THE ANALYSIS OF HIGH TEMPERATURE CREEP IN SHALLOW DOMES

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This report is concerned with the time dependent deformational analysis of a clamped shallow dome by means of an analogous framework. The dome is subjected to axially symmetric pressures which can be varied in time, and to constant temperatures which cause creep.

The analysis uses a Tresca type of steady creep law to determine the creep flow conditions in the bars of the framework. Creep strains are computed from the stresses in the bars, and vary in magnitude and direction from one time interval to the next by Shanley's step-by-step procedure. The creep strains introduced at the end of each time interval disturb the equilibrium and continuity conditions at the node points of the framework. These are re-established by means of a relaxation process which involves a physical adjustment scheme for the deflections and rotations at the node points.

The procedure, as in the Cross moment distribution method for frameworks, employs distribution, carry-over, and stiffness coefficients, and has been programmed for the ILLIAC, the high speed digital computer of the University of Illinois. Numerical results for stresses and deflections at node points can be obtained at regular time intervals as the dome creeps. The results of some illustrative cases are summarized in a set of tables and some trends are indicated graphically.
I INTRODUCTION

1. Object and Scope

The investigation is concerned with the analysis by means of an analogous framework of the deformation behavior as a function of time of a clamped shallow spherical shell of constant thickness subjected to rotationally symmetrical pressures acting radially inward, and to high temperatures causing creep.

For any given case the temperature is assumed to be constant throughout the shell and constant in time so that the effects of temperature induced strain rates can be considered to be negligible. Also the nature of the clamped edge is such that no stresses are introduced when the shell is heated to a given temperature level. Since creep strains of the order of magnitude of 1 or 2 percent are developed in the structure under study, the effects of time dependent plastic behavior are also assumed to be negligible [1]. Thus, in the analysis the effects of elastic and steady or secondary creep strain rates alone are considered. The effects of transient or primary creep strain rates are classified under time dependent plastic behavior.

A practical example of the type of structure considered in this analysis is the blunt end of the nose cone of a missile in supersonic flight as it is subjected to radially inward pressures and to high temperature or rapid creep over a time span of only minutes. Actual pressure distributions and heating temperatures for this type of structure may be obtained from a NASA Memorandum by Cooper and Mayo. [2]

Analytical tools such as variational methods are available for the solution of the creep problem [3]. However, the steady creep of thin-walled structures still remains a problem of considerable analytical difficulty [4].

* Numbers in brackets refer to the References at the end of the text.
In view of these difficulties and also to make the problem more suitable for computer use, analysis by means of a substitute mechanism is suggested in order that the creep solution of the shell may be simplified. The mechanism which represents a shell with a Poisson's ratio of zero was first proposed by Duberg in a recent interim report to the Air Force Special Weapons Center [5].

In an experimental investigation, Dorn [6] showed that for pure aluminum and its dilute solid-solution alloys at low stresses, the high temperature steady creep rate $\dot{\varepsilon}$ is related to the $\sigma$ by $\dot{\varepsilon} = \sigma^n$. At high stresses the steady creep rate depended on a more complicated exponential function of $\sigma$ and the time $t$. In general, however, the first stress-strain relationship may be used to approximate the high temperature creep behavior of metal at all stress levels. It was found to be particularly useful in column theory because it permitted accurate mathematical formulation [7], [8].

Tables have been prepared by Dorn and Tietz [9] which give for a variety of aluminum alloys numerical values of constants which are pertinent to the power function relationship. Additional data may be obtained from a report prepared by Simmons and Cross for the ASTM-ASME Joint Committee on Effect of Temperature on the Properties of Metals [10].

The idea to approximate non-linear stress-strain curves by a power function relationship is not new. One of the earlier empirical stress-strain equations of this type that has proved to be of practical use to the structural engineer was the Ramberg-Osgood formula in plasticity [11]. Hoff [12] and Odqvist [13] extended the relationship to include the effects of creep strain rates in uniaxial and multiaxial problems.

McVetty was the first to plot the isochronous stress-strain diagrams for secondary or steady creep [14]. He suggested that this would be a very
useful way to present creep data. It was from this type of diagram that Shanley [15] postulated his engineering hypothesis in order that creep data obtained from constant uniaxial tensile stress tests may be generalized to creep problems which are subjected to variable stresses.

Because of the incompleteness of the experimental information on creep under multiaxial stress, the use of a simplified flow law seemed to be warranted at the present time. Wahl [16] has recently suggested a law which is related to the Tresca yield criterion of the theory of plasticity and the power function stress-strain relationship of uniaxial creep. The law greatly simplified the analytical treatment of the creep of rotating disks.

The purpose of this investigation is to develop a procedure of analysis which can be conveniently applied to shells with a large variety of parameters. This study involves the following two phases: the development of a numerical procedure that is suitable for use on a high speed digital computer, and the preparation of programs for the ILLIAC, the digital computer of the University of Illinois.

The adopted method of analysis for the creep in domes is based on a mechanism which is fundamentally the same as that used by Duberg, but it differs in the way convergence for the iterative cycling of the deflections is obtained, and also in the way it is extended for solution of the creep problem. Duberg's mechanism was developed to handle the dynamic analysis of domes subjected to blast loads. The convergence of the solution of the deflections associated with the dynamic equilibrium of the mechanism at a certain instant of time was achieved by means of a cyclic integration procedure of the accelerations of concentrated masses at panel points. The convergence of the creep displacement solutions of the mechanism which are obtained by a quasi-static relaxation
process of moments and thrusts at panel points is based essentially on a forward finite difference approximation of the Newton-Raphson iteration process [17]. The simplified creep flow law as suggested by Wahl, together with Stanley's engineering hypothesis are used to determine the creep behavior in terms of multiaxial stress resultants along top and bottom surfaces of the bar elements of the mechanism.

A program has been developed for the ILLIAC, the high speed digital computer of the University of Illinois, which handles the creep response of the shell subjected to radially inward directed pressures. The distribution shape of the rotationally symmetrical pressures along a meridian of the shell may be varied in time, the restriction being that the variations are not excessive so that dynamic action may be neglected. The pressure variations on the shell are proposed to be introduced into the analysis in a stepwise manner in accordance with Shanley's engineering hypothesis regarding creep. An additional program has been developed which handles the elastic response of the shell only. It is very useful for the study of the elastic buckling phenomenon of the shell subjected to different rotationally symmetrical pressures. In their present form, these programs are restricted to shells with clamped edges; however without much difficulty they can be generalized to include cases having edges of variable fixity. There are virtually no restrictions placed on the values of the physical properties of the metals used in the creep program, although a metal requiring a power constant in the stress-creep strain rate relationship of large magnitude, say 50 or 60, may require a special study of the problem of scaling the numbers used in the digital computer program. Some numerical solutions have been obtained for various types of loading representative of the different cases the programs are capable of solving. The metal considered for these cases is 24S-T3 aluminum at 600°F.
The details of the method of analysis are discussed in Chapter II. A description of the computational procedure and general organization of the creep program is given in Chapter III. Chapter IV deals with the numerical solutions.

2. **Notation**

The symbols are defined as they are first introduced in the analysis. They are summarized here in alphabetical order for easy reference.

- \( d \) = thickness of a face sheet of a shell.
- \( D = \frac{Eh^3}{12(1-\nu^2)} \) = flexural rigidity per unit width of the homogeneous shell.
- \( E \) = Young's modulus of elasticity of the metal.
- \( F_{\theta j} \) = meridional membrane force in panel \( j \).
- \( F_{\varphi j} \) = circumferential membrane force at joint \( j \).
- \( F_{\theta j} \) = meridional fixed-end membrane force in panel \( j \).
- \( F_{\varphi j} \) = circumferential fixed-end membrane force at joint \( j \).
- \( \Delta F_{\theta j} \) = meridional "fixed-end" creep membrane force at panel \( j \).
- \( \Delta F_{\varphi j} \) = circumferential "fixed-end" creep membrane force at joint \( j \).
- \( h \) = thickness of the homogeneous shell.
- \( H \) = distance between centers of the face sheets.
- \( i \) = suffix indicating the number of the iteration performed in the modified Newton-Raphson procedure.
- \( j \) = suffix indicating the joint or panel in consideration.
- \( M_{\theta j}^L, M_{\theta j}^R \) = meridional bending moments on the left and right side, respectively, of joint \( j \).
- \( M_{\varphi j} \) = circumferential bending moment at joint \( j \).
\( M_{ij}^{\text{LF}}, M_{ij}^{\text{RF}} \) = meridional fixed-end moments on the left and right sides, respectively, of joint \( j \).

\( M_{ij}^{\text{F}} \) = circumferential fixed-end moment at joint \( j \)

\( \Delta M_{ij}^{\text{LF}}, \Delta M_{ij}^{\text{RF}} \) = meridional "fixed-end" creep moments on the left and right sides, respectively, of joint \( j \).

\( \Delta M_{ij}^{\text{F}} \) = circumferential "fixed-end" creep moment at joint \( j \).

\( n \) = creep power constant.

\( N \) = number of meridional beam segments.

\( p \) = normal pressure per unit area of shell.

\( \bar{p} = p(R^2) \) = non-dimensional intensity of normal pressure on the shell.

\( R_j \) = concentrated applied radial force at joint \( j \).

\( r \) = radial coordinate to any point in the shell.

\( R \) = radius of middle surface of the shell.

\( r_j \) = total resisting force at joint \( j \).

\( r_{ij} \) = total resisting force at joint \( j \) for the \( i \)th iteration in the modified Newton-Raphson procedure.

\( S \) = value of the equation of the v. Mises ellipse.

\( \tau \) = length of time in which creeping took place.

\( \Delta \tau \) = interval of time in which creeping took place.

\( T \) = length of time in which the loads acted.

\( \Delta T \) = interval of time in which a certain loading configuration acted.

\( u \) = meridional displacement.

\( u^o \) = meridional displacement of the middle surface.

\( u_j \) = meridional displacement of joint \( j \).
\[ V_j = \text{radial shear force at joint } j. \]
\[ w = \text{radial displacement} \]
\[ w_j = \text{radial displacement of joint } j. \]
\[ w_{j,i} = \text{radial displacement of joint } j \text{ for the } i^{th} \text{ iteration in the modified Newton-Raphson procedure.} \]
\[ x = \text{coordinate along the axis of a meridional beam segment of length } R\Delta\theta. \]
\[ z = \text{radial coordinate in the shell as measured from the middle surface.} \]
\[ \dot{e} = \text{resultant strain rate vector of the principal strain rates.} \]
\[ \dot{\varepsilon}_1, \dot{\varepsilon}_2 = \text{principal strain rates in the directions 1 and 2, respectively.} \]
\[ \varepsilon_\theta, \varepsilon_\varphi = \text{strains in the meridional and circumferential directions, respectively.} \]
\[ \varepsilon^0_\theta, \varepsilon^0_\varphi = \text{middle surface strains in the meridional and circumferential directions, respectively.} \]
\[ \Delta\varepsilon^T_j, \Delta\varepsilon^B_j = \text{incremental creep strains associated with an increment of time } \Delta t \text{ in the top and bottom face sheets, respectively, in the meridional beam } j. \]
\[ \theta = \text{meridional coordinate of the shell.} \]
\[ \theta_j = \text{meridional coordinate of joint } j. \]
\[ \bar{\theta}_j = \text{meridional coordinate of the midpoint of panel } j. \]
\[ \Delta\theta = \text{angle opening associated with the meridional beam length } R\Delta\theta. \]
\[ \lambda = \text{creep constant.} \]
\[ \Lambda = \text{half angle opening of the shell.} \]
\[ \nu = \text{Poisson's ratio.} \]
\[ \bar{\sigma} = \text{resultant stress vector of the principal stresses.} \]
\[ \sigma_1, \sigma_2 = \text{principal stresses in the directions 1 and 2, respectively.} \]
\[ \sigma_\theta, \sigma_\varphi = \text{stresses in the meridional and circumferential directions, respectively.} \]
\( \sigma_{\theta}^{O}, \sigma_{\varphi}^{O} \) = middle surface stresses in the meridional and circumferential directions, respectively.

\( \sigma_{\theta}^{T}, \sigma_{\varphi}^{T} \) = average total stresses as determined from the total meridional stresses due to both bending and direct forces on the right and left side of joint \( j \), in the top and bottom face sheets, respectively.

\( \sigma_{\theta}^{J}, \sigma_{\varphi}^{J} \) = total circumferential stresses due to both bending and direct forces at joint \( j \), in the top and bottom face sheets, respectively.

\( \sigma_{\theta}^{LT}, \sigma_{\theta}^{LB} \) = meridional stresses due to bending on the left side of joint \( j \) in the top and bottom face sheets, respectively.

\( \sigma_{\theta}^{RT}, \sigma_{\theta}^{RB} \) = meridional stresses due to bending on the right side of joint \( j \) in the top and bottom face sheets, respectively.

\( \sigma_{\varphi}^{T}, \sigma_{\varphi}^{J} \) = circumferential stresses due to bending at joint \( j \) in the top and bottom face sheets, respectively.

\( \sigma_{\theta}^{c} \) = meridional stress due to direct forces in panel \( j \).

\( \sigma_{\varphi}^{c} \) = circumferential stress due to direct forces at joint \( j \).

\( \varphi \) = circumferential or longitudinal coordinate of the shell.

\( \Delta \varphi \) = circumferential angle opening associated with the meridional beam width, \( R \sin \phi_j \Delta \varphi \), in the \( j \)th panel.

\( \theta_j \) = rotation of joint \( j \) in the meridional plane.

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II. METHOD OF SOLUTION

4. Statement of Problem

The shell considered is shown in Fig. 1. It can be subjected to any type of varying external pressure which is symmetrical with respect to the axis of rotation of the shell. The edge of the shell is clamped in such a manner that no temperature stresses are developed and the induced strains are negligible. The shell is subjected to high temperatures which cause creep. Pressures can be varied in time; however, the pressure changes should be small since the effects of dynamics are not treated in the analysis. Temperatures are held constant in time and are also held constant throughout the shell. This restriction rules out temperature induced stresses and strains.

The method used in solving the creep behavior of shallow domes can be broken down into two distinct computational parts. The first is the elastic solution in which the dome is approximated by a mechanism which essentially follows the elastic behavior of shallow domes. The second, the creep solution, is a step-by-step procedure by which deflections and corresponding stresses due to creeping of the material after each successive short time interval can be obtained.

In the following description of the method of analysis an attempt has been made to bring out and emphasize the physical significance of the steps involved. Assumptions made to simplify the analysis will also be indicated with each step.

5. Theoretical Analysis of Shallow Spherical Shells

The differential equations for the symmetrical deformation of shallow domes were originally obtained by Chien [18]. The theory is based upon assumptions of small strains and a linear stress-strain relationship. These equations can be
derived also from a variational principle. The potential energy of the shell is

\[ P.E. = \int \frac{1}{2} s_{ij} e_{ij} \, dV + \int p w dS \]  

(1)

where \( s_{ij} \) are the components of the stress tensor, \( e_{ij} \) are the components of the strain tensor, \( w \) is the radial displacement, and \( p \) is the external pressure, positive inward. See Fig. 1.

The stress-strain relations for the rotationally symmetrical dome are

\[ s_{11} = \sigma_\theta = \frac{E}{1-\nu^2} (\varepsilon_\theta + \nu \varepsilon_\phi) \]  

(2)

\[ s_{22} = \sigma_\phi = \frac{E}{1-\nu^2} (\varepsilon_\phi + \nu \varepsilon_\theta) \]

where \( \nu \) is Poisson's ratio, and \( E \) is Young's modulus of elasticity.

Let \( r \) be the radial distance to any point in the shell, and \( R \) the radius of the middle surface. For the case of spherical shells of small curvature with small displacements, \( u \) and \( w \), and letting \( \sin \theta \approx \theta \) and \( \cos \theta \approx 1 \), the non-linear strain displacement relations for rotationally symmetric deformations in a spherical coordinate system \((\theta, \phi, r)\) are

\[ \varepsilon_\theta = \frac{1}{r} (u_\theta - w) + \frac{1}{2} \left( \frac{w_\theta}{r} \right)^2 \]

\[ \varepsilon_\phi = \frac{1}{r} (u_\phi - w) \]  

(3)

where subscripts on \( u \) and \( w \) denote differentiation.

In thin shell theory the following assumptions are made in connection with the distribution of the displacements through the thickness of the shell,

\[ u = u^0 + \frac{z}{R} u^0_\theta; \quad w = w^0 \]  

(4)
where \( u^0 \) and \( w^0 \) are the displacements of the middle surface, and \( z = R-r \).

Substituting Eqs. (2), (3), and (4) into Eq. (1), and performing the \( z \) and \( \phi \) integration, the potential energy may be written as

\[
P.E. = \frac{\pi \varepsilon_c h R}{1 - \nu^2} \int_0^\Lambda \left( \varepsilon_\theta^0 + \varepsilon_\phi^0 + 2 \varepsilon_\theta^0 \varepsilon_\phi^0 \right) \theta d\theta \]

\[
+ \frac{\pi \varepsilon_c^3}{12(1 - \nu^2) R^2} \int_0^\Lambda \left( w_\theta^2 + \frac{1}{\theta^2} w_\phi^2 + \frac{2\nu}{\theta} w_\theta w_\theta \right) \theta d\theta
\]

\[
- 2\pi \int_0^\pi \phi w d\theta d\theta
\]

where \( \varepsilon_\theta^0 \) and \( \varepsilon_\phi^0 \) are the strains on the middle surface, \( \Lambda \) is the half angle opening of the shell, and \( h \) the thickness of the shell.

Setting the first variation of Eq. (5) equal to zero results in the following two nonlinear equations of equilibrium.

\[
\frac{\partial}{\partial \theta} \left( \sigma_\theta^0 \right) = \frac{\partial}{\partial \phi} \left( \sigma_\phi^0 \right)
\]

\[
- \frac{h}{r} \left( \left( \sigma_\theta^0 + \sigma_\phi^0 \right) \theta + \frac{1}{R} \frac{d}{d \theta} \left( \theta w_\theta \right) \right)

+ \frac{D}{R} \left[ \left( \frac{d}{d \theta} \left( \frac{1}{\theta} \frac{d}{d \theta} \left( \theta w_\theta \right) \right) \right) \right]
\]

In addition to yielding Eqs. (6), the variational problem gives the following boundary conditions

\[
\left[ \frac{h}{R} \left( \sigma_\theta^0 \right) \delta u^0 + \frac{\theta}{R^2} \left[ - \frac{d}{d \theta} \left[ \frac{d}{d \theta} \left( \theta w_\theta \right) \right] + \nu w_\theta \right] \frac{D}{R^2} \frac{d}{d \theta} \left( h \sigma_\phi^0 w_\phi \right) \right]_{\theta = 0} = 0
\]

\[
+ \frac{D}{R} \left( \theta w_\theta + \nu w_\phi \right) \delta w_\theta \bigg]_{\theta = 0} = 0
\]
Eqs. (6) are two equations with three unknowns \( \sigma^{O}_{\theta}, \sigma^{O}_{\phi}, \) and \( w. \) It is also necessary that the compatibility equation for the middle surface is satisfied. This third equation can be obtained by eliminating \( u \) from Eqs. (3) for \( z = 0. \) Thus

\[
\frac{\partial}{\partial \theta} \left[ \sigma_{\phi} \phi - \nu \sigma_{\theta} \right] + \sigma_{\theta}^{O} + \nu \sigma_{\phi} + \frac{E}{R} (\phi \psi_{\theta}) + \frac{E}{2} \left( \frac{\phi^{O}}{R} \right)^{2} = 0 \tag{6}
\]

The various components of the equilibrium equations, Eqs. (6) can be interpreted in physical terms. The first of Eqs. (6) states that the total radial component of the circumferential force should be in equilibrium with the rate of change of the total radial component of the force in the meridional direction. The second states that the normal load should be in equilibrium with (1) the change in shear due to bending in the meridional plane, (2) the normal or radial component of the membrane forces in the undeflected shape of the shell, and (3) the additional normal component of the meridional membrane forces which is developed when the shell is deflected. By neglecting the effect of Poisson's ratio a mechanism is constructed which behaves according to these four basic actions. This mechanism obeys the boundary conditions, Eq. (7). They are simply; at \( \theta = 0, w_{\theta} = 0, \phi_{\theta} = 0 \) and at \( \theta = \Lambda, w_{\theta} = 0, w = 0, u^{O} = 0 \). The compatibility equation of the middle surface, Eq. (8) is automatically satisfied since all the quantities of the mechanism will be expressed explicitly in terms of the radial deflections, \( u \).

6. Mechanism for the Shell

Since the geometry and the loading of the shell are symmetrical with respect to the rotational axis, it is sufficient to investigate only a typical wedge of the dome. It is assumed that the wedge consists of a number of tapered beams of constant depth and varying width as shown in Fig. 2. This wedge is fixed.
along its bottom edge, and while its apex is allowed to translate radially, it is not permitted to rotate and translate meridially.

In Fig. 1 the spherical coordinates are depicted. Arrowheads indicate the positive directions of the meridional coordinate \( \theta \), the longitudinal or circumferential coordinate \( \varphi \), and the radial coordinate \( z \), the meridional displacement \( u \), the radial displacement \( w \), and the normal pressure \( p \). In Fig. 2, the numbering of the joints and panels are shown. The coordinate \( \theta_j \) is the angle opening to the midpoint of panel \( j \); coordinate \( \theta_j \) is the angle opening to joint \( j \). The joints are numbered 0, 1, 2 to \( N \) from the apex on down in the positive direction of \( \theta \). Similarily, the panels are numbered 0, 1, 2 to \( N-1 \) in the positive direction of \( \theta \).

On account of the non-linearity of the creep stress-strain rate relationships, an important simplication in the analysis is made by assuming that the total stress resultants due to bending and direct forces are carried along the top and bottom boundaries of the beams. This permits the direct addition of linear elastic stresses with nonlinear creep stresses. Fig. 3 shows one of the meridional beams with stress resultants being transmitted across the boundaries or face sheets. Arrow heads indicate positive directions. The beams have a light weight core of constant thickness \( H-d \) between two face sheets of a given material of thickness \( d \). The material obeys a non-linear creep law which will be discussed in a subsequent section. The radial shear is supposed to be carried completely by the core.

The sandwich beams of a typical wedge form the basic components of meridional resistance against loading of the shell; the circumferential resistance against loading may be visualized as a system of torsion and extension
springs fastened to the node points at one end and to a fixed ground position at the other. The interaction between the individual wedges can be likened to the behavior of springs acting at node points, since the deformation of the shell are assumed to be symmetrical about the axis of rotation. The meridional beams act as if they are placed over continuous supports at the junctions. These beams resist the rotations and translations of the supports as these supports are adjusted in order to reach equilibrium. The combined action of the meridional beams and the springs at the joints approximates then the behavior of a spherical shell.

The tapered meridional beam of length $R\theta$ is the basic element in the mechanism. At the junction or node point of each of the beams a concentrated radial force $P$ is applied which is resisted by membrane forces $F_\theta$ and moments $M_\theta$ in the beams, and membrane forces $A_F$ and moments $A_M$ in the springs. $A_F$ and $A_M$ are the components along a meridian of the forces in the circumferential direction. Forces $F_\theta$ and $M_\theta$ may be visualized as acting in circumferential beams. The torsion and extension springs represent the action of the components of these forces along a meridian. The magnitude of the external force $P$ is the product of the pressure and the area of the shell surface of half a beam above and below a junction. The circumferential forces and moments are concentrated in a similar fashion.

Figs. 4 and 5 show the membrane forces, the moments, and the shears which act as an element. The arrowhead indicates the positive directions of these quantities. The sign convention is as follows. The normal loads are positive if they act radially inward. Membrane forces are positive if they produce tension.
in the beams. Moments in the beams are positive if they produce tension in the top fiber. Moments about the joints are positive if they act in a counterclockwise direction. Shears are positive if they cause a counterclockwise rotation of the beams.

In Fig. 4 the meridional membrane force $F_{\theta_j}$ and $F_{\theta_{j-1}}$ are shown as they act on the $j^{th}$ joint. Rather than to act tangent to the middle surface of the original shell, the forces are assumed to act along a straight line between the deflected joints. The rotations of the lines of action of the forces with respect to the normal to the radius $R$ at joint $j$ can be computed from the radial deflections of the joints $j+1$ and $j-1$. (See Eq. 19)

The method used to find the elastic response of the shell to loading is based on a physical relaxation process of the membrane forces and the moments at the joints until equilibrium is reached. For this procedure so-called "fixed-end" forces and moments are established in terms of the radial deflections $w_j$ at the joints. Stiffness factors of the beams and the springs are obtained by considering unit translations and rotations of the joints. Carry-over factors of the membrane forces from the relaxed end to the fixed end of the beams are assumed to be unity; those of the moments as minus one-half since small beam elements are considered.

7. Tresca Type of Steady Creep Law

It is assumed in the following that the principal axes of the strain rates coincide at all times with those of the stresses. It is further assumed that during creep flow the yield hexagon remains centered at the origin and merely changes in size.

The Kirchoff-Love assumption states that normals to the undeformed middle surface of a thin shell remain straight, normal, and inextensional under
deformation, so that the state of stress in a thin shell can be assumed to be essentially two-dimensional. For plane stress the steady creep behavior is most conveniently discussed in terms of a rectangular coordinate system (see Figs. 6 and 7), where the principal stresses $\sigma_1$ and $\sigma_2$ are used as the coordinates.

In Fig. 6, an arbitrary state of plane stress represented by a point Q is shown. The vector $\hat{\varepsilon}$ is plotted at Q. The components of $\hat{\varepsilon}$ are the principal strain rates $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_2$ produced by $\sigma_1$ and $\sigma_2$, respectively. The creep law for the state of plane stress requires that the vector $\hat{\varepsilon}$ is normal to the v. Mises ellipse. The magnitude of the strain-rate vector is specified by the equation

$$\sigma_1 \dot{\varepsilon}_1 + \sigma_2 \dot{\varepsilon}_2 = \tilde{\sigma} \cdot \hat{\varepsilon} = f(\sigma)g(t)S_0 \quad (9)$$

where the function $f(\sigma)g(t)$ can be determined from uniaxial creep rate tests [19], and where $S_0$ can be determined by the equation for the v. Mises ellipse,

$$\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2 = S_0^2 \quad (10)$$

The quadratic terms in Eqs. (9) and (10) constitute a source of difficulty in the solution of most problems in plasticity and creep. Prager [20] and Koiter [21] handled this difficulty by approximating the v. Mises ellipse by a polygon.

In this study such an approximation was applied to the solution of the creep deformation of spherical shells. This approximation is better known as the Tresca hexagon.

Fig. 7 shows the Tresca hexagon passing through the stress point Q. The stress parameter $S_0$ now defines the size of the hexagon, and is related to the state of stress by the following

$$S_0 = \max \left( |\sigma_1 - \sigma_2|, |\sigma_1|, |\sigma_2| \right) \quad (11)$$
As in the case of the v. Mises flow condition, the direction of the strain rate vector of the Tresca flow polygon is also normal to the boundaries. For instance, for a state of stress represented by the point Q in Fig. 7 the strain rate vector is normal to the side AB.

As in Eq. (9) the magnitude of the strain rate vector is also determined by

\[ \sigma_1 \dot{\varepsilon}_1 + \sigma_2 \dot{\varepsilon}_2 = \ddot{\varepsilon} \cdot \ddot{\varepsilon} = f(\sigma)g(t)S_0 \]  

(12)

where \( f(\sigma)g(t) \) is the same as in Eq. (11), but \( S_0 \) is now determined by the conditions in Eq. (11). Eq. (12) may be written in the following form by observing the normality of the strain rate vector to the Tresca hexagon.

\[ \max. \dot{\varepsilon} = \frac{1}{2} f(\sigma)g(t)S_0 = f(\sigma)g(t) \]  

(13)

where \( \dot{\varepsilon} \) stands for \( \dot{\varepsilon}_1, \dot{\varepsilon}_2, - (\dot{\varepsilon}_1 + \dot{\varepsilon}_2) \).

The locus of the end points of the strain rate vectors corresponding to the states of stress represented by the Tresca hexagon is indicated in Fig. 7. For a state of stress represented by a corner of the polygon the strain rate vector is not uniquely determined. It can be situated anywhere between the normals drawn to adjacent sides of the hexagon.

In this analysis the directions and magnitudes of strain rate vectors at the corners are determined as follows. When the state of stress is represented by the corners A, C, D or F, the strain rate vector is considered to act in a normal direction to the sides AF or CD. The magnitude of the vector is the vector sum of the components \( \max. \dot{\varepsilon} \). At the corners B and E the direction of the strain rate vector is assumed to bisect the directions of the two normals of the adjacent sides AB and BC or the adjacent sides DE and EF, and furthermore the magnitude of the vector is assumed to be the vector sum of the two strain
rates perpendicular to the adjacent sides.

8. **Step-By-Step Procedure**

The empirical relationship for uniaxial creep most commonly applied to the high temperature creep analysis of more complex metal structures such as columns [12] is

\[
\dot{\varepsilon} = \frac{1}{E} \frac{d\sigma}{dt} + \frac{\sigma^n}{\lambda}
\]  

(14)

where \( \dot{\varepsilon} \) is the steady or secondary creep rate, \( t \) is time, \( \sigma \) is the uniaxial tensile stress constant throughout testing, \( \lambda \) is a constant, and \( n \) is a number equal to the slope of the straight line plot of stress versus creep strain rate. A few typical values \( \lambda \) and \( n \) are given in Table 1. They are taken from data presented in a paper by Dorn and Tietz [9].

In this investigation the effects of primary or transient creep are neglected, and secondary or constant rate creep is considered only. The function on the right hand side of Eq. (13) depends now on \( \sigma \) alone. The equation for plane stress creep may be written then as

\[
\max. \dot{\varepsilon} = f(\sigma) = \frac{1}{E} \frac{d\sigma}{dt} + \frac{\sigma^n}{\lambda}
\]  

(15)

where \( \dot{\varepsilon} \) stands for \( \dot{\varepsilon}_1, \dot{\varepsilon}_2, - (\dot{\varepsilon}_1 + \dot{\varepsilon}_2) \).

From Eq. (14) when integrated with respect to time a set of isochronous stress-strain curves can be drawn. Fig. 8 shows a typical set of these curves as they are plotted for different time instants. These iso-stress-strain curves provide information on the amount of deformation that will occur in a selected time interval under a steady stress. Shanley [15] has suggested a procedure whereby the behavior under changing stress can be predicted from iso-stress-strain curves.
In this method it is assumed that a constant stress rate curve can be approximated by a stepped curve representing a series of instantaneous changes in stress followed by a time interval of constant stress. The rate of stressing is equal to the stress increment $\Delta \sigma$ divided by the horizontal increment $\Delta t$. It is suggested that during the instantaneous stress increment $\Delta \sigma$, the point representing stress and strain moves parallel to the elastic line. During the constant stress increment, the point moves on a horizontal line for a distance equal to the creep strains involved for a time interval $\Delta t$. The justification for applying the creep strain this way is based on the grounds that for a constant stress rate, the elastic strain rate is also constant.

The method as presented by Shanley for the creep buckling of columns has been modified to suit the shell problem. Besides being governed by the Tresca-flow conditions as indicated in Eq. (15), rather than just basing the computation of the creep strains on the intensity of stress in the face sheets at the beginning of a time interval, it is based on average strains due to stress intensities which occur at the beginning and the end of each time interval.

To justify this point, consider the integration of Eq. (15) over the interval $\Delta t$.

$$\max.\Delta \varepsilon = \frac{\Delta \sigma}{E} + \int_{\Delta t} \left(\frac{\varepsilon}{\chi}\right)^{\frac{n}{2}} \, dt$$  \hspace{1cm} (16)

where $\Delta \varepsilon = \varepsilon_2 - \varepsilon_1$, $\Delta \varepsilon = -(\Delta \varepsilon_0 + \Delta \varepsilon_1)$, if $\varepsilon_1 = \varepsilon_0$ and $\varepsilon_2 = \varepsilon_1$. The sign of the second term in Eq. (16) must be suitably chosen for each condition of loading and for each value of $n$. For instance, the negative sign is needed when a compressive (negative) stress is applied and $n$ is an even power; thus both terms (the elastic and creep parts) of Eq. (16) make negative contributions to the strain $\varepsilon$. Consequently, the positive sign is needed for the application of any tensile stress.
Eq. 16 is approximately equal to

\[
\text{max. } \Delta \varepsilon = \frac{\Delta \sigma}{E} + \frac{1}{2} \left[ \left( \frac{\sigma_{t+\Delta t}}{\lambda} \right)^n + \left( \frac{\sigma_{\Delta t}}{\lambda} \right)^n \right] \Delta t
\]

(17)

where \( \Delta \varepsilon \) stands for \( \Delta \varepsilon_{\theta}, \Delta \varepsilon_{\phi}, - (\Delta \varepsilon_{\theta} + \Delta \varepsilon_{\phi}) \).

The computational process of taking the average of the creep strains at the beginning and end of a time interval is equivalent to a first approximation integration with respect to time of the actual creep strain rate. For small time intervals which involve incremental creep strains of the order of magnitude of about 25 per cent of the actual strains, it was found that this approximation was a very good one.

9. Creep Computational Procedure

The computational process which takes place in a typical incremental time interval is outlined in this section to bring out the basic assumptions which were made, and also to show the connection of the step-by-step procedure and the Tresca flow law with this process. The overall computational procedure is discussed in section 13 of Ch. III.

At the beginning of a time interval \( \Delta t \), the total stress resultants in the face sheets due to bending and direct thrusts are known at each of the joints. The Tresca flow law is used to determine the direction of the strain rate vector in the face sheets at each of the joints. The face sheets of the meridional and circumferential beams then creep according to the components of the prescribed strain rate vectors. Note that the torsion and extension springs mentioned in section 6 represent the action of the components of the forces and moments in the circumferential beams in the direction of the meridional beams.
In order that the Tresca flow law can be applied to the mechanism so that it creeps in the two-dimensional manner of the original shell, the following very necessary assumption is made. It is assumed that one-half of the meridional component of the circumferential stresses in the face sheets due to both bending and direct forces acts at one side of a joint, the other one-half at the other side. This is equivalent to assuming that the meridional components of the stresses in circumferential beams start to feed gradually into corresponding meridional beams at the half-way point of a meridional beam at the one side of a joint and will be transferred completely by the time the half-way mark is reached of the meridional beam at the other side of the joint. This eliminates the abrupt jumps in the values of the meridional forces and moments which will exist at the joints if concentrated circumferential forces and moments were considered to act directly at the joints. The assumption insures a much closer approximation to the actual shell behavior in which the components of the circumferential stresses gradually transfer into the meridional stresses as indicated by the first of Eqs. (6).

In the next step, in order to find the total strains at joints the incremental creep strains, which are functions of the stresses in the top and bottom face sheets, are integrated over the length of the beams in the meridional direction and, if indicated by the Tresca hexagon, they are also integrated over the length of the beams in the circumferential direction. In this operation the beams are considered to be freed at one of their neighboring joints, and fixed at the other joint. To re-establish continuity membrane forces, moments, and shears are applied at the liberated ends. These forces which are needed to push the ends back into their original positions may be described as "fixed-end" creep trusses, moments, or shears. Equilibrium which now has been dis-
turbed at the joints is re-established by the relaxation procedure which involves the adjusting of the joints as indicated in section 6. This step is equivalent to an instantaneous elastic stress change in Shanley's step-by-step method.

In the last step the strains and their "fixed-end" creep forces due to the stress intensities at the end of the time interval are computed and averaged with the "fixed-end" creep forces computed originally from the stress intensities at the beginning of the time interval as indicated by the integration formula Eq. (17), and then balanced again for equilibrium at the joints.

Finally, the above procedure is repeated for the next time interval and so on. The loads are assumed to remain constant for the duration of a time interval. They may be varied in a step-wise fashion with respect to time, and their consequent stress changes are introduced elastically into the mechanism in the manner prescribed by Shanley. (See step 8 of the overall procedure in section 13 of Ch. III).

10. Mode of Buckling

Special note should be made of the fact that in this investigation the case of symmetrical buckling only has been considered. It is possible that the lowest buckling mode is antisymmetrical; however, the framework which is constructed to represent the behavior of the shallow dome is not capable of assuming such a mode.

In defense of the symmetrical buckling mode assumption, the following could be said, however, (see the discussion on this subject by Reiss in the introduction of [22]); in the case of a thin elastic hollow sphere subjected to external pressures, it was observed experimentally that as the pressure is increased gradually from zero, the shell deforms uniformly by contracting into
smaller and smaller spheres. This mode of deformation continues until a certain critical pressure intensity is reached. At this intensity the dome suddenly "snaps through" or buckles into a nonspherical buckling configuration. This symmetric shallow "dimple" which shows itself at some point on the surface of the shell. The rest of the shell remains apparently spherical.

The argument that a cap or clamped shallow dome is but a part of the total sphere, led to the general acceptance of the symmetrical buckling assumption for the shallow elastic dome. If this type of reasoning may be extended to include the justification of a symmetrical buckling mode for the spherical shell which is subjected to creep remains to be seen. Further study in this area is needed.

11. Elastic Formulas

The internal resistance of the mechanism to normal forces at each panel arises from; (1) the change in shear at the joints due to bending in the meridional direction, (2) the radial components of the membrane forces $F_\theta$ and $F_\varphi$.

The first contribution to the resistance at joint $j$ can be written in terms of moments as

$$V_j - V_{j-1} = \frac{M_{\theta}^R + M_{\theta}^L}{R \Delta \theta} + \frac{M_{\theta}^R + M_{\theta}^L}{R \Delta \theta}$$

where the change of shear $V_j - V_{j-1}$ is positive in the radially inward direction, and $M_{\theta}^R$ and $M_{\theta}^L$ are the moments acting on the right side and left side, respectively, of joint $j$, positive in a counterclockwise direction.
The second contribution to the resistance at joint $j$ is, assuming small angles,

$$F_{\theta j} \left[ \frac{\Delta \theta}{2} + \frac{\dot{w}_{j+1} - \dot{w}_{j}}{R \Delta \theta} \right] + F_{\theta j-1} \left[ \frac{\Delta \theta}{2} - \frac{\dot{w}_{j} - \dot{w}_{j-1}}{R \Delta \theta} \right] + \Delta \Phi F_{\phi j} \theta (19)$$

where $F_{\theta j}$ and $F_{\phi j}$ are the membrane forces at joint $j$ in the meridional and circumferential directions, respectively. The radial resultant of the membrane forces is positive in the radially inward direction (see Fig. 4). It is assumed here that an adequate approximation to the average slope of a beam due to radial deformations is given by the difference in the deflections of its neighboring joints divided by beam length.

The "fixed-end" forces are introduced into the mechanism by non-rotational deflections $w$.

$$F_{\theta j}^{F} = -\Delta \Phi R \theta_{j} 2d E \left[ \frac{\dot{w}_{j+1} + \dot{w}_{j}}{2R} - \frac{1}{2} \frac{\dot{w}_{j+1}^2}{R \Delta \theta} \right] \tag{20}$$

where $F_{\theta j}^{F}$ is the fixed-end force acting over an average width of beam, $\Delta \Phi R \theta_{j}$, along the axis in the $j$th meridional beam, and

$$\Delta \Phi F_{\phi j}^{F} = \Delta \Phi R \Delta \theta 2d E \frac{\dot{w}_{j}}{R} \tag{21}$$

where $\Delta \Phi F_{\phi j}^{F}$ is the meridional component of the fixed-end circumferential force acting at joint $j$ over a width $R \Delta \theta$. Eqs. (20) and (21) are a finite difference approximation of the $w$ terms in the combined Eqs. (2) and (3) for $\nu = 0$. Note that positive displacements of $w$ produce compressive forces $F_{\theta j}^{F}$ and $F_{\phi j}^{F}$.

Equilibrium of the membrane forces is obtained by relaxing the tangential displacements $u$ at the joints. For this procedure the relative stiffness factors for the beams and extension springs that meet at a joint are required.
A positive displacement $u_j$ of the $j$th joint produces forces along the meridian of

$$F_{\theta j} = -\Delta \varphi R\dot{\theta}_j \frac{2dE}{R\Delta \theta} u_j$$

$$F_{\theta j-1} = \Delta \varphi R\dot{\theta}_{j-1} \frac{2dE}{R\Delta \theta} u_j$$

$$\Delta \varphi F_j = \Delta \varphi R\Delta \theta \frac{2dE}{R\Delta \theta} u_j$$

These equations are a finite difference approximation of the $u$ terms in the combined Eqs. (2) and (3) for $\nu = 0$. Note that a positive displacement $u_j$ yields a compressive force $F_{\theta j}$, and tensile forces $F_{\theta j-1}$ and $F_j$.

The relative extensional stiffnesses for a unit meridional displacement of joint $j$ are determined from Eqs. (24). They are for $F_{\theta j}$, $F_{\theta j-1}$, and $F_j$, respectively,

$$\frac{\ddot{\vartheta}_j}{\ddot{\vartheta}_j + \ddot{\vartheta}_{j-1} + \frac{(\Delta \theta)^2}{\theta_j}}, \quad \frac{\ddot{\vartheta}_{j-1}}{\ddot{\vartheta}_j + \ddot{\vartheta}_{j-1} + \frac{(\Delta \theta)^2}{\theta_j}}, \quad \frac{(\Delta \theta)^2/\theta_j}{\ddot{\vartheta}_j + \ddot{\vartheta}_{j-1} + \frac{(\Delta \theta)^2}{\theta_j}}$$

The carry-over factor of forces is positive and equal to unity.

The "fixed-end" moments acting over a meridional beam width, $\Delta \varphi R\dot{\theta}_j$, are determined by non-rotational deflections $w$ of the joints in a manner similar to that in framework analysis [23]. Assuming a positive relative displacement $w_{j+1} - w_j$ in the positive direction of the meridional coordinate $\theta$,

$$M_{\theta j}^{LF} = 6E \frac{\Delta \varphi R\dot{\theta}_j}{(R\Delta \theta)^2} \frac{dF_j}{2} (w_{j+1} - w_j)$$

$$M_{\theta j}^{RF} = 6E \frac{\Delta \varphi R\dot{\theta}_{j-1}}{(R\Delta \theta)^2} \frac{dF_j}{2} (w_j - w_{j-1})$$
where \( M_{ij}^{LF} \) and \( M_{ij}^{RF} \) are the fixed-end moments in the meridional plane about the left and right hand side respectively, of joint \( j \). The moment of inertia, \( \Delta \rho R_{ij} \frac{d H^2}{2} \), of a sandwich beam element in the meridional direction is computed from the average width of that beam element. The smaller the beam element the better this approximation will be. The non-rotational displacements \( w \) do not develop moments in the circumferential beam elements.

Rotational equilibrium is obtained by relaxing the "clamped joints". The bending stiffnesses of the beams and the torsion springs are computed by applying a rotation \( \psi \) at the joint \( j \).

A clockwise rotation \( \psi \) of the \( j^{th} \) joint produces the following positive moments about the joint of

\[
\begin{align*}
M_{ij}^{L} &= 4 \varepsilon E \frac{\Delta \rho R_{ij} \frac{d H^2}{2}}{R^2 \theta} \psi_j \\
M_{ij}^{R} &= 4 \varepsilon E \frac{\Delta \rho R_{ij-l} \frac{d H^2}{2}}{R^2 \theta} \psi_j \\
-C_{ij} M_{ij} &= E \frac{\Delta \rho R^2 \frac{d H^2}{2}}{R^2 \theta} \psi_j
\end{align*}
\]

Consequently, the relative bending stiffness for a unit meridional rotation of joint \( j \) are \( \frac{\Delta \rho M_{ij}}{M_{ij}} \), \( \frac{\Delta \rho M_{ij}}{M_{ij}} \), and \( \frac{\Delta \rho M_{ij}}{M_{ij}} \), respectively,

\[
\begin{align*}
\frac{\Delta \rho \psi_j}{4 \theta_j} + \frac{\Delta \rho \psi_{j-l}}{4 \theta_j} \frac{(\Delta \theta)^2}{\theta_j} + \frac{\Delta \rho \psi_{j-l}}{4 \theta_j} \frac{(\Delta \theta)^2}{\theta_j} = \frac{\Delta \rho \psi_j}{4 \theta_j} + \frac{\Delta \rho \psi_{j-l}}{4 \theta_j} \frac{(\Delta \theta)^2}{\theta_j} + \frac{\Delta \rho \psi_{j-l}}{4 \theta_j} \frac{(\Delta \theta)^2}{\theta_j}
\end{align*}
\]

Since small meridional beam elements are used, the carry-over factors of moments are assumed as \( \frac{1}{2} \). There is no moment or force carry-over in the circumferential beam elements; these elements behave symmetrically.
12. Creep Formulas

Fig. 9 shows a typical meridianal beam element. The right hand end of the beam has been released and allowed to creep, while the left hand end has been fixed. The so-called "fixed-end" creep forces, moments, and shears which are needed to push back the liberated end to restore continuity are indicated. Moments are positive when acting about the joint in a counterclockwise direction. Moments in the beam are positive when they produce tension in the top fibers.

First, the component along the meridian of the creep displacement of the liberated end is equated with the displacement \( u_j \) of the meridianal membrane force required to push the end back.

\[
\int_{0}^{R \Delta \theta} \frac{\Delta \varepsilon_{\theta}^T + \Delta \varepsilon_{\theta}^B}{2} d\xi = \frac{\Delta F_{\theta}^F}{\Delta \varepsilon R \theta_j} 2d E \tag{27}
\]

where \( x \) is a coordinate measured from the joint \( j \) along the meridianal beam in a direction of positive \( \theta \), \( \Delta \varepsilon_{\theta}^T \) and \( \Delta \varepsilon_{\theta}^B \) are the incremental creep displacements per unit length of beam in the top and bottom face sheets, respectively, of the \( j^{th} \) meridianal beam, and \( \Delta F_{\theta}^F \) is the incremental middle surface force along the \( j^{th} \) beam required to restore continuity in the meridianal direction.

Second, the amount of creep rotation in the meridianal direction at the liberated end is equated with the rotation at that end \( \psi_j \) of the moments \( \Delta M_{\psi}^{LF} \) and \( \Delta M_{\psi}^{RF} \).

\[
\int_{0}^{R \Delta \theta} \frac{\Delta \varepsilon_{\theta}^T - \Delta \varepsilon_{\theta}^B}{H} d\xi = \left( -\Delta M_{\psi}^{LF} \right)_{j} - \left( -\Delta M_{\psi}^{RF} \right)_{j+1} R \Delta \theta. \frac{\Delta \varepsilon_{\theta}^F}{\Delta \varepsilon R \theta_j} d H \tag{28}
\]
where $\Delta M_{\theta j}^{LF}$ and $\Delta M_{\theta j+1}^{RF}$ are the incremental meridional moments at the right and left end, respectively, of the $j$th meridional beam which are required to restore rotational continuity of the freed end, and as will be indicated presently to restore radial continuity as well.

Third, the component of creep deflection in the radial direction is equated with the deflection $w_j$ of the moments $\Delta M_{\theta j}^{LF}$ and $\Delta M_{\theta j+1}^{RF}$ to push the end back radially,

$$
\int_0^{R} \frac{\Delta \theta_j^T - \Delta \theta_j^B}{H} \, dx = \left( -\Delta M_{\theta j}^{LF} - 2\Delta M_{\theta j+1}^{RF} \right) \frac{(R \Delta \theta_j)^2}{E \Delta \phi R \phi_j \, d \theta_j} \, dx 
$$

(29)

Eqs. (28) and (29) are two simultaneous equations with two unknowns, $\Delta M_{\theta j}^{LF}$ and $\Delta M_{\theta j+1}^{RF}$.

In the mechanism, the variation of the stress in the top and bottom face sheets along a meridional beam consists of two straight line distributions, one going from one joint to mid panel point, the other from mid panel point to the other joint, since one half of the meridional component of the stresses in the circumferential beam is assumed to feed in on one side of a joint, the other half on the other side (see section 9). However, due to the fact that small beam elements are considered, and also because of the fact that changes in variations of the components along the meridional beams of the circumferential forces are small and gradual, a linear distribution of the meridional stress is assumed in the derivations of the following equations.

Assuming $\sigma_{\theta j+1}^{T} > \sigma_{\theta j}^{T}$ and $\sigma_{\theta j+1}^{B} > \sigma_{\theta j}^{B}$, the incremental creep strain contribution to the total strain increment over the interval $\Delta t$ may be written as (See Eq. 16),

$$
\Delta \varepsilon_{\theta j}^T = k \left[ \left( \frac{\sigma_{\theta j}^{T} - \sigma_{\theta j}^{B}}{R \Delta \theta_j} \right) \frac{1}{H} \right] \Delta t
$$
\[ \Delta \varepsilon_{\theta_j}^B = k \left\{ \left[ \frac{B}{\sigma_{\theta_j}} + \left( \sigma_{\theta_j}^{B} - \sigma_{\theta_j}^{T} \frac{x}{R \Delta \theta} \right) \right] \right\}^n \Delta t \]  

(30)

where \( n \) is even, \( k = 1 \) when \( \left[ \sigma_{\theta_j}^{T} + \left( \sigma_{\theta_j+j+1}^{T} - \sigma_{\theta_j}^{T} \right) \frac{x}{R \Delta \theta} \right] > 0 \); \( k = -1 \), when \( \left[ \sigma_{\theta_j}^{T} + \left( \sigma_{\theta_j+j+1}^{T} - \sigma_{\theta_j}^{T} \right) \frac{x}{R \Delta \theta} \right] < 0 \) for \( n \) odd, \( k = 1 \), always. The \( k \) term in the second equation of Eqs. (30) has similar conditions. \( \sigma_{\theta_j}^{T} \) and \( \sigma_{\theta_j}^{B} \) are the average total stresses as determined from the meridional stresses due to both bending and direct forces at the right and left side of joint \( j \) in the top and bottom face sheets, respectively. Similarly, \( \sigma_{\theta_{j+1}}^{T} \) and \( \sigma_{\theta_{j+1}}^{B} \) are the average total stresses at joint \( j+1 \) in the top and bottom face sheets, respectively. To take the average value of the total left end and right hand stresses in a meridional beam is the same as assuming that one-half of the meridional component of the total circumferential stress acts at one side of a joint, and the other one-half at the other side (see section 9).

Substitution of Eqs. (30) into Eq. (27) and integration with respect to \( x \) of the resulting equation yields the equation for the so-called "fixed-end" creep forces

\[ \Delta F_{\theta_j}^F = \frac{\Delta \varphi \ R \theta_j \ 2 \pi \ E}{2(n+1)} \left\{ \left[ \frac{1}{\lambda \Delta \theta} \right]^n \Delta t \left[ \left( \sigma_{\theta_j}^{T} \right)^{n+1} - \left( \sigma_{\theta_j}^{T} \right)^{n+1} \right] \right\} + \frac{\Delta \varphi \ R \theta_j \ 2 \pi \ E}{2(n+1)} \left\{ \left[ \frac{1}{\lambda \Delta \theta} \right]^n \Delta t \left[ \left( \sigma_{\theta_{j+1}}^{B} \right)^{n+1} - \left( \sigma_{\theta_{j+1}}^{B} \right)^{n+1} \right] \right\} \]  

(31)
Note that the terms on the right hand side of Eq. (33) have been multiplied by 
-1 in order that $\Delta N_{\theta j+1}^{RF}$ will have the proper sign for the rotational sign con­
vention of the joints; i.e. positive moments are counterclockwise about a 
joint. The absolute sign brackets about the stress terms are a substitute for 
the conditions set by the term $k$ in Eqs. (30). The use of these brackets in 
the above equations automatically yields the correct signs for the "fixed-end" 
creep forces and moments. (also see Eqs. 16).

Special attention should be given to the case where $\sigma_{\theta j+1}^{T} = \sigma_{\theta j}^{T}$ and 
also where $\sigma_{\theta j+1}^{B} = \sigma_{\theta j}^{B}$. In these cases the Eqs. (31), (32), and (33) are inde-
terminate. Let the stress variations in the meridional direction between two 
joints $j$, and $j+1$ be constant in both face sheets. Then by going back to Eqs. 
(30) and eliminating the second or variable terms of these equations, and substi-
tuting the resulting equations in to Eqs. (27), (26), and (29) the following 
equations are obtained.

$$\Delta F_{\theta j} = \frac{\Delta \rho R^3}{2} \sqrt{\frac{2a}{E}} \left(\frac{1}{\lambda}\right)^n \Delta t \left[(\sigma_{\theta j}^{T})^2 + (\sigma_{\theta j}^{B})^2 \right]^{n-1}$$
Similarly, by substitution of Eqs. (30) into Eqs. (28) and (29), integrating with respect to \(x\), and solving the two resulting simultaneous equations for the "fixed-end" creep moments, one obtains,

\[
\Delta M_{\theta j}^{LF} = E \frac{\Delta \varphi \tilde{R}_{\theta j} \delta H}{n+1} \left\{ \frac{(\frac{1}{\lambda})^n \Delta t}{(\sigma_{\theta j+1}^{F \theta} - \sigma_{\theta j}^{F \theta})} \right\} \left[ \left| (\sigma_{\theta j+1}^{T \theta}) \right|^{n+1} + 2 \left| (\sigma_{\theta j}^{T \theta}) \right|^{n+1} \right]
\]

\[
- \frac{(\frac{1}{\lambda})^n \Delta t}{(\sigma_{\theta j+1}^{B \theta} - \sigma_{\theta j}^{B \theta})} \left[ \left| (\sigma_{\theta j+1}^{B \theta}) \right|^{n+1} + 2 \left| (\sigma_{\theta j}^{B \theta}) \right|^{n+1} \right]
\] (32)

\[
\frac{3(1)^n \Delta t}{(n+2)(\sigma_{\theta j+1}^{F \theta} - \sigma_{\theta j}^{F \theta})^2} \left[ \left| (\sigma_{\theta j+1}^{T \theta}) \right| \left| (\sigma_{\theta j+1}^{T \theta}) \right|^{n+1} - \left| (\sigma_{\theta j}^{T \theta}) \right| \left| (\sigma_{\theta j}^{T \theta}) \right|^{n+1} \right]
\]

\[
+ \frac{3(1)^n \Delta t}{(n+2)(\sigma_{\theta j+1}^{B \theta} - \sigma_{\theta j}^{B \theta})^2} \left[ \left| (\sigma_{\theta j+1}^{B \theta}) \right| \left| (\sigma_{\theta j+1}^{B \theta}) \right|^{n+1} - \left| (\sigma_{\theta j}^{B \theta}) \right| \left| (\sigma_{\theta j}^{B \theta}) \right|^{n+1} \right]
\]

and

\[
\Delta M_{\theta j+1}^{RF} = E \frac{\Delta \varphi \tilde{R}_{\theta j+1} \delta H}{n+1} \left\{ \frac{(\frac{1}{\lambda})^n \Delta t}{(\sigma_{\theta j+1}^{F \theta} - \sigma_{\theta j}^{F \theta})} \right\} \left[ 2 \left| (\sigma_{\theta j+1}^{T \theta}) \right|^{n+1} + \left| (\sigma_{\theta j}^{T \theta}) \right|^{n+1} \right]
\]

\[
- \frac{(\frac{1}{\lambda})^n \Delta t}{(\sigma_{\theta j+1}^{B \theta} - \sigma_{\theta j}^{B \theta})} \left[ 2 \left| (\sigma_{\theta j+1}^{B \theta}) \right|^{n+1} + \left| (\sigma_{\theta j}^{B \theta}) \right|^{n+1} \right]
\] (33)
The right hand side of the third equation of Eqs. (34) has been multiplied by \(-1\) so that \(\Delta M_{\theta j}^{RF}\) has the proper sign for the sign convention for moments about the joints. Should in the course of computations the meridianal stress variation be constant in one of the face sheets only, the terms of Eqs. (34) which concern that face sheet should be substituted for their counterparts in Eqs. (31), (32), and (33).

Since the variation of total stress resultants along the circumferential beams is always constant, the component "fixed-end" forces in the circumferential direction are,

\[
\Delta \varphi F_{\theta j}^{F} = E \frac{\Delta \varphi}{2 \Delta t} \frac{R \Delta \theta}{2} \left[ \left( \sigma_{\theta j}^{T} \right) \left| \frac{\sigma_{\theta j}^{T}}{\varphi_{j}} \right|^{n-1} + \left( \sigma_{\theta j}^{B} \right) \left| \frac{\sigma_{\theta j}^{B}}{\varphi_{j}} \right|^{n-1} \right]
\]

(34)

\[
\Delta \varphi M_{\theta j+1}^{RF} = E \frac{\Delta \varphi R \Delta \theta}{2 \Delta t} \left[ \left( \frac{\sigma_{\theta j}^{T}}{\varphi_{j}} \right) \left| \frac{\sigma_{\theta j}^{T}}{\varphi_{j}} \right|^{n-1} - \left( \sigma_{\theta j}^{B} \right) \left| \frac{\sigma_{\theta j}^{B}}{\varphi_{j}} \right|^{n-1} \right]
\]

where \(\Delta \varphi F_{\theta j}^{F}\) and \(\Delta \varphi M_{\theta j}^{F}\) are the incremental meridianal components of the direct force and moment, respectively, at joint \(j\), required to restore circumferential continuity at the joint. \(\sigma_{\theta j}^{T}\) and \(\sigma_{\theta j}^{B}\) are the total circumferential stresses due to both bending and direct forces at joint \(j\) in the top and bottom face sheets, respectively. Note that the right hand side of the second equation of Eqs. (35)
has been multiplied by -1, so that \( \Delta \varphi M^F \) has the proper sign for the sign convention for moments about joints.

Referring to Fig. 7, which is the Tresca flow hexagon, it has been assumed then that the states of stress, \( \sigma_\varphi > \sigma_\theta \) and \( -\sigma_\varphi > -\sigma_\theta \), represented by the sides BC and EF are never actually developed; where \( \sigma_1 = \sigma_\theta \), and \( \sigma_2 = \varphi \).

In other words, it has been suggested that the face sheets of the meridional beams creep continually, and that the face sheets of the circumferential beams will creep when the states of stress are represented by the corners B or E, or the sides AF or CD.

The assumption greatly simplifies the application of Eqs. (31), (32), (33), and (35). Eqs. (35) impose no difficulty when the flow hexagon prescribes that there is no flow in the face sheets at a joint in the circumferential direction (\( \sigma_\varphi < \sigma_\theta \), or \( -\sigma_\varphi < -\sigma_\theta \)). The incremental creep strain in that direction is then simply zero at that joint. In the meridional direction this is another matter. A state of stress in the face sheets at joint \( j \) represented by the side BC or EF of the flow diagram means that there is no incremental creep in the meridional direction. Assuming that at the neighboring joint \( j+1 \) the flow hexagon does prescribe an incremental creep strain, the straight line approximation of Eqs. (31), (32) and (33), connecting a state of zero stress at joint \( j \) to a state of stress at \( j+1 \) is a poor representation of the actual stress distribution along a meridional beam, and a more complicated formulation of the problem is called for. However, an assumption stating that such a state of stress cannot exist, even if the circumferential stresses in the face sheets at a joint were to be the larger at the end of a time interval, permits the use of Eqs. (31), (32), and (33) for every time interval, simplifying the computational process for the creep strains. The numerical values of stress used
in Eqs. (31), (32), (33), and (35) for this case are the values of the actual stresses computed for that instant; that is, the larger values \( \sigma_{\varphi} \) are not substituted for the values \( \sigma_{\theta} \) in Eqs. (31), (32), and (33); they are used in Eqs. (35) only.

A study of data showed that when \( \sigma_{\varphi} \) is larger than \( \sigma_{\theta} \) in a face sheet at a joint in one instant of time, it would not be the larger in one instant of time, it would not be the larger in the very next instant. The existence of the case of the joint for which \( \sigma_{\varphi} \) has become larger than \( \sigma_{\theta} \) may be attributed to the fact that in the course of re-establishing equilibrium and continuity of joints at the end of a time interval, the procedure has actually overshot its mark. By letting the face sheet or face sheets of the circumferential beam in which this \( \sigma_{\varphi} \) occurs creep in the next time interval, the situation is corrected, since stresses in this beam are relieved. Apparently the assumption that \( \sigma_{\varphi} \) could never actually be larger than \( \sigma_{\theta} \) is correct.
13. The ILLIAC* Programs

The ILLIAC programs have been written to handle radially inward directed pressures only. The pressure distributions considered are uniform over the shell area or they can vary along a meridian in a rotationally symmetrical fashion. The pressures can be varied in a step-wise manner with respect to time; that is, the distribution profile of the pressures along the meridian does not have to stay constant; it can be changed from one time interval to the next in the course of creep computations of a given shell.

A separate program has been developed which solves elastic problems only. However, since the elastic program is but a part of the total creep program it is not discussed separately. The problem of elastic buckling can be studied with it, thereby giving an upper limit to the loads that should be considered in the creep buckling problem. This program will also give the number of iterative cycles which are required for the convergence of the elastic solutions. From this the geometric buckling criterion for the creep problem is determined. At present the criterion is assumed as being twice the number of cycles required for elastic convergence.

The quantities evaluated are the meridional bending stresses in the top fibers on both sides of the joints, the circumferential bending stresses in the top fibers at the joints, the direct stresses in the meridional as well as the circumferential directions, and the deflections at all joints. All results are expressed in dimensionless form. These results are printed out at regular time intervals describing the time history behavior of the mechanism under high

*The ILLIAC is an automatic electronic digital computer of the University of Illinois [24].
temperature creep.

The parameters which have to be specified in the use of the creep program are the ratio of the creep constant over the modulus of elasticity of the material \( \frac{\lambda}{E} \) for a given temperature, the creep power constant of the material \( n \) for this temperature, the half angle of the half angle of the shell \( \Lambda \), the number of meridional beam segments \( N \), the ratio of the distance between the stress resultants in the face sheets over the middle surface radius of the shell \( \frac{R}{R} \), a creep time interval in terms of hours \( \Delta t \), the non-dimensionlized intensity of loading at each joints \( \tilde{P} = p \left( \frac{R^2}{4EdH} \right) \) and a predetermined geometric buckling criterion. This criterion is discussed in section 16. The time interval \( \Delta t \) is selected on the basis that the "fixed-end" creep forces are to be approximately 25 per cent of the first elastically induced total forces. The program in itself will refine the magnitude of the time interval so that no "fixed-end" creep force will exceed the 25 per cent limit at any time during the creep process. When an additional time limit \( \Delta t \) which specifies the duration through which the loads act, is included with the parameters, a new set of loads for the next load time interval should be added. The time duration of a set of loads is infinity when there is no load time interval \( \Delta t \) given with the set of load parameters.

The creep program utilizes the entire Williams (fast) memory of the ILLIAC and part of the magnetic drum (slow) memory. There is sufficient storage space in the high-speed memory to handle the analysis of a tapered beam section with 20 segments. Actual computations however, suggested that computer time-wise the most economical number of divisions in the tapered section is from 4 to 8.
The computer time necessary to obtain a complete creep record, that is when the solution is carried through to buckling, depends mostly upon the number of meridional beam segments involved, \( N \), the half angle opening \( \Lambda \), and the load intensity \( \overline{p} \). The remaining parameters do not affect the computational time as extensively. Assuming that the 25 per cent creep stress criterion determining the duration of the \( \Delta t \) interval is a satisfactory one, it was found that for a uniformly distributed non-dimensionalized pressure of \( \overline{p} = p\left(\frac{R^2}{EJH}\right) = 0.2 \), the computational time is approximately 2 1/2 hours for each of the following cases; when 4 segments are used for a section with half angle opening of 15 degrees, 6 segments are used for a section with a half angle opening of 30 degrees, and 8 segments for a section with a half angle opening of 45 degrees. When considering the same breakdown of segments for the respective half angle openings, but applying now a uniform pressure of \( \overline{p} = 0.4 \), it was found that the time required is about 1 1/2 hours.

With relatively minor modifications the present programs can be extended to handle shells with simply supported or partially fixed edges. Program changes could be also made to include shells of variable thicknesses.

14. Outline of Computational Procedure

The purpose of this section is to present an outline of the computer program, and in doing so to present also the overall computational procedure.

The procedure used consists of the following basic steps;

(1) Compute the shell constants and stiffness constants from the given parameters.

(2) Assume a trial set of radial displacements \( \Delta w_j \).
(3) From the radial displacements $\Delta w_j$, compute the incremental fixed-end forces and moments, and add them to the total forces and moments. In the first cycle, the elastic cycle, the total quantities are zero.

(4) For given $w_j$ displacements, two simultaneous relaxations are made. The unbalanced moments at each joint are distributed according to the rotational stiffnesses, and then carried over to adjacent joints. Similarly, the unbalanced forces at each joint are distributed according to the extensional stiffnesses, and then carried over to adjacent joints. By relaxing the membrane forces the joints translate in the $u$ direction until force equilibrium is obtained; by relaxing the moments the joints rotate meridianally until moment equilibrium is obtained.

(5) Using the total balanced moments and forces, compute the radial resisting force $R_j$ at joint $j$.

(6) Test if the difference between the applied force at joint $j$ and the resistance at that joint is smaller than a preset convergence criterion; $P_j - R_j < \varepsilon$. If smaller, go to (7); if not smaller, assume the next $\Delta w_j$, add this value to the total displacement $w_j$, and repeat steps (3) to (6) inclusive. After a maximum of $h$ iterations, regardless of whether $P_j - R_j < \varepsilon$, go to (7).

(7) Assume the next $\Delta w_{j+1}$, and add this to the total displacement $w_{j+1}$ at joint $j+1$. Repeat steps (3) to (7) inclusive until all joints are in equilibrium with the applied forces, then go to (8). The interpolation procedure used in the $\Delta w_j$ and $\Delta w_{j+1}$ assumption cycles is discussed in detail in section 14.

(8) Test if the end of the load time interval is reached. If it is reached read in the next set of load parameters, then repeat steps (2) to (8) inclusive and proceed to (9). If it is not reached proceed to (9) directly.
In this step the mechanism is allowed to creep over a time interval \( \Delta t \) in the manner described in section 9 of Ch. II. The average "fixed-end" creep forces and moments due to stresses at the beginning and at the end of the time interval are computed and added to the total moments and forces. The now unbalanced moments and forces at the joints are in turn distributed and carried over by the relaxation procedure of steps (4) to (7) inclusive.

The steps (3) to (9) inclusive are repeated until the mechanism will either fail by buckling geometrically or by excessive stress changes for small changes in time. These buckling criteria are discussed in detail in section 15. Note that the steps (1) to (7) inclusive made up the elastic program.

Since this computational procedure for the creeping of the shell with small curvature is based on step-by-step creep calculations over incremental time intervals as governed by the Tresca flow rule, the method of loading of the shells (loading history) plays an important role in the outcome of the results. For instance, loading the shell to a certain level and then unloading it by an equal amount over the same length of time does not cause the shell to reassume its first deflection configuration, but will cause a certain amount of set to have formed together with certain residual stresses. The residual stresses in turn cause the shell to continue creeping, or as it is then called, to relax.

The basic steps of the procedure are summarized in Fig. 10 in the form of a flow diagram. A complete description of the program has been deposited in the ILLIAC library of the Structural Research Laboratory of the Department of Civil Engineering at the University of Illinois.
15. **Interpolation Procedure**

The procedure is essentially a forward finite difference approximation of the Newton-Raphson interpolation formula,

\[ w_{j,i+1} = w_{j,i} - (P_j - R_{j,i}) \frac{\Delta w_{j,i}}{R_{j,i} - R_{j,i-1}} \]  

(36)

where \( P_j \) is the concentrated applied radial load at joint \( j \), \( R_{j,i} \) is the total resisting force of the \( i^{th} \) iteration at joint \( j \), \( w_{j,i} \) is the radial displacement of the \( i^{th} \) iteration at joint \( j \), and \( \Delta w_{j,i} \) is the incremental radial displacement, from \( w_{j,i-1} \) to \( w_{j,i} \).

For the interpolation method to be effective, the first assumed value \( w_{j,1} \) at the joint \( j \) should be close to the correct value \( w_j \). The procedure proposed to give these initial values which are in the vicinity of the correct values or roots is based on interpolation by means of the tangent or cotangent ratios of the absolute values of the total resisting force \( R_{j,i} \) and the total radial displacement \( w_{j,i} \) depending on whether the ratio \( \frac{|R_{j,i}|}{|w_{j,i}|} \) is smaller or larger than unity. The direction of interpolation depends also on whether the response of the resisting force to increasing values of deflection is of an ascending or descending nature.

Plotting \( R_j \) versus \( w_j \) of the joint \( j \), and assuming a response curve with positive slope for the dependent variable \( R_j \) and the independent variable \( w_j \) of joint \( j \), while holding all other joints fixed against radial translations, the following interpolation formulas may be written,

for \( R_{j,i} < w_{j,i} \) as,

\[ w_{j,i+1} = w_{j,i} + (P_j - R_{j,i}) \frac{R_{j,i}}{w_{j,i}} \]  

(37)
for $R_{j,i} > w_{j,i}$ as,

$$w_{j,i+1} = w_{j,i} + (P_j - R_{j,i}) \frac{w_{j,i}}{R_{j,i}}$$

These formulas apply then to the case in which positive changes in $w_j$ give positive changes in the resisting force $R_j$ (positive slope).

For a response curve with negative slope the formulas become

for $R_{j,i} < w_{j,i}$,

$$w_{j,i+1} = w_{j,i} - (P_j - R_{j,i}) \frac{R_{j,i}}{w_{j,i}} \quad (38)$$

for $R_{j,i} > w_{j,i}$,

$$w_{j,i+1} = w_{j,i} - (P_j - R_{j,i}) \frac{w_{j,i}}{R_{j,i}}$$

These formulas apply to the case in which positive changes in $w_j$ give negative changes in the resisting force $R_j$ (negative slope). This type of response for a joint is not likely to occur; however, it is possible and should be considered.

For the formulas Eqs. (37) and (38) convergence is rapid in the early stages, but it is very slow in the later stages of iterative cycling. However, the fact that these formulas are so rapidly convergent in the first few iterations makes them very useful to the Newton-Raphson formula Eq. (36) which is powerfully convergent once the assumed values are in the vicinity of the root. Furthermore, the formulas Eqs. (37) and (38) give a consecutive set of $\Delta w$'s of decreasing magnitude and same sign which is beneficial to the moment and force distribution and carry-over procedure. Gradual and reasonable changes in $\Delta w$'s result in a smaller number cycles required for the convergence of the distribution procedure thereby reducing computer time.
For the Newton-Raphson formula to become effective in the overall interpolation process, it was observed that only two iterations were required from the interpolation method of Eqs. (37) and (38). These first two iterations are then followed by two iterations of the Newton-Raphson method. The total of four iterations are the maximum number of iterations performed at each joint in one cycle of interpolations of all joints. The process is repeated at each joint until there is no joint left which needs to be iterated; that is, until all joints are in equilibrium. See step (6) of section 13.

16. Buckling Criteria

The geometric buckling criterion may be defined as follows: An element of the shell is said to be in the process of geometric buckling when deflections increase uncontrollably with no apparent increase in the loads.

The typical nonlinear loading curve of the pressure $p$ versus the deflection $w$ of the crown of the shallow dome, when plotted starts out at the origin of the axes $p$ and $w$ and culminates into a first maximum after which there will be a sudden break downward in the curve representing the so called "snap-through" phenomenon. Buckling is represented by the first maximum. In the vicinity of that point on the curve, small changes in pressure cause large changes in deformation, which means that at this point of maximum pressure the slope is zero, and consequently in the limit Eq. (56) tends to infinity. In other words, the Newton-Raphson procedure is not able to attain convergence. The post buckling behavior of a shallow dome can be represented by another loading curve which starts out beyond the break at some point of minimum pressure intensity and corresponding deflection and then proceeds to climb to a new maximum. The behavior after "snap-through" has not been considered in the investigation.
In the problems considered for this report the geometric buckling criterion is assumed to take place when the number of iterative cycles necessary for the balancing of all joints in the creep process is larger than two times the amount of cycles required to establish the elastic equilibrium of the problem of a dome which was tested previously under similar loading conditions by the computer program which solves elastic problems only. It has been observed that the number cycles needed for the elastic convergence of the solution at the end of each creep time interval remains fairly constant throughout creeping, and that when excessive deflections become apparent they tend to increase only slightly. When buckling is apparent, convergence is suddenly unattainable from one time interval to the next. From these observations buckling appears to be adequately detected by testing for excessive cycling against a counter which is specified as being two times the predetermined cycling counter of the elastic problem.

For certain metals at certain temperature levels geometric buckling does not control. The phenomenon which controls in these cases is referred to as creep buckling. The creep buckling criterion may be defined as follows: An element of the shell is considered to be in the process of creep buckling when the stress in the element increases very rapidly for only a small increment of time. Creep buckling is an asymptotic process where the increase of stress with respect to time tends to infinity when a certain time limit is approached.

For the computer program at present, the shell is assumed to have creep buckled when the creep time interval which is determined by the 25 percent limit imposed on the "fixed-end" creep forces with respect to the total forces becomes less than one fourth the magnitude of the first creep time interval of the computational process. In other words, the shell is considered to
have buckled when the largest stress rate of all joints in a certain time inter-

val becomes about four times the size of the largest stress rate in the

first creep time interval. The determination of the creep buckling time may

be refined by taking the asymptote to an extrapolated stress-time plot.
17. **Illustrative Problems**

This chapter is concerned with the presentation of some numerical results which were obtained with the aid of the ILLIAC for illustrative purposes. The intention is to indicate with these solutions the variety of problems that can be solved by the ILLIAC programs. For convenience the solutions are limited to a shell with a fixed set of geometric and material parameters. Only the pressure parameter is varied. The shell has the following two fixed geometric parameters; the half angle opening $\Lambda = 30^\circ$, and the ratio of the distance between stress resultants in the face sheets over the middle surface radius $\frac{H}{R} = .01$. The creep temperature assumed is $600^\circ F$. The material used in the face sheets of the shell is 24S-T3 aluminum for which at $600^\circ F$, $\lambda = 87,500$ lb.-in.$^{-2}$ hr.$^{1/3}$, $E = 7.4 \times 10^6$ lb.-in.$^{-2}$, and $n = 3$. The various problems considered include also the case when $t = 0$; the elastic case. For these problems the various convergence criteria of the computational process have been set such that a numerical accuracy of 5 decimal places is obtained for the results.

18. **Cases Considered**

To indicate that solutions for the elastic case, $t = 0$, can be obtained separately from the creep solutions, numerical results for stresses and deflections at the joints are listed in Table 2 for shells subjected to the uniform pressures $\overline{P} = 0.2$ and $\overline{P} = 0.4$. Figs. 11(a) and 11(b) show the trends of the meridional total stresses in the bottom face sheets and their corresponding radial deflections at the joints along a meridian of the elastic shell ($t = 0$) when it is subjected to a series of uniform pressures which range almost up to buckling. The meridional total stress at a joint is the average of
the meridional stress resultants of the bending and direct stresses at both sides of the joint.

Solutions for the creep case when pressures are rotationally symmetrical, but which vary along a meridian are listed in Table 3 for different instances of time. A parabolic pressure distribution is selected to illustrate this case. At the apex of the shell a maximum pressure intensity of $\tilde{p} = 0.2$ is chosen which tapers off parabolically along the meridian to an intensity of $\tilde{p} = 0.12$ at the edge. The slope to the parabolic pressure distribution curve is zero at the apex. This pressure is kept constant during creeping. Note that at the creep time 0.72 hours the shell has not yet buckled. At that instant the problem was taken off the ILLIAC since it had exceeded the permitted two hour computer time limit.

In Table 4, the numerical results are listed for the case where an uniformly distributed pressure $\tilde{p} = 0.4$ is applied on the shell and kept constant during creeping. By the criteria described in section 15 the shell has buckled geometrically at the time 0.0873 hours. To depict trends, maximum radial deflections which occur at the apex, and maximum total stresses which occur in the bottom face sheet at the fixed edge of the shell are plotted as a function of time in Fig. 12. Note the increase in the deflections and stress rates when buckling is imminent for $\tilde{p} = 0.4$. The solution for $\tilde{p} = 0.2$ has not been carried through to buckling.

In Figs. 13(a) and 13(b), the redistribution trends during creep of the meridional bending and direct stresses along a meridian of the shell are plotted at different time instants for the uniform pressure $\tilde{p} = 0.4$. In Fig. 13(a), the meridional bending stresses in the top face sheet are given as a fraction of their corresponding maximum meridional bending stresses in the top
face sheet are given as a fraction of their corresponding maximum meridional bending stresses in the top face sheet which occur at the fixed edge. Similarly, in Fig. 13(b) the meridional direct stresses are shown as fractions of corresponding meridional direct stresses at \( \theta/\Lambda = 0.5 \), which is the location of joint 3 when a shell with 6 beam segment divisions for its half angle opening is considered. The stresses are the averages of the stresses at both sides of the joints.

In Table 5, the results are listed for the case where the load is varied in time. The step-wise loading which in this case is kept uniformly distributed throughout creeping has the following schedule; from \( t = 0 \) to \( t = 0.03 \) hours, \( \bar{p} = 0.2 \); from \( t = 0.03 \) to \( t = 0.065 \) hours, \( \bar{p} = 0.25 \); from \( t = 0.065 \) to \( t = 0.105 \) hours, \( \bar{p} = 0.3 \); from \( t = 0.105 \) to \( t = 0.145 \) hours, \( \bar{p} = 0.35 \); and from \( t = 0.145 \) to \( t = \infty \) hours, \( p = 0.4 \). Geometric buckling occurred at 0.188 hours.

Pressures having changing distribution shapes along a meridian as a function of time also may be applied to the shell. However, the presentation of this case would only be an extended example of the parabolic pressure case when combined with the step-wise pressure case. It was thought that the separate presentation of the two cases give sufficient indication of the ability of the ILLIAC program to handle the shell subjected to any kind of time dependent loading condition for rotationally symmetrical pressures.
V SUMMARY AND CONCLUSION

19. Concluding Remarks

In this report a numerical procedure was developed which has been programmed for the ILLIAC, the high speed digital computer of the University of Illinois, and with which the creep deformation behavior of symmetrically loaded shallow spherical shells can be studied for any given constant temperature level. Some numerical results are presented in the form of tables and graphs in order to show the number of cases that may be handled by the computer programs.

The procedure has great flexibility in that any type of rotationally symmetrical pressure distribution can be applied to the shell, and also in that the pressure distribution can be changed at any given time in the course of computations.

The main disadvantage of a method of solution of this type is the face that the accuracy of the numerical results can not be determined by analytical means. Only by actually performing creep experiments on shallow dome specimens can a comparison study of results be made.

One way to check the accuracy of results at least partially would be to modify the computer program for the elastic case ($t = 0$) which was written for the sandwich shell (elasticity is concentrated in face sheets) such that it will include the case of the shell which is homogeneously elastic throughout its thickness. Results obtained with the modified program may be compared with results of other investigations which have been concerned with the stresses and deflections in the elastic unbuckled state as well as the load intensities of the buckled states of shallow spherical shells. In these studies approximate solutions of the boundary value problem (Eqs. 6, 7, and 8) have been obtained
by applying power series calculations [22], perturbation methods [25], linearizations [26], and finite difference techniques [27]. In this way, at least the accuracy of the elastic framework can be tested. Lack of time and funds prevented the author from performing the modifications in the computer programs necessary to make this check. However, the numerical results for the elastic case of the sandwich shell do compare favorably with the results obtained by Reiss and collaborators [22] [26] [27] for shells of similar size and loading.

The above mentioned methods of analysis are all limited to spherical shells which are subjected to uniformly distributed external pressures only. The analogous framework procedure on the other hand, enables us to study the nonlinear behavior of domes for which the external pressure can have any rotationally symmetrical distribution, which is decidedly an advantage.

In accordance with the Tsien, von Karman theory for the buckling of hollow-spheres [28], investigators so far have assumed that the lowest buckling mode for elastic domes is rotationally symmetrical about the main axis. As has been indicated in Section 10 of the report, the assumed buckling mode of the shallow dome subjected to creep at high temperatures is also symmetrical. It should be pointed out, however, that in the case of creep buckling of shallow spherical shells such a deflection configuration may not be the lowest possible mode. This possibility could be investigated with the aid of a framework which is able to deal with the antisymmetrical mode of deformation.
REFERENCES


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### TABLE 1
VALUES OF EXPONENT \( n \) AND CONSTANT \( \lambda \) IN
CREEP LAW FROM TESTS BY DORN AND TIETZ (9)

<table>
<thead>
<tr>
<th>MATERIALS</th>
<th>( 90^\circ F )</th>
<th>( 212^\circ F )</th>
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### TABLE 2
VALUES OF STRESSES AND DEFLECTIONS AT JOINTS FOR
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### TABLE 1 (continued)
VALUES OF EXPONENT \( n \) AND CONSTANT \( \lambda \) IN
CREEP LAW FROM TESTS BY DORN AND TIETZ (9)

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### TABLE 2 (continued)
VALUES OF STRESSES AND DEFLECTIONS AT JOINTS FOR
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### TABLE 3
VALUES OF STRESSES AND DEFLECTIONS AT JOINTS AS A
FUNCTION OF TIME FOR SHELL WITH $P = .2$ (PARABOLIC),
$\lambda/E = .0118243$ hrs.$^{1/2}$, $n = 3$, $\Lambda = 30^\circ$, $H/R = .01$, AND $N = 6$

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TABLE 4  VALUES OF STRESSES AND DEFLECTIONS AT JOINTS
AS A FUNCTION OF TIME FOR SHELL WITH \( p = .4 \) (UNIFORM),
\( \lambda / E = .0118243 \) hrs. \( 1/3 \), \( n = 3 \), \( \Delta = 30^\circ \), \( H/R = .01 \), and \( N = 6 \)

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FIG. 1 - SHELL GEOMETRY

FIG. 2 - SHELL NOTATION
FIG. 3 - STRESSES ON BEAM ELEMENT

FIG. 4 - MEMBRANE FORCES ON JOINT

FIG. 5 - MOMENTS AND SHEARS ON BEAM ELEMENT
FIG. 6 - V. MISES ELLIPSE

FIG. 7 - TRESCA HEXAGON
FIG. 8 - ISO-STRESS-STRAIN CURVES

FIG. 9 - CREEP FORCES ON BEAM ELEMENT
Read in Parameters and First Set of Loads of Jts.

Compute Shell Constants and Stiffnesses

Assume Initial W's or next ΔW's at All Jts.

Read in Next Set of Loads at Jts.

Add New ΔT to ΣΔT

Compute Fixed-End Quantities

Distribute and Carry Over at All Jts.

Add New Δt to ΣΔt

Add Creep Quantities to Total Quantities of Beginning of Creep Time Interval

Average Creep Quantities at the Beginning with Creep Quantities at the End of Creep Time Interval

All Joints in Equilibrium?

No. of Current Cycles > Buckling Criterion?

Yes

Assume Next ΔWj of Jt.j

Yes

Assume Next ΔWj+1 of Jt.j+1

No

End of Load Time Interval ΣΔT = ΣΔt?

Print Out and STOP

Print Out at End of Every 3 Creep Time Intervals or Each Load Time Interval

Compute Fixed-End Creep Quantities from Current Stress and Given Δt or Δt'
FIG. 11—MERIDIONAL TOTAL STRESSES AND RADIAL DEFLECTIONS ALONG A MERIDION FOR DIFFERENT UNIFORM PRESSURES ON A SHELL AT TIME \( t = 0 \), WITH \( \Delta = 30^\circ \), \( H/R = 0.01 \), AND \( N = 8 \).
FIG. 12 — MAXIMUM TOTAL STRESS AND DEFLECTION AS A FUNCTION OF TIME $t$ FOR SHELL WITH $\bar{p} = .2$ (UNIFORM), AND $\bar{p} = .4$ (UNIFORM), $\lambda/E = .0118243$ hrs$^{1/3}$, $n=3$, $\Delta = 30^\circ$, $H/R = .01$, AND $N = 6$. 
FIG. 13 - DISTRIBUTION OF MERIDIONAL BENDING AND DIRECT STRESSES ALONG A MERIDION AT DIFFERENT TIME INSTANCES FOR A SHELL WITH $\bar{p} = 0.4$ (UNIFORM), $\lambda/E = 0.118243$ hrs.$^{1/3}$, $n = 3$, $\Delta = 30^\circ$, $H/R = 0.01$, AND $N = 6$. 

(a) RATIO OF BENDING STRESSES $\sigma_0^T / \sigma_0^{T_{\text{max}}}$

(b) RATIO OF DIRECT STRESSES $\sigma_0 / \sigma_{0\text{tolv}}$