\end{equation}

This will be rigorously shown in the next theorem. Here, having this convergence concept in mind, we try to find a bound on function $\zeta^\circ$. Let’s first define some constants:
\begin{equation}
K_1 \overset{\text{def}}{=} \|\partial_x L\|_{C(\mathbb{R}^2)}, \quad K_2 \overset{\text{def}}{=} \|A^\dagger\|_{C(\mathbb{R})},
K_3 \overset{\text{def}}{=} \|\partial_x L\|_{C(\mathbb{R}^2)} |\bar{A}|, \quad K_4 \overset{\text{def}}{=} \alpha |\bar{A}|
R \overset{\text{def}}{=} \frac{1}{2} K_3 T^2 + K_4 T, \quad M \overset{\text{def}}{=} \alpha + \frac{K_1}{\alpha},
\end{equation}

where the norms are defined in section 2.2.1. We note that by definition of $\partial_x L$ and boundedness of $A^\dagger$ from lemma 2.2.1 both norms are well-defined. Using the Gronwall-Bellman type of inequality (see Appendix D for the details) we get the following bound on $\zeta^\circ$ in (2.32):
\begin{equation*}
\kappa \overset{\text{def}}{=} R \left(1 + \frac{\alpha^2}{\alpha^2 + K_1} e^{MT}\right)
\end{equation*}

We fix $\kappa' > \kappa$ and define
\begin{equation*}
\tau^\circ \overset{\text{def}}{=} \inf\{t > 0 : |\zeta^\circ_t| \geq \kappa'\},
\end{equation*}

We assume $\tau^\circ \leq T$. In other words, we assume that $\zeta^\circ$ exceeds $\kappa'$ before time $T$ and $\tau^\circ$ is the first time this happens.

For any $t \in [0, \tau^\circ]$, from (2.31) and (2.33), we will have
\begin{equation*}
|\zeta^\circ_t| \leq \alpha \int_0^t |\zeta^\circ_s| ds + K_1 \int_0^t \int_0^s |\zeta^\circ_r| dr ds + \frac{1}{2} K_3 T^2 + K_4 T
+ \varepsilon \left\{K_1 K_2 + \alpha K_2 + \|\mathcal{D}^{(2,0)} L\|_{C(\mathbb{R}^2)} \left(\kappa' + \|A\|_{C(\mathbb{R})}\right) T^3\right\}.
\end{equation*}
By applying the Gronwall-Bellman theorem in appendix D we get:

$$|\zeta_\varepsilon^t| \leq \kappa + \varepsilon N \left(1 + \frac{\alpha^2}{\alpha^2 + K_1} e^{MT}\right), \quad t \in [0, \tau^\varepsilon]$$

where

$$N \overset{\text{def}}{=} \left\{ K_1 K_2 + \alpha K_2 + \|D^{(2,0)}L\|_{C(\mathbb{R}^2)} \left(\kappa' + \|A\|_{C(\mathbb{R})}\right) T^3 \right\}.$$ 

Therefore, there exists an \( \varepsilon_0 < \varepsilon \) such that

$$\varepsilon_0 N \left(1 + \frac{\alpha^2}{\alpha^2 + K_1} e^{MT}\right) < \kappa' - \kappa.$$ (2.34)

This implies that for \( 0 < \varepsilon < \varepsilon_0 \):

$$|\zeta^\varepsilon_{\tau^\varepsilon}| < \kappa',$$

which contradicts the definition of \( \tau^\varepsilon \). Therefore, \( \tau^\varepsilon > T \) for \( \varepsilon \in (0, \varepsilon_0) \). In other words,

$$\sup_{0 < \varepsilon < \varepsilon_0} \|\zeta^\varepsilon\|_{C_0, T} \leq \kappa',$$

and this completes the proof. \( \square \)

Next, we employ the result of the previous theorem to show the main result of this section. In particular, in the next theorem the rate of convergence of the \( z^\varepsilon \) to \( z^0 \) will be shown.

**Theorem 2.2.3.** Let \( \zeta^\varepsilon \) be defined in the statement of theorem 2.2.2. Then \( \zeta^\varepsilon \to \zeta^0 \) as \( \varepsilon \to 0 \) in \( \left(C([0, T], \mathbb{R}), \mathcal{F}_{\|\cdot\|_{C_0, T}}\right) \) where

$$\zeta_t^0 = -\alpha \int_0^t \zeta_s^0 ds$$

$$+ \int_0^t \partial_x L(x^\varepsilon(s), y^\varepsilon(s)) \int_0^s \zeta_r^0 dr ds$$

$$+ \int_0^t \partial_x L(x^\varepsilon(s), y^\varepsilon(s)) \bar{A} s ds - \alpha \bar{A} t.$$

**Proof.** By the definition of \( \zeta_t^\varepsilon \) in theorem 2.2.2 and the fact that \( y^\varepsilon \) and \( y^0 \) are the flows
of the equations of (2.1) and (2.6) respectively, function \( t \in [0, T] \mapsto \zeta^\varepsilon_t \) is continuous. We consider a collection of continuous functions of the form

\[
\mathcal{F} \overset{\text{def}}{=} \{ \zeta^\varepsilon : \varepsilon \in (0, \varepsilon_0) \},
\]

such that \( \zeta^\varepsilon \) satisfies (2.31) and \( \varepsilon_0 \) is defined in (2.34). First, we note that

\[
\mathcal{F} \subset \left( C([0, T], \mathbb{F}_{\| \cdot \|_{C_0,T}}) \right).
\]

Considering boundedness of \( \zeta^\varepsilon \) by theorem 2.2.2, using the definition of \( A^\dagger \) from (2.22) and finally employing equation (2.31), we notice that for any \( t, t' \in [0, T] \)

\[
\sup_{\varepsilon \in (0, \varepsilon_0)} |\zeta^\varepsilon_t - \zeta^\varepsilon_s| \leq C|t - s|,
\]

for some constant \( C > 0 \), which implies the equicontinuity of \( \mathcal{F} \). In addition, uniform boundedness of \( \zeta^\varepsilon \) forces the boundedness of \( \mathcal{F} \). Therefore, by Arzela-Ascoli theorem \( \mathcal{F} \) is totally bounded (precompact). Hence, any sequence in \( \mathcal{F} \) has a convergent subsequence in \( \bar{\mathcal{F}} \subset C([0, T], \mathbb{R}) \). Let \( \{ \zeta^{\varepsilon_n} \}_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{F} \) such that \( \varepsilon_n \to 0 \) as \( n \to \infty \). With slight abuse of notation we show such subsequence by \( \{ \zeta^{\varepsilon_n} \}_{n \in \mathbb{N}} \) as well. Suppose now that \( \zeta^* \) is the limit point of this subsequence in the space \( C([0, T], \mathbb{R}) \). Considering dominated convergence theorem and the fact that \( (x^\varepsilon, y^\varepsilon) \to (x^0, y^0) \) as \( \varepsilon \to 0 \) in \( C([0, T], \mathbb{R}) \), equation (2.31) implies that \( \zeta^* \) should satisfy:

\[
\zeta^*_t = -\alpha \int_0^t \zeta^*_s ds \\
+ \int_0^t \partial_x L(x^\circ(s), y^\circ(s)) \int_0^s \zeta^*_r dr ds \\
+ \int_0^t \partial_x L(x^\circ(s), y^\circ(s)) A ds - \alpha A t.
\]
Equivalently, $ζ^\ast$ is the solution of an ODE of the form
\[
\begin{align*}
\dot{ζ}(t) &= η(t) \\
\dot{η}(t) &= -αη(t) + \partial_x L(x^\circ(t), y^\circ(t))ζ(t) \\
&\quad + \partial_x L(x^\circ(t), y^\circ(t))\bar{A}t - α\bar{A} \\
(ζ(0), η(0)) &= (0, 0).
\end{align*}
\]

Existence and uniqueness of solution for this ODE and the fact that the limit point of any convergent subsequence satisfies this ODE implies that $ζ^\circ$ is the unique limit point of $ζ^\varepsilon$ as $\varepsilon \to 0$ in $C([0,T], \mathbb{R})$ and this completes the proof. \hfill \Box

\textbf{Remark 2.2.4.} Considering (2.28) we notice that the same rate of convergence applies to $x^\varepsilon$.

\section{Stochastic Perturbation}

In this section, we study the the optimal velocity dynamical model which is perturbed by some random noises. As mentioned before, the stochastic perturbation of dynamical systems and the averaging principles has been studied in seminal works of [34, 42, 43, 35, 37, 44] among others. We apply some of these classic results to study the deviation of the solution of the optimal velocity model (1.6) from the unperturbed trajectory.

Let’s first look at the case in which the perturbation in the system is created by a bounded and paths continuous random process.

\textbf{Theorem 2.3.1.} Consider the dynamical model\footnote{A similar statement can be found in [34, Theorem 1.1] without a detailed proof.}
\[
\begin{align*}
\dot{x}_t^\varepsilon &= y_t^\varepsilon \\
\dot{y}_t^\varepsilon &= b(x_t^\varepsilon, y_t^\varepsilon) + \varepsilon η(t, ω).
\end{align*}
\]

where $η(t, ω)$ is a bounded and pathwise continuous random process in $\mathbb{R}^N$ and the function
is as (1.13) with $\beta = 0$. Let $t \mapsto z^\varepsilon_t$ be the integral curve of these dynamics, and $t \mapsto z_t$ denotes the integral curve of the corresponding unperturbed dynamics (when $\varepsilon = 0$). Then

$$
\mathbb{P} \left\{ \lim_{\varepsilon \to 0} \sup_{t \in [0,T]} |z^\varepsilon_t - z_t| \to 0 \right\} = 1.
$$

**Proof.** Existence and uniqueness of the solution can be constructed as follows. Let us start with the uniqueness. Suppose $z^\varepsilon_t$ and $z^{\varepsilon'}_t$ are two solutions. Then for a.e. $\omega \in \Omega$ we have

$$
|z^\varepsilon_t - z^{\varepsilon'}_t| \leq \int_0^t |B(z^\varepsilon_s) - B(z^{\varepsilon'}_s)| ds
\leq \tilde{\kappa} \int_0^t |z^\varepsilon_s - z^{\varepsilon'}_s| ds,
$$

where,

$$
B(z) = B(x, y) = \begin{pmatrix} y \\ b(x, y) \end{pmatrix}
$$

Gronwall’s inequality then implies the uniqueness.

Next we construct the solution. Let

$$
z^{(0),\varepsilon}_t = z_0,
$$

and then iteratively we define

$$
z^{(1),\varepsilon}_t = z_0 + \int_0^t B(z^\varepsilon_0) ds + \varepsilon \int_0^t \eta(s, \omega) ds
\quad \text{and} \quad
z^{(n+1),\varepsilon}_t = z_0 + \int_0^t B(z^{(n),\varepsilon}_s) ds + \varepsilon \int_0^t \eta(s, \omega) ds.
$$

It should be noted that for each $n \in \mathbb{N}$, $z^{(n),\varepsilon}$ has continuous sample paths with respect to $t$ and $\varepsilon$. Let’s define

$$
g_n(t) \overset{\text{def}}{=} \sup_{\varepsilon \in [0,1]} \sup_{t \in [0,T]} |z^{(n),\varepsilon}_s - z^{(n-1),\varepsilon}_s|.
$$

Let

$$
\sup_{t \in [0,T]} |\eta(t, \omega)| \leq C'
$$

(2.35)
for \(\mathbb{P}\text{-a.s.} \ \omega\). Then we can calculate

\[
g_1(t) = \sup_{\varepsilon \in [0,1]} \sup_{t \in [0,t]} |z^{(1),\varepsilon}_t - z^{(0),\varepsilon}_t| \leq (|B(z_0)| + C') T
\]

\[
g_n(t) \leq \tilde{k} \int_0^t g_{n-1}(r) dr
\]

Therefore, by induction we have that for a.e. \(\omega \in \Omega\) and \(t \in [0,T]\)

\[
g_n(t) \leq C_T \tilde{k}^{n-1} \frac{t^{n-1}}{(n-1)!}
\]

where \(C_T \overset{\text{def}}{=} (|B(z_0)| + C') T\) and \(C'\) is as of (2.35). Therefore,

\[
\sum_{n \geq 1} g_n(t) < \infty,
\]

or in other words,

\[
\sum_{n \geq 0} \sup_{\varepsilon \in [0,1]} \sup_{t \in [0,T]} |z^{(n+1),\varepsilon}_t - z^{(n),\varepsilon}_t| < \infty, \ \text{a.s.}
\]

This implies that the sequence \((z^{(n),\varepsilon}_t)\) converges uniformly on \(\varepsilon \in [0,1]\) and \(t \in [0,T]\) to a limiting process \((z^\varepsilon_t)\) with a continuous sample paths jointly with respect to \(t\) and \(\varepsilon\). In particular, This shows that

\[
\sup_{\varepsilon \in [0,1]} \sup_{t \in [0,T]} |z^\varepsilon_t| < \infty.
\]

Finally, given the Lipschitz continuity of \(B(\cdot,\cdot)\), for every \(t \in [0,T]\) and \(\varepsilon \in [0,1]\) we have

\[
\lim_{n \to \infty} \left( \int_0^t B(z^\varepsilon_s) ds - \int_0^t B(z^{(n),\varepsilon}_s) \right) = 0,
\]

in probability. Therefore, \((z^\varepsilon_t)\) solves the dynamics. The next part of the theorem in which we let \(\varepsilon \to 0\) is by Arzela-Ascoli theorem similar to what we discussed in previous section. \(\Box\)

Next we look at the deviation of the solution when the perturbation is as the result of a Brownian noise. We consider the optimal velocity dynamics of (1.8) when the drift term
b (as in (1.13)) contains no singular term; i.e. $\beta = 0$. First we discuss the results in two-dimension (Theorem 2.3.2) which reveals the structure of our problem more explicitly and then the generalization of the results to higher dimensions will be discussed (Theorem 2.3.3). The proofs (when $\beta = 0$) follow from some classic results in the literature (see [34, Theorem 1.2]).

**Theorem 2.3.2.** Let $(X^\varepsilon(t), Y^\varepsilon(t))^T$ be the flow of dynamics of (1.8) and $(x(t), y(t))^T$ be the flow of the deterministic dynamical model (1.7). Then

$$
P\left\{ \sup_{t \in [0, T]} |Y^\varepsilon(t) - y(t)| > L \right\} \leq \varepsilon^2 L^{-2} u(T),
$$

for a continuous function $u : [0, T] \rightarrow \mathbb{R}$.

**Proof.** From (1.7) and (1.8), we have

$$
\sup_{t \in [0, T]} |Y^\varepsilon(t) - y(t)| \leq \alpha \int_0^T \left| V \left( \frac{X^\varepsilon(s)}{d} \right) - V \left( \frac{x(s)}{d} \right) \right| ds + \alpha \int_0^T |Y^\varepsilon(s) - y(s)| ds + \varepsilon \sup_{t \in [0, T]} |W(t)|.
$$

Therefore we can write

$$
P\left\{ \sup_{t \in [0, T]} |Y^\varepsilon(t) - y(t)| > L \right\} \leq P\left\{ \alpha \int_0^T \left| V \left( \frac{X^\varepsilon(s)}{d} \right) - V \left( \frac{x(s)}{d} \right) \right| ds > \frac{L}{4} \right\}
$$

$$
+ P\left\{ \alpha \int_0^T |Y^\varepsilon(s) - y(s)| ds > \frac{L}{4} \right\}
$$

$$
+ P\left\{ \varepsilon \sup_{t \in [0, T]} |W(t)| > \frac{L}{2} \right\}
$$

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Now let’s bound the first term.

\[ \mathbb{P} \left\{ \alpha \int_0^T |Y_\varepsilon(s) - y(s)| \, ds > \frac{L}{4} \right\} \leq 16\alpha^2 L^{-2} \mathbb{E} \left( \int_0^T |Y_\varepsilon(s) - y(s)|^2 \, ds \right)^{\frac{1}{2}} \]

\[ \leq 16\alpha^2 L^{-2} T \mathbb{E} \int_0^T (Y_\varepsilon(s) - y(s))^2 \, ds \]

\[ = 16\alpha^2 L^{-2} \int_0^T \mathbb{E} (Y_\varepsilon(s) - y(s))^2 \, ds, \tag{2.36} \]

where the first inequality is by Chebyshev and the second one is by applying the Holder’s inequality and the last equality is by Fubini’s theorem. To calculate (2.36) we apply the Ito’s formula on \( F(Y_\varepsilon(t), y(t)) \), where \( F(v, w) \stackrel{\text{def}}{=} (v - w)^2 \)

\[ (Y_\varepsilon(t) - y(t))^2 = -2\alpha \int_0^t \left\{ (Y_\varepsilon(s) - y(s)) \left\{ V \left( \frac{X_\varepsilon(s)}{d} \right) - V \left( \frac{x(s)}{d} \right) \right\} \right\} \, ds \]

\[ - 2\alpha \int_0^t (Y_\varepsilon(s) - y(s))^2 \, ds + 2\varepsilon \int_0^t (Y_\varepsilon(s) - y(s)) \, dW(s) + \varepsilon^2 t. \]

Dropping the negative term, taking expectation from both sides and the fact that expectation of the martingale term is zero, we obtain

\[ \mathbb{E} [Y_\varepsilon(t) - y(t)]^2 \leq -2\alpha \int_0^t \mathbb{E} \left\{ (Y_\varepsilon(s) - y(s)) \left[ V \left( \frac{X_\varepsilon(s)}{d} \right) - V \left( \frac{x(s)}{d} \right) \right] \right\} \, ds + \varepsilon^2 T. \]

Applying Young’s inequality, we have

\[ \mathbb{E} [Y_\varepsilon(t) - y(t)]^2 \leq \alpha \int_0^t \mathbb{E} [Y_\varepsilon(s) - y(s)]^2 \, ds + \alpha \int_0^t \mathbb{E} \left[ V \left( \frac{X_\varepsilon(s)}{d} \right) - V \left( \frac{x(s)}{d} \right) \right]^2 \, ds \]

\[ + \varepsilon^2 T \]

\[ \tag{2.37} \]

The function \( V \) satisfies Lipschitz continuity by its definition and suppose \( \kappa_V \) denotes the
Lipschitz constant. Thus we can write:

$$\left| V\left(\frac{X^\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right) \right| \leq \kappa V |X^\varepsilon(s) - x(s)|$$

$$\leq \kappa V \int_0^s |Y^\varepsilon(r) - y(r)| \, dr.$$ 

Therefore, by applying Holder’s inequality on the right hand side, we obtain:

$$\left[ V\left(\frac{X^\varepsilon(s)}{d}\right) - V\left(\frac{x(s)}{d}\right) \right]^2 \leq \kappa^2 V T \int_0^s (Y^\varepsilon(r) - y(r))^2 \, dr. \quad (2.38)$$

Replacing this in (2.37), for any \( t \in [0, T] \) we have

$$\mathbb{E} [Y^\varepsilon(t) - y(t)]^2 \leq \alpha \int_0^t \mathbb{E} [Y^\varepsilon(s) - y(s)]^2 \, ds$$

$$+ \alpha \kappa^2 V T \int_0^t \int_0^s \mathbb{E} [Y^\varepsilon(r) - y(r)]^2 \, dr \, ds + \varepsilon^2 T$$

The Gronwall-Bellman type inequality [40, Appendix] implies that for any \( t \in [0, T] \):

$$\mathbb{E} [Y^\varepsilon(t) - y(t)]^2 \leq \varepsilon^2 T \left( 1 + \frac{\alpha}{\alpha + \kappa^2 V T} e^{(\alpha + \kappa^2 V T) t} \right) \quad (2.39)$$

Therefore, from (2.36) we may write:

$$\mathbb{P} \left\{ \alpha \int_0^T |Y^\varepsilon(s) - y(s)| \, ds > \frac{1}{4} \right\} \leq \varepsilon^2 L^{-2} u_1(T), \quad (2.40)$$

where

$$u_1(T) \defeq 16\alpha^2 T^2 \int_0^T \left\{ 1 + \frac{\alpha}{\alpha + \kappa^2 V T} e^{(\alpha + \kappa^2 V T) t} \right\} dt.$$
Now we consider the next probability term:

\[
\mathbb{P}\left\{ \alpha \int_0^T \left| V\left( \frac{X^\varepsilon(s)}{d} \right) - V\left( \frac{x(s)}{d} \right) \right| ds > \frac{L}{4} \right\} \\
\leq 16 \alpha^2 L^{-2} \mathbb{E}\left\{ \int_0^T \left| V\left( \frac{X^\varepsilon(s)}{d} \right) - V\left( \frac{x(s)}{d} \right) \right| ds \right\}^2 \\
\leq 16 \alpha^2 L^{-2} T \mathbb{E}\left\{ \int_0^T \left( V\left( \frac{X^\varepsilon(s)}{d} \right) - V\left( \frac{x(s)}{d} \right) \right)^2 ds \right\},
\]

where the first inequality is by Chebyshev and the second one is by applying the Holder’s inequality. Using (2.38) and (2.39) we may write

\[
\mathbb{P}\left\{ \alpha \int_0^T \left| V\left( \frac{X^\varepsilon(s)}{d} \right) - V\left( \frac{x(s)}{d} \right) \right| ds > \frac{L}{4} \right\} \leq \varepsilon^2 L^{-2} u_2(T)
\]

where

\[
u_2(T) \overset{\text{def}}{=} 16 \alpha^2 \kappa_\nu^2 T^3 \int_0^T \int_0^s \left\{ 1 + \frac{\alpha}{\alpha + \kappa_\nu^2 T} e^{(\alpha + \kappa_\nu^2 T)r} \right\} dr ds.
\]

Finally applying Doob’s maximal inequality, we have that

\[
\mathbb{P}\left\{ \varepsilon \sup_{t \in [0,T]} |W(t)| > \frac{L}{2} \right\} \leq 4 \varepsilon^2 L^{-2} \mathbb{E}|W(T)|^2 \\
= 4 \varepsilon^2 L^{-2} T = \varepsilon^2 L^{-2} u_3(T)
\]

Therefore letting \( u(T) \overset{\text{def}}{=} u_1(T) + u_2(T) + u_3(T) \) completes the proof.

By the definition of the dynamics of optimal velocity model, the same rate applies for \( X^\varepsilon \).

**Theorem 2.3.3 (Generalization to Higher Dimension).** Let us consider (1.18) in which \( t \mapsto Z^\varepsilon_t = (X^\varepsilon_t, Y^\varepsilon_t)^T \in \mathbb{R}^{2N} \) is the solution of

\[
dZ^\varepsilon_t = B(Z^\varepsilon_t) dt + \varepsilon \Xi dW_t \\
Z^\varepsilon_0 = z_0,
\]

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where the constant matrix $\Xi$ is defined as (1.19) and the drift term $B$ is defined in (1.17). In addition, let $t \mapsto z_t = (x_t, y_t)^T \in \mathbb{R}^{2N}$ be the solution of unperturbed dynamic, i.e. when $\varepsilon = 0$. Then we have that

$$\mathbb{P}\left\{ \sup_{t \in [0,T]} |Z^\varepsilon_t - z_t| > L \right\} \leq L^{-2}\varepsilon^2 u(T),$$

for a continuous function $u : [0, T] \to \mathbb{R}$.

**Proof.** We have that

$$\mathbb{P}\left\{ \sup_{t \in [0,T]} |Z^\varepsilon_t - z_t| > L \right\} \leq \mathbb{P}\left\{ \int_0^T |B(Z^\varepsilon_t) - B(z_t)| dt > \frac{L}{2} \right\} + \mathbb{P}\left\{ \sup_{t \in [0,T]} |\varepsilon W_t| > \frac{L}{2} \right\}.$$

Like before, let $F(u, v) = |u - v|^2$, for $u, v \in \mathbb{R}^N$. We apply the Ito’s formula on $F(Z^\varepsilon_t, z_t)$ and by taking the expectation from both sides, we get

$$\mathbb{E}|Z^\varepsilon_t - z_t|^2 \leq 2\mathbb{E}\int_0^t (X^\varepsilon_s - x_s, Y^\varepsilon_s - y_s) ds + 2\mathbb{E}\int_0^t (Y^\varepsilon_s - y_s, b(X^\varepsilon_s, Y^\varepsilon_s) - b(x_s, y_s)) ds + \varepsilon^2 Nt$$

$$\leq 2(\kappa_b \vee 1) \int_0^t \mathbb{E}|Z^\varepsilon_s - z_s|^2 ds + \varepsilon^2 NT,$$

where $\kappa_b$ is the Lipschitz constant of $b$, and we note that the expectation of the martingale term vanishes. Therefore, by Gronwall’s inequality, we have

$$\mathbb{E}|Z^\varepsilon_t - z_t|^2 \leq \varepsilon^2 NT e^{2(\kappa_b \vee 1)t}.$$

Using Chebychev inequality

$$\mathbb{P}\left\{ \int_0^T |B(Z^\varepsilon_t) - B(z_t)| dt > L/2 \right\} \leq 4\kappa_b^2 L^{-2} \int_0^T \mathbb{E}|Z^\varepsilon_t - z_t|^2 dt \leq L^{-2}\varepsilon^2 u(T),$$

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where
\[ u(t) \overset{\text{def}}{=} K_{(2.41)} \int_0^t e^{2(\kappa_b \sqrt{1})^s} ds, \] (2.41)
and constant \( K_{(2.41)} \) depends on \( \kappa_b, N \) and \( T \). Similarly, applying the Chebychev and then Doob’s maximal inequalities, we have

\[
P \left\{ \sup_{t \in [0,T]} |\epsilon W_t| > \frac{L}{2} \right\} \leq 4L^{-2}\epsilon^2 \mathbb{E} \left\{ \sup_{t \in [0,T]} |W_t|^2 \right\} \leq 16L^{-2}\epsilon^2 \mathbb{E}|W_T|^2 = 16NL^{-2}\epsilon^2T \leq \epsilon^2 L^{-2} u(T),
\]
for appropriately defined constant \( K_{(2.41)} \). Putting these results all together, the claim follows.

This result is valid for more general diffusion term if Lipschitz continuity and sublinear growth rate hold (see [34, 36]).

### 2.4 Conclusion and Future Works

In this section, using tools from perturbation theory and averaging principle, we studied the deviation of the perturbed solution from the trajectory of averaged problem. In the deterministic case, we applied a fast perturbation into the dynamics. In the stochastic case we discussed perturbation by a bounded and paths continuous random process as well as perturbation by a Brownian noise.

The dynamical model we considered in our analysis did not include the singularity term (i.e. \( \beta = 0 \)) and hence the function defining the dynamics had nice properties (smoothness and global Lipschitz continuity). On the other hand, as we discussed before and will elaborate this in the Chapter 3, a more interesting dynamical model (from application point of view) should contain the singularity term (i.e. \( \beta > 0 \)). However, in this case the defining function loses its nice regularity conditions. An interesting extension to the results of this section is to investigate the behavior of the dynamics under different perturbation forces while many classical assumptions fail.
CHAPTER 3

ASYMPTOTIC COLLISION ANALYSIS OF STOCHASTIC OPTIMAL VELOCITY DYNAMICAL MODEL

In this chapter we will focus on analyzing the collision in (stochastic) optimal velocity dynamics with singularity term. Some research articles address collision in the deterministic case and from various standing points. For human driven vehicles, Davis [45] studies the optimal velocity and its modifications to show the effect of reaction time on collision in the platoon through simulations. Hamdar and Mahmassani [46] simulate several microscopic acceleration models and investigate the accident-prone behaviors by relaxing the safety conditions. Tordeux and Syfried [14] propose a modification of optimal velocity model and investigate collision by study of stability. More relevantly, Davis [47] considers the dynamics of autonomous vehicles, and shows that under the assumption of stability there are some domain for parameters in which the collision does not happen.

Our approach in this note is different. Given a platoon of vehicle, the collision can be analyzed based on the interaction of any two vehicles. Let’s consider the dynamics of such interaction explicitly. Let $x$ be the position of a following vehicle, which is following a (rightward-travelling) preceding vehicle whose position is $x^{(p)}$ with the dynamics

$$
\ddot{x}_t = -\alpha \left\{ \dot{x}_t - V \left( \frac{x_t^{(p)} - x_t}{d} \right) \right\} - \beta \frac{\dot{x}_t - x_t^{(p)}}{\left( x_t^{(p)} - x_t \right)^2} + \varepsilon \dot{W}_t
$$

where $\alpha, \beta, d$ and $V$ are defined before. The preceding vehicle car could in fact be the lead car in a platoon, or one in a sequence of platooning vehicles (which would correspond to a recursive sequence of equations of the type (3.1)).

We are interested in collision; i.e. $x_t = x_t^{(p)}$. In this chapter we consider the case that $x^{(p)}$ is the position of the leading vehicle. We study collision by investigating the behavior
of the dynamics locally near the collision singularity. We show the probability of collision in the stochastic case with a provable bound.

### 3.1 Deterministic Collision Dynamics

We consider the system (1.7) on the state space

\[ S \overset{\text{def}}{=} (0, \infty) \times \mathbb{R} \]  

(3.2)

The boundary

\[ \partial S = \{0\} \times \mathbb{R} \]  

(3.3)

where \( x = 0 \) corresponds to collision. Defining \( B_0 : S \to \mathbb{R}^2 \) by

\[ B_0(x, y) \overset{\text{def}}{=} \begin{pmatrix} y \\ -\alpha \left\{ V \left( \frac{x}{d} \right) + y - v_o \right\} - \beta \frac{y}{x^2} \end{pmatrix} \]  

\[ (x, y) \in S \]

consider the ODE

\[ \begin{pmatrix} \dot{x}_t \\ \dot{y}_t \end{pmatrix} = B_0(x_t, y_t) \]  

(3.4)

starting at \( (x_0, y_0) = (x_o, y_o) \in S \), standard ODE theory ensures that (3.4) has a unique solution on a maximal interval

\[ [0, t_c) \subset \mathbb{R} \]  

(3.5)

of definition. This formalism transfers questions of collision in (1.7) (and (3.4)) to questions of hitting the boundary \( \partial S \) of \( S \).

Since \( V : (0, \infty) \to (0, V(\infty)) \) is an invertible transformation, the dynamics (3.4) has a unique equilibrium point at \( (x_\infty, 0) \), where

\[ x_\infty = d \cdot V^{-1}(v_o) = d \left\{ 2 + \tanh^{-1}(v_o + \tanh(-2)) \right\} \]  

(3.6)
which is well-defined under the assumption that $v_o \in (0, 1 + \tanh(2))$. Namely, if the preceding vehicle has constant velocity $v_o$, the system (1.7) selects $x_o$ as the proper following distance, and maintains velocity $v_o$ at distance $x_\infty$ behind the preceding vehicle. Define

$$H(x, y) \overset{\text{def}}{=} \frac{1}{2} y^2 + P(x) \quad (x, y) \in \mathbb{R}^2$$

where

$$P(x) \overset{\text{def}}{=} \alpha \int_{x_\infty}^{x} \left\{ V\left( \frac{x'}{d} \right) - v_o \right\} dx' \quad x \in \mathbb{R}. \quad (3.8)$$

Since

$$P'(x) = \alpha \left\{ V\left( \frac{x}{d} \right) - v_o \right\}, \quad x > 0 \quad (3.9)$$

we can write the dynamics (3.4) as

$$\dot{x}_t = y_t = \frac{\partial H}{\partial y}(x_t, y_t)$$

$$\dot{y}_t = -P'(x_t) - \beta \frac{y_t}{x_t^2} = -\frac{\partial H}{\partial x}(x_t, y_t) - \beta \frac{y_t}{x_t^2} \quad (3.10)$$

for $t \in [0, t_c)$.

We can explicitly calculate

$$P(x) = -\frac{\alpha d}{2} \log \left( \frac{1 - \tanh^2 \left( \frac{2 - x}{d} \right)}{1 - \tanh^2 \left( \frac{2 - x_\infty}{d} \right)} \right) + \alpha (\tanh(2) - v_o) (x - x_\infty).$$
Figure 3.2: Function $P(x)$ illustrated for $v_o = 0.5, \alpha = 2, d = 5$. It should be noted that $P(x_\infty) = 0$.

Remark 3.1.1. We have that $V(x/d) - v_o < 0$ if $x < x_\infty$ and $V(x/d) - v_o > 0$ if $x > x_\infty$. Thus $P$ is increasing on $(x_\infty, 0)$ and decreasing on $(0, x_o)$ and thus has a minimum at $x_o$.

See Figure 3.2.

Proposition 3.1.2. The point $(x_\infty, 0)$ is the unique equilibrium point of (3.4). Secondly, the function $H$ is nonincreasing along the trajectory of (3.4), and strictly decreasing if and only if $(x_o, y_o) \neq (x_\infty, 0)$

Proof. An explicit calculation shows that $(x_\infty, 0)$ is indeed the unique equilibrium point of (3.4).

For $t \in [0, t_c)$,

$$
\frac{dH}{dt}(x_t, y_t) = \dot{x}_t \frac{\partial H}{\partial x}(x_t, y_t) + \dot{y}_t \frac{\partial H}{\partial y}(x_t, y_t) = -\left\{\alpha + \frac{\beta}{x_t^2}\right\}y_t^2 \leq 0 \quad (3.11)
$$

so $H$ is nonincreasing.

If $H$ is strictly decreasing. This is impossible if $(x_o, y_o)$ is the fixed point $(x_\infty, 0)$, so necessarily $(x_o, y_o) \neq (x_\infty, 0)$. On the other hand, assume that $(x_o, y_o) \neq (x_\infty, 0)$, and then assume that $H$ is not strictly decreasing. Then there is a nonempty $(t_1, t_2) \subset (0, \infty)$ on which $H(x_t, y_t)$ is constant for $t \in (t_1, t_2)$. From (3.11), this means that $y_t = 0$ for $t \in (t_1, t_2)$. Thus $x$ takes on a constant value $x_c$ on $(t_1, t_2)$. So $(x_c, 0)$ is a fixed point and thus must be the
unique fixed point \((x_\infty, 0)\). This violates our assumption, so necessarily \(H\) must be strictly decreasing.

The level set

\[
S_h \overset{\text{def}}{=} \{(x, y) \in S : H(x, y) \leq h\}
\]  

(3.12)
is invariant under the dynamics of (3.4). Also define

\[
h_o \overset{\text{def}}{=} H(x_o, y_o)
\]  

(3.13)

which is the initial value of \(H(x_t, y_t)\); then \((x_t, y_t) \in S_{h_o}\) for all \(t \geq 0\).

We now can start to understand what happens at \(t_c\). Standard calculations [48] imply that if \(t_c < \infty\), the trajectory of (3.4) must escape every compact subset of \(S\). Proposition 3.1.2 narrows this down a bit more. Define

\[
\bar{y} \overset{\text{def}}{=} \sqrt{2(h_o + 1)}
\]

\[
\bar{x} \overset{\text{def}}{=} P^{-1}\big|_{(x_\infty, \infty)}(h_o + 1)
\]

(3.14)

and fix \((x, y) \in S\). If \(x > \bar{x}\), then

\[
H(x, y) \geq P(x) > P(\bar{x}) = h_o + 1
\]

Similarly, if \(|y| > \bar{y}\),

\[
H(x, y) \geq \frac{1}{2}y^2 > \bar{y}^2 = h_o + 1.
\]

Thus

\[
\{(x, y) \in S : H(x, y) \leq h_o\} \subset \{(x, y) \in S : H(x, y) \leq h_o + 1\} \subset (0, \bar{x}] \times [-\bar{y}, \bar{y}]
\]

(3.15)

(we will here only use the fact that the leftmost set of (3.15) is contained in the rightmost set; in Section 3.3, the extra margin of \(h_o + 1\) (as opposed to \(h_o\)) will give us a margin to allow for noise).
Proposition 3.1.2 hence implies that

\[(x_t, y_t) \in (0, \bar{x}] \times [-\bar{y}, \bar{y}]\]  \hspace{1cm} (3.16)

for all \(0 \leq t < t_c\).

This gives us

**Lemma 3.1.3.** If \(t_c < \infty\),

\[\lim_{t \nearrow t_c} x_t = 0.\]  \hspace{1cm} (3.17)

**Proof.** Using Proposition 3.1.2

\[y_t^2 \leq 2H(x_t, y_t) \leq 2h_o\]

\[P(x_t) \leq H(x_t, y_t) \leq h_o\]

for \(t \in [0, t_c)\). Thus

\[|y_t| \leq \bar{y} \quad \text{and} \quad x_t \leq \bar{x},\]

for \(t \in [0, t_c)\).

Suppose now that \(t_n \nearrow t_c\); then \((x_{t_n}, y_{t_n}) \in (0, \bar{x}] \times [-\bar{y}, \bar{y}]\) for all \(n\). Since \([0, \bar{x}] \times [-\bar{y}, \bar{y}]\) is a compact subset of \(\mathbb{R}^2\), we further assume that \((x^*, y^*) = \lim_{n \to \infty} (x_{t_n}, y_{t_n})\). If \(x^* > 0\), a solution of (3.4) can be constructed from \((x^*, y^*)\), contradicting the definition of \(t_c\). Consequently \(x^* = 0\). Thus if \(t_n \nearrow t_c\), there is a subsequence such that \(\lim_{n \to \infty} x_{t_n} = 0\). The claim follows.

A boundary-layer analysis near \(\partial S\) will preclude this case and as a result we will in fact have

**Theorem 3.1.4.** \(t_c = \infty\) and the equilibrium solution \((x_0, y_0) = (x_\infty, 0)\) (defined in (3.6)) of the dynamical model (3.4) is globally and asymptotically stable.
3.1.1 Collision Analysis

Let’s now understand the dynamics of (3.4) near the $\partial S$, defined as of (3.3). Let’s assume now that

$$ t_c < \infty \quad (3.18) $$

Let’s also fix

$$ x_- \overset{\text{def}}{=} \min\{x_0, x_\infty\}, \quad (3.19) $$

define

$$ \tilde{t} \overset{\text{def}}{=} \sup\{t < t_c : x_t \geq \frac{1}{2} x_-\} \quad (3.20) $$

so $x_t < \frac{1}{2} x_-$ for $t \in (\tilde{t}, t_c)$ (see Figure 3.3). We should note that $\tilde{t} < \infty$ by assumption (3.18).

Let’s partition $(0, \frac{1}{2} x_-) \times \mathbb{R}$ into “upper” and “lower” parts, writing

$$ (0, \frac{1}{2} x_-) \times \mathbb{R} = \{(0, \frac{1}{2} x_-) \times (0, \infty)\} \cup \{(0, \frac{1}{2} x_-) \times (-\infty, 0]\} $$

**Lemma 3.1.5.** The set $(0, \frac{1}{2} x_-) \times [0, \infty)$ is invariant for $\{(x_t, y_t) ; t \in (\tilde{t}, t_c)\}$; if $(x_{t'}, y_{t'}) \in (0, \frac{1}{2} x_-) \times [0, \infty)$ for some $t' \in (\tilde{t}, t_c)$, then $(x_t, y_t) \in (0, \frac{1}{2} x_-) \times (0, \infty)$ for $t \in [t', t_c)$. 

\[53\]
Proof. Assume not; then there is a nonempty \([s'', t''] \subset (\tilde{t}, t_c)\) such that \((x_{s''}, y_{s''}) \in (0, \frac{1}{2} x_-) \times [0, \infty)\) but \((x_r, y_r) \not\in (0, \frac{1}{2} x_-) \times [0, \infty)\) for \(r \in (s'', t'')\). By definition of \(\tilde{t}\) and \(t_c\), \(0 < x_r < \frac{1}{2} x_-\) for \(r \in [s'', t'']\). Letting \(r \searrow s''\), so

\[
(x_{s''}, y_{s''}) \in \partial \left((0, \frac{1}{2} x_-) \times [0, \infty)\right) \cap \left((0, \frac{1}{2} x_-) \times \mathbb{R}\right) = (0, \frac{1}{2} x_-) \times \{0\}.
\]

and

\[
(x_r, y_r) \in \left((0, \frac{1}{2} x_-) \times \mathbb{R}\right) \setminus \left((0, \frac{1}{2} x_-) \times [0, \infty)\right) = (0, \frac{1}{2} x_-) \times (-\infty, 0)
\]

for \(r \in (s'', t'')\). In other words, \(y_{s''} = 0\) but \(y_r < 0\) for \(r \in (s'', t'')\). This contradicts, however, the fact that \(\dot{y}_{s''} > 0\) by (3.4).

We also have \(\Box\)

**Lemma 3.1.6.** If \((x_{t'}, y_{t'}) \in (0, \frac{1}{2} x_-) \times (0, \infty)\), then \(x_t > x_{t'}\) for \(t \in (t', t_c)\).

**Proof.** For any such \(t\),

\[
x_t = x_{t'} + \int_{s=t}^{t'} y_s \, ds > x_{t'}
\]

\(\Box\)

Taken together, we get \(\Box\)

**Lemma 3.1.7.** Under the assumption of (3.18),

\[
(x_t, y_t)^T \in (0, \frac{1}{2} x_-) \times (-\infty, 0)
\]

(3.21)

for all \(t \in [\tilde{t}, t_c)\), and

\[
\lim_{t \searrow t_c} y_t \leq 0
\]

(3.22)

**Proof.** If \((x_t, y_t)^T \in (0, \frac{1}{2} x_-) \times [0, \infty)\), Lemmas 3.1.5 and 3.1.6 contradict the fact that \(\lim_{t \nearrow t_c} x_t = 0\). Equation (3.22) follows from (3.21).

\(\Box\)

We furthermore have that

54
Lemma 3.1.8. We have that $y$ is strictly increasing on $(\tilde{t}, t_c)$.

Proof. If $y_t < 0$ and $x_t \in (0, x_-)$, then $P'(x_t) < 0$ and

$$\dot{y}_t = -P'(x_t) - \left\{ \alpha + \frac{\beta}{x_t^2} \right\} y_t \geq 0.$$

Lemma 3.1.8 implies that

$$y_t = y_{t_-} - \int_{s=t}^{t_c} \dot{y}_s ds < y_{t_-} \leq 0. \quad (3.23)$$

Thus if $t_c < \infty$, we must in fact have that

$$\lim_{t \uparrow t_c} (x_t, y_t) = (0, y^*)$$

for some $y^* \in (y_t, 0)$ with $y$ strictly increasing on $(\tilde{t}, t_c)$.

Let’s now try to parametrize the integral curve $t \in (\tilde{t}, t_c) \mapsto (x_t, y_t)^T$. In view of Lemma 3.1.8, we should be able to locally write

$$x_t = \varphi(y_t), \quad t \in (\tilde{t}, t_c) \quad (3.24)$$

and if so

$$\dot{x}_t = \varphi'(y_t) \dot{y}_t = \varphi'(y_t) \left\{ -\alpha \left[ V \left( \frac{x_t}{d} \right) + y_t - v_0 \right] - \beta \frac{y_t}{x_t^2} \right\}$$

$$= \varphi'(y_t) \left\{ -\alpha \left[ V \left( \frac{\varphi(y_t)}{d} \right) + y_t - v_0 \right] - \beta \frac{y_t}{\varphi^2(y_t)} \right\} \quad (3.25)$$

Let’s formalize this by considering function $f : (-\infty, 0) \times (0, x_-) \to \mathbb{R}$, defined by

$$f(y, Y) \overset{\text{def}}{=} \frac{y}{-\alpha \left[ V \left( \frac{Y}{d} \right) + y - v_0 \right] - \beta \frac{y}{Y^2}} = \frac{y}{-P'(x) - \left\{ \alpha + \beta/Y^2 \right\} y}.$$  

Since $P' > 0$ on $(0, x_-)$ and $\alpha$ and $\beta$ are also positive, $f$ is well-defined.

Since $(y_t, x_t) \in (-\infty, 0) \times (0, x_-)$, there is a maximal interval $(\eta_-, \eta_+)$ containing $y_t$ such
that the ODE
\[ \phi'(y) = f(y, \phi(y)) \]
\[ \phi(y_i) = x_i \]  \hspace{1cm} (3.26)
has a unique solution on \((\eta_-, \eta_+)\).

**Lemma 3.1.9.** We have that
\[ \inf_{y \in [y_t, 0)} \varphi(y) > 0. \]  \hspace{1cm} (3.27)

**Proof.** Let’s define
\[ R(y) \overset{\text{def}}{=} \left( \left( \frac{1}{2} x_- \right)^{-1} + \frac{1}{\beta} (y - y_t) \right)^{-1} \]
for \(y \in [y_t, 0)\). Then
\[ \frac{\dot{R}(y)}{R^2(y)} = -\frac{1}{\beta} \]
for \(y \in (y_t, 0)\).

Let’s now define
\[ \xi(y) \overset{\text{def}}{=} \left( 1 - \frac{1}{\varphi(y)} \right) \chi_{\{\varphi(y) < R(y)\}} = \left( 1 - \frac{1}{\varphi(y)} \right) \chi_{\{\varphi(y) > 1/R(y)\}} \]
for \(y \in [y_t, \eta_+)\) If \(y \in (y_t, \eta_+)\) and \(\varphi(y) \neq R(y)\), then
\[ \dot{\xi}(y) = \left( \frac{\dot{R}(y)}{R^2(y)} - \frac{\dot{\varphi}(y)}{\varphi^2(y)} \right) \chi_{\{\varphi(y) < R(y)\}} \]
\[ = \left( \frac{1}{\beta} - \frac{\alpha \varphi^2(y) \left\{ V \left( \frac{\varphi^2(y)}{d} \right) + y - v_o \right\} + \beta y}{\beta (-y) + \alpha \varphi^2(y) (-y) - \alpha \varphi^2(y) \left\{ V \left( \frac{\varphi^2(y)}{d} \right) - v_o \right\} - \frac{1}{\beta}} \right) \chi_{\{\varphi(y) < R(y)\}} \]
If \(y \in [y_t, \eta_+)\), then \((-y) > 0\). Noting that \(R(y) < \varphi(y_i)\) for \(y \in [y_t, 0)\) we have that
\[ \varphi(y) < \varphi(y_i) = x_i < x_\infty \]
if $y \in (y_t, \eta_+)$ and $\varphi(y) < R(y)$. Thus
\[
V \left( \frac{\varphi(y)}{d} \right) - v_\circ \leq V \left( \frac{x_\infty}{d} \right) - v_\circ = 0
\]
if $y_t < y < \eta_+$. Thus
\[
\alpha \varphi^2(y)(-y) - \alpha \varphi^2(y) \left\{ V \left( \frac{\varphi^2(y)}{d} \right) - v_\circ \right\} > 0
\]
if $y_t < y < \eta_+$ and $\varphi(y) < R(y)$ so
\[
\frac{(-y)}{\beta(-y) + \alpha \varphi^2(y)(-y) - \alpha \varphi^2(y) \left\{ V \left( \frac{\varphi^2(y)}{d} \right) - v_\circ \right\}} < \frac{1}{\beta}
\]
and hence
\[
\dot{\xi}(y) \leq 0
\]
if $y \in (y_t, \eta_+)$ and $\varphi(y) < R(y)$. Thus $\xi$ is decreasing on $[y_t, \eta_+)$. Since
\[
\xi(y_t) = \left( \frac{1}{x_t} - \frac{1}{x_t} \right)^+ = 0,
\]
so $\xi \leq 0$ on $[y_t, \eta_+]$ and hence $\varphi \geq R$ on $[y_t, \eta_+]$ so
\[
\inf_{[y_t, \eta_+)} \varphi(y) \geq \inf_{[y_t, 0)} R(y) \geq \left\{ (\frac{1}{2} x_-)^{-1} + \frac{1}{\beta} (-y_t) \right\}^{-1} > 0,
\]
so the claim follows.

We can now see that collision is impossible.

Proof of Theorem 3.1.4. Uniqueness allows us to formalize (3.24). Since $y$ is continuous and furthermore strictly increasing, $y$ is a bijection from $[\bar{t}, t_c)$ to $[y_t, y_{c-})$ with continuous inverse. Inverting this and noting that the continuous image of a connected set is connected, we get that
\[
[t, t_c'] = y \big|_{[t, t_c)}^{-1} \left( (y_t, \eta_+) \cap [y_t, y_{c-}) \right)
\]
for some \( t'_c \leq t_c \). From (3.25),

\[
(\phi(y_t), y_t)
\]

satisfies (3.4) for \( t \in [\tilde{t}, t'_c] \) and thus agrees with \((x_t, y_t)\) in \([\tilde{t}, t'_c] \).

We will show that under (3.18), Lemma 3.1.9, Lemma 3.1.7 and the definition (3.20) of \( \tilde{t} \) contradict each other. Assume first that \( t'_c < t_c \). Then

\[
(x_{t'_c}, y_{t'_c}) \in \partial \left( (0, \frac{1}{2}x_-) \times (-\infty, 0) \right) \setminus \partial S = \left( (0, \frac{1}{2}x_-) \times \{0\} \right) \cup \left( \{\frac{1}{2}x_-\} \times (-\infty, 0) \right).
\]

Lemma 3.1.7 precludes \((x_{t'_c}, y_{t'_c}) \in (0, \frac{1}{2}x_-) \times \{0\}, \) and the definition (3.20) of \( \tilde{t} \) precludes \((x_{t'_c}, y_{t'_c}) \in \{\frac{1}{2}x_-\} \times (-\infty, 0) \). If \( t'_c = t_c \), then

\[
\lim_{t \nearrow t_c} \phi(y_t) = \lim_{t \nearrow t_c} x_t = 0,
\]

which is precluded by Lemma 3.1.9.

The next step is to show the global and asymptotic stability of the equilibrium solution. We recall the set \( S_h \) from (3.12). Firstly, \( P(\pm \infty) = +\infty \) and \( P'' > 0 \) imply that \( S_h \) is bounded and closed and hence compact. By Proposition 3.1.2, \( S_h \) is invariant and shrinks to the equilibrium point as \( h \to 0 \). This immediately implies the stability of the equilibrium point.

Next, we show the global asymptotic stability. We assume that \( z_0 \neq z_\infty \) (otherwise the claim is obvious). We show that \( t \mapsto H(x_t, y_t) \) is monotonically decreasing. Considering (3.11), the equality holds if and only if \( y_t \equiv 0 \) on a non-trivial interval. But in this case the first equation of dynamics of (3.10) forces that \( \dot{x}_t \equiv 0 \) on such interval. The second equation of this dynamic implies that in fact \( x_t = x_\infty \). This is a contradiction with our assumption on the initial value \( z_0 \neq z_\infty \). Therefore, \( t \in \mathbb{R}_+ \mapsto H(x_t, y_t) \) is monotonically decreasing.

We consider \( S_{h_0} \) as of (3.12) with \( h_0 \) defined in (3.13). The orbit of the flow \((x_t, y_t)^T\) through the initial value \( z_0 \) is forward complete and the forward orbit is contained in the compact set \( S_{h_0} \). Hence, it has a compact closure. Therefore, the \( \omega \)-limit set \( \omega(z_0) \) of the orbit through \( z_0 \) is non-empty, compact and invariant. By Ponicaré-Bendixon theorem,
Figure 3.4: For the top figures: $x_\infty = 1.5$, $(x_0, y_0) = (0.5, -1)$ and for the bottom figures: $x_\infty = 5.5$, $(x_0, y_0) = (6, 3)$.

$\omega(z_0)$ is either the equilibrium point $z_\infty$ or a periodic orbit. But the monotonically decreasing behavior of the $t \mapsto H(x_t, y_t)$ precludes the latter. In other words,

$$\lim_{t \to \infty} (x_t, y_t)^T = (x_\infty, 0)^T.$$  

This completes the proof.

Figure 3.4 illustrates monotonic behavior of function $H(x, y)$ and the asymptotic stability when the initial point is to the left and right of the equilibrium point respectively.

### 3.2 Properties of Stochastically Perturbed Dynamical Model

Next we consider the effect of small Brownian perturbation in (1.8). We showed that in the deterministic case, i.e. for the dynamics of (1.7) where $\varepsilon = 0$, collision does not happen. In this section, we study similar results for stochastically perturbed dynamics. We prove an analogue of Theorem 3.1.4 and show that collision is unlikely over quantifiably long time
intervals.

In order to be able to use the standard tools from stochastic analysis (in particular existence and uniqueness), we start by regularizing the collision singularity in (1.8). For any fixed $\delta > 0$, we define

$$c_\delta(y) \overset{\text{def}}{=} \begin{cases} \frac{1}{\delta} & \text{if } y > \frac{1}{\delta} \\ y & \text{if } -\frac{1}{\delta} \leq y \leq \frac{1}{\delta} \\ -\frac{1}{\delta} & \text{if } y < -\frac{1}{\delta}, \end{cases} \quad (3.29)$$

which is a Lipshitz-continuous truncation of the map $y \mapsto y$ at $\pm 1/\delta$ (see Figure 3.5). Using the notation $x \lor y \overset{\text{def}}{=} \max\{x, y\}$ for $x$ and $y$ in $\mathbb{R}$, the map $(x, y) \mapsto c_\delta(y)/x^2 \lor \delta^2$ is then a bounded and Lipshitz-continuous function on $\mathbb{R}^2$.

For any $\varepsilon > 0$ and $\delta > 0$, consider

$$dX^{\varepsilon, \delta}_t = Y^{\varepsilon, \delta}_t dt$$
$$dY^{\varepsilon, \delta}_t = \left\{-\alpha \left\{ V \left( \frac{X^{\varepsilon, \delta}_t}{d} \right) + Y^{\varepsilon, \delta}_t - v_o \right\} - \beta \frac{c_\delta \left( Y^{\varepsilon, \delta}_t \right)}{(X^{\varepsilon, \delta}_t)^2 \lor \delta^2} \right\} dt + \varepsilon dW_t \quad t \geq 0$$
$$(X^{\varepsilon, \delta}_0, Y^{\varepsilon, \delta}_0) = (x_o, y_o). \quad (3.30)$$

For each $\delta > 0$, we set $\mathbf{B}(\delta) : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\mathbf{B}(\delta)(z) \overset{\text{def}}{=} \begin{pmatrix} y - \alpha \left\{ V \left( \frac{x}{d} \right) + y - v_o \right\} - \beta \frac{c_\delta(y)}{x^2 \lor \delta^2} \end{pmatrix}^T ; z = (x, y)^T \in \mathbb{R}^2 \quad (3.31)$$

then $Z^{\varepsilon, \delta}_t \overset{\text{def}}{=} \begin{pmatrix} X^{\varepsilon, \delta}_t \ Y^{\varepsilon, \delta}_t \end{pmatrix}^T$ satisfies

$$Z^{\varepsilon, \delta}_t = z_0 + \int_0^t \mathbf{B}(\delta)(Z^{\varepsilon, \delta}_s) ds + \varepsilon \mathbf{e} W_t \quad t \geq 0 \quad (3.32)$$
for all $t > 0$, where $z_0 = (x_o, y_o)^T$ and $\mathbf{e} \overset{\text{def}}{=} (0, 1)$.

Before we start our main argument, let’s look at some numerical results which will be insightful in the theory that follows. Firstly, if $\delta$ is not sufficiently small then the singularity
Figure 3.5: Cutoff $c_\delta$

Figure 3.6: If $\delta$ is not sufficiently small, the collision cannot be prevented. The initial condition is considered to be $(x_0, y_0) = (0.1, -10)$ (very close to the collision singularity and approaches to the collision boundary very fast). Other parameters of the model are $\alpha = 10$, $\beta = 2$, $v_0 = 0.8$.

is not strong enough to prevent the collision, see Figure 3.6. This means, we are mainly interested in the case where $\delta$ vanishes to zero. Next we observe that the result of simulations (Figures 3.7 and 3.8) suggest that the trajectories of stochastic dynamics remain close to the deterministic trajectory for sufficiently small noise intensities. This implies that for sufficiently small noises, we should be able to show that the collision is unlikely. This is in fact what we will rigorously prove in the rest of this chapter.
Figure 3.7: Variation of stochastic trajectory from the deterministic dynamic over $[0, T]$. Here, the rest point is $z_\infty = (1.5, 0)$. We consider $\alpha = 10$, $\beta = 2$ and $v_0 = 0.5$. The initial value is considered to be $(x_o, y_o) = (4.5, -3)$ for the top figures and $(x_o, y_o) = (0.5, -3)$ for the bottom figures, and $\delta = 0.05$. In the left figures, $\varepsilon = 0.05$ and in the right ones $\varepsilon = 0.2$. In all figures 50 random trajectories are created within time $T = 50$.

Figure 3.8: The initial point is considered to coincide with the equilibrium point $z_\infty = (1.497, 0)$. All other parameters are the same as the previous figures. In particular, $\varepsilon = 0.05, 0.2$ for the left and right figure respectively.
3.3 Collisions in Stochastic Dynamics

We carefully defined the vector field $B^{(\delta)}$ to agree with $B$ of (3.1) on the set

$$R_\delta \equiv [\delta, \infty) \times [-1/\delta, 1/\delta]. \quad (3.33)$$

Let’s define

$$r_\delta \equiv [2\delta, \infty) \times [-1/2\delta, 1/2\delta]; \quad (3.34)$$

see Figure 3.9. Note that $r_{\delta_+} \subset r_{\delta_-}$ if $\delta_- < \delta_+$ (i.e., $\delta \mapsto r_\delta$ is decreasing in $\delta$), and

$$S = \lim_{\delta \searrow 0} r_\delta = \bigcup_{\delta > 0} r_\delta.$$

We should be able to construct a solution of (1.8) by gluing the solutions of (3.32). We expect, however, that a result similar to Proposition 3.1.2 should allow us to effectively restrict our calculations to a set like (3.16). We are ultimately interested in $\varepsilon \searrow 0$ asymptotics, but want to carry out calculations on the regularized process $Z^{\delta, \varepsilon}$. 

Figure 3.9: Sets where relaxed dynamics agree with desired ones.

\[ \delta_- = 0.2, \delta_+ = 0.3 \]
To organize our thoughts, let’s define

\[ \tau_{\epsilon,\delta} \overset{\text{def}}{=} \inf \left\{ t \geq 0 : Z_{t}^{\epsilon,\delta} \notin r_{\delta} \right\}. \]

Since \( \tau_{\epsilon,\delta} \) is the first time that \( Z_{t}^{\epsilon,\delta} \) enters an open set, it is a stopping time with respect to \( \{ \mathcal{F}_{t} \}_{t \geq 0} \) (which is assumed to be right-continuous). In particular, [49, Proposition 3.9].

**Proposition 3.3.1.** Suppose \( (Z_{t}) \) is an adapted process with continuous sample paths and takes values in \( \mathbb{R}^{d} \). Furthermore, suppose \( \mathcal{O} \) is an open subset of \( \mathbb{R}^{d} \). Then

\[ T_{\mathcal{O}} \overset{\text{def}}{=} \inf \left\{ t \geq 0 : Z_{t} \in \mathcal{O} \right\}, \]

is a stopping time of \( (\mathcal{F}_{t+}) \).

**Proof.** For any \( t > 0 \) we have

\[ \{ T_{\mathcal{O}} < t \} = \bigcup_{s \in [0,t] \cap \mathbb{Q}} \{ Z_{s} \in \mathcal{O} \} \in \mathcal{F}_{t}. \]

On the other hand, \( T \) is a stopping time of \( (\mathcal{F}_{t+}) \) if and only if \( \{ T < t \} \in \mathcal{F}_{t} \) for any \( t > 0 \). Therefore, the result follows. \( \square \)

We in fact have a consistency result;

**Proposition 3.3.2.** For \( 0 < \delta_{-} < \delta_{+} \), \( \mathbb{P} \)-a.s.

\[ \tau_{\epsilon,\delta_{+}} < \tau_{\epsilon,\delta_{-}} \quad (\infty < \infty) \]

\[ Z_{t \wedge \tau_{\epsilon,\delta_{+}}}^{\epsilon,\delta_{+}} = Z_{t \wedge \tau_{\epsilon,\delta_{-}}}^{\epsilon,\delta_{-}} \quad t \geq 0 \]

\[ \tau_{\epsilon,\delta_{+}} = \inf \left\{ t \in [0, \tau_{\epsilon,\delta_{-}}) : Z_{t}^{\epsilon,\delta_{-}} \notin r_{\delta_{+}} \right\}. \] \hfill (3.35)

**Proof.** The paths \( Z_{t}^{\epsilon,\delta_{+}} \) and \( Z_{t}^{\epsilon,\delta_{-}} \) have common Brownian fluctuations and hence their difference is differentiable. Furthermore, we have

\[ r_{\delta_{+}} \subset R_{\delta_{+}} \cap r_{\delta_{-}} \subset R_{\delta_{-}}; \] \hfill (3.36)
Define
\[ \bar{\nu} \overset{\text{def}}{=} \min \left\{ \delta_+ - \delta_- , \frac{1}{\delta_-} - \frac{1}{\delta_+} \right\} . \]

Suppose that \( z_+ = (x_+, y_+) \in R_{\delta_+} \cap r_{\delta_-} \) and \( z_- = (x_-, y_-) \) in \( \mathbb{R}^2 \) is such that \( \| z - z' \| \leq \bar{\nu} \). Then
\[
x' \geq x - \| z - z' \| \geq \delta_+ - \bar{\nu} \geq \delta_-
\]
\[
|y'| \leq |y| + \| z - z' \| \leq \frac{1}{\delta_+} + \bar{\nu} \leq \frac{1}{\delta_-}
\]
so \( (x_-, y_-) \in R_{\delta_-} \). Then \( B^{(\delta_+)}(z_+) = B(z_+) \) and \( B^{(\delta_-)}(z_-) = B(z_-) \). Define
\[
K_{(3.37)} \overset{\text{def}}{=} \sup_{z_+ \in R_{\delta_+} \cap r_{\delta_-}} \frac{\| B(z_+) - B(z_-) \|}{\| z_+ - z_- \|} \]
(3.37)

(which is bounded from above by the Lipshitz coefficient of \( B \) on \( R_{\delta_-} \)).

Fix now \( \nu \in (0, \bar{\nu}) \) and define
\[
\sigma_1 = \inf \left\{ t \geq 0 : Z_t^{\varepsilon, \delta_+} \notin R_{\delta_+} \cap r_{\delta_-} \right\}
\]
\[
\sigma_{2,\nu} = \inf \left\{ t \geq 0 : \| Z_t^{\varepsilon, \delta_+} - Z_t^{\varepsilon, \delta_-} \| > \nu \right\} \wedge \sigma_1
\]
\[
E_t^{\nu} \overset{\text{def}}{=} \left\{ \| Z_t^{\varepsilon, \delta_+} - Z_t^{\varepsilon, \delta_-} \|^2 + \left( \frac{\nu}{2} \right)^2 \right\}^{1/2} e^{-K_{(3.37)}t} \quad t \geq 0.
\]

Thus
\[
\dot{E}_t^{\nu} = \left[ \left\{ \frac{\left< Z_t^{\varepsilon, \delta_+} - Z_t^{\varepsilon, \delta_-}, B^{(\delta_+)}(Z_t^{\varepsilon, \delta_+}) - B^{(\delta_-)}(Z_t^{\varepsilon, \delta_-}) \right>}{\left\{ \| Z_t^{\varepsilon, \delta_+} - Z_t^{\varepsilon, \delta_-} \|^2 + \left( \frac{\nu}{2} \right)^2 \right\}^{1/2}} \right] - K_{(3.37)} \left\{ \| Z_t^{\varepsilon, \delta_+} - Z_t^{\varepsilon, \delta_-} \|^2 + \left( \frac{\nu}{2} \right)^2 \right\}^{1/2} e^{K_{(3.37)}t}
\]
If $0 < t < \sigma_{2,\nu}$,

$$
\dot{E}_t^{\nu} \leq \left\{ \frac{K_{(3.37)} \left\| Z_t^{\epsilon,\delta_+} - Z_t^{\epsilon,\delta_-} \right\|^2}{\left\{ \left\| Z_t^{\epsilon,\delta_+} - Z_t^{\epsilon,\delta_-} \right\|^2 + (\nu/2)^2 \right\}^{1/2}} - \frac{K_{(3.37)} \left\{ \left\| Z_t^{\epsilon,\delta_+} - Z_t^{\epsilon,\delta_-} \right\|^2 + (\nu/2)^2 \right\}^{1/2}}{e^{-K_{(3.37)}t}} \leq 0
$$

so

$$
\left\| Z_{\sigma_{2,\nu} \wedge L}^{\epsilon,\delta} - Z_{\sigma_{2,\nu} \wedge L}^{\epsilon,\delta} \right\| \leq E_{\sigma_{2,\nu}}^{\nu} \leq E_0^{\nu} = \frac{\nu}{2},
$$

for every $L > 0$. Letting $L \nearrow \infty$,

$$
\left\| Z_{\sigma_{2,\nu}}^{\epsilon,\delta_+} - Z_{\sigma_{2,\nu}}^{\epsilon,\delta_-} \right\| \leq E_{\sigma_{2,\nu}}^{\nu} \leq E_0^{\nu} = \frac{\nu}{2},
$$

on the set where $\sigma_{2,\nu} < \infty$. This is impossible if $\sigma_{2,\nu} < \sigma_1$, so $\sigma_{2,\nu} = \sigma_1$ ($\mathbb{P}$-a.s.), so

$$
\sup_{0 \leq t < \sigma_1} \left\| Z_t^{\epsilon,\delta_+} - Z_t^{\epsilon,\delta_-} \right\| \leq \nu.
$$

Letting $\nu \searrow 0$, we have that

$$
\sup_{0 \leq t < \sigma_1} \left\| Z_t^{\epsilon,\delta_+} - Z_t^{\epsilon,\delta_-} \right\| = 0. \tag{3.38}
$$

We next claim that

$$
\tau^{\epsilon,\delta_+} < \sigma_1 \leq \tau^{\epsilon,\delta_-}. \tag{3.39}
$$

If $t < \tau^{\epsilon,\delta_+}$, then $Z_t^{\epsilon,\delta_+} \in r_{\delta_+}$. In light of the first inclusion in (3.38), we have that $\sigma_1 \geq \tau^{\epsilon,\delta_+}$. If $\tau^{\epsilon,\delta_+} < \infty$, then $Z_{\tau^{\epsilon,\delta_+}}^{\epsilon,\delta_+} = Z_{\tau^{\epsilon,\delta_+}}^{\epsilon,\delta_-} \in r_{\delta_+}$. In fact, $r_{\delta_+}$ is contained in the interior of $R_{\delta_+} \cap r_{\delta_-}$, so the continuity of $Z^{\epsilon,\delta_+}$ implies that there is a $\nu > 0$ such that $Z_t^{\epsilon,\delta_+} \in R_{\delta_+} \cap r_{\delta_-}$ for $t \in [\tau^{\epsilon,\delta_+}, \tau^{\epsilon,\delta_+} + \nu)$; thus $\sigma_1 \geq \tau^{\epsilon,\delta_+} + \nu$. The left-hand inequality of (3.39) follows. If $t < \sigma_1$, then $Z_t^{\epsilon,\delta_-} = Z_t^{\epsilon,\delta_+} \in R_{\delta_+} \cap r_{\delta_-} \subset r_{\delta_-}$ and the right-hand claim of (3.39) follows. The chain (3.39) of inequalities directly proves the first claim of (3.35), and, when combined with (3.38), also gives us the second claim of (3.35).
Define
\[ \hat{\tau} \overset{\text{def}}{=} \inf \left\{ t \in [0, \tau^{\epsilon,\delta_-}) : Z_t^{\epsilon,\delta_-} \not\in r_{\delta_+} \right\}. \]

If \( t < \tau^{\epsilon,\delta_+} \), \( Z_t^{\epsilon,\delta_-} = Z_t^{\epsilon,\delta_+} \in r_{\delta_+} \); thus \( \hat{\tau} \geq \tau^{\epsilon,\delta_+} \). From (3.39),
\[ \tau^{\epsilon,\delta_+} = \inf \left\{ t \in [0, \sigma_1) : Z_t^{\epsilon,\delta_-} \not\in r_{\delta_+} \right\} = \inf \left\{ t \in [0, \sigma_1) : Z_t^{\epsilon,\delta_-} \not\in r_{\delta_+} \right\} \geq \hat{\tau}. \]

This gives us the last claim of (3.35). \( \Box \)

Let’s now define
\[ \tau^{\epsilon} \overset{\text{def}}{=} \sup_{m \geq 1} \tau^{\epsilon,1/m}, \tag{3.40} \]
since \( \tau^{\epsilon} \) is a supremum of a countable collection of stopping times, it too is a stopping time.

In light of the first claim of (3.35), \( \tau^{\epsilon} = \lim_{m \to \infty} \tau^{\epsilon,1/m}, \) \( \mathbb{P} \)-a.s. Let’s next piece together the \( Z^{\epsilon,\delta} \)'s on \([0, \tau^{\epsilon})\). Let’s add a cemetery state to formalize what happens after \( \tau^{\epsilon} \). Fix a point \( * \) not in \( S \), and define
\[ S^* \overset{\text{def}}{=} S \cup \{ * \}. \]

and endow \( S \) with the standard topology of one-point compactifications. For \( m > 1 \), let’s then define
\[ \hat{Z}_t^{\epsilon,1/m} \overset{\text{def}}{=} \begin{cases} Z_t^{\epsilon,1/m} & \text{if } t < \tau^{\epsilon,1/m} \\ * & \text{if } t \geq \tau^{\epsilon,1/m} \end{cases} \]

**Theorem 3.3.3.** Fix \( \epsilon > 0 \). For each \( t \geq 0 \), \( \hat{Z}_t^{\epsilon} \overset{\text{def}}{=} \lim_{m \to \infty} \hat{Z}_t^{\epsilon,1/m} \) is \( \mathbb{P} \)-a.s. well-defined (in the topology of \( S^* \)). For each \( \delta > 0 \), we \( \mathbb{P} \)-a.s. have that

\[ \hat{Z}_{t\wedge \tau^{\epsilon,\delta}} = Z_{t\wedge \tau^{\epsilon,\delta}}, \quad t \geq 0 \]
\[ \hat{Z}_{t\wedge \tau^{\epsilon,\delta}} = z_0 + \int_{s=0}^{t\wedge \tau^{\epsilon,\delta}} B(\hat{Z}_s) ds + \epsilon eW_{t\wedge \tau^{\epsilon,\delta}} \tag{3.41} \]
\[ \tau^{\epsilon,\delta} = \inf \left\{ t \in [0, \tau^{\epsilon}) : \hat{Z}_t^{\epsilon} \not\in r_{\delta} \right\}. \quad (\infty = \infty) \]

**Proof.** If \( t < \tau^{\epsilon} \), then \( t < \tau^{\epsilon,1/m} \) for some positive integer \( m \). For \( m' \geq m \), \( t < \tau^{\epsilon,1/m} < \tau^{\epsilon,1/m'} \)
\[ \hat{Z}_{t}^\varepsilon, m' = Z_t^{\varepsilon,1/m'} = Z_t^{\varepsilon,1/m} \]

\( \mathbb{P} \)-a.s. Thus \( \hat{Z}_t^\varepsilon \) is well-defined if \( t < \tau^\varepsilon \). If \( t \geq \tau^\varepsilon \), then \( t > \tau^\varepsilon,1/m \mathbb{P} \)-a.s. for all positive integers \( m \), then \( \hat{Z}_t^\varepsilon, m = * \mathbb{P} \)-a.s. Thus \( \hat{Z}_t^\varepsilon \) is well-defined if \( t \geq \tau^\varepsilon \).

For \( \delta > 0 \) and any positive integer \( m \) with \( m > 1/\delta \) then

\[ \hat{Z}_{t \wedge \tau^\varepsilon, \delta}^\varepsilon, m = Z_{t \wedge \tau^\varepsilon, \delta}^{\varepsilon,1/m} = Z_{t \wedge \tau^\varepsilon, \delta}^{\varepsilon, \delta} \quad (3.42) \]

for each \( t \geq 0 \), \( \mathbb{P} \)-a.s. Taking limit in \( m \), we get the first claim of (3.41). From (3.42) and the dynamics of \( Z_{t \wedge \tau^\varepsilon, \delta}^{\varepsilon, \delta} \),

\[ \hat{Z}_{t \wedge \tau^\varepsilon, \delta}^\varepsilon = Z_{t \wedge \tau^\varepsilon, \delta}^{\varepsilon, \delta} = z_0 + \int_{s=0}^{t \wedge \tau^\varepsilon, \delta} \mathbf{B}(Z_s^{\varepsilon, \delta}) ds + \varepsilon \mathbf{e} W_{t \wedge \tau^\varepsilon, \delta} \]

which is the second claim of (3.41).

Define

\[ \hat{\tau} = \inf \left\{ t \in [0, \tau^\varepsilon) : \hat{Z}_t^\varepsilon \not\in r_\delta \right\} \]

If \( t < \tau^\varepsilon, \delta \), then the first claim of (3.41) implies that \( \hat{Z}_t^\varepsilon = Z_{t \wedge \tau^\varepsilon, \delta}^{\varepsilon, \delta} \in r_\delta \) and hence \( \hat{\tau} \geq \tau^\varepsilon, \delta \). Fix an integer \( m > 1/\delta \). From the third claim of (3.35) and the first claim of (3.41))

\[ \tau^\varepsilon, \delta = \inf \left\{ t \in [0, \tau^\varepsilon,1/m) : Z_t^{\varepsilon,1/m} \not\in r_\delta \right\} = \inf \left\{ t \in [0, \tau^\varepsilon,1/m) : \hat{Z}_t^\varepsilon \not\in r_\delta \right\} \geq \hat{\tau} \]

This gives us the last claim of (3.41). \( \square \)

Since the limit \( \hat{Z}_t^\varepsilon \) is well-defined \( \mathbb{P} \)-a.s. for each \( t \), \( \hat{Z}^\varepsilon \) is adapted. For \( t \in [0, \tau^\varepsilon) \), let’s write \( \hat{Z}_t^\varepsilon = (X_t^\varepsilon, Y_t^\varepsilon) \). Using Ito’s formula to function \( H \in C^2(\mathbb{R}^2) \) at \( (X_t^\varepsilon, Y_t^\varepsilon) \), we get the
semimartingale

\[ h^\varepsilon_{t \wedge \tau^{\varepsilon,\delta}} \overset{\text{def}}{=} H(X^\varepsilon_{t \wedge \tau^{\varepsilon,\delta}}, Y^\varepsilon_{t \wedge \tau^{\varepsilon,\delta}}) = h_0 - \int_0^{t \wedge \tau^{\varepsilon,\delta}} \left[ \alpha(Y^\varepsilon_s)^2 + \beta(Y^\varepsilon_s)^2 \right] ds + \varepsilon \int_0^{t \wedge \tau^{\varepsilon,\delta}} Y^\varepsilon_s dW_s + \frac{1}{2} \varepsilon^2 (t \wedge \tau^{\varepsilon,\delta}), \]  

(3.43)

for \( t \in [0, T] \) and where,

\[ h_0 = H(-x_1, v_0 - v_1). \]

By definition of function \( H(x, y) \) and the fact that \( P(x) \geq 0 \), it is clear that \( h_0 \geq 0 \). By monotone increasing property of \( P(x) \), there exists \( \bar{x} > x_\infty \) such that

\[ h_0 + 1 = P(\bar{x}), \]

or rather,

\[ \bar{x} \overset{\text{def}}{=} P^{-1}(h_0 + 1). \]

Since we know \( P(x_\infty) = 0 \) and \( P \) is decreasing on \( (-\infty, x_\infty) \) therefore, \( P(0) > 0 \). In addition, we define

\[ \bar{y} \overset{\text{def}}{=} \sqrt{2(h_0 + 1)}. \]  

(3.44)

If \( x > \bar{x} \), then

\[ H(x, y) \geq P(x) > P(\bar{x}) = h_0 + 1. \]

If \( |y| > \bar{y} \) then

\[ H(x, y) \geq \frac{1}{2} y^2 \geq \frac{1}{2} \bar{y}^2 \geq h_0 + 1. \]

Let us define

\[ \mathfrak{B} = [0, \bar{x}] \times [-\bar{y}, \bar{y}]. \]

Then, we have

\[ \inf_{(x, y) \in (\mathbb{R}_+ \times \mathbb{R}) \setminus \mathfrak{B}} H(x, y) \geq h_0 + 1. \]  

(3.45)
We define
\[ \tau_B^{\varepsilon, \bar{\varepsilon}} \triangleq \inf \{ t \in [0, \tau^\varepsilon) : Z_t^\varepsilon \not\in \mathcal{B} \} \]  
with \( \inf \emptyset = \infty \). We recall that on \([0, \tau^\varepsilon)\) by definition \( X_t^\varepsilon > 0 \). Therefore, \( \tau_B^{\varepsilon, \bar{\varepsilon}} \) is the first time before \( \tau^\varepsilon \) that either \( X_t^\varepsilon > \bar{x} \) or \( |Y_t^\varepsilon| > \bar{y} \).

It should be noted that the random variable \( \tau_B^{\varepsilon, \bar{\varepsilon}} : \Omega \to [0, \infty] \) defined in (3.46) is a stopping time of the canonical filtration \((\mathcal{F}_t^+)\). To see this, we define
\[ s^\varepsilon \triangleq \inf \{ t \geq 0 : Z_t^\varepsilon \not\in \mathcal{B} \} . \]

Noting that the complement of \( \mathcal{B} \) is an open set, and hence \( s^\varepsilon \) is a stopping times of filtration \((\mathcal{F}_t^+)\). In addition, we define
\[ A \triangleq \{ \omega \in \Omega : s^\varepsilon(\omega) < \tau^\varepsilon(\omega) \} \in \mathcal{F}_{s^\varepsilon \wedge \tau^\varepsilon} . \]  
(3.47)

Therefore, \( \tau_B^{\varepsilon, \bar{\varepsilon}} \) can be rewritten as
\[ \tau_B^{\varepsilon, \bar{\varepsilon}}(\omega) = \begin{cases} s^\varepsilon(\omega) &; \omega \in A \\ \infty &; \omega \not\in A \end{cases} . \]

Then we have
\[ \{ \tau_B^{\varepsilon, \bar{\varepsilon}} \leq t \} = A \cap \{ s^\varepsilon \leq t \} \in \mathcal{F}_{t^+} \]
and since \( A \in \mathcal{F}_{s^\varepsilon} \) by (3.47), \( \tau_B^{\varepsilon, \bar{\varepsilon}} \) is a stopping time of the filtration \((\mathcal{F}_{t^+})\) .

In the next theorem we show that for sufficiently small \( \varepsilon \), with high probability it will take a long time before \( Z_t^\varepsilon \) leaves the domain \( \mathcal{B} \).

**Theorem 3.3.4.** For any fixed \( L > 0 \), we have
\[ \lim_{\varepsilon \sqrt{L} \to 0} \mathbb{P} \{ \tau_B^{\varepsilon, \bar{\varepsilon}} < L \} = 0 . \]  
(3.48)

**Proof.** Let’s fix \( m \in \mathbb{N} \). It should be noted that by definition (3.46), if \( \tau_B^{\varepsilon, \bar{\varepsilon}} \) is finite then it
should be strictly less that \( \tau^\varepsilon \). Hence, considering the definition of \( \tau^{B,\varepsilon} \), (3.45), (3.43) and assuming that \( \tau^{B,\varepsilon} < L \land \tau^{\varepsilon,1/m} \) implies that

\[
\tau^{B,\varepsilon} = \tau^{B,\varepsilon} \land \tau^{\varepsilon,1/m} \land L.
\]

Therefore,

\[
h_\phi + 1 \leq h^{\varepsilon}_{\tau^{B,\varepsilon} \land \tau^{\varepsilon,1/m} \land L} \leq h_\phi + \varepsilon \int_{\tau^{B,\varepsilon} \land \tau^{\varepsilon,1/m} \land L} Y^\varepsilon_s \, dW_s + \frac{1}{2} \varepsilon^2 (\tau^{B,\varepsilon} \land \tau^{\varepsilon,1/m} \land L)
\]

\[
\leq h_\phi + \varepsilon \int_{\tau^{B,\varepsilon} \land \tau^{\varepsilon,1/m} \land L} Y^\varepsilon_s \, dW_s + \frac{1}{2} \varepsilon^2 (\tau^{B,\varepsilon} \land \tau^{\varepsilon,1/m} \land L)
\]

\[
\leq h_\phi + \varepsilon \int_{\tau^{B,\varepsilon} \land \tau^{\varepsilon,1/m} \land L} Y^\varepsilon_s \, dW_s + \frac{1}{2} \varepsilon^2 L.
\]

If \( \varepsilon^2 L < 1 \), then

\[
\{ \tau^{B,\varepsilon} < L \land \tau^{\varepsilon,1/m} \} \subset \left\{ h_\phi + \varepsilon \int_{\tau^{B,\varepsilon} \land \tau^{\varepsilon,1/m} \land L} Y^\varepsilon_s \, dW_s + \frac{1}{2} \geq 1 + h_\phi \right\}
\]

\[
\subset \left\{ \varepsilon \int_{\tau^{B,\varepsilon} \land \tau^{\varepsilon,1/m} \land L} Y^\varepsilon_s \, dW_s \geq \frac{1}{2} \right\}
\]

Therefore,

\[
P \{ \tau^{B,\varepsilon} < L \land \tau^{\varepsilon,1/m} \} \leq P \left\{ \varepsilon \int_{\tau^{B,\varepsilon} \land \tau^{\varepsilon,1/m} \land L} Y^\varepsilon_s \, dW_s \geq \frac{1}{2} \right\} \leq 4\varepsilon^2 \mathbb{E} \left\{ \left( \int_{\tau^{B,\varepsilon} \land \tau^{\varepsilon,1/m} \land L} Y^\varepsilon_s \, dW_s \right)^2 \right\}
\]

\[
\leq 4\varepsilon^2 \mathbb{E} \left\{ \int_{\tau^{B,\varepsilon} \land \tau^{\varepsilon,1/m} \land L} (Y^\varepsilon_s)^2 \, ds \right\}
\]

\[
\leq 4\varepsilon^2 \bar{y}^2 L,
\]

where the inequality in second line is by Ito's isometry, and the last inequality is by the fact that \( |Y^\varepsilon_t| \leq \bar{y} \), for \( t \in [0, \tau^{B,\varepsilon}) \). Now, letting \( m \to \infty \) completes the proof. \( \square \)
3.3.1 Barrier Function

In this section by introducing a barrier function we rigorously analyze the collision in stochastically perturbed system of (3.30). Let’s define the smooth function

\[ Y \in (0, \infty) \mapsto f_1(Y) = \frac{-1}{\alpha + \beta/Y^2}, \]

which is well-defined for \( \alpha, \beta > 0 \). In addition, we define

\[ x_+ \overset{\text{def}}{=} \min \left\{ x_o, dV^{-1}(v/2) \right\}; \quad (3.49) \]

Let’s consider the initial value problem

\[ \phi'(y) = f_1(\phi(y)) \]
\[ \phi(-\bar{y}) = x_+. \quad (3.50) \]

**Theorem 3.3.5.** Let’s also define

\[ \phi \overset{\text{def}}{=} \left( \frac{1}{x_+} + \frac{2\bar{y}}{\beta} \right)^{-1}. \quad (3.51) \]

The equations of (3.50) has a unique complete solution. Furthermore, we have that

\[ \inf_{y \in [-\bar{y}, \bar{y}]} \phi(y) \geq \phi. \]

**Proof.** Local existence of a unique solution \( y \mapsto \phi(y) \) follows by smoothness of function \( Y \mapsto f_1(Y) \). Let us denote the maximal interval of existence by \((\eta_-, \eta_+)\). In the view of the initial value of (3.50), \( \eta_- < 0 \). In addition, we have that

\[ \phi'(y) = \frac{-1}{\alpha + \beta/\phi^2(y)} = \frac{-\phi^2(y)}{\alpha \phi^2(y) + \beta}, \quad \eta_- < y < \eta_. \quad (3.52) \]
This immediately implies that
\[
\phi'(y) < 0, \quad \text{for} \quad \eta_- < y < \eta_+. \quad (3.53)
\]

Let’s suppose that \( \eta_+ < \infty \).

**Lemma 3.3.6.** We have that
\[
\lim_{y \to \eta_+} \phi(y) < \infty.
\]

**Proof of Lemma 3.3.6.** Considering the initial value and the fact that function \( \phi \) is decreasing (see (3.53)), the claim follows.

**Lemma 3.3.7.** We have that
\[
\lim_{y \to \eta_+} \phi(y) > 0.
\]

**Proof of Lemma 3.3.7.** Using the second presentation of the dynamics of (3.52) we have that
\[
-\frac{\phi'(y)}{\phi^2(y)} = \frac{1}{\beta + \alpha \phi^2(y)} \leq \frac{1}{\beta}
\]
for \( \eta_- < y < \eta_+ \). Integrating both sides,
\[
\frac{1}{\phi(y)} - \frac{1}{\phi(-\bar{y})} \leq \frac{1}{\beta} (y + \bar{y}) \quad (3.55)
\]
for \(-\bar{y} \leq y < \eta_+ \), or rather
\[
\frac{1}{\phi(y)} \leq \frac{1}{\phi(-\bar{y})} + \frac{1}{\beta} (y + \bar{y}) \leq \frac{1}{\phi(-\bar{y})} + \frac{1}{\beta} (\eta_+ + \bar{y})
\]
\[
= \frac{1}{x_{\dagger}} + \frac{1}{\beta} (\eta_+ + \bar{y})
\]
for \( y \in [-\bar{y}, \eta_+] \). Thus by assumption (3.54)
\[
\lim_{y \to \eta_+} \frac{1}{\phi(y)} < \infty.
\]
and in fact
\[
\lim_{y \nearrow y_+} \phi(y) \geq \left( \frac{1}{x^+} + \frac{1}{\beta(y_+ + \bar{y})} \right)^{-1} > 0
\]
This proves the desired result.

But Lemmas 3.3.6 and 3.3.7 together with extensibility theorem precludes \( y_+ < \infty \); i.e. \( y_+ = \infty \).

Let’s next suppose that
\[
y_+ > -\infty.
\] (3.56)

By (3.52) we note that
\[
|\phi'(y)| \leq \frac{1}{\alpha},
\] (3.57)
and hence under the assumption (3.56) we have that
\[
\lim_{y \searrow y_-} \phi(y) < \infty.
\] (3.58)

In addition, the decreasing behavior of \( \phi \) (see (3.53)) and Lemma 3.3.7 imply that
\[
\lim_{y \searrow y_-} \phi(y) > 0.
\] (3.59)

Then (3.58), (3.59) and the extensibility theorem suggest that \( y_+ > -\infty \) cannot characterize the maximal interval of definition and hence \( y_+ = -\infty \) and thus the first claim follows.

The second statement of the theorem follows from (3.55) for \( y \in [-\bar{y}, \bar{y}] \).

Figure 3.10 shows the behavior of trajectory \( \phi \) if the ODE (3.50) and Figure 3.11 is an illustration of the behavior of the first and the second derivatives of the integral curve \( \phi \).

Next we show that \( \phi' \) and \( \phi'' \) are bounded (see figure 3.11 for illustration).

**Theorem 3.3.8.** Let \( y \mapsto \phi(y) \) be the solution of ODE (3.50). We have that
\[
\phi'(y) \left\{ -\alpha y - \beta \frac{y}{\phi^2(y)} \right\} - y = 0
\] (3.60)
Figure 3.10: Illustration of solution $\phi$ for parameters $\alpha = 2$ and $\beta = 5$. The initial value is considered to be $\phi(-3) = 2$.

Figure 3.11: Illustration of boundedness of the $\phi'$ and $\phi''$ for $a = 2$, $b = 5$ and the initial value $\phi(-3) = 2$.

for $y \in [-\bar{y}, \bar{y}]$.

\[
|\phi'(y)| \leq \frac{1}{\alpha} \quad \text{and} \quad |\phi''(y)| \leq \frac{2x_{\infty}}{\alpha \beta}
\] (3.61)

for $y \in [-\bar{y}, \bar{y}]$.

Proof. The statement (3.60) is in fact the ODE (3.52). Consequently, we can calculate

\[
|\phi'(y)| = \frac{1}{\alpha + \beta/\phi^2(y)} \leq \frac{1}{\alpha}.
\]

We can compute the second derivative of $\phi$ using the ODE (3.52);

\[
\phi''(y) = \frac{1}{(\alpha + \beta/\phi^2(y))^2} \times \frac{-2\beta}{\phi^3(y)} \phi'(y)
\]

\[
= \frac{1}{(\alpha + \beta/\phi^2(y))^2} \times \frac{2\beta}{\phi^3(y)} \frac{1}{\alpha + \beta/\phi^2(y)}
\]

\[
= \frac{2\beta\phi(y)}{(\beta + \alpha\phi^2(y))^2 (\alpha + \beta/\phi^2(y))}.
\]
Figure 3.12: The desired dynamics $\varphi$ in comparison with the $\phi$ dynamics.

Using the initial value problem (3.50) and the fact that $\phi(y) \leq \phi(-\bar{y}) = x_\dagger \leq x_\infty$ for $y \in [-\bar{y}, \bar{y}]$ ($\phi$ is monotonically decreasing), we get the second bound of (3.61) for $y \in [-\bar{y}, \bar{y}]$. 

Let’s fix a constant $\delta_\circ$ (it will be defined more accurately later). Roughly speaking, for the analytical reasons that become clear in Subsection 3.3.2, we need to define dynamics

$$\varphi'(y) = f(y, \varphi(y)) \quad (3.62)$$

such that it behaves like Figure 3.12; $\varphi(y) \equiv \phi(y)$ on $(-\infty, 0)$ and then decreases monotonically on $[0, \delta_\circ)$ and becomes flat on $[\delta_\circ, \infty)$. It should be noted that $\phi$ as in (3.50) cannot be instantaneously flatten out to right of any point in a differentiable way (it is monotonically decreasing function on $\mathbb{R}$ and hence by flating it out to the right, the differentiability will be lost). In other words, we need to construct the $\varphi$ dynamics such that it has a two times differentiable solution and is flat to the right of $\delta_\circ$.

In particular, we construct $\varphi$ by gluing several dynamics together. By previous remarks, we consider $\varphi \equiv \phi$ (as of (3.50)) on $(-\infty, 0)$. Let’s define the initial value problem

$$\psi'(y) = f_2(y, \psi(y))$$

$$\psi(0) = \phi(0); \quad (3.63)$$
where $f_2 : \left( -\frac{\delta}{\sqrt{2}}, \frac{\delta}{\sqrt{2}} \right) \times (0, \infty) \to \mathbb{R}$ is defined by

$$f_2(y, Y) \overset{\text{def}}{=} \frac{-\delta_0^{-2}(\delta_0^2 - 2y^2)}{\alpha + \beta y^2}. \quad (3.64)$$

By smoothness of the function $f_2$ on $( -\delta_0/\sqrt{2}, \delta_0/\sqrt{2} ) \times (0, \infty)$ the local solution of the dynamics exists with maximal interval of existence

$$(\eta_-, \eta_+) \subset ( -\frac{\delta_0}{\sqrt{2}}, \frac{\delta_0}{\sqrt{2}} ) \quad (3.65)$$

with

$$\eta_- < 0. \quad (3.66)$$

By (3.64) we have

$$\psi'(y) \leq 0, \quad y \in (\eta_-, \eta_+). \quad (3.67)$$

Let’s have a closer look at the maximal interval of existence for the integral curve $y \mapsto \psi(y)$.

**Lemma 3.3.9.** We have that

$$\lim_{y \to \eta_+} \psi(y) < \infty.$$

*Proof.* Considering that the $\psi'(y) \leq 0$ on $(\eta_-, \eta_+)$ (monotone decreasing function) and the initial value $\psi(0) = \phi(0) \in (0, \infty)$, the result follows. \qed

**Lemma 3.3.10.** We have that

$$\lim_{y \to \eta_+} \psi(y) > 0.$$

*Proof.* We can rewrite the dynamics of $\psi$ as

$$-\frac{\psi'(y)}{\psi^2(y)} = \frac{(\delta_0^2 - 2y^2)}{\delta_0^2(\alpha \psi^2(y) + \beta)} \leq \frac{\delta_0^2}{\delta_0^2 \beta} = \frac{1}{\beta}, \quad y \in (\eta_-, \eta_+).$$

We integrate both sides. Keeping (3.66) in mind, we have

$$\frac{1}{\psi(y)} - \frac{1}{\psi(0)} \leq \frac{1}{\beta} y \quad (3.68)$$
for any $y \in [0, \eta_+)$. Rearranging the terms and considering (3.65), we have

$$\frac{1}{\psi(y)} \leq \frac{1}{\beta} y + \frac{1}{\psi(0)} \leq \frac{1}{\beta} \eta_+ + \frac{1}{\phi(0)} < \infty, \quad y \in [0, \eta_+).$$

In other words

$$\psi(y) \geq \left( \frac{1}{\beta} \eta_+ + \frac{1}{\phi} \right)^{-1} > 0, \quad y \in [0, \eta_+), \quad (3.69)$$

where the last inequality uses the fact that $\phi(0) \geq \bar{\phi}$ and hence the claim follows. □

**Lemma 3.3.11.** The integral curve $y \mapsto \psi(y)$ of the initial value problem (3.63) is well-defined and smooth on $[0, \frac{\delta_o}{2}]$. In addition,

$$\inf_{y \in [0, \frac{\delta_o}{2}]} \psi(y) \geq \left( \frac{\delta_o}{2\beta} + \frac{1}{\bar{\phi}} \right)^{-1} \overset{\text{def}}{=} \psi. \quad (3.70)$$

**Proof.** The results of Lemma 3.3.9 and 3.3.10 and the classic extensibility theorem imply that the integral curve is well defined and smooth on the domain of definition and consequently on $[0, \frac{\delta_o}{2}]$. The second statement of the lemma follows immediately from (3.68) for $y \in [0, \frac{\delta_o}{2}]$. □

Next we consider the initial value problem

$$\eta'(y) = f_3(y, \eta(y))$$

$$\eta\left(\frac{\delta_o}{2}\right) = \psi\left(\frac{\delta_o}{2}\right) \quad (3.71)$$

where $f_3 : (0, \delta_o) \times (0, \infty) \to \mathbb{R}$ is defined by

$$f_3(y, Y) \overset{\text{def}}{=} -\frac{2\delta_o^{-2}(\delta_o - y)^2}{\alpha + \frac{\beta}{Y^2}}. \quad (3.72)$$

We can study this dynamics similar to the previous case. Smoothness of the function $f_3$ implies the existence of unique solution $\eta$ on maximal interval of existence $(\tau_-, \tau_+) \subset (0, \delta_o)$. We should note that

$$\tau_- < \frac{\delta_o}{2} \quad (3.73)$$
We have the following results on the maximum interval of existence \((r_-, r_+)\).

**Lemma 3.3.12.** We have that
\[
\lim_{y \searrow r_+} \eta(y) < \infty.
\]

*Proof.* Since \(\eta'(y) < 0\) and \(\eta'(\delta/2) \equiv \psi'(\delta/2) \in (0, \infty)\), the result follows. \(\square\)

**Lemma 3.3.13.** We have that
\[
\lim_{y \nearrow r_+} \eta(y) > 0.
\]

*Proof.* Considering (3.73) and the initial value problem (3.71), the proof is similar to the proof of Lemma 3.3.10. \(\square\)

**Remark 3.3.14.** The previous results and the basic extensibility theorem imply that the integral curve \(y \mapsto \eta(y)\) of the initial value problem (3.71) is well-defined and smooth on \([\delta/2, \delta]\). In addition,
\[
\inf_{y \in [\delta/2, \delta]} \eta(y) \geq \left( \frac{\delta}{\beta} + \frac{1}{\psi} \right)^{-1} \left( \frac{\delta}{\beta} \right)^{-1} \psi \quad \text{(3.74)}
\]
and \(\psi\) is as of (3.70).

We can explicitly calculate the second derivatives
\[
\psi''(y) = \frac{4\delta_0^{-2}y}{\alpha + \beta/\psi^2(y)} - 2\beta\delta_0^{-2}(\delta_0^2 - 2y^2)\psi'(y)\psi^2(y) \times \frac{1}{(\alpha + \beta/\psi^2(y))^2} \quad \text{(3.75)}
\]
\[
\eta''(y) = \frac{4\delta_0^{-2}(\delta_0 - y)}{\alpha + \beta/\eta^2(y)} - 4\beta\delta_0^{-2}(\delta_0 - y)^2 \frac{\eta'(y)}{\eta^3(y)} \frac{1}{(\alpha + \beta/\eta^2(y))^2} \quad \text{(3.76)}
\]

We show that these dynamics are second order differentiable at the critical points of \(y = 0, y = \delta/2\) and \(y = \delta_0\). In particular, the values of the dynamics coincide at these points by the initial value of each dynamic. For the values of first order derivatives at these critical...
points, we calculate
\[ \psi'(0) = \frac{-1}{\alpha + \psi^2(0)} = \frac{-1}{\alpha + \beta \psi^2(0)} = \phi'(0) \]
\[ \psi'(\delta_2) = \frac{-1}{2}{\frac{1}{\alpha + \psi^2(\delta_2/2)}} = \eta'(\delta_2) \] (3.77)
\[ \eta'(\delta_0) = 0. \]

For the values of second derivatives at the critical points, we have
\[ \psi''(0) = 2\beta \psi'(0) \times \frac{1}{(\alpha + \psi^2(0))^2} = \phi''(0). \]
\[ \psi''(\delta_2) = \frac{2\delta_2^{-1}}{\alpha + \psi^2(\delta_2/2)} - \beta \psi'(\delta_2/2) \times \frac{1}{(\alpha + \psi^2(\delta_2/2))^2} = \eta''(\delta_2). \] (3.78)
\[ \eta''(\delta_0) = 0. \]

By smoothness of integral curve \( y \mapsto \phi(y) \) of initial value problem (3.52), Lemma 3.3.11, Remark 3.3.14, (3.77) and (3.78), we can define a smooth function which behaves in the way we explained in (3.62). In particular,

**Definition 3.3.1 (Barrier Function).** We define the barrier function \( \varphi \) by

\[ \varphi(y) \overset{\text{def}}{=} \begin{cases} 
\phi(y) & ; y \in [-\bar{y}, 0] \\
\psi(y) & ; y \in [0, \frac{\delta_2}{2}] \\
\eta(y) & ; y \in [\frac{\delta_2}{2}, \delta_0] \\
\eta(\delta_0) & ; y \in [\delta_0, \bar{y}] 
\end{cases} \] (3.79)

**Proposition 3.3.15.** We have that for \( y \in (-\bar{y}, \bar{y}) \)

\[ |\varphi'(y)| \leq \frac{1}{\alpha} \] (3.80)
\[ |\varphi''(y)| \leq \frac{2}{\alpha \delta_0} + \frac{2x_\infty}{\alpha \beta} \] (3.81)
\[
\inf_{y \in [-\bar{y}, \bar{y}]} \varphi(y) \geq \eta. \tag{3.82}
\]

**Proof.** For \(y \in (-\bar{y}, 0)\) the claim follows from Theorem 3.3.8. For \(y \in (0, \frac{\delta}{2})\) the bound on the first derivative follows immediately from (3.64) and (3.72). For second derivatives, keeping (3.75) and (3.76) in mind, on \((0, \frac{\delta}{2})\) we have

\[
\frac{4\delta_o^{-2}y}{\alpha + \frac{\beta}{\varphi^2(y)}} \leq \frac{2\delta_o^{-1}}{\alpha + \frac{\beta}{\varphi^2(y)}} \leq \frac{2}{\delta_o \alpha}.
\]

In addition,

\[
\left| 2\beta \delta_o^{-2}(\delta_o^2 - 2y^2) \frac{\varphi'(y)}{\varphi^3(y)} \times \frac{1}{\alpha + \beta/\varphi^2(y)^2} \right| \leq \left| 2\beta \frac{1}{\alpha \varphi^2(y)^2} \left( \frac{-\delta_o^{-2}(\delta_o^2 - 2y^2)}{\alpha + \beta/\varphi^2(y)} \right) \frac{\varphi'(y)}{\alpha \varphi^2(y)^2 + \beta} \right| \leq \frac{2x_\infty}{\alpha \beta}
\]

where the last inequality is by

\[
\varphi(y) = \psi(y) \leq \psi(0) = \phi(0) \leq \phi(-\bar{y}) = x_1 \leq x_\infty, \quad \text{for} \ y \in [0, \frac{\delta}{2}].
\]

Therefore

\[
|\varphi''(y)| \leq \frac{2}{\alpha \delta_o} + \frac{2x_\infty}{\alpha \beta}, \quad y \in (0, \frac{\delta}{2}).
\]

Similarly, we can calculate

\[
|\varphi''(y)| \leq \frac{2}{\alpha \delta_o} + \frac{x_\infty}{2\alpha \beta}, \quad y \in (\frac{\delta}{2}, \delta_o).
\]

Therefore putting things together the result follows.

The last claim of the proposition follows from the definition (3.79) of the dynamics of \(\varphi\) and the fact that by (3.70) and (3.74) we have that

\[
\eta \leq \psi \leq \phi.
\]

\[\square\]
Let’s next define the danger function

\[ D(x, y) \overset{\text{def}}{=} \varphi(y) - x \]

which measures distance to the left of the graph of \( \varphi \). If \( X^\varepsilon_t \leq \frac{1}{2} \eta \) at time \( t \in [0, \tau^{\text{B,} \varepsilon} \wedge \tau^\varepsilon) \) and \( \eta \) is defined in (3.74) (i.e., the following vehicle is close to the leading vehicle), then

\[ D(X^\varepsilon_t, Y^\varepsilon_t) = \varphi(Y^\varepsilon_t) - X^\varepsilon_t \geq \frac{1}{2} \eta, \]

since (as we recall from (3.46)), \( Y^\varepsilon_t \in (-\bar{y}, \bar{y}) \) on \([0, \tau^{\text{B,} \varepsilon} \wedge \tau^\varepsilon)\). Therefore on \([0, \tau^{\text{B,} \varepsilon} \wedge \tau^\varepsilon)\) we have that

\[ \left\{ X^\varepsilon_t \leq \frac{1}{2} \eta \right\} \subset \left\{ D(X^\varepsilon_t, Y^\varepsilon_t) \geq \frac{1}{2} \eta \right\}. \tag{3.83} \]

### 3.3.2 A Stochastic Argument

We want to prove that collision is unlikely. Define

\[ \tau^{D, \varepsilon} \overset{\text{def}}{=} \inf \{ t \in [0, \tau^{\text{B,} \varepsilon} \wedge \tau^\varepsilon) : D(X^\varepsilon_t, Y^\varepsilon_t) > \frac{1}{2} \eta \} \]

(with \( \inf \emptyset \overset{\text{def}}{=} \infty \)); \( \tau^{D, \varepsilon} \) is the first time (before \( \tau^{\text{B,} \varepsilon} \wedge \tau^\varepsilon \)) at which \( D(X^\varepsilon_t, Y^\varepsilon_t) \) exceeds \( \frac{1}{2} \eta \).

By (3.83), collision can occur after \( \tau^{D, \varepsilon} \). We want to show that \( \tau^{D, \varepsilon} \) should be large. In particular, this shows that \( \tau^\varepsilon \) should be large.

**Theorem 3.3.16.** We have that

\[ \lim_{\varepsilon \sqrt{L} \to 0} \mathbb{P}\{ \tau^{D, \varepsilon} < L \} = 0. \tag{3.84} \]

Informally, this claims that it will take a long time for \( D(X^\varepsilon_t, Y^\varepsilon_t) \) to exceed level \( \frac{1}{2} \eta \); (3.83)
thus means that it will take a long time for collision to occur.

If \( D(X_t^\varepsilon, Y_t^\varepsilon) < 0 \) (for some \( t \in [0, \tau^{B,\varepsilon} \wedge \tau^\varepsilon) \)), then \( X_t^\varepsilon > \phi(Y_t^\varepsilon) \geq \eta \), meaning that the following car is at least at distance \( \eta \) from the lead vehicle at time \( t \). We are not interested in the dynamics of \( D(X_t^\varepsilon, Y_t^\varepsilon) \) in this case, so we use a cutoff function. Define

\[
\Delta_t^\varepsilon \overset{\text{def}}{=} (\max\{D(X_t^\varepsilon, Y_t^\varepsilon), 0\})^2, \quad 0 \leq t < \tau^{B,\varepsilon} \wedge \tau^\varepsilon.
\]

We note that the function \((x, y) \mapsto (\max\{D(x, y), 0\})^2\) is in \( C^1(\mathbb{R}) \) and in fact has a piecewise-continuous second derivative. Figure 3.13a illustrates this cut-off function as well as the level set \( D(x, y) = 0 \). Figure 3.13b shows the continuous derivatives of the cut-off function \( D(x, y) \) and the bend in these derivatives along the level set \( D(x, y) = 0 \) (characteristic curve). Finally, Figure 3.13c depicts the second derivative of the cut-off function with a jump along the level set \( D(x, y) = 0 \).

Let’s apply Ito’s rule to \( \Delta_t^\varepsilon \). Define

\[
\begin{align*}
\zeta_t^{(1),\varepsilon} &\overset{\text{def}}{=} \varphi'(Y_t^\varepsilon) \left\{-\alpha Y_t^\varepsilon - \alpha \left\{ V \left( \frac{X_t^\varepsilon}{d} \right) - v_o \right\} \right. - \beta \frac{Y_t^\varepsilon}{(X_t^\varepsilon)^2} \left\} - Y_t^\varepsilon \varepsilon \\
\zeta_t^{(2),\varepsilon} &\overset{\text{def}}{=} \varphi''(Y_t^\varepsilon) \\
\zeta_t^{(3),\varepsilon} &\overset{\text{def}}{=} \varphi'''(Y_t^\varepsilon) \\
\zeta_t^{(4),\varepsilon} &\overset{\text{def}}{=} (\varphi'(Y_t^\varepsilon))^2
\end{align*}
\]

for \( t \in [0, \tau^{B,\varepsilon} \wedge \tau^\varepsilon) \). Formally, for \( t \in [0, \tau^{B,\varepsilon} \wedge \tau^\varepsilon) \)

\[
\begin{align*}
d\Delta_t^\varepsilon &= 2D(X_t^\varepsilon, Y_t^\varepsilon) \chi_{[0,\infty)}(D(X_t^\varepsilon, Y_t^\varepsilon)) \left\{ \zeta_t^{(1),\varepsilon} dt + \varepsilon \zeta_t^{(2),\varepsilon} dW_t + \frac{1}{2} \varepsilon^2 \zeta_t^{(3),\varepsilon} dt \right\} \\
&\quad + \varepsilon^2 \chi_{[0,\infty)}(D(X_t^\varepsilon, Y_t^\varepsilon)) \zeta_t^{(4),\varepsilon} dt.
\end{align*}
\]

**Proof of Theorem 3.3.16.** We start by bounding \( \zeta_t^{(1),\varepsilon} \). Let’s define the parameter \( \delta_o \) in (3.79) by

\[
\delta_o \overset{\text{def}}{=} \frac{\alpha v_o/2}{\alpha + \frac{4\beta}{\eta^2}}
\]

for \( t \in [0, \tau^{B,\varepsilon} \wedge \tau^\varepsilon) \).
(a) An illustration of Cut-off function \( (\max\{D(x,y),0\})^2 \) and the level set \( D(x,y) = 0 \).

(b) First order derivatives of the cut-off function. It can be noticed that the first order derivatives of the cut-off function are continuous but not differentiable along the level set \( D(x,y) = 0 \).

(c) Second order derivatives of cut-off function that are piecewise continuous and there is a jump along the level set \( D(x,y) = 0 \).

Figure 3.13: The function \( \Delta \) and its first and second derivatives. The second partial derivatives have jump along the level sets.
where dynamic of \( \phi \) is as of (3.50). We note that for \( t \in [0, \tau^{23,\varepsilon} \land \tau^\varepsilon) \) and on the event 

\[ \Xi \overset{\text{def}}{=} \{ Y_t^\varepsilon \leq 0 \} \cap \{ D(X_t^\varepsilon, Y_t^\varepsilon) > 0 \} \]

we have

\[ X_t^\varepsilon \leq \varphi(Y_t^\varepsilon) \leq \varphi(-\bar{y}) = x_\dagger \leq dV^{-1}(v_o/2) \]  \hfill (3.87)

In addition, \( \varphi' < 0, \alpha > 0, V \) is nondecreasing. Thus, (3.87) implies that for \( t \in [0, \tau^{23,\varepsilon} \land \tau^\varepsilon), \) on the event \( \Xi \) we have

\[
\varphi'(Y_t^\varepsilon)(-\alpha) \left\{ V \left( \frac{X_t^\varepsilon}{d} \right) - v_o \right\} \leq \varphi'(Y_t^\varepsilon)(-\alpha) \left\{ V \left( \frac{x_\dagger}{d} \right) - v_o \right\} \\
\leq \varphi'(Y_t^\varepsilon)(-\alpha)(-v_o/2) \leq 0
\]

Therefore, by (3.85) and (3.60) on the event \( \Xi \) we have that

\[
\zeta_t^{(1),\varepsilon} \leq \varphi'(Y_t^\varepsilon)(-Y_t^\varepsilon) \left\{ \alpha + \beta \frac{1}{(X_t^\varepsilon)^2} \right\} - Y_t^\varepsilon \\
\leq \varphi'(Y_t^\varepsilon)(-Y_t^\varepsilon) \left\{ \alpha + \beta \frac{1}{\varphi^2(Y_t^\varepsilon)} \right\} - Y_t^\varepsilon = 0
\]

for \( t \in [0, \tau^{D,\varepsilon}) \).

Now, let’s bound \( \zeta_t^{(1),\varepsilon} \) on the event 

\[ \Gamma \overset{\text{def}}{=} \{ Y_t^\varepsilon \in (0, \delta_o) \} \cap \{ D(X_t^\varepsilon, Y_t^\varepsilon) \geq 0 \} \]

\[
\varphi'(Y_t^\varepsilon) \left\{ -\alpha \left\{ V \left( \frac{X_t^\varepsilon}{d} \right) - v_o \right\} - \alpha Y_t^\varepsilon - \beta \frac{Y_t^\varepsilon}{(X_t^\varepsilon)^2} \right\} \\
= \varphi'(Y_t^\varepsilon)(-Y_t^\varepsilon) \left( \alpha + \frac{\beta}{(X_t^\varepsilon)^2} \right) + \varphi'(Y_t^\varepsilon)(-\alpha) \left\{ V \left( \frac{X_t^\varepsilon}{d} \right) - v_o \right\}
\]

On the event \( \Gamma \) we have

\[
\varphi'(y) < 0; \\
X_t^\varepsilon \leq \varphi(Y_t^\varepsilon) \leq \varphi(-\bar{y}) = x_\dagger \leq dV^{-1}(v_o/2);
\]
Therefore, like the previous case we have

$$V \left( \frac{X_t^\varepsilon}{d} \right) - v_o \leq -v_o/2.$$  

On the other hand, for any $t \in [0, \tau^{D,\varepsilon})$ we have that

$$X_t^\varepsilon \geq \varphi(Y_t^\varepsilon) - \frac{1}{2} \eta \geq \frac{1}{2} \eta.$$  

Therefore,

$$\alpha + \frac{\beta}{(X_t^\varepsilon)^2} \leq \alpha + \frac{\beta}{\eta^2/4}.$$  

Putting them all together, by definition of $\delta_o$ as of (3.86), and the assumption $Y_t^\varepsilon \leq \delta_o$ we get that

$$\zeta^{(1),\varepsilon}_t \leq -\varphi'(Y_t^\varepsilon) \left\{ (-\alpha v_o/2) + Y_t^\varepsilon \left( \alpha + \frac{\beta}{(1/2)^2} \right) \right\} - Y_t^\varepsilon \leq 0.$$  

Finally, on the event $\{Y_t^\varepsilon \geq \delta_o \} \cap \{D(X_t^\varepsilon, Y_t^\varepsilon) > 0\}$ we have that

$$\varphi'(y) = 0,$$

and hence it follows that

$$\zeta^{(1),\varepsilon}_t \leq 0.$$  

By Proposition 3.3.15, we have

$$\left| D(X_t^\varepsilon, Y_t^\varepsilon) \chi_{[0,\infty)}(D(X_t^\varepsilon, Y_t^\varepsilon)) \zeta^{(3),\varepsilon}_t \right| \leq \frac{1}{\eta} \left( \frac{1}{\delta_o \alpha} + \frac{x_\infty}{\alpha \beta} \right)$$

$$\left| \chi_{[0,\infty)}(D(X_t^\varepsilon, Y_t^\varepsilon)) \zeta^{(4),\varepsilon}_t \right| \leq \frac{1}{\alpha^2},$$

if $0 \leq t < \tau^{D,\varepsilon}$.

Similar to our previous discussion, $\tau^{D,\varepsilon}$ is a stopping time of canonical filtration ($\mathcal{F}_t^\varepsilon$).
Furthermore, if $\tau^{D,\varepsilon} < L$, then $\tau^{D,\varepsilon} = \tau^{D,\varepsilon} \wedge L$ and we have that

\[
(\frac{1}{2})^2 \leq \Delta_{\tau^{D,\varepsilon} \wedge L} \leq 2\varepsilon \int_{s=0}^{\tau^{D,\varepsilon} \wedge L} D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon}) \chi_{[0,\infty)}(D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon})) \varphi'(Y_{s}^{\varepsilon}) dW_{s} \\
+ \varepsilon^2 K_{(3.88)}(\tau^{D,\varepsilon} \wedge L) \\
\leq 2\varepsilon \int_{s=0}^{\tau^{D,\varepsilon} \wedge L} D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon}) \chi_{[0,\infty)}(D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon})) \varphi'(Y_{s}^{\varepsilon}) dW_{s} + \varepsilon^2 K_{(3.88)} L
\]

(3.88)

where

\[
K_{(3.88)} \overset{\text{def}}{=} \frac{\eta}{2} \left( \frac{1}{\delta_{0} \alpha} + \frac{x_{\infty}}{\alpha \beta} \right) + \frac{1}{\alpha^2}.
\]

If

\[
\varepsilon^2 L < \frac{1}{K_{(3.88)}} \frac{1}{2} \left( \frac{1}{2\eta} \right)^2 = \frac{\eta^2}{8K_{(3.88)}},
\]

then

\[
\{ \tau^{D,\varepsilon} < L \} \subset \left\{ \left( \frac{1}{2\eta} \right)^2 \leq 2\varepsilon \int_{s=0}^{\tau^{D,\varepsilon} \wedge L} D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon}) \chi_{[0,\infty)}(D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon})) \varphi'(Y_{s}^{\varepsilon}) dW_{s} + \frac{1}{2} \left( \frac{1}{2\eta} \right)^2 \right\}
\]

\[
\subset \left\{ 2\varepsilon \int_{s=0}^{\tau^{D,\varepsilon} \wedge L} D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon}) \chi_{[0,\infty)}(D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon})) \varphi'(Y_{s}^{\varepsilon}) dW_{s} \geq \frac{1}{2} \left( \frac{1}{2\eta} \right)^2 \right\}.
\]

Thus

\[
\mathbb{P} \{ \tau^{D,\varepsilon} < L \} \leq \mathbb{P} \left\{ 2\varepsilon \int_{s=0}^{\tau^{D,\varepsilon} \wedge L} D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon}) \chi_{[0,\infty)}(D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon})) \varphi'(Y_{s}^{\varepsilon}) dW_{s} \geq \frac{\eta^2}{8} \right\}
\]

\[
\leq \frac{256\varepsilon^2}{\eta^4} \mathbb{E} \left[ \int_{s=0}^{\tau^{D,\varepsilon} \wedge L} D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon}) \chi_{[0,\infty)}(D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon})) \varphi'(Y_{s}^{\varepsilon}) dW_{s} \right]^2 
\leq \frac{256\varepsilon^2}{\eta^4} \mathbb{E} \left[ \int_{s=0}^{\tau^{D,\varepsilon} \wedge L} (D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon}) \chi_{[0,\infty)}(D(X_{s}^{\varepsilon}, Y_{s}^{\varepsilon})) \varphi'(Y_{s}^{\varepsilon}) \right]^2 ds 
\leq \frac{256\varepsilon^2}{\eta^4} \left( \frac{1}{2\eta} \right)^2 \frac{1}{\alpha^2} L = \frac{64}{\eta^2 \alpha^2} (\varepsilon^2 L)
\]

Therefore, we have that

\[
\mathbb{P} \{ \tau^{D,\varepsilon} < L \} \leq \frac{64}{\eta^2 \alpha^2} (\varepsilon^2 L),
\]
and the claim follows.

3.4 Summary and Future Works

In this section we considered a car-following optimal velocity model as dynamics of autonomous vehicles which is stochastically perturbed by a small Brownian noise. The behavior of such stochastic problem can be studied from different aspects.

We mainly focused on collision analysis of the optimal dynamical model. We rigorously investigated the behavior of the dynamical model rigorously near the collision boundary layer. To do so, first we showed the boundedness of the solutions by some using ideas from dissipative Hamiltonian systems. Then by carefully introducing a barrier dynamic, we studied the behavior of the dynamical model.

In presence of the noise, we needed more involving analysis. More accurately, using tools from stochastic analysis we rigorously studied the probability of collision. In the stochastic case, we needed to define the barrier function more carefully. We introduced and studied an appropriate barrier function and then we defined a danger function which is in fact a measure of closeness of the trajectory to the collision boundary. In this way we could efficiently show that the probability of collision is small depending on the intensity of noise. This result was expressed in the form of a provable bound on the probability of collision.

Under the current settings, we assumed constant leading velocity. One possible extension would be considering a time-dependent leading vehicle. The dynamics in this case address the interaction between any two consecutive vehicles in a platoon. For this purpose, we need to reevaluate some of the properties; e.g. the energy dissipation and consequently the boundedness of the solutions. In particular, the velocity of the leading vehicle in presence of the noise will be a random process. Therefore, analyzing the interaction between the two consecutive vehicles in this case will reveal the effect of attenuation of the noise in collision.

Another important direction would be study of statistical properties of the solutions, presented by random processes. Detailed analysis of the statistical properties of solutions
explains many properties of the stochastic trajectory, and in particular propagation of the noise in the system.
CHAPTER 4

PROPAGATION OF NOISE: TRANSITION DENSITY FUNCTION

4.1 Introduction

The goal of this section is a brief discussion on the concept of propagation of noise in stochastic optimal velocity dynamical models. One way to look at this problem is by understanding the behavior of the solution of the system through studying the existence and explicit form of the transition density functions of the associated Markov process. The existence of the transition density function depends on regularity of the coefficients (see [50] and we will review some of the works in this domain later the following sections). The optimal velocity dynamical model (1.15) does not satisfy the well-known regularities in the literature that are required for the existence of the transition density function [51] which is as a result of the presence of singularity in this dynamical model.

For this reason, we consider an approximation of the optimal velocity dynamical model. In particular, for a fixed $\delta > 0$ we consider the dynamical model

$$
\begin{align*}
\frac{dX_t}{dt} &= Y_t dt \\
\frac{dY_t}{dt} &= b(\delta)(X_t, Y_t) dt + \Lambda dW_t \\
(X_0, Y_0) &= (x_0, y_0),
\end{align*}
$$

(4.1)

where the approximate function $b(\delta) : (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$
b^{(\delta), n}(x, y) = -\alpha \left\{ V \left( \frac{x^{(n)} - x^{(n-1)}}{d} \right) + y^{(n)} - v_o \right\} - \beta c_d \frac{y^{(n)} - y^{(n-1)}}{d(x^{(n)} - x^{(n-1)})},
$$

(4.2)
for \( n \in \{1, \cdots, N\} \) and for a constant coefficient matrix \( \Lambda = \text{Diag}(\lambda_n)_{n=1}^N \), with \( \lambda_n \neq 0 \).

In addition, \((W_t)_{t \geq 0}\) is an \( N \)-dimensional Wiener process defined on some abstract filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) which is considered to be complete throughout the section.

The functions \( c_\delta \) and \( d_\delta \) will be defined in the next section.

We will show that by properties of the functions \( c_\delta \) and \( d_\delta \) the coefficients of the dynamical model (4.1) have sufficient regularities for existence of transition density function.

In particular, by employing the method of [52] we discuss the construction of the explicit transition density function of the associated Markov process.

More importantly, we show that the solutions of the approximating dynamics of (4.1) are consistent with respect to \( \delta \) in the sense of Chapter 3 and hence we can define the solution of the (1.15) in a limiting sense. This implies that the solution of the dynamical model (4.1) coincides with the solution of (1.15) up to some random time.

\section*{4.2 Regularities of the Drift Term}

Let’s start with defining the functions \( c_\delta \) and \( d_\delta \) which explain the regularity of the drift term.

\textbf{Definition 4.2.1.} Let function \( r \in \mathbb{R} \mapsto \varrho^+(r) \in [0,1] \) be non-increasing and smooth with bounded first derivative, defined by \( \varrho^+(r) = 1 \) for \( r \in (-\infty, 1] \) and \( \varrho^+(r) = 0 \) for \( r \in [2, \infty) \). Furthermore, let function \( r \in \mathbb{R} \mapsto \varrho^-(r) \in [0,1] \) be non-decreasing and smooth with bounded first derivative, defined by \( \varrho^-(r) = 1 \) for \( r \in [-1, \infty) \) and \( \varrho^-(r) = 0 \) for \( r \in (-\infty, -2] \). Then we define

\[
c_\delta(r) \overset{\text{def}}{=} \begin{cases} 
\varrho^+(r\delta)r + (1 - \varrho^+(r\delta)) \left( \frac{2}{\delta} \right) & \text{if } r \geq 0 \\
\varrho^-(r\delta)r + (1 - \varrho^-(r\delta)) \left( \frac{-2}{\delta} \right) & \text{if } r \leq 0
\end{cases} \quad (4.3)
\]

In particular, by this definition the function \( c_\delta(r) = r \) for \( |r| \leq \frac{1}{\delta} \), \( c_\delta(r) = \frac{2}{\delta} \) for \( r \geq \frac{2}{\delta} \) and \( c_\delta(r) = -\frac{2}{\delta} \) for \( r \leq -\frac{2}{\delta} \) and smooth in between (see Lemma 4.2.1).

\textbf{Definition 4.2.2.} Let function \( \varrho : \mathbb{R} \rightarrow [0,1] \) be non-increasing and smooth with bounded first derivative function defined by \( \varrho(r) = 1 \) for \( r \in (-\infty, 1] \) and \( \varrho(r) = 0 \) for \( r \in [2, \infty) \).
Then we define
\[ d_\delta(r) \overset{\text{def}}{=} \varrho(\frac{r}{\delta})\delta^2 + (1 - \varrho(\frac{r}{\delta})) r^2. \] (4.4)

By this definition, \( d_\delta(r) = r^2 \) for \( r \geq 2\delta \), \( d_\delta(r) = \delta^2 \) for \( r \leq \delta \) and smooth in between (see Lemma 4.2.2).

Lemma 4.2.1. The function \( r \mapsto c_\delta(r) \) is smooth, bounded and have bounded first derivative over \( \mathbb{R} \).

Proof. Smoothness of \( c_\delta \) at \( r \neq 0 \) follows from the smoothness of \( \varrho^+ \) and \( \varrho^- \) in Definition 4.2.1. From (4.3) we can calculate the derivative

\[ c'_\delta(r) = \begin{cases} \varrho^+(r\delta) + (r\delta - 2) \frac{d}{du} \varrho^+(u) \bigg|_{u=r\delta} & \text{if } r \geq 0 \\ \varrho^-(r\delta) + (r\delta + 2) \frac{d}{du} \varrho^-(u) \bigg|_{u=r\delta} & \text{if } r \leq 0 \end{cases} \] (4.5)

Therefore,

\[ \lim_{r \to 0^+} c'_\delta(r) = 1 - 2 \frac{d}{du} \varrho^+(0) = 1 \]
\[ \lim_{r \to 0^-} c'_\delta(r) = 1 - 2 \frac{d}{du} \varrho^-(0) = 1 \]

In a similar way for all derivatives, the first statement of the lemma follows. On the other hand, from (4.5) we have that

\[ c'_\delta(r) > 0, \quad \text{for } 1/\delta < r < 2/\delta. \]

The increasing behavior of \( c_\delta \) on \((1/\delta, 2/\delta)\) implies that \( c_\delta \) is bounded and in fact

\[ |c_\delta(r)| \leq \frac{2}{\delta} \] (4.6)

which proves the second assertion of the lemma. Furthermore, assuming that \( \delta < 2 \), by (4.5) we can write

\[ |c'_\delta(r)| \leq 2 \sup_{u \in [1,2]} \left| \frac{d}{du} \varrho^+(u) \right| < \infty, \quad r \in \mathbb{R} \]
where we are using the fact that by Definition 4.2.1, $\varrho^+$ and $\varrho^-$ have bounded first order derivatives.

Lemma 4.2.2. The function $r \mapsto d_\delta(r)$ is smooth and increasing. Furthermore, the inverse function $r \mapsto d_\delta^{-1}(r)$ is bounded and Lipschitz continuous.

Proof. The first statement of the lemma follows from the smoothness of the function $\varrho$ in Definition 4.2.2. In addition,

$$d'_\delta(r) = \left( \delta - \frac{r^2}{\delta} \right) \varrho'(r/\delta) + 2r(1 - \varrho(r/\delta))$$

$$= \frac{\delta^2 - r^2}{\delta} \varrho'(r/\delta) + 2r(1 - \varrho(r/\delta))$$

By the definition of the function $d_\delta$ (as of (4.4)), we have that $d_\delta(r) \geq 0$ for $r \leq \delta$ or $r \geq 2\delta$. For $\delta < r < 2\delta$, considering that $\varrho'(r) < 0$ ($\varrho$ is a decreasing function) and $\delta^2 - r^2 < 0$, we get that

$d'_\delta(r) > 0$.

Putting all together, we conclude that $d_\delta$ is an increasing function over $\mathbb{R}$. This proves the second statement of the lemma. In fact,

$$d_\delta(r) \geq \delta^2, \quad r \in \mathbb{R}$$

or rather

$$d_\delta^{-1}(r) \leq 1/\delta^2$$

which proves the third claim. Finally,

$$\left| \frac{d}{dr} d_\delta^{-1}(r) \right| = \left| -\frac{d'_\delta(r)}{(d_\delta(r))^2} \right| \leq \frac{\delta^2 - r^2}{\delta} \varrho'(r/\delta) + 2r(1 - \varrho(r/\delta)) \frac{\varrho(r/\delta)}{\varrho(\zeta)} \delta^2 + (1 - \varrho(\zeta)) r^2$$
Let's consider several cases. For $r < \delta$,

$$\left| \frac{d}{dr} d_{\delta}^{-1}(r) \right| = 0.$$

For $r > 2\delta$

$$\left| \frac{d}{dr} d_{\delta}^{-1}(r) \right| \leq \frac{2r}{r^2} \leq \frac{1}{\delta}.$$

For $\delta < r < 2\delta$, considering that $g'(r) \leq M$ for some $M > 0$, and the previous result on increasing behavior of the $d_{\delta}$, we have that

$$\left| \frac{d}{dr} d_{\delta}^{-1}(r) \right| \leq \frac{3\delta^2 M + 4\delta}{\delta^2} = 3M + \frac{4}{\delta}.$$

This proves that $\left| \frac{d}{dr} d_{\delta}^{-1}(r) \right|$ is bounded on $\mathbb{R}$ and hence the Lipschitz continuity follows. \(\square\)

**Theorem 4.2.3.** The function $b^{(\delta)}$ is smooth with bounded derivative and satisfies Lipschitz continuity condition.

**Proof.** Let's consider $b^{(\delta), n}$ in (4.2). Then

$$\partial_{z_i} b^{(\delta), n}(z) = 0, \quad \text{for } i \neq n, i \neq n - 1$$

and for $n \in \{1, \cdots, N\}$

$$\partial_{x_n} b^{(\delta), n} = -\alpha \left\{ \frac{1}{d} V' \left( \frac{x^{(n)} - x^{(n-1)}}{d} \right) \right\} + \beta \frac{d_{\delta}'(x^{(n)} - x^{(n-1)}) c_{\delta}(y^{(n)} - y^{(n-1)})}{d_{\delta}^2(x^{(n)} - x^{(n-1)})},$$

$$\partial_{y_n} b^{(\delta), n} = -\alpha - \beta \frac{c_{\delta}'(y^{(n)} - y^{(n-1)})}{d_{\delta}(x^{(n)} - x^{(n-1)})}.$$

This proves the smoothness of the drift term $b^{(\delta)}$. The first term (pre-multiplied by $\alpha$) is smooth and Lipschitz continuous by properties of the optimal velocity function $V$. Considering the result of Lemmas 4.2.1 and 4.2.2, the second term (pre-multiplied by $\beta$) is a product
of two smooth bounded and Lipschitz function and hence

\[ |c_\delta(y^{(n)}) - y'^{(n-1)}|_{\delta^{-1}}(x^{(n)} - x^{(n-1)}) - c_\delta(y'^{(n)}) - y'^{(n-1)}|_{\delta^{-1}}(x'^{(n)} - x'^{(n-1)})| \]

\[ \leq |c_\delta(y^{(n)}) - y'^{(n-1)}|_{\delta^{-1}}(x^{(n)} - x^{(n-1)}) - c_\delta(y'^{(n)}) - y'^{(n-1)}|_{\delta^{-1}}(x'^{(n)} - x'^{(n-1)})| \]

\[ + |c_\delta(y'^{(n)}) - y'^{(n-1)}|_{\delta^{-1}}(x^{(n)} - x^{(n-1)}) - c_\delta(y'^{(n)}) - y'^{(n-1)}|_{\delta^{-1}}(x'^{(n)} - x'^{(n-1)})| \]

\[ \leq k |d_{\delta^{-1}}(x^{(n)} - x^{(n-1)})| |y - y'| + k |c_\delta(y'^{(n)}) - y'^{(n-1)}| |x - x'| \]

where in the second inequality we have used the Lipschitz continuity of \( c_\delta \) and \( d_{\delta^{-1}} \), and in the last inequality we employed the boundedness of these functions (see Lemmas 4.2.1 and 4.2.2).

Therefore, the drift term \( b^{(\delta)} \) satisfies Lipschitz continuity and hence it’s first derivative is bounded. This completes the proof. \( \square \)

**Remark 4.2.4 (Existence and Uniqueness).** Lipschitz continuity of the drift term for any fixed \( \delta > 0 \) (see Theorem 4.2.3) ensures the existence and uniqueness of a solution for stochastic dynamics of (4.1) over \( \mathbb{R}^{2N} \).

### 4.3 Consistency of the Solution

In section 4.2, we discussed the regularity of the coefficients for problem (4.1) for a fixed \( \delta > 0 \). Let’s take a closer look at the dependency of the solution of stochastic dynamical model (4.1) to the parameter \( \delta \). In particular, we like to show some kind of consistency in the solution of (4.1) with respect to \( \delta \) (see Theorem 4.3.1). First, we introduce some notations. Let

\[ A_{1,\delta} \overset{\text{def}}{=} \{ x \in \mathbb{R}^N : x^{(n)} - x^{(n-1)} \geq 2\delta, \forall n \in \{1, \cdots, N\} \} \]
which is set of points over which all the vehicles are at least $2\delta$-apart from each other.

Similarly we define

\[ A_{2,\delta} \overset{\text{def}}{=} \{ y \in \mathbb{R}^N : |y^{(n)} - y^{(n-1)}| \leq \frac{1}{\delta}, \forall n \in \{1, \cdots, N\} \}, \]

which is set of points (representing velocities of vehicles) for which the difference of velocities do not exceed a threshold of $\frac{1}{\delta}$. Let

\[ A_\delta \overset{\text{def}}{=} A_{1,\delta} \times A_{2,\delta}. \]

Employing these notations, we define

\[ \tau_\delta \overset{\text{def}}{=} \inf \{ t \in [0, T] : Z_\delta t = (X_\delta t, Y_\delta t)^T \not\in A_\delta \}, \]

where, $Z_\delta t$ is the solution of (4.1). In fact, the $(\mathcal{F}_t)$-stopping time $\tau_\delta$ (the first time that solution enters an open set, and we are assuming the right-continuous filtration) is the first time that the solution violates either $X_\delta^{n,n} - X_\delta^{n,n-1} < 2\delta$ for some $n \in \{1, \cdots, N\}$ or $Y_\delta^{n,n} - Y_\delta^{n,n-1} > \frac{1}{\delta}$ for some $n \in \{1, \cdots, N\}$.

Therefore, if we consider fixed $\delta_1, \delta_2 > 0$ such that $\delta_1 < \delta_2$, then

\[ A_{\delta_2} \subset A_{\delta_1}. \] (4.7)

Similar to the result of previous chapter, we expect that $\tau_{\delta_1} > \tau_{\delta_2}$. However, first we need to show that the solution is consistent. This will be the subject of the next theorem.

**Theorem 4.3.1.** For fixed $\delta_1, \delta_2 > 0$, the solutions $Z_{\delta_1} t$ and $Z_{\delta_2} t$ of (4.1) on $[0, \tau_{\delta_1} \wedge \tau_{\delta_2})$ are indistinguishable.

**Proof.** Let

\[ \tau_R \overset{\text{def}}{=} \inf \{ t \in [0, T] : |Z_{\delta_1} t| > R, \text{ or } |Z_{\delta_2} t| > R \} \] (4.8)
for some $R > 0$. For $z = (x, y)^T$, we define

$$
B^\delta(x, y) \overset{\text{def}}{=} \begin{pmatrix} y \\ b^\delta(x, y) \end{pmatrix} \in \mathbb{R}^N \times \mathbb{R}^N,
$$

and so the solution $Z^\delta t$ of (4.1) can be written as

$$
Z^\delta t = (X^\delta t, Y^\delta t)^T = (x_0, y_0)^T + \int_0^t B^\delta(X^\delta s, Y^\delta s)ds + \Xi_A \begin{pmatrix} 0_{N \times 1} \\ W_t \end{pmatrix}.
$$

where

$$
\Xi_A \overset{\text{def}}{=} \begin{pmatrix} 0_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & \Lambda_{N \times N} \end{pmatrix}.
$$

Since both solutions have the same Brownian fluctuations, for $\mathbb{P}$-a.s $\omega \in \Omega$ we have that

$$
\left| Z^\delta_{t_1 \wedge \tau_{\delta_1} \wedge \tau_{\delta_2} \wedge \tau_R} - Z^\delta_{t_2 \wedge \tau_{\delta_1} \wedge \tau_{\delta_2} \wedge \tau_R} \right|^2 \leq \left( \int_0^{t \wedge \tau_{\delta_1} \wedge \tau_{\delta_2} \wedge \tau_R} \left| B^{\delta_1}(Z^\delta_s) - B^{\delta_2}(Z^\delta_s) \right| ds \right)^2
$$

Therefore, for $\mathbb{P}$-a.s $\omega \in \Omega$

$$
\left| Z^\delta_{t_1 \wedge \tau_{\delta_1} \wedge \tau_{\delta_2} \wedge \tau_R} - Z^\delta_{t_2 \wedge \tau_{\delta_1} \wedge \tau_{\delta_2} \wedge \tau_R} \right|^2 \leq \left( \int_0^{t \wedge \tau_{\delta_1} \wedge \tau_{\delta_2} \wedge \tau_R} \left| B^{\delta_1}(Z^\delta_s) - B^{\delta_2}(Z^\delta_s) \right| ds \right)^2 \leq T \int_0^{t \wedge \tau_{\delta_1} \wedge \tau_{\delta_2} \wedge \tau_R} \left| B^{\delta_1}(Z^\delta_s) - B^{\delta_2}(Z^\delta_s) \right|^2 ds \tag{4.9}
$$

where in the last inequality we have used the H"older's inequality. By definition, we can write

$$
\left| B^{\delta_1}(Z^\delta_s) - B^{\delta_2}(Z^\delta_s) \right|^2 = \left| Y^{\delta_1}_s - Y^{\delta_2}_s \right|^2 + \left| b^{(\delta_1)}(Z^\delta_s) - b^{(\delta_2)}(Z^\delta_s) \right|^2 \\
\leq \left| Z^{\delta_1}_s - Z^{\delta_2}_s \right|^2 + \sum_{n=1}^N \left| b^{(\delta_1),n}(Z^\delta_s) - b^{(\delta_2),n}(Z^\delta_s) \right|^2 \tag{4.10}
$$
By definition of \( b^{(\delta)} \) as of (4.2) and on the time interval of \([0, \tau^{\delta_1} \wedge \tau^{\delta_2}]\) we have

\[
|b^{(\delta_1),n}(Z_{\delta_1}^{\delta_1}) - b^{(\delta_2),n}(Z_{\delta_2}^{\delta_2})|^2 \leq 2\alpha C |Z_{\delta_1}^{\delta_1} - Z_{\delta_2}^{\delta_2}|^2 + 2\beta \left| \frac{Y_{\delta_1}^{\delta_1,n} - Y_{\delta_1}^{\delta_1,n-1}}{(X_{\delta_1}^{\delta_1,n} - X_{\delta_1}^{\delta_1,n-1})^2} - \frac{Y_{\delta_2}^{\delta_2,n} - Y_{\delta_2}^{\delta_2,n-1}}{(X_{\delta_2}^{\delta_1,n} - X_{\delta_2}^{\delta_1,n-1})^2} \right|^2
\]

\[
\leq \alpha C |Z_{\delta_1}^{\delta_1} - Z_{\delta_2}^{\delta_2}|^2
\]

\[
+ \beta C \left| (X_{\delta_1}^{\delta_1,n} - X_{\delta_1}^{\delta_1,n-1})^{-2} \right| \left| (Y_{\delta_1}^{\delta_1,n} - Y_{\delta_1}^{\delta_1,n-1}) - (Y_{\delta_2}^{\delta_2,n} - Y_{\delta_2}^{\delta_2,n-1}) \right|^2
\]

\[
+ \beta C \left| Y_{\delta_2}^{\delta_2,n} - Y_{\delta_2}^{\delta_2,n-1} \right|^2 \left| (X_{\delta_1}^{\delta_1,n} - X_{\delta_1}^{\delta_1,n-1})^{-2} - (X_{\delta_2}^{\delta_1,n} - X_{\delta_2}^{\delta_1,n-1})^{-2} \right|^2
\]

\[
\leq C(1 + \delta_1^{-4} + \delta_2^{-2}) |Z_{\delta_1}^{\delta_1} - Z_{\delta_2}^{\delta_2}|^2
\]

(4.11)

Therefore, from (4.9), (4.10) and (4.11) we get

\[
\mathbb{E} \left[ |Z_{t\wedge\tau^{\delta_1}\wedge\tau^{\delta_2}\wedge\tau^R}^{\delta_1} - Z_{t\wedge\tau^{\delta_1}\wedge\tau^{\delta_2}\wedge\tau^R}^{\delta_2}|^2 \right] \leq CT \left( 1 + CN(1 + \delta_1^{-4} + \delta_2^{-2}) \right)
\]

\[
\times \int_0^t \mathbb{E} \left[ |Z_{s\wedge\tau^{\delta_1}\wedge\tau^{\delta_2}\wedge\tau^R}^{\delta_1} - Z_{s\wedge\tau^{\delta_1}\wedge\tau^{\delta_2}\wedge\tau^R}^{\delta_2}|^2 \right] ds
\]

We note that on the time interval of \([0, \tau^R]\) both solutions \( Z_t^{\delta_1} \) and \( Z_t^{\delta_2} \) are bounded (see the definition (4.8)) and hence by Gronwall’s inequality we can conclude that

\[
\mathbb{E} \left[ |Z_{t\wedge\tau^{\delta_1}\wedge\tau^{\delta_2}\wedge\tau^R}^{\delta_1} - Z_{t\wedge\tau^{\delta_1}\wedge\tau^{\delta_2}\wedge\tau^R}^{\delta_2}|^2 \right] = 0.
\]

Letting \( R \to \infty \), the statement of the theorem follows.

By the result of Theorem 4.3.1 (consistency of the solutions), the inclusion of (4.7) and \( \mathbb{P}\)-a.s paths continuity of the solution \( Z_t^{\delta} \), we conclude that

\[
\tau^{\delta_1} > \tau^{\delta_2}, \quad \text{if } 0 < \delta_1 < \delta_2
\]

(4.12)

assuming that \((x_o, y_o)^T \in A_{\delta_2}\). Equivalently, for any \( m_1, m_2 \in \mathbb{N} \) with \( m_1 > m_2 \) we have
that
\[ \tau^{1/m_1} > \tau^{1/m_2}. \]

In other words, the \((\mathcal{F}_t)\)-stopping time \(\tau^{1/m}\) is monotonically increasing with respect to \(m\) and so
\[ \tau_\infty \overset{\text{def}}{=} \lim_{m \to \infty} \tau^{1/m} \quad (4.13) \]
is well-defined. As a limit of monotonically increasing stopping times, \(\tau_\infty\) is an \((\mathcal{F}_t)\)-stopping times as well. In addition, by Theorem 4.3.1, if \(m_1 > m_2\)

\[ Z_t^{1/m_1} = Z_t^{1/m_2}, \quad t \in [0, \tau^{1/m_2}). \quad (4.14) \]

Therefore, (4.13) and (4.14) imply that
\[ Z_t \overset{\text{def}}{=} \lim_{m \to \infty} Z_t^{1/m} \quad (4.15) \]
is well-defined on \([0, \tau_\infty)\). The process \(Z_t\) is characterized as the solution of (1.15) such that
\[ Z_t = Z_{t^\delta}, \quad t \in [0, \tau^\delta). \]

In other words, for any \(\delta > 0\) we can identify this solution as
\[ Z_{t \wedge \tau^\delta} = Z_{t \wedge \tau^\delta} = z_0 + \int_0^{t \wedge \tau^\delta} B(Z_s^\delta) ds + \Xi_\Lambda \begin{pmatrix} 0_{N \times 1} \\ W_t \end{pmatrix} \quad (4.16) \]

where the drift term \(B(z)\) is defined as (1.17).

**Remark 4.3.2.** The study of random time \(\tau^\delta\) is outside the scope of these notes. However, this is an interesting problem which shows for instance under what conditions we can expect that the solution \(Z_t\) and \(Z_t^\delta\) coincide in some probabilistic sense (see results of Chapter 3...
when the noise is small). Such results will help us understanding the behavior of the solution $Z_t$ of (1.15) when the noise intensity is small.

**Remark 4.3.3.** The choice of the constant diffusion matrix in dynamics of (4.1) is for simplicity of the notations and based on the particular application which was discussed in previous sections. The results of next section can be immediately extended to more general diffusion terms under some regularities.

### 4.4 Transition Density Function

In section 4.3, we showed that the solutions of the dynamics of (4.1) are consistent and we defined the solution $Z_t$ of (1.15) as the limit of the solutions of these approximate dynamics. In this section, we will discuss the construction of the transition density function for the approximate problem (4.1) for a fixed $\delta > 0$.

In the presence of the *degeneracy* in the diffusion term (this concept will be explained later in this section), the existence of the transition density function requires stronger regularities on coefficients [50, 51]. In section 4.2, we discussed the regularity of the drift term $b^{(\delta)}$ of the stochastic dynamical model (4.1). In this section by applying the method of [52], we discuss that the transition density function corresponding to the solution of stochastic differential equations of (4.1) exists and show its explicit form. The transition density, if it exists, can be considered as the solution of backward Kolmogorov equations

$$
Lu(t, z) = \partial_t u(t, z), \quad z = (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, t \in (0, T] \\
u(0^+, z) = \delta_\zeta(z),
$$

for any $\zeta \in \mathbb{R}^{2N}$. The $\delta_\zeta$ notation is the Dirac distribution concentrated at $\zeta$ and for any

---

In the present section, we detail the method of [52] to customize the results and the notations to the case of our problem.
$f \in C_c^2(\mathbb{R}^{2N})$; two times continuously differentiable in $\mathbb{R}^{2N}$ with compact support, we define

$$L_f \overset{\text{def}}{=} \frac{1}{2} \sum_i \lambda_i^2 \partial_{y_i}^2 f + \sum_i y^{(i)} \partial_{x_i} f + \sum_i b^{(\delta)\cdot i}(z) \partial_{y_i} f.$$  \hspace{1cm} (4.18)

Conventionally, we define the operator $\mathcal{L} \overset{\text{def}}{=} L - \partial_t$. The differential operator $\mathcal{L}$ is the generator of the semigroup associated with solution of the stochastic dynamic (as a Markov process). From the PDE point of view, the solution of (4.17), if it exists, is called *fundamental solution* of operator $\mathcal{L}$ and is in this case the transition density function of the corresponding Markov process. The operator $\mathcal{L}$ includes some degeneracy in the derivatives of highest order. In particular, it does not contain the second order derivatives with respect to $x_i$'s. This type of degenerate models arises in different applications and mainly in probabilistic frameworks; for example stochastic forces in the velocity of particles in physics.

Let’s take a closer look at the concept of degeneracy of stochastic differential equations of the form (4.1). We explain this degeneracy through the classical example of Kolmogorov which in fact invoked the research on the subject.

### 4.4.1 Kolmogorov Example

Let’s consider a simple 2-dimensional stochastic dynamic of the form

$$\begin{align*}
    dX_t &= Y_t dt \\
    dY_t &= \sigma dW_t \\
    (X_0, Y_0) &= (x, y),
\end{align*}$$

which corresponds to operator

$$\mathcal{L}_o u = \frac{1}{2} \sigma^2 \partial_{yy} u + y \partial_x u - \partial_t u.$$ \hspace{1cm} (4.20)

For the first time, Kolmogorov \[53\] posed and solved this problem explicitly and in fact proved the existence of transition density kernel in degenerate case. This stimulated a huge
body of work in this domain to show the existence and construction of the explicit transition
density function for various assumptions on regularity of the coefficients. The solution of
(4.19) can be found directly as a Gaussian process of the form
\[
\begin{pmatrix}
X_t \\
Y_t
\end{pmatrix} = \begin{pmatrix}
x + yt + \sigma \int_0^t W_s ds \\
y + \sigma W_t
\end{pmatrix}
\]
with mean vector and covariance matrix
\[
\mu_z(t) = \begin{pmatrix} x + yt \\ y \end{pmatrix}, \quad A(t) = \sigma^2 \begin{pmatrix} \frac{1}{3} t^3 & \frac{1}{2} t^2 \\ \frac{1}{2} t^2 & t \end{pmatrix}.
\]
It is clear that
\[
det A(t) = \sigma^4 t^4 / 12,
\]
and the inverse of the covariance matrix is
\[
\hat{A}(t) = (\hat{A}_{ij}(t)) = \sigma^{-2} \begin{pmatrix} \frac{12}{t^2} & -\frac{6}{t^2} \\ -\frac{6}{t^2} & \frac{2}{t} \end{pmatrix}
\]
Therefore, the density function with respect to $L_\phi$ is
\[
p^G(t, z, \zeta) \overset{\text{def}}{=} \frac{1}{2\pi \sqrt{\det A(t)}} \exp \left\{ -\frac{1}{2}(\zeta - \mu_z(t))^T \hat{A}(t)(\zeta - \mu_z(t)) \right\}.
\]
If we define
\[
Q_t(u, v) \overset{\text{def}}{=} \frac{12}{t^3 \sigma^2} u^2 - \frac{12}{t^2 \sigma^2} uv + \frac{4}{t \sigma^2} v^2
\]
\[
= \frac{12}{t^3 \sigma^2} \left\{ u^2 - uv t \right\} + \frac{4}{t \sigma^2} v^2
\]
\[
= \frac{12}{t^3 \sigma^2} \left\{ u - \frac{1}{2} vt \right\}^2 + \frac{1}{t \sigma^2} v^2,
\]
(4.22)
for any \((u, v) \in \mathbb{R}^2\), then we have

\[
(\zeta - \mu_z(t))^T \hat{A}(t) (\zeta - \mu_z(t)) = Q_t (\zeta_1 - x - yt, \zeta_2 - y) = \frac{12}{t^3 \sigma^2} \left\{ \zeta_1 - x - \frac{1}{2} t (\zeta_2 + y) \right\}^2 + \frac{1}{t \sigma^2} (\zeta_2 - y)^2
\]  

(4.23)

Therefore, from (4.21) and (4.23) the transition density function with respect to \(L\) has the specific form of

\[
p^G(t, z, \zeta) = \frac{6}{\pi t^2 \sigma^2} \exp \left\{ -\frac{(\zeta_2 - y)^2}{2t \sigma^2} - \frac{6 \{ \zeta_1 - x - \frac{1}{2} t (\zeta_2 + y) \}^2}{t^3 \sigma^2} \right\}.
\]  

(4.24)

In addition, \((t, z, \zeta) \mapsto p^G(t, z, \zeta)\) is smooth outside the diagonal \((t = 0, z = \zeta)\). It can further be shown that such transition kernel satisfies

\[
\lim_{t \searrow 0} \int_{\mathbb{R}^2} p^G(t, z, \zeta) f(\zeta) d\zeta = f(z),
\]

for any continuous and bounded function \(f\). We note that, as a result of degeneracy in operator \(L\), that there are two different time scales \(t^{1/2}\) and \(t^{3/2}\) for standard deviations (c.f. the non-degenerate case with time scale of \(t^{1/2}\)). From the probabilistic point of view, in (4.19) the noise in the dynamic of velocity is as a result of Brownian perturbation while the noise in the dynamics of the position is as a result of perturbation in the velocity. In fact, different time scales in (4.24) can be also interpreted as different ways that noise can be propagated in the system and hence this makes the analysis of the degenerate systems more complex. In fact, the main difference between the degenerate and non-degenerate stochastic systems is that in the non-degenerate ones the noise can only propagate in the system through the diffusion term, while in the degenerate case the drift term also can be the source of propagation.
4.4.2 Hörmander Condition for Existence and Smoothness of Density Function

In this section we show that under certain regularities on the coefficients, the degenerate diffusion processes has density function. The main result in this context is suggested by Hörmander [50]. Let’s start with some definitions.

**Definition 4.4.1.** A linear differential operator $L$ with $C^\infty$ coefficients in an open set $U \subset \mathbb{R}^N$ is called hypoelliptic if for any distribution $u$, the fact that $Lu$ is smooth in $U$ implies the smoothness of $u$ in $U$ (see Hörmander [54]) or by notation

$$\text{sing supp}(u) = \text{sing supp}(Lu),$$

where $\text{sing supp}(f)$ denotes the closure of set of points on which $f$ is $C^\infty$ function.

**Theorem 4.4.1.** The stochastic flow $(X_t, Y_t)$ of (4.1) when the coefficients are smooth, has a smooth density if the corresponding infinitesimal operator $L$ of the flow is Hypoelliptic.

**Proof.** See [50] for proof and some more results. \qed

Inspired by the Kolmogorov example, the necessary and sufficient condition for hypoellipticity for second order differential operators has been extended by Hörmander [50]. In particular, suppose we can write a differential operator as

$$L = \frac{1}{2} \sum_{i=1}^N A_i^2 + A_0,$$

where $A_i$’s are smooth vector fields representing the first order differential operators. If $A_1, \cdots, A_N, [A_r, A_u]_{(r,u)\in[0,N]^2}, [A_r, [A_u, A_k]]_{(r,u,k)\in[0,N]^3}, \cdots$, where $[,]$ is the Lie Algebra, spans $\mathbb{R}^{2N}$, then the differential operator $L$ is hypoelliptic. For the Kolmogorov example one can define

$$A_1 = \sigma \partial_y, \quad A_0 = y \partial_x$$
and hence
\[ [A_1, A_0] = \sigma \partial_x \]
and hence \( \{ A_1, [A_1, A_0] \} \) spans \( \mathbb{R}^2 \) and therefore, differential operator \( \mathcal{L}_o \) as in (4.20) is hypoelliptic. The smoothness of the fundamental solution in Kolmogorov’s example outside the diagonal \( (t = 0, z = \zeta) \) can also be verified as a result of the Hörmander’s condition in Theorem 4.4.1.

### 4.4.3 Existence of Density Functions Under Weaker Regularities

The existence of fundamental solution of operator \( \mathcal{L} \) was extended in several works. Hörmander [50] generalized Kolmogorov’s result to the case of constant coefficients. Freidman [55] and Il’in [56] employed Lévy’s iterative method to construct the fundamental solution in non-degenerate case. Later, Weber [57], Sonin [52] and Il’in [58] extended this result by finding an explicit solution when the operator \( \mathcal{L} \) is degenerate under some regularity assumptions on coefficients.

#### Construction of Density Function

As mentioned before, for the rest of this section we employ the method of [52] to construct the fundamental solution of operator \( \mathcal{L} \). We start by defining
\[ \mathcal{L}_o u(t, z) \overset{\text{def}}{=} \frac{1}{2} \sum_{i=1}^{N} \lambda_i^2 \partial_{y_i} u + \sum_{i=1}^{N} y^{(i)} \partial_{x_i} u - \partial_t u. \]  
which is also known as Kolmogorov operator. Note that in two dimensions, \( \mathcal{L}_o \) corresponds to the solution of the Kolmogorov example in Subsection 4.4.1. To find the fundamental
solution of \( \mathcal{L}_c \), denoted by \( p^G(t, z, \zeta) \) in these notes, we start with the corresponding SDE

\[
\begin{align*}
\begin{align*}
    dX_t &= Y_t \, dt \\
    dY_t &= \Lambda \, dW_t \\
    (X_0, Y_0) &= (x, y).
\end{align*}
\end{align*}
\]

Component-wise the equations of this dynamics read

\[
\begin{align*}
    X_t^{(n)} &= x^{(n)} + y^{(n)} t + \lambda_n \int_0^t W_s^{(n)} \, ds \\
    Y_t^{(n)} &= y^{(n)} + \lambda_n W_t^{(n)}
\end{align*}
\]

which defines a Gaussian process with expectation

\[
\mu_a(t) = \mathbb{E} \left( \begin{pmatrix} X_t \\ Y_t \end{pmatrix} \right) = \begin{pmatrix} x + y t \\ y \end{pmatrix} \in \mathbb{R}^2.
\]

Let’s calculate the covariance matrix. First, we note that the independence of \( W_t^{(n)} \) and \( W_t^{(m)} \) for any \( m \neq n \), implies that

\[
\begin{align*}
    \text{Cov}(X_t^{(n)}, X_t^{(m)}) &= 0, \quad m \neq n \\
    \text{Cov}(X_t^{(n)}, Y_t^{(m)}) &= 0, \quad m \neq n \\
    \text{Cov}(Y_t^{(n)}, Y_t^{(m)}) &= 0, \quad m \neq n
\end{align*}
\]

and for any \( n \in \{1, \cdots, N\} \)

\[
\begin{align*}
    \text{Cov}(X_t^{(n)}, X_t^{(n)}) &= \lambda_n^2 \mathbb{E} \left\{ \int_0^t \int_0^t W_s^{(n)} W_r^{(n)} \, ds \, dr \right\} = \frac{1}{3} \lambda_n^2 t^3 \\
    \text{Cov}(X_t^{(n)}, Y_t^{(n)}) &= \frac{1}{2} \lambda_n^2 t^2 \\
    \text{Cov}(Y_t^{(n)}, Y_t^{(n)}) &= \lambda_n^2 t.
\end{align*}
\]
Therefore, the covariance matrix can be calculated
\[
A(t) = \begin{pmatrix}
\frac{1}{2} t^3 \Lambda^2 & \frac{1}{2} t^2 \Lambda^2 \\
\frac{1}{2} t^2 \Lambda^2 & t \Lambda^2
\end{pmatrix}_{2N \times 2N}, \quad \det(A(t)) = \frac{4^N - 3^N}{12^N} \lambda_1^4 \cdots \lambda_N^4 t^{4N}.
\]

The covariance matrix across all \(n\)’s is block diagonal. The inverse of the covariance matrix will be
\[
\hat{A}(t) \overset{\text{def}}{=} A^{-1}(t) = \begin{pmatrix}
\frac{12}{t^2} \Lambda^{-2} & -\frac{6}{t^2} \Lambda^{-2} \\
-\frac{6}{t^2} \Lambda^{-2} & \frac{4}{t^2} \Lambda^{-2}
\end{pmatrix},
\]
where \(\Lambda^{-2} = (\Lambda^2)^{-1}\). Therefore, the fundamental solution of \(L_0\), as a Gaussian density, can be written in the form of
\[
p^G(t, z, \zeta) = \frac{c}{t^{2N}} \exp \left\{-\frac{1}{2} \left( \hat{A}(t)(\zeta - \mu_z(t)), (\zeta - \mu_z(t)) \right) \right\}
\]
with constant
\[
c \overset{\text{def}}{=} \left( \frac{4^N - 3^N}{6^N \pi^N} \right)^{-1/2} \lambda_1^{-2} \cdots \lambda_N^{-2}.
\]

We need some algebraic manipulation for the exponent term of \(p^G(t, z, \zeta)\). Let \(\zeta = (\zeta_1, \zeta_2)^T \in \mathbb{R}^N \times \mathbb{R}^N\). We use \(\zeta_{1,i}\) (resp. \(\zeta_{2,i}\)) for the entries of vectors \(\zeta_1\) (resp. \(\zeta_2\)); i.e.
\[
\zeta_j = (\zeta_{1,j}, \cdots, \zeta_{N,j}), \quad j = 1, 2.
\]

In addition, we consider the notation \(\zeta^{(i)}\) to present the entries of vector \(\zeta \in \mathbb{R}^{2N}\) without distinction in \(\zeta_1\) and \(\zeta_2\) elements; i.e.
\[
\zeta = (\zeta^{(1)}, \cdots, \zeta^{(2N)}).
\]
Then the exponent can be written in the form of

$$
\sum_{i,j=1}^{2N} \hat{a}_{i,j}(t)(\zeta^{(i)} - \mu_{z,i})(\zeta^{(j)} - \mu_{z,j}) = \sum_{i=1}^{N} \frac{12}{t^3 \lambda_i^2} (\zeta_{1,i} - x^{(i)} - y^{(i)} t)^2 \\
- \sum_{i=1}^{N} \frac{12}{t^2 \lambda_i^2} (\zeta_{1,i} - x^{(i)} - y^{(i)} t)(\zeta_{2,i} - y^{(i)}) \\
+ \sum_{i=1}^{N} \frac{4}{t \lambda_i^2} (\zeta_{2,i} - y^{(i)})^2 \\
= \sum_{i=1}^{N} \frac{12}{t^3 \lambda_i^2} \left\{ \zeta_{1,i} - x^{(i)} - \frac{1}{2} t (\zeta_{2,i} + y^{(i)}) \right\}^2 \\
+ \sum_{i=1}^{N} \frac{1}{t \lambda_i^2} (\zeta_{2,i} - y^{(i)})^2
$$

Therefore, if we put everything together, we will have

$$
p^G(t, z, \zeta) = \frac{c}{t^{2N}} \exp \left\{ - \frac{1}{t^3} \sum_{i=1}^{N} \frac{6}{\lambda_i^2} \left\{ \zeta_{1,i} - x^{(i)} - \frac{1}{2} t (\zeta_{2,i} + y^{(i)}) \right\}^2 - \frac{1}{t} \sum_{i=1}^{N} \frac{1}{2 \lambda_i^2} (\zeta_{2,i} - y^{(i)})^2 \right\}
$$

for $z = (x, y)^T$ and $\zeta = (\zeta_1, \zeta_2)^T \in \mathbb{R}^{2N}$. Similar to the two dimensional case, it can be shown that $p^G(t, z, \zeta)$ is the fundamental solution of $L_0$, or equivalently, it solves

$$
L_0 u(t, z) = 0, \quad t \in (0, T], \quad z \in \mathbb{R}^{2N} \\
u(0+, z) = \delta_\zeta(z),
$$

(4.26)

for any $\zeta \in \mathbb{R}^{2N}$. Let’s introduce some notations for simplicity. First, we denote the exponent of $p^G(t, z, \zeta)$ by $K(t, z, \zeta)$;

$$
p^G(t, z, \zeta) = \frac{c}{t^{2N}} \exp \{ -K(t, z, \zeta) \}.
$$
In addition, we write

\[ p^G(t, z, \zeta) = \frac{c}{t^{2N}} \exp \left\{ -\frac{1}{t^3} \sum_{i=1}^{N} R_i^2(t, z, \zeta) \right\} \exp \left\{ -\frac{1}{t} \sum_{i=1}^{N} S_i^2(t, z, \zeta) \right\} \]

\[ = \frac{c}{t^{2N}} \exp \left\{ -\frac{1}{t^3} R(t, z, \zeta) \right\} \exp \left\{ -\frac{1}{t} S(t, z, \zeta) \right\}, \tag{4.27} \]

where

\[ R_i(t, z, \zeta) \equiv \sqrt{6} \left\{ \zeta_{1,i} - x^{(i)} - \frac{1}{2} t \left( \zeta_{2,i} + y^{(i)} \right) \right\} \]

\[ S_i(t, z, \zeta) \equiv \frac{1}{\sqrt{2\lambda_i}} \left( \zeta_{2,i} - y^{(i)} \right) \]

\[ R(t, z, \zeta) \equiv \left( \sum_{i}^N R_i^2(t, z, \zeta) \right)^{1/2} \]

\[ S(t, z, \zeta) \equiv \left( \sum_{i}^N S_i^2(t, z, \zeta) \right)^{1/2}. \]

We denote the fundamental solution of operator \( L \) (defined as in (4.18)) by \( p(t, z, \zeta) \). Next the idea is to construct \( p(t, z, \zeta) \) using the Gaussian kernel \( p^G(t, z, \zeta) \).

The goal is to show that the Lévy iterative scheme in the case of non-degenerate operators (see [55]), can be similarly applied to show

\[ p(t, z, \zeta) = p^G(t, z, \zeta) + \int_0^t \int_{\mathbb{R}^{2N}} p^G(t-s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds \]

\[ \equiv p^G(t, z, \zeta) + H(t, z, \zeta) \tag{4.28} \]

where the kernel \( \Phi \) is to be determined. This result implies that the transition density function \( p(t, z, \zeta) \) is constructed from \( p^G(t, z, \zeta) \) and its convolution with a function \( \Phi(t, z, \zeta) \).

Suppose that (4.28) holds. Let’s apply the differential operator \( L \) to (4.28) and further suppose that by the regularity condition of the drift term \( z \mapsto b^{(\delta)}(z) \) the differentiation
under the integral is justified (we will verify this assumption later). Then we must have

$$LH(t, z, \zeta) = \int_0^t \int_{\mathbb{R}^{2N}} \mathcal{L}p^G(t - s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds - \Phi(t, z, \zeta), \quad (4.29)$$

This implies that $\Phi$ should be the solution of a Volterra equation of the form

$$\Phi(t, z, \zeta) = \mathcal{L}p^G(t, z, \zeta) + \int_0^t \int_{\mathbb{R}^{2N}} \mathcal{L}p^G(t - s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds,$$

and can be calculated recursively by

$$\Phi(t, z, \zeta) \overset{\text{def}}{=} \phi_1(t, z, \zeta) + \sum_{r \geq 1} (\phi_{r+1}(t, z, \zeta) - \phi_r(t, z, \zeta)), \quad (4.30)$$

where

$$\phi_1(t, z, \zeta) \overset{\text{def}}{=} \sum_{i=1}^N b^{(\delta)_z}(z) \partial_{y_i} p^G(t, z, \zeta)$$

$$\phi_{r+1}(t, z, \zeta) \overset{\text{def}}{=} \phi_1(t, z, \zeta) + \int_0^t \int_{\mathbb{R}^{2N}} \phi_1(t - s, z, \xi) \phi_r(s, \xi, \zeta) d\xi ds, \quad (4.31)$$

for any $(t, z) \in (0, T] \times \mathbb{R}^{2N}$.

**Lemma 4.4.2.** For any $\zeta \in \mathbb{R}^{2N}$ the series $\Phi(t, z, \zeta)$ as of (4.30) converges absolutely for $(t, z) \in (0, T] \times \mathbb{R}^{2N}$ and uniformly on any compact subset outside the diagonal of $(t = 0, z = \zeta)$.

**Proof.** From (4.27), we can calculate

$$\partial_{y_i} p^G(t, z, \zeta) = \frac{C}{t^{2N}} \exp \left\{-\mathcal{K}(t, z, \zeta)\right\} \left\{ \frac{1}{t^2} \frac{\sqrt{6}}{\lambda_i} R_i(t, z, \zeta) + \frac{2}{t} \frac{1}{\sqrt{2} \lambda_i} S_i(t, z, \zeta) \right\}. \quad (4.32)$$
Let’s consider the first term more carefully. In fact, we can write this term as
\[
\frac{C_i \lambda_i^{-1}}{t^{2N+\frac{1}{2}}} \left( \frac{\mathcal{R}_i^2(t, z, \zeta)}{t^3} \right)^{1/2} t^{3/2} \exp \left\{ -K(t, z, \zeta) \right\} \leq \frac{C_i \lambda_i^{-1}}{t^{2N+\frac{1}{2}}} \left( \frac{\mathcal{R}_i^2(t, z, \zeta)}{t^3} \right)^{1/2} \exp \left\{ -\frac{1}{t^3} \mathcal{R}_i^2(t, z, \zeta) \right\} \times \exp \left\{ -\frac{1}{t} S_i^2(t, z, \zeta) \right\} \leq \frac{C_i \lambda_i^{-1}}{t^{2N+\frac{1}{2}}} \exp \left\{ -\alpha \cdot K(t, z, \zeta) \right\}.
\]
for any $\alpha \in (0, 1)$. The last inequality is by the fact that for any $r, k > 0$, there exists a constant $C > 0$ such that for $x > 0$
\[
x^r e^{-kx} \leq C,
\]
and hence
\[
x^r e^{-x} = x^r \exp \left\{ -\alpha x - (1 - \alpha) x \right\} \leq C e^{-\alpha x}, \quad \alpha \in (0, 1).
\]
By a similar reasoning for the second term, we have
\[
\left| \partial_{y_i} p^G(t, z, \zeta) \right| \leq \frac{C \lambda_i^{-2}}{t^{2N+1/2}} \exp \left\{ -\alpha \cdot K(t, z, \zeta) \right\}
\leq \frac{C \lambda_i^{-2}}{t^{1/2}} p^G(t, \hat{z}, \hat{\zeta}),
\]
where $\hat{z} = \alpha^{1/2} z$ and $\hat{\zeta} = \alpha^{1/2} \zeta$. Also,
\[
\Delta \overset{\text{def}}{=} \min\{\lambda_i : i \in \{1, \cdots, N\}\}
\]
(4.34)

Therefore,
\[
\left| D_{y_i} p^G(t, z, \zeta) \right| \leq \frac{CN^{1/2} \lambda_i^{-2}}{t^{1/2}} p^G(t, \hat{z}, \hat{\zeta}).
\]
(4.35)

By the sublinear growth rate of $b^{(i)}(z)$, (4.31) and bound (4.35) we can calculate
\[
\left| \phi_1(t, z, \zeta) \right| \leq CN^{1/2} \lambda^{-2} (a + a' |z|) t^{-1/2} p^G(t, \hat{z}, \hat{\zeta})
\leq CN^{1/2} \lambda^{-2} (b + b' |\zeta|) t^{-1/2} p^G(t, \hat{z}, \hat{\zeta}), \quad (t, z) \in (0, T] \times \mathbb{R}^{2N},
\]
(4.36)
for $\hat{z} = \alpha_{1/2}^1 = z, \hat{\zeta} = \alpha_{1/2}^2 \zeta$ and $\alpha_o \in (0, 1)$. The last inequality is by Appendix A.

Employing the bound (4.36) and the Chapman-Kolmogorov equation for $p^G(t, z, \zeta)$, we can calculate

$$|\phi_2(t, z, \zeta) - \phi_1(t, z, \zeta)| \leq C^2 N \Lambda^{-4} (a + a' |z|) (b + b' |\zeta|) \times \int_0^t (t - s)^{-1/2} s^{-1/2} ds \int_{\mathbb{R}^{2N}} p^G(t - s, \hat{z}, \hat{\zeta}) p^G(s, \hat{\zeta}, \hat{\zeta}) d\zeta.$$  

Using the Chapman-Kolmogorov equations we have that

$$\int_{\mathbb{R}^{2N}} p^G(t - s, \hat{z}, \hat{\zeta}) p^G(s, \hat{\zeta}, \hat{\zeta}) d\zeta = p^G(t, \hat{z}, \hat{\zeta}),$$

and

$$\int_0^t (t - s)^{-1/2} s^{-1/2} ds = B(\frac{1}{2}, \frac{1}{2}) = \frac{\Gamma^2(1/2)}{\Gamma(1)}.$$

Therefore, putting all together we have

$$|\phi_2(t, z, \zeta) - \phi_1(t, z, \zeta)| \leq C^2 N \Lambda^{-4} \alpha_o^{-N} \Gamma^2(1/2) (b + b' |\zeta|)^2 p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1)}.$$  

(4.37)

By induction on (4.37) we can conclude that for $r \geq 1$

$$|\phi_{r+1}(t, z, \zeta) - \phi_r(t, z, \zeta)| \leq (CN^{1/2} \Lambda^{-2} \alpha_o^{-N/2} \Gamma(1/2))^{r+1} p^G(t, \hat{z}, \hat{\zeta}) (b + b' |\zeta|)^{r+1} t^{-\frac{r+1}{2}} \frac{1}{\Gamma(\frac{r+1}{2})}.$$  

(4.38)

Therefore, the series defined in (4.30) converges absolutely for any $(t, z) \in (0, T] \times \mathbb{R}^{2N}$ and uniformly convergent on any compact subset outside the diagonal $(t = 0, z = \zeta)$. Hence $\Phi(t, z, \zeta)$ is well-defined and continuous at any $(t, z)$ outside this diagonal. In particular, we can explicitly calculate (see Appendix B)

$$|\Phi(t, z, \zeta)| \leq k (b + b' |\zeta|) t^{-1/2} p^G(t, \hat{z}, \hat{\zeta}) \exp \{k |\zeta|^2 t\}.$$  

(4.39)
Next we need to verify (4.29). We start with first derivatives.

**Lemma 4.4.3.** Let $H$ be as in (4.28). Then we have

$$\partial_y H(t, z, \zeta) = \int_0^t \int_{\mathbb{R}^{2N}} \partial_y p^G(t - s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds \quad (4.40)$$

**Proof.** Let

$$H_{(yi)}(t, z, \zeta) \overset{\text{def}}{=} \int_0^t \int_{\mathbb{R}^{2N}} \partial_y p^G(t - s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds.$$

By (4.33) and (4.39) we have that

$$|H_{(yi)}(t, z, \zeta)| \leq C(b + b' |\zeta|) \exp \{k |\zeta|^2 T \} \int_0^t (t - s)^{-1/2} s^{-1/2} \int_{\mathbb{R}^{2N}} p^G(t - s, \hat{z}, \hat{\xi}) p^G(s, \hat{\xi}, \hat{\zeta}) d\xi ds$$

$$\leq C(b + b' |\zeta|) \exp \{k |\zeta|^2 T \} \left( B\left(\frac{1}{2}, \frac{1}{2}\right) p^G(t, \hat{\xi}) \right)$$

(4.41)

This implies that $H_{(yi)}(t, z, \zeta)$ is well-defined. In addition, we have

$$\partial_y H(t, z, \zeta) = \lim_{\theta \to 0} \frac{1}{\theta} [H(t, (x, y + \theta e_i), \zeta) - H(t, (x, y), \zeta)]$$

$$= \lim_{\theta \to 0} \int_0^t \int_{\mathbb{R}^{2N}} \frac{1}{\theta} \left[ p^G(t - s, (x, y + \theta e_i), \xi) - p^G(t - s, (x, y), \xi) \right] \Phi(s, \xi, \zeta) d\xi ds$$

$$= \lim_{\theta \to 0} \int_0^t \int_{\mathbb{R}^{2N}} \left\{ \int_0^1 \partial_y p^G(t - s, (x, y + \lambda \theta e_i), \xi) d\lambda \right\} \Phi(s, \xi, \zeta) d\xi ds$$

Now Fubini theorem, the integrability result of (4.41) and consequently dominated convergence theorem complete the proof.

**Lemma 4.4.4.** We have that

$$\partial_{x_i} H(t, z, \zeta) = \int_0^t \int_{\mathbb{R}^{2N}} \partial_{x_i} p^G(t - s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds \quad (4.42)$$

**Proof.** By definition of $p^G(t, z, \zeta)$ we can calculate

$$|\partial_{x_i} p^G(t, z, \zeta)| \leq t^{-3/2} p^G(t, \hat{z}, \hat{\zeta}).$$

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Then using the same approach as in the previous lemma, we can proof the claimed result.

While the singularity of the first partial derivatives is integrable in $p^G(t,z,\zeta)$ and hence integration under the integral sign is justified in these cases, we need more work to show that the singularity of the second partial derivatives are integrable. In doing so, first we show that the kernel $\Phi(t,z,\zeta)$ is differentiable. This result helps us proving the integrability of the second order derivatives.

**Theorem 4.4.5.** *The function* $z \mapsto \Phi(t,z,\zeta)$ *defined as in (4.30) is differentiable for any* $t > 0$ *and furthermore, for any* $i \in \{1, \cdots, N\}$

$$\left| \frac{\partial}{\partial y_i} \Phi(t,z,\zeta) \right| \leq C(\text{Const.} + t^{-1})p^G(t,\hat{z},\hat{\zeta}) \exp \{k|\zeta|^2t \}. \tag{4.43}$$

*Proof.* To show the result, we differentiate the terms of the series of (4.30) and show that the resulting series converges. In particular, for a fixed $i \in \{1, \cdots, N\}$ we show that

$$\partial_{y_i} \Phi(t,z,\zeta) = \partial_{y_i} \phi_1(t,z,\zeta) + \sum_{r \geq 1} \frac{\partial}{\partial y_i} (\phi_{r+1}(t,z,\zeta) - \phi_r(t,z,\zeta)). \tag{4.43}$$

is convergent and hence well-defined. First we calculate

$$\partial_{y_i} p^G(t,z,\zeta) = \frac{c}{t^{2N}} \exp \{-K(t,z,\zeta)\} \left\{ \frac{1}{t^2} \sqrt{\frac{6}{\lambda_j}} R_j(t,z,\zeta) + \frac{2}{t} \frac{1}{\sqrt{2\lambda_j}} S_j(t,z,\zeta) \right\}. $$

In addition,

$$\partial_{z_{2,j}} p^G(t,z,\zeta) = \frac{c}{t^{2N}} \exp \{-K(t,z,\zeta)\} \left\{ \frac{1}{t^2} \sqrt{\frac{6}{\lambda_j}} - \frac{2}{t} \frac{1}{\sqrt{2\lambda_j}} S_j(t,z,\zeta) \right\}. $$

This implies that

$$\partial_{y_i} p^G(t,z,\zeta) = -\partial_{z_{2,j}} p^G(t,z,\zeta) + B_j(t,z,\zeta), \tag{4.44}$$

where

$$B_j(t,z,\zeta) \overset{\text{def}}{=} \frac{2 \sqrt{6} \lambda_j^{-1}}{t^{2N+2}} R_j(t,z,\zeta) \exp \{-K(t,z,\zeta)\}.$$
By (4.30) we have

\[
\frac{\partial y_i}{\partial \phi_1(t, z, \zeta)} = \sum_{j=1}^{N} \left\{ \frac{\partial y_i}{\partial b^{(\delta)}(z)} \partial y_j \frac{\partial f^G(t, z, \zeta)}{\partial y_j} + b^{(\delta)}(z) \frac{\partial y_i}{\partial \delta} \left( -\partial_{\zeta^2} f^G(t, z, \zeta) + B_j(t, z, \zeta) \right) \right\}
\]

(4.45)

and

\[
\frac{\partial \zeta^2}{\partial \phi_1(t, z, \zeta)} = \sum_{j=1}^{N} b^{(\delta)}(z) \frac{\partial \zeta^2}{\partial y_j} \frac{\partial f^G(t, z, \zeta)}{\partial y_j}.
\]

Therefore,

\[
\frac{\partial y_i}{\partial \phi_1(t, z, \zeta)} = -\frac{\partial \zeta^2}{\partial \phi_1(t, z, \zeta)} + A_i(t, z, \zeta),
\]

(4.46)

where

\[
A_i(t, z, \zeta) \overset{\text{def}}{=} \sum_{j=1}^{N} \left\{ \frac{\partial y_i}{\partial b^{(\delta)}(z)} \partial y_j \frac{\partial f^G(t, z, \zeta)}{\partial y_j} + b^{(\delta)}(z) B_j(t, z, \zeta) \right\}.
\]

(4.47)

By Theorem 4.2.3, the drift term \( b^{(\delta)} \) is smooth and has bounded derivative. Therefore, we can immediately note that \( A_i(t, z, \zeta) \) has an integrable singularity. In the second iteration, we show

\[
\frac{\partial}{\partial y_i} \left( \phi_2(t, z, \zeta) - \phi_1(t, z, \zeta) \right) = \frac{\partial}{\partial y_i} \int_{0}^{t} \int_{\mathbb{R}^{2N}} \phi_1(t - s, z, \xi) \phi_1(s, \xi, \zeta) d\xi ds = \int_{0}^{t} \int_{\mathbb{R}^{2N}} \frac{\partial y_i}{\partial \phi_1(t - s, z, \xi) \phi_1(s, \xi, \zeta)} d\xi ds.
\]

and we bound this term. First, let’s show that the integral on the right hand side is well-defined. To do this, we need to break the time integral to \((0, \frac{t}{2}) \cup (\frac{t}{2}, t)\). On the first interval the result is immediate by smoothness and boundedness of \( p^G(t - s, z, \xi) \) and its derivatives.
In particular, using the first presentation of (4.45) we can write
\[
\int_0^{t/2} \int_{\mathbb{R}^{2N}} |\partial_y \phi_1(t - s, \mathbf{z}, \zeta)| \phi_1(s, \xi, \zeta) \, d\xi \, ds \\
\leq \int_0^{t/2} \int_{\mathbb{R}^{2N}} \left\{ \sum_{j=1}^N \left( |\partial_y b^{(\delta)}(\mathbf{z})| |\partial_y p^{G}(t - s, \mathbf{z}, \xi)| + |b^{(\delta)}(\mathbf{z})| |\partial_{y,y_j} p^{G}(t - s, \mathbf{z}, \xi)| \right) \right\} \\
\quad \times |\phi_1(s, \xi, \zeta)| \, d\xi \, ds
\]
(4.48)

Let’s bound each of these terms separately. By the fact that \( b^{(\delta)} \) has a bounded first derivatives, (4.33) and (4.36) we have
\[
\int_0^{t/2} \int_{\mathbb{R}^{2N}} \left\{ \sum_{j=1}^N \left( |\partial_y b^{(\delta)}(\mathbf{z})| |\partial_y p^{G}(t - s, \mathbf{z}, \xi)| \right) \right\} |\phi_1(s, \xi, \zeta)| \, d\xi \, ds \leq C(b + b' |\zeta|) p^{G}(t, \hat{\mathbf{z}}, \hat{\zeta}) \frac{\Gamma^2(1/2)}{\Gamma(1)}
\]
where constant \( C \) depends on \( N, A \) and \( \alpha_0 \).

For the second term, using the fact that \( b^{(\delta)} \) has a sublinear growth rate and the fact that on the time interval \((0, \frac{t}{2})\) the kernel \( p^{G}(t - s, \mathbf{z}, \zeta) \) is smooth and bounded with all its derivatives, we can write
\[
\int_0^{t/2} \int_{\mathbb{R}^{2N}} \left\{ \sum_{j=1}^N \left( |b^{(\delta)}(\mathbf{z})| |\partial_{y,y_j} p^{G}(t - s, \mathbf{z}, \xi)| \right) \right\} |\phi_1(s, \xi, \zeta)| \leq C t^{-1/2} (b + b' |\zeta|)^2 p^{G}(t, \hat{\mathbf{z}}, \hat{\zeta})
\]
where here also \( C \) depends on \( N, A \) and \( \alpha_0 \).

Therefore, putting these results together, on the interval of \((0, \frac{t}{2})\) we have that
\[
\int_0^{t/2} \int_{\mathbb{R}^{2N}} |\partial_y \phi_1(t - s, \mathbf{z}, \zeta)| |\phi_1(s, \xi, \zeta)| \, d\xi \, ds \leq C(b + b' |\zeta|) p^{G}(t, \hat{\mathbf{z}}, \hat{\zeta}) B\left(\frac{1}{2}, \frac{1}{2}\right) \left( 1 + t^{-1/2}(b + b' |\zeta|) \right) \leq C(b + b' |\zeta|)^2 p^{G}(t, \hat{\mathbf{z}}, \hat{\zeta}) B\left(\frac{1}{2}, \frac{1}{2}\right) (c + t^{-1/2})
\]
(4.49)

On the time interval \((\frac{t}{2}, t)\) on the other hand, \( \partial_{y,y_j} p^{G}(t - s, \mathbf{z}, \zeta) \) does not enjoy integrable singularities. In this case we apply the change of variable (4.46). More accurately, on the
time interval \((\frac{t}{2}, t)\) the kernel \(\phi_1(s, \xi, \zeta)\) has no singularity. So, we write

\[
\int_{t/2}^{t} \int_{\mathbb{R}^2N} \partial_{y_i} \phi_1(t-s, z, \xi) \phi_1(s, \xi, \zeta) d\xi ds = -\int_{t/2}^{t} \int_{\mathbb{R}^2N} \partial_{\xi_2} \phi_1(t-s, z, \xi) \phi_1(s, \xi, \zeta) d\xi ds \\
+ \int_{t/2}^{t} \int_{\mathbb{R}^2N} A_i(t-s, z, \xi) \phi_1(s, \xi, \zeta) d\xi ds \\
= \int_{t/2}^{t} \int_{\mathbb{R}^2N} \phi_1(t-s, z, \xi) \partial_{\xi_2} \phi_1(s, \xi, \zeta) d\xi ds \\
+ \int_{t/2}^{t} \int_{\mathbb{R}^2N} A_i(t-s, z, \xi) \phi_1(s, \xi, \zeta) d\xi ds
\]

Following the same calculations as of \(I(t, z, \zeta)\) we have

\[
\int_{t/2}^{t} \int_{\mathbb{R}^2N} \phi_1(t-s, z, \xi) \partial_{\xi_2} \phi_1(s, \xi, \zeta) d\xi ds \leq C(b + b' |\zeta|)p^G(t, \tilde{z}, \hat{\zeta})B(\frac{1}{2}, \frac{1}{2}) (1 + t^{-1/2}(b + b' |\zeta|)) \\
\leq C(b + b' |\zeta|)^2p^G(t, \tilde{z}, \hat{\zeta})B(\frac{1}{2}, \frac{1}{2})(c + t^{-1/2})
\]

(4.50)

for some constant \(c > 0\). By the definition (4.47) of \(A_i(t, z, \zeta)\) we have

\[
\int_{t/2}^{t} \int_{\mathbb{R}^2N} |A_i(t-s, z, \xi)| |\phi_1(s, \xi, \zeta)| d\xi ds \leq CB(\frac{1}{2}, \frac{1}{2})(b + b' |\zeta|)p^G(t, \tilde{z}, \hat{\zeta}) (1 + b + b' |\zeta|) \\
\leq CB(\frac{1}{2}, \frac{1}{2})(b + b' |\zeta|)^2p^G(t, \tilde{z}, \hat{\zeta})
\]

(4.51)

for proper constant \(C > 0\). Therefore, (4.49), (4.50) and (4.51) imply that

\[
\left| \frac{\partial}{\partial y_i} (\phi_2(t, z, \zeta) - \phi_1(t, z, \zeta)) \right| \leq CB(\frac{1}{2}, \frac{1}{2})(b + b' |\zeta|)^2p^G(t, \tilde{z}, \hat{\zeta})(c + t^{-1/2})
\]

(4.52)

Let’s calculate another iteration. Using the definition (4.30) of \(\Phi\) series, we like to show that

\[
\frac{\partial}{\partial y_i} (\phi_3(t, z, \zeta) - \phi_2(t, z, \zeta)) = \frac{\partial}{\partial y_i} \int_{0}^{t} \int_{\mathbb{R}^2N} \phi_1(t-s, z, \xi)(\phi_2 - \phi_1)(s, \xi, \zeta) d\xi ds \\
= \int_{0}^{t} \int_{\mathbb{R}^2N} \partial_{y_i} \phi_1(t-s, z, \xi)(\phi_2 - \phi_1)(s, \xi, \zeta) d\xi ds
\]

(4.53)
Similar to calculations of the previous iteration, we show that the second presentation of (4.53) is well-defined (integrable). Considering (4.45) and (4.37), on the time interval \((0, \frac{t}{2})\)

\[
\int_0^{t/2} \int_{\mathbb{R}^2N} |\partial_y \phi_1(t-s, z, \xi)| |(\phi_2 - \phi_1)(s, \xi, \zeta)| d\xi ds \leq C^2 p^G(t, \hat{z}, \hat{\zeta})(b + b' |\zeta|)^2 \frac{1}{\Gamma(1)} \times (t^{1/2} + (b + b' |\zeta|))
\]

By (4.46), on the time interval \((\frac{t}{2}, t)\) we have

\[
\int_{t/2}^{t} \int_{\mathbb{R}^2N} \partial_y \phi_1(t-s, z, \xi) (\phi_2 - \phi_1)(s, \xi, \zeta) d\xi ds = \int_{t/2}^{t} \int_{\mathbb{R}^2N} \phi_1(t-s, z, \xi) \frac{\partial(\phi_2 - \phi_1)}{\partial \xi_{2,i}}(s, \xi, \zeta) d\xi ds
\]

\[
+ \int_{t/2}^{t} \int_{\mathbb{R}^2N} A_i(t-s, z, \xi)(\phi_2 - \phi_1)(s, \xi, \zeta) d\xi ds
\]

For the first term

\[
\int_{t/2}^{t} \int_{\mathbb{R}^2N} |\phi_1(t-s, z, \xi)| \left| \frac{\partial(\phi_2 - \phi_1)}{\partial \xi_{2,i}}(s, \xi, \zeta) \right| d\xi ds \leq C^2 p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1)} (b + b' |\zeta|) \times (t^{1/2} + (b + b' |\zeta|) + t^{1/2}(b + b' |\zeta|))
\]

By (4.47) the second term can be bounded by

\[
\int_{t/2}^{t} \int_{\mathbb{R}^2N} |A_i(t-s, z, \xi)| |(\phi_2 - \phi_1)(s, \xi, \zeta)| d\xi ds \leq C^3(b + b' |\zeta|)^3 p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1)}.
\]

Putting all together, we have

\[
\left| \frac{\partial}{\partial y_i} (\phi_3(t, z, \zeta) - \phi_2(t, z, \zeta)) \right| \leq \int_0^{t} \int_{\mathbb{R}^2N} |\partial_y \phi_1(t-s, z, \xi)| |(\phi_2 - \phi_1)(s, \xi, \zeta)| d\xi ds
\]

\[
\leq C^3(b + b' |\zeta|)^2 p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1)} (t^{1/2} + (b + b' |\zeta|) + t^{1/2}(b + b' |\zeta|))
\]

(4.55)
By induction on (4.55) we get that

\[
\left| \frac{\partial}{\partial y_i} (\phi_{r+1} - \phi_r) (t, z, \zeta) \right| \leq C^{r+1} (b + b' |\zeta|)^r p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma\left(\frac{r}{2}\right)} \\
\times \left( \frac{r-1}{2} + t^{\frac{r-2}{2}} (b + b' |\zeta|) + t^{\frac{r-1}{2}} (b + b' |\zeta|) \right)
\] (4.56)

for \( r \geq 1 \). Using (4.56) we show that the series (4.43) of derivative of \( \Phi \) converges. We need to consider several sub-series and prove their convergence. From (4.45) and (4.56) we consider the first series

\[
S_1 \overset{\text{def}}{=} Ct^{-1/2} p^G(t, \hat{z}, \hat{\zeta}) + C^2 (b + b' |\zeta|) p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1/2)} + C^3 (b + b' |\zeta|)^2 t^{1/2} p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1)} + \cdots.
\] (4.57)

Now we define two sub-series of \( S_1 \). First

\[
Ct^{-1/2} p^G(t, \hat{z}, \hat{\zeta}) \left\{ 1 + \frac{1}{\Gamma(1)} C^2 (b + b' |\zeta|)^2 + t + \frac{1}{\Gamma(2)} (C^2 (b + b' |\zeta|)^2 + t)^2 + \cdots \right\} \leq Ct^{-1/2} p^G(t, \hat{z}, \hat{\zeta}) \exp \left\{ k |\zeta|^2 t \right\}.
\] (4.58)

The second sub-series of \( S_1 \) can be rearranged as

\[
C^2 (b + b' |\zeta|) \frac{1}{\Gamma(1/2)} p^G(t, \hat{z}, \hat{\zeta}) + C^4 (b + b' |\zeta|)^3 p^G(t, \hat{z}, \hat{\zeta}) t \left\{ \frac{1}{\Gamma(3/2)} + \frac{C^2 (b + b' |\zeta|)^2 t}{\Gamma(5/2)} + \cdots \right\} \leq \left\{ C^2 (b + b' |\zeta|) \frac{1}{\Gamma(1/2)} + C^4 (b + b' |\zeta|)^3 \frac{1}{\Gamma(3/2)} + C^4 (b + b' |\zeta|)^3 t \exp \left\{ k |\zeta|^2 t \right\} \right\} \times p^G(t, \hat{z}, \hat{\zeta})
\]

Adding both sub-series and choosing constant \( C \) properly, we have

\[
S_1 \leq C (\text{Const.} + t^{-1/2}) p^G(t, \hat{z}, \hat{\zeta}) \exp \left\{ k |\zeta|^2 t \right\}.
\] (4.59)
The second series of (4.56) which needs to be considered will be

\[ S_2 \overset{\text{def}}{=} C t^{-1} (b + b' |\zeta|) p^G(t, \hat{z}, \hat{\zeta}) + C^2 t^{-1/2} (b + b' |\zeta|)^2 p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1/2)} \]

\[ + C^3 (b + b' |\zeta|)^3 p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1)} + \ldots. \] (4.60)

Similar to the previous case, we consider two sub-series to prove the convergence of \( S_2 \). The first sub-series is selected as

\[ C(b + b' |\zeta|) t^{-1} p^G(t, \hat{z}, \hat{\zeta}) \left\{ 1 + \frac{C^2 (b + b' |\zeta|)^2 t}{\Gamma(1)} + \frac{(C^2 (b + b' |\zeta|)^2 t)^2}{\Gamma(2)} + \ldots \right\} \]

\[ \leq C(b + b' |\zeta|) t^{-1} p^G(t, \hat{z}, \hat{\zeta}) \exp \{ k |\zeta|^2 t \}. \]

The second sub-series will be

\[ C^2(b + b' |\zeta|)^2 t^{-1/2} p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1/2)} + C^4(b + b' |\zeta|)^4 t^{1/2} \left\{ \frac{1}{\Gamma(1/3)} + \frac{C^2(b + b' |\zeta|)^2 t}{\Gamma(5/2)} + \ldots \right\} \]

\[ \leq C^2(b + b' |\zeta|)^2 t^{-1/2} p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1/2)} + C^4(b + b' |\zeta|)^4 t^{1/2} p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1/3)} \]

\[ + C^4(b + b' |\zeta|)^4 t^{1/2} p^G(t, \hat{z}, \hat{\zeta}) \exp \{ k |\zeta|^2 t \} \]

Adding the sub-series together and properly redefining the constant \( C \), we have

\[ S_2 \leq C(\text{Const.} + t^{-1})(b + b' |\zeta|)^4 p^G(t, \hat{z}, \hat{\zeta}) \exp \{ k |\zeta|^2 t \}. \] (4.61)

Finally, the third series can will be

\[ S_3 \overset{\text{def}}{=} C^2(b + b' |\zeta|)^2 p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1/2)} + C^3 t^{1/2} (b + b' |\zeta|)^3 p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1)} + \ldots. \]

The first convergent sub-series of \( S_3 \) will be

\[ C^3 t^{1/2} p^G(t, \hat{z}, \hat{\zeta})(b + b' |\zeta|)^3 \left\{ \frac{1}{\Gamma(1)} + \frac{C^2(b + b' |\zeta|)^2 t}{\Gamma(2)} + \frac{(C^2(b + b' |\zeta|)^2 t)^2}{\Gamma(3)} + \ldots \right\} \]

\[ \leq C^3(b + b' |\zeta|)^3 p^G(t, \hat{z}, \hat{\zeta})^{3 t^{1/2}} \exp \{ k |\zeta|^2 t \}. \]
The second sub-series will be

\[
C^2(b + b' |\zeta|)^2p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1/2)} + C^4(b + b' |\zeta|)^4p^G(t, \hat{z}, \hat{\zeta})t \\
\times \left\{ \frac{1}{\Gamma(3/2)} + \frac{C^2(b + b' |\zeta|)^2t}{\Gamma(5/2)} + \frac{(C^2(b + b' |\zeta|)^2t)^2}{\Gamma(7/2)} \cdots \right\} \\
\leq C^2(b + b' |\zeta|)^2p^G(t, \hat{z}, \hat{\zeta}) \frac{1}{\Gamma(1/2)} + C^4(b + b' |\zeta|)^4p^G(t, \hat{z}, \hat{\zeta})t \frac{1}{\Gamma(3/2)} \\
+ C^4(b + b' |\zeta|)^4p^G(t, \hat{z}, \hat{\zeta})t \exp\left\{ k |\zeta|^2 t \right\}
\]

Adding these sub-series, we can write

\[
S_3 \leq C(\text{Const.} + t)(b + b' |\zeta|)^4p^G(t, \hat{z}, \hat{\zeta}) \exp\left\{ k |\zeta|^2 t \right\}. \quad (4.62)
\]

From (4.59), (4.61) and (4.62) and by redefining the constant $C$ appropriately we conclude that

\[
\left| \frac{\partial}{\partial y_i} \Phi(t, z, \zeta) \right| \leq C(\text{Const.} + t^{-1})(b + b' |\zeta|)^4p^G(t, \hat{z}, \hat{\zeta}) \exp\left\{ k |\zeta|^2 t \right\} \\
\leq C(\text{Const.} + t^{-1})p^G(t, \hat{z}, \hat{\zeta}) \exp\left\{ k |\zeta|^2 t \right\}
\]

Using this result we can show

**Theorem 4.4.6.** We have

\[
\partial^2_{yi} H(t, z, \zeta) = \int_0^t \int_{\mathbb{R}^{2N}} \partial^2_{yi} p^G(t-s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds.
\]

**Proof.** By using the result of Theorem 4.4.5, the proof of this theorem is straightforward. We break the time interval to $(0, \frac{t}{2}) \cup (\frac{t}{2}, t)$. On the first interval by smoothness and boundedness of $p^G(t, z, \zeta)$ we will have the result. On the second interval, using (4.44) and Theorem 4.4.5 we apply integration by parts and the result will follow.
Theorem 4.4.7. Let function $H$ be as in (4.28). Then
\[
\frac{dH}{dt}(t, z, \zeta) = \int_0^t \int_{\mathbb{R}^{2N}} \frac{d}{dt} p^G(t - s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds.
\]

Proof. By definition,
\[
\frac{d}{dt} H(t, z, \zeta) = \lim_{h \to 0} \frac{H(t + h, z, \zeta) - H(t, z, \zeta)}{h} = \lim_{h \to 0} \frac{1}{h} \left\{ \int_0^{t+h} \int_{\mathbb{R}^{2N}} p^G(t + h - s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds \right. \\
- \int_0^t \int_{\mathbb{R}^{2N}} p^G(t - s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds \left\}
\]
\[
= \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{R}^{2N}} p^G(t + h - s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds \\
+ \lim_{h \to 0} \frac{1}{h} \int_0^t \int_{\mathbb{R}^{2N}} \left[ p^G(t + h - s, \xi) - p^G(t - s, \xi) \right] \Phi(s, \xi, \zeta) d\xi ds \\
= \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{R}^{2N}} p^G(t + h - s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds \\
+ \lim_{h \to 0} \int_0^t \int_{\mathbb{R}^{2N}} \left[ \int_0^1 \frac{d}{dt} p^G(t + (1 + \lambda)h - s, z, \xi) d\lambda \right] \Phi(s, \xi, \zeta) d\xi ds
\]

By the Dirac behavior of $p^G(t, z, \zeta)$, the first integral term is
\[
\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \int_{\mathbb{R}^{2N}} p^G(t + h - s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds = \Phi(t, z, \zeta).
\]

For the second term, we note that by (4.26)
\[
L_0 p^G(t, z, \zeta) = \frac{dp^G}{dt}(t, z, \zeta).
\]

Therefore, using Fubini theorem (we suppose Fubini holds first and then the final result confirms that this assumption is valid), breaking the time interval, applying the result of Theorem 4.4.6 and finally dominated convergence theorem, the proof follows. □

Putting all of the previous results on differentiability of $H$ together, the claim (4.29)
Theorem 4.4.8. For any continuous function $f : \mathbb{R}^{2N} \rightarrow \mathbb{R}$

$$\lim_{t \to 0} \int_{\mathbb{R}^{2N}} p(t, z, \zeta) f(\zeta) d\zeta = f(z), \quad z \in \mathbb{R}^{2N}.$$ (4.63)

Proof. Without loss of any generality, we assume that $f \geq 0$. We directly use the definition of kernel $p(t, z, \zeta)$ to prove the theorem. In particular, we have

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^{2N}} p(t, z, \zeta) f(\zeta) d\zeta = \lim_{t \downarrow 0} \int_{\mathbb{R}^{2N}} \left( p^G(t, z, \zeta) + \int_0^t \int_{\mathbb{R}^{2N}} p^G(t - s, z, \xi) \Phi(s, \xi, \zeta) d\xi ds \right) f(\zeta) d\zeta$$

But we have

$$\lim_{t \downarrow 0} \int_{\mathbb{R}^{2N}} p^G(t, z, \zeta) f(\zeta) d\zeta = f(z).$$

So to prove the result, we need to show that

$$\mathcal{I} \overset{\text{def}}{=} \lim_{t \downarrow 0} \int_{\mathbb{R}^{2N}} \int_0^t \int_{\mathbb{R}^{2N}} p^G(t - s, z, \xi) \Phi(s, \xi, \zeta) f(\zeta) d\xi ds d\zeta = 0.$$ 

It is sufficient to prove that the absolute value of the integral expression converges to 0. By the bound (4.39) we have that

$$|\mathcal{I}| \leq \lim_{t \downarrow 0} \int_{\mathbb{R}^{2N}} f(\zeta) \int_0^t \int_{\mathbb{R}^{2N}} p^G(t - s, z, \zeta) |\Phi(s, \xi, \zeta)| d\xi ds d\zeta$$

$$\leq \lim_{t \downarrow 0} \int_{\mathbb{R}^{2N}} f(\zeta) k(b + b' |\zeta|) t^{1/2} \exp \left\{ k |\zeta|^2 T \right\} p^G(t, \alpha_0 z, \alpha_0 \zeta) d\zeta$$

$$\leq C \lim_{t \downarrow 0} t^{1/2} (a + a' |z|) \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^{2N}} p^G(t, \alpha_1 z, \alpha_1 \zeta) \exp \left\{ k |\zeta|^2 T \right\} d\zeta$$

$$\leq C \lim_{t \downarrow 0} t^{1/2} (a + a' |z|) \exp \left\{ k |z|^2 T \right\} \int_{\mathbb{R}^{2N}} f(\zeta) p^G(t, \alpha_2 z, \alpha_2 \zeta) d\zeta.$$ 

for $\alpha_2 < \alpha_1 < \alpha_0 \in (0, 1)$. Note that for the last two inequalities we use the results proven in the Appendices A and C. By properties of density function $p^G(t, \alpha_2 z, \alpha_2 \zeta)$, the last integral is well-defined and hence the claim follows. \qed
4.5 Summary and Future Works

In this section, we briefly discussed the concept of the propagation of the noise in the stochastic optimal velocity dynamical model. The propagation of the noise in this model can be studied by investigating the explicit form of the transition density function of the Markov process corresponding to the solution.

The existence of the transition density function depends on the regularity of the coefficients. Therefore, we approximated the optimal velocity model and we proved that in the approximating dynamics the coefficients are sufficiently regular. Furthermore, we proved the consistency of the solutions in the approximating dynamics with respect to parameter $\delta$, which leads to defining the solution of optimal velocity dynamical model up to some random time. Investigating the properties of this random time is an interesting extension of this work. Finally, applying the method of [52], we showed that transition density function can be explicitly constructed by Lévy iterative method.
REFERENCES


APPENDIX A

THE LINEAR GROWTH OF THE DRIFT TERM

Lemma A.1. We have that

\[(a + a' |z|)p^G(t, \alpha z, \alpha_\circ \zeta) \leq C(T, \bar{\lambda})(b + b' |\zeta|)p^G(t, \alpha_1 z, \alpha_1 \zeta), \tag{A.1}\]

for \(\alpha_1 < \alpha_\circ \in (0, 1)\), and \(C(T, \bar{\lambda})\) is a constant depends on \(T\) and the maximum eigenvalue denoted by \(\bar{\lambda}\).

Proof.

\[|z| \leq |(x, y) - (x, \zeta_2)| + |(x, \zeta_2) - (x, 0)| + |(x, 0) - (\zeta_1, 0)| + |\zeta_1| \]

\[= |y - \zeta_2| + |\zeta_2| + |x - \zeta_1| + |\zeta_1| \]

One the other hand,

\[|x - \zeta_1| \leq |\zeta_1 - x - \frac{1}{2}t(\zeta_2 + y)| + \frac{1}{2}t|\zeta_2 + y| \leq \frac{\bar{\lambda}}{\sqrt{6}} \left( \sum_{i=1}^{N} R^2_i(t, z, \zeta) \right)^{1/2} + \frac{1}{2}t |y - \zeta_2| + t |\zeta_2| . \]

Putting all together we have

\[|z| \leq (1 + \frac{1}{2}t) \frac{\bar{\lambda}}{\sqrt{6}} \left( \sum_i S^2_i(t, z, \zeta) \right)^{1/2} + \frac{\bar{\lambda}}{\sqrt{6}} \left( \sum_{i=1}^{N} R^2_i(t, z, \zeta) \right)^{1/2} + (1 + t)|\zeta| \]
Therefore, it is easy to calculate that

\[
|z| p^G(t, \alpha_0 z, \alpha_0 \zeta) \leq C(T, \bar{\lambda}) t^{1/2} \left( \frac{\sum_i S_i^2(t, z, \zeta)}{t} \right)^{1/2} p^G(t, \alpha_0 z, \alpha_0 \zeta)
\]

\[
+ C(T, \bar{\lambda}) t^{3/2} \left( \frac{\sum_{i=1}^N R_i^2(t, z, \zeta)}{t^3} \right)^{1/2} p^G(t, \alpha_0 z, \alpha_0 \zeta)
\]

\[
+ C(T) |\zeta| p^G(t, \alpha_0 z, \alpha_0 \zeta)
\]

\[
\leq C(T, \bar{\lambda})(1 + |\zeta|) p^G(t, \alpha_1 z, \alpha_1 \zeta)
\]

for any \(\alpha_1 < \alpha_0 \in (0, 1)\). Therefore, this proves the claimed result. \qed
APPENDIX B

BOUNDING THE KERNEL $\Phi$

We have

$$|\Phi(t, z, \zeta)| \leq C(b + b'|\zeta|)t^{-1/2}p^G(t, \hat{z}, \hat{\zeta})$$

$$+ \sum_{r \geq 1} C^{r+1}(b + b'|\zeta|)^{r+1}t^{r-1} \frac{1}{\Gamma\left(\frac{r+1}{2}\right)}$$

We need to split the summation into the following sums

$$A = C^2(b + b'|\zeta|)^2p^G(t, \hat{z}, \hat{\zeta}) \sum_{r \geq 0} \frac{(C^2(b + b'|\zeta|)^2t)^r}{\Gamma(r + 1)}$$

$$\leq C^2(b + b'|\zeta|)^2p^G(t, \hat{z}, \hat{\zeta}) \exp\{C^2(b + b'|\zeta|)^2t\}.$$

The second sum will be

$$B = C^3(b + b'|\zeta|)^3t^{1/2}p^G(t, \hat{z}, \hat{\zeta}) \left[ \frac{1}{\Gamma(3/2)} + \frac{C^2(b + b'|\zeta|)^2t}{\Gamma(5/2)} + \frac{(C^2(b + b'|\zeta|)^2t)^2}{\Gamma(7/2)} + \cdots \right]$$

$$\leq C^3(b + b'|\zeta|)^3t^{1/2}p^G(t, \hat{z}, \hat{\zeta}) \exp\{C^2(b + b'|\zeta|)^2t\}$$

where the last inequality is by noting that

$$\Gamma\left(\frac{r}{2}\right) = \sqrt{\pi} \left(\frac{r - 2}{2}\right)!!.$$}

Therefore, adding all together, we have

$$|\Phi(t, z, \zeta)| \leq C(b + b'|\zeta|)p^G(t, \hat{z}, \hat{\zeta})t^{-1/2} \left[ 1 + C(b + b'|\zeta|)t^{1/2} + C^2(b + b'|\zeta|)^2t \right] \times$$

$$\exp\{C^2(b + b'|\zeta|)^2t\}.$$
Using the fact that by selecting the right constants

\[ 1 + C(b + b'|\zeta|)t^{1/2} + C^2(b + b'|\zeta|)^2 t \leq \exp\left\{ C(b + b'|\zeta|)t^{1/2} \right\}, \]

we can show that

\[ |\Phi(t, z, \zeta)| \leq k(b + b'|\zeta|)t^{-1/2} p^G(t, \hat{z}, \hat{\zeta}) \exp\left\{ k|\zeta|^2 t \right\} \]
APPENDIX C

EXPONENTIAL TRANSFORMATION

Let $z = (x, y)^T, \zeta = (\zeta_1, \zeta_2)^T \in \mathbb{R}^2$, and $t \in (0, T]$. We like to show that

$$\exp \left\{ k_1 |z|^2 t \right\} p^G(t, \alpha_0 z, \alpha_0 \zeta) \leq C(T) \exp \left\{ k_2 |\zeta|^2 t \right\} p^G(t, \alpha_1 z, \alpha_2 \zeta), \quad (C.1)$$

for any $\alpha_1 < \alpha_0 \in (0, 1)$, and constant $k_1, k_2$ and $C(T)$ may depends on $T$. We first consider the problem in two-dimension and then formally generalize the results to higher dimension.

We start by rewriting the left hand side as

$$p^G(t, \alpha_0 z, \alpha_0 \zeta) \left\{ 1 + \frac{k|z|^2 t}{1!} + \frac{(k|z|^2 t)^2}{2!} + \ldots \right\} \quad (C.2)$$

The constant $k$ here is generic and may depends on $T$. Now, we consider each of these terms separately. By the calculations in Appendix A, for the first term we will have

$$|z|^2 p^G(t, \hat{z}, \hat{\zeta}) \leq C \left\{ (1 + t) |\zeta_2 - y| + |\mathcal{R}(t, z, \zeta)| + (1 + t) |\zeta| \right\}^2 p^G(t, \hat{z}, \hat{\zeta})$$

We need some basic tools before we proceed. For any $a, b \in \mathbb{R}_+$ we have the binomial expansion as

$$(a + b)^n = a^n + b^n + \sum_{k=1}^{n-1} \binom{n}{k} a^{n-k} b^k.$$

By Young inequality each of there terms will become

$$a^{n-k} b^k \leq \frac{(a^{n-k})^p}{p} + \frac{(b^k)^q}{q}$$
where \( p, q \) are conjugate exponents. Let \( kq = n \), then

\[
q = \frac{n}{k}, \quad p = \frac{n}{n-k}.
\]

This means that

\[
\binom{n}{k} a^{n-k} b^k \leq \binom{n}{k} \left( \frac{a^n}{n-k} + \frac{b^n}{k} \right).
\]

Therefore, we have

\[
(a + b)^n \leq \left( 1 + \sum_{k=1}^{n-1} \binom{n-1}{k} \right) a^n + \left( 1 + \sum_{k=1}^{n-1} \binom{n-1}{k-1} \right) b^n
\]

\[
= \sum_{k=0}^{n-1} \binom{n-1}{k} a^n + \sum_{k=1}^{n-1} \binom{n-1}{k-1} b^n.
\]

Using Binomial theorem, we can calculate

\[
\sum_{k=0}^{n-1} \binom{n-1}{k} = \sum_{k=1}^{n} \binom{n-1}{k-1} = (1 + 1)^{n-1} = 2^{n-1}.
\]

Moreover, by repeating the previous calculations, it is easy to see that for any \( a, b, c \in \mathbb{R}_+ \) we have

\[
(a + b + c)^n \leq (2^2)^{n-1}(a^n + b^n + c^n).
\]

Now, we get back to calculations of (C.2). For the second term we have

\[
|z|^2 p^G(t, \hat{z}, \hat{\zeta}) \leq 2^2 C \{ (1 + t)^2 |\zeta_2 - y|^2 + R^2(t, z, \zeta) + (1 + t)^2 |\zeta|^2 \} p^G(t, \hat{z}, \hat{\zeta}).
\]

Bounding each of these terms separately, we have

\[
|z|^2 p^G(t, \hat{z}, \hat{\zeta}) \leq 2^2 C(1 + T)^2 Tp^G(t, \alpha_1 z, \alpha_1 \zeta)
\]

\[
+ 2^2 C T^3 p^G(t, \alpha_1 z, \alpha_1 \zeta)
\]

\[
+ 2^2 C(1 + t)^2 |\zeta|^2 p^G(t, \alpha_1 z, \alpha_1 \zeta)
\]

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Similarly,

\[ |z|^4 p^G(t, \hat{z}, \hat{\zeta}) \leq (2^2)^3 \left( CT(1 + T)^2 \right)^2 p^G(t, \alpha_1 z, \alpha_1 \zeta) \]

\[ + (2^2)^3 \left( CT^3 \right)^2 p^G(t, \alpha_1 z, \alpha_1 \zeta) \]

\[ (2^2)^3 \left( C(1 + t)^2 |\zeta|^2 \right)^2 p^G(t, \alpha_1 z, \alpha_1 \zeta) \]

Therefore, considering (C.2) and by induction we will have three different exponential series on the right hand side

\[ I \overset{\text{def}}{=} (2^{2k}(T)t)T(1 + T)^2 \left\{ 1 + \frac{1}{1!} \left[ (2^4k(T)t) (T(1 + T)^2) \right] + \frac{1}{2!} \left[ (2^4k(T)t) (T(1 + T)^2) \right]^2 + \cdots \right\} \]

\[ \leq k_1(T) \exp \{ k_1(T) \} \]

\[ II \overset{\text{def}}{=} (2^4k(T)t)T^3 \left( 1 + \frac{1}{1!} \left( 2^4k(T)tT^3 \right) + \frac{1}{2!} \left( 2^2k(T)tT^3 \right)^2 + \cdots \right) \]

\[ \leq k_2(T) \exp \{ k_2(T) \} \]

\[ III \overset{\text{def}}{=} 1 + \left[ 2^4k(T)(1 + t)^2 \right] (t |\zeta|^2 + \left[ 2^4k(T)(1 + t)^2 \right]^2 (t |\zeta|^2)^2 + \cdots \]

\[ \leq \exp \left\{ k_3(T) t |\zeta|^2 \right\} . \]

Therefore, putting all together, we have that

\[ \exp \left\{ k_1 |z|^2 t \right\} p^G(t, \alpha_0 z, \alpha_0 \zeta) \leq (I + II + III) p^G(t, \alpha_1 z, \alpha_1 \zeta), \]

and hence (C.1) follows.

Now, we generalize the previous result to the multi dimension case.

**Lemma C.1.** Let \( z = (x, y)^T, \zeta = (\zeta_1, \zeta_2)^T \in \mathbb{R}^{2N}, \) and \( t \in (0, T]. \) We have that

\[ \exp \left\{ k_1 |z|^2 t \right\} p^G(t, \alpha_0 z, \alpha_0 \zeta) \leq (I + II + III) p^G(t, \alpha_1 z, \alpha_1 \zeta), \]

for any \( \alpha_1 < \alpha_0 \in (0, 1), \) and all constants may depend on \( T. \)
Proof. We start by rewriting the left hand side as

\[ p^G(t, \alpha_\circ z, \alpha_\circ \zeta) \left\{ 1 + \frac{k |z|^2 t}{1!} + \frac{(k|z|^2 t)^2}{2!} + \cdots \right\} \]  
\[ (C.4) \]

The constant \( k \) here is generic and may depend on \( T \). Now, we consider each of these terms separately. By the calculations in Appendix A, for the first term we will have

\[ |z|^2 p^G(t, \hat{z}, \hat{\zeta}) \leq C(T, \bar{\lambda}) \left\{ \left( \sum_i S_i^2(t, z, \zeta) \right)^{1/2} + \left( \sum_i R_i^2(t, z, \zeta) \right)^{1/2} + |\zeta| \right\}^2 p^G(t, \hat{z}, \hat{\zeta}) \]

We need some basic tools before we proceed. For any \( a, b \in \mathbb{R}_+ \) we have the binomial expansion as

\[ (a + b)^n = a^n + b^n + \sum_{k=1}^{n-1} \binom{n}{k} a^{n-k} b^k. \]

By Young inequality each of these terms will become

\[ a^{n-k} b^k \leq \frac{(a^{n-k})^p}{p} + \frac{(b^k)^q}{q} \]

where \( p, q \) are conjugate exponents. Let \( kq = n \), then

\[ q = \frac{n}{k}, \quad p = \frac{n}{n-k}. \]

This means that

\[ \binom{n}{k} a^{n-k} b^k \leq \binom{n}{k} \left( \frac{a^n}{n-k} + \frac{b^n}{n} \right). \]

Therefore, we have

\[ (a + b)^n \leq \left( 1 + \sum_{k=1}^{n-1} \binom{n-1}{k} \right) a^n + \left( 1 + \sum_{k=1}^{n-1} \binom{n-1}{k-1} \right) b^n \]

\[ = \sum_{k=0}^{n-1} \binom{n-1}{k} a^n + \sum_{k=1}^{n} \binom{n-1}{k-1} b^n. \]
Using Binomial theorem, we can calculate
\[ \sum_{k=0}^{n-1} \binom{n-1}{k} = \sum_{k=1}^{n} \binom{n-1}{k-1} = (1 + 1)^{n-1} = 2^{n-1}. \]
Moreover, by repeating the previous calculations, it is easy to see that for any \( a, b, c \in \mathbb{R}_+ \) we have
\[ (a + b + c)^n \leq (2^{n-1})(a^n + b^n + c^n). \]
Now, we get back to calculations of (C.4). For the second term we have
\[
|z|^2 p^G(t, \hat{z}, \hat{\zeta}) \leq 2^2 C^2(T, \bar{\lambda}) \left\{ \sum_i S_i^2(t, z, \zeta) + \sum_i R_i^2(t, z, \zeta) + |\zeta|^2 \right\} p^G(t, \hat{z}, \hat{\zeta})
\leq 2^2 C^2(T, \bar{\lambda}) \left\{ t \left( \sum_i S_i^2(t, z, \zeta) \right) + t^3 \left( \sum_i R_i^2(t, z, \zeta) \right) + |\zeta|^2 \right\}
\leq 2^2 C^2(T, \bar{\lambda})(1 + |\zeta|^2) p^G(t, \alpha_1 z, \alpha_1 \zeta) \]
Similarly,
\[
|z|^4 p^G(t, \hat{z}, \hat{\zeta}) \leq (2^2)^3 C^4(T, \bar{\lambda})(1 + |\zeta|^4) p^G(t, \alpha_1 z, \alpha_1 \zeta). \]
Therefore, considering (C.2) and by induction we will have the following exponential series
\[
I \overset{\text{def}}{=} \left( 1 + \frac{2^2 C^2(T, \bar{\lambda}) t}{1!} + \frac{(2^2)^3 C^4(T, \bar{\lambda}) t^2}{2!} + \cdots \right) p^G(t, \alpha_1 z, \alpha_1 \zeta)
\leq \exp \left\{ 2^2 C^2(T, \bar{\lambda}) t \right\}
\]
\[
II \overset{\text{def}}{=} \left( 1 + \frac{2^2 C^2(T, \bar{\lambda}) |\zeta|^2 t}{1!} + \frac{(2^2)^3 C^4(T, \bar{\lambda}) |\zeta|^4 t^2}{2!} + \cdots \right) p^G(t, \alpha_1 z, \alpha_1 \zeta)
\leq \exp \left\{ 2^2 C^2(T, \bar{\lambda}) |\zeta|^2 t \right\}.
\]
Adding these terms, the claim of (C.3) follows. \( \square \)
In this section we bring a Gronwall-Bellman inequality type without the proof which is used in the proof of theorem 2.2.2. The detailed proof and more similar inequalities can be found in [59].

**Theorem D.1.** Let $u(t), f(t)$ and $g(t)$ be real-valued non-negative continuous functions defined on set $I = [0, \infty)$, such that

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s) \int_0^s g(r)u(r)dr \, ds, \quad t \in I,$$

holds. Then,

$$u(t) \leq u_0 \left(1 + \int_0^t f(s) \exp \left\{ \int_0^s (f(r) + g(r))dr \right\} \, ds \right).$$

In the case of our problem, and considering the constants defined in (2.33) we have:

$$|\zeta^\circ_t| \leq \alpha \int_0^t |\zeta^\circ_s| ds + K_1 \int_0^t \int_0^s |\zeta^\circ_r| dr \, ds + R.$$

Therefore if we define

$$f(s) = \alpha, \quad g(s) = \frac{K_1}{\alpha}, \quad u_0 = R.$$

then in the view of theorem D.1 we have

$$|\zeta^\circ_t| \leq R \left(1 + \frac{\alpha^2}{\alpha^2 + K_1} \exp \left\{ \left(\alpha + \frac{K_1}{\alpha}\right) t \right\} \right).$$