SUMMARY OF INVESTIGATIONS IN NUMERICAL AND APPROXIMATE METHODS OF STRESS ANALYSIS

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Approved by
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CHAPTER I

INTRODUCTION

This report is in fulfillment of the requirements of Contract N6ori-07l(06), Task Order VI, Project Designation Numbers NR-055-l83 and NR-064-183, between the Mechanics Branch of the Office of Naval Research and the University of Illinois. The purpose of this contract was to conduct research in the field of numerical and approximate methods of analysis of structural and machine elements and in related fields of mechanics.

This research program included studies of the range of applicability and the relative merits of available numerical and approximate methods of stress analysis, the development of new and more efficient numerical methods, the preparation of programs for use on the ILLIAC, the high speed digital computer of the University of Illinois, the solution of important practical problems of stress analysis, instability, vibration, impact, etc., and the publication of the results of these studies. In general, two different types of numerical procedures have been considered. The first concerns procedures which are rapid and reasonably accurate for preliminary design. Attention was devoted in this aspect of the work to procedures which do not require complicated computing equipment. These procedures are capable of application with only a slide rule or at most an ordinary desk computing machine. The second type of procedure that has been investigated is intended primarily for research, to give solutions of any desired accuracy for problems for which classical methods of analysis are difficult to apply. This type of procedure is especially well suited for use with high speed electronic computing equipment.
The contract for this program was initiated on June 1, 1946 and terminated on November 30, 1955. Since the termination date research on numerical and approximate methods of analysis has been continued under Contract Nonr-1834(03), Task Order 3, Project Designation NR-064-183.

Quarterly status reports have been issued regularly since December 1948. These reports contain a brief description of the progress of the work in the various phases of the project, plans for future investigations, and relevant administrative details such as changes in personnel.

The investigations of this program were carried on by men working for their M.S. and Ph.D. degrees. A list of thirty-five theses prepared under the sponsorship of this project is given in Table I. Of these, twenty-six were for the Ph.D. degree and nine were for the M.S. degree. In addition to the students whose names appear in Table I, there have been a number of graduate students who worked for a time on Task VI but are not listed in this report because they were transferred to other projects where they obtained their degrees. Also there have been a number of undergraduate students who worked as assistants but whose names are not included herein.

Technical reports have been issued when a specific phase of the work was completed or when important results meriting wide distribution were obtained. Twenty-six technical reports have been issued on this project. A listing of these technical reports is given in Table II. Also included in Table II are two reports which are of the nature of technical reports but were issued before the system of technical reports was begun.
Twenty-one formal publications, including nineteen papers and two discussions, have resulted to date from the investigations of this program. The titles, references to the journal, etc., are given in Table III.

The studies in this program have been conducted under the supervision of Dr. N. M. Newmark with the assistance of Drs. W. J. Austin, L. E. Goodman, T. P. Tung, and A. S. Veletsos.

The writers appreciate greatly the enlightened attitude of the sponsors in giving complete academic freedom in this study of fundamental problems in numerical methods applied to engineering. The studies have not been dictated by external demands nor restricted by time limitations. It has been possible to make use of many graduate students thereby increasing greatly their advanced educational training and at the same time enabling the students to earn part of their expenses. The aim of the writers to strengthen the fundamental scientific research program in structural engineering has been made possible by the support of the Office of Naval Research. The influence of this support has been felt on the whole field of structural mechanics.

The purpose of this report is to summarize the major investigations conducted under this research contract. This work is presented in the following four chapters. Chapter II is concerned with investigations in stress analysis, Chapter III with studies of buckling and vibration, Chapter IV with studies of the response of structures under transient loading, and Chapter V with miscellaneous investigations of a general mathematical nature. A short summary is given of each investigation
including the object of the study, the methods employed, the nature of the results, and any important conclusions.
CHAPTER II
INVESTIGATIONS IN STRESS ANALYSIS

In this chapter are described the investigations conducted in the field of stress analysis. Included are investigations in numerical analysis of problems of plane stress, radially symmetric stress, generalized plane stress, torsion, curved beams, beams on elastic supports, plates, shells, and domes. Much study has been expended in this field and this effort has resulted in sixteen advanced degree theses.

1. Plane Stress

As is well known, the solution of any plane stress problem must satisfy the equations of equilibrium and compatibility for all interior points of the structure considered and also the specified conditions at the boundaries. Available methods of analysis may be classified from a physical point of view as follows:

(1) Equilibrium procedures, in which the conditions of equilibrium are expressed in terms of unknowns which automatically satisfy the compatibility requirements, and

(2) Compatibility procedures, in which the conditions of compatibility are expressed in terms of unknowns which automatically satisfy the conditions of equilibrium.

In either case, the values of the unknowns along the boundary are chosen so as to comply with the specified boundary conditions.
In problems of plane stress, the choice of displacements to express the equations of equilibrium and the boundary conditions is an example of the use of the equilibrium procedure. Any set of displacements automatically satisfies the compatibility requirements. An illustration of the application of the compatibility procedure to the same class of problems is provided by the use of the Airy stress function. In this case the biharmonic equation expresses the conditions of compatibility in terms of a stress function which automatically satisfies the equations of equilibrium.

Numerical solutions may be made in both the equilibrium and the compatibility procedures. Numerical methods fall into two general categories:

1. Framework analogy methods or physical analogies in which the solid material is replaced by a system of bars, usually pin-connected, the proportions of which are chosen so as to represent the physical behavior of the solid as closely as possible;

2. Finite difference methods in which approximate solutions to the appropriate differential equations are obtained.

Four numerical procedures of the framework analogy type were studied. Two frameworks were considered, a hexagonal framework and a square lattice network. The hexagonal framework consists of pin-connected bars formed into equilateral triangles. The six triangular elements having vertices at a joint form a hexagon, which represents the basic unit considered in the analysis. The square lattice framework consists of pin-connected bars formed into squares with crossing diagonal bars connecting
opposite corners. Both equilibrium and compatibility procedures were studied for the solution of each type of framework. In the equilibrium procedure the two equations of equilibrium for each joint are expressed in terms of the displacement of that joint and the displacements of the adjacent joints. The two components of displacement at each joint are found by the solution of a set of linear algebraic equations. In the compatibility procedure a set of statically balanced forces are assumed in the bars of the framework. These stresses must, of course, be consistent with the boundary stresses and any internal body forces. In general these assumed stresses give elongations of the bars which do not satisfy continuity requirements; that is, the bars cannot fit together with such deformations. By introducing statically self-balanced sets of stresses in the bars the continuity requirements can be satisfied without disturbing the equilibrium of the joints. Standard correction patterns are worked out for each framework and the required values may be found by the solution of simultaneous algebraic equations.

This study was conducted by E. L. Dauphin and reported in his M.S. thesis (Item II of Table I). Both framework analogies were found to be practical methods for plane stress analysis. The compatibility procedure was found to possess the following advantages over the equilibrium procedure for the solution of the bar stresses in the frameworks. The compatibility equations are simpler and less changeable near boundaries than are the equilibrium equations. This feature is very helpful if relaxation or iterative procedures of solution are used. Another great advantage of the compatibility procedures is that they result in
fewer simultaneous equations. For hexagonal networks less than one half as many simultaneous equations are needed for a compatibility solution as are needed for an equilibrium solution, since the compatibility solution requires only one equation for each interior joint and no equations for boundary joints. Body forces do not affect the compatibility equations but add an extra term in the equilibrium equations. In the compatibility procedure, when the solutions of the simultaneous equations are not exact, as is often the case for iteration or relaxation solutions, one ends up with a statically balanced set of bar stresses with small discrepancies in continuity. This result seems preferable to ending up with a set of statically unbalanced stresses, which is the case with the equilibrium procedure. On the other hand, the compatibility procedures have the following two disadvantages: (1) they cannot be used when displacements are specified at the boundary and (2) the physical significance of the correction quantities in the compatibility computations is not as apparent as that in the equilibrium procedure.

Further studies of numerical procedures for plane stress were conducted by Z. K. Lee and reported in his M.S. thesis (Item 18 of Table I). In this investigation the use of finite differences in solving for the Airy stress function was studied and this procedure was compared with the square lattice framework method. Several aspects of the Airy function problem were considered. In addition to the derivation of a basic procedure based on first order differences, a structural analogy consisting of a system of bars was devised which leads to identically the same equations as the finite difference approximation to Airy's function. This
analogy is helpful in understanding the nature of the approximations involved in the finite difference procedure. In addition, the use of higher order differences was investigated for a few cases and was found to result in a considerable increase in accuracy. Finally, as a special case, a procedure for plates with holes was derived. An important aspect of this investigation was concerned with a comparison between the Airy function procedure and the square lattice framework method. It was found that for the same size network both methods give about the same accuracy. However, the framework analogy method leads to about twice as many equations as the Airy stress function method. Mr. Lee concludes that between these two methods the choice is likely to be in favor of the Airy function method in most cases.

The fundamental ideas of these studies were discussed by Dr. N. M. Newmark in a paper entitled, "Numerical Methods of Analysis of Bars, Plates, and Elastic Bodies" presented at a symposium held at Illinois Institute of Technology and published in a book entitled, "Numerical Methods of Analysis in Engineering" (Item 2 of Table III).

2. Radially Symmetric Stress Problems

The object of this study was to develop satisfactory approximate numerical methods for the solution of radially symmetric stress problems. Two general methods were considered. One is the lattice analogy method in which it is imagined that the solid body can be replaced by a framework of bars. The second scheme involves the well known procedure of expressing a differential equation in its finite difference
form. In both methods the displacements at a discrete number of points are considered, and a solution is obtained by a process of iteration.

In the lattice analogy method it is assumed that the behavior of a solid radially symmetric body under a radially symmetric load can be approximated by that of a framework of bars subjected to the same load. The area of each bar is determined from a consideration of the deformation of an element of the solid body and the corresponding element of the framework, when both are subjected to certain simple force systems. Equations of equilibrium are written at each joint in terms of the displacements at the joints. Therefore, two equations are obtained at each nodal point, and the analysis is made by the solution of a set of linear, algebraic equations. From the components of displacements the forces in the bars of the framework are computed. From the forces in the bars of the framework one can obtain an approximation to the actual stress distribution in the solid body.

The lattice analogy method was found suitable for the solution of mixed boundary value problems. However, the method appears to be limited, due to time consuming computational work, to bodies whose geometric shape is formed of straight radial and axial lines. Curved surfaces in the framework are represented by a series of steps which follow in an approximate manner the outline of the boundary. Problems solved with such irregular boundaries in the analogy method showed high stress concentrations at the re-entrant corners. The solutions in these cases were not representative of the actual curved boundary problem. Solutions with finer subdivisions in the region of the curved surface were attempted,
but the results did not appear to justify the lengthy computational work involved.

In an effort to obtain a more suitable method of defining a curved surface by a discrete number of points and of simplifying the solution of curved boundary problems to a point where a good approximation could be obtained with a coarse net in a short period of time, the lattice analogy method was replaced by the method of finite differences.

In the finite difference method the differential equations of equilibrium in terms of displacements were approximated by finite differences. Several coordinate systems were used. The use of cylindrical coordinates gave good results for rectangular boundaries. However, curved boundary surfaces cannot be suitably treated with a system of cylindrical coordinates because it is difficult to define the boundary accurately with a discrete number of network points. An attempt was made to find the stresses around a notch, but accurate answers could not be obtained at the root of the notch where the stress gradient is steep.

To obtain a better solution for the notched bar problem an orthogonal curvilinear system of coordinates was adopted. Such a system has certain inherent properties which make it desirable for the solution of curved boundary problems. No difficulties arise in the delineation of a curved surface by a discrete number of points or in the determination of the finite difference expressions for the boundary conditions in terms of stress that apply to the curved surface.

The particular coordinate system used was the ellipsoidal. The network of intersecting traces of hyperboloids and ellipsoids on an axial
plane yielded a pattern which had the desirable feature of the finer and denser subdivisions occurring in the region of high stress concentration at the root of the notch. Good results were obtained with the curvilinear coordinate solution.

Finally, a method was developed for solving inelastic radially symmetric stress problems. The yield and flow condition used was assumed to be a function only of the second invariant of the stress deviation. To define the start of plastic yielding or to define the flow condition for perfectly plastic materials the Huber-Mises-Hencky condition was used. It was assumed that the material was under a state of gradually increasing load, and the stress analysis was made for a particular stage in the loading process. It was also assumed that: (a) the axes of principal stress and strain coincide, (b) the deviations of stress and strain are proportional, and (c) the intensity of stress is a completely determined function of the intensity of strain for all complex states of stress.

The basic equations that arise from these fundamental relations and the equations of statics are linear in form when expressed in terms of displacement components. This characteristic of linearity of the differential equations is a desirable feature as it permits one to obtain a solution of an inelastic problem by solving a sequence of elastic problems through a process of successive corrections.

This work is reported in a Ph.D. dissertation by Elio D'Appolonia (Item 10 of Table I). Part of this work was published in an article by D'Appolonia and Newmark, (Item 7 of Table III) and this paper was reprinted and issued in combination with two other papers published at the same time.
3. Generalized Plane Stress

The two dimensional solution of plane problems in the theory of elasticity is a mathematical idealization. It only gives reasonable stresses either in a plate which is very thin or in the neighborhood of the mid-plane of a very thick plate. For a plate with moderate thickness neither the plane stress nor plain strain solution furnishes adequate results.

In an investigation conducted under this program, a numerical procedure was developed for the three dimensional analysis of a plate. The method of finite differences was used to obtain a correction to the existing two dimensional solution, such that the complete solution satisfied all the equations of equilibrium and compatibility as well as the prescribed boundary conditions. The solution was made using displacements as unknowns.

The numerical procedure was applied to the analysis of an infinite plate with a circular hole under uniaxial tension, where the diameter of the hole is equal to the thickness of the plate. Approximate solutions to this problem had been published by Green and by Sadowsky and Sternberg, and these solutions were compared with the numerical results of this study. The problem was solved numerically with a three dimensional rectangular network, applicable to any shape of hole, and also with a two dimensional rectangular network applicable only to this problem in which
the variation of displacements, stresses, etc., about the hole is known. Different size networks and higher order differences were used and a study of the accuracy of the various possibilities was made.

Although this study was restricted to an elastic, isotropic material, it appears that it can be readily generalized to account for conditions such as inelastic deformation and non-isotropy.

This investigation was conducted by T. P. Tung and reported in his Ph.D. dissertation (Item 29 of Table I). Dr. Tung concludes that the method of finite differences is an adequate method for solving problems which otherwise would be very tedious by classical methods. Reasonable results may be obtained without introducing very fine networks. It is suggested that in general improved results might be obtained if the derivatives in the boundary equations are approximated by finite differences including higher order terms than are needed for the equations of the interior points.

4. Torsion

The purpose of this investigation was to develop an approximate numerical procedure for the solution of Laplace's equation and Poisson's equation, with special emphasis on the stress function equation for torsion. The procedure which was developed depends upon replacing the differential equation with an equivalent difference equation. The equation is expressed in terms of the differences between ordinates to a surface rather than in terms of the ordinates themselves, which is the usual procedure. For any particular problem this procedure results essentially
in a set of simultaneous, linear, algebraic equations. The equations are never actually written but the solution is made on a simple diagram by a process of iteration or relaxation. The solution is presented as a flow analogy in a network of pipes. An important aspect of this investigation was the development of a convenient procedure for the problem of torsion of a bar of constant cross-section having one or more internal longitudinal holes.

Throughout this study an effort was made to present the procedures in engineering terms so that they would be readily understood and used in practice.

The findings of this investigation were presented in an M.S. thesis by E. C. Colin (Item 9 of Table I), in two technical reports by Colin and Newmark (Items 27 and 28 of Table II), and in a published paper by Colin and Newmark (Item 1 of Table III). The procedures developed in this study are also described in a general paper on numerical methods of stress analysis by N. M. Newmark (Item 2 of Table III).

5. Curved Beams

This study was concerned with three aspects of the stress analysis of curved beams. First, a theory of flexure of curved beams was developed, by which the normal stresses in curved beams due to direct thrust and due to bending about two axes, one radial and the other perpendicular to the plane of curvature, may be readily calculated. This method employs the concept of the transformed section, as originally suggested by Hardy Cross. The method is based upon a generalization of the
assumptions of the Winkler-Bach theory for curved beams. With this procedure the computations are almost identical to the corresponding ones for straight beams. Stresses in curved beams due to shears and twisting moments were not considered in this investigation.

A second aspect of the study was the determination of the shear center for unsymmetrical thin-walled open cross-sections of curved beams. Formulas were derived for the shear flow in a thin-walled open cross-section and for the location of the shear center for certain types of channel sections and other cross-sections.

Finally, the transformed section method was applied to the problem of secondary bending of flanges of thin-walled symmetrical I and T sections. A different method for dealing with this problem had been presented previously by H. H. Bleich.

This study was made by H. K. Yuan and reported in his Ph.D. dissertation (Item 35 of Table I).

6. Beams on Elastic Supports

Two investigations were conducted on the development of numerical methods for the analysis of beams on elastic supports. The first study was concerned with beams resting on an elastic foundation of the Winkler type, whereas the second study was concerned with beams resting on an elastic continuum.

In a foundation of the Winkler type, the reaction provided by the foundation on the beam at any point is proportional to the deflection of the beam at that point. Two different numerical procedures were investigated for the analysis of beams resting on such a foundation. The most
versatile procedure was found to be a combination of the Stodola successive approximations method with Newmark's numerical integration method for finding deflections of beams. With this procedure one may consider in a straightforward manner any desired variations in loading, stiffness of beam, and modulus of foundation. Also one may conveniently take into account the effect of spring reactions from supporting members, and of non-linear or plastic behavior of the beam and/or supporting medium. Better accuracy is obtained with this procedure than can be obtained by most other numerical methods, for the same number of divisions in the length of the beam.

The success of this method depends entirely on the rate of convergence of the successive approximations to the desired solution. When rapid convergence is attained the method is superior to other numerical procedures. On the other hand, when the successive approximations diverge badly a solution can be obtained by this method only with great difficulty. Because of the importance of this feature, much time was spent in a study of convergence and in attempts to devise schemes to force convergence for naturally divergent problems. It was found that the rate of convergence or divergence depends upon a single dimensionless parameter, which is the ratio of the foundation modulus to the stiffness of the beam. The range of this parameter for which the successive cycles converge has been clearly delineated and, based upon the theoretical results, a method was developed for obtaining a good initial approximation.

An alternate method of analysis was developed for problems which are so divergent that the successive approximations procedure is not feasible. This procedure makes use of a step-by-step solution of the equations
in the manner analogous to that of an initial value problem. This pro-
cedure does not involve difficulties of convergence, but is restricted
to linear problems.

This investigation was reported in an M.S. thesis by F. W. Schutz (Item 25 of Table I).

In the second study a numerical procedure was developed for
the analysis of beams resting on a semi-infinite elastic-solid medium.
For this problem the force exerted by the supporting material at any
point depends not only on the deflection at that point but also on the
deflection at every other point. When the subgrade is treated as a
true elastic continuum characterized by the modulus of elasticity and
Poisson's ratio, the problem is governed by an integro-differential
equation of which very few exact solutions are known.

In this investigation the beam is treated as an elastic strip
following the elementary beam theory. The subgrade is considered either
as a plane plate or as a three-dimensional elastic body. Two procedures
were developed. The first approach uses the method of successive approx-
imations in conjunction with Newmark's numerical integration procedure.
This procedure is very similar to that described by Schutz in the first
investigation, except for the determination of the subgrade reactions.
An alternate procedure utilizing a solution of simultaneous equations is
proposed for beams of uniform section, and especially for symmetrically
loaded beams resting on a very stiff subgrade. In this alternate method
influence coefficients are used.
The key to these methods is the determination of the subgrade reaction induced by a given set of deflections. This is found by approximating the actual subgrade reaction by block loadings. The deflection at each of a series of node points along the surface of the semi-infinite elastic solid due to block loads at those points is found by the theory of elasticity. These deflections form a matrix which may be inverted to give the forces on the subgrade required to produce the given set of deflections. This method was found to be accurate enough for practical purposes.

The method developed is quite general. It has been applied to the problem of buckling of beams on elastic-solid subgrades, and can easily be extended for solving problems of rectangular plates supported on an elastic solid. In the latter problem the ordinary plate theory is assumed to be valid and the subgrade is treated as a three dimensional elastic body. The plate is divided into a convenient network, with the node point considered to be the center of each element. The calculus of finite differences may be used to express the differential equation of the plate. The subgrade reaction distribution is approximated by block loads over each element. The deflection at the node point due to the block loads on the elements is found by the theory of elasticity. These deflections form a matrix which is inverted to give reactions in terms of deflections. The remaining steps are the same as those for the beam-subgrade problem.

This investigation was conducted by D. H. Lee and reported in his Ph.D. dissertation (Item 17 of Table I).
7. **Flexure of Plates**

Two investigations were conducted to determine the state of stress and deflection in medium-thick elastic plates supported by flexible beams. The specific problems considered are discussed in the following sections.

(a) **Analysis of Plates Continuous Over Flexible Beams**

This study was concerned with the analysis of an interior panel of a plate continuous over a rectangular grid of flexible beams which are supported at their intersections by columns. It is considered that the plate is uniformly loaded over its entire area, that it has a large number of panels in both directions, and that parallel beams are of equal stiffness and uniform spacing. Under these conditions, the distribution of moments in all interior panels may be assumed to be identical and only one interior panel need be considered. In this investigation the ordinary theory of medium thick plates was used. In addition, it was assumed that the widths of the beams and the cross-sectional dimensions of the columns are small compared with the panel dimensions. While this latter simplification has little effect on the deflections and moments near the center of the panel, it does, in some cases, lead to excessively large bending moments near the columns.

The system was analyzed by means of the Rayleigh-Ritz energy procedure, using a set of polynomial functions due to W. J. Duncan. The second derivatives of these functions are normalized Legendre polynomials, and the first function of the set is proportional to the deflection of a
uniformly loaded fixed-ended beam. The properties of these functions were found to be very appropriate for the present problems.

Numerical calculations have been made for panels with side ratios equal to 0.5, 0.8, and 1.0, and various ratios of beam rigidities. For the square panels, the value of the ratio of rigidities was varied from zero to infinity. For the rectangular panels, it was assumed either that all beams are of equal rigidity or that all beams have equal moments of inertia. For each structure considered, numerical values were computed for deflections at three locations (midpoint of the panel, and midpoint of the beam in each direction), for longitudinal and transverse moments in the plate at four locations (midpoint of the panel, corner of the panel, and midpoint of the beam in each direction), for the average moments in the column and the middle strips at the center lines of the panel and over the beams in each direction, and for the moments in the beams at midspan and at the columns. The values given are for Poisson's ratio equal to zero; however, they can be easily converted to any desired value of Poisson's ratio. The numerical data obtained have been presented in both tabular and graphical forms.

In general, the results presented are those obtained using nine terms of the series for the deflection function. Thus each case involves the solution of a set of nine simultaneous equations. However, in order to test the accuracy of the results obtained, two additional solutions were obtained for each case considered in the present work, using only one and four terms, respectively, in the approximating series of the deflection function. By inspecting the rate of convergence of a sequence
of such solutions, some indication of the accuracy of the results may be obtained. It has been found that the convergence of the sequence of solutions is very good for the clamped plate, and it becomes even better as the ratio of beam rigidity to plate rigidity approaches unity. As this ratio is decreased, the convergence becomes poorer. The solutions show the poorest convergence for the plate supported on columns only. Also the convergence is better for square or nearly square panels than for long and narrow panels.

Comparisons were also made between the energy solutions and known analytic solutions for the two limiting cases of a plate clamped against both rotation and deflection along its edges, and a plate supported by a rectangular array of columns without connecting beams. The agreement of the results has been found to be fairly good. Comparisons were also made with solutions obtained by means of finite differences. It was found that the energy solutions obtained with only four terms in the series (four simultaneous equations) are generally more accurate than the finite difference solutions obtained with 24 simultaneous equations.

The solutions obtained may be applied in the analysis of practical structures such as two-way concrete floor slabs, provided that proper account is taken of the dissimilarity between the actual conditions and the idealized conditions assumed in the present study. These solutions, by appropriate superpositions with other known solutions, may be used also to find the influence of loads which are not uniformly distributed over the entire area of the floor slab.
This study has been described in a Ph.D. dissertation by J. G. Sutherland (Item 27 of Table I) and in a technical report by Sutherland, Goodman, and Newmark (Item 11 of Table II). An exact solution found in this study was published by Sutherland (Item 9 of Table III).

(b) Skew Slab-and-Girder Floor Systems

This investigation was concerned with the behavior of simple-span skew slabs supported on flexible girders. The girders were considered to be identical, uniformly spaced, parallel to one pair of sides of the slab, and simply supported at their ends. No analytical solutions for this type of structure had been available. In this study, the numerical method of finite differences was used.

Difference equations were derived for a general system of skew coordinates to permit the analyses of the structures for any angle of skew, \( \varphi \), ratio of girder spacing to span, \( b/a \), and relative stiffness of the girders and slab, \( H \). The equations derived are applicable to structures having any number of girders.

As an application of the equations to a practical problem, a special type of structure often used in highway bridge construction, the skew I-beam bridge, was analyzed. A total of 18 skew bridges, each having five girders, were considered. The physical characteristics of the bridges were defined by combinations of the following variables: \( \varphi = 30, 45, \) and \( 60 \) deg.; \( H = 2, 5 \) and 10 for \( b/a = 0.1 \); and \( H = 1, 2, \) and 5 for \( b/a = 0.2 \). With the use of difference equations, influence values for moments in the slab and girders at various locations produced by a moving unit concentrated load were determined. The solution of simultaneous equations was
carried out by the ILLIAC. Numerical data were presented in tabular as well as graphical form.

Maximum live load moments at mid-span of girders produced by standard highway truck loadings were then computed for 72 structures having beam spacings of 5, 6, 7, and 8 ft. for different values of $b/a$, $H$, and $\varphi$. Based on these results, empirical relationships were developed for estimating the maximum live load moments at mid-span of any skew bridge having dimensions within the ranges of the variables considered.

The network of points used in the numerical calculations was formed by two sets of parallel lines. The first set of lines drawn parallel to the abutments divides the length of the span into eight equal spaces. The second set of lines drawn parallel to the girders divides each slab panel into two equal segments. To test the accuracy of the computed data using this network of points, analyses were made for the right I-beam bridge ($\varphi = 0$), for which known exact solutions are available for comparison. Comparisons of the exact and difference solutions showed that the method of finite differences is very satisfactory for the determination of girder moments; even for a coarse network of points as used in this study, the agreement between the exact and difference solutions is quite good. However, rather serious errors are to be expected in the difference solutions for slab moments in the floor system for certain load positions, unless a much finer network than used in this investigation is chosen. For the structures considered, it was found necessary to apply certain corrections to some of
the influence values in order to make up for the coarseness of the network.

This investigation was made by T. Y. Chen and is reported in his doctoral dissertation (Item 8 of Table I). This study was in part supported by a different project. The results of the investigation have been prepared for publication as a University of Illinois Engineering Experiment Station Bulletin which is now in press.

8. **Stress Analysis of Stiffened Shell Structures**

This investigation was concerned with the stress analysis of stiffened shell type structures, such as are used in aircraft construction. A simple numerical procedure was developed which requires but a single viewpoint to handle the analysis of stiffened shells subjected to both bending and torsion. This method is based upon the same simplifying assumptions as are found in the previously developed analyses in this field, but the procedure described herein is more general. For example, the procedure may be readily applied to the analysis of shells with unsymmetrical cross-sections and to multi-celled box structures.

In essence the analysis requires the solution of a set of simultaneous linear equations in which the unknowns are the displacements at a discrete number of points in the structure. These equations are never actually written out but instead are solved indirectly by an iterative procedure, working directly on a drawing of the structure. Because the convergence of a straight iterative procedure is very slow for this problem, a special method was developed for making a good
initial estimate of the displacements and also a special group correction
pattern was derived for occasional use to make large adjustments. The
iterative procedure is set up so that advantage is taken of the process
of continuous multiplication on an automatic calculating machine.

This investigation is reported in a Ph.D. dissertation by J. E.
Duberg (Item 12 of Table I).

9. Analysis of Shells

Five investigations have been conducted with the object of
developing convenient numerical procedures for the analysis of various
shell problems. Two studies have been made on shells of revolution
loaded symmetrically, one study on the analysis of cylindrical roof
shells, one study on hipped plate structures, and one study of the
stresses caused by initial irregularities in tubes subjected to external
pressures. These investigations are discussed in the order named above.

(a) Pressure Vessel Heads

A numerical procedure for the analysis of pressure vessel heads
was developed by T. Au. Of course, the pressure vessel head is a shell
of revolution loaded with uniform internal pressure. The procedure dis-
cussed herein is based on the general theory of thin elastic shells due
to A. E. H. Love. The method is approximate in the sense that the govern-
ing mathematical equations are satisfied only in finite difference form.
However, it has the merit of simplicity and of directly providing numeri-
cal values of normal forces, moments, and shears needed in design. The
appropriate finite difference equations were first developed and then
applied to the particular cases of pressure vessels having spherical
heads, conical heads, torispherical heads, toriconical heads, hemispherical heads, and flat circular heads. For the test problems considered in this study, close agreement was obtained between the results found by "classical" analysis and those obtained by the numerical method.

This investigation is reported in a Ph.D. thesis (Item 2 of Table I) and in a technical report by Au, Goodman, and Newmark (Item 2 of Table II).

(b) Domes Under Symmetrical Loading

Another study of the analysis of domes having the form of a surface of revolution was made by R. Schmidt. One phase of this investigation was concerned with studies of the application of the method of finite differences and a second phase was concerned with the development of a procedure referred to as the method of finite elements. This method is based upon considerations of the equilibrium and distortion of finite size elements. The simplifying assumptions made in this procedure are the same as those underlying A. E. H. Love's theory of shells.

The common feature of the methods used in this study is that they reduce the problem of analysis of domes to the solution of a system of simultaneous linear equations. Since, in order to achieve good accuracy, a large number of equations is required, the use of high-speed electronic computers is indispensable for the majority of problems. However, with the rapid development of electronic computing, this is not considered to be a great handicap.

In this study primary emphasis was placed on the elastic analysis of domes of such proportions that the resulting deflections may be
considered to be small in comparison with the thickness of the dome. However, in addition, a procedure was suggested which makes it possible to analyze domes exhibiting moderately large deflections.

This study is reported in a doctoral dissertation (Item 24 of Table I).

(c) Cylindrical Shell Roofs

Cylindrical shell roof structures, often called barrel vaults, are of great importance in civil engineering construction. A numerical method for the analysis of such structures was developed by W. S. Schnobrich and reported in his M.S. thesis (Item 23 of Table I).

In this method the shell is replaced by an analogous framework which has a finite number of degrees of indeterminacy. A distributed load on the shell is transformed into equivalent concentrated forces acting on the joints of the framework. The analogous framework is analyzed by solving a set of simultaneous, linear algebraic equations involving the displacements of the joints as unknowns. From the displacements the forces in the framework may be found readily. If the framework is suitably chosen, its behavior will closely approximate the behavior of the shell. The displacements of the original shell will be approximately equal to the displacements of the analogous framework at corresponding points, and approximate values of the direct stresses and moments in the original shell may be found from the forces and moments in the bars of the framework.

It is possible to approximate a cylindrical shell with many different types of framework arrangements. The framework arrangement developed in this study consists of a system of rigid bars arranged in squares,
joined at their intersections by elastic joints, and internally linked by torsion springs.

The value of the bar-spring system, like that of all numerical procedures, lies in its complete versatility and adaptability. However, the solution of a large number of simultaneous equations may be required to obtain good accuracy.

(d) Hipped Plates

In this investigation a moment distribution type analysis was developed for structures composed of rectangular flat plate elements connected side by side along their longitudinal edges and simply supported on opposite edges. The method of analysis considers the flexural action of the plate under normal components of loading, the plane stress action of the plates under planar loads, and the effects due to joint displacements. The exterior longitudinal edges of the structure may be free, simply supported, fixed, or elastically restrained.

The fundamental basis of this procedure is as follows. The load is expanded into a sine series and each term is considered separately. Due to each loading term it is assumed that all plates in the structure deform in a sinusoidal shape in the longitudinal direction. In a manner similar to Newmark's procedure for rectangular slabs continuous over intermediate supports, the moments and direct stresses are found by a moment distribution type, relaxation procedure. The procedure is not exact but is based upon certain assumptions, especially with regard to the distribution of the direct stresses in the structure.
This study is described in a Ph.D. thesis by G. S. Wu (Item 32 of Table I).

(e) Stresses Caused by Initial Irregularities

In a thin circular cylindrical shell subjected to uniform external pressure, only direct stresses are present in the center portion of the shell. However, bending moments and corresponding stresses are introduced if the cross-section deviates from the circular shape due to defects of manufacture, imperfect fabrication, improper handling, or the action of external forces or shock. The purpose of this investigation was twofold: (a) to develop a method of analysis of the stresses caused by an initial irregularity which is uniform along the length of the tube, and (b) to determine the stresses in a cylindrical shell with a dent which varies in magnitude along the longitudinal axis of the shell.

When the initial irregularity in a long tube is uniform along its length, the moments, stresses, and deformations are also uniform and, therefore, the stresses may be found from the solution of an analogous two-dimensional ring. A numerical method of analysis was developed for the determination of the stresses and deformations in rings with initial imperfections and uniform external load. This procedure is a combination of the Stodola method of successive approximations and the numerical integration method of Newmark. In this procedure both large deflections and extensions of the center line of the ring are taken into consideration. Due to the fact that the procedure involves a considerable amount of numerical computation, a simplified method for predicting the maximum bending moment developed in a ring subjected to uniform pressure was developed.
The simplified method is approximate, but it is direct. The results obtained by these two methods check extremely well.

The second objective achieved in this investigation was the determination of the stresses in a thin cylindrical shell with a dent varying in magnitude along the longitudinal axis of the cylinder which is subjected to uniform pressure load. The dent is assumed to have the shape of one of the buckling modes on a cross-section and to vary longitudinally as a sine wave. By assuming that the initial irregularity is small and that its square can be neglected, all fundamental quantities such as curvatures, strain and stress components, which generally have very complicated mathematical expressions, are expressed in comparatively simple forms. Three governing displacement equations of equilibrium for the case of a circular cylindrical shell under uniform pressure are obtained. They contain correction terms due to the assumed initial irregularities.

This investigation is described in a doctoral dissertation by T. S. Wu (Item 33 of Table I) and in a technical report by Wu, Goodman, and Newmark (Item 14 of Table II).
CHAPTER III
INVESTIGATIONS IN BUCKLING AND VIBRATIONS

In this chapter are described a variety of investigations in buckling of plates and beams, and in steady-state forced vibration and free vibration of elastic systems of various kinds.

10. **Numerical Procedures for Vibration Problems**

Two of the most powerful numerical methods for solving problems of free vibration and steady-state forced vibration are: (a) the Stodola-Vianello method of successive approximations, and (b) the step-by-step method developed by Holzer.

A review of these methods with some generalizations and applications to a variety of problems was made by S. Aisawa who developed a convenient scheme for arranging the computations for both methods. This scheme is similar to that developed by Newmark and originally used in connection with the computation of moments, deflections, and buckling loads of bars.

Both methods were found to be well suited for the determination of the natural frequencies of spring-mass systems having several degrees of freedom. Studies were made of three different procedures for evaluating the higher natural frequencies and modes by the method of successive approximations. For the analysis of the steady-state forced vibrations of simply-connected spring-mass systems, the step-by-step method was found to be more advantageous than the successive approximations procedure. The step-by-step method was extended also to the case of viscously damped systems.
The results of this study have been reported by Aisawa in his M.S. thesis (Item 1 of Table I).

A second investigation concerned with the solution of characteristic value problems in general, and specifically with the determination of the natural frequencies and modes of vibration of beams, was conducted by Myron L. Gossard. Dr. Gossard demonstrated the use of the so-called iterative transformation method.

In NACA Report No. 1073, Gossard describes as follows the general features of this method: "The principle of the iterative transformation procedures is similar in form to that of the standard iteration procedure for solving characteristic-value problems. Both procedures require the determination of the solutions in the order of the magnitudes of the eigenvalues, beginning with the fundamental. Both procedures require assumptions of modes, integrations which generally must be done numerically, and sweeping operations for higher-order-mode determinations. The distinguishing features of the iterative transformation procedure occur in the determination of solutions higher than the fundamental and are as follows: (1) The immediate aim is to determine not the true nth mode, as in the standard iteration procedure, but a particular linear combination composed of all modes from the fundamental to the nth. This linear combination is referred to as the transformed nth mode. The transformed nth mode can be made to have nodal (zero) points at specified stations of the wing; such a feature is highly desirable in numerical work. (2) The sweeping operations, which consist of subtractions of lower-order-mode shapes from the function obtained by integrating the
assumed mode, do not employ the orthogonality relations as in the standard iteration method but make use of forcing functions that, in numerical work, greatly simplify the sweeping operations and increase the over-all accuracy of the results by making the sweeping operations more consistent with the rest of the process. (3) Although the true nth eigenvalue is determined directly in the iterative transformation procedure, the true nth mode must be computed from quantities within the iteration cycle after the transformed nth mode is found."

The application of the iterative transformation procedure to the problem of calculating modes and frequencies of natural vibration of beams in flexure was first described by Gossard in his doctoral dissertation (Item 14 of Table I).

11. Natural Frequencies of Elastically Restrained Bars

The transcendental equation for the natural frequencies of elastically restrained bars is well known. However, this equation is not only difficult to evaluate but also it does not exhibit the relative importance of the various parameters involved. The purpose of this investigation was to develop a simple approximate procedure for determining the natural frequencies of elastically end-restrained prismatic bars. The restraints may be due to elements such as coil springs or they may result from the continuity of the bar with adjoining members.

The natural frequencies of the elastically restrained bar have been expressed as the product of an end-fixity coefficient $C_n$ multiplied by the fundamental frequency of the same bar simply supported at the
ends. From the results of numerical calculations based on the exact solutions a simple empirical formula was developed for the coefficient \( C_n \). This formula is applicable only to bars for which the stiffnesses of the end restraints are positive. The formula is valid for the fundamental as well as the higher natural frequencies and is accurate to within a maximum error of 4 per cent, the maximum error occurring in the fundamental frequency.

The results of this study can be used to calculate also the fundamental frequencies of two-span beams and of particular arrangements of three-span beams continuous over non-deflecting supports and elastically restrained against rotation at the ends. As before, the stiffnesses of the end restraints are assumed to be positive.

The problem of determining the natural frequencies of a continuous beam is basically the same as that of determining the frequencies of one of its spans only, provided proper account is taken of the actual restraints existing at the ends of the single span. A procedure was developed which consists of progressively reducing a continuous beam to its elastically restrained, one span analogue. In this procedure the dynamic end restraints are expressed as the product of the restraints which are provided under static conditions multiplied by a diminution factor which takes account of the inertia effects. Once this one-span analogue is defined, its fundamental frequency, and therefore the fundamental frequency of the original continuous beam, is evaluated from the approximation formula referred to previously.
This investigation has been described in two technical reports by Newmark and Veletsos (Items 5 and 13 of Table II) and in two published papers by the same authors (Items 10 and 13 of Table III).

12. Vibration of Continuous Flexural Systems

This investigation was concerned with the free vibration and the steady-state forced vibration of continuous flexural systems having distributed mass and elasticity. A method was developed for calculating the undamped natural frequencies and the corresponding natural modes of continuous beams on rigid or flexible supports, continuous frames without sidesway, symmetrical single-bay multi-story frames for which the joints are free to translate, and continuous plates having two opposite edges simply supported. The method can be used also to determine the steady-state forced vibration of these systems. A system with distributed mass and elasticity has an infinite number of natural frequencies. With the method developed one is capable of determining as many of the natural frequencies as he desires.

The method is a generalization of the well known Holzer method for calculating natural frequencies of torsional vibration of shafts. Like Holzer's method, it has been reduced to a routine scheme of computations which, when repeated a sufficient number of times will give the natural frequencies of the system to any desired degree of accuracy.

The method is based on the fact that, in the absence of damping, the exciting force (the term force is used in a generalized sense to indicate a force or a couple) which is necessary to maintain a
dynamical system in steady-state forced vibration with finite amplitudes becomes equal to zero at a natural frequency. Briefly, the method consists of (a) assuming a frequency of vibration, (b) determining the magnitude of the exciting force which when applied at some appropriately selected joint of the structure will produce a vibration configuration having a fixed amplitude of displacement at some other joint, (c) repeating these steps for a number of frequencies, and (d) plotting the exciting force as a function of the frequency of vibration. The frequencies for which the exciting force vanishes represent the desired natural frequencies of the system.

For an assumed frequency of vibration, the magnitude of the exciting force can be determined by a number of different procedures. The conditions to be satisfied are simply those of equilibrium and continuity for each joint of the system considered. To satisfy the condition of equilibrium, the sum of the moments and forces at the ends of the members meeting at a joint must be, respectively, equal to zero. To satisfy the condition of continuity, the slopes of the members meeting at a joint must be equal and also the deflections of the members meeting at the joint must have the same magnitude. These conditions may be expressed in equation form in a number of different ways and the equations may be solved by a number of procedures. In the method adopted, these conditions are expressed in the form of a generalized slope-deflection equation, and the distortions of the structure and the exciting force are computed by the repeated application of this equation, working progressively from one end of the structure to the other.
Extensive tables of numerical values have been computed for
the various quantities which are necessary in the analysis by this method. With these tables the calculations required in the application of the method to particular problems are simplified greatly. The tabulated values may be used also with other analytical techniques as well as for the analysis of the steady-state forced vibration of structures.

This study was made by A. S. Veletsos and is reported in his doctoral dissertation (Item 31 of Table I). The results have been summarized in three technical reports (Items 9, 16, and 20 of Table II), in three published papers (Items 12, 15, and 17 of Table III), and in a discussion of a paper (Item 21 of Table III).

13. Effects of Rotatory Inertia and Shearing Distortion on Vibration of Bars

The natural frequencies of beams as predicted by the classical Bernoulli-Euler theory are known to be higher than those obtained in carefully controlled experiments. The difference between theory and experiment is due mainly to the fact that the classical theory does not consider the influence of the rotatory inertia and shearing distortion. This difference may become practically significant when the cross-sectional dimensions of the beam are small compared to the length of the beam between nodal sections.

The first correction to the classical equation of motion was made by Lord Rayleigh. Recognizing that the elements of a vibrating bar perform not only translatory motion but also a rotatory motion, Rayleigh included the inertia load due to the rotatory motion and studied its
influence on the response of the beam. Later, Timoshenko showed that a still more accurate differential equation of motion can be obtained if one includes also the influence of the deflection due to shear.

In the present study Timoshenko's differential equation was derived starting from the equations of motion of the theory of elasticity. Proceeding in this way, it is possible to exhibit the importance of the various approximations and parameters used for purposes of simplification. A procedure was developed for estimating the magnitude of the shear deflection coefficient, and the values of this coefficient were computed for various cross sections. These results, partly original and partly due to other investigators, were then used to calculate the natural frequencies of simply supported and cantilever beams.

This study was conducted by J. G. Sutherland and L. E. Goodman and was described in a technical report (Item 4 of Table II), and in a discussion of a paper (Item 20 of Table III).

14. Vibration of an Elastic Solid

This investigation was concerned with the free vibration of an elastic solid bounded by two parallel planes. It was found that two fundamental radially symmetric solutions of the linear theory of elasticity describe the free vibrations of an elastic plate of finite thickness. These solutions correspond to "extensional" and "flexural" motions and are analogous to the plane strain waves discovered by Lamb and Rayleigh.

Following the treatment of Lamb, the transcendental equation relating the frequencies of vibration to the wave shape were developed,
and the values of the frequencies were evaluated as a function of the wave length for the fundamental and for the higher modes. From a study of the higher modes of vibration a family of surfaces was found across which no stress is transmitted. These surfaces define a class of solids of revolution for which the motions in question represent normal modes of free vibration.

For very thick plates the solutions developed correspond to Rayleigh surface waves, while, for thin plates, they approach asymptotically, well known results of the elementary plate theory.

This investigation was conducted by L. E. Goodman and was described by him in a paper (Item 8 of Table III) which was reprinted and issued as a technical report (Item 13 of Table II).

15. **Buckling of Elastically Restrained Non-Prismatic Bars**

This investigation was concerned with the elastic buckling of bars for which the moment of inertia of the cross section varies linearly from a minimum at the ends to a maximum at midspan. It is assumed that the bars are compressed by concentric end forces, that they rest on non-deflecting supports and are elastically restrained against rotation at the ends.

The characteristic equation for the critical buckling load was derived and was used to evaluate the lowest critical loads for various degrees of end fixity and various ratios of \( I_{\text{max}}/I_{\text{min}} \). The buckling load for the elastically restrained bar was expressed as the product of an end fixity coefficient \( C \) multiplied by the corresponding buckling load.
for a bar hinged at both ends. From the results of numerical calculations based on the exact solution empirical equations were next developed for the coefficient C and the buckling load of a bar hinged at both ends. The maximum error in these equations is of the order of 2 1/2 per cent. The equation for C is valid only for positive end restraints.

The strength of bars with linearly varying "I" was finally compared with that of prismatic bars having the same volume. In general, hinged bars or bars subjected to moderate end restraints are stronger than prismatic bars of the same volume. However, fixed bars and bars subjected to large end restraints are weaker than prismatic bars of the same volume.

The results of this study were summarized by A. S. Veletsos in his M.S. thesis (Item 30 of Table I).

16. Buckling of Plates

A study of numerical procedures for problems of plate buckling was made under the sponsorship of this program. A relaxation-iteration type procedure was developed for solving the homogeneous linear, simultaneous equations which result from the use of finite difference approximations to the pertinent differential equations. A study was made of methods to speed up the convergence of the calculations. This feature has considerable importance if electronic computing machinery is not available.

A special procedure was developed for cases where Levy's solution may be used to reduce the partial differential equation to an
ordinary differential equation. For this case a solution may be obtained very easily. Of course, most analytical solutions fall in this class.

The problem of buckling in the large deflection range has been treated very briefly. A numerical procedure was developed for solving the two simultaneous, non-linear partial differential equations by means of finite differences. However, the solution is quite laborious for hand computation.

The studies described above are reported by W. J. Austin in a doctoral dissertation (Item 3 of Table I).

After the thesis was completed an improved method was developed for the solution of plate buckling problems under conditions for which the usual differential equation can be reduced to an ordinary differential equation. This method is a combination of the Stodola method of successive approximations with the numerical procedure of integration by Newmark. Plates reinforced with rib stiffeners were considered as well as unreinforced plates. The general procedure developed in this work is applicable to many other problems, since it permits a relatively simple and accurate numerical integration of a class of differential equations. This study is published in a technical report by Austin and Newmark (Item 1 of Table II) and in a paper (Item 5 of Table III). The paper was reprinted and issued as one of three papers in a technical report by Newmark, D'Appolonia, and Austin (Item 12 of Table II).

17. Torsional-Flexural Buckling of Beams and Columns

Four studies have been made on the problem of torsional-flexural buckling of beams and columns. The first two studies were concerned
primarily with the development of simple numerical procedures and the last two studies were concerned with solutions of practical problems of importance.

(a) Development of Numerical Procedures

The first study was concerned with the development of a numerical procedure for determining the critical torsional-flexural load of beams and columns of open section and with an investigation of the effect of web deformations on the torsional buckling strength of I-section columns with thin webs.

The procedure developed is of the Stodola type of successive approximations. In this procedure the necessary integrations are performed numerically using first Newmark's integration technique and then an initial value type, step-by-step procedure. The solution converges quickly to the lowest critical load.

The study of the effect of deformation of the cross-section during buckling on the torsional buckling load of columns of I-section was carried out by means of the calculus of variations. Two distorted configurations of the web of the twisted I-section column were assumed and the minimum load required to maintain each in its buckled configuration was found. One consideration leads to the critical stress producing torsional buckling and web deformation. The other yields the critical stress which causes local failure in the web accompanied by twisting of the flanges without rotation of the cross-section as a whole. It is called "buckling by flange twist." The critical loads of several columns were computed and compared with the critical loads given by
Euler's and Wagner's theories. In general, it was found that the reduction in the torsional buckling strength of columns due to deformations of the web is negligible in the case of I-sections with proportions commonly used in engineering practice.

This investigation was conducted by Mounir Badir and is reported in his Ph.D. thesis (Item 4 of Table I).

A second numerical procedure for the solution of torsional-flexural problems was developed by C. P. C. Tung and is reported in a doctoral dissertation (Item 28 of Table I). This procedure is a combination of the Stodola method of successive approximations and the numerical procedure by Newmark. This procedure is more versatile and more accurate than that developed by Badir, but is subject to convergence difficulties in some cases. The procedure is illustrated for problems of lateral buckling of beams, torsional buckling of columns, and lateral-torsional buckling of beam-columns due to combined transverse and longitudinal loads. In addition to the development of the analytical procedure, Dr. Tung worked a series of important practical problems and in his dissertation he discusses these solutions as they relate to current steel design formulas.

(b) Elastically End-Restrained I-Beams

The design of beams subject to lateral buckling is based upon formulas for the critical stress of simply supported beams which are uniformly loaded on the top flange. In a large investigation at the University of Washington these formulas were verified by experiments conducted on beams with simple supports. In addition, some tests were
also run on beams with more practical end conditions, such as clip angles or seat connections, and it was found that the restraint offered by these simple connections increased considerably the ultimate load.

The purpose of this investigation was to provide a theoretical study of the critical loads and corresponding stresses for I-beams with rotational, elastic end restraints. In practical structures these restraints are, of course, provided by the supports. In this study the beam was assumed at both ends to be elastically restrained against rotation about both principal axes and to be held against rotation about the longitudinal axis. Both uniform load and concentrated load were considered, with the load assumed to be applied at the top flange, the centroid, and the bottom flange. Fifteen hundred solutions were made for the practical range of the variables. The solutions were made numerically using finite differences. The solutions were obtained on the ILLIAC in about 10 hours.

The solutions have been presented in the form of tables and charts in a paper by Austin, Yegian, and Tung (Item 14 of Table III).

(c) Beams Supported by Cables

The purpose of this investigation was to find the critical weight of slender girders when suspended in the air during erection procedures. The girders are assumed to be lifted by two vertical cables attached by clamps to the top flange. The two lifting cables may be attached at the ends or they may be at some point inside so that the beam is divided into a central span with overhangs at each end.
The critical weight of I-beams, both bi-symmetrical and mono-symmetrical, were found for a wide range of parameters covering all practical conditions. In addition, a study was made of the maximum stresses which occur in bi-symmetrical I-beams when the cables are attached with small initial eccentricities. The study of stresses was made to determine the effect of small deviations from idealized conditions on the ultimate strength. The solutions were made numerically on the ILLIAC. Two analytical procedures were used, a successive approximations and integration procedure, and a finite difference method.

The solutions, in the form of tables and curves, and interpretations of the solutions in practical terms are presented in a thesis by S. Yegian (Item 34 of Table I). It is planned to prepare this work for a technical report and formal publication.

This investigation was begun under the sponsorship of Task VI and has been concluded under sponsorship of Contract Nonr-1834(03), Task III.
CHAPTER IV
RESPONSE OF STRUCTURES UNDER TRANSIENT LOADING

The investigations described in this Chapter concern the development of numerical procedures for structural dynamics problems and the subsequent application of these procedures and of other information in the solution of important practical problems connected with the response of structures to earthquake and blast loading.

18. Development of Numerical Procedures of Analysis

There have been developed under this program several numerical procedures for the analysis of the response of structures subjected to dynamic loading. An objective held constantly in view during the course of the investigations has been to develop procedures which, with a reasonable amount of work, can give approximate answers sufficiently accurate for engineering purposes and which, with additional work, yield results to any desired degree of accuracy. Attention has been focused on simplicity of concept and generality of procedure.

The procedures developed in this investigation, which are described in more detail in the following articles, have several features in common. First, the procedures were derived primarily for structures which are considered to consist of separate concentrated masses supported on a flexible but weightless system of framing. As in an actual structure the mass and elasticity are usually not concentrated but distributed, the structure which is analyzed is only an approximation to
the actual structure. But with sound judgment a reasonable approximation can generally be made so as to result in fairly good accuracy in the calculations.

Second, the procedures described here all involve a process of step-by-step integration. That is, the time coordinate is first divided into a number of short intervals. The acceleration, velocity, and displacement of a mass at the beginning of any particular time interval are either given or known from previous history of the structural behavior. From these are computed the acceleration, velocity and displacement of the mass at the end of the interval by devices which differ for different procedures and which distinguish one procedure from another. By repeating this process of integration successively for all the intervals of time within the range of interest, a complete solution for the dynamic response of the structure is obtained.

Any type of loading such as that due to blast of a bomb, impact from a moving object, or foundation motion due to an earthquake, can be considered.

(a) Procedure Using Taylor's Expansion

In this procedure the displacement and velocity of the mass at the end of the time interval are computed by means of a Taylor's series whose terms involve the displacement and its time derivatives at the beginning of the interval.

The procedure was developed primarily for the study of flexural vibration of beams, but it applies to any transient problem for which the governing equation of motion is linear and has constant
coefficients. While only simple problems of vibration within the elastic region have been considered, the procedure may be extended to problems involving plastic action, to beams resting on elastic or rigid supports, to simply-supported or continuous spans, to beams subjected to axial as well as lateral loads, and to cases with damping.

Several examples were solved by this procedure and conclusions were deduced from comparisons with exact solutions and solutions obtained by other methods.

The details of this procedure were presented in full in a Ph.D. thesis by Z. K. Lee (Item 19 of Table I).

(b) Newmark's $\beta$ Method

This is a generalized method of numerical integration which emphasizes the physical aspects of the problem. For each time interval considered the solution is carried out by an iterative scheme. A trial value is estimated for the acceleration of each mass at the end of the time interval. From this value and from the values of the acceleration, velocity, and displacement at the beginning of the time interval, the velocity and displacement of each mass at the end of the time interval are computed by means of appropriate quadrature formulas. From these quantities new accelerations at the end of the interval are evaluated by use of Newton's second law of motion applied to each mass. If the derived accelerations do not agree with the assumed values, another trial is made using the derived values as the initial assumed values. The procedure is repeated until derived and assumed values agree within a desired degree of accuracy.
The quadrature formulas used involve a quantity $\beta$ which defines the nature of the variation of the acceleration within the time interval. By a proper choice of $\beta$, the method reduces to any of a number of classical numerical procedures. For example, the method reduces to the constant acceleration method for $\beta = 0$; to Newmark's linear acceleration method for $\beta = 1/6$; to Timoshenko's modified method for $\beta = 1/4$; and to Fox's method for $\beta = 1/12$.

Two important aspects of this work were concerned with (a) the convergence of the successive iterations for each time interval and (b) the evaluation of the errors inherent in the numerical procedure. Particular attention has been given to defining the conditions under which the various procedures will either fail to converge or lead to exceedingly inaccurate (unstable) solutions.

The method may be used to determine the dynamic response of structures having any relationship between force and displacement ranging from linear behavior through various degrees of inelastic action up to failure. The same general technique may be used also when damping is present.

A complete discussion of this method has been given in a paper by N. M. Newmark (Item 4 of Table III). This paper has been issued also as a technical report (Item 10 of Table II).

(c) Extension of $\beta$ Method - Use of Dynamic Load Factor

This procedure is an extension of the $\beta$ method and represents an attempt to improve the accuracy of the $\beta$ method for certain conditions. The method of analysis is exactly the same as the $\beta$ method,
except that here the external loads are modified by selected factors in the course of the computation.

The derivation of the expressions for the load factors and the illustration of their application to several examples have been included in an M.S. thesis by T. C. Hu (Item 16 of Table I). It was found that this method of dynamic load factors has merit under the following conditions:

(1) An approximate value of the fundamental natural frequency of vibration of the system is known.

(2) In a given time interval the loading either is discontinuous or lasts over a portion of the time interval.

(3) The maximum response of the structure occurs at the primary stage of the loading.

19. Review of Numerical Procedures of Analysis

There are as many different procedures for the numerical analysis of dynamic structural response as there are methods for the numerical integration of differential equations. The objectives of this investigation were to study the accuracy and range of applicability of various methods of numerical integration now available and often used in dynamics problems, and to obtain data that will enable one to make a judicious choice of a suitable technique for a specific problem at hand.

Two systematic and fairly comprehensive studies have been made and the conclusions have been summarized in two technical reports, the first by Newmark and Chan (Item 7 of Table II) and the second by Tung
and Newmark (Item 7 of Table II). The first study is also reported in the doctoral dissertation of S. P. Chan (Item 7 of Table I). In both studies comparisons have been made between the various techniques with respect to the accuracy of solutions at any stage of the integration process, the nature of errors, the limitations imposed by considerations of stability and convergence, time consumption, and self-checking provisions of the procedures. The studies have been confined to a single-degree-of-freedom system. However, the comparisons made can be used equally well for more complicated systems, since the motion of multi-degree-of-freedom systems can be considered as being made up of the motions of the eigenmodes, each mode acting as a single-degree-of-freedom system.

The methods considered include the constant acceleration method, Timoshenko's modified acceleration method, Newmark's $\beta$ methods, the various methods of finite differences due to Levy, Salvadori, Houbolt, Adams and others, Euler's and modified Euler's methods, the third order rule of Runge-Heun-Kutta, Kutta's fourth order rule, and methods involving Taylor expansions.

One of the conclusions drawn from these studies is that the $\beta$ method is particularly well suited for the analysis of dynamic structural response because of its simplicity and flexibility in application. Other significant conclusions may be found in the reports cited above.
20. Response of Structures to Earthquake Motions

Present methods of earthquake resistant design of structures are mainly based on engineering judgment, experience, and practice. The studies reported herein are intended to furnish information which will help to remove some of the indeterminacy associated with the determination of the magnitude of the forces for which provision must be made in design. These studies are summarized in the following articles. It should be noted that these studies have a wider field of application than the field of earthquake resistant design. With proper modification they can be used to predict the response of other dynamical systems subjected to "ground" disturbances which possess characteristics of randomness.

(a) Influence of Ductility on the Response of Simple Structures to Earthquake Motions

The object of this investigation was to estimate the influence of ductility on the ability of steel frame mill buildings to withstand earthquakes. A simplified one-story steel frame symmetrical about its center line was considered. It was assumed that the mass of the structure was concentrated at the tops of the columns and that the roof truss was infinitely rigid as compared to the columns. The structure may then be analyzed as a system with a single-degree-of-freedom. The material of the structure was assumed to have an idealized elasto-plastic behavior, that is, a "flat-top" stress-strain, or load-deformation curve.

The studies made by G. W. Housner show that the records of past strong-motion earthquakes exhibit characteristics of randomness.
and that in so far as their effect on the response of structures is concerned, the actual earthquakes may be replaced by a set of fictitious random ground motions. Based on this principle, five possible ground motions were constructed for the purpose of this investigation. Each of the motions considered consisted of 20 stepped acceleration pulses which were selected so as to be random in direction, in amplitude, and in duration.

The principal parameters considered were: the natural frequency of vibration of the structure; the root mean square amplitude of the ground motion; the maximum dynamic deflection of the top of a column with respect to the base; and the maximum deflection of the top of a column relative to its base which can take place before the onset of plastic action in the column. The last quantity is a measure of the ductility of the structure; its value is infinite for a perfectly elastic structure and zero for a perfectly plastic structure.

A graphical method of analysis, similar to the gyrogram or phase-plane diagram, was developed for the response of an elasto-plastic single-degree-of-freedom system subjected to a complex base motion. The method may be extended to include the effect of strain-hardening by the introduction of an additional parameter. The application of the method is not restricted to structures subjected to stepped acceleration pulse functions as considered in this study; any ground motion for which the acceleration is a linear function of time may be handled with equal ease.
Using this method, the responses of a number of structures covering a wide range of the parameters described above were determined for each of the five random ground motions considered. From these data conclusions were drawn regarding the effect of ductility on the maximum response of structures due to earthquakes.

This study is described in a Ph.D. thesis by S. L. Pan (Item 20 of Table I) and in a technical report by Pan, Goodman, and Newmark (Item 3 of Table II).

(b) Aseismic Design of Elastic Structures Founded on Firm Ground

The object of this investigation was to arrive at a rational basis for the design of earthquake resistant structures. The scope of this work was limited to structures whose behavior is essentially elastic and which are founded on firm ground. The structures may have any number of degrees of freedom, and both structure and ground may have viscous damping.

The method which has been developed is based on inferences drawn from accelerograms of strong-motion earthquakes and on the theory of probability. Structures designed to withstand the responses computed by this method will be of uniform strength in the sense that all their parts will have equal probability of successfully withstanding an earthquake. Different parts of a structure will not be relatively overdesigned or underdesigned.

While the sources of excitation considered in this study have been ground tremors, they may, with slight changes in the analysis, be wind gusts or other dynamic disturbances.
This investigation is described in a doctoral dissertation by E. Rosenblueth (Item 22 of Table I), in a technical report by Goodman, Rosenblueth, and Newmark (Item 6 of Table II), and in a published paper by the same authors (Item 11 of Table III).

(c) **Response of a Typical Tall Building to Actual Earthquakes**

The dynamic response of a 10-story structure subjected to the action of strong-motion earthquakes has been evaluated with the aid of the ILLIAC. The ground motions considered included twelve different strong-motion earthquakes recorded by the U.S. Coast and Geodetic Survey. The structure was assumed to react as an elastic shear-beam. Two different configurations of the building were considered: (a) a building with uniform distribution of mass and stiffness, and (b) a building with a linear distribution of mass and stiffness.

The calculations were performed for three different damping conditions corresponding to (a) no damping; (b) 2 per cent of critical damping; and (c) 10 per cent of critical damping. The damping coefficient was computed on the basis of the fundamental frequency and average weight of the building. In these analyses each accelerogram was approximated by a series of polygonal lines. The time interval used in the numerical integration process was 0.0075 sec. Each problem required from 20 to 30 minutes of machine time with about one third of the time used in punching results on the output tape.

The magnitude and distribution of the maximum dynamic shears throughout the building were evaluated and the results were compared with those obtained on the basis of the procedure recommended by the Joint
Committee on Lateral Forces. The maximum shears determined by these elastic analyses are found to be significantly larger than those predicted on the basis of the design recommendations of the Joint Committee on Lateral Forces.

This study has been described by Tung and Newmark in a technical report (Item 25 of Table II) and in two papers (Items 18 and 19 of Table III).

(d) Distribution of Extreme Shear in a Tall Building Subject to Earthquakes

This study was made to determine statistically the relative distribution of the extreme shear developed at different stories of a tall building which is subjected to the shocks of an earthquake. Such knowledge is helpful in achieving a unified aseismic design of tall buildings.

For the purpose of analysis, a tall uniform shear beam was assumed as an idealized model of a tall building frame. An obvious advantage of this simulation lies in the simplification obtained in the governing equation of motion which is the one-dimensional wave equation. The solution to this equation may be expressed in two parts, one representing a forward wave starting at the base and propagating toward the top, and the other representing a wave traveling from the top to the base, or a reflected wave. Both parts are functions of the velocity of propagation of the disturbance along the beam.

For making a statistical estimate, the ground motion was assumed to consist of a large number of random pulses, each having an
equal order of magnitude. The state of shear in the structure due to the random pulses is that produced by the shear waves traveling from base to top and reflecting back to the base. An analogy may then be drawn between this problem and the problem of random walks in which a particle may move forward or backward, or may stand still on a line, according to specified probabilities.

Based on this principle and the use of statistical methods, a frequency distribution function for the story shear developed at different levels of the structure was determined. The results obtained indicate that the dynamic story shear which is a random variable in the present analysis, possesses a normal distribution with a zero mean, and that the ratio of the expected extreme shear at any story to the maximum base shear varies parabolically with the height of the story from the base.

This study is described in a technical report by Tung and Newmark (Item 26 of Table II).

21. **Response of Structures to Blast Loading**

The field of design of structures to withstand the shock waves produced by the detonation of high explosive and atomic bombs is relatively new and is one in which available unclassified literature has been sparse. While an increasing number of analytical techniques have been developed for the determination of the responses of structures to blast loading, there still remain many practical problems of design on which information is lacking. These problems are of fundamental importance from the designer's point of view.
The aim of this phase of the project therefore has been to obtain pertinent data which may serve as a basis toward establishing rational design criteria for structures subjected to blast loading. Work accomplished along this line includes: (1) a study of the response of simple systems to dynamic loads and of the effect on that response of variations in the characteristics of the excitation and in the properties of the structure; (2) an investigation of the blast loading transmitted from wall coverings to the building frames; and (3) studies of the effect on dynamic response of foundation coupling and vertical loads. In addition, an engineering design procedure for blast resistant structures was developed. These studies are described briefly in the following articles. In general, only the simplest types of structures were considered, that is, those which may be simulated by single-degree-of-freedom systems. While the conclusions made in these studies pertain only to simple structures, it is believed that the general trend of the results obtained will throw light on the dynamic behavior of more complicated structures as well.

A general and concise discussion of the fundamental principles and various aspects of dynamic analysis and design has been presented in the paper "Analysis and Design of Structures Subjected to Dynamic Loading" by N. M. Newmark (Item 3 of Table III).

(a) Response of Simple Structures to Dynamic Loads

The objectives of this investigation were (a) to study the response of simple structures to the action of dynamic forces such as those arising from the explosion of bombs, and (b) to determine the effect
on that response of variations in the values of each of the parameters entering into the problem. The importance of the second objective becomes apparent when one recalls that properties of a structure and the characteristics of a dynamic disturbance are seldom known accurately. In order to interpret the significance of calculated responses, it is essential for a designer to know to what extent variations in the fundamental physical parameters may affect the response of the structure.

This study was restricted to the action of undamped systems possessing a single degree of freedom. It was assumed that the structure can deform elastically up to a point and that for deformations in excess of this limit it behaves in an ideally plastic manner. The types of forces considered included the rectangular pulse and triangular pulses with their peak value at the beginning, the end, or the middle of the interval of time over which the disturbance is spread. Extensive numerical calculations have been carried out to determine the maximum response of the structure, and the results obtained have been summarized in a set of useful design charts. Also, graphs have been prepared for the effect upon the maximum response of small variations in the magnitude of each of the quantities entering into the problem. With these charts, the determination of the maximum response of a structure subjected to a blast load and the sensitivity of the response to changes in the magnitude or shape of the pulse, or in the characteristics of the structure, may be determined readily.
From the results of these studies, a simple semi-empirical formula was developed by Newmark for the maximum response of a single-degree-of-freedom elastoplastic system subjected to a triangular pulse with an initial peak. The accuracy of this formula has been tested for a wide range of the variables involved and it has been found to be extremely satisfactory.

This study is described in a M.S. thesis by N. E. Brooks (Item 6 of Table I) and in a technical report by Brooks and Newmark (Item 15 of Table II).

(b) An Engineering Approach to Blast Resistant Design

Nearly all previous studies on the subject of the resistance of structures to atomic blast have been concerned primarily with the analysis of the response of a particular structure to a given loading. However, this approach is at best of academic interest only and is not very satisfactory from a designer's point of view, because the structure to be analyzed does not yet exist, and the real problem that confronts the designer is the preliminary choice of the structure. Furthermore, many uncertainties exist in the details of the blast loading and in the properties of the structure. It can be demonstrated that even minor variations in these details may cause major changes in the computed responses. Elaborate analytical techniques are time consuming and may even be misleading if the results are applied indiscriminately to practical designs.

On the other hand, a procedure which involves rather crude approximations but permits the designer to arrive at a preliminary
choice of sections directly, intelligently and fairly rapidly, should prove of great practical value. The purpose of this study was to develop such a procedure.

The procedure developed is believed to be particularly useful for preliminary design of a new structure or for strengthening an existing structural frame or its component parts. In this procedure one needs to estimate (a) the natural period of vibration of the frame or component parts in question relative to the duration of the loading on the structure, and (b) the ductility factor, defined as the ratio of the permissible limiting deflection to the yield-point deflection. Using these estimates and certain approximate relations which have been developed, one can determine readily the required yield resistance of the structure for a specified pressure. In essence this procedure leads to the computation of either the yield-point resistance of the structure necessary to resist a given overpressure or, alternatively, the peak overpressure which can be resisted by a given structure.

This study is described in a paper by Newmark (Item 16 of Table III). This paper has been reprinted and issued as a project technical report (Item 17 of Table II).

(c) **Blast Loading Transmitted from Walls to Building Frames**

When a structure having wall coverings is subjected to blast pressure from the detonation of bombs, the load that is transmitted from the walls to the structural frame may have a pulse-time relation quite different from that of the pulse which impinges on the walls. It
was the purpose of this investigation to study the load-time relationship of the reaction exerted on the frames.

A typical bay of a one-story frame building was considered. The wall covering of the bay was assumed to behave as a series of closely-spaced beams which are simply supported between the adjacent frames and act independently of one another. It was further assumed that the motion of the frames had no effect on the load transmitted to them. The pressure loading on the walls was considered to be a triangular pulse with an initial peak. This conventionalized pulse has been found to be a good approximation of the actual pressure pulse.

Both elastic and plastic analyses were made. In the plastic analysis, it was assumed that a plastic hinge forms at the center of the beam. From these two analyses expressions for the dynamic end reactions of the walls on the frames were derived, and charts were prepared giving the variation of the dynamic reactions as a function of time for different beam characteristics and different durations of the load pulse.

This study is described in a doctoral dissertation by W. J. Francy (Item 13 of Table I) and in a technical report by Francy and Newmark (Item 21 of Table II).

(d) **Influence of Foundation Coupling on the Dynamic Response of Simple Structures**

This study was concerned with the determination of the effect on the dynamic response of a structure of the interaction between the structure and the foundation. The structure considered was a rectangular
box-shaped frame. The following two conditions of motion were investigated: (1) the frame moves only as a rigid body, and (2) the frame distorts as a shear beam. The exciting force was assumed to be a pulse of rectangular shape. The quantities necessary to define this force are the point of application of the force, its magnitude, and its duration.

The flexibilities of the soil in the vertical and horizontal directions were represented by the action of two vertical springs and a horizontal spring. The springs were assumed to be linearly elastic and weightless.

For various combinations of the parameters considered, numerical values of the responses of the structure were determined, from which conclusions were deduced regarding the effect of interaction between foundation and structure.

This investigation is described in the doctoral dissertation by J. G. Hammer (Item 15 of Table I).

(e) Effect of Vertical Loads on the Dynamic Response of Structures

This study was concerned with the effect of vertical loads on the dynamic response of a simple structure. The vertical load may be the dead weight of the structure or may be produced by the vertical component of blast forces acting on the roof of the structure. Vertical load decreases the resistance of the frame to horizontal loads and also causes the initiation of yielding in the columns at a reduced value of the horizontal forces.
In this investigation a method was developed for the analysis of simple portal frames subjected to both vertical and horizontal loading. The procedure considers the behavior of the structure before and after yielding and up to complete collapse. The load-deformation curve neglecting vertical load is assumed to consist of a linear elastic segment and a linear plastic segment having a slope different from that of the elastic segment. This investigation included also a study of the validity of some of the approximations used by Newmark in his design procedure for structures exposed to atomic blast.

This study is described in a doctoral dissertation by H. E. Stephens (Item 26 of Table I).
In this chapter are presented descriptions of the following three unrelated investigations: (1) bounds and convergence of relaxation procedures; (2) a method of numerical integration for transient problems of heat conduction; and (3) two studies on the use of high speed digital computing machines.

22. **Bounds and Convergence of Relaxation and Iteration Procedures**

In connection with the solution of the simultaneous, linear, algebraic equations which arise in studies of stress analysis, considerable effort was expended in the projects reported herein on methods of solution of the equations and, particularly, on the bounds and convergence of relaxation and iteration procedures. Some of the ideas developed relative to these problems are summarized in a paper by N. M. Newmark (Item 6 of Table III). This paper was reprinted and issued as one of three reprints in a project technical report (Item 12 of Table II).

To provide an indication of what is contained in this paper, the abstract is quoted below:

"This paper discusses a number of general concepts relating to relaxation procedures, methods of "steepest descent," and iteration procedures. The relationship between these is indicated for certain systematic ways in which relaxation patterns may be developed. An
extrapolation procedure is developed for problems in which the successive approximations technique, in its usual form, diverges. The procedure is a generalization of an observation by Hartree and others that in many cases such divergent problems can be treated by considering the divergence to be a geometric one. The study of upper and lower bounds to the errors in relaxation procedures is a second major part of the paper. General procedures are developed for determining the maximum and minimum errors in any of the quantities for which bounds to an influence function can be determined. Use of the theorems derived makes it possible, for example, to estimate the error due to all sources including "round off" at any stage in the numerical solution of Laplace's or Poisson's equation."

23. **A Method of Numerical Integration for Transient Problems of Heat Conduction**

In this study, a numerical method of integration was developed for studying transient heat flow in solids. The canonical form of the governing differential equation is of parabolic type, for which various methods of numerical integration have been studied. The method developed in connection with this investigation is more general and versatile than other existing methods. The two methods which give stable solutions, studied by Hyman, Kaplan, and Brien, are found to be two special cases of the present method.

In this procedure, the differential operator with respect to the spatial coordinates is replaced by its eigenvalues. This simplifies
the development, and the results can be interpreted in terms of eigen modes of the differential operator, which are found to exist for a great many physical problems, even with time-dependent conditions. A spectrum for the truncational error is obtained with respect to the eigenvalues of the system.

The procedure is essentially a method of step-by-step integration. The time coordinate is first divided into a number of short intervals. For each step the temperature and its first time derivative at the end of the step are calculated from the corresponding quantities at the beginning of the step and the governing equation. By repeating this process successively for all intervals of time within the range of interest, a complete solution is obtained. The time interval chosen for this method must lie within certain limits to give stable solutions, and these bounds were derived in this work.

Within each step the calculation of the temperature at the end of the interval may be done either by the straightforward solution of simultaneous equations or by an iterative procedure. The criterion for convergence of the iterative procedure was derived.

The differential equation which governs the flow of heat in solids is of the same type as that defining several other physical phenomena. For instance, the equation for consolidation of clay, the kinetic equation of rate process, the diffusion equation in chemical reactions, and the Fokker-Planck Equation in the theory of probability, are all of the parabolic type treated in this note.

This work is described in a technical report by Tung and Newmark (Item 22 of Table II).

24. Studies on the Use of High Speed Digital Computing Machines

In addition to the studies of the use of the digital computer which were made in connection with the applications reported in other
sections herein, the two investigations described below were concerned solely with the development of efficient computer programs for problems of common occurrence.

A study of procedures for the solution of large numbers of simultaneous linear equations was made by A. R. Robinson. The main characteristics of different methods were compared. A new type of iterative procedure was developed. This method has the advantage that it is applicable to any system of equations having a unique solution. However, the new procedure shares with all other known iterative methods the disadvantage of converging extremely slowly for very badly conditioned equations. For such sets of equations, which are by no means rare in applications, it seems that the Gauss elimination procedure, if properly used, is the best computational scheme for obtaining results in a reasonable amount of time. Yet the storage requirements of the Gauss method seriously limit the number of equations which may be solved on present-day digital computers.

This study is reported in a Ph.D. dissertation (Item 21 of Table I).

A study of procedures for the solution of structural mechanics problems on high speed digital computing machines was made by J. A. Brooks. The primary object of the investigation was the development of computer programs for the solution of complex structural dynamics problems such as those associated with effects of explosions, vehicle impact, and earthquakes. As part of this general investigation the problems of determining static deflections, buckling loads, natural frequencies, and
transient response have been treated separately. The whole investigation of problems of this type may be subdivided into three phases according to the type of work involved.

The first part of this work is the derivation of equations of motion for the structure under study, where these equations are required to fit a form for which methods of solution are available. Preferably, the methods would already exist in coded form.

The second part of this work is the preparation of computer programs for the resisting forces of the structure, the external forces, and the residual forces. The residual force routine for the equations of equilibrium is merely a combination of the resisting force routine and the external force routine. The residual force routine that enters into computations of the matrices in the buckling problem is the same as that for the static deflection problem.

The third part of the work is the construction of the equation solving codes and library routines. Ideally, this phase of the work should be completed before the first two parts are undertaken.

The study is reported in a Ph.D. thesis (Item 5 of Table I). In this thesis is outlined an approach to the solution of structural mechanics problems in which the only coding work required for study of an individual structure is the preparation of a resisting force routine and an external force routine. The remainder of the codes can be prepared as library routines which can be used for other structural problems. In support of this approach, programs have been prepared for
finding the static deflections, buckling loads, and transient response of an elastic arch which was used as an example.

As part of this study a numerical procedure for the solution of the following determinental equation,

\[ |A + \lambda B + \lambda^2 C \ldots| = 0 \]

was prepared. The matrices A, B, C, D, \ldots are symmetrical and real. The quantity \( \lambda \) is a scalar representing the magnitude of the load on the structure. The solution of the above equation gives the buckling loads and buckling modes. The method of solution is based on Lanczos' method of minimized iterations. Also a program for the solution of slightly non-linear simultaneous algebraic equations was prepared. The method is based on the method of conjugate gradients.
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* The notation SRS refers to the Structural Research Series of reports of the Civil Engineering Department of the University of Illinois.
TABLE II (CONTINUED)

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    47, 1952.

13. "Circular-Crested Vibrations of An Elastic Solid Bounded by Two

14. "Effect of Small Initial Irregularities on the Stresses in Cylindri-
    cal Shells," by T. S. Wu, L. E. Goodman, and N. M. Newmark,
    SRS 50, April 1953.

    Brooks and N. M. Newmark, SRS 51, April 1953.

16. "A Method for Calculating the Natural Frequencies of Continuous
    Beams, Frames, and Certain Types of Plates," by A. S. Veletsos
    and N. M. Newmark, SRS 58, June 1953.

17. "An Engineering Approach to Blast Resistant Design," by N. M.
    Newmark, Reprint, SRS 63, October 1953.

18. "A Simple Approximation for the Fundamental Frequencies of Two-
    Span and Three-Span Continuous Beams," by A. S. Veletsos and N. M.
    Newmark, SRS 66, February 1954.


20. "Tables of Deflection and Moment Coefficients for the Steady-State
    Vibration of Uniform Bars," by A. S. Veletsos and N. M. Newmark,
    SRS 71, May 1954.

    Francy and N. M. Newmark, SRS 89, December 1954.


23. "Lateral Buckling of Elastically End-Restrained I-Beams," by W. J.
TABLE II (CONTINUED)


In addition to the reports listed above, the following are of the nature of technical reports, but were issued before the system of technical reports was begun.


TABLE III

LIST OF PUBLICATIONS PREPARED UNDER SPONSORSHIP OF CONTRACT N6ori-71, TASK ORDER VI


TABLE III (CONTINUED)


<table>
<thead>
<tr>
<th>Administrative Reference and Liaison Activities</th>
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<tbody>
<tr>
<td><strong>Chief of Naval Research</strong></td>
</tr>
<tr>
<td>Department of the Navy</td>
</tr>
<tr>
<td>Washington 25, D.C.</td>
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<tr>
<td>ATTN: Code 438</td>
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<td>Code 432</td>
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<tr>
<td>Code 423</td>
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<tr>
<td><strong>Director</strong></td>
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<tr>
<td>Naval Research Laboratory</td>
</tr>
<tr>
<td>Washington 25, D.C.</td>
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<td>ATTN: Tech. Info. Officer</td>
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<td>Technical Library (6)</td>
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<tr>
<td>Mechanics Division (1)</td>
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<tr>
<td><strong>Commanding Officer</strong></td>
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<tr>
<td>Office of Naval Research</td>
</tr>
<tr>
<td>Branch Office</td>
</tr>
<tr>
<td>495 Summer Street</td>
</tr>
<tr>
<td>Boston 10, Massachusetts (1)</td>
</tr>
<tr>
<td><strong>Commanding Officer</strong></td>
</tr>
<tr>
<td>Office of Naval Research</td>
</tr>
<tr>
<td>Branch Office</td>
</tr>
<tr>
<td>346 Broadway</td>
</tr>
<tr>
<td>New York 13, New York (1)</td>
</tr>
<tr>
<td><strong>Office of Naval Research</strong></td>
</tr>
<tr>
<td>The John Crerar Library Bldg.</td>
</tr>
<tr>
<td>10th Floor, 86 E. Randolph St.</td>
</tr>
<tr>
<td>Chicago 1, Illinois (2)</td>
</tr>
<tr>
<td><strong>Commander</strong></td>
</tr>
<tr>
<td>U.S. Naval Ordnance Test Station</td>
</tr>
<tr>
<td>Inyokern, China Lake, California</td>
</tr>
<tr>
<td>ATTN: Code 501</td>
</tr>
<tr>
<td><strong>Commander</strong></td>
</tr>
<tr>
<td>U.S. Naval Proving Grounds</td>
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<tr>
<td>Dahlgren, Virginia</td>
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<tr>
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<td>Documents Service Center</td>
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<tr>
<td>Knott Building</td>
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<td>Dayton 2, Ohio</td>
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**Department of Defense Other Interested Government Activities**

**GENERAL**

- Research and Development Board
  - Department of Defense
  - Pentagon Building
  - Washington 25, D.C.
  - ATTN: Library (Code 3D-1075)

- Armed Forces Special Weapons Project
  - P.O. Box 2610
  - Washington, D.C. (1)
  - ATTN: Chief, Weapons Effects Division
  - Chief, Blast Branch (2)
### ARMY

<table>
<thead>
<tr>
<th>Position</th>
<th>Contact Information</th>
</tr>
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<tbody>
<tr>
<td>Chief of Staff</td>
<td>Department of the Army Research and Development Div., Washington 25, D.C. ATTN: Chief of Research and Development</td>
</tr>
<tr>
<td>Office of the Chief of Engineers</td>
<td>Assistant Chief for Public Works, Department of the Army, Bldg. T-7, Gravelly Point, Washington 25, D.C. ATTN: Structural Branch (R. L. Bloor)</td>
</tr>
<tr>
<td>Chief of Engineers</td>
<td>Engineering Division, Military Construction, Washington 25, D.C. ATTN: ENGEB</td>
</tr>
<tr>
<td>The Commanding General</td>
<td>Sandia Base, P.O. Box 5100, Albuquerque, New Mexico ATTN: Structures Branch</td>
</tr>
<tr>
<td>Corps of Engineers, U.S. Army</td>
<td>Fort Belvoir, Virginia ATTN: F. M. Mellingier</td>
</tr>
<tr>
<td>Operations Research Officer</td>
<td>The John Hopkins University, 6410 Connecticut Avenue, Chevy Chase, Maryland ATTN: ORDTE</td>
</tr>
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### NAVY

<table>
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<tr>
<th>Position</th>
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<tbody>
<tr>
<td>Director</td>
<td>David Taylor Model Basin, Washington 7, D.C. ATTN: Structural Mechanics Division</td>
</tr>
<tr>
<td>Director</td>
<td>Naval Engineering Experiment Station, Annapolis, Maryland</td>
</tr>
</tbody>
</table>
Director
Materials Laboratory
New York Naval Shipyard
Brooklyn 1, New York (1)

Chief, Bureau of Ordnance
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ATTN: Ad-3, Technical Lib. Rec., T. N. Giraud (1)

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ATTN: Mechanics Division (2)

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DE-22, C. W. Hurley (1)
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Research Assistants  
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