STUDIES OF PROBABILISTIC SAFETY ANALYSIS OF STRUCTURES AND STRUCTURAL SYSTEMS

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I. INTRODUCTION

1.1 General

The question of "how should a structure be designed for safety?" is not an easy one to answer. In the first place the measurement of structural safety is not clear cut. Engineers have been designing supposedly safe and economical structures through a "factor of safety." One way of doing this is basically to specify an allowable working stress which is equal to the yield stress, or some other limiting stress of a material, divided by the factor of safety. Every part of a structure is then proportioned such that no where in the structure is the working stress exceeded. An alternative procedure is to proportion the material of a structure such that its ultimate capacity (which may be determined on a conservative basis) is at least equal to the design load times the factor of safety. Clearly, the factor of safety approach assumes that both the structural capacity and the applied load are completely predictable quantities. In reality, however, this is often not the case. More likely, the load and structural capacity cannot be determined with certainty.

When information relating to loads and structural strengths, including maximum loads and minimum strengths, are not completely predictable, absolute safety cannot be assured. Statements of structural safety should then take into consideration the remote possibility of structural failure, which may mean partial damage, unservicability, or outright collapse. The notion of a probability of failure is then unavoidable, and any rational measure of structural safety ought to be based on the acceptance of this notion. When such a state of uncertainty exists and the load and strength variables can be described in probabilistic sense, the safety of a structure can truthfully be specified only with an associated probability.
The safety of structures in terms of probability of survival (reliability), or conversely of probability of failure, has been treated before \(^1,2,3,4,5,6,7^\); foremost among these include the pioneering work of Freudenthal\(^1\), Pugsley\(^4\), and Prot\(^6\). Needless to say and in spite of these works, the probabilistic consideration of structural safety remains shrouded with a number of problems; among these are the absence of data or insufficiency of data for determining the probabilistic descriptions of the applied loads and structural strengths. Also, a number of theoretical questions relating to certain fundamental concepts and the significance of certain physical assumptions remain unanswered.

Of relevance to the consideration of structural safety are probability statements pertaining to a structure surviving a sequence of load applications without damage, and also to the possibility of collapse under a severe load. For purposes of design, both of these probabilities may be of interest and importance.

1.2 Object and Scope

The purposes of the present report are (1) to review and clarify a number of principal assumptions that are necessary for the evaluation of the reliabilities of structures and structural systems, and (2) to establish reliable mathematical formulations for the analysis of structural safety consistent with realistic physical assumptions. On this basis the highest lower bound and the lowest upper bound reliabilities of linear structures and structural systems are identified and derived mathematically in terms of the probability of failure under the first load.

*Numbers refer to the items in the list of references on page 48.*
Every effort is made to avoid ambiguity; all results and conclusions are derived on the basis of first principles. In Chapter II, the hazard or risk function for structures subjected to repeated loadings is examined for a class of structures that are of considerable importance in civil engineering. The safety and reliability of structural systems are discussed in Chapter III. The upper and lower bound reliabilities of determinate and indeterminate systems are identified in this chapter. In Chapter IV, a mathematical model for evaluating the probability of collapse of general indeterminate systems is described. The systems may be assemblages of ductile or brittle members. Expressions are developed for the probability distribution functions of the ultimate capacities of simple indeterminate systems in terms of the resistance distribution functions of the individual components.

1.3 Notation

The symbols used in this report are defined in the text where they first appear. For convenience of reference, these are summarized here in alphabetical order.

\begin{align*}
A & = \text{quantity defined by Eq. (20)} \\
A_i & = \text{the set: } \left\{ \frac{R_{ij}}{\alpha_i} \leq S \right\} \\
B, B^0 & = \text{quantities defined by Eqs. (55) and (56), respectively} \\
B_i & = \text{the set: } \{ S > 0 \} \\
C_i & = \text{the set: } \left\{ \frac{R_{ni}}{\alpha_i} \geq S \right\} \\
D_i & = \text{the set: } \{ S \leq 0 \} \\
E & = \text{event of surviving a prescribed loading sequence consisting of positive and negative loads} \\
E_{N,n} & = \text{event of surviving a sequence of } N \text{ loads, among which } n \text{ are positive} \\
E_i & = \text{failure of member } i
\end{align*}
\( E^P_i \) = failure of member \( i \) under positive load
\( E^n_i \) = failure of member \( i \) under negative load
\( f_{X_i}(x) \) = probability distribution function of \( X \); wherever no confusion results \( X \) has been omitted
\( f_{X_i}(x) \) = probability density function of \( X \); wherever no confusion results \( X \) has been omitted
\( f_{X_1, X_2, \ldots , X_n} \) = joint probability density function of \( X_1, X_2, \ldots , X_n \)
\( f_{X_1, \ldots , X_n | Y_1, \ldots , Y_m} \) = conditional probability density function of \( X_1, \ldots , X_n \) given \( Y_1 = y_1, \ldots , Y_m = y_m \)
\( G_i \) = subset of sample space of \( R \) such that if \( R \) is in \( G_i \) failure occurs through the \( i^{th} \) path of failure
\( h(N) \) = risk function
\( l_1(s), l_2(s), J_1(s), J_2(s) \) = functions defined after Eq. (47a)
\( k_s, k_p, k_n \) = positive scale parameters in the extreme value distributions
\( L(N) \) = reliability function
\( L_H(N) = (1-p_f^H)^N \)
\( L_s(N) = (1-p_f^s)^N \)
\( N \) = number of load applications
\( n \) = number of loads among \( N \) which are positive
\( P(E), P[E], P(E) \) = probability of event \( E \)
\( p_f \) = probability of system failure under the first load
\( p_f^I \) = value of \( p_f \) when member resistances are perfectly correlated
\( p_f^{II} \) = value of \( p_f \) when member resistances are statistically independent
\( p_f^* \) = approximation to \( p_f \) obtained by assuming that member failures are statistically independent
\[
Q(r_p, r_n) = \int_{r_n}^{r_p} f_S(s) \, ds
\]

\[
Q_i(s) = 1 - F_{p_i}(\alpha_i, s)
\]

\[
\bar{Q}_i(s) = F_{n_i}(\alpha_i, s)
\]

\[R_p, R_n = \text{random variables for positive and negative resistance of a structure, respectively}\]

\[R_i = \text{random variable for resistance of member } i\]

\[\bar{R} = \text{random vector the components of which are the member resistances.}\]

\[R_0 = \text{median of } R\]

\[\hat{R} = \text{mode of } R\]

\[S = \text{random variable for load}\]

\[S_i = \text{random variable for the } i^{th} \text{ load of a loading sequence.}\]

\[S_0 = \text{median of } S\]

\[\hat{S} = \text{mode of } S\]

\[V_i = \text{random resistance of an indeterminate system given that failure occurs through the } i^{th} \text{ path of failure}\]

\[V_S, V_p, V_n = \text{characteristic values in the extreme value distributions}\]

\[w = \text{load}\]

\[\alpha_i = \text{absolute value of the force induced in member } i \text{ due to a unit load on the system}\]

\[\delta_R, \delta_S = \text{standard deviations of } \ln R \text{ and } \ln S\]

\[\mu = \text{mean}\]

\[\nu = \text{central factor of safety}\]

\[\sigma = \text{standard deviation}\]

\[\Omega = \text{certain event}\]
$A \subseteq B$ = A is a subset of B

$x \in A$ = $x$ belongs to set $A$
II. SAFETY OF STRUCTURES TO REPEATED LOADS

2.1 Probability of Failure to First Load

Structures or structural components subjected to loads may be susceptible to several modes of failure; for instance, axial, bending, shear, or torsional mode. The structural capacity in each of these modes may also vary over two distinct ranges of strengths, such as tension and compression of a member in an axial mode or clockwise and counterclockwise moments in a bending mode. For convenience, these will be denoted as positive and negative strengths or vice versa. Depending on the applied load, which will be similarly denoted, failure may occur in one region or the other. Clearly, the calculation of failure probabilities should reflect the fact that positive loads are resisted only by positive strengths and negative loads by negative strengths. For most structures, the positive and negative strengths are likely to be at least partially correlated.

Let \( S \) be a continuous random variable whose values represent possible loads applied on a structure; also let \( R_p \) and \( R_n \) be random variables representing structural resistances to positive and negative loads, respectively. For obvious reasons, the resistances are expressed in terms of the same unit as the load, and \(-\infty < s < \infty, 0 < r_p < \infty, \) and \(-\infty < r_n \leq 0 \). Accordingly, the failure of a structure is defined as the event,

\[
\left[ (R_p \leq S, S > 0) \cup (R_n \geq S, S \leq 0) \right]
\]

where the symbol \( \cup \) denotes the occurrence of at least one of the two joint events. Since the events \((S > 0)\) and \((S \leq 0)\) are mutually exclusive, the above two joint events are also mutually exclusive. Therefore, if \( f_{S, R_p, R_n}(s, r_p, r_n) \) is the joint density function of \( S, R_p, R_n \), the probability of failure is
\[
p_f = \int_{0}^{\infty} \int_{r_p}^{\infty} f_{S,R_p,R_n}(s,r_p,r_n) ds dr_p dr_n \\
+ \int_{-\infty}^{0} \int_{r_n}^{\infty} f_{S,R_p,R_n}(s,r_p,r_n) ds dr_p dr_n \\
= \int_{0}^{\infty} \int_{r_p}^{\infty} f_{S,R_p}(s,r_p) ds dr_p + \int_{-\infty}^{0} \int_{r_n}^{\infty} f_{S,R_n}(s,r_n) ds dr_n \\
\tag{1}
\]

where the regions of integration are the cross-hatched areas shown in Fig. 1.

A special case of Eq. (1) is given when the load is statistically independent of the structural resistances, such that

\[
f_{S,R_p}(s,r_p) = f_S(s) f_{R_p}(r_p) \\
\tag{2}
\]

and

\[
f_{S,R_n}(s,r_n) = f_S(s) f_{R_n}(r_n) \\
\tag{3}
\]

In this case, Eq. (1) becomes

\[
p_f = \int_{0}^{\infty} \int_{r_p}^{\infty} f_S(s) f_{R_p}(r_p) ds dr_p + \int_{-\infty}^{0} \int_{r_n}^{\infty} f_S(s) f_{R_n}(r_n) ds dr_n \\
= \int_{0}^{\infty} [1 - F_S(r_p)] f_{R_p}(r_p) dr_p + \int_{-\infty}^{0} F_S(r_n) f_{R_n}(r_n) dr_n \\
\tag{4}
\]

This can also be written as,

\[
p_f = \int_{0}^{\infty} F_{R_p}(s) f_S(s) ds + \int_{-\infty}^{0} [1 - F_{R_n}(s)] f_S(s) ds \\
\tag{4a}
\]

The physical significance of \( p_f \) as given by Eq. (4) is illustrated in Fig. 2.
2.2 Probability of Failure to Repeated Load Applications

Structures are seldom designed for the purpose of resisting only one application of a possible loading. More often structures are designed to withstand repeated applications of loads. For instance, a building located in a region that is susceptible to seismic disturbance is intended to withstand repeated earthquakes; a tower is designed to withstand repeated high wind velocities for some definite or indefinite period of time. Also, aircrafts are designed to withstand repeated gust loads. Therefore, the probability that a structure will survive a series of load applications during its intended useful life is a significant measure of its safety.

Although a loading record during any one disturbance (for instance, the ground motions during an earthquake) may well be a continuous function of time, the peak load magnitude during each disturbance is most likely to cause failure of a structure. Hence, a discrete loading sequence consisting of the maximum load amplitudes, or the maximum effects (including dynamics), is of primary concern in the evaluation of structural safety to repeated loads. For a structure to survive $N$ load applications, it must survive the first $(N-1)^{th}$ loads as well as the $N^{th}$ load of the loading sequence. Denoting this probability as $L(N)$,

$$L(N) = L(N-1)[1 - h(N)]$$

(5)

where $h(N)$ is the conditional probability that the structure will fail at the $N^{th}$ load application on the assumption that it survives through the $(N-1)^{th}$ load of the loading sequence. $h(N)$ is the risk of failure to the $N^{th}$ load.

From Eq. (5) this is

$$h(N) = 1 - \frac{L(N)}{L(N-1)}$$

(6)
The risk of failure under the following assumptions is of special significance in the consideration of safety of structures and structural systems.

(1) The loading sequence $S_1, S_2, ..., S_N$ consists of independent identically distributed random variables that are also independent of the structural resistances $R_p$ and $R_n$.

(2) The structural resistances $R_p$ and $R_n$ remain invariant with $N$.

Denote $E_{N,n}$ as the event that a structure survives $N$ load applications, among which $n$ are positive. The survival of a structure to $N$ load applications then is defined by the event $\bigcup_{n=0}^{N} E_{N,n}$. The events $E_{N,n}$ are clearly mutually exclusive; therefore, the survival probability of the structure to $N$ load applications is

$$L(N) = \sum_{n=0}^{N} P(E_{N,n})$$

In general, the joint density function of the $N$ loads $S_1, S_2, ..., S_N$ and the structural resistances $R_p$ and $R_n$ is of the form

$$f_{S_1, S_2, ..., S_N, R_p, R_n}(S_1, S_2, ..., S_N, R_p, R_n)$$

For a particular loading sequence where the applications of $n$ positive loads and $(N-n)$ negative loads occur in a specified order, the probability of surviving such a sequence, denoted by $E_s$, is

$$P(E) = \int_{0}^{r_n} \int_{0}^{r_p} \int_{0}^{r_n} \int_{0}^{r_p} \int_{0}^{r_n} \int_{0}^{r_p} \int_{0}^{r_n} \int_{0}^{r_p} \int_{0}^{r_n} \int_{0}^{r_p} f_{S_1, ..., S_N, R_p, R_n}(S_1, ..., S_N, R_p, R_n) ds_1 ... ds_N f_{R_p, R_n} dr_p dr_n$$
where the identical limits of integration $r_p$ and $r_n$ in Eq. (8) imply the assumption that the structural resistances remain invariant with the number of load applications. If the loading sequence consists of independent random variables that are identically distributed and are also independent of $R_p$ and $R_n$, then

$$f_{S_1, \ldots, S_N|R_p, R_n}(s_1, \ldots, s_N|r_p, r_n) = \left[ f_S(s) \right]^N$$

(9)

In this case,

$$P(E) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \left[ \int f_S(s) ds \right] \left[ \int f_S(s) ds \right] f(r_p, r_n) dr_p dr_n$$

(10)

Since there are $\binom{N}{n}$ possible sequences of $N$ loads among which $n$ are positive,

$$P(E_{N,n}) = \binom{N}{n} P(E)$$

(11)

Hence, the probability of surviving $N$ loads, as given by Eq. (7) becomes,

$$L(N) = \sum_{n=0}^{N} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \binom{N}{n} \left[ \int f_S(s) ds \right] \left[ \int f_S(s) ds \right] f_{R_p, R_n}(r_p, r_n) dr_p dr_n$$

(12)

Interchanging the order of integration and summation in Eq. (12) and recognizing that the resulting sum is a binomial expansion yields,

$$L(N) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \left[ \int f_S(s) ds \right] \left[ \int f_S(s) ds \right] \sum_{n=0}^{N} \binom{N}{n} f_{R_p, R_n}(r_p, r_n) dr_p dr_n$$

$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \left[ \int f_S(s) ds \right] f_{R_p, R_n}(r_p, r_n) dr_p dr_n$$

(13)

This result is intuitively reasonable; it means that all $N$ loads applied on a structure are within the range $r_p$ and $r_n$, which are, respectively, the negative and positive resistances of the structure.
It may be emphasized that Eq. (13) expresses the survival probability of one structure subjected to \( N \) repeated loads, in which the \( N \) loads are statistically independent and have a common probability law, and the structural resistances do not change with \( N \). Using

\[
Q(r_p, r_n) \equiv \int_{r_n}^{r_p} f_S(s)ds
\]  

(14)

\[
L(N) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q^N(r_p, r_n)f_{R_p, R_n}(r_p, r_n)dr_p dr_n
\]

(15)

Similarly,

\[
L(N-1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q^{N-1}(r_p, r_n)f_{R_p, R_n}(r_p, r_n)dr_p dr_n
\]

(16)

The risk function, Eq. (6), then becomes

\[
h(N) = 1 - \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q^N(r_p, r_n)f_{R_p, R_n}(r_p, r_n)dr_p dr_n}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q^{N-1}(r_p, r_n)f_{R_p, R_n}(r_p, r_n)dr_p dr_n}
\]

(17)

It should be emphasized that Eq. (17) gives the risk function of a structure whose resistances are invariant with \( N \), and the loading sequence consists of independent identically distributed random variables.

An important property of this risk function is the following: The risk function of Eq. (17) is a monotonically decreasing function of \( N \), or

\[
h(N) < h(N-1) \text{ for all } N > 1
\]

(18)

The proof of Eq. (18) is equivalent to proving the following inequality:
where the limits of integration, subscripts of \( f \), and the arguments \((r_p, r_n)\) have been purposely omitted. Let

\[
A = \iint_{Q^N} f \, dr \, dr_n \iint_{Q^{N-1}} f \, dr \, dr_n - \left[ \iint_{Q^{N-1}} f \, dr \, dr_n \right]^2
\]

(20)

Eq. (20) can be written as

\[
A = \iiint_{Q_1^N} Q_2^{N-2} \, f f' \, dr \, dr_n \, dr_1 \, dr_n - \iiint_{Q_1^{N-1}} Q_2^{N-1} \, f f' \, dr \, dr_n \, dr_1 \, dr_n
\]

(21)

Likewise,

\[
A = \iiint_{Q_1^{N-2}} Q_2^N \, f f' \, dr \, dr_n \, dr_1 \, dr_n - \iiint_{Q_1^{N-1}} Q_2^{N-1} \, f f' \, dr \, dr_n \, dr_1 \, dr_n
\]

(22)

Adding Eqs. (21) and (22) yields

\[
2A = \iiint \left[ Q_1^N Q_2^{N-2} + Q_1^{N-2} Q_2^N - 2Q_1^{N-1} Q_2^{N-1} \right] f f' \, dr \, dr \, dr_n \, dr_n
\]

(23)

and

\[
A = \frac{1}{2} \iiint \left[ \frac{N}{2} \frac{N-2}{2} Q_1^2 Q_2^2 - Q_1^2 Q_2^2 \right]^2 f f' \, dr \, dr \, dr_n \, dr_n
\]

(24)

Since \( f \) and \( f' \) are non-negative functions, the integrand and hence the integral of Eq. (24) are clearly non-negative. Thus,

\[
A \geq 0
\]

(25)

The equality in Eq. (25) holds only if

\[
Q_2^2 (r_p, r_n) Q_2^{N-2} (r_p^1, r_n^1) = Q_2^2 (r_p^1, r_n) Q_2^{N-2} (r_p, r_n) = 0
\]

which means that

\[
Q(r_p, r_n) = Q(r_p^1, r_n)
\]
for all \((r_p, r_n)\) and \((r'_p, r'_n)\). This is clearly not possible for any \(f_S(s)\), where \(S\) is a continuous random variable. Hence,

\[ A > 0 \] (26)

Since the terms in Eq. (19) are all non-negative this result, therefore, proves Eq. (18).

If the loads and structural resistance are strictly positive or strictly negative, the result proved above is still valid. The proof, however, is simplified which can be outlined as follows: In this case, the probability of surviving \(N\) loads that are independent and identically distributed is immediately,

\[ L(N) = \int_0^\infty \left[ \int_{f_S(s)}^r \right]^N f_R(r) dr \] (13a)

and the risk function, Eq. (6), is

\[ h(N) = 1 - \frac{\int_0^{Q^{N-1}(r)} f_R(r) dr}{\int_0^{Q^N(r)} f_R(r) dr} \] (17a)

where,

\[ Q(r) = \int_r^\infty f_S(s) ds \] (14a)

Then the quantity \(A\) becomes,
\[ A = \int_{0}^{\infty} Q^{N} f(r) dr \int_{0}^{\infty} Q^{N-2} f(r) dr - \left( \int_{0}^{\infty} Q^{N-1} f(r) dr \right)^2 \] (20a)

\[ = \int_{0}^{\infty} Q_{1}^{N-2} Q_{2}^{N} f(r) dr - \int_{0}^{\infty} Q_{1}^{N-1} Q_{2}^{N-2} f(r) dr \] (21a)

Likewise,

\[ A = \int_{0}^{\infty} Q_{1}^{N-2} Q_{2}^{N} f(r) dr - \int_{0}^{\infty} Q_{1}^{N-1} Q_{2}^{N-2} f(r) dr \] (22a)

and the rest of the proof follows as before.

The physical implication of Eq. (18) has been previously recognized\(^9\) intuitively and illustrated approximately for normal and lognormal variates. It should be emphasized that Eq. (18) holds for all probability densities as shown above; however, the rate of decrease of \( h(N) \) with \( N \) depends on the underlying distributions. Figure 3 shows this for specific distributions as follows:

In Fig. 3a, the load \( S \) and resistance \( R \) are both Rayleigh distributed. The risk function in this case is,

\[ h(N) = \frac{\sum_{k=0}^{N-1} (-1)^{k} \binom{N-1}{k} \left[ 1 + (k+1)\nu_{o}^2 \right]}{\sum_{k=0}^{N-1} (-1)^{k} \binom{N-1}{k} \left[ 1 + k\nu_{o}^2 \right]} \]

where, \( \nu_{o} = \frac{R}{S} \) is the central factor of safety, and \( R \) and \( S \) are the modes of \( R \) and \( S \), respectively. Values of \( h(N)/p_f \) are plotted against \( N \) with \( \nu_{o} = 10 \) in Fig. 3a.

If \( S \) and \( R \) are both logarithmic normal variates, the risk function becomes
where $\delta_R$ and $\delta_S$ are the standard deviations of $\ln R$ and $\ln S$, respectively; $\nu_o = R_o / S_o$ in which $R_o$ and $S_o$ are the corresponding medians; and

$$h(N) = 1 - \frac{\int_{-\infty}^{\infty} \phi^N \left( \frac{\delta_R u + \ln \nu}{\delta_S} \right) e^{-\frac{1}{2}u^2} du}{\int_{-\infty}^{\infty} \phi^{N-1} \left( \frac{\delta_R u + \ln \nu}{\delta_S} \right) e^{-\frac{1}{2}u^2} du}$$

Values of $h(N)/p_f$ for this case with $\delta_R = \delta_S = 0.2$ are shown in Fig. 3b.

In Fig. 3c the load $S$ consists of both positive and negative values described by the following density function,

$$f_S(s) = \frac{1}{2} \frac{k_s}{v_s} \left( \frac{v_s}{s} \right)^{k_s+1} \exp \left[ -\left( \frac{v_s}{s} \right)^{k_s} \right], \text{ for } s \geq 0$$

and

$$f_S(s) = f_S(-s), \text{ for } s \leq 0$$

which is obtained from the second asymptotic distribution of largest values by distributing the probability density from $-\infty$ to $+\infty$ symmetrically about $s = 0$. In the terminology of extreme values, $k_s$ is a positive scale factor and $v_s$ is a positive parameter indicating the central value of $|S|$. The positive and negative resistances, $R_p$ and $R_n$, are assumed to have the third asymptotic distribution of smallest values and largest values, respectively; or

$$f_R(r_p) = \frac{k_p}{v_p} \left( \frac{r_p}{v_p} \right)^{k_p-1} \exp \left[ -\left( \frac{r_p}{v_p} \right)^{k_p} \right], \text{ for } r_p \geq 0$$

$$f_R(r_n) = \frac{k_n}{v_n} \left( \frac{r_n}{v_n} \right)^{k_n-1} \exp \left[ -\left( \frac{r_n}{v_n} \right)^{k_n} \right], \text{ for } r_n \leq 0$$
where \( k_p \) and \( k_n \) are positive scale parameters, and \( \nu_p > 0 \) and \( \nu_n < 0 \) are the characteristic smallest and largest values, respectively, of \( R_p \) and \( R_n \).

Using

\[
I(i, \nu, k_1, k_2) = \int_0^\infty \exp\left[ -\left( u + \nu \right) \frac{k_2 - k_2}{k_1} \right] du
\]

the risk function in this case is,

\[
h(N) = 1 - \sum_{i=0}^{N-1} \left( \begin{array}{c} N \\ i \end{array} \right) I(i, \nu_p, k_p, k_s) I(N-i, \nu, k_n, k_s)
\]

where \( \nu_p = \frac{\nu_p}{\nu_s} \) and \( \nu_n = \frac{\nu_n}{\nu_s} \). Values of \( h(N)/p_f \) are plotted in Fig. 3c for \( k_s = 5.8 \), and \( k_p = k_n = 7.4 \) and two sets of values of \( \nu_p \) and \( \nu_n \).

**2.3 Reliability Function**

From Eq. (5), the reliability function can be shown to be

\[
L(N) = \prod_{n=1}^{N} \left[ 1 - h(n) \right]
\]

Clearly \( L(N) \) is a non-increasing function of \( N \), which means that there is more likelihood of a failure when the number of load applications increases. Using Eq. (18) sequentially,

\[
h(N) < h(N-1) < \ldots < h(2) < h(1)
\]

where it can be verified from Eqs. (1), (6), and (13) that

\[
h(1) = p_f
\]

Hence,

\[
h(N) < p_f \quad \text{for all } N > 1
\]
Therefore, the probability of failure is highest when a structure is loaded for the first time. This stands to reason since the greatest uncertainty exists during the first loading; if a structure survives the first load application without damage (or any damage is repaired such that the original system is restored), the same structure will surely not fail under the same or lesser loads on subsequent loadings.

By virtue of Eq. (3),
\[ L(N) > (1-p_f)^N, \text{ for all } N > 1 \]  
(31)

Therefore, assuming a constant risk \( h(n) = p_f \), the resulting function
\[ L_M(N) = (1 - p_f)^N \]  
(32)
is a conservative estimate of the reliability of a structure to repeated loads. Furthermore, if \( Np_f \) is small compared with 1.0, expanding \( (1-p_f)^N \) in a binomial expansion will show that,
\[ L_M(N) \approx 1 - Np_f \]  
(32a)

It should be emphasized that the preceding results are valid only if the assumptions leading to Eq. (17) are applicable. For structures that are subject to cumulative damage effect or if the load magnitudes have a tendency to increase with time, these results may not hold.
III. RELIABILITY OF STRUCTURAL SYSTEMS

3.1 Symbolic Definition of Safety

The general results presented in Chap. II are applicable for the determination of the reliability of a structural system if the appropriate probability law describing the system resistance capacity is used; that is, the density function \( f_{R_p} (r_p) \) and \( f_{R_n} (r_n) \) in Eq. (4), or \( f_{R_p', R_n'} (r_p, r_n) \) in Eq. (13), must be those of the system. Specifically, Eq. (32) also provides a conservative estimate of the system reliability if \( p_f \) is the failure probability of a system to the first load application. Clearly, the strength or capacity of a structural system depends on the strengths of its components. Moreover, statistical information is usually available only for the strengths of the structural components; consequently, the system failure probability \( p_f \) must be determined on the basis of the probability information of the components.

Let \( E_i \) represent the failure of member \( i \) when a system is subjected to a load \( S \); thus,

\[
E_i = \left[ \frac{R_p}{\alpha_i} \leq S, S > 0 \right] \cup \left[ \frac{R_n}{\alpha_i} \geq S, S \leq 0 \right] = E_i^p \cup E_i^n \tag{33}
\]

where \( \alpha_i \) is the absolute value of the force induced in member \( i \) due to a unit load, \( s = 1 \), applied on the system; \( R_p \) is the appropriate generalized resistance of member \( i \) to the generalized force induced in the member resulting from a positive load applied on the system, and \( R_n \) is the corresponding resistance of the same member to the generalized force resulting from a negative load. The positive and negative senses of \( S \) can be arbitrarily specified. For instance, the loading on a structure by wind forces on one side of the structure may be positive, while wind forces on the opposite side is negative.
Write,

\[ E_i = A_i B_i \cup C_i D_i \tag{33a} \]

where the sets \( A_i, B_i, C_i, D_i \) are defined in Eq. (33) and are as shown in Fig. 4. The survival of member \( i \) when the system is loaded with \( S \) then is the complementary event of \( E_i \), or

\[
\overline{E_i} = \overline{A_i B_i} \cup \overline{C_i D_i} = (A_i B_i \cup C_i D_i)
\]

\[
= (B_i - A_i B_i) \cup (D_i - C_i D_i)
\]

\[
= A_i B_i \cup C_i D_i \tag{34}
\]

where \( \Omega \) is the certain event. Hence, the survival of member \( i \) is,

\[
\overline{E_i} = \left( \frac{R_{i}}{\alpha_i} > S, S > 0 \right) \cup \left( \frac{R_{i}}{\alpha_i} < S, S \leq 0 \right) \subseteq \overline{E_i} \cup \overline{E_i} \tag{35}
\]

3.2 Statically Determinate Systems

The survival of a statically determinate structural system to an applied load requires the survival of all of its component members to the forces induced in the members. For an \( m \)-member determinate system its survival then is given by the event \( \overline{E_1 \overline{E_2} \cdots \overline{E_m}} \). Because \( (S > 0) \) and \( (S \leq 0) \) are mutually exclusive events, the probability of this event, which is the probability of system survival, can be shown to be

\[
p_s = P(\overline{E_1 \overline{E_2} \cdots \overline{E_m}}) = P(\overline{E_1 \overline{E_2} \cdots \overline{E_m}}) + P(\overline{E_1 \overline{E_2} \cdots \overline{E_m}}) \tag{36}
\]

Eq. (36) represents a fundamental result in the reliability analysis of structural systems; all subsequent results can be derived from this equation. The probability of system failure, therefore, is

\[
p_f = 1 - p_s \tag{37}
\]
By de Morgan's Law
\[ E_1 \cap E_2 \cap \ldots \cap E_m = E_1 \cup E_2 \cup \ldots \cup E_m \] (38)

Hence, the probability of system failure is also given by
\[ p_f = P(E_1 \cup E_2 \cup \ldots \cup E_m) \] (39)

Eq. (39) expresses the fact that the failure of one or more members is tantamount to failure of the entire system. This also implies that the strength of a determinate system depends on the weakest member in the system.

For a prescribed loading, the forces in the members must satisfy statical requirements. Consequently, the member forces are perfectly correlated and, in fact, are linearly related if geometry effect is neglected. The applied loads, however, are statistically independent of the structural resistances; it is difficult to conceive that the statistical distribution of an external load applied on a structure should depend on the strength of the structure.

In considering the reliability or probability of failure of determinate systems, there are two extreme cases that should be recognized. These depend on the assumption of the joint probability descriptions of the strengths of the different members. On the one extreme, the member strengths are assumed to be perfectly correlated; that is, the strengths of the different members are functionally related. In this case, the weakest member in a m-member system is the member with the largest force-to-strength ratio and the system probability of failure to the first load application is equal to the probability of failure of this member. This means that the sets \( E_i \) are nested; therefore Eq. (39) becomes

\[ p_f^i = P[\max(E_1, E_2, \ldots, E_m)] \] (40)
\[ = \max[P(E_1), P(E_2), \ldots, P(E_m)] \] (40a)
On the other extreme, the strengths of different members as well as the positive and negative strengths of a member are assumed to be statistically independent. In this case, the positive strength $R_p$ of an $m$-member system can be described as:

$$ P(R_p > s) = \prod_{i=1}^{m} \int_{\alpha_i s}^{\infty} f_{p_i}(x) dx = \prod_{i=1}^{m} \left[ 1 - F_{p_i}(\alpha_i s) \right] $$  \hspace{1cm} (41)

Thus,

$$ F_{R_p}(s) = 1 - \prod_{i=1}^{m} \left[ 1 - F_{p_i}(\alpha_i s) \right] $$  \hspace{1cm} (42)

Similarly, the negative strength $R_n$ of the system is,

$$ F_{R_n}(s) = \prod_{i=1}^{m} F_{n_i}(\alpha_i s) $$  \hspace{1cm} (43)

Using Eqs. (42) and (43) in Eq. (4a), or on the basis of Eqs. (36) and (37), the probability of failure of the system to the first load application is

$$ p_f^{(1)} = 1 - \int_{-\infty}^{\infty} \left[ \prod_{i=1}^{m} Q_i \right] f_S(s) ds - \int_{-\infty}^{0} \left[ \prod_{i=1}^{m} \bar{Q}_i \right] f_S(s) ds $$  \hspace{1cm} (44)

where

$$ Q_i = \left[ 1 - F_{p_i}(\alpha_i s) \right] $$  \hspace{1cm} (45)

and

$$ \bar{Q}_i = F_{n_i}(\alpha_i s) $$  \hspace{1cm} (46)

In an actual system, the strengths of the individual members are likely to be partially correlated; hence, Eqs. (36) and (37) for the general case is
\[ p_f = 1 - \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{p_1, \ldots, p_m}(r_{p_1}, \ldots, r_{p_m}) dr_{p_1} \cdots dr_{p_m} \right] f_S(s) ds \]

\[ \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{n_1, \ldots, n_m}(r_{n_1}, \ldots, r_{n_m}) dr_{n_1} \cdots dr_{n_m} \right] f_S(s) ds \quad (47) \]

It is observed that Eq. (44) may be obtained as a special case of Eq. (47). It should also be recognized that if there is any correlation between the member strengths, it will be positive; that is, the members will tend to be jointly strong or jointly weak. This is because any partial correlation will invariably be the result of a similar or identical manufacturing process.

The quantities \( p_f^l, p_f^r, \) and \( p_f^{ll} \) are related as follows:

\[ p_f^l \leq p_f \leq p_f^{ll} \quad (48) \]

With Eq. (48), \( p_f^l \) and \( p_f^{ll} \) are then respectively, the highest lower bound and lowest upper bound failure probabilities of a statically determinate system.

The first inequality of Eq. (48) is proved by recognizing from Eqs. (39) and (40) that,

\[ \max(E_{1}, E_{2}, \ldots, E_{m}) \subseteq (E_{1} \cup E_{2} \cup \ldots \cup E_{m}) \]

thus,

\[ p_f^l \leq p_f \]

The second inequality of Eq. (48) can be established as follows:

Write Eqs. (44) and (47), respectively, as

\[ p_f^{ll} = 1 - \int_{0}^{\infty} I_{1}(s)f_{S}(s) ds - \int_{-\infty}^{0} I_{2}(s)f_{S}(s) ds \quad (44a) \]

and,

\[ p_f = 1 - \int_{0}^{\infty} J_{1}(s)f_{S}(s) ds - \int_{-\infty}^{0} J_{2}(s)f_{S}(s) ds \quad (47a) \]
where

\[ J_1(s) = P[R_{p1} > \alpha_1 s, R_{p2} > \alpha_2 s, \ldots, R_{pm} > \alpha_m s]; \]

\[ J_2(s) = P[R_{n1} < \alpha_1 s, R_{n2} < \alpha_2 s, \ldots, R_{nm} < \alpha_m s]; \]

\[ I_1(s) = P[R_{p1} > \alpha_1 s] P[R_{p2} > \alpha_2 s] \ldots P[R_{pm} > \alpha_m s]; \]

\[ I_2(s) = P[R_{n1} < \alpha_1 s] P[R_{n2} < \alpha_2 s] \ldots P[R_{nm} < \alpha_m s]. \]

Consider first \( m = 2 \). In this case,

\[ J_1(s) = P[R_{p1} > \alpha_1 s | R_{p2} > \alpha_2 s] P[R_{p2} > \alpha_2 s]. \]

But,

\[ P[R_{p1} > \alpha_1 s] = P[R_{p1} > \alpha_1 s | R_{p2} > \alpha_2 s] P[R_{p2} > \alpha_2 s] + P[R_{p1} > \alpha_1 s | R_{p2} \leq \alpha_2 s] \alpha[R_{p2} = \alpha_2 s] \]

because \((R_{p2} > \alpha_2 s)\) and \((R_{p2} \leq \alpha_2 s)\) are mutually exclusive. For small values
of \( s \), for instance \( s \leq \frac{\mu_2 - 2\sigma_2}{\alpha_2} \), where \( \mu_2 \) and \( \sigma_2 \) are the mean and standard deviation
of \( R_2 \),

\[ P[R_{p2} > \alpha_2 s] \gg P[R_{p2} \leq \alpha_2 s] \]

Furthermore, if \( R_{p1} \) and \( R_{p2} \) are positively correlated random variables,

\[ P[R_{p1} > \alpha_1 s | R_{p2} > \alpha_2 s] > P[R_{p1} > \alpha_1 s | R_{p2} \leq \alpha_2 s] \]

Therefore, for small values of \( s \),

\[ P[R_{p1} > \alpha_1 s] \geq P[R_{p1} > \alpha_1 s | R_{p2} > \alpha_2 s] P[R_{p2} > \alpha_2 s] > P[R_{p1} > \alpha_1 s] P[R_{p2} > \alpha_2 s] \]

Hence,

\[ J_1(s) > I_1(s) \quad (49) \]

By mathematical induction, Eq. (49) is established for all \( m \) as follows:

\[ P[R_{p1} > \alpha_1 s, \ldots, R_{pm} > \alpha_m s] = P[R_{p1} > \alpha_1 s, \ldots, R_{pm} > \alpha_m s | R_{m+1} > \alpha_{(m+1)} s] P[R_{p(m+1)} > \alpha_{m+1} s] \]

\[ + P[R_{p1} > \alpha_1 s, \ldots, R_{pm} > \alpha_m s | R_{m+1} \leq \alpha_{(m+1)} s] P[R_{p(m+1)} \leq \alpha_{m+1} s] \]
Similarly, it can be shown that for small values of $|s|$,

$$J_2(s) > I_2(s)$$  \hspace{1cm} (49a)

Hence, for small $|s|$,

$$J_1(s) - I_1(s) > 0; \text{ for } s > 0$$

and

$$J_2(s) - I_2(s) > 0; \text{ for } s \leq 0$$

Because $J(0) = I(0) = 1.0$ and both $J(s)$ and $I(s)$ are monotonically decreasing functions of $|s|$, a possible variation of $[J(s) - I(s)]$ is as shown in Fig. 5.

For structural systems that are adequately designed for safety, the members will be chosen such that the central factors of safety are at least 2 or 3; that is, $R_{oi}/\alpha_i S_o \geq 2$ or 3, where $R_{oi}$ and $S_o$ are the central values of $R_i$ and $S$, respectively. Therefore, $S_o \leq \frac{R_{oi}}{2\alpha_i}$ and thus, the major portion of $f_S(s)$ lies in the positive regions of $[J(s) - I(s)]$ as illustrated in Fig. 5.

Hence,

$$P_f \leq P_f^{''}$$

where the equality holds only if the structural strengths of the members are statistically independent.

It might be emphasized that the second inequality of Eq. (48) is valid only for structures where the probabilities of failure of the members are individually small. For structures with large member failure probabilities ($> 0.1$), the second inequality may not always hold (at least the proof given here is not sufficient). However, it has been shown that for a number of bivariate
distributions including the bivariate normal with positive correlations, 

\[ [J(s) - I(s)] > 0 \text{ for all } s. \]

The probability of failure of statically determinate systems are usually determined on the assumption that the failure of the different members are statistically independent events. This implies or would require that the forces induced in the different members are statistically independent, in addition to the assumption of independent member resistances. The assumption of independence of the member forces is clearly not possible for determinate systems since these forces are always perfectly correlated through the requirements of statics and in fact are linearly related for linear systems. However, the calculation of the system failure probability on this basis is easier to carry out than that of Eq. (44) or (47). Therefore, in spite of the implausibility of the underlying assumption, the system failure probability estimated on this basis is of practical computational interest. Denote this probability as \( p^f \); then through Eqs. (36) and (37),

\[
p_f^* = 1 - \prod_{i=1}^{m} \int_{0}^{\infty} \tilde{Q}_i f_S(s) ds - \prod_{i=1}^{m} \int_{-\infty}^{0} \tilde{Q}_i f_S(s) ds
\]

(50)

The quantities \( p_f^* \) and \( p_f^* \) are related by the following inequality,

\[
p_f^* \leq p_f^* \text{ for all } m > 1
\]

(51)

Eq. (51) is proved by showing that for all \( m > 1 \)

\[
\int_{0}^{\infty} \prod_{i=1}^{m} \tilde{Q}_i f_S(s) ds \geq \prod_{i=1}^{m} \int_{0}^{\infty} \tilde{Q}_i f_S(s) ds
\]

(52)

and

\[
\int_{-\infty}^{0} \prod_{i=1}^{m} \tilde{Q}_i f_S(s) ds \geq \prod_{i=1}^{m} \int_{-\infty}^{0} \tilde{Q}_i f_S(s) ds
\]

(53)
Consider Eq. (52) with $m = 2$, which is

\[
\int_0^\infty Q_1 Q_2 f_S(s)ds \geq \int_0^\infty Q_1 f_S(s)ds \int_0^\infty Q_2 f_S(s)ds
\]  

(54)

Let

\[
B = \int_0^\infty Q_1 Q_2 f_S(s)ds - \int_0^\infty Q_1 f_S(s)ds \int_0^\infty Q_2 f_S(s)ds
\]  

(55)

Multiply the first integral in Eq. (55) by $\int_0^\infty f_S(s)ds \leq 1.0$, and denote the result as $B'$, which can be expressed as

\[
B' = \int \int Q_1(s_1)Q_2(s_1)f_1f_2 ds_1 ds_2 - \int \int Q_1(s_1)Q_2(s_2)f_1f_2 ds_1 ds_2
\]  

(56)

Also,

\[
B' = \int \int Q_1(s_1)Q_2(s_2)f_1f_2 ds_1 ds_2 - \int \int Q_1(s_2)Q_2(s_1)f_1f_2 ds_1 ds_2
\]  

(57)

where the limits of integration are all from $0$ to $\infty$, and $f_1 = f_S(s_1)$, $f_2 = f_S(s_2)$. Adding Eqs. (56) and (57) yields

\[
B' = \frac{1}{2} \int \int \left[ Q_1(s_1) - Q_1(s_2) \right] \left[ Q_2(s_1) - Q_2(s_2) \right] f_1f_2 ds_1 ds_2
\]  

(58)

From Eq. (45), it should be observed that $Q_1$ and $Q_2$ are both non-increasing functions, hence,

\[
\left[ Q_1(s_1) - Q_1(s_2) \right] \left[ Q_2(s_1) - Q_2(s_2) \right] \geq 0 \text{ for } s_1 \neq s_2
\]  

(59)

Since $f_1$ and $f_2$ are non-negative functions,

\[
B' \geq 0
\]
Therefore, $B \geq B^t$ and $B \geq 0$; Eq. (52) is then proved for $m = 2$, and by mathematical induction it is proved also for all $m > 1$. The proof of Eq. (53) follows similarly; in this case $\tilde{Q}_1$ are non-decreasing functions, hence, the inequality of Eq. (59) holds also for $\tilde{Q}_1$ and $\tilde{Q}_2$. Thus, Eq. (51) is proved.

By virtue of Eq. (51), the quantity $p^*_f$, therefore, is at least as large as the upper bound value of the system failure probability; hence, it may be used in place of $p_f$ to determine a conservative measure of the reliability of a given statically determinate system.

3.3 Statically Indeterminate Systems

In contrast to a determinate system in which failure of one component member is tantamount to collapse of the system, the failure of one or more members may not necessarily constitute collapse of an indeterminate structural system. The collapse of the system will usually be preceded by the successive failures of the reduced intermediate systems. Because of this built-in redundancy which provides reserved strength against sudden catastrophic failure, statically indeterminate systems are structurally more desirable than determinate systems. Nevertheless, the failure or malfunction of even one member of an indeterminate system is to be avoided for most designs. Also, for structures with limited degrees of redundancy, the probability of subsequent additional failures may be high after the first component failure.

Consequently, for practical purposes the reliability of an indeterminate system may be measured by the probability of conserving the original system; that is, the probability that all members in the system survive. Symbolically, this is given also by Eq. (36). However, it should be recognized and emphasized that in this case $p^*_f$, as given by Eq. (37), is not necessarily
the probability of system collapse but is the probability of at least one member failing; in other words, it is the probability of damage to the system. In an indeterminate structural system, the forces induced in the members are functions of the stiffness properties of the members. Ordinarily, these properties are much more predictable than the structural strength and may be assumed deterministic; the coefficients \( \alpha_i \), representing the absolute forces in the members resulting from a unit applied load, therefore, are deterministic constants.

Therefore, with the change in the interpretation of \( p_f \) as indicated above, all the results and conclusions pertaining to determinate systems remain valid for statically indeterminate systems, including Eqs. (40), (44), and (47) through (51). In particular, the system reliability to the first load is also bounded below by \( (1-p_f') \), where \( p_f'' \) is given by Eq. (44).

3.4 Reliability Functions of Structural Systems

The determination of the survival probabilities of determinate and indeterminate systems, therefore, involves the same probabilistic formulation. Although the system probabilities, \( p_s \) and \( p_f \) of Eqs. (36) and (37), are generally not obtainable precisely the highest lower-bound of the system reliability to the first load application is given by \( (1 - p_f') \). Hence, on the basis of Eq. (32), a conservative estimate of the reliability function of a structural system is

\[
L_M(N) = (1-p_f')^N
\]

(60)

where \( p_f'' \) is given by Eq. (44) for both determinate and indeterminate systems. Furthermore, by virtue of Eq. (51) a convenient and even more conservative estimate of the system reliability function is given by

\[
L^*(N) = (1 - p_f^*)^N
\]

(61)
where $P_f^*$ is given by Eq. (50) for both determinate and indeterminate systems. Again if $N_p^*$ is small compared with 1.0,

$$L^*(N) \approx 1 - N_p^* \quad (61a)$$

Eq. (61) or (61a) is perhaps an overly conservative estimate of the reliability function of a structural system. However, in view of the difficult computations that are invariably required to obtain more precise reliability measures, these conservative estimates are worthwhile in spite of the underlying error, which is on the safe side. More precise values, of course, may be determined using the equations formulated earlier.

### 3.5 An Illustration

A statically determinate truss and a similar indeterminate system are illustrated in Fig. 6. The loads and structural strengths are both assumed to be logarithmic normal variates and the bar areas are assumed deterministic. It is observed that the survival probabilities of the two systems are, respectively, $(1 - 1.8 \times 10^{-4})$ and $(1 - 9.6 \times 10^{-4})$ showing that in this case the probability of no damage is smaller for the indeterminate system than that of the corresponding determinate system. However, it should be emphasized that this does not necessarily mean that the determinate system is to be preferred, since any damage to the determinate structure is tantamount to total collapse of the system. For a fair comparison, the corresponding probability of the collapse of the indeterminate system must be computed; the determination of such probabilities, however, is much more involved. Chapter IV considers a possible approach for the evaluation of the probability of collapse of indeterminate systems.
IV. PROBABILITY OF COLLAPSE OF INDETERMINATE SYSTEMS

4.1 Introductory Remarks

In contrast to a determinate system in which failure of the system requires only the failure of the weakest member, the complete collapse of a statically indeterminate system generally requires the failure of more than one member. In this chapter the term failure is restricted to mean collapse under a single application of a severe load and the term "path of failure" is symbolic of the fact that the failure of an indeterminate system is caused by the successive failures of its components; the different orders in which the components fail may constitute different failure paths. Also the system may collapse as a result of the failure of different components. Accordingly, all failure paths must be considered in the determination of the probability of collapse of an indeterminate structural system.

The reliability function and probability of failure of special and simple indeterminate systems have recently been considered.\textsuperscript{14,15} An approximate technique for estimating the probability of failure of indeterminate frames in which the plastic moment capacities at every critical section of the members are considered as independent random variables has been proposed.\textsuperscript{16} For general indeterminate systems, the precise determination of the collapse probability remains unsolved.

The present treatment of the probability of collapse of statically indeterminate systems is based on the premise that the failure of a structural system occurs whenever the capacity of the system is less than the applied loading. The problem, therefore, lies in the determination of the probability distribution function of the ultimate capacity of a structure. Having this and the load distribution, the probability of failure may then be determined.
through Eq. (4a). A general model for the determination of the distribution function of system capacity in terms of the component capacities, is described symbolically. The model is illustrated for structures with both brittle and ductile types of behavior.

The development of a conceptual model is a necessary first step toward the determination of the probability of collapse of statically indeterminate structural systems. It constitutes the basis for a correct formulation of the relevant problem.

4.2 Distribution Function of System Capacity

The formulation of the probability distribution function of the capacity of a statically indeterminate structural system in terms of the distribution functions of the component capacities has not been clarified. Up to now there has been no model that is suitable for formulating the problem for a general indeterminate system. A general model, described in terms of set theory, is presented herein for this purpose.

Let \( \mathbf{R} \) be a random vector whose components \( R_1, R_2, \ldots, R_m \) are random variables representing the resistances of the individual members. For simplicity and clarity, the following presentation will be limited to strictly positive loads and structural capacities (or strictly negative load and capacities). Define \( G_i \), as a set of vectored sample values of \( \mathbf{R} \) such that if \( \mathbf{R} \) is in \( G_i \) failure occurs through the \( i^{th} \) "path of failure."

Clearly, the sets \( G_i \) must be mutually exclusive and exhaustive sets, or

\[
G_1 \cup G_2 \cup \cdots \cup G_k = \Omega
\]  

(62)

where \( \Omega \) is the vectored sample space of \( \mathbf{R} \) and \( k \) is the total number of possible paths of failure.

Let \( \mathbf{R} \) denote the ultimate capacity of a system; then the distribution
The function of \( R \) is

\[
F_R(r) = P(R \leq r) = \sum_{i=1}^{k} P[(V_i \leq r) \cap (\bar{R}eG_i)]
\]

(63)

in which \( V_i \) is the resistance of the system if failure occurs through path \( i \).

The event \([(V_i \leq r) \cap (\bar{R}eG_i)]\), therefore, is the failure of the system in the \( i \)th failure path under a load \( r \). Because the sets \( G_i \) are disjoint and satisfy Eq. (62),

\[
F_R(0) = \sum_{i=1}^{k} P[(V_i \leq 0) \cap (\bar{R}eE_i)] = 0
\]

and

\[
F_R(\infty) = \sum_{i=1}^{k} P[(V_i \leq \infty) \cap (\bar{R}eE_i)] = 1.0
\]

For an indeterminate system with a single path of failure, \( G_1 = \Omega \), and \( V_1 = R \).

The symbolic model described above is a general model applicable to any indeterminate system. Physically, the meaning of the sets \( G_i \) may depend on the nature of structural failure. In particular, these mean different things for structures with brittle behavior from those with ductile behavior. The determination of the sets \( G_i \) requires an analysis of the structural system which can best be amplified through illustrative examples. Where statistical independence or identical distribution of member strengths can be assumed, considerable simplification usually results.
4.3 Illustrative Examples

4.3.1 Example 1

Consider the system of three cables shown in Fig. 7 which is intended to carry loads that are applied vertically downward. Assume that the individual members are brittle; that is, when a member fails it can no longer carry any load. Let \( \alpha_i, i = 1, 2, 3 \), be the force in member \( i \) resulting from a unit load. Then

\[
\alpha_1 + \alpha_2 + \alpha_3 = 1.0
\]

Member 1 will rupture first if

\[
\frac{R_2}{\alpha_2} > \frac{R_1}{\alpha_1} \quad \text{and} \quad \frac{R_3}{\alpha_3} > \frac{R_1}{\alpha_1}
\]

and the rupture of member 2 will follow that of member 1, if in addition

\[
\frac{R_3}{\alpha_3} > \frac{R_2}{\alpha_2}
\]

The failure path denoted by the above sequence of failures, therefore, will occur if values of \( \mathbf{R} = \{R_1, R_2, R_3\} \) belongs to the set \( G_1 = \left\{ \frac{R_1}{\alpha_1} < \frac{R_2}{\alpha_2} < \frac{R_3}{\alpha_3} \right\} \).

In this case there are a total of \( 3! = 6 \) paths of failure corresponding to the number of permutations of the orders of rupture of the three members. For example, \( G_6 = \left\{ \frac{R_3}{\alpha_3} < \frac{R_2}{\alpha_2} < \frac{R_1}{\alpha_1} \right\} \). Then the distribution function of the system resistance becomes,

\[
F_R(r) = \sum_{i=1}^{6} P[(V_i \leq r) \cap (\mathbf{R} \in G_i)]
\]  

(64)
where for instance,

\[
P[(V_i \leq r) \cap (\tilde{R} \in \mathbb{G}_1)] \iiint_D f(r_1, r_2, r_3) \, dr_3 \, dr_2 \, dr_1
\]

in which \(D\) is the region \((0 < r_1 \leq \alpha_1 r; \frac{\alpha_2}{\alpha_1} r_1 < r_2 \leq \frac{\alpha_2 + \alpha_3}{\alpha_2} r; \frac{\alpha_3}{\alpha_2} r_2 < r_3 \leq r)\) of the \(r_1 r_2 r_3\) space.

If the random variables are statistically independent with individual distribution functions \(F_i(r_i)\), Eq. (64) yields,

\[
F_R(r) = \sum_{i=1}^{3} F_i(\alpha_i r) \left\{ F_k(r) F_j \left( \frac{\alpha_i}{\alpha_k + \alpha_j} r \right) + F_j(r) F_k \left( \frac{\alpha_j}{\alpha_k + \alpha_j} r \right) \right. \\
- F_k \left( \frac{\alpha_k}{\alpha_k + \alpha_j} r \right) F_j \left( \frac{\alpha_j}{\alpha_k + \alpha_j} r \right) \right\} - F_i(r) F_j(\alpha_i r) F_k(\alpha_i r)
\]

\[
+ \prod_{i=1}^{3} F_i(\alpha_i r) \tag{65}
\]

in which \(j, k = 1, 2, 3\) but \(j \neq k \neq i\). This specific problem has been treated before using a different formulation.\(^{15}\) It is readily verified that the two solutions are identical.

If the member resistances have identical probability distributions, \(F(r)\), and the applied load is carried equally by the members such that \(\alpha_i = \frac{1}{3}\) for all \(i\), Eq. (65) reduces to

\[
F_R(r) = F^3 \left( \frac{F}{3} \right) - 3F^2 \left( \frac{F}{2} \right) F \left( \frac{F}{3} \right) - 3F(r) F^2 \left( \frac{F}{3} \right) + 6F(r) F \left( \frac{F}{2} \right) F \left( \frac{F}{3} \right) \tag{66}
\]

which has been derived previously\(^{14}\) using a method that is limited only to special cases with symmetry (identical members).
4.3.2 Example II

Consider the system of Example 1 shown in Fig. 7, in which the cables are now of ductile behavior, specifically are of elastic-perfectly plastic material. In this case, as a member reaches its yield capacity it continues to carry its share of the load corresponding to this yield capacity. Clearly, the ultimate capacity of the system corresponds to the state in which all three members have yielded. Therefore,

\[ F_R(r) = P[\sum_{i=1}^{3} R_i \leq r] \quad (67) \]

The event \( \bigcap_{i=1}^{3} R_i \leq r \) can be depicted as the volume in \( r_1 r_2 r_3 \)-space as shown in Fig. 8.

In contrast to the brittle system of Example 1, it might be intuitively clear that in the present case, the order in which the members yield is immaterial since the system capacity is determined when all the members have yielded. Consequently, for this example problem there is only one path of failure. This can be verified as follows.

Consider the different orders in which the members yield such that the sets \( G_i, i = 1, 2, \ldots, 6 \), are the same as those of Example 1 and Eq. (64), therefore, remains applicable. However, because the members do not rupture, a typical term in Eq. (64) now becomes, for instance

\[ P[(V_1 \leq r) \cap (\overline{R} \in E_1)] = \iiint_{D_1} f(r_1, r_2, r_3) \, dr_3 \, dr_2 \, dr_1 \quad (68) \]
where $D_1$ is the region $[0 < r_1 \leq \alpha_1 r; \frac{\alpha_2}{\alpha_1} r_1 \leq r_2 \leq \frac{\alpha_2}{\alpha_2 + \alpha_3} (r - r_1); \frac{\alpha_3}{\alpha_2} r_2 \leq r_3 \leq r - (r_1 + r_2)]$. In Fig. 8, this region is shown cross-hatched with vertices at $(0,0,0),(0,0,r), (\alpha_1 r, \alpha_2 r, \alpha_3 r)$ and $(0, \frac{\alpha_2}{\alpha_2 + \alpha_3} r, \frac{\alpha_3}{\alpha_2 + \alpha_3} r)$.

It can be shown that the sum of the regions of integration for all the terms in Eq. (64) is equal to the volume described by the region $0 \leq r_1 + r_2 + r_3 \leq r$ as shown in Fig. 8; thus verifying Eq. (67) and demonstrating the fact that for systems with elastic perfectly plastic properties the order in which the different members yield need not be considered provided the event of interest requires yielding of all the members.

In this example problem, if the yield capacities of the three cables are statistically independent and are identically distributed with distribution function $F(r)$, and the applied load is carried equally among the three members such that $\alpha_i = \frac{1}{3}$ for all $i$, then Eq. (67) yields

$$F_R(r) = \iiint_{0 \leq r_1 + r_2 + r_3 \leq r} f(r_1)f(r_2)f(r_3)dr_3dr_2dr_1$$

Referring to Fig. 8,

$$F_R(r) = \int_0^r \int_0^{r-r_1} \int_0^{r-(r_1+r_2)} f(r_1)f(r_2)f(r_3)dr_3dr_2dr_1$$

In the present case, because of symmetry about the line $r_1 = r_2 = r_3 = \frac{r}{3}$, the above integral is equal to 6 times the integral evaluated inside the shaded volume shown in Fig. 8; thus,
\[ F_R(r) = 6\int_0^{r/3} f(r_1)dr_1 + \int_{r_1}^{1/2(r-r_1)} f(r_2)[F(r_1 - r_2) - F(r_2)]dr_2 \]

This result has also been derived before using a method which is limited to problems with symmetry as in the latter case of the present example.

4.3.3 Example III

Consider the prismatic fixed-fixed beam shown in Fig. 9a. The load is applied at the distance of 3/5 \( L \) from the left end. Failure is assumed to occur through ruptures at the points of high moment, which are indicated as points 1, 2, and 3. The resisting moment capacity at points 1 and 3 is \( M_1 \) and the corresponding resistance at point 2 is \( M_2 \). \( M_1 \) and \( M_2 \) are treated as non-negative random variables and it is assumed that upon occurrence of failure at a point, the moment resistance at that point is reduced to zero. The moment diagram prior to occurrence of any failure is shown in Fig. 9b.

The three possible paths of failure and the associated moment diagrams are given in parts a, b, and c of Fig. 10. The sets \( G_1 \) and the contribution of each path to the distribution function of the ultimate strength are determined in much the same way as in Example I; therefore, details are given only for path 1.

Path 1 --- Failure occurs through the sequence 2-3-1; that is, point 2 ruptures first and point 1 last. From the moment diagram of Fig. 9b, point 2 will fail first if

\[ M_1 > \frac{90}{72} M_2 \]

The resulting moment diagram will then be as shown in Fig. 10a, which shows that point 3 will fail before point 1. Therefore, failure will
occur through path 1 if values of $\tilde{R} = \{M_1, M_2\}$ belong to the set $G_1 = \{M_1 > \frac{90}{72} M_2\}$. Within this path, failure occurs whenever

$$M_2 \leq \frac{72}{625} \& r \text{ and } M_1 \leq \frac{54}{175} \& r$$

Therefore, the corresponding collapse probability is,

$$P\{(V_1 \leq r) \cap (\tilde{R} \in G_1)\} = P\{(M_1 \leq \frac{54}{175} \& r, M_2 \leq \frac{72}{625} \& r) \cap (M_1 > \frac{90}{72} M_2)\}$$

$$= \int_{0}^{\frac{72}{625} \& r} \int_{\frac{90}{72} M_2}^{\frac{54}{175} \& r} f(m_1, m_2) \, dm_1 \, dm_2 \quad (69)$$

Path 2 --- Failure occurs through the sequence 3-2-1. Using the information given in the moment diagrams of Figs. 9b and 10b, the collapse probability in this path is,

$$P\{(V_2 \leq r) \cap (\tilde{R} \in G_2)\} = P\{(M_1 < \frac{90}{625} \& r, M_1 < \frac{3}{5} \& r) \cap (\frac{105}{108} M_2 < M_1 < \frac{90}{72} M_2)\}$$

$$= \int_{0}^{\frac{90}{625} \& r} \int_{\frac{72}{90} M_1}^{\frac{106}{105} M_1} f(m_1, m_2) \, dm_2 \, dm_1 \quad (70)$$

Path 3 --- Failure occurs through sequence 3-1-2. The corresponding collapse probability is

$$P\{(V_3 \leq r) \cap (\tilde{R} \in G_3)\} = P\{(M_1 < \frac{90}{625} \& r, M_2 < \frac{6}{25} \& r) \cap (M_1 < \frac{105}{108} M_2)\}$$

$$= \int_{0}^{\frac{90}{625} \& r} \int_{\frac{6}{25} \& r}^{\frac{106}{105} M_1} f(m_1, m_2) \, dm_2 \, dm_1 \quad (71)$$
The respective regions of integration of the above integrals are shown in solid lines in Fig. 11. The probability of collapse of the system, on the basis of Eq. (63), therefore, is the sum of Eqs. (69) through (71).

If \( M_1 \) and \( M_2 \) are statistically independent random variables, the distribution function of the beam capacity is,

\[
F_R(r) = F_1\left(\frac{90}{625} \& r\right)\left[F_2\left(\frac{6}{25} \& r\right) - F_2\left(\frac{72}{625} \& r\right)\right] + F_1\left(\frac{54}{175} \& r\right)F_2\left(\frac{72}{625} \& r\right)
\]

In the above problem, if \( M_1 \) and \( M_2 \) are the yield moment capacities such that failure will occur through a plastic yield mechanism, the distribution function of the capacity of the beam is, according to Eq. (67),

\[
F_R(r) = P\left\{M_1 + M_2 \leq \frac{6}{25} \& r\right\}.
\]

The region of integration for this case is shown bounded by the lines \( m_1 = 0 \), \( m_2 = 0 \), and \( m_1 + m_2 = \frac{6}{25} \& r \) in Fig. 11. From this figure, it is clear that

\[
F_{R_{\text{plastic}}} (r) < F_{R_{\text{brittle}}} (r)
\]

4.3.4 Example IV

The next example is a two-span continuous beam as shown in Fig. 12a. The geometrical properties of the beam are assumed to be deterministic. The yield moment of each span is assumed to be a random variable; however, within one span the moment capacity is uniform throughout the span length. The structure is subjected to two concentrated loads of random magnitudes \( W \) and \( 2W \).

The structure is statically indeterminate to the first degree; hence, two plastic hinges are required to cause collapse of the structure. Furthermore, since the moment capacity is uniform within each span, plastic
hinges will occur only at the points of high statical moment; that is, at point 1, 2, 3, 4 as shown in Fig. 12a. Consequently, there are a total of \( \binom{4}{2} = 6 \) conceivable hinge formations which must be considered. These hinge formations are summarized in Fig. 13. An examination will reveal that the probability of occurrence of some of these mechanisms is zero; specifically, the last three shown in Fig. 13 are not possible to occur. For example, the mechanism shown in Fig. 13e is impossible because it is not possible to have a plastic hinge at point 1 and not at point 2 since 2 is stressed higher than point 1 in this span. Similar arguments apply to the hinge formations of Figs. 13d and 13f.

Considering that the order of hinge formation of each mechanism is immaterial in the evaluation of the system capacity, the only possible paths of failure of this structure correspond to the three mechanisms shown in Figs. 13a, through 13c. The sets \( G_i \) associated with these failure paths and the corresponding collapse loads are also given in Fig. 13. For example, the collapse load of Fig. 13c is obtained from

\[
\frac{V_3 \ell}{4} - \frac{M_1}{2} = M_2
\]

or

\[
V_3 = \frac{M_1 + 2M_2}{\ell}
\]

(72)

For this particular mechanism to occur, the moment capacities at points 1 and 3 must not be exceeded. Therefore, at point 1

\[
\frac{V_3 \ell}{4} - \frac{M_1}{2} < M_1
\]

and using Eq. (72), this becomes

\[
M_2 < 2.5 M_1
\]
Also, at point 3,

\[ M_1 < M_2 \]

It follows, therefore, that the structure will fail through the mechanism shown in Fig. 13c if

\[ \bar{R} \in G_3 \]

(73)

where \( \bar{R} = \{M_1, M_2\} \)

\[ G_3 = \{M_1 < M_2 < 2.5 M_1\} \]

Similarly, the structure will fail through one of the other two possible mechanisms shown in Figs. 13a and 13b, respectively, if

\[ \bar{R} \in G_1 \text{ or } \bar{R} \in G_2 \]

where \( G_1 = \{M_2 > 2.5 M_1\} \)

\[ G_2 = \{M_2 < M_1\} \]

Using the information contained in Fig. 13, the distribution function of the system resistance is obtained through Eq. (63) as follows:

\[
F_R(r) = P[(M_1 < \frac{\bar{R}}{6}) \cap (M_2 > 2.5M_1)] + P[(M_2 < \frac{\bar{R}}{3}) \cap (M_2 < M_1)]
+ P[(M_1 + 2M_2 < r) \cap (M_1 < M_2 < 2.5M_1)]
\]

\[
= \int_{0}^{\frac{\bar{R}}{6}} \int_{0}^{2.5m_1} f(m_1, m_2) \, dm_2 \, dm_1 + \int_{0}^{\frac{\bar{R}}{3}} \int_{0}^{\infty} f(m_1, m_2) \, dm_1 \, dm_2,
\]

\[
+ \int_{0}^{\frac{\bar{R}}{6}} \int_{m_1}^{2.5m_1} f(m_1, m_2) \, dm_2 \, dm_1 + \int_{m_1}^{\frac{\bar{R}}{3}} \int_{m_1}^{\infty} f(m_1, m_2) \, dm_2 \, dm_1,
\]

(74)
If the moment capacities are statistically independent with individual density and distribution functions $f_i(m)$ and $F_i(m)$, $i = 1, 2$, Eq. (74) then becomes

$$F_R(r) = \frac{1}{3} \left[ 1 - F_1\left( \frac{\delta_r}{3} \right) \right] \left[ 1 - F_2\left( \frac{\delta_r}{3} \right) \right] - \int \frac{\delta_r}{6} \left[ f_1(m) \left[ 1 - F_2\left( \frac{\delta_r-m}{2} \right) \right] \right] dm \quad (75)$$

It is easily verified that Eq. (75) satisfies the following requirements of a distribution function:

\[
\frac{dF_R(r)}{dr} > 0
\]

\[
F_R(0) = 0
\]

and

\[
F_R(\infty) = 1.0
\]

4.4 Discussions of Proposed Model

The mathematical model described in Sect. 4.2 provides a conceptual basis on which the probability distribution function of the system capacity of any statically indeterminate structure may be formulated. Basic in the application of the proposed model is the identification of the sets $G_i$ corresponding to the different possible paths of failure. For structures that fail by plastic yielding of the members, there are certain conditions under which not all conceivable mechanisms are possible paths of failure; this is because the formation of the plastic hinges are not statistically independent. This is illustrated, in particular, for structures where the yield capacity of each member is random but uniform over the member length. The distinction between the possible failure paths from those that are impossible is of central importance in the precise modeling of the probability of collapse of general indeterminate structural systems.
From the illustrative problems, it may be clear that the identification and physical meaning of failure paths are distinctly different and depend on the nature of failure of the components. For structures that fail through yielding of its components a failure path corresponds to a possible collapse mechanism (Examples II, IIIb and IV) regardless of the order of yielding of the component members or hinges in a particular mechanism. However, for structures that fail through successive ruptures of its components, the capacity of a structure depends on the order in which the component members fail and a failure path then corresponds to a particular order in which the component failures occur (Examples I and IIIa).

A major practical difficulty will invariably be present in the evaluation of the resulting integrals for the purpose of determining the distribution function of the system capacity. Numerical evaluation of these integrals appears to be generally necessary. Another difficulty is concerned with the identification of $Q_i$ for a given problem; a more systematic procedure should be found for this purpose.

The model described herein is applicable also for structures where collapse can be caused by several modes of failure such as the combination of bending, axial, shear, and torsional modes.
V. SUMMARY AND CONCLUSIONS

A number of fundamental questions pertaining to the probabilistic analysis of safety of structures and structural systems are examined. The major points expounded in the report can be summarized as follows:

(1) Under suitable assumptions which are appropriate for many civil engineering structures it is shown that the risk function \( h(N) \) is a monotonically decreasing function of the number of past loads sustained, or

\[
\begin{align*}
\text{h}(N) &< \text{h}(N-1) \quad \text{for all } N > 1 \\
\text{h}(N) &< p_f \\
\text{hence, the reliability function is, } \text{L}(N) > (1 - p_f)^N
\end{align*}
\]

(2) For a general determinate system subjected to external loads the forces induced in the members are always perfectly correlated, while the member strengths are likely to be partially correlated. The failure probability for this general case is,

\[
p_f^\dagger = 1 - \int_0^\infty \left[ \int_{\alpha_{l1}^s}^{\alpha_{l2}^s} \cdots \int_{\alpha_{l1}^s}^{\alpha_{l2}^s} f_{p1} \cdots f_{pm} \int_{r_1}^{r_2} \cdots \int_{r_1}^{r_2} f_s(s) ds \right] \text{f}_s(s) ds
\]

\[
- \int_0^\infty \left[ \int_{\alpha_{m1}^s}^{\alpha_{m2}^s} \cdots \int_{\alpha_{m1}^s}^{\alpha_{m2}^s} f_{n1} \cdots f_{nm} \int_{r_1}^{r_2} \cdots \int_{r_1}^{r_2} f_s(s) ds \right] \text{f}_s(s) ds
\]

If the member resistances are assumed to be statistically independent, Eq. (47) becomes

\[
p_f^\dagger = 1 - \int_0^\infty \left[ \int_{r_1}^{r_2} \cdots \int_{r_1}^{r_2} f_{p1} \cdots f_{pm} \int_{r_1}^{r_2} \cdots \int_{r_1}^{r_2} f_s(s) ds \right] \text{f}_s(s) ds
\]

\[
- \int_0^\infty \left[ \int_{r_1}^{r_2} \cdots \int_{r_1}^{r_2} f_{n1} \cdots f_{nm} \int_{r_1}^{r_2} \cdots \int_{r_1}^{r_2} f_s(s) ds \right] \text{f}_s(s) ds
\]
which has been proved to be the largest possible failure probability for a determinate system.

(3) Although the generally accepted assumption (that failures of the members are statistically independent events) is physically impausible, the necessary calculations are simpler. The resulting probability of failure \( p^*_f \) given as Eq. (50) is always larger than \( p''_f \), or

\[
P_f \leq p''_f \leq p^*_f
\]

(4) The reliability function for a determinate system, therefore,

\[
L(N) > (1-p_f)^N \geq (1-p''_f)^N \geq (1-p^*_f)^N
\]

consequently, \( (1-p^*_f)^N \) which can be approximated, for small \( p^*_f \), by

\[
(1-p^*_f)^N \approx 1 - NP^*_f
\]

is a very conservative estimate of the true reliability of a statically determinate system.

(5) The determination of the reliability of a statically indeterminate system (that is, the probability of no damage to the original system) involves the same formulation as that of a determinate system. Hence, the procedure for calculating the survival probability and all related conclusions stated above pertaining to determinate systems apply also to statically indeterminate systems.

(6) The determination of the probability of failure (collapse) of an indeterminate system, however, is much more difficult than that of a determinate system. A conceptual model is presented for this purpose in symbolic terms and its application is illustrated for ductile and brittle structures.
(7) From the illustrative problems, it seems clear that the physical meaning of the paths of failure of an indeterminate system are different between a brittle system and a ductile system. For brittle systems, the failure paths depend on the sequence of member failures, while for yielding structures the failure paths correspond only to different possible failure mechanisms regardless of the order in which the individual hinges are formed within each mechanism.
REFERENCES

FIG. 1  FAILURE REGIONS

FIG. 2  STATISTICALLY INDEPENDENT RANDOM VARIABLES
FIG. 3. SPECIFIC RISK FUNCTIONS
\[ A_i = \left( \frac{R_{p_i}}{\alpha_i} \leq S \right) \quad ; \quad C_i = \left( \frac{R_{n_i}}{\alpha_i} \geq S \right) \]

\[ B_i = \left( S > 0 \right) \quad ; \quad D_i = \left( S \leq 0 \right) \]

**FIG. 4** SUBSETS OF THE CERTAIN EVENT \( \Omega \)
Example: \( \frac{\sigma_r}{R_o} = 0.10 \)
\( \frac{\sigma_r}{R_o} = 2.0 \)

FIG. 5: FUNCTION \( [J(s)-I(s)] \) AND \( f_s(s) \) FOR MOST STRUCTURES
Top Chord Members: 8 in. 
Bottom Chord and Diagonal Members: 3.5 in.

Logarithmic Normal Load and Strengths

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Load</th>
<th>Strength</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Tension</td>
</tr>
<tr>
<td>Median</td>
<td>50 kips</td>
<td>40 ksi</td>
</tr>
<tr>
<td>Std. Deviation of Logarithm</td>
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<td>0.04</td>
</tr>
</tbody>
</table>

Probabilities of Damage $p_f^*$:

Determinate System = $1.8 \times 10^{-4}$
Indeterminate System = $9.6 \times 10^{-4}$

FIG. 6. A COMPARISON OF DAMAGE PROBABILITIES.
FIG. 7 SYSTEM CONSIDERED IN EXAMPLE I

FIG. 8 REGIONS OF INTEGRATION FOR EXAMPLE II
FIG. 9 SYSTEM CONSIDERED IN EXAMPLE III

FIG. 10 POSSIBLE PATHS IN EXAMPLE III

\[ x = \frac{w\ell}{625} \]
FIG. II REGIONS OF INTEGRATION FOR EXAMPLE III
(b) Elastic Moment Diagram

FIG. 12 SYSTEM CONSIDERED IN EXAMPLE IV

Path i  \( G_i \)  Collapse Load

1 \((M_2 > 2.5M_1)\)  \( V_1 = 6M_1/l \)

2 \((M_2 < M_1)\)  \( V_2 = 3M_2/l \)

3 \((M_1 < M_2 < 2.5M_1)\)  \( V_3 = (2M_2 + M_1)/l \)

Not Possible Mechanisms of Failure

FIG. 13 CONCEIVABLE HINGE FORMATIONS FOR EXAMPLE IV