A NUMERICAL PROCEDURE FOR THE ANALYSIS
OF PRESSURE VESSEL HEADS

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By
T. Au,
L. E. Goodman
and
N. M. Newmark

DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS
A NUMERICAL PROCEDURE FOR THE ANALYSIS
OF PRESSURE VESSEL HEADS.

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T. Au, L. E. Goodman and N. M. Newmark

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The need for more nearly exact methods of analysis of pressure vessel heads is supplied in this report by a numerical technique of wide applicability. The analysis is based on the general theory of thin elastic shells due to A. E. H. Love. While the method is approximate in the sense that the governing mathematical equations are satisfied only in finite difference form, it has the merit of simplicity and of directly providing numerical values of normal forces, moments and shears needed in design. The general finite-difference method is first developed and then applied to the particular cases of pressure vessels having spherical heads, conical heads, torispherical heads, toriconical heads, hemispherical heads and flat circular heads. Close agreement is gotten between the results obtained by "classical" analysis and those given by the numerical method, for the spherical-segment head chosen as a test problem.
A NUMERICAL PROCEDURE FOR THE ANALYSIS OF PRESSURE VESSEL HEADS.

I. INTRODUCTION.

1. Object and Scope of Investigation

The design and analysis of pressure vessels have for many years proceeded along separate lines. On the one hand, a mathematical theory, due in the first instance to A. E. H. Love (15), has been applied to the analysis of a few simple shapes. Industry, on the other hand, has been forced to rely on rule-of-thumb design formulas because the shapes of pressure vessels encountered in practice are not amenable to analysis by the exact theory. So long as large factors of safety could be used, this dichotomy produced no great difficulties; but today, as the requirements of high pressures and economy of design become more stringent, the need for more nearly exact analyses of pressure vessel heads is pressing. This need is here supplied by a numerical technique of wide applicability. While it is approximate in the sense that the governing mathematical equations are satisfied only in finite difference form, it has the merit of simplicity and of directly providing numerical values of normal forces, moments and shears needed in design.

The analysis is based on the general theory of thin elastic shells due to Love. Although Love's theory has been re-examined by many others who have questioned its adequacy in some respects, it represents a systematic first approximation for most practical purposes. Reduction of Love's theory to two differential equations of second order for shells of revolution having constant radius of curvature along the meridian has been made by E. Meissner (38). Solutions of the Love-Meissner equations even for shells of simple geometrical shapes, with the notable exception of the cylindrical shell and flat circular pressure vessel head, involve the use of the hypergeometric series which usually exhibits slow convergence. Several simplified methods have been developed as approximations to Love's theory. Most of these analyses follow the "approximate theory" of J. W. Geckeler (49) which neglects some of the terms in the governing differential equations. The idea of seeking solutions of the governing differential equations by the finite difference analogy is not entirely new, but little

1. Numbers in parentheses refer to the Annotated Bibliography at the end of the report.
progress has been made along this line since the pioneer work of P. Pasternak (52).

The present investigation consists in the development of a general procedure and its application to different types of pressure vessel heads including:

1. Spherical head
2. Conical head
3. Torispherical head
4. Toriconical head
5. Hemispherical head
6. Flat circular head

The accuracy of the numerical method is examined by means of a comparative study of a test problem, the spherical-segment head, analyzed by three different methods: Love-Meissner's classical analysis, Geckeler's approximate solution, and the method of finite differences.

2. Acknowledgment

This report is based on the doctoral dissertation in the Department of Civil Engineering, University of Illinois, prepared by T. Au, under the supervision of L. E. Goodman and the general direction of N. M. Newmark. Part of the material is the outgrowth of a research program "Numerical and Approximate Methods of Stress Analysis" sponsored by the Office of Naval Research as Task VI of Contract N6or-71, Project Designation NR-035-183, in the Engineering Experiment Station of the University of Illinois. The writers also acknowledge the advantages of association with the Design Division of the Pressure Vessel Research Committee of the Welding Research Council, and the Copper and Brass Research Association.
II. RÉSUMÉ OF FUNDAMENTAL EQUATIONS BASED ON LOVE'S THEORY OF THIN SHELLS.

3. Nomenclature

The following nomenclature has been used in the derivation of equations throughout the report. Insofar as practicable, this nomenclature conforms to conventional symbols for applied mechanics, following the treatment in "Theory of Plates and Shells" by S. Timoshenko (61).

\[ P = \text{Uniform internal pressure} \]
\[ c, h, k = \text{Subscripts denoting cylindrical shell, head proper and knuckle, respectively.} \]
\[ t = \text{Thickness of the shell.} \]
\[ \phi = \text{Angle between axis of revolution and a perpendicular to middle surface at any point of a shell of revolution, measured along the meridian.} \]
\[ \theta = \text{Angle between a reference meridian and any other meridian passing through a point of a shell of revolution, measured along the parallel circle.} \]
\[ \psi = (90^\circ - \phi). \]
\[ r_m = \text{Radius of curvature of the meridional curve formed by the intersection of the middle surface of a shell with a plane passing through the axis of revolution and the point in question.} \]
\[ r_c = \text{Distance from the point in question to the axis of revolution measured along a line normal to the middle surface of a shell at the point in question.} \]
\[ r_0 = \text{Radius of the circumferential circle formed by the intersection of the middle surface of a shell with a plane normal to the axis of revolution and passing through the point in question.} \]
\[ R = \text{Radius of cylindrical shell (} r_c = r_0). \]
\[ \rho = \text{Radius of spherical head (} r_m = r_c). \]
\[ K = \text{Radius of toroidal knuckle (} r_m = r_m). \]
\[ \lambda = \text{A constant such that} \lambda K \text{ is the distance from the axis of revolution to the center of the generating circle of a toroidal knuckle.} \]
S = Slant height of conical head.
\( \psi \) = One half of the apex angle for conical head.
x = Distance from head-to-shell juncture to any point in question along a cylindrical shell.
y = Distance from the cone apex to any point on the middle surface of a conical head, measured in meridional direction.
r = Radial distance from the center of a flat circular head to any point on the middle surface of the head.
s. = Distance from head-to-shell juncture to any point on a spherical head, measured in meridional direction. (A quantity arising in connection with the approximate solution of spherical heads).
v = Displacement in the meridional direction for any point of the shell, measured along a tangent to the undeformed middle surface at that point, positive in direction of increasing \( \phi \).
w = Displacement in the radial direction for any point of the shell, measured along a normal to the undeformed middle surface at that point, positive outward.
w = Displacement normal to the axis of revolution for any point on the middle surface of the shell, positive outward.

\( N_m \) = Normal force at any point, per unit length of section, acting in meridional direction, positive for tension.
\( N_c \) = Normal force at any point, per unit length of section, acting in circumferential direction, positive for tension.
\( M_m \) = Moment at any point, per unit length of section, acting in meridional direction, positive for tensile stress at outer surface.
\( M_c \) = Moment at any point, per unit length of section, acting in circumferential direction, positive for tensile stress at outer surface.
\( Q_m \) = Shear at any point, per unit length of section, acting along a plane normal to the meridian, positive inward on the section of the element near the pole of the axis of revolution.
\[ \sigma_m = \text{Component of direct stress in meridional direction, positive for tensile stress.} \]

\[ \sigma_c = \text{Component of direct stress in circumferential direction, positive for tensile stress.} \]

\[ \tau = \text{Shearing stress acting on a section normal to the meridian, positive in the direction of positive shearing force.} \]

\[ \varepsilon_m = \text{Unit strain in meridional direction, positive for elongation.} \]

\[ \varepsilon_c = \text{Unit strain in circumferential direction, positive for elongation.} \]

\[ \chi_m = \text{Curvature in the meridian plane passing through a point in question.} \]

\[ \chi_c = \text{Curvature in the plane perpendicular to the meridian passing through a point in question.} \]

\[ \Lambda = \text{Angle through which the tangent to a meridian at any point rotates during deformation of a shell [ } = r^{-1}_m (v - dw/d\phi) \text{], a variable in the Love-Meissner equations.} \]

\[ U = \text{Product of meridional shear and circumferential radius } \]

\[ (= r_c q_m), \text{another variable in the Love-Meissner equations.} \]

\[ L(\ldots) = \text{Meissner's differential operator, defined in Section 5.} \]

\[ \eta = \text{A new variable related to } \Lambda, \text{for the family of curved surfaces.} \]

\[ \chi = \text{A new variable related to } U, \text{for the family of curved surfaces.} \]

\[ w = \text{A function introduced to simplify the Love-Meissner equations for the family of curved surfaces.} \]

\[ \Delta = \text{A unit division of a mesh, denoting either a unit angle in the case of spherical or toroidal shells, or a unit length in the case of a conical shell or flat circular head.} \]

\[ a_n, b_n, c_n, \ldots i_n, j_n = \text{Numerical constants for the derivation of the difference equations.} \]

\[ n = \text{A general point of a mesh.} \]

\[ c = \text{A point denoting the crown of a sphere, the apex of a cone, or the center of a flat circular head.} \]

\[ p = \text{A point denoting the inner boundary of a torus.} \]
\(q\) = A point denoting the outer boundary of a sphere, a cone, a torus or a flat circular head.

\(\beta_0\) = A constant in the "exact" solution of cylindrical shells.

\(e\) = Base of natural logarithms. \((2.718...)\)

\(\mu\) = Poisson's ratio.

\(E\) = Young's modulus of elasticity.

\(E' = E (1 - \mu^2)^{-1}\)

\(I\) = Moment of inertia per unit length of the section \(( = t^3/12)\).

\(D\) = Flexural rigidity of the section \(( = E' I)\).

\(N, Q\) = Particular values of \(N_m\) and \(Q_m\) respectively, acting on the cylindrical shell at the head-to-shell juncture.

\(M\) = Particular value of \(M_m\) at the head-to-shell juncture.

\(N', Q'\) = Particular values of \(N_m\) and \(Q_m\) at the head-to-knuckle juncture, respectively.

\(M'\) = Particular value of \(M_m\) at the head-to-knuckle juncture.

\(\alpha, \delta\) = Particular values of \(A\) and \(\bar{w}\) at the head-to-shell juncture, respectively.

\(\sigma^{(o)}, \sigma^{(i)}\) = Total stresses at outer and inner surfaces of pressure vessel, respectively.

4. **Fundamental Quantities**

The fundamental quantities entering into the derivation of the equations of equilibrium and deformation are defined in this section.

A. Normal forces, moments and shears.

The type of shell under consideration is in the form of a surface of revolution and is subjected to loads symmetrical with respect to the axis of revolution and normal to the surface of the shell. The form of a shell is defined geometrically by its middle surface. In order to analyze the state of stress in a small element which is cut from the shell by two adjacent meridians and two parallel circles, a system of coordinates, parallel and perpendicular to the edges of the 2. The left-handed system of coordinate axes here adopted follows the convention of Reference (77).
element, X, Y and Z is introduced as shown in Fig. 1. The position of a meridian is defined by an angle $\theta$ measured from a reference meridian; and the position of a parallel circle is defined by the angle $\phi$ made by the normal to the surface and the axis of revolution. The meridian plane and the plane perpendicular to the meridian are the planes of principal curvature.

The normal stresses acting on the sections of the element can be resolved in the directions of the coordinate axes. Shearing stresses act on the cross-sections lying in the parallel circles but vanish on meridian sections because of the axial symmetry of the shell. The magnitudes of the normal forces per unit length of the section acting in meridional and circumferential directions are given by Equations [1].

\[ N_m = \int_{-\psi z}^{+\psi z} \sigma_m (1 + \frac{z}{r_c}) dz \quad [1a] \]
\[ N_c = \int_{-\psi z}^{+\psi z} \sigma_c (1 + \frac{z}{r_m}) dz \quad [1b] \]

The bending moments of the normal stresses per unit length of the section acting in meridional and circumferential directions are given by Equations [2].

\[ M_m = \int_{-\psi z}^{+\psi z} \sigma_m (1 + \frac{z}{r_c}) dz \quad [2a] \]
\[ M_c = \int_{-\psi z}^{+\psi z} \sigma_c (1 + \frac{z}{r_m}) dz \quad [2b] \]

The shearing force per unit length of the section lying in the plane normal to the meridian is given by Equation [3].

\[ Q_m = \int_{-\psi z}^{+\psi z} \tau (1 + \frac{z}{r_c}) dz \quad [3] \]

Since the thickness of a thin shell is small in comparison with the radii of curvature, the terms $\frac{z}{r_m}$ and $\frac{z}{r_c}$ in Equations [1] to [3] approach zero, and are negligible in the evaluation of the resultant normal forces, bending moments and shearing forces.

**B. Curvatures, strains and stresses.**

The small displacement of an element in a shell of revolution can be resolved into two components, the tangential component along the meridian and the radial component normal to the middle surface.

\[ \text{See Reference (50), p. 53, for the derivation of Equations [1] to [3].} \]
SIGN CONVENTION:
POSITIVE AS INDICATED

FIG. 1 EQUILIBRIUM OF A SHELL ELEMENT
The tangential component along the parallel circle vanishes because of the axial symmetry of the shell. (See Fig. 2). Then, the components of strain in the meridional and circumferential directions can be expressed by Equations [4].

\[ \varepsilon_m = r_m^{-1} (\frac{dv}{d\phi} + w) \]  
\[ \varepsilon_c = r_c^{-1} (v \cot \phi + w) \]

The changes of principal curvatures of the middle surface can be expressed by Equations [5].

\[ \kappa_m = r_m^{-1} (\frac{d}{d\phi}) [r_m^{-1} (v - \frac{dw}{d\phi})] \]  
\[ \kappa_c = r_c^{-1} (\cot \phi) [r_m^{-1} (v - \frac{dw}{d\phi})] \]

By using Hooke's law, the elastic relations between direct stress at any point and the strains can be obtained as Equations [6].

\[ \sigma_m = E(1 - \mu^2)^{-1} \left( \varepsilon_m + \mu \varepsilon_c \right) \]  
\[ \sigma_c = E(1 - \mu^2)^{-1} \left( \varepsilon_c + \mu \varepsilon_m \right) \]

Similarly, the elastic relations between bending stresses at any point at a distance \( z \) from the middle surface and changes of curvatures can be obtained as Equations [7].

\[ \sigma_m = Ez(1 - \mu^2)^{-1} \left( \kappa_m + \mu \kappa_c \right) \]  
\[ \sigma_c = Ez(1 - \mu^2)^{-1} \left( \kappa_c + \mu \kappa_m \right) \]


4. See Reference (60), p. 56, for the derivation of Equations [4] to [7].
SIGN CONVENTION:
POSITIVE AS INDICATED

FIG. 2 DEFORMATION OF A SHELL ELEMENT
5. Development of Love-Meissner Equations

The equations of equilibrium for an infinitely small element of a thin shell can be obtained through the summation of forces and moments in the directions of the coordinate axes. Owing to the symmetry of the shell with respect to its axis of revolution, there are only three equations of static equilibrium. In the derivation of these equations, quantities of second order pertaining to the changes of curvature are neglected. The system of equations resulting from the summation of forces in the direction of the Y- and Z-axes, and the summation of moments about the X-axis is given in Equations [8].

\[
\frac{d}{d\phi}(N_m r_c \sin \phi) - N_c r_m \cos \phi + Q_m r_c \sin \phi = 0 \quad [8a]
\]

\[
\frac{d}{d\phi}(Q_m r_c \sin \phi) - N_c r_m \sin \phi - N_m r_c \sin \phi + P_m r_c \sin \phi = 0 \quad [8b]
\]

\[
\frac{d}{d\phi}(M_m r_c \sin \phi) - M_c r_m \cos \phi - Q_m r_m r_c \sin \phi = 0 \quad [8c]
\]

In these three equations of static equilibrium, there appear five unknowns, \(N_m\), \(N_c\), \(M_m\), \(M_c\) and \(Q_m\). However, the resultant normal forces \(N_m\) and \(N_c\) are related to the displacements \(v\) and \(w\) through Hooke's law. So also are the bending moments \(M_m\) and \(M_c\). These relationships, which we term the equations of deformations, are derived by substituting Equations [6] into [1] and Equations [7] into [2] respectively. Thus,

\[
N_m = E' r_m^{-1}(\frac{d}{d\phi} v + w) + \mu r_c^{-1}(v \cot \phi + w) \quad [9a]
\]

\[
N_c = E' r_c^{-1}(v \cot \phi + w) + \mu r_m^{-1}(\frac{d}{d\phi} v + w) \quad [9b]
\]

\[
M_m = D \left\{ r_m^{-1}(\frac{d}{d\phi}) [r_m^{-1}(v - \frac{dw}{d\phi})] + \mu (r_c^{-1} \cot \phi) [r_m^{-1}(v - \frac{dw}{d\phi})] \right\} \quad [10a]
\]

\[
M_c = D \left\{ (r_c^{-1} \cot \phi) [r_m^{-1}(v - \frac{dw}{d\phi})] + \mu r_m^{-1}(\frac{d}{d\phi}) [r_m^{-1}(v - \frac{dw}{d\phi})] \right\} \quad [10b]
\]

5. See Reference (60), p. 54, for derivation of Equations [8], and p. 59, for derivation of Equations [9] and [10].
The three equations of static equilibrium and four equations of deformations will make it possible to determine the seven unknowns.

The work of reducing these differential equations can be simplified considerably by the proper choice of two new dependent variables $A$ and $U$, the relations of which with respect to $v$, $w$ and $Q_m$ are defined in Equations [11].

$$A = r_m^{-1} (v - dw/d\phi)$$  \[11a\]

$$U = r_c Q_m$$  \[11b\]

Then, the fundamental equations of equilibrium and deformations can be reduced to two simultaneous differential equations of second order. By the introduction of the Meissner differential operator $L(x)$, which is defined as follows:

$$L(x) = (r_m \sin \phi)^{-1} (d/d\phi) \left[ r_m^{-1}(r_c \sin \phi) (dx/d\phi) \right] - x r_c^{-1} \cot^2 \phi,$$

the system of equations for the general case of shells of uniform thickness can be expressed by Equations [12] which represent the simplest form of the Love-Meissner equations.\(^6\)

$$L(A) - \mu A r_m^{-1} = U D^{-1}$$  \[12a\]

$$L(U) + \mu U r_m^{-1} = -E \mu \alpha + \Phi(\phi)$$  \[12b\]

where $\Phi(\phi)$ depends only on the shape of the shell and the magnitude of the load. In the case of shells of uniform thickness, the function becomes:

$$\Phi(x) = r_m^{-1} (d/d\phi) \left[ (\mu + r_c r_m^{-1} (\sin^{-2} \phi) F(\phi) - r_c^2 F \right]$$

$$+ (\mu r_m^{-1} + r_c^{-1} (\sin^{-2} \phi) (\cot \phi) F(\phi) +$$

$$(\mu r_m^{-1} + r_c r_m^{-2} \left[ (\sin^{-2} \phi) F(\phi) - r_m r_c F \right]) (\cot \phi)$$

in which

$$F(\phi) = \int Fr_m r_c \sin \phi \cos \phi d\phi.$$  

---

\(^6\). See Reference (60), p. 63, for derivation.
There are several quantities in which we are particularly interested. Normal forces, moments and shears are vital because design stresses are computed from these quantities. The meridional rotation and the component of deflection normal to the axis of revolution are also important in the solution of the boundary conditions. They should be expressed in terms of the new variables or some other quantities related to the new variables. Such relations are given in Equations [13].

\[ Q_m = U r_c^{-1} \]  \[ Q_m = U r_c^{-1} \cot \phi + r_c^{-1} (\sin^{-1} \phi) F(\phi) \]  \[ N_c = r_m^{-1} \left( \frac{dU}{d\phi} \right) + P_r - r_m^{-1} (\sin^{-2} \phi) F(\phi) \]  \[ M_m = D \left[ r_m^{-1} \left( \frac{dA}{d\phi} \right) + \mu A r_c^{-1} \cot \phi \right] \]  \[ M_c = D \left[ A r_c^{-1} \cot \phi + \mu r_m^{-1} \left( \frac{dA}{d\phi} \right) \right] \]  \[ A = A \]  \[ \bar{w} = (r_c \sin \phi)(Et)^{-1} (N_c - \mu N_m) \]

For the case of spherical shells, conical shells and toroidal shells, it is convenient to use the differential equations based on the new variables \( A \) and \( U \). In the case of cylindrical shells and flat circular heads, however, it is preferable to choose the equations based on the original variables \( v \) and \( w \), because they can be reduced to equations which contain only one dependent variable in each equation.

6. Reduction of Equations for Shells of Special Geometrical Shape

The Love-Meissner equations for the general case of shells of revolution can be simplified for shells of special geometrical shapes by the proper choice of dependent and independent variables. The differential equations, Equations [13], relating normal forces, moments, shears, etc., to the dependent variables of \( A \) and \( U \) can also be expressed in simpler form for particular geometrical shapes.
The special geometrical relations for shells of various shapes under consideration can be summarized as follows:

1. Cylindrical shell
   \[ \phi = 90^\circ, \quad r_m \, d\phi = dx, \]
   \[ r_m = \infty, \quad r_c = R. \]

2. Spherical shell
   \[ \phi = 90^\circ - \psi, \quad d\phi = -d\psi, \]
   \[ r_m = r_c = \rho. \]

3. Conical shell
   \[ \phi = 90^\circ - \psi_{o}, \quad r_m \, d\phi = dy, \]
   \[ r_m = \infty, \quad r_c = y \tan \psi_{o}. \]

4. Toroidal shell
   \[ \phi = 90^\circ - \psi, \quad d\phi = -d\psi, \]
   \[ r_m = K, \quad r_c = K(1 + \lambda \csc \phi). \]

5. Flat circular head
   \[ \phi = 0, \quad r_m \, d\phi = dr, \quad r_c \sin \phi = r, \]
   \[ r_m = r_c = \infty. \]

The use of these relations, including the introduction of the quantities such as \( x \) in the case of cylindrical shells, \( y \) in the case of conical shells, etc., permits the reduction of the governing differential equations to the following simpler forms:

1. Cylindrical shell
   \[ \frac{d^4 w}{dx^4} + 4\beta_0^4 w = P D^{-1} \quad [14] \]
   where
   \[ \beta_0 = [3(1 - \mu^2)] (Et)^{-1} \]

2. Spherical shell
   \[ L(A) - \mu A \rho^{-1} = U D^{-1} \quad [15a] \]
   \[ L(U) + \mu U \rho^{-1} = -EtA \quad [15b] \]

7. The treatment of this type of shell is confined to a particular portion of the shell which serves as the head knuckle and is not generalized to apply to a complete torus.

8. For derivation of Equations [14] to [18], see Reference (57).
where
\[ L(x) = \rho^{-1} \left[ \frac{d^2x}{d\phi^2} + \left( \frac{d\phi}{dx} \right) \left( \cot \phi \right) - x \cot^2 \phi \right] \]

3. Conical shell
\[ L(A) = U D^{-1} \]  \[ L(U) = - \beta A + \Phi(\phi) \]  \[ \Phi(\phi) = - \left( \frac{3}{2} \right) (\beta_0 \tan \gamma_0) \]

4. Toroidal shell
\[ L(A) - \mu A K^{-1} = U D^{-1} \]  \[ L(U) + \mu U K^{-1} = - \beta A + \Phi(\phi) \]

where
\[ L(x) = \kappa^{-1} \left[ \left( \frac{d^2x}{d\phi}\right)(1 + \lambda \csc \phi) + \left( \frac{dx}{d\phi} \right)(\cot \phi) \right. 
- \left. x \left( \cot \phi \cos \phi \right) \left( \lambda + \sin \phi \right)^{-1} \right] \]
and
\[ \frac{\Phi(\phi)}{\phi} = \lambda PK(\cos \phi)(\lambda + \sin \phi)(\sin \phi - 2\lambda)(2 \sin^4 \phi)^{-1} \]

5. Flat circular head
\[ \frac{d^3w}{dr^3} + r^{-1} \left( \frac{d^2w}{dr^2} \right) - r^{-2} \left( \frac{dw}{dr} \right) = P r (2D)^{-1} \]  \[ \frac{d^2v}{dr^2} + r^{-1} \left( \frac{dv}{dr} \right) - r^{-2} v = 0 \]

The relationships for normal forces, moments, shear, meridional rotation and deflection measured normal to the axis of revolution, with respect to the foregoing dependent and independent variables, can be given for shells of special geometrical shapes as follows:

1. Cylindrical shell
\[ Q_m = - D \left( \frac{d^3w}{dx^3} \right) \]  \[ N_m = \frac{PR}{2} \]  \[ N_c = +PR - RD \left( \frac{d^4w}{dx^4} \right) \]  \[ M_m = - D \left( \frac{d^2w}{dx^2} \right) \]  \[ M_c = - \mu D \left( \frac{d^2w}{dx^2} \right) \]
\( A = - \frac{dw}{dx} \) [19f]

\( \bar{w} = + w \) [19g]

2. Spherical shell

\( Q_m = U \rho^{-1} \) [20a]

\( N_m = U \rho^{-1} \cot \phi + P \rho/2 \) [20b]

\( N_c = \rho^{-1} (dU/d\phi) + P \rho/2 \) [20c]

\( M_m = D \rho^{-1} [(dA/d\phi) + \mu A \cot \phi] \) [20d]

\( M_c = D \rho^{-1} [A \cot \phi + \mu (dA/d\phi)] \) [20e]

\( A = A \) [20f]

\( \bar{w} = (\rho \sin \phi)(Et)^{-1}(N_c - \mu N_m) \) [20g]

3. Conical shell

\( Q_m = U(y \tan \gamma_o)^{-1} \) [21a]

\( N_m = U y^{-1} + (P y \tan \gamma_o)/2 \) [21b]

\( N_c = dU/dy + P y \tan \gamma_o \) [21c]

\( M_m = D [dA/dy + \mu A y^{-1}] \) [21d]

\( M_c = D [A y^{-1} + \mu (dA/dy)] \) [21e]

\( A = A \) [21f]

\( \bar{w} = (y \sin \gamma_o)(Et)^{-1}(N_c - \mu N_m) \) [21g]

4. Toroidal shell

\( Q_m = U[K (1 + \lambda \csc \phi)]^{-1} \) [22a]

\( N_m = PK(1 + \lambda \csc \phi)/2 + Q_m \cot \phi \) [22b]

\( N_c = PK(1 - \lambda^2 \csc^2 \phi)/2 + (1 + \lambda \csc \phi)(dQ_m/d\phi) \) [22c]

\( M_m = DK^{-1}[dA/d\phi + \mu A(\cot \phi)(1 + \lambda \csc \phi)^{-1}] \) [22d]

\( M_c = DK^{-1}[A(\cot \phi)(1 + \lambda \csc \phi)^{-1} + \mu (dA/d\phi)] \) [22e]

\( A = A \) [22f]

\( \bar{w} = K(1 + \lambda \csc \phi)(\sin \phi)(Et)^{-1}(N_c - \mu N_m) \) [22g]
5. **Flat circular head**

\[ Q_m = - \Pr/2 \quad [23a] \]

\[ N_m = E' t \ (dv/dr + \mu v r^{-1}) \quad [23b] \]

\[ N_c = E' t \ [v r^{-1} + \mu (dv/dr)] \quad [23c] \]

\[ M_m = -D \ [d^2 w/dr^2 + \mu r^{-1} (dw/dr)] \quad [23d] \]

\[ M_c = -D \ [r^{-1} (dw/dr) + \mu (d^2 w/dr^2)] \quad [23e] \]

\[ A = -dw/dr \quad [23f] \]

\[ \tilde{w} = + v \quad [23g] \]
III. GENERAL CONSIDERATIONS IN PRESSURE VESSEL ANALYSIS.

7. Basic Assumptions of the Analysis

The pressure vessels under investigation are assumed to have the following properties and to be loaded in the following manner:

1. The vessel consists of a cylindrical shell with a rigidly attached head at each end in the form of a surface of revolution. Both the cylindrical shell and the head of the vessel are assumed to be of uniform thickness, but the two thicknesses may be different. In all cases, both the thickness of the cylindrical shell and that of the head are taken to be small in comparison with the respective radii of curvature of the unloaded middle surface in the meridional and circumferential directions.

2. The only load on the vessel considered in this presentation is a uniform internal pressure. Stresses due to dead weight, support reactions and other concentrated loads are to be added to those obtained by this analysis. The method here developed can, however, be extended to cases in which the pressure, while symmetrical with respect to the axis of revolution and normal to the surface of the vessel, is non-uniform.

3. The cylindrical shell of the vessel is assumed to have a length exceeding at least twice its diameter so that it can be considered as semi-infinite. The heads of the vessel are taken to be free from holes or any geometrical stress raisers other than those inherent in the shapes of the heads themselves.

4. The radial component of stress in the cylindrical shell and in the head is small in comparison with the stresses developed in the meridional and circumferential directions. It is neglected in the analysis just as, in conventional flat plate theory, the component of stress normal to the bounding planes is neglected.

5. The displacements of the vessel are assumed to be so small that the equilibrium conditions for an element in the cylindrical shell or in the head are the same before and after
deformation. Therefore, quantities of second order pertaining to the changes in curvature can be neglected.

6. Lines normal to the undeformed middle surface of the vessel are assumed to remain normal to the middle surface after deformation.

7. The condition at the head-to-shell juncture of the vessel is assumed to be such that the juncture point is the intersection of the middle surfaces of the cylindrical shell and the head.

8. Types of Heads under Investigation

The types of heads under investigation have been selected with a view to conformity between the assumptions underlying the theory and agreement with conditions existing in industrial practice. With the aim of checking the numerical procedure against the results of other methods of analysis, consideration has first been given to a form of head (the spherical segment) which all of these other methods of analysis are capable of handling. Attention has then been turned to certain geometrical surfaces which are encountered in industrial practice. The limited number of types of heads here presented does not, however, exhaust the variety which can be handled by the proposed numerical procedure which is, in fact, quite general.

The six types of heads under investigation are as follows:

(See Fig. 3).

1. Spherical head
2. Conical head
3. Torispherical head
4. Toriconical head
5. Hemispherical head
6. Flat circular head

These heads are built up of one or more of the four basic elements: the spherical shell, conical shell, toroidal shell and flat circular head. Numerical solutions by finite difference equations are first developed for each of these fundamental cases. Then, any problem involving a combination of these elements can be solved by the proper treatment of the boundary conditions.
FIG. 3  TYPES OF HEADS UNDER INVESTIGATION

(A) SPHERICAL HEAD

(B) CONICAL HEAD

(C) TORISPHERICAL HEAD

(D) TORICONICAL HEAD

(E) HEMISPHERICAL HEAD

(F) FLAT CIRCULAR HEAD
9. **Treatment of Boundary Conditions**

The pressure vessels under consideration are built up of a cylindrical shell with a rigidly attached head at each end. This head in turn may be a single geometrical shape or a combination of geometrical surfaces. A fundamental difficulty in the analysis is the adjustment of statical and deformational conditions at points where different shapes merge. This problem may arise from the presence of a head-to-shell juncture, head-to-knuckle juncture, or knuckle-to-shell juncture. In any case, static equilibrium and continuity of deformation at the particular juncture must be maintained. The former conditions require the equality of moments and shears in the meridional direction; the latter conditions call for the equality of meridional rotation and deflections measured normal to the axis of revolution.

Of the possible alternative treatments of the boundary conditions, it is found convenient to equate the meridional rotations and the deflections measured normal to the axis of revolution on both sides of the juncture. These quantities are readily expressed in terms of meridional moments and shears at the juncture.

For the sake of convenience in computation, the boundary conditions of the types of vessel heads under investigation are classified into four main categories as shown in Fig. 4.

1. **Group One** - The head and the cylindrical shell show discontinuity at the head-to-shell juncture, and their meridians intersect at an arbitrary angle less than ninety degrees. This group includes the cases in which spherical or conical heads are joined to the cylindrical shell.

2. **Group Two** - The head and the cylindrical shell show discontinuity at the head-to-shell juncture, but their meridians intersect at right angles. This group is a special case of Group One and includes only the case of flat circular heads joining the cylindrical shell.

3. **Group Three** - The head proper or the knuckle of the head meets the cylindrical shell in a smooth, continuous, head-

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9. It is sufficient to satisfy this single deformation condition, since deformations parallel to the axis of revolution can be made compatible by a rigid body translation not including any stresses.
FIG. 4 CLASSIFICATION OF BOUNDARY CONDITIONS

(A) GROUP ONE

(B) GROUP TWO

(C) GROUP THREE

(D) GROUP FOUR
to-shell juncture. This group includes the cases in which the hemispherical head or the toroidal knuckle joins the cylindrical shell.

4. Group Four - The head proper meets the knuckle in a smooth, continuous, head-to-knuckle juncture. This includes the cases in which the spherical or conical heads join the toroidal knuckle.

The basic difference in these four groups of junctures lies in the conditions of static equilibrium at the junctures. As is noted in Fig. 4, the moments on either side of a juncture are equal and opposite in each group, but the relations between the shears and normal forces in the meridional direction at the various junctures are different. These relations can be summarized as follows:

1. Group One
   \[ Q = H \sin \phi_h - V \cos \phi_h \]
   \[ N = H \cos \phi_h + V \sin \phi_h \]
   where
   \[ V = (Pp \sin \phi_h)/2 \quad \text{for spherical head} \]
   \[ V = (PS \cos \phi_h)/2 \quad \text{for conical head} \]

2. Group Two
   \[ Q = -V, \quad N = H \]
   where
   \[ V = FR/2 \]

3. Group Three
   \[ Q = H, \quad N = V \]
   where
   \[ V = Pp/2 \quad \text{for hemispherical head} \]
   \[ V = PK(1 + \lambda)/2 \quad \text{for toroidal knuckle} \]

4. Group Four
   \[ Q' = Q', \quad N' = N'. \]

The meridional rotation and the component of deflection normal to the axis of revolution are linearly related to the meridional moment and shear at the juncture by the solutions for the cylindrical shell and the head. For example, at the head-to-shell juncture, the rotation \( \alpha \) and the
deflection $\delta$ can be obtained for the head in the form:

$$\alpha_h = a_1 M + a_2 H + a_3 P$$
$$\delta_h = b_1 M + b_2 H + b_3 P$$

and, for the cylindrical shell,

$$\alpha_c = c_1 M + c_2 H + c_3 P$$
$$\delta_c = d_1 M + d_2 H + d_3 P$$

Similarly, at the head-to-knuckle juncture, the rotation $\alpha$ and deflection $\delta$ can be obtained for the head proper in the form:

$$\alpha'_h = e_1 M + e_2 H + e_3 M' + e_4 Q' + e_5 P$$
$$\delta'_h = f_1 M + f_2 H + f_3 M' + f_4 Q' + f_5 P$$

and for the knuckle,

$$\alpha'_k = g_1 M + g_2 H + g_3 M' + g_4 Q' + g_5 P$$
$$\delta'_k = h_1 M + h_2 H + h_3 M' + h_4 Q' + h_5 P$$

The numerical determination of the constants $a_n, b_n, \ldots$ etc. are discussed later. For example, in Section 11 of this chapter, $\alpha$ and $\delta$ at the juncture of a cylindrical shell are obtained in terms of $M, H$ and $P$. The determination of these constants for various types of heads is given in Chapter IV. For the present it suffices to note that by equating $\alpha$ and $\delta$ at the joint, the values of $M$ and $H$ at the joint are readily found in the case of vessel heads without knuckles. Similarly, by equating $\alpha'$ and $\delta'$ at the head-to-knuckle juncture as well as $\alpha$ and $\delta$ at the knuckle-to-shell juncture, the values of $M, H, M'$ and $Q'$ are obtained for vessel heads with toroidal knuckles.

While the foregoing discussion is confined to the boundary conditions encountered in the analysis of the particular types of heads which we propose to examine in detail, it does not limit the application to these particular types of heads. Not uncommonly, pressure vessel heads in actual service are better approximated by several segments of toroidal shells having different radii along the meridian. The introduction of each additional shell segment brings into consideration
four more unknowns and a like number of boundary conditions. Such problems can always be handled although the computation is, of course, more tedious.

10. **Nature of Critical Stresses**

For engineering purposes, the magnitudes of the stress components are of primary importance. In particular, designers must be able to estimate the critical stresses which will occur. It is therefore highly desirable that the results of analysis be expressible in terms of stress.

As a first step, the quantities $N_m$, $N_c$, $M_m$, $M_c$ and $Q_m$ are computed by means of the previously developed formulas, Equations [19] to [23] of Chapter II, after the solution of the governing difference equations is obtained. These quantities are computed for points at regular intervals apart for which the equations of finite difference are established.

Since the thicknesses of both the cylindrical shell and the head of the vessel are assumed to be small in comparison with the respective radii in meridional and circumferential directions, assumptions regarding the distribution of stress through the thickness similar to those employed in elementary beam theory may be used. The direct stresses are assumed to be uniformly distributed over the thickness of the shell; the bending stresses are assumed to vary linearly through the thickness of the shell, with zero value at the middle surface and numerical maxima at outer and inner surfaces; and the shearing stress is assumed to have a parabolic distribution across the thickness of the shell, with zero values at the extreme fibers and a maximum at the middle surface equal to $3/2$ the average shearing stress. These relations are conventional and, in fact, were introduced in the development of the basic theory.

The direct and bending stresses in the same direction are additive algebraically. The sum of these two is designated the "total" stress. The total stresses at the outer and inner surfaces in each direction are principal stresses because the shearing stress is zero at extreme fibers. The components of stress in both meridional and circumferential directions vary from point to point. In general, numerical maxima occur in the vicinity of the juncture either on the outer or inner surface, and decrease rapidly both in the head and in the cylindrical shell until they reach the value of membrane stresses. The critical stress may occur either in the head or the cylindrical shell depending on the geometry of the head.
The distribution of stress through the thickness assumed in the previous paragraph leads directly to the following expressions which relate the components of total stresses to the normal forces and moments. At the outer surface:

$$\sigma_m^{(0)} = \frac{N_m}{t} + \frac{6M_m}{t^2} \quad [24a]$$
$$\sigma_c^{(0)} = \frac{N_c}{t} + \frac{6M_c}{t^2} \quad [24b]$$

At the inner surface:

$$\sigma_m^{(1)} = \frac{N_m}{t} - \frac{6M_m}{t^2} \quad [24c]$$
$$\sigma_c^{(1)} = \frac{N_c}{t} - \frac{6M_c}{t^2} \quad [24d]$$

The shearing stresses are proportional to the meridional shears. The critical shearing stress may also occur either in the head or the cylindrical shell, and is related to the maximum meridional shear, since

$$\tau = \frac{32m}{2t} \quad [24e]$$

11. Analysis of the Cylindrical Shell

At this point we develop for the cylindrical shell some of the relations which will be of use in the later chapters. For this case, an "exact" solution is well known. The governing differential equation for cylindrical shells has been given in Equation [14] as follows:

$$\frac{d^4 w}{dx^4} + 4\beta_0^4 w = P D^{-1} \quad [14]$$

where

$$\beta_0 = \sqrt{\frac{3(1 - \mu^2)}{(Rt)}}^{-1}$$

The classical solution of this equation has the well known form:

$$w = e^{\beta_0 x} (c_1 \cos \beta_0 x + c_2 \sin \beta_0 x) +$$
$$e^{-\beta_0 x} (c_3 \cos \beta_0 x + c_4 \sin \beta_0 x) + PR^2(2 - \mu)(St)^{-1}$$

where $c_1, c_2, c_3$ and $c_4$ are constants of integration. With given boundary conditions as in the case of the cylindrical shell of a pressure vessel, the expression for the deflection $w$ and its consecutive derivatives can be represented by the following equations. 10

10. See Reference (61), p. 392, for derivation.
where \( M \) and \( H \) are respectively the meridional moment and meridional shear acting on the cylindrical shell at the juncture.

The relationships of normal forces, moments, shear, meridional rotation and deflection normal to the axis of revolution can also be expressed in terms of \( M \) and \( H \) as follows:

\[
\begin{align*}
Q_m &= - e^{-\beta_0 x} [2\beta_0 M \sin \beta_0 x - H(\cos \beta_0 x - \sin \beta_0 x)] \\
N_m &= \frac{PR}{2} \\
N_c &= \frac{PR}{2} - R(2 e^{-\beta_0 x} \beta_0^3)^{-1} [\beta_0 M(\cos \beta_0 x - \sin \beta_0 x) + H \cos \beta_0 x] \\
M_m &= (2 e^{-\beta_0 x} \beta_0)^{-1} [2\beta_0 M(\cos \beta_0 x - \sin \beta_0 x) + H \sin \beta_0 x] \\
M_c &= \mu M_m \\
A &= -(2 e^{-\beta_0 x} \beta_0^2)^{-1} [2\beta_0 M \cos \beta_0 x + H(\cos \beta_0 x + \sin \beta_0 x)] \\
w &= (2 e^{-\beta_0 x} \beta_0^3)^{-1} [\beta_0 M(\cos \beta_0 x - \sin \beta_0 x) + H \cos \beta_0 x] + \frac{PR^2}{2} (2 - \mu)(2\beta_0 t)^{-1}
\end{align*}
\]

At the head-to-shell juncture, where \( x = 0 \), the rotation \( \alpha \) and the deflection \( \delta \) can be obtained as follows:
$$\alpha = A \quad x = 0 = - (28^2 M + H)(28^2 D)^{-1}$$  \[27a\]

$$\delta = w \quad x = 0 = - (B_0 M + H)(28^3 D)^{-1} + FR^2 (2 - \mu)(2t)^{-1}$$  \[27b\]

These quantities are to be dealt with later in the consideration of the conditions of continuity at the head-to-shell juncture. Equations [27] are examples of the linear relationship between edge moment and edge shear on the one hand and the rotation and deflection on the other, to which reference was made in Section 9 of this chapter.
IV. METHOD OF FINITE DIFFERENCES.

12. General Concepts

The solution of the governing differential equations, Equations [12], for shells of various shapes by the method of finite differences consists in replacing the differential coefficients by finite difference analogues which are simply algebraic relationships between the values of the dependent variables at points on the middle surface of the shell. There results a set of simultaneous algebraic equations, one equation for each point of a mesh. The unknowns in this set are the wanted quantities from which may be derived the values of the normal forces, bending moments and shears.

To a certain extent, the procedure is analogous to the finite difference method which has long been applied to the solution of flat plate design problems. The principal additional obstacles which arise in connection with this extension have their source in the proper treatment of the boundary conditions. These difficulties are overcome through the introduction of fictitious points of the mesh. Although this idea is well known, the technique of its systematic application to the analysis of pressure vessel heads is believed to be presented here for the first time.

For the family of curves surfaces which includes spherical, conical, and toroidal shells, the governing differential equations are simplified by the introduction of a pair of new variables. The flat circular head closing a cylindrical pressure vessel is a special member of this class. Its analysis has been included for the purpose of comparison in spite of the fact that the "exact" solution for this case is relatively simple. The types of shell for which the method of finite differences is to be developed are shown in Fig. 5.

The basic formulas for the finite difference analogues are obtained by taking the first term of Taylor's series representing the respective differential coefficients. These formulas are approximations because the higher order terms in the series are omitted. As a matter of fact, the finite difference formulas presented in this section represent the derivatives of a function \( y = f(x) \) for which \( x \) exists at values in integer multiples of \( \Delta \), so that \( y = y_1, y_2, \ldots, y_n \) for \( x = \Delta, 2\Delta, \ldots, n\Delta \).
FIG. 5  BASIC ELEMENTS IN NUMERICAL SOLUTION
Let us consider a general point \( n \). The central difference formulas for the derivatives of the function \( y \) at the point \( n \) are as follows:

\[
\frac{dy}{dx}_n = \frac{y_{n+1} - y_{n-1}}{2\Delta} \quad \text{[28a]}
\]

\[
\frac{d^2y}{dx^2}_n = \frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta^2} \quad \text{[28b]}
\]

\[
\frac{d^3y}{dx^3}_n = \frac{y_{n+2} - 2y_{n+1} + 2y_n - y_{n-1}}{2\Delta^3} \quad \text{[28c]}
\]

\[
\frac{d^4y}{dx^4}_n = \frac{y_{n+2} - 4y_{n+1} + 6y_n - 4y_{n-1} + y_{n-2}}{\Delta^4} \quad \text{[28d]}
\]

The formulas for forward and backward differences are used in this work instead of central differences only when the case must call for such substitution (e.g., at the edge of the shell). Their use is required only for the first derivative of the function, as given here.

\[
\frac{dy}{dx}_n = -\frac{3y_n - 4y_{n+1} + y_{n+2}}{2\Delta} \quad \text{[28e]}
\]

\[
\frac{dy}{dx}_n = +\frac{3y_n - 4y_{n-1} + y_{n-2}}{2\Delta} \quad \text{[28f]}
\]

Cases do arise in which the differential coefficients can not be replaced directly by their finite difference analogues. Notable examples are the equations for points at the crown of a sphere, the apex of a cone, or the center of a flat circular head. Due to the fact that \( \phi = 0 \) in the case of a sphere, \( y = 0 \) in the case of a cone, or \( r = 0 \) in the case of a flat circular head, indeterminate forms appear in the equations for these points. Under these circumstances, the differential coefficients must first be transformed by de L'Hospital's rule which gives:

\[
\lim_{x \to 0} \frac{\varphi(x)}{\theta(x)} = \lim_{x \to 0} \frac{\varphi'(x)}{\theta'(x)} \quad \text{[29]}
\]

in which \( \varphi(x) \) and \( \theta(x) \) are functions of \( x \), satisfying certain conditions, and \( \varphi'(x) \) and \( \theta'(x) \) are their respective first derivatives.

11. For derivation of Equations [28], see "Numerical Mathematical Analysis" by J. B. Scarborough. Johns Hopkins Press, Baltimore, Maryland, 1930, p. 38.

12. For derivation of the expression, see "Treatise on Advanced Calculus" by P. Franklin. John Wiley and Sons, Inc., 1940, p. 121.
13. **Spherical Shell**

The governing differential equations for a spherical shell are given in Equations [15], Chapter II, which state:

\[
L(A) - \mu A p^{-1} = U D^{-1} \tag{15a}
\]

\[
L(U) + \mu U p^{-1} = - E t A \tag{15b}
\]

where

\[
L(x) = \rho^{-1} \left[ \frac{d^2 x}{d \phi^2} + (\cot \phi)(dx/d\phi) - x \cot^2 \phi \right].
\]

We introduce two new variables, $\eta$ and $\xi$, defined by the equations:

\[
\Delta = \rho^2 D^{-1} \omega \eta \tag{30a}
\]

\[
U = \rho \omega \xi \tag{30b}
\]

where $\omega(\phi)$ is a function so determined as to eliminate the terms involving the derivatives of first order in the governing differential equations. This is accomplished by taking $\omega$ in the form:

\[
\omega = + (\sin \phi)^{-1/2} \tag{31a}
\]

\[
\frac{d\omega}{d\phi} = - (\omega/2)(\cot \phi) \tag{31b}
\]

\[
\frac{d^2\omega}{d\phi^2} = + (\omega/4)(\cot \phi + 2 \csc^2 \phi) \tag{31c}
\]

Then, the governing differential equations become:

\[
\frac{d^2\eta}{d\phi^2} - [a(\phi) + \mu] \eta = \xi \tag{32a}
\]

\[
\frac{d^2\xi}{d\phi^2} - [a(\phi) - \mu] \xi = - 12(1 - \mu^2)(\rho/t)^2 \eta \tag{32b}
\]

in which

\[
a(\phi) = (3/4)(\cot^2 \phi) - 1/2
\]

For simplicity of presentation, several numerical constants are introduced and defined here. The subscript $n$ in each term refers to the particular point (i.e., the particular value of $\phi_n = n\Delta$) under consideration.

\[
a_n = (3/4)(\cot^2 n\Delta) - 1/2 \tag{33a}
\]

\[
b_n = 2 + a_n \Delta^2 \tag{33b}
\]

\[
c_n = 2 + b_n \Delta^2 \left[ 12(1 - \mu)(\rho/t)^2 - \mu^2 \right] \Delta^4 \tag{33c}
\]

\[
d_n = (1 - 2\mu) \Delta \cot n\Delta \tag{33d}
\]

\[
e_n = (b_n + \mu \Delta^2) - d_n \tag{33e}
\]

\[
f_n = (b_{n-1} + b_n - d_n)/e_n \tag{33f}
\]
The difference equations for a general point, the analogues of Equations [32], are as follows:

\[ \eta_{n-1} + (b_n + \mu \Delta^2) \eta_n + \eta_{n+1} = \Delta^2 \gamma_n \]  
\[ \gamma_{n-1} - (b_n + \mu \Delta^2) \gamma_n + \gamma_{n+1} = -12(1 - \mu^2) (\rho / t)^2 \Delta^2 \eta_n \]  

On separating the variables \( \eta \) and \( \gamma \) in these equations, we have the following form:

\[ \eta_{n-2} - (b_{n-1} + b_n) \eta_{n-1} + c_n \eta_n - (b_n + b_{n+1}) \eta_{n+1} + \eta_{n+2} = 0 \]  
and

\[ \gamma_n = \Delta^{-2} [\eta_{n-1} - (b_n + \mu \Delta^2) \eta_n + \eta_{n+1}] \]

The difference equations for normal forces, moments, shear, meridional rotation and deflection normal to the axis of revolution at the general point \( n \) can be obtained in terms of \( \eta \).

\[ (Q_m)_n = w \Delta^{-2} [\eta_{n+1} - (b_n + \mu \Delta^2) \eta_n + \eta_{n-1}] \]  
\[ (N_m)_n = \frac{P}{2} + (Q_m)_n \cot n\Delta \]  
\[ (N_c)_n = \frac{P}{2} + (2\Delta)^{-1} [(Q_m)_{n+1} - (Q_m)_{n-1}] \]  
\[ (M_m)_n = \rho w_n (2\Delta)^{-1} [\eta_{n+1} - (1 - 2\mu \Delta (\cot n \Delta)) \eta_n - \eta_{n-1}] \]  
\[ (M_c)_n = \mu \rho w_n (2\Delta)^{-1} [\eta_{n+1} + (2/\mu - 1) \Delta (\cot n \Delta) \eta_n - \eta_{n-1}] \]  
\[ (A)_n = \rho_0 \Delta^{-1} w_n \eta_n \]  
\[ (\tilde{w})_n = (\rho \sin n\Delta) (\delta t)^{-1} [(N_c)_n - \mu (N_m)_n] \]

It must be noted that Equation [35] for a general point is not readily applicable to points near the crown or the edge of the spherical shell. So are some of Equations [37] which introduce terms involving points beyond the boundary. Fictitious points are introduced in these instances and the appropriate values of \( \eta \) are determined through the conditions of static equilibrium at the boundary, as is explained in the succeeding paragraphs of this section.
At the crown of the spherical shell, it is known from Equation [31a] that for \( n = 0, w \to \infty \). Then, from Equations [30], we must have

\[
\eta_0 = 0 \quad \text{[38a]}
\]

\[
\gamma_0 = 0 \quad \text{[38b]}
\]

Therefore, at the points \( n = 1 \) and \( n = 2 \), the difference equations become

\[
(c_1 - 1)\eta_1 - (b_1 + b_2)\eta_2 + \eta_3 = 0 \quad \text{[39a]}
\]

\[
- (b_1 + b_2)\eta_1 + c_2\eta_2 - (b_2 + b_3)\eta_3 + \eta_4 = 0 \quad \text{[39b]}
\]

At the edge of the spherical shell for which \( n = q \), it is known from Equations [37d] and [37a] respectively:

\[
M = \rho w_q (2\Delta)^{-1} [\eta_{q+1} - d_q \eta_q - \eta_{q-1}] 
\]

\[
Q = w_q \Delta^2 [\eta_{q+1} - (b_q + \mu \Delta^2)\eta_q + \eta_{q-1}] 
\]

By solving the above equations for \( \eta_q \) and \( \eta_{q+1} \), we obtain

\[
\eta_q = 2 e_q^{-1} \eta_{q-1} + 2\Delta (\rho w_q e_q)^{-1} M - \Delta^2 (w_q e_q)^{-1} Q \quad \text{[40a]}
\]

\[
\eta_{q+1} = (1 + 2d_q e_q^{-1})\eta_{q-1} + 2\Delta (1 + d_q e_q^{-1})(\rho w_q)^{-1} M \\
- \Delta^2 d_q (w_q e_q)^{-1} Q \quad \text{[40b]}
\]

Then, the difference equations at points \((q - 2)\) and \((q - 1)\) become

\[
\eta_{q-4} - (b_{q-3} + b_{q-2})\eta_{q-3} + c_{q-2}\eta_{q-2} + (b_{q-2} + b_{q-1} - 2 e_{q-1}^{-1})\eta_{q-1} \\
+ 2\Delta (\rho w_q e_q)^{-1} M - \Delta^2 (w_q e_q)^{-1} Q = 0 \quad \text{[41a]}
\]

\[
\eta_{q-3} - (b_{q-2} + b_{q-1})\eta_{q-2} + (c_{q-1} - 2 f_{q-1} + 1)\eta_{q-1} \\
- 2\Delta (f_m - 1)(\rho w_q)^{-1} M + \Delta^2 f_q w_q^{-1} Q = 0. \quad \text{[41b]}
\]

The equations expressing normal forces, moments and shear for points at the crown and the edge are modified similarly. At the crown, the value
of shear \((Q_m)_0\) is zero as it must be from considerations of symmetry.

For the normal forces \((N_m)_0\) and \((N_c)_0\), de L'Hospital's rule is first applied to the indeterminate form and then the forward difference formula is used in obtaining the equations.

\[
(Q_m)_0 = 0 \quad \text{[42a]}
\]
\[
(N_m)_0 = Pp/2 + (2\Delta)^{-1}[4(Q_m)_1 - (Q_m)_2] \quad \text{[42b]}
\]
\[
(N_c)_0 = Pp/2 + (2\Delta)^{-1}[4(Q_m)_1 - (Q_m)_2] \quad \text{[42c]}
\]

Since the function \(w\) has a singularity at the crown, the values of the moments at that point, \((M_m)_0\) and \((M_c)_0\) have to be obtained by extrapolation from adjacent points. This introduces no serious error as these moments are, in general, so small as to be quite unimportant.

Another exceptional point requiring special attention is that which coincides with the edge of the spherical shell \((n = q)\). Here all of Equations [37] are valid, but the use of [37c] to compute \((N_c)_q\) would introduce mesh points two units beyond the edge. This equation is therefore to be replaced by its backward difference analogue.

\[
(N_c)_q = Pp/2 + (2\Delta)^{-1}[3(Q_m)_q - 4(Q_m)_{q-1} + (Q_m)_{q-2}] \quad \text{[43]}
\]

14. Conical Shell

The governing differential equations for a conical shell are given in Equations [16], Chapter II, which state:

\[
L(A) = AD^{-1} \quad \text{[16a]}
\]
\[
L(U) = - EtA + \frac{1}{2}(f) \quad \text{[16b]}
\]

where

\[
L(x) = y(\tan \gamma_0)[d^2x/dy^2 + dx/dy - x/y]
\]

and

\[
\frac{1}{2}(f) = - (3/2)(Py \tan \gamma_0).
\]

We introduce two new variables \(\eta\) and \(\phi\) defined by the equations:

\[
A = D^{-1} \omega \eta \quad \text{[44a]}
\]
\[
U = (w \tan \gamma_0) \phi \quad \text{[44b]}
\]

where \(\omega(y)\) is a function so determined as to eliminate the terms involving the derivatives of first order in the governing differential equations. This is accomplished by taking \(\omega\) in the form:
\[ w = + y^{-1/2} \]  
\[ \frac{dw}{d\phi} = - \frac{w}{2y} \]  
\[ \frac{d^2w}{d\phi^2} = + \frac{3w}{4y} \]

Then the governing differential equations become:

\[ \frac{d^2\eta}{d\phi^2} - \frac{3\eta}{4y^2} = \frac{\delta}{y} \]

\[ \frac{d^2\eta}{d\phi^2} - \frac{3\eta}{4y^2} = -12(1 - \mu^2)(\eta/y)(\cot^2\psi_0) \frac{t^{-2}}{2w}(\tan \psi_0) \]

For simplicity of presentation, several numerical constants are introduced and defined here. The subscript \( n \) in each term refers to the particular point (i.e. the particular value of \( y_n = n\Delta \)) under consideration.

\[ a_n = 2 + \frac{3}{4n^2} \]  
\[ b_n = \frac{3}{2}(\omega_n^{-1} \tan \psi_0) \]  
\[ c_n = 2 + a_n^2 + 12(1 - \mu^2)(\Delta/nt)^2(\cot^2\psi_0) \]  
\[ d_n = (1 - 2\mu)/n \]  
\[ e_n = a_n - (1 - 2\mu)/n \]  
\[ f_n = \frac{[\eta - (n - 1)a_{n-1} + na_n - nd_n]e_n}{e_n} \]

The difference equations for a general point, the analogues of Equations \([46]\), are as follows:

\[ \eta_{n-1} - a_n \eta_n + \eta_{n+1} = n^{-1} \Delta \eta_n \]  
\[ \eta_{n-1} - a_n \eta_n + \eta_{n+1} = -12(1 - \mu^2)(\Delta/nt)(\cot^2\psi_0)\eta_n - b_n \eta_n \]

On separating the variables \( \eta \) and \( \eta \) in these equations, we have the following form:

\[ (n - 1)\eta_{n-2} - [(n - 1)a_{n-1} + na_n]\eta_{n-1} + nc_n \eta_n - [na_n + (n + 1)a_{n+1}]\eta_{n+1} + (n + 1)\eta_{n+2} + b_n \Delta^2 \eta = 0 \]

and

\[ \eta_n = n\Delta^{-1} [\eta_{n+1} - a_n \eta_n + \eta_{n-1}] \]
The difference equations for the solution of normal forces, moments, shear, meridional rotation and deflection normal to the axis of revolution at the general point can be obtained in terms of \( \eta \).

\[
(q_m)_n = \omega_n A^{-2} [\eta_{n+1} - a_n \eta_n + \eta_{n-1}] \tag{51a}
\]

\[
(N_m)_n = (PN/2) \tan \psi_o + (Q_m)_n \tan \psi_o \tag{51b}
\]

\[
(N_c)_n = PN \tan \psi_o + [(\tan \psi_o)/2][(n + 1)(Q_m)_{n+1} - (n - 1)(Q_m)_{n-1}] \tag{51c}
\]

\[
(M_m)_n = \omega_n (2A)^{-1} [\eta_{n+1} - (1 - 2\mu)n^{-1} \eta_n - \eta_{n-1}] \tag{51d}
\]

\[
(M_c)_n = \mu \omega_n (2A)^{-1} [\eta_{n+1} + (2/\mu - 1)n^{-1} \eta_n - \eta_{n-1}] \tag{51e}
\]

\[
(A)_n = D^{-1} \omega_n \eta_n \tag{51f}
\]

\[
(\bar{\omega})_n = nA \sin \psi_o \left( \frac{E}{1 - \nu} \right)^{-1} [(N_c)_n - \mu (N_m)_n] \tag{51g}
\]

It is noted, as in the case of the spherical shell, that Equation [49] for a general point is not readily applicable to the points near the apex or the edge of the conical shell. So also are some of Equations [51] which introduce terms involving points beyond the boundary.

At the apex of the conical shell, it can be seen from Equation [45a] that for \( n = 0 \), \( \omega \to \infty \). Then, from Equations [44] we have

\[
\eta_o = 0 \tag{52a}
\]

\[
\psi_o = 0 \tag{52b}
\]

Therefore, the difference equations for the points \( n = 1 \) and \( n = 2 \) become

\[
c_1 \eta_1 = (a_1 + 2a_2) \eta_2 + 2n_3 + b_1 A^2 p = 0 \tag{53a}
\]

\[
- (a_1 + 2a_2) \eta_1 + 2c_2 \eta_2 - (2a_2 + 3a_3) \eta_3 + 3n_4 + b_2 A^2 p = 0 \tag{53b}
\]

At the edge of the conical shell for which \( n = q \), it is known from Equations [51d] and [51a] respectively:
\[ M = w_q (2\Delta)^{-1} \left[ \eta_{q+1} - \delta_q \eta_q - \eta_{q-1} \right] \]

\[ Q = w_{q} \Delta^{-2} \left[ \eta_{q+1} - \alpha_q \eta_q + \eta_{q-1} \right] \]

By solving the above equations for \( \eta_q \) and \( \eta_{q+1} \), we obtain

\[ \eta_q = 2 e_q^{-1} \eta_{q-1} + 2\Delta (w_{q} e_q)^{-1} M - \Delta^2 (w_{q} e_q)^{-1} Q \]

\[ \eta_{q+1} = (1 + 2d_q e_q^{-1}) \eta_{q-1} + 2\Delta (1 + d_q e_q^{-1}) w_q^{-1} M - \Delta^2 d_q (w_{q} e_q)^{-1} Q \]

Then, the difference equations at points \((q - 2)\) and \((q - 1)\) become

\[ (q - 3) \eta_{q-4} - [(q - 3)a_{q-3} + (q - 2)a_{q-2}] \eta_{q-3} + (q - 2)c_{q-2} \eta_{q-2} \]

\[- [(q - 2)a_{q-2} + (q - 1)a_{q-1} - 2(q - 1)e_q^{-1}] \eta_{q-1} \]

\[ + 2\Delta (q - 1)(w_{q} e_q)^{-1} M - \Delta^2 (q - 1)(w_{q} e_q)^{-1} Q + b_{q-2} \Delta^2 P = 0 \]

\[ (q - 2) \eta_{q-3} - [(q - 2)a_{q-2} + (q - 1)a_{q-1}] \eta_{q-2} + \]

\[ [(q - 1)c_{q-1} - 2f_q + q] \eta_{q-1} - 2\Delta (f_q - q) w_q^{-1} M \]

\[ + \Delta^2 f_q w_q^{-1} Q + b_{q-1} \Delta^2 P = 0 \]

The equations expressing normal forces, moments and shear for points at the apex and the edge are modified similarly. At the apex, the value of shear \((Q_m)_0\) is zero as it must be from considerations of symmetry. For the normal forces \((N_m)_0\) and \((N_c)_0\), de L'Hospital's rule is used in obtaining their values.

\[ (Q_m)_0 = 0 \]

\[ (N_m)_0 = 0 \]

\[ (N_c)_0 = 0 \]

Since the function \(\omega\) has a singularity at the apex, the values of the moments at that point \((M_m)_0\) and \((M_c)_0\) have to be obtained by extrapolation.
from adjacent points. This introduces no serious error as these moments are, in general, so small as to be quite unimportant.

Another exceptional point requiring special attention is that which coincides with the edge of the conical shell \( n = q \). Here all of Equations [51] are valid, but the use of [51c] to compute \( (N_q)_c \) would introduce mesh points two units beyond the edge. This equation is therefore to be replaced by its backward difference analogue.

\[
(N_q)_m = PqA \tan \psi_o + \left[ (\tan \psi_o)/2 \right] [3q(Q_m)_q - 4 (q - 1)(Q_m)_{q-1} + (q - 2)(Q_m)_{q-2}]
\]

[57]

15. Toroidal Shell

The governing differential equations for a toroidal shell are given in Equations [17], Chapter II, which state

\[
L(A) - \mu A K^{-1} = UD^{-1}
\]

[17a]

\[
L(U) + \mu U K^{-1} = -EtA + \Phi(\psi)
\]

[17b]

where

\[
L(x) = K^{-1} \left[ (d^2 x/\partial \phi^2)(1 + \lambda \csc \phi) + (dx/\partial \phi)(\cot \phi) - x(\cot \phi \cos \phi)(\lambda + \sin \phi)^{-1} \right]
\]

and

\[
\Phi(\phi) = K(\cos \phi)(\lambda + \sin \phi)(\sin \phi - 2\lambda)(2 \sin^4 \phi)^{-1}
\]

We introduce two new variables \( \eta \) and \( \zeta \) defined by the equations:

\[
A = K^2 D^{-1} \omega \eta
\]

[58a]

\[
U = K \omega \zeta
\]

[58b]

where \( \omega(\phi) \) is a function so determined as to eliminate the terms involving the derivatives of first order in the governing differential equations. This is accomplished by taking \( \omega \) in the form:

\[
\omega = \tau (\lambda + \sin \phi)^{-1/2}
\]

[59a]

\[
\frac{d \omega}{d \phi} = - (\omega^3/2) \cos \phi
\]

[59b]

\[
\frac{d^2 \omega}{d \phi^2} = \frac{1}{2} (\omega^5/4)(3 + 2 \lambda \sin \phi - \sin^2 \phi)
\]

[59c]

Then the governing differential equations become

\[
a(\phi)(d^2 \eta/d\phi^2) - [b(\phi) + \mu] \eta = \zeta
\]

[60a]
\[ a(\phi)(\dddot{\phi}/\dot{\phi}^2) - [b(\phi) - \mu]\dot{\phi} = -12(1 - \mu^2)(\kappa/t)^2 \eta + \]
\[ \omega^{-1} \Phi(\phi). \]

where
\[ a(\phi) = 1 + \lambda \csc \phi \]
\[ b(\phi) = (\omega^4/4)(3 - 2\lambda \sin \phi - 5 \sin^2 \phi) \]

For simplicity of presentation, several numerical constants are introduced and defined here. The subscript \( n \) in each term refers to the particular point under consideration. It should be noted (See Fig. 5) that \( \phi_n \) is the angle between the axis of revolution and a perpendicular to the middle surface at any point of the toroidal shell, measured along the meridian. Therefore, the angle at the inner boundary of the torus is \( \phi_n = \phi_n \) and that at the outer boundary is \( \phi_n = \phi_n + \phi_k \).

\[ a_n = (\lambda + \sin \phi_n)(\sin^{-1} \phi_n) \]
\[ b_n = (\omega_n^4/4)(3 - 2\lambda \sin \phi_n - 5 \sin^2 \phi_n) \]
\[ c_n = \lambda \kappa_n \cot \phi_n (\sin \phi_n - 2\lambda)(2 \omega_n \sin^2 \phi_n)^{-1} \]
\[ d_n = 2a_n + b_n \Delta^2 \]
\[ e_n = d_n^2/a_n + 12(1 - \mu^2)(\kappa/t)^2 \Delta^4/a_n - \mu^2 \Delta^4/a_n \]
\[ + (a_{n-1} + a_{n+1}) \]
\[ f_n = (1 - 2\mu)(\cot \phi_n)\Delta/a_n \]
\[ g_n = (d_n + \mu \Delta^2)/a_n + f_n \]
\[ h_n = (d_n + d_{n+1} + a_n f_n)/g_n \]
\[ i_n = (d_n + \mu \Delta^2)/a_n - f_n \]
\[ j_n = (d_{n-1} + d_n - a_n f_n)/i_n \]

The difference equations for a general point, the analogues of Equations [60], can be written as follows:
\[ a_n \eta_{n-1} - (d_n + \mu \Delta^2)\eta_n + a_n \eta_{n+1} = \Delta^2 \Psi_n \]
\[ a_n \bar{\eta}_{n-1} - (d_n - \mu \Delta^2) \eta_n + a_n \bar{\eta}_{n+1} = -12 \left(1 - \mu^2 \right) (X/t)^2 \Delta^2 \eta_n + c_n \Delta^2 \phi \quad [62b] \]

On separating the variables \( \eta \) and \( \phi \) in these equations, we have the following form:

\[ a_{n-1} \eta_{n-2} - (d_{n-1} + d_n) \eta_{n-1} + a_n \eta_n - (d_n + d_{n+1}) \eta_{n+1} + a_{n+1} \eta_{n+2} - c_n \Delta^2 \phi = 0 \quad [63] \]

and

\[ \bar{\eta}_n = a_n \Delta^{-2} \left[ \eta_{n-1} - (d_n + \mu \Delta^2) \eta_n \right] \quad [64] \]

The difference equations for the solution of normal forces, moments, shear, meridional rotation and deflection normal to the axis of revolution at the general point can be obtained in terms of \( \eta \).

\[ (Q_n)_n = \omega_n \Delta^{-2} \left[ \eta_{n+1} - (d_n + \mu \Delta^2) \eta_n \right] \quad [65a] \]

\[ (N_n)_n = \frac{\Phi a_n}{2} + (Q_m)_n \cot \phi_n \quad [65b] \]

\[ (N_c)_n = \frac{\Phi a_n}{2} \left[ 1 - a_n/2 \right] + a_n \left(2 \Delta \right)^{-1} \left[ (Q_m)_{n+1} - (Q_m)_{n-1} \right] \quad [65c] \]

\[ (M_n)_n = \frac{Kw_n}{2} \left(2 \Delta \right)^{-1} \left[ \eta_{n+1} - (1 - 2 \mu) \Delta a_n^{-1} (\cot \phi_n) \eta_n - \eta_{n-1} \right] \quad [65d] \]

\[ (M_c)_n = \frac{Kw_n}{2} \left(2 \Delta \right)^{-1} \left[ \eta_{n+1} + (2/\mu - 1) \Delta a_n^{-1} (\cot \phi_n) \eta_n - \eta_{n-1} \right] \quad [65e] \]

\[ (A)_n = K^2 \Delta^{-1} \omega_n \eta_n \quad [65f] \]

\[ (\bar{\omega})_n = a_n \Delta (\sin \phi_n) (2t)^{-1} \left[ (N_c)_n - \mu (N_m)_n \right] \quad [65g] \]

It must be noted that Equation [65] for a general point is not readily applicable to the points near the inner and outer boundaries. So are some of Equations [65] which may involve terms related to points beyond the boundaries. Fictitious points are introduced in these instances and appropriate values of \( \eta \) are determined through the conditions of equilibrium at the boundaries.
At the inner edge of the shell for which \( n = p \), it is known from Equations [65d] and [65a] respectively:

\[
M' = K_{wp} (2\Delta)^{-1}(\eta_{p+1} - f_{p+1} \eta_p - \eta_{p-1})
\]

\[
Q' = w_p \Delta^{-2} [\eta_{p+1} - (d_p + \mu \Delta^2) d_p^{-1} \eta_p + \eta_{p+1}]
\]

By solving the above equations for \( \eta_p \) and \( \eta_{p-1} \), we obtain

\[
\eta_p = 2 e_p^{-1} \eta_{p+1} - 2\Delta (K_{wp} e_p)^{-1} M' - \Delta^2 (w_p e_p)^{-1} Q'
\]  

[66a]

\[
\eta_{p-1} = (1 - 2f_p e_p^{-1}) \eta_{p+1} - 2\Delta (1 - f_p e_p^{-1}) (K_{wp})^{-1} M'
\]

+ \Delta^2 f_p (w_p e_p)^{-1} Q'.

[66b]

Then, the difference equations for points \((p + 1)\) and \((p + 2)\) will be as follows:

\[
(a_p - 2h_p + e_{p+1}) \eta_{p+1} - (d_{p+1} + d_{p+2}) \eta_{p+2} + a_{p+2} \eta_{p+3} +
\]

\[
2\Delta (h_p - a_p) (K_{wp})^{-1} M' + \Delta^2 h_p w_p^{-1} Q' - c_{p+1} \Delta^4 P = 0
\]  

[67a]

\[
(2a_{p+1} e_p^{-1} - d_{p+1} - d_{p+2}) \eta_{p+1} + e_{p+2} \eta_{p+2}
\]

\[
- (d_{p+2} + d_{p+3}) \eta_{p+3} + a_{p+3} \eta_{p+4} - 2\Delta a_{p+1} (K_{wp} e_p)^{-1} M'
\]

- \Delta^2 a_{p+1} (w_p e_p)^{-1} Q' - c_{p+2} \Delta^4 P = 0

[67b]

At the outer edge of the shell for which \( n = q \), it is again known from Equations [65d] and [65a] respectively:

\[
M = K_{wq} (2\Delta)^{-1}(\eta_{q+1} - f_q \eta_q - \eta_{q-1})
\]

\[
Q = w_q \Delta^{-2} [\eta_{q+1} - (d_q + \mu \Delta^2) a_q^{-1} \eta_q - \eta_{q-1}]
\]

By solving the above equations for \( \eta_q \) and \( \eta_{q+1} \), we obtain:

\[
\eta_q = 2 i_q^{-1} \eta_{q-1} + 2\Delta (K_{wq} i_q)^{-1} M - \Delta^2 (w_q i_q)^{-1} Q
\]  

[68a]

\[
\eta_{q+1} = (1 + 2f_q i_q^{-1}) \eta_{q-1} + 2\Delta (1 + f_q i_q^{-1}) (K_{wq})^{-1} M
\]

- \Delta^2 f_q (w_q i_q)^{-1} Q.

[68b]
Then, the difference equations at points \((q - 2)\) and \((q - 1)\) will be as follows:

\[
a_{q-3}q_{q-4} - (d_{q-3} + d_{q-2})q_{q-2} + e_{q-2}q_{q-2} - \\
(d_{q-2} + d_{q-1} - 2a_{q-1})q_{q-1} + 2\Delta a_{q-1} (\Xi_1^i q_{q})^{-1} M \\
- \Delta^2 a_{q-1} (\Xi_i q_{q})^{-1} Q - c_{q-2}q_{q+1} P = 0 \quad [69a]
\]

\[
a_{q-2}q_{q-3} - (d_{q-2} + d_{q-1})q_{q-2} + (e_{q-1} - 2j_{q} + a_{q})q_{q-1} - \\
2\Delta (j_{q} - a_{q}) (\Xi_1^i q_{q})^{-1} M + \Delta^2 j_{q} (\Xi_i q_{q})^{-1} Q - c_{q-1}q_{q+1} P = 0 \quad [69b]
\]

Equations [65] for normal forces, moments and shear are all valid, but the use of Equation [65e] to compute \((N_c)_{p}\) or \((N_c)_{q}\) would introduce mesh points two units beyond each boundary of the toroidal knuckle. The central difference analogue in this equation should therefore be replaced by either forward or backward differences. For the inner edge, we have

\[
(N_c)_{p} = PKa_{p} (1 - \frac{a_{p}}{2}) \\
- a_{p} (2\Delta)^{-1} [3(q_{m})_{p} - 4(q_{m})_{p+1} + (q_{m})_{p+2}] \quad [70]
\]

And, for the outer edge, we have

\[
(N_c)_{q} = PKa_{q} (1 - \frac{a_{q}}{2}) \\
+ a_{q} (2\Delta)^{-1} [3(q_{m})_{q} - 4(q_{m})_{q-1} + (q_{m})_{q-2}] \quad [71]
\]

16. Flat Circular Head

The governing differential equations of a flat circular head are given in Equations [18], Chapter II, which state:

\[
d^2w/dr^2 + r^{-1}(d^2w/dr^2) - r^{-2}(dw/dr) = Pr(2D)^{-1} \quad [18a]
\]

\[
d^2v/dr^2 + r^{-1}(dv/dr) - r^{-2} v = 0 \quad [18b]
\]

These two equations are independent of each other as far as dependent variables are concerned. They are treated separately herein.
For simplicity of presentation, several numerical constants are introduced and defined here. The subscript \( n \) in each term refers to the particular point under consideration.

\[
\begin{align*}
a_n &= 2 + \frac{2}{n} + \frac{1}{n^2} \quad &[72a] \\
b_n &= 2 - \frac{2}{n} + \frac{1}{n^2} \quad &[72b] \\
c_n &= \frac{2n}{(2n + \mu)} \quad &[72c] \\
d_n &= \frac{(2n - \mu)}{(2n + \mu)} \quad &[72d] \\
e_n &= \frac{n}{(3n + 2\mu)} \quad &[72e]
\end{align*}
\]

Then, the difference equations for normal forces, moments, shear, axial rotation and deflection normal to the axis can be obtained for a general point \( n \).

\[
\begin{align*}
(Q_m)_n &= - \frac{Pn\Delta}{2} \quad &[73a] \\
(N_m)_n &= - E\epsilon t(2\Delta)^{-1}(v_{n-1} - 2\mu v_n/n - v_{n+1}) \quad &[73b] \\
(N_c)_n &= - E\epsilon t(2\Delta)^{-1}(v_{n-1} - 2\mu v_n/n - v_{n+1}) \quad &[73c] \\
(K_m)_n &= - \frac{\mu D\Delta^{-2}}{2} [(1 - \mu/2\mu)w_{n-1} - 2w_n + (1 + \mu/2\mu)w_{n+1}] \quad &[73d] \\
(K_c)_n &= - \frac{\mu D\Delta^{-2}}{2} [(1 - 1/2\mu)w_{n-1} - 2w_n + (1 + 1/2\mu)w_{n+1}] \quad &[73e] \\
(A)_n &= + (w_{n-1} - w_{n+1})(2\Delta)^{-1} \quad &[73f] \\
(\bar{w})_n &= + v_n \quad &[73g]
\end{align*}
\]

Considering now the governing differential equation for \( w \), we obtain the difference equation for a general point \( n \) in the form:

\[
w_{n-2} - a_n w_{n-1} + \frac{4}{n} w_n - b_n w_{n+1} - w_{n+2} + n\Delta^4 P^{-1} = 0 \quad [74]
\]

It is noted that at the center of the head, for which \( n = 0 \), this equation cannot be applied because \( r \) vanishes. However, by differentiating the original equation with respect to \( r \), we obtain

\[
d^4 w/dr^4 + r^{-1} (d^3 w/dr^3) - 2 r^{-2} (d^2 w/dr^2) + 2 r^{-3} (dw/dr) = P(2D)^{-1}
\]

The above equation can be expressed in another form which can be simplified by de L'Hôpital's rule.
For $r = 0$, the second term of the above equation has the indeterminate form $\varphi(r)/\varphi'(r) = 0/0$. Therefore

$$\lim_{x \to 0} \frac{\varphi(r)}{\varphi'(r)} = \lim_{x \to 0} \frac{\varphi'(r)}{\varphi''(r)} = \frac{1}{3} \left( \frac{d^4 w}{dr^4} \right).$$

From this, we obtain

$$\frac{d^4 w}{dr^4} = \frac{1}{3} \left( \frac{d^4 w}{dr^4} \right).$$

Then, the difference equation for $n = 0$ can be written as follows:

$$w_2 - 4w_1 + 6w_0 - 4w_{-1} + w_{-2} - \frac{3}{8} \left( \Delta^4 D^{-1} P \right) = 0.$$

From the symmetrical properties of the flat circular head, we must have $w_1 = w_{-1}$, $w_2 = w_{-2}$. Then, the difference equations for $n = 0$ and $n = 1$ become

$$6w_0 - 8w_1 + 2w_2 - \frac{3}{8} \left( \Delta^4 D^{-1} P \right) = 0 \quad [75a]$$

$$5w_0 - 5w_1 - w_2 + w_3 - \frac{1}{3} \left( \Delta D^{-1} P \right) = 0 \quad [75b]$$

For points near the boundary of the head, the equations for a general point should also be modified. It is known from the condition of static equilibrium at the boundary that

$$M = -D \Delta^2 \left[ \left( 1 + \mu/2q \right) w_{q-1} - 2w_q + \left( 1 - \mu/2q \right) w_{q+1} \right]$$

From this expression, we obtain

$$w_{q+1} = -c \frac{\Delta^2 D^{-1} M}{q} - \frac{d^2 w_{q-1}}{q} - 2c_w q.$$

Still another equation is needed to express the relation between $w_q$ and the other unknowns at points inside the boundary. We are only interested in the relative values of $w$'s because only these relative values affect the meridional rotation at the edge which we want to obtain. We can therefore assume any arbitrary value for $w_q$, say $w_q = 0$. Then,

$$w_q = 0 \quad [76a]$$

$$w_{q+1} = -c \frac{\Delta^2 D^{-1} M}{q} - \frac{d^2 w_{q-1}}{q} \quad [76b]$$

Therefore, the difference equations for points $(q - 2)$ and $(q - 1)$ become
\[w_{q-4} - a_{q-2}w_{q-3} + 4(q - 2)^{-1} w_{q-2} + b_{q-2}w_{q-1}\]
\[+ (q - 2)\Delta^H_{D-1} P = 0 \quad [77a]\]
\[w_{q-3} - a_{q-1}w_{q-2} + [4(q - 1)^{-1} + d_{q}]w_{q-1}\]
\[+ c_{q} \Delta^2_{D-1} M + (q - 1)\Delta^H_{D-1} P = 0 \quad [77b]\]

The second of the governing differential equations involving \(v\) can be treated in the same way. The difference equations for a general point can be written as follows:
\[(2 - n^{-1})v_{n-1} - (4 + 2n^{-2})v_n + (2 + n^{-1})v_{n+1} = 0. \quad [78]\]

At the center of the head where \(n = 0\), it is known from the physical conditions that \(v_0 = 0\). Then, for points \(n = 0\) and \(n = 1\), the equations become:
\[v_0 = 0 \quad [79a]\]
\[-6v_1 + 3v_2 = 0 \quad [79b]\]

At the edge of the flat circular head, it is known from the condition of static equilibrium
\[N = \mathbb{E}^t [(2\Delta)^{-1}(3v_q - 4v_{q-1} + v_{q-2}) + \mu(q\Delta)^{-1}v_q].\]

Here the backward difference analogue has been used instead of the central difference analogue as shown in Equation \([73b]\). From the above expression, we obtain
\[v_q = 2\Delta e_q (\mathbb{E}^t)^{-1} N - e_q v_{q-2} + 4 e_q v_{q-1}. \quad [80]\]

Difference equations for points \((q - 2)\) and \((q - 1)\) can be expressed as follows:
\[\begin{align*}
[2 - (q - 2)^{-1}] v_{q-3} - [4 + 2(q - 2)^{-2}] v_{q-2} + \\
[2 + (q - 2)^{-1}] v_{q-1} &= 0 \\
[2(1 - e_q) - (1 + e_q)(q - 1)^{-1}] v_{q-2} + \\
- [4(1 - 2e_q) - 4e_q (q - 1)^{-1} + 2(q - 1)^{-2}] v_{q-1} + 2\Delta e_q (\mathbb{E}^t)^{-1} [2 - (q - 1)^{-1}]N &= 0 \quad [81a]\end{align*}\]
Now, it is possible to set up two sets of simultaneous different equations, one set involving w's only and the other set v's only. The solutions of these sets of equations give w's in terms of M and P, and v's in terms of N.

The expressions for normal forces, moments and shears at the center of the head can be obtained by de L'Hospital's rule.

\[
\begin{align*}
(Q_m)_o &= 0 \\
(N_m)_o &= (2\Delta)^{-1} (3v_o - 4v_1 + v_2)(1 + \mu)Em' \\
(N_c)_o &= (2\Delta)^{-1} (3v_o - 4v_1 + v_2)(1 + \mu)Em' \\
(M_m)_o &= D\Delta^{-2} (2w_{n+1} - 2w_n)(1 + \mu) \\
(M_c)_o &= D\Delta^{-2} (2w_{n+1} - 2w_n)(1 + \mu).
\end{align*}
\]

At the edge of the flat circular head, Equations [73] are applicable, and there is no need for modification.

17. Application of the Procedure

For the family of curved surfaces which includes spherical, conical and toroidal shells, the applications of the procedure are identical. The flat circular head closing a cylindrical pressure vessel is a special member of this class. The application of the procedure to this type of head is slightly modified.

A. Summary of the procedure.

The procedure developed in this chapter is summarized here. The following steps are given for the family of curved surfaces:

1. Set up simultaneous difference equations for the values of \( \eta \) at various points in integral values of \( \Delta \), excluding the points at the boundaries, which are related to the points inside the boundary and can be obtained later. The number of equations will be equal to the number of unknowns.

2. The solution of the simultaneous equations gives the unknown values of \( \eta \) in terms of the meridional moment and meridional shear at the boundary, and the uniform internal pressure.
3. The meridional rotation and deflection normal to the axis at the boundary can also be expressed in terms of these quantities. By consideration of the conditions of continuity at the juncture, the edge moment and edge shear can be evaluated. Then, the values of $\eta$ are obtained explicitly.

4. Having the values of $\eta$ at discrete points on the shell, compute normal forces, moments and shears at these points. The total (direct plus bending) stresses and shearing stresses are obtained from the resultant normal forces, moments and shears by Equations [24] given in Chapter III.

For flat circular heads, the steps are only slightly modified:

1. Set up two sets of simultaneous difference equations, one involving only $w_i$'s and the other only $v_i$'s. With the number of equations equal to the number of unknowns in each set, the equations can be solved.

2. The solution of the two sets of equations gives $w_i$'s in terms of meridional moment and the uniform internal pressure, and $v_i$'s in terms of meridional normal force and pressure.

3. The axial rotation and deflection normal to the axis at the boundary are next computed, using Equations [37f] and [37g]. After the meridional moment and meridional normal force are evaluated from the conditions of continuity, the values of the $w_i$'s and $v_i$'s can be obtained explicitly.

4. The normal forces, moments and shears at various points are readily related to $w_i$'s and $v_i$'s as given by Equations [37a] to [37e]. They are first computed and the stresses are obtained from them.

B. Further approximation of the procedure.

While the formulas of this chapter lend themselves readily to numerical computations, it is possible, with experience, to cut certain corners in the work. In particular, the formulas are much simplified if we take, for instance,

1. Spherical shell
   
   \[ e_q = f_q = 2 \]

2. Conical shell
   
   \[ e_q = 2, \quad f_q = 2q - 1 \]
3. Toroidal shell
\[ p = \frac{i}{q} = 2 \]
\[ h_p = a_p + a_{p+1} \]
\[ j_q = a_{q-1} + a_q \]

4. Flat circular head
\[ c_q = d_q = 1. \]

These approximations prove to be reasonably good if the mesh is sufficiently fine. They are not recommended for general use until experience has been acquired in problems by the straightforward numerical method outlined in the previous sections.

C. Accuracy of the method.

Although the accuracy of this procedure depends to a certain extent on the fineness of the mesh, results sufficiently accurate for engineering purposes may be obtained with a coarse network requiring only a moderate amount of labor. To be specific, it is shown that, in general, a set of approximately ten points along the meridian provides accuracy sufficient for most purposes. The computation of coefficients for these equations, the solution of the equations themselves (with the aid of a desk calculator), and the computation of the normal forces, moments and shears has required a total of approximately sixteen hours in each case. If a large number of problems are to be solved, the time requirements may be shortened for each problem as there will be a number of constants common to all the problems which need not be recomputed each time.
V. COMPARISON OF METHODS OF SOLUTION.

18. Statement of the Test Problem

Before proceeding with the analyses of specific types of pressure vessel heads by means of the numerical procedure developed in the preceding chapter, the accuracy of the method is here examined through a comparative study of the spherical head, analyzed by three different methods. First, the classical analysis based on the Love-Meissner equations is considered as an "exact" solution. Next, an approximate solution is sought by neglecting some of the terms of the governing differential equations as suggested by Geckeler. Lastly, a numerical solution is obtained by the finite difference method.

The vessel under consideration consists of a cylindrical shell with a spherical segment at each end. It is subjected to a uniform internal pressure of 100 psi. The choice of dimensions in this and the other numerical examples in the next chapter has been motivated by a desire to compare the results with possible future tests of model size vessels. The thicknesses of the cylindrical shell and the spherical heads are taken to be different in this case. The dimensions are given as follows: (See Fig. 3A).

1. Spherical head
   \[ t_h = 1/4 \text{ in.}, \quad \rho = 14 \text{ in.}, \quad \phi_h = 30^\circ \]

2. Cylindrical shell
   \[ t_c = 3/8 \text{ in.}, \quad R = 7 \text{ in.} \]

The material from which the vessel is constructed is assumed to have the following mechanical properties:

\[ E = 30 \times 10^6 \text{ psi.} \]
\[ \mu = 1/3 \]

These are reasonable values in practice and resemble the properties of structural steel. The quantity \( E \) is needed only in the computation of deformations and is immaterial as far as stresses are concerned if the heads and the cylindrical shell of the vessel are made of the same material as is assumed here.

13. We need not, of course, adopt a specific value of pressure. All normal forces, moments and shears will be proportional to \( P \) in the elastic range and \( P \) can therefore be carried as a common factor.
The classical solution of this problem is well known. It is reproduced in Appendix II of this report for the sake of completeness as is the approximate solution due to Geckeler. Only the method of finite differences is explained in detail in the illustrative example of this chapter.

It is appreciated that the results of this numerical example should not be used as the basis of general conclusions as to the relative accuracy of these methods. Many factors may affect the final accuracy and all of them cannot be included in a single example.

19. Illustrative Example of the Method of Finite Differences

The numerical example given in this section has been worked out by the method of finite differences developed in the preceding chapter. In selecting the mesh size for the application of the numerical procedure in this case, 12 divisions have been chosen. The point for which \( n = 0 \) is at the crown; \( n = 12 \) at the head-to-shell juncture. This simply means \( \Delta = 2.5^\circ \) for \( \beta_n = 30^\circ \).

The first step in the application of the method is to derive the simultaneous difference equations by substituting proper numerical values into the formulas developed in Section 13. With \( \eta_0 = 0 \) as given in Equation [38], the algebraic relationships at points \( n = 1 \) and \( n = 2 \) can be obtained from Equations [39] and are as follows:

\[
\begin{align*}
+ 9.6747 \eta_1 - 4.9339 \eta_2 + \eta_3 &= 0 \\
- 4.9339 \eta_1 + 6.8981 \eta_2 - 4.2671 \eta_3 + \eta_4 &= 0.
\end{align*}
\]

Those for the intermediate points from \( n = 3 \) to \( n = 9 \) inclusive are obtainable from Equations [35].

\[
\begin{align*}
+ \eta_1 - 4.2671 \eta_2 + 6.4538 \eta_3 - 4.1264 \eta_4 + \eta_5 &= 0 \\
+ \eta_2 - 4.1264 \eta_3 + 6.3032 \eta_4 - 4.0731 \eta_5 + \eta_6 &= 0 \\
+ \eta_3 - 4.0731 \eta_4 + 6.2345 \eta_5 - 4.0470 \eta_6 + \eta_7 &= 0 \\
+ \eta_4 - 4.0470 \eta_5 + 6.1973 \eta_6 - 4.0323 \eta_7 + \eta_8 &= 0 \\
+ \eta_5 - 4.0323 \eta_6 + 6.1752 \eta_7 - 4.0232 \eta_8 + \eta_9 &= 0 \\
+ \eta_6 - 4.0232 \eta_7 + 6.1607 \eta_8 - 4.0152 \eta_9 + \eta_{10} &= 0 \\
+ \eta_7 - 4.0152 \eta_8 + 6.1503 \eta_9 - 4.0130 \eta_{10} + \eta_{11} &= 0.
\end{align*}
\]
The equations for points $n = 10$ and $n = 11$ are represented by Equations [41].

\[ + \eta_8 - 4.0130 \eta_9 + 6.1438 \eta_{10} - 3.0099 \eta_{11} = \\
- 2.2038 \times 10^{-3} M + 0.6732 \times 10^{-3} Q \]

\[ + \eta_9 - 4.0099 \eta_{10} + 3.1309 \eta_{11} = \\
+ 4.4076 \times 10^{-3} M - 2.6926 \times 10^{-3} Q. \]

We now have 11 equations in the 11 unknowns $\eta_1, \eta_2, \ldots, \eta_{10}, \eta_{11}$. The constant term vanishes from these equations except for the equations written for the edge points 10 and 11. These two equations have constant terms linear in the edge moment $M$ and the edge shear $Q$. Setting $M = 1,000$ and $Q = 0$, the equations can be solved by any one of a number of routine methods. One procedure which has been found satisfactory is the Doolittle or Gauss method\(^{14}\) which has been used with the aid of a desk calculator. Then we obtain as follows:

\[ \begin{align*}
\eta_1 &= -0.013253 \\
\eta_2 &= -0.046541 \\
\eta_3 &= -0.101409 \\
\eta_4 &= -0.177067 \\
\eta_5 &= -0.261517 \\
\eta_6 &= -0.321011 \\
\eta_7 &= -0.288504 \\
\eta_8 &= -0.055229 \\
\eta_9 &= +0.526481 \\
\eta_{10} &= +1.614474 \\
\eta_{11} &= +3.307355.
\end{align*} \]

Setting $M = 0$ and $Q = 1,000$, the solution is repeated.

\[
\begin{align*}
\eta_1 &= + 0.016965 \\
\eta_2 &= + 0.053085 \\
\eta_3 &= + 0.097784 \\
\eta_4 &= + 0.134771 \\
\eta_5 &= + 0.134598 \\
\eta_6 &= + 0.049148 \\
\eta_7 &= - 0.189091 \\
\eta_8 &= - 0.657115 \\
\eta_9 &= - 1.412445 \\
\eta_{10} &= - 2.432863 \\
\eta_{11} &= - 3.524766.
\end{align*}
\]

The two sets of solutions directly give each of $\eta_1, \eta_2, \ldots, \eta_{11}$ as linear functions of $M$ and $Q$. For example:

\[
\begin{align*}
\eta_1 &= (- 0.013253 M + 0.016965 Q) \times 10^{-3} \\
\eta_2 &= (- 0.046541 M + 0.053085 Q) \times 10^{-3}
\end{align*}
\]

and so on. Lastly, the quantities $\eta_{12}$ and $\eta_{13}$ can also be expressed in terms of $M$ and $Q$ by substituting the values of $\eta_{11}$ into Equations [40].

\[
\begin{align*}
\eta_{12} &= (+ 5.511155 M - 4.197865 Q) \times 10^{-3} \\
\eta_{13} &= (+ 7.714955 M - 3.524766 Q) \times 10^{-3}.
\end{align*}
\]

The next step is to obtain $\alpha$ and $\delta$ at the head-to-shell juncture. These quantities are given by Equations [37f] and [37g] as follows:

\[
\begin{align*}
\alpha_h &= \rho^2 \omega_{12} \eta_{12} \\
\delta_h &= (\rho \sin 30^\circ)(\omega t)^{-1}[\langle N_c \rangle_{12} - \mu \langle N_m \rangle_{12}].
\end{align*}
\]

To find $\alpha_h$, we simply substitute the value of $\eta_{12}$ into the proper equation. The solution for $\delta_h$ involves first the determination of $\langle N_m \rangle_{12}$ and
It is shown in Equations [37b] and [43] that

\[(N_c)_{12} = \frac{Pp}{2} + (Q_m)_{12} (\cot 30^\circ)\]

\[(N_c)_{12} = \frac{Pp}{2} + (2\Delta)^{-1}[3(Q_m)_{12} - 4(Q_m)_{11} + (Q_m)_{10}]\]

and from Equation [37a]

\[(Q_m)_{12} = \omega_{12} \Delta^{-2} [\eta_{11} - 2.00393 \eta_{12} + \eta_{13}]\]

\[(Q_m)_{11} = \omega_{11} \Delta^{-2} [\eta_{10} - 2.00496 \eta_{11} + \eta_{12}]\]

\[(Q_m)_{10} = \omega_{10} \Delta^{-2} [\eta_{9} - 2.00626 \eta_{10} + \eta_{11}]\]

Since \(\eta_{9}, \eta_{10}, \eta_{11}, \eta_{12}\) and \(\eta_{13}\) are readily expressed in terms of \(M\) and \(Q\), the quantities \(\alpha_h\) and \(\delta_h\) can also be expressed in the same way.

The expressions for \(\alpha_h\) and \(\delta_h\) for the spherical-segment head, in terms of the edge moment and shear, are found to be as follows:

\[\alpha_h = D_h^{-1} ( + 1.527600 M - 1.163606 Q)\]

\[\delta_h = D_h^{-1} (- 0.507735 M + 0.748347 Q + 19.140625)\]

where \(Q = H/2 - 303.1105\).

The solution for \(\alpha_c\) and \(\delta_c\) for the cylindrical shell is given by Equations [27] in Chapter III as follows:

\[\alpha_c = D_c^{-1} ( - 1.267855 M - 0.803733 H)\]

\[\delta_c = D_c^{-1} (- 0.803733 M - 1.019017 H + 53.833008)\]

From the conditions of continuity at the juncture, we have \(\alpha_h = \alpha_c\) and \(\delta_h = \delta_c\). Solving for \(M, Q\) and \(H\),

\[M = - 134.4785 \text{ in.} \text{lb./in.}\]

\[Q = - 162.3410 \text{ lb./in.}\]

\[H = + 251.5390 \text{ lb./in.}\]

The last step is to find the numerical values of the \(\eta\)'s by substituting the above values of \(M\) and \(Q\) into the solutions of the simultaneous equations. Then, the normal forces, moments and shears at
various points are obtained by Equations [37].

20. Comparison of Results by Different Methods of Solution

The results of the analysis by three different methods of solution are given in tabulated form. The normal forces, moments and shears at various points on the spherical head are summarized in Table 1A. The total stresses (direct plus bending) have been computed as multiples of the membrane stress and are given in Table 1B. These computation results are also plotted in Figs. 6A to 6E. While the results given by Geckeler's approximate solution do not differ greatly from those of the classical solution, they are inferior to those obtained by the numerical method.

It is to be noted in Fig. 6A that Geckeler's approximate theory assumes the normal force in the meridional direction to be uniform throughout the head. This assumption would be nearly correct if the meridian were continuous from the cylindrical shell to the head at the head-to-shell juncture (e.g. as in a hemispherical head), but in the present example, it is in error. For the normal force in the circumferential direction, the results of the three different methods very nearly coincide.

For the moments in both meridional and circumferential directions, the results of Geckeler's approximate solution show discrepancies at various points when compared with those of the classical solution, as shown in Fig. 6B. The shears obtained by the three different methods agree very closely. (Fig. 6C).

The total stresses (direct plus bending) at the outer and inner surfaces of the vessel are plotted in Figs. 6D and 6E. They are given in terms of multiples of the membrane stress. Again, the results obtained by the finite difference method are close to those of classical solution. The greatest advantage of the numerical method over Geckeler's approximation is that its accuracy does not depend on the thickness of the head, whereas in the latter, the terms neglected in the governing differential equations will have more effect in thicker shells than in thinner shells.
### Table 1.

**Comparison of Results Obtained by Different Methods of Solution for a Spherical Head.**

**A. Normal Forces, Moments and Shears at Various Points on the Spherical Head.**

<table>
<thead>
<tr>
<th>Method of Solution</th>
<th>Point</th>
<th>( N_m )</th>
<th>( N_c )</th>
<th>( M_m )</th>
<th>( M_c )</th>
<th>( Q_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical</td>
<td>0</td>
<td>+738.90</td>
<td>+738.90</td>
<td>2.25</td>
<td>2.25</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>+741.49</td>
<td>+744.02</td>
<td>1.84</td>
<td>2.15</td>
<td>+1.81</td>
</tr>
<tr>
<td></td>
<td>2</td>
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### Table 1

**Comparison of Results Obtained by Different Methods of Solution for a Spherical Head.**

**B. Total Stress, Direct and Bending, at Various Points on the Spherical Head.**

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Fig. 6 Comparison of Methods of Solution

(A) Normal forces at various points on spherical head

Legend
- Classical Analysis
- Approximate Solution
- Numerical Method
(B) Moments at various points on spherical head

Fig. 6 Comparison of methods of solution
FIG. 6 COMPARISON OF METHODS OF SOLUTION
(D) TOTAL STRESSES ALONG OUTER SURFACE OF SPHERICAL HEAD

FIG. 6 COMPARISON OF METHODS OF SOLUTION
LEGEND
CLASSICAL ANALYSIS
APPROXIMATE SOLUTION
NUMERICAL METHOD

(E) TOTAL STRESSES ALONG INNER SURFACE OF SPHERICAL HEAD
Fig. 6 COMPARISON OF METHODS OF SOLUTION
VI. NUMERICAL ANALYSES OF VARIOUS TYPES OF PRESSURE VESSEL HEADS.

21. Dimensions of the Pressure Vessel Heads

The purpose of the numerical analyses of various types of heads is twofold: firstly, to illustrate the application of the numerical procedure developed in Chapter IV and secondly, to compare the stress distribution in various types of pressure vessel heads. The vessels to be compared are taken to be made of plates of the same thickness. They are subjected to a uniform internal pressure of 100 psi. The material from which these vessels are constructed is assumed to have the following mechanical properties:

\[ E = 30 \times 10^6 \text{ psi} \]
\[ \mu = \frac{1}{3} \]

As previously noted, these are reasonable values in practice and resemble the properties of structural steel. The quantity \( E \) is needed only in the computation of deformations and is immaterial as far as stresses are concerned if the head and cylindrical shell are made of the same material as assumed in these examples.

The dimensions of the cylindrical shells, to which the various types of heads are attached, are identical. The dimensions of the vessel heads are given as follows: (See Fig. 3).

1. Spherical head
   \[ t = \frac{1}{8} \text{ in.}, \quad R = 7 \text{ in.} \]
   \[ \phi_h = 30^\circ, \quad \rho = 14 \text{ in.} \]
2. Conical head
   \[ t = \frac{1}{8} \text{ in.}, \quad R = 7 \text{ in.} \]
   \[ \psi_o = 60^\circ, \quad S = 7 \csc \psi_o \text{ in.} \]
3. Torispherical head
   \[ t = \frac{1}{8} \text{ in.}, \quad R = 7 \text{ in.} \]
   \[ \phi_h = 30^\circ, \quad \phi_k = 60^\circ, \]
   \[ \rho = 11 \text{ in.}, \quad K = 3 \text{ in.}, \quad \lambda = \frac{4}{3} \]
4. Toriconical head
   \[ t = \frac{1}{8} \text{ in.}, \quad R = 7 \text{ in.} \]
   \[ \phi_h = 30^\circ, \quad \psi_o = \phi_k = 60^\circ \]
$S = 5.5 \csc \phi \text{ in.}$

$K = 3 \text{ in.}$, $\lambda = 4/3$

5. Hemispherical head

$t = 1/8 \text{ in.}$, $R = 7 \text{ in.}$

$\phi_h = 90^\circ$, $\rho = 7 \text{ in.}$

6. Flat circular head

$t = 1/8 \text{ in.}$, $R = 7 \text{ in.}$

22. Analyses of Six Types of Heads

The numerical procedure developed in Chapter IV provides directly the normal forces, moments and shears in the vessel heads. An example of this computation has been given in detail in the preceding chapter. Here, only the final results of analyses are shown in tabulated form (Tables 2 to 7). The total (direct plus bending) stresses are computed from the resultant normal forces and moments and the shearing stresses from the resultant shears.

In selecting the mesh size for the application of the numerical procedure to various types of heads, the number of points has been chosen with a view not only of providing the necessary accuracy but also of facilitating the computation of the coefficients of the difference equations. This simply means, for example, that the number of mesh points on the toroidal knuckle of torispherical head is taken to be an integral divisor of the total meridian angle subtended by the knuckle. The total stresses and the shearing stresses are obtained for these points. In the cylindrical shell, for which the "exact" solution is well known, stresses are computed by means of this solution at arbitrary regular intervals along the shell from the juncture toward the center of the shell until the local effect of bending moments and shears disappears and the stresses reduce to membrane stresses.

While the distances along the cylindrical shells of the vessels are measured from the juncture at increments of $\Delta = 1/2 \text{ in.}$, the size of mesh for various types of heads is summarized as follows:

1. Spherical head

$\Delta = 2.5^\circ$, $n = 12$ ($\phi_h = 30^\circ$)
The point $n = 0$ is at the crown; $n = 12$ at the head-to-shell juncture.

2. Conical head

\[ \Delta = (\ell/2) \csc \psi_0 \text{ in.}, \quad n = 14 \quad (s = 7 \csc \psi_0 \text{ in.}) \]

The point $n = 0$ is at the apex; $n = 14$ at the head-to-shell juncture.

3. Torispherical head

a. Spherical head proper

\[ \Delta = 2.5^\circ, \quad n = 12 \quad (\phi_h = 30^\circ) \]

The point $n = 0$ is at the crown; $n = 12$ at the head-to-knuckle juncture.

b. Toroidal knuckle

\[ \Delta = 5^\circ, \quad n = 12 \quad (\phi_k = 60^\circ) \]

The point $n = 12$ is at the head-to-knuckle juncture; $n = 24$ at the knuckle-to-shell juncture.

4. Toriconical head

a. Conical head proper

\[ \Delta = (\ell/2) \csc \psi_0 \text{ in.}, \quad n = 11 \quad (s = 5.5 \csc \psi_0 \text{ in.}) \]

The point $n = 0$ is at the apex; $n = 11$ is at the head-to-knuckle juncture.

b. Toroidal knuckle

\[ \Delta = 5^\circ, \quad n = 12 \quad (\phi_k = 60^\circ) \]

The point $n = 11$ is at the head-to-knuckle juncture; $n = 23$ at the knuckle-to-shell juncture.

5. Hemispherical head

\[ \Delta = 5^\circ, \quad n = 18 \quad (\phi_h = 90^\circ) \]

The point $n = 0$ is at the crown; $n = 18$ at the head-to-shell juncture.

6. Flat circular head

\[ \Delta = 1/2 \text{ in.}, \quad n = 14 \quad (R = 7 \text{ in.}) \]

The point $n = 0$ is at the center of the head; $n = 14$ at the head-to-shell juncture.
23. **Comparison of Results for Various Types of Heads**

The general patterns of stress distribution in various types of heads are plotted from the results of the total stresses (direct plus bending stress) for the purpose of comparison. They are shown in Figs. 7A to 7F. The stresses at various points along the meridian of each vessel are plotted on the normals to the middle surface at those points. Stresses plotted outside the contour of the vessel are tensile; those inside the contour represent compressive stresses.

While the results for spherical and conical heads (Figs. 7A and 7B) reveal that the critical stresses occur at the head-to-shell juncture, these stresses are of a localized character. A factor which may reduce this high stress concentration is the commercial practice of "streamlining" spherical and conical heads by the replacement of sharp corners with rounded fillets. This is borne out by a comparison with the torispherical and toriconical heads (Figs. 7C and 7D). Also, the stress concentration at the juncture is partly relieved after the material reaches its yield strength in that locality. The theoretical analysis, in the end, gives a qualitative rather than a quantitative overall picture of the stress distribution in this case.

In the torispherical and toriconical heads, the stress concentration at the junctures has been partly reduced by the introduction of toroidal knuckles. Although there exist peaks in the stress patterns along the knuckles, the order of magnitude of stresses in the head proper and in the knuckle in each case is very nearly the same. It should be pointed out that the torispherical head has been frequently used to approximate an ellipsoidal head. The distribution of stress in these two types of head differs somewhat, in theory, due to the basic difference in the geometrical properties (i.e. due to the theoretical sharp change in radius of curvature in one case and not in the other), but the magnitudes of the critical stresses for the two types of heads are known not to differ greatly. Since the ellipsoidal head has a gradually changing radius along the meridian, it can be expected that the variation of stress along the ellipsoidal head will be smoother without the peaks which exist in the stress pattern along the torispherical head. Commercially fabricated dished heads are usually neither in the form of a true ellipsoidal head nor a true torispherical head, but rather some combination of the two. After a period of service
under internal pressure, however, it seems natural to infer that even a true torispherical head will gradually deform into an ellipsoidal shape. Since engineers are primarily interested in the critical stresses, the differences in the distribution of stress due to these slight changes in shape do not materially affect the design.

The hemispherical head shows a very smooth pattern of stress distribution both in meridional and circumferential directions. (Fig. 72). Of the types of heads examined, it is the only one for which bending moment and shearing force at the juncture do not play an important part in influencing the critical stress. This type of head, however, is not particularly desirable from the standpoint of fabrication or economy.

The flat circular head is seen to introduce large bending stresses at the center as well as at the juncture of the head (Fig. 7F). The stresses obtained for an internal pressure of 100 psi. far exceed the yield strength of the material in some localities. The example given here is intended to compare various types of heads of same thickness and under same load. In the case of a thicker head, these high bending stresses can partly be reduced. Increase of the thickness of a flat circular head should introduce no complication in analysis or in fabrication. In spite of this, the stress concentration at the head-to-shell juncture would probably still be high enough to discourage the use of this type of head.
TABLE 2.

STRESSES IN SPHERICAL HEAD OF PRESSURE VESSEL.

<table>
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<th>Part of Vessel</th>
<th>Point</th>
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<th>Shearing Stresses in psi.</th>
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<td>Shearing Stresses in psi.</td>
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## Table 4.

**Stresses in Torispherical Head of Pressure Vessel.**

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## TABLE 5.

**STRESSES IN TORICONICAL HEAD OF PRESSURE VESSEL.**

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<th>Part of Vessel</th>
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<th>Shearing Stresses in psi.</th>
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### TABLE 6.
STRESSES IN HEMISPHERICAL HEAD OF PRESSURE VESSEL.

<table>
<thead>
<tr>
<th>Part of Vessel</th>
<th>Point</th>
<th>Total Stresses in psi.</th>
<th>Shearing Stresses in psi.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \sigma_m )</td>
<td>( \sigma_c )</td>
</tr>
<tr>
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<td>+ 2,800</td>
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<td></td>
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<td>+ 2,800</td>
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<td></td>
<td>2</td>
<td>+ 2,800</td>
<td>+ 2,800</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>+ 2,800</td>
<td>+ 2,800</td>
</tr>
<tr>
<td>Head</td>
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<td>+ 2,800</td>
<td>+ 2,800</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>+ 2,800</td>
<td>+ 2,800</td>
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<tr>
<td></td>
<td>6</td>
<td>+ 2,800</td>
<td>+ 2,800</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>+ 2,800</td>
<td>+ 2,800</td>
</tr>
<tr>
<td></td>
<td>8</td>
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<tr>
<td></td>
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<td></td>
<td>14</td>
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<td></td>
<td>17</td>
<td>+ 2,001</td>
<td>+ 3,046</td>
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<tr>
<td></td>
<td>18</td>
<td>+ 3,047</td>
<td>+ 4,111</td>
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<td>+ 4,111</td>
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<tr>
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<td>9</td>
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</tr>
<tr>
<td></td>
<td>12</td>
<td>+ 2,800</td>
<td>+ 5,600</td>
</tr>
</tbody>
</table>
## TABLE 7.

**STRESSES IN FLAT CIRCULAR HEAD OF PRESSURE VESSEL.**

<table>
<thead>
<tr>
<th>Part of Vessel</th>
<th>Point</th>
<th>Total Stresses in psi.</th>
<th>Shearing Stresses in psi.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(\sigma_m)</td>
<td>(\sigma_c)</td>
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<tr>
<td></td>
<td>4</td>
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<td>Circular</td>
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<td>Head</td>
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<td>+135,243</td>
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<td></td>
<td>7</td>
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<td></td>
<td>4</td>
<td>+16,024</td>
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<tr>
<td></td>
<td>12</td>
<td>+2,800</td>
<td>+5,600</td>
</tr>
</tbody>
</table>
FIG. 7  STRESS DISTRIBUTION IN PRESSURE VESSEL

LEGENDS
OUTER SURFACE
INNER SURFACE

SCALE: 1 IN. = 20,000 P.S.I.
0 10,000 20,000 P.S.I.

(A) SPHERICAL HEAD
FIG. 7 STRESS DISTRIBUTION IN PRESSURE VESSEL

(legend) outer surface ——— inner surface ———

scale: 1 in. = 20,000 p.s.i.

0 10,000 20,000 p.s.i.

(b) conical head
FIG. 7  STRESS DISTRIBUTION IN PRESSURE VESSEL

LEGENDS

OUTER SURFACE
INNER SURFACE

SCALE: 1 IN = 20,000 P.S.I.
0 10,000 20,000 P.S.I.

(c) TORISPHERICAL HEAD
FIG. 7  STRESS DISTRIBUTION IN PRESSURE VESSEL

LEGENDS
OUTER SURFACE
INNER SURFACE

SCALE: 1 IN. = 20,000 P.S.I.
0 10,000 20,000 P.S.I.

(D) TORICONICAL HEAD
FIG. 7  STRESS DISTRIBUTION IN PRESSURE VESSEL

LEGENDS
OUTER SURFACE
INNER SURFACE

SCALE: 1 IN. = 20,000 P.S.I.
0 10,000 20,000 P.S.I.

(E) HEMISPHERICAL HEAD
FIG. 7 STRESS DISTRIBUTION IN PRESSURE VESSEL

LEGENDS
OUTER SURFACE
INNER SURFACE

SCALE: 1 IN. = 200,000 P.S.I.
0 100,000 200,000 P.S.I.

NOTE: THIS SCALE IS 10 X THOSE OF FIGS. 7A, B, C, D, & E.

(F) FLAT CIRCULAR HEAD
VII. CONCLUSION.

24. Summary

The present investigation has developed a numerical method which can be used to analyze the state of stress in pressure vessel heads. While it is approximate in the sense that the governing mathematical equations are satisfied only in finite difference form, it has the merit of simplicity and of directly providing numerical values of normal forces, moments and shears needed in design.

A. The numerical procedure.

The first step in the solution of the governing differential equations for shells of various shapes by the method of finite differences consists in replacing the differential coefficients by finite difference analogues which are simply algebraic relationships between the values of the dependent variables at points on the middle surface of the shell. There results a set of simultaneous algebraic equations, one equation for each point of a mesh. The unknowns in this set are the wanted quantities from which may be derived the values of the normal forces, bending moments and shears.

While the accuracy of this procedure depends to a certain extent on the fineness of the mesh, results sufficiently accurate for engineering purposes may be obtained with a coarse network requiring only a moderate amount of labor. To be specific, it is shown that, in general, a set of approximately ten points along the meridian provides accuracy sufficient for most purposes. The computation of the coefficients for these equations, the solution of the equations themselves (with the aid of a desk calculator), and the computation of the normal forces, moments and shears has required a total of approximately sixteen hours in each case.

For the family of curved surfaces which includes spherical, conical, and toroidal shells, the governing differential equations are simplified by the introduction of a pair of new variables. The flat circular head closing a cylindrical pressure vessel is a special member of this class. Its analysis has been included for the purpose of comparison in spite of the fact that the "exact" solution for this case is relatively simple.
The principal difficulties which arise in connection with the foregoing procedure lie in the treatment of the boundary conditions. These difficulties are overcome through the introduction of fictitious points of the mesh. Although this idea is well known, the technique of its systematic application to the analysis of pressure vessel heads is believed to be presented here for the first time.

B. Analyses of six types of heads.

The numerical procedure here developed has been applied to the solution of different types of pressure vessel heads including:

1. Spherical head
2. Conical head
3. Torispherical head
4. Toriconical head
5. Hemispherical head
6. Flat circular head

The purpose of these numerical analyses is twofold: firstly, to illustrate the application of the procedure, and secondly, to compare the stress distribution in various types of pressure vessel heads. The results of the analyses are given in terms of total stresses (direct plus bending stress). The general patterns of stress distribution in various types of heads are plotted for the purpose of comparison.

The results for spherical and conical heads reveal that the critical stresses occur at the head-to-shell juncture, but these stresses are shown to be of a localized character. The introduction of knuckles as in torispherical and toriconical heads is shown to release part of this stress concentration. Of those examined, the hemispherical head is the only case in which bending moment and shearing force at the juncture do not play an important part in influencing the critical stress. The flat circular head is shown to be accompanied by large bending stresses. While the stresses in the cylindrical shell are also high at the juncture for this type of head, they decrease rapidly to membrane stresses.

C. Comparison of methods of solution.

A spherical-segment head has been chosen as a test problem. The classical analysis based on the Love-Meissner equations is considered as an "exact" solution. Then, an approximate solution is sought by neglect-
ing some terms of the governing differential equations as suggested by Geckeler. Lastly, a numerical solution is obtained by the finite difference method.

It is appreciated that the results of this numerical example should not be used as the basis of general conclusions as to the relative accuracy of these methods. Many factors may affect the final accuracy and all of them cannot be included in a single example.

Although the results given by Geckeler's approximation do not differ greatly from those of the classical solution, they are inferior to those obtained by the numerical method. The greatest advantage of the numerical method over Geckeler's approximation is that its accuracy does not depend on the thickness of the head, whereas in the latter, the terms neglected in the governing differential equations will have more effect in thicker shells than in thinner shells.

25. Concluding Remarks

The method developed in this report is a numerical technique of wide applicability in the analysis of pressure vessel heads. It can be applied to many practical design problems with an accuracy and speed which make it attractive for engineering purposes. At the same time, all the essential sources of stresses in a vessel (membrane stress, bending stress and shearing stress) are retained in the analysis. Although the examples cited are confined to pressure vessels under internal pressure, the method is equally applicable to vessels subjected to uniform external pressure. The buckling phenomena which may occur in the latter case are, however, outside the scope of this work.

To a certain extent, the procedure is analogous to the finite difference method which has long been applied to the solution of flat plate design problems. In fact, the circular head closing a cylindrical pressure vessel is considered in detail as a special case. In general, however, the curvature of the middle surface introduces complications, the solution of which is carefully explained. This numerical method is capable of extension to the case of pressure vessel heads of variable thickness. In the extension, of course, several terms must be added to the governing differential equations for shells of uniform thickness. In dealing with this type of problem, numerical solutions can be obtained
by considering the average thickness of the head in each division of the mesh to be uniform. This numerical solution, while clearly not exact, would have the advantage of being applicable to many problems for which classical solutions have not been heretofore available. Further extension along this line will require a more thorough investigation.

While the analysis in this report is based on the general theory of thin elastic shells, some preliminary consideration has been given to the phenomenon of plasticity. This line of approach is still in an immature stage and has not been reported herein. Recently, the modification of Geckeler's approximation for pressure vessel analysis through the introduction of considerations based on a theory of plasticity has been suggested by F. K. G. Odqvist (81). In view of the complications introduced by the consideration of plastic behavior, it has seemed to be desirable to limit the present discussion to the elastic case.
APPENDIX I. ANNOTATED BIBLIOGRAPHY.

1. Brief Review of Literature

This brief review is intended as a general survey rather than as a critical study of the references listed in the bibliography. Only those references most important for the purposes of this report are mentioned explicitly.

A. Early works on thin elastic surfaces.

Work on this problem can be traced back to Navier (1,2) whose memoir on elastic plates inspired the prize paper of Germain (4, 5) and the elaborate study of Poisson (6) on thin elastic shells. The general concepts of these early works are invaluable, but the treatments of the problem in these memoirs are incomplete and sometimes erroneous. It was Lamé and Clapeyron (8) who first threw light on the limits of internal pressure which could be applied to metal vessels. In addition, Lamé (9) made an admirable study of the equilibrium of a spherical shell which also caught the attention of Navier (3) and Poisson (7). The work of Kirchhoff (10) is important, not only for his analysis of plates and cylindrical surfaces, but also for his critical review of Poisson's work, pointing out the problems which arise in connection with the boundary conditions for thin shells. Later, Aron (12) and others tried to deduce the general equations of thin elastic shells without success.

B. General theory of thin elastic shells and plates.

A general theory of thin elastic shells was first established by A. E. H. Love (15). This theory has been reproduced in his "Treatise on the Mathematical Theory of Elasticity," better known in Continental Europe in the German translation of A. Timpe (16). Since the publication of Love's work in 1888, many papers have been written on the general theory of thin shells and plates. These latter theories differ from Love's in certain respects, and can be classified into three main categories:

1. Deviation in fundamental assumptions underlying the theory.
2. Retention or omission of certain terms in connection with the fundamental assumptions.
3. Different methods used for the derivation of the equations of equilibrium and deformations.
It is not the purpose of this brief review to discuss any of these aspects in detail. They are examined here in order to justify the use of Love's theory as a basis for the numerical analysis of pressure vessels. The usual assumptions made for the general theory of thin elastic shells are as follows:

1. The thickness of the shell is small in comparison with its other dimensions.
2. The stress component normal to the middle surface is small in comparison with the other stress components.
3. The displacements are small enough to permit the neglect of products of the displacements, compared with the displacements themselves. Quantities of second order pertaining to changes in curvature are also neglected.
4. The normals to the middle surface of the undeformed shell become normals to that of the deformed shell.

These assumptions on which Love's theory is based seem to be reasonable and, with or without slight modifications, have been used extensively by others. Only the "intrinsic theory" developed by W. Z. Chien (21,22), which seems to be unique for its complete generality, avoids these assumptions. However, Chien's theory introduces considerable complications which make it unsuitable for the present practical application.

A new derivation of the general differential equations of equilibrium of shells by means of Castigliano's theory, with the help of the calculus of variations, has been presented by E. Trefftz (19). A strain-energy expression for thin elastic shells has also been given by H. L. Langhaar (26) who notes some inaccuracies in Love's theory of thin shells and particularly in Love's formulas for the energy of bending. E. Reissner (23,24) modified the equations of deformation of Love's theory by introducing the assumption that the displacement components vary linearly with the distance along the normal from the middle surface of the shell. W. R. Osgood and J. A. Joseph (27) pointed out inconsistencies in Love's equations of equilibrium and presented a new set of equations. However, these equations reduce to Love's equations when the deformation of the middle surface is inextensional and irrotational. While these comments on Love's theory are in general sound and indeed convincing, it may also be noted that such modifications and corrections may be of little practical importance. In
particular, in the case of pressure vessels, the extension and rotation of the middle surface are small.

Theories based on general curvilinear coordinates (as contrasted with those using the lines of curvatures of the middle surface as coordinates) have been developed by I. N. Rabotov (30), W. Zerna (35) and others. The theory of thin elastic shells given by A. E. Green and W. Zerna (36) has the added advantage of not employing the usual assumption that normals to the middle surface of the undeformed shell become normals to that of the deformed shell.

C. Solutions based on Love's theory.

On the basis of Love's theory, E. Reissner (37) succeeded in simplifying the problem of the spherical shell by the proper choice of two dependent variables. E. Meissner (38, 39) further generalized the solution for the most important types of shells of revolution, those having constant curvature along the meridian of the shell. In a series of dissertations under the direction of E. Meissner, shells of various geometrical shapes were treated by L. Bolle (40), F. Dubois (41), E. Honegger (42) and H. Wissler (43).

There are many approximate methods for the solution of shell problems, which, in one way or another, attempt to avoid the application of "classical" solutions which depend on the rapidity of convergence of the infinite series entering into the solution. The first type of approximate method seeks a solution of the problem by asymptotic integration of the governing differential equations. This technique was first suggested by O. Blumenthal (47) for the spherical shell and then extended to the general case of shells in the form of surfaces of revolution by E. Steuermann (48).

The second type of approximate method simplifies the governing differential equations by neglecting some of the terms which are considered to be unimportant. The most elaborate work along this line has been given by J. W. Geckeler (49, 50). With slight modification, a closer approximation for spherical shells was obtained by M. Hetényi (51). The third type of approximate method solves the problem through a finite difference analogy. The work of P. Pasternak (52, 53) has treated the subject in this way, and deserves special mention for having inspired the present investigation.
In recent years, investigations of classical solutions of shell problems have been revived. A rather complete survey of the classical solutions for shells of simple geometrical shapes was made by W. R. Burrows (57). E. Reissner (58) reduced Love's fundamental equations in a modified manner for shallow parabolic shells, the solution of which is expressed in terms of Bessel functions.

There are a number of books in which are presented didactic treatments of the subject of shells. The most important ones are those given by W. Flügge (59), T. Pöschl (60) and S. Timoshenko (61).

D. Special treatises on pressure vessel analysis.

The concepts underlying the treatment of the analysis of pressure vessels as a problem in the theory of thin elastic shells were given by E. Meissner (64). He developed the solution for a vessel with cylindrical shell and hemispherical heads. This approach was further extended by J. W. Geckeler (50), A. Huggenburger (65), E. Höhn (67) and F. Schultz-Grunow (68) to apply to different shapes of symmetrical vessel heads.

Applications of these solutions based on the classical theory of shells to the analysis of pressure vessels are not at all common. Apart from the crude treatments based on membrane theory, most analyses follow the "approximate theory" of J. W. Geckeler (50) which neglects some terms in the governing differential equations. Much work has been done along this line for various geometrical shapes of heads. Notable among these contributions are the method of calculating stresses in full heads due to W. M. Coates (69), the analysis of stresses in dished heads by C. O. Rhys (70), the investigation on conical heads by H. C. Boardman (72), and others.

Recently, increasing attention has been paid to the application of classical solutions to pressure vessel heads. The most remarkable summary of such applications has been outlined in a paper by G. W. Watt and W. R. Burrows (77). Due to its mathematical complexity, however, the classical approach seems to be limited in usefulness.

None of the treatments discussed above was devoted to the application of the finite difference method for the pressure vessel problems. The present report is partly intended to serve that purpose.
2. Bibliography

A. Early Works on Thin Elastic Surfaces.


In the historical review, a brief account of the attempts made by Navier, Lamé and others in approaching the problem of elastic surfaces is given.


The author presents for the first time the general equations of equilibrium and the equations which must hold at the boundary of the elastic plate.


The investigation includes the results of experiments on circular tubes and other vessels subjected to internal pressure.


This work is in four parts: (1) Exposition of principles which can serve as basis for the theory of elastic surfaces; (2) Investigation of terms which constitute the equations of elastic surfaces; (3) Equations of the vibrating cylindrical surface and circular ring; and (4) Comparison of the results of the theory and experiment.


The work consists of some preliminary observations and then is divided into two sections: (1) Exposition of the conditions which characterize the surface; and (2) General equation of vibrating elastic surfaces.


This memoir is divided into two sections: (1) Equation of equilibrium of non-elastic surface; (2) Equation of the elastic surface in equilibrium.


The note includes some comments on experiments on spherical shells of constant thickness subjected to internal pressure.

The memoir consists of an introduction and four sections: (1) Differential equations; (2) Theorems of pressures; (3) Simple cases; and (4) General cases. In the third section, the authors deduce formulas limiting the pressure which can safely be applied to the interior of metal vessels.


The memoir contains an investigation of the conditions for the equilibrium of a spherical elastic envelope or shell subjected to a given distributed load.


The memoir consists of five sections preceded by a short historical introduction. The points raised by the author on Poisson's work have led to problems with regard to boundary conditions for thin shells.


This work is divided into three parts: (1) Bodies having all their dimensions finite; (2) Bodies which have one dimension or two dimensions infinitely small; and (3) Applications. In the annotations to the work of Clebsch, de Saint-Venant treats various vigorous solutions for the bending of plates. It is most complete account yet given for the bending of circular plates.


This paper deals with arbitrary curved surfaces or thin elastic shells which are subjected to finite deformation. The general equations of equilibriums are given.


The papers contain five sections: (1) General formulas; (2) Study of a surface around one of its points; (3) Lines traced on a surface; (4) Element of arc; and (5) Various systems of coordinates on surface.

The book, in two volumes, consists of an extensive survey and review of literature on the theory of elasticity in chronological order. The early works on the problem of elastic surfaces by Navier, Germain, Poisson, Lamé, Clapeyron, Kirchhoff and others are mentioned by the author.

B. General Theory of Thin Elastic Shells and Plates.


In the paper, the potential energy of deformation of an isotropic elastic shell is investigated by the same method employed by Kirchhoff in his discussion on the vibration of thin plates. As the geometrical theory of the deformation of extensible surfaces appeared not to have been hitherto made out, it was necessary to give the elements of such a theory for small deformations.


A reproduction of Love's general theory of thin plates and shells is given in Chapter XXIV of the book.


The author has a somewhat different definition of the symbols for the deformation of the middle surface from that given by Love. He points out that it is not possible to draw an absolute line of demarcation between the deformations in which the governing feature is the extension of the middle surface and those which involve flexure with little or no extension.


This paper includes an introduction and four sections: (1) On tensor analysis, differential geometry, and theory of deformation; (2) The resulting terms and the resulting conditions of equilibrium; (3) Relation between strains and stresses; and (4) The complete equilibrium conditions and the integration of the fundamental equations.

This paper deals with the equilibrium of the shell element by means of the Castigliano's principle of least work with the aid of the calculus of variations. For simplicity, only the case of constant thickness is treated.


The author raises the point whether some terms of second order in the theory of shells should be neglected. He presents a theory of first approximation independent from the ordinary hypothesis that normals to the undeformed middle surface of a shell remain normals after deformation.


The authors systematize the theory of shells by developing a basic exact theory into which various approximations may be fitted. All empirical assumptions are avoided in the derivation of the basic equations.

22. "The Intrinsic Theory of Thin Shells and Plates" by W. Z. Chien. Quarterly of Applied Mathematics, v. 1, 1944, p. 297; v. 2, 1944, p. 43 and p. 120.

The paper gives a systematic treatment of the general problem of the thin shell which includes the problem of the thin plate as a special case. It is divided into three parts: (1) General theory; (2) Application to thin plates; and (3) Application to thin shells. The fundamental equations of the theory include three equations of equilibrium and three equations of compatibility. The displacement does not appear explicitly in the argument. A complete classification of all shell and plate problems is obtained.


The paper presents a modified derivation of stress-strain relations in a shell which uses directly the known expression for strain components with respect to orthogonal systems of coordinates and utilizes the assumption that the displacement components vary linearly with the distance along the normal from the middle surface of the shell.


This note is to complete the discussion of the author's previous paper by deriving expressions for the components of strains for the stress-strain relations.
In this paper, a theory of small deformations of a thin elastic shell having any shape is developed. This theory is valid on the assumption that the deformations are small enough so that terms of second order and higher in the deformation components may be neglected, but it is exact to the other terms in consideration.

A derivation is given for the strain energy of an isotropic elastic shell whose radii of curvature are sufficiently large so that the strains may be assumed to vary linearly through-out the thickness. The effects of the tangential displacements upon the energy due to bending are found to differ appreciably from Love's results in the first order terms. Special forms of the general energy expression are given for shells in the shape of flat plates, circular cylinders, etc.

The authors re-examine certain parts of Love's general theory of thin shells. Expressions are obtained for the changes of curvature and the twist, and new equations of equilibrium are introduced.

The paper gives the fundamental equations of shell problems in a rather simple way by means of vector analysis.

A system of equations is given for each of the three selected groups of the integrals describing the membrane, bending and boundary effect, defining them with all precision which may be attained by the theory of shells based on the Kirchhoff-Love hypothesis.

A system of equations is given for each of the three selected groups of the integrals describing the membrane, bending and boundary effect, defining them with all precision which may be attained by the theory of shells based on the Kirchhoff-Love hypothesis.

The paper presents the equations of equilibrium and deformations of thin shells referred to curvilinear coordinates. Kirchhoff's hypothesis is used for the derivation of the relationship between strain and stress.


The author establishes a system of differential equations of the sixth order for problems of bending of thin plates. On the basis of his equations, it is possible and necessary to satisfy three boundary conditions along the edges of a plate while the classical theory leads to two boundary conditions only. This is due to the fact that Reissner's theory includes the shear deflection and therefore embraces an additional degree of freedom.


A system of equations is developed for the theory of bending of thin elastic plates which takes into account the transverse shear deformability of the plate. This system of equations is of such a nature that three boundary conditions can and must be prescribed along the edge of the plate.


This paper gives an account of the author's earlier derivations in simpler and more general form. Plates of homogeneous and sandwich constructions are considered.


This paper shows that Reissner's equations can be obtained directly from the equations of equilibrium and the stress-strain relations. Moreover, by the consistent use of complex variable notation, the form of the results is simplified.


In this paper, a new set of fundamental equations for shell problems is developed, using a different coordinate system. It treats some special problems in a simpler way.


A theory of thin elastic shells is developed as a first approximation without using the usual assumption that the normals to the middle surface of the undeformed shell remain normals in the deformed shell. It includes Reissner's theory of bending of elastic plates as a special case.
C. Solutions Based on Love's Theory.


On the basis of Love's theory, a solution for the differential equations pertaining to spherical shells has been obtained by the proper choice of two variables related to those used by Love.


A generalized solution of the differential equations for shells of revolution having constant curvature along the meridian of the shell is given in this paper.


This paper presents the solutions of the differential equations for shells of revolution, including those for spherical shells, conical shells and toroidal shells.


This dissertation deals with the problems of spherical shells. It gives examples of different combinations of loading and boundary condition.


This dissertation treats conical shells under various loadings and with various boundary conditions.


This dissertation deals with problems related to conical shells of linearly variable thickness.


This dissertation includes investigations of problems in connection with toroidal shells.

This work treats a number of problems related to spherical and conical shells. The solutions include the integration of differential equations of the fifth order by means of infinite series.


The author presents the governing differential equations of elastic arched surfaces in terms of components of displacements.


This is a summary of Keller's work on the problem of elastic surfaces.


In this paper, the author presents an approximate solution of the Love-Weiszner equations for spherical shells by the method of asymptotic integration.


This paper gives an approximate solution for the general case of shells in the form of surfaces of revolution, following the asymptotic integration method for the spherical shell suggested by Blumenthal.


This paper presents a theoretical analysis of axially symmetric shells. The simplified differential equation and its application to the computation of stresses in vessel heads are given. An experiment made to support the theory is also presented.


In this paper, the author presents an approximate theory of thin elastic shells, and points out the possibility of further approximations.

This paper presents formulas for a more accurate approximation to the Love-Weissner equations for spherical shells. The method takes into account the first derivatives in addition to the second derivatives and neglects only the functions themselves in the resulting governing differential equation.


This paper presents the method of finite differences for problems connected with the spherical shell in bending, the circular plate on an elastic foundation, and the circular cylindrical shell with built-in ends.


Formulas are given in the paper for the numerical solution of bending stresses in spherical and conical heads rigidly connected to a cylindrical wall.


The work includes the problems of elastic isotropic shells in the form of surfaces of revolution, but extends to the case of anistropic shells.


This paper presents a solution of the problem of a spherical shell which is axisymmetrically loaded, when the variation in wall thickness is taken as a quadratic function of the angle along the meridian. The Love-Weissner equations for the case of non-uniform wall thickness are derived.


This paper presents an approximate solution for conical shells of linearly varying wall thickness. The integral of the differential equation of 4th order is obtained.


This report is a summary of work on problems of thin elastic shells based on Love's theory, showing the interrelation of the formulas pertaining to various types of shell.
58. "On the Theory of Thin Elastic Shells" by E. Reissner. Reissner
Anniversary Volume, Contributions to Applied Mechanics, Edwards
Brothers, Inc., Ann Arbor, Michigan, 1949, p. 231.

This paper deals with the subject of axially symmetrical deforma-
tions of thin elastic shells of revolution. The simplification
of Love's fundamental equations is accomplished by the author in
a manner modified from that of E. Meissner.

59. "Statik und Dynamik der Schalen" by W. Flügge. Springer,
Berlin, 1934.

The bending theory of cylindrical shell is given in Chapter VI,
p. 110, and the bending theory of shells of revolution is given
in Chapter VII, p. 145.

60. "Berechnung von Behältern" by T. Pöschl. Second edition, Springer,
Berlin, 1926.

This book deals extensively with the subject of elastic thin
shells. The bending theory of shells is given in Chapter III,
p. 50.

61. "Theory of Plates and Shells" by S. Timoshenko. McGraw-Hill,

The general theory of cylindrical shells is given in Chapter XI,
p. 389. Shells having the form of a surface of revolution and
loaded symmetrically with respect to their axes are treated in
Chapter X, p. 450.

62. "Drang und Zwang" by A. and L. Förpl. Oldenbourg, München and

A full chapter of the book is devoted to problems of thin
eselastic shells. Examples for various types of shells and differ-
ent loading conditions are given.

63. "Die Dampfturbinen" by A. Stodola. Fourth edition, Springer,
Berlin, 1910, p. 597.

This is one of the early works which present the governing differ-
etial equations of shells in terms of components of displacements,
as a result of reduction from Love's equations of equilibrium and
deformation.

D. Special Treatises on Pressure Vessel Analysis.

64. "Zur Festigkeitsberechnung von Hochdruck-Kesseltrömmeln" by

This paper introduces the concept of the boundary problems of
thin shells in the analysis of pressure vessels. The case of a
vessel consisting of a cylindrical shell with a hemispherical
head at each end serves as an example.
This paper presents the important properties and the approximate solutions of the problem of deformations and stresses in a vessel. The analytical results are checked with experiments.

This paper presents an investigation of the strength and deformation of pressure vessel heads. Test results are given for six types of ellipsoidal heads.

This book deals with the calculation of stresses in vessel heads. It also gives results of experimental investigations which are in good agreement with the approximate analytical solution.

This paper presents the general theory for the calculation of stresses in axially symmetrical pressure vessel heads and its application to different types of heads.

The author derives expressions for stresses and displacements of a thin walled pressure vessel, following the "approximate theory" of Geckeler.

This paper develops methods for determining the proper profile of dished heads in the design of pressure vessels. The results of tests made on a number of vessels are also given.

The paper derives simplified formulas for calculating stresses in cone shaped heads. The proposed method seems to give more accurate results than those given by the differential equation based on Geckeler's "approximate theory".

The author presents an approximate method for computation of stresses at the junction of cone and cylinder of uniform thickness.

73. "Cylinder Shell Loaded Symmetrically with Respect to its Axis" by E. Wetterstrom, with an introduction by N. Little. Report to Pressure Vessel Research Committee, April 1, 1947.

An approximate method following the "approximate theory" of Geckeler is given for the conical head problem.


The author derives a simplified differential equation for cylindrical shells in the same form as that suggested by Geckeler, except that a different set of variables is used. An approximate method for substituting a cylinder for a cone is also given.


The author applies the method proposed in the previous paper to thin spherical shells.


The same general procedure proposed by the author for conical heads is applied to hemispherical dished heads.


Means are outlined for calculating internal pressure stresses and deformations of semi-infinite cylindrical vessel shells with hemispherical heads, ellipsoidal heads, torispherical heads, toriconical heads, conical heads, and flat heads. The classical analysis by Love is used in the paper.


In this report, the author presents the solution of ellipsoidal heads based on Love's theory.

The paper presents a solution for a torus of uniform thickness which is a part of the head knuckle of a pressure vessel.


The report gives a summary of the results of Love's equations applied to the cone-cylinder juncture.


In this paper, plasticity is applied to analysis of pressure vessels consisting of thin shells with rotational symmetry. Geckeler's approximation has been used as the basis of analysis.
APPENDIX II. CLASSICAL AND APPROXIMATE SOLUTIONS FOR A SPHERICAL-SEGMENT HEAD.

1. Classical Analysis

The governing differential equations of a spherical shell are given by Equations [15], in Chapter II as follows:

\[ L(A) + \mu A \rho^{-1} = UD^{-1} \]  \[ \text{[15a]} \]
\[ L(U) - \mu U \rho^{-1} = -EtA \]  \[ \text{[15b]} \]

in which

\[ L(x) = \rho^{-1} \left[ \frac{d^2x}{d\phi^2} + (\frac{dx}{d\phi})(\cot \phi) - x \cot^2 \phi \right] \]

On substituting the value of \( A \) given by the second of these equations into the first, a fourth order differential equation is obtained.

\[ LL(U) + \frac{a^4}{a^2 U} = 0 \]  \[ \text{[83]} \]

in which

\[ a^4 = Et U^{-1} - (\mu/\rho)^2. \]

This equation is satisfied by the solutions of either of the two second order equations,\(^{15}\)

\[ L(U) + ia^2 U = 0 \]  \[ \text{[84a]} \]
\[ L(U) - ia^2 U = 0 \]  \[ \text{[84b]} \]

in which \( i \) is an imaginary number denoting the quantity \( \sqrt{-1} \). Assuming that the two linearly independent solutions of Equation [84a] are

\[ U_1(\phi) = I_1 + iI_2, \quad U_2(\phi) = I_3 + iI_4, \]

then, the solutions of Equation [84b] are

\[ U_3(\phi) = I_1 - iI_2, \quad U_4(\phi) = I_3 - iI_4, \]

in which \( I_1 \) and \( I_2 \) are functions of \( \phi \).

Let us first consider Equation [84a] which can be written in the form:

\[ \frac{d^2U}{d\phi^2} + (\frac{dU}{d\phi})(\cot \phi) - U(\cot^2 \phi - 2ib^2) = 0 \]  \[ \text{[85]} \]

in which

\[ b^2 = \rho a^2/2 = \left[ 3(\rho/t)^2(1 - \mu^2) - \mu^2/4 \right]^{1/2}. \]

\(^{15}\) See Reference (61), p. 415 for derivation.
By introducing the new variables
\[ x = \sin^2 \phi \quad [86a] \]
\[ z = U(\phi \sin \phi)^{-1} \quad [86b] \]

Equation [85] is further simplified as follows:
\[ x(x - 1)(d^2z/dx^2) + (5x/2 - z)(dz/dx) - (1 - 2ib^2)z/4 \quad [87] \]

This equation belongs to a known type of differential equation of the second order, the solution of which can be obtained in the form of a hypergeometric series. The first and second integrals of Equation [87] are as follows:

\[ z_1 = c_1 \sum_{n=1}^{\infty} \frac{(\sin^2 - d^2)(\sin^2 - d^2) \ldots [(4n - 5)^2 - d^2]}{(n - 1)! n!} \frac{(x/16)^n}{n} \]

in which \( c_1 \) is an arbitrary constant of integration and \( d \) is a constant defined as:
\[ d = (5 + 2ib^2)^{1/2} \]

and
\[ z_2 = z_1 \log x + \phi(x) \]

in which \( \phi(x) \) is a power series.

The series of the first integral can be divided into real and imaginary parts, designated by \( S_1 \) and \( S_2 \) respectively. These are power series which are convergent for \(|x| < 1\). The term \( \phi(x) \) in the second integral also represents a power series which is convergent at \(|x| < 1\). Due to the fact that \( z_2 \to \infty \) for \( \phi = 0 \), this integral does not exist for the present example when there is no hole at the crown of the sphere. Therefore,
\[ x_1 = c_1 S_1 + ic_1 S_2 \]
\[ z_2 = 0 \]

The corresponding solutions of Equation [84a] are obtained by the

---

relation of \( z \) and \( U \) in Equation [86b].

\[
U_1 = c_1 I_1 + ic_1 I_2
\]

\[
U_2 = 0
\]

in which

\[
I_1 = S_1 \rho \sin \phi
\]

\[
I_2 = S_2 \rho \sin \phi.
\]

Similarly, the solutions of Equation [84b] can readily be expressed as follows:

\[
U_3 = c_2 I_1 - ic_2 I_2
\]

\[
U_4 = 0
\]

in which \( c_2 \) is another arbitrary constant of integration. Thus the general solution of Equation [83] can be expressed in the form:

\[
U = k_1 I_1 + k_2 I_2
\]  

for which

\[
k_1 = c_1 + c_2
\]

\[
k_2 = i(c_1 - c_2).
\]

The unknown \( A \) can be determined from the function \( U \) which is now known. By substituting \( U_1 \) into Equation [84a], we have

\[
L(c_1 I_1 + ic_1 I_2) = -ia^2 (c_1 I_1 + ic_1 I_2).
\]

Therefore

\[
L(I_1) = a^2 I_2, \quad L(I_2) = -a^2 I_1.
\]

From the second of the governing differential equations, we have:

\[
-\frac{EtA}{2} = L(U) + \mu U \rho^{-1}
\]

Thus, we obtain

\[
A = -((\mu \rho)^2[p a^2 (k_1 I_2 - k_2 I_1) + \mu (k_1 I_1 + k_2 I_2)]).
\]

By introducing the constants
the unknown \( A \) can be written in the form
\[
A = K_1 I_1 + K_2 I_2.
\]  

Denoting the first derivatives of \( I_1 \) and \( I_2 \) with respect to \( \phi \) by \( I_1' \) and \( I_2' \) respectively, the expressions of normal forces, moments, shear, meridional rotation, and deflection normal to the axis of revolution at any point on the spherical head are obtained from Equations \([20]\) in Chapter II.

\[
Q_m = \rho^{-1}(k_1 I_1 + k_2 I_2) \quad [91a]
\]
\[
N_m = F \rho/2 + \rho^{-1}(k_1 I_1 + k_2 I_2)(\cot \phi) \quad [91b]
\]
\[
N_c = F \rho/2 + \rho^{-1}(k_1 I_1' + k_2 I_2') \quad [91c]
\]
\[
M_m = D \rho^{-1}[(k_1 I_1' + k_2 I_2') + \mu(\cot \phi)(k_1 I_1 + k_2 I_2)] \quad [91d]
\]
\[
M_c = D \rho^{-1}[(\cot \phi)(k_1 I_1 + k_2 I_2) + \mu(k_1 I_1' + k_2 I_2')] \quad [91e]
\]
\[
\mathbf{A} = [K_1 I_1 + K_2 I_2] \quad [91f]
\]
\[
\bar{w} = \rho(\sin \phi)(E_t)^{-1}[(N_m)_n - \mu(N_c)_n] \quad [91g]
\]
These quantities can be computed after \( k_1, k_2, K_1 \) and \( K_2 \) are determined from the boundary conditions.

At the head-to-shell juncture, the conditions of static equilibrium and continuity must be met. The edge moment \( M \) and the edge shear \( Q \) are known from Equations \([91]\) for \( \phi = \phi_h \).

\[
M = D \rho^{-1}[(k_1 I_1' + k_2 I_2') + \mu(\cot \phi)(k_1 I_1 + k_2 I_2)] \quad [92a]
\]
\[
Q = \rho^{-1} [k_1 I_1 + k_2 I_2] \quad [92b]
\]
in which \( I_1 \) and \( I_2 \) are functions of \( \phi \) for particular values of
\( \phi = \dot{\phi}_h \) at the head-to-shell juncture; so are the derivatives \( I_1' \) and \( I_2' \). After the values of \( I_1, I_2, I_1' \) and \( I_2' \) are obtained for \( \phi = \dot{\phi}_h \), Equations \([89]\) and \([92]\) together will give \( k_1, k_2, K_1 \) and \( K_2 \) in terms of \( M \) and \( Q \).

Also, at the head-to-shell juncture, the meridional rotation and deflection normal to the axis of revolution are known to be as follows:

\[
\alpha = [K_1 I_1 + K_2 I_2] - \dot{\phi}_h \tag{93a}
\]
\[
\delta = \rho (\sin \dot{\phi}_h)(\dot{q}_t)^{-1} [ (k_1 I_1' + k_2 I_2') - \mu (\text{cot } \phi) (k_1 I_1' + k_2 I_2') \dot{\phi} = \dot{\phi}_h
\]
\[
+ P_0^2 (1 - \mu)(\sin \dot{\phi}_h)(2\dot{q}_t)^{-1}. \tag{93b}
\]

Since \( k_1, k_2, K_1 \) and \( K_2 \) are readily represented in terms of \( M \) and \( Q \), the quantities \( \alpha \) and \( \delta \) can also be expressed in terms of \( M \) and \( Q \), which are evaluated from the conditions of continuity at the juncture.

Even after the formal completion of the classical analysis of the governing differential equations, the process of obtaining numerical results useful for engineering purposes is not simple. The amount of work entailed in the classical solution depends to a great extent on the rapidity of convergence of the series \( I_1 \) and \( I_2 \) entering into the solution. This convergence becomes slower, and more terms of the series must be calculated, as the thickness of the shell becomes smaller in comparison with its least radius of curvature. For the example under consideration, twenty terms of the series have been required in order to obtain an accuracy comparable with that obtained by the numerical procedure.

The computation for the present specific case has been carried out with a precision unnecessary except for purposes of illustration. For the solution of the problem, the series \( I_1 \) and \( I_2 \) and their derivatives at the juncture \( (\phi = \dot{\phi}_h) \) are first to be determined. Then, from the solution of Equations \([89]\) and \([92]\), we obtain:

\[
k_1 = -0.292290 M - 0.017131 Q
\]
\[
k_2 = -0.320979 M + 0.441869 Q
\]
\[
K_1 = \mathbb{B}^{-1} (-17.205600 M + 23.091846 Q)
\]
\[
K_2 = \mathbb{B}^{-1} (-15.304414 M + 0.853111 Q).
\]
Substituting these values into Equations [93], we obtain the following for the spherical head:

\[
\alpha_h = D_h^{-1}(+ 1.522402 M - 1.090881 Q)
\]

\[
\delta_h = D_h^{-1}(- 0.545440 M + 0.778121 Q + 19.140625)
\]

where \( Q = R/2 - 303.1105 \).

From the solution of the cylindrical shell, as given by Equations [27] in Chapter III, we obtain

\[
\alpha_c = D_c^{-1}(- 1.267855 M - 0.803733 H)
\]

\[
\delta_c = D_c^{-1}(- 0.803733 M - 1.019017 H + 53.833008).
\]

The required quantities \( M, Q \) and \( H \) are obtained from the conditions of continuity at the head-to-shell juncture, i.e., \( \alpha_h = \alpha_c \) and \( \delta_h = \delta_c \). Then, from the solution of these equations,

\[
M = -128.2325 \text{ in.
\text{l bureaucr.}}
\]

\[
Q = + 283.9734 \text{ lb./in.}
\]

\[
H = -161.1238 \text{ lb./in.}
\]

After \( M \) and \( Q \) are evaluated, the quantities \( k_1, k_2, K_1 \) and \( K_2 \) can be obtained as follows:

\[
k_1 = + 40.241282
\]

\[
k_2 = - 30.035680
\]

\[
K_1 = - E^{-1} (1514.329288)
\]

\[
K_2 = - E^{-1} (2099.979387).
\]

Then, the normal forces, moments and shears at various points on the head are obtained from Equations [93].
2. **Approximate Method**

An approximate solution of the problem may be sought by neglecting some of the terms of the governing differential equations. This was first suggested by Geckeler whose original suggestion was later modified and extended to the solutions of many types of heads. The approximate solution presented here is not the original version of Geckeler's work, but rather represents a modified form, following the treatment of W. M. Coates. 17

This approximate solution is reasonably correct for comparatively thin shells. It is obtained by neglecting, in comparison with terms proportional to the higher order derivatives, some terms in the Love-Meissner equations proportional to the functions \( A \) and \( U \) and their first derivatives. By introducing a new independent variable \( s \) so that \( ds = \rho \, d\phi = \rho \, d\psi \), the governing differential equation of the Geckeler-Coates approximation becomes:

\[
\frac{d^4w}{ds^4} + 4\beta^2w = \rho \, D^{-1}
\]  

in which, for the spherical head, \( \beta \) is a constant.

\[
\beta = [3(1 - \frac{2}{3})]^{1/2} \left( pt \right)^{-1}
\]

On solving this differential equation and expressing the constants of integration in terms of the edge moment and edge shear, we have

\[
w = (2e^{3\beta^2})^{-1} \left[- \beta M(\cos \beta s - \sin \beta s) + Q \cos \beta s \right]
\]

\[
+ \frac{Pp^2}{2(1 - \mu)(2E)} (2Et)^{-1}
\]

in which \( Q = H \cos \psi - (Pp/4)(\sin 2\psi) \).

The relationships of normal forces, moments, shear, etc. with respect to the new independent variable are obtained as follows:

\[
Q_m = + D\frac{d^3w}{ds^3}
\]  

\[
N_m = + \frac{Pp}{2}
\]  

\[
N_c = + \frac{Pp}{2} - \rho D\frac{d^4w}{ds^4}
\]  

\[
M_m = - D\frac{d^2w}{ds^2}
\]

17. See Reference (69) for the derivation of the method.
\[ M_c = - \mu D (d^2 w / ds^2) \]  
\[ A = + \frac{dw}{ds} \]  
\[ \bar{w} = + w \cos \psi \]  

By substituting the expression for \( w \) in Equation [95] into the above equations, we have

\[ q_m = e^{-\beta s} [2BM \sin \beta s + Q (\cos \beta s - \sin \beta s)] \]  
\[ N_m = Pp/2 \]  
\[ N_c = Pp/2 \rho (2e^{\beta s} \beta^3)^{-1} [- 2BM (\cos \beta s - \sin \beta s) + Q \cos \beta s] \]  
\[ n_m = (2e^{\beta s}) (2BM \cos \beta s + Q (\cos \beta s + \sin \beta s)) \]  
\[ M_c = \mu m \]  
\[ A = (2e^{\beta s} \beta^2 D)^{-1} [2BM \cos \beta s + Q (\cos \beta s + \sin \beta s)] \]  
\[ \bar{w} = (2e^{\beta s} \beta^3 D)^{-1} [- 2BM (\cos \beta s - \sin \beta s) + Q \cos \beta s] (\cos \psi) \]  
\[ + Pp^2 (1 - \mu) (2Et)^{-1} (\cos \psi) \]  

At the head-to-shell juncture where \( s = 0 \), the quantities \( \alpha \) and \( \delta \) become:

\[ \alpha = A \bigg|_{s=0} = (2\beta^2 D)^{-1} (- 2BM + Q) \]  
\[ \delta = \bar{w} \bigg|_{s=0} = (2BD)^{-1} (- 2BM + Q) (\cos \psi) + Pp^2 (1 - \mu) (2Et)^{-1} (\cos \psi) \]  

The quantities \( M \) and \( Q \) are evaluated from the conditions of continuity at the juncture.

In the present numerical example, we obtain from Equations [98] for the spherical head

\[ \alpha_h = D_c^{-1} (1.463948 M - 1.071572 Q) \]  
\[ \delta_h = D_c^{-1} (- 0.535786 M + 0.784363 Q + 19.140625) \]
in which

\[ Q = \frac{H}{2} - 303.1105 \]

For the cylindrical shell, we obtain from Equations [27] in Chapter III that

\[ \alpha_c = D_c^{-1} \left( -1.267855 M - 0.803733 H \right) \]

\[ \delta_c = D_c^{-1} \left( -0.803733 M + 1.019017 H + 53.833008 \right) \]

The conditions of continuity at the juncture, i.e., \( \alpha_h = \alpha_c \) and \( \delta_h = \delta_c \), lead to the solution of \( M, Q \) and \( H \). Then,

\[ M = -130.2314 \text{ in.} \text{lb./in.} \]

\[ H = +286.3491 \text{ lb./in.} \]

\[ Q = -159.9360 \text{ lb./in.} \]

The normal forces, moments and shears at various points of the spherical head can be obtained by Equations [97].
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FPO, New York City, N. Y. (5)

Commanding Officer
Office of Naval Research
Branch Office
244 N. Rush Street
Chicago 11, Illinois (2)

Library of Congress
Washington 25, D. C.
Attn: Navy Research Section (2)

Over
Department of Defense
Other Interested Government Activities

GENERAL

Research and Development Board
Department of Defense
Pentagon Building
Washington 25, D. C.
Attn: Library (Code 3D-1075) (1)

Armed Forces Special Weapons Project
P.O. Box 2610
Washington, D. C.
Attn: Lt. Col. G. F. Blunda (1)

Joint Task Force 3
12 St. and Const. Ave., N.W. (Temp.U)
Washington 25, D. C.
Attn: Major B. D. Jones (1)

ARMY

Chief of Staff
Department of the Army
Research and Development Division
Washington 25, D. C.
Attn: Chief of Res. and Dev. (1)

Office of the Chief of Engineers
Assistant Chief for Public Works
Department of the Army
Bldg. T-7, Gravelly Point
Washington 25, D. C.
Attn: Structural Branch
(R. L. Bloor) (1)

Office of the Chief of Engineers
Asts. Chief for Military Construction
Department of the Army
Bldg. T-7, Gravelly Point
Washington 25, D. C.
Attn: Structures Branch
(H. F. Carey) (1)
: Protective Construction
Branch (I. C. Thorley) (1)

Office of the Chief of Engineers
Asts. Chief for Military Operations
Department of the Army
Bldg. T-7, Gravelly Point
Washington 25, D. C.
Attn: Structures Development
Branch (W. F. Woollard) (1)

Engineering Research and
Development Laboratory
Fort Belvoir, Virginia
Attn: Structures Branch (1)

The Commanding General
Sandia Base, P.O. Box 5100
Albuquerque, New Mexico
Attn: Col. Canterbury (1)

Operations Research Officer
Department of the Army
Ft. Lesley J. McNair
Washington 25, D. C.
Attn: Howard Bracket (1)

Office of Chief of Ordnance
Research and Development Service
Department of the Army
The Pentagon
Washington 25, D. C.
Attn: ORDTB (2)

Ballistic Research Laboratory
Aberdeen Proving Ground
Aberdeen, Maryland
Attn: Dr. C. W. Lampson (1)

Commanding Officer
Watertown Arsenal
Watertown, Massachusetts
Attn: Laboratory Division (1)

Commanding Officer
Frankford Arsenal
Attn: Laboratory Division (1)

Commanding Officer
Squier Signal Laboratory
Fort Monmouth, New Jersey
Attn: Components and
Materials Branch (1)
Other Interested Government Activities

NAVY

Chief of Bureau of Ships
Navy Department
Washington 25, D. C.
Attn: Director of Research
: Code 440 (1)
: Code 430 (1)
: Code 421 (1)

Director,
David Taylor Model Basin
Washington 7, D. C.
Attn: Structural Mechanics Div. (2)

Director,
Naval Engr. Experiment Station
Annapolis, Maryland (1)

Director,
Materials Laboratory
New York Naval Shipyard
Brooklyn 1, New York (1)

Chief of Bureau of Ordnance
Navy Department
Washington 25, D. C.
Attn: Ad-3, Technical Library
: Rec., P. R. Girouard (1)

Superintendent,
Naval Gun Factory
Washington 25, D. C. (1)

Naval Ordnance Laboratory
White Oak, Maryland
RFD 1, Silver Spring, Maryland
Attn: Mechanics Division (2)

Naval Ordnance Test Station
Inyokern, California
Attn: Scientific Officer (1)

Naval Ordnance Test Station
Underwater Ordnance Division
Pasadena, California
Attn: Structures Division
: Physics Division (1)

Chief of Bureau of Aeronautics
Navy Department
Washington 25, D. C.
Attn: TD-41, Technical Library
: DE-22, C. W. Hurley (1)
: DE-23, E. M. Ryan (1)

Naval Air Experimental Station
Naval Air Material Center
Naval Base
Philadelphia 12, Pa.
Attn: Head, Aeronautical Materials Laboratory (1)

Chief of Bureau of Yards and Docks
Navy Department
Washington 25, D. C.
Attn: Code P-314
: Code C-313 (1)

Officer in Charge
Naval Civil Engr. Research and Eval. Laboratory
Naval Station
Port Hueneme, California (1)

Superintendent
Post Graduate School
U. S. Naval Academy
Annapolis, Maryland (1)

AIR FORCES

Commanding General
U. S. Air Forces
The Pentagon
Washington 25, D. C.
Attn: Research and Development Division (1)

Commanding General
Air Materiel Command
Wright-Patterson Air Force Base
Dayton, Ohio
Attn: MCREX-B (E. H. Schwartz) (2)

Office of Air Research
Wright-Patterson Air Force Base
Dayton, Ohio
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OTHER GOVERNMENT AGENCIES

U. S. Atomic Energy Commission
Division of Research
Washington, D. C.

Argonne National Laboratory
P.O. Box 5207
Chicago 80, Illinois

Director,
National Bureau of Standards
Washington, D. C.
Attn: Dr. W. R. Ramberg

U. S. Coast Guard
1300 E Street, N.W.
Washington, D. C.
Attn: Chief, Testing and Development Division

Forest Products Laboratory
Madison, Wisconsin
Attn: L. J. Markwardt

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National Advisory Committee for Aeronautics
1724 F Street, N.W.
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National Advisory Committee for Aeronautics
Langley Field, Virginia
Attn: Mr. E. Lundquist

National Advisory Committee for Aeronautics
Cleveland Municipal Airport
Cleveland, Ohio
Attn: J. R. Collins, Jr.

U. S. Maritime Commission
Technical Bureau
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Library, Engineering Foundation
29 West 39th Street
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