DISCRETE ANALYSIS OF
CONTINUOUS FOLDED PLATES

by
A. KARIM CONRADO
and
W. C. SCHNOBRICH

A Report on a Research Program Carried Out under
National Science Foundation Grant No. GK-538

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS
MARCH, 1967
DISCRETE ANALYSIS OF CONTINUOUS FOLDED PLATES

by
A. Karim Conrado
and
W. C. Schnobrich

University of Illinois
Urbana, Illinois
March, 1967
ACKNOWLEDGMENT

This report was prepared as a doctoral dissertation by Mr. A. Karim Conrado. The work was under the general supervision of Dr. A. R. Robinson and the immediate supervision of Dr. W. C. Schnobrich. The research has been partially supported by the National Science Foundation under Research Grant NSF-GK 538.

The authors wish to express their thanks to the staff of the Department of Computer Science for their cooperation in the use of the IBM 7094-1401 computer system (supported partially by a grant from the National Science Foundation under Grant NSF-GP700).
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>vi</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>vii</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1. General Remarks</td>
<td>1</td>
</tr>
<tr>
<td>1.2. Previous Work.</td>
<td>3</td>
</tr>
<tr>
<td>1.3. Object and Scope</td>
<td>5</td>
</tr>
<tr>
<td>1.4. Nomenclature</td>
<td>5</td>
</tr>
<tr>
<td>METHOD OF ANALYSIS</td>
<td>8</td>
</tr>
<tr>
<td>2.1. Introductory Remarks</td>
<td>8</td>
</tr>
<tr>
<td>2.2. Assumptions and Limitations</td>
<td>8</td>
</tr>
<tr>
<td>2.3. Coordinate Systems</td>
<td>10</td>
</tr>
<tr>
<td>2.4. Description of the Discrete System Model</td>
<td>10</td>
</tr>
<tr>
<td>2.5. Sign Convention</td>
<td>12</td>
</tr>
<tr>
<td>2.5.1. Displacements</td>
<td>12</td>
</tr>
<tr>
<td>2.5.2. External Forces</td>
<td>13</td>
</tr>
<tr>
<td>2.5.3. Internal Forces</td>
<td>13</td>
</tr>
<tr>
<td>2.5.4. Eccentricity of Stiffeners</td>
<td>13</td>
</tr>
<tr>
<td>2.5.5. Angle of Inclination of Plate</td>
<td>13</td>
</tr>
<tr>
<td>2.6. Equilibrium Equations</td>
<td>14</td>
</tr>
<tr>
<td>2.6.1. U Equation</td>
<td>14</td>
</tr>
<tr>
<td>2.6.2. V Equation</td>
<td>16</td>
</tr>
<tr>
<td>2.6.3. W Equation</td>
<td>16</td>
</tr>
<tr>
<td>2.7. Boundary Forces</td>
<td>21</td>
</tr>
<tr>
<td>2.7.1. In-Plane Force $N_y$</td>
<td>21</td>
</tr>
<tr>
<td>2.7.2. Longitudinal Shear Force $N_{yx}$</td>
<td>22</td>
</tr>
<tr>
<td>2.7.3. Transverse Moment $M_y$</td>
<td>23</td>
</tr>
<tr>
<td>2.7.4. Transverse Reaction $R_y$</td>
<td>24</td>
</tr>
<tr>
<td>2.8. Description of Method of Solution</td>
<td>25</td>
</tr>
<tr>
<td>NUMERICAL RESULTS</td>
<td>32</td>
</tr>
<tr>
<td>3.1. General Remarks</td>
<td>32</td>
</tr>
<tr>
<td>3.2. Simply-Supported Trough-Shaped Folded Plate</td>
<td>32</td>
</tr>
<tr>
<td>3.3. Two-Span Continuous V Folded Plate</td>
<td>33</td>
</tr>
</tbody>
</table>
TABLE OF CONTENTS (Continued)

3.4. Two-Span Continuous Saw-Tooth Folded Plate ........ 35
   3.4.1. Comparison of Transverse Moments $M_y$ ........ 36
   3.4.2. Comparison of In-Plane Shears $N_{xy}$ ......... 37
   3.4.3. Comparison of Forces $N_y$ ................... 38
   3.4.4. Comparison of Longitudinal Stresses $N_x$ .... 39

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE STUDY .......... 41
   4.1. Conclusions ......................................... 41
   4.2. Recommendations for Future Study ................. 42

LIST OF REFERENCES ..................................... 43

TABLES ....................................................... 45

FIGURES ..................................................... 52

APPENDIX A. OPERATORS FOR INTERIOR POINTS ............... 76

APPENDIX B. EQUILIBRIUM EQUATIONS AND OPERATORS
   FOR THE TRANSVERSE STIFFENERS ....................... 81
   B.1. General Remarks .................................. 81
   B.2. Strain-Displacement Relations ................... 82
   B.3. Axial Forces and V Operators ................... 83
   B.4. Bending Moments and W Operators ................ 85
   B.5. Determination of Eccentricity and
        Depth of Equivalent Beam ....................... 87
   B.6. Coupling of Linear Equations due to
        Eccentricity of Stiffeners ................. 88

APPENDIX C. DESCRIPTION OF COMPUTER PROGRAM ........... 91
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Right-Hand Side of Equilibrium Equations for Unit Displacements at the Boundaries. No Stiffener Present.</td>
<td>45</td>
</tr>
<tr>
<td>2</td>
<td>Terms to be Added to Right-Hand Side of Equilibrium Equations for Unit Boundary Displacements When a Rectangular Stiffener is Present.</td>
<td>46</td>
</tr>
<tr>
<td>3</td>
<td>Solution of Simply Supported Folded Plate. Problem 1</td>
<td>47</td>
</tr>
<tr>
<td>4</td>
<td>Properties and Loading of Two-Span Continuous V Folded Plate. Problem 2.</td>
<td>48</td>
</tr>
<tr>
<td>5</td>
<td>Relative Distribution of Total Vertical Reaction Between End and Center Supports. Problem 2.</td>
<td>49</td>
</tr>
<tr>
<td>6</td>
<td>Solution of Two-Span Continuous Saw-Tooth Folded Plate. Problem 3.</td>
<td>50</td>
</tr>
<tr>
<td>7</td>
<td>Values of $N_x$ Over Central Supports. Problem 3.</td>
<td>51</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>1</td>
<td>Some Common Folded Plate Cross-Sections</td>
<td>52</td>
</tr>
<tr>
<td>2</td>
<td>Global and Local Systems of Coordinates</td>
<td>53</td>
</tr>
<tr>
<td>3</td>
<td>Discrete System Model</td>
<td>53</td>
</tr>
<tr>
<td>4</td>
<td>Contributing Areas and Location of Displacement Points.</td>
<td>54</td>
</tr>
<tr>
<td>5</td>
<td>Sign Convention for Displacements</td>
<td>55</td>
</tr>
<tr>
<td>6</td>
<td>Positive Bending Moments and Normal Forces</td>
<td>55</td>
</tr>
<tr>
<td>7</td>
<td>Positive Twisting Moments and Shear Forces</td>
<td>55</td>
</tr>
<tr>
<td>8</td>
<td>Identification of Interior Points on Plate</td>
<td>56</td>
</tr>
<tr>
<td>9</td>
<td>In-Plane Forces Acting on U Bar (J,n)</td>
<td>56</td>
</tr>
<tr>
<td>10</td>
<td>In-Plane Forces Acting on V Bar (m,K)</td>
<td>57</td>
</tr>
<tr>
<td>11</td>
<td>Transverse Forces Acting on W Point (J,K)</td>
<td>57</td>
</tr>
<tr>
<td>12</td>
<td>Moments and Transverse Shears Acting on U Bar (J,n)</td>
<td>57</td>
</tr>
<tr>
<td>13</td>
<td>Displacements and Rotations of Bar (J,n)</td>
<td>58</td>
</tr>
<tr>
<td>14</td>
<td>Forces and Displacements at Boundary</td>
<td>59</td>
</tr>
<tr>
<td>15</td>
<td>Identification of Points Near Edge of Plate</td>
<td>60</td>
</tr>
<tr>
<td>16</td>
<td>In-Plane Forces Acting Upon Longitudinal Bar (J,n) at Edge</td>
<td>60</td>
</tr>
<tr>
<td>17</td>
<td>Transverse Forces on W Point at Edge</td>
<td>61</td>
</tr>
<tr>
<td>18</td>
<td>Local and Global Displacements at Edge of Plate</td>
<td>62</td>
</tr>
<tr>
<td>19</td>
<td>Forces Acting Upon Longitudinal Joints of Plate</td>
<td>62</td>
</tr>
<tr>
<td>20</td>
<td>$N_x$ Values of Midspan. Problem 1</td>
<td>63</td>
</tr>
<tr>
<td>21</td>
<td>Variation of $M_y$ at Midspan. Problem 1</td>
<td>64</td>
</tr>
<tr>
<td>22</td>
<td>Transverse Displacements at Midspan. Problem 1</td>
<td>64</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES (Continued)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>$N_{xy}$ at Simple Support. Problem 1</td>
<td>65</td>
</tr>
<tr>
<td>24</td>
<td>$N_x$ at Free Edge. Problem 2-DB</td>
<td>66</td>
</tr>
<tr>
<td>25</td>
<td>$N_x$ at Fold. Problem 2-DB</td>
<td>66</td>
</tr>
<tr>
<td>26</td>
<td>$M_x$ Along Free Edge. Problem 2-DB</td>
<td>67</td>
</tr>
<tr>
<td>27</td>
<td>$M_y$ at Center of Each Span. Problem 2-DB</td>
<td>67</td>
</tr>
<tr>
<td>28</td>
<td>Variation of $N$ Across Several Sections of Folded Plate. Problem 2-NS</td>
<td>68</td>
</tr>
<tr>
<td>29</td>
<td>$N_x$ and $N_{xy}$ Values. Problem 2</td>
<td>69</td>
</tr>
<tr>
<td>30</td>
<td>Variation of $M_y$ Along First Valley. Problem 3</td>
<td>70</td>
</tr>
<tr>
<td>31</td>
<td>Variation of $M_y$ Along Central Ridge. Problem 3</td>
<td>70</td>
</tr>
<tr>
<td>32</td>
<td>Moment in Transverse Stiffener. Problem 3</td>
<td>71</td>
</tr>
<tr>
<td>33</td>
<td>Transverse Variation of $N_{xy}$. Problem 3-BO</td>
<td>72</td>
</tr>
<tr>
<td>34</td>
<td>Axial Forces in Stiffener. Problem 3</td>
<td>73</td>
</tr>
<tr>
<td>35</td>
<td>Variation of $N_y$ Along Valley Line B</td>
<td>73</td>
</tr>
<tr>
<td>36</td>
<td>Transverse Variation of $N_x$. Problem 3-NB</td>
<td>74</td>
</tr>
<tr>
<td>37</td>
<td>Distribution of Longitudinal Stresses in Central Plate. Problem 3-CP</td>
<td>75</td>
</tr>
<tr>
<td>B-1</td>
<td>Dimensions of Actual and Equivalent Stiffener</td>
<td>89</td>
</tr>
<tr>
<td>B-2</td>
<td>Identification of Points Along Stiffener.</td>
<td>89</td>
</tr>
<tr>
<td>B-3</td>
<td>V Displacements at Point (m,k)</td>
<td>90</td>
</tr>
<tr>
<td>B-4</td>
<td>Element of Bent Stiffener</td>
<td>90</td>
</tr>
</tbody>
</table>
1.1. General Remarks

Much has been written in recent years on the subject of folded plate structures. The majority of this literature has dealt with prismatic folded plates of single span and simple supports.

A prismatic folded plate structure consists of a series of rectangular plates connected along their longitudinal edges and supported transversely by two or more diaphragms or frames. The main field of application of this type of construction is for structures where relatively large spans are required such as hangars, auditoriums and industrial buildings. Some of the most commonly used cross sections are shown in Fig. 1.

The material quantities in folded plate construction are slightly higher than the quantities required when singly or doubly curved shells are used. Offsetting this increase in material usage, easier formwork and concreting frequently make the application of folded plates more economical than curved shell systems. In addition, the corrugations formed in the folded plate system may be used for ducts and utility troughs. An excellent discussion of the economics and practical aspects of folded plate construction is given by Whitney, Anderson and Birnbaum. (23)*

Longitudinally continuous folded plates are usually stiffened by means of transverse frames at the column lines. These frames prevent

*Numbers in parentheses refer to entries in the List of References.
distortion of the cross-section of the folded plate, thus eliminating the high transverse bending moments that would be present at the column lines if no frame existed there. In addition, the plates transmit the external loads to the frames which in turn are supported by the columns. When no frames are used, the loads are transmitted directly from the plate to the column supports resulting in high stress concentrations in that zone. Simple-span folded plates are sometimes provided with transverse frames spaced at intervals along the span either to induce beam-like behavior or to serve as anchors for tie-rods.

The methods of analysis of prismatic folded plates may be divided into four categories:

(a) Beam method.

(b) Folded plate theory neglecting relative joint displacements.

(c) Folded plate theory considering relative joint displacements.

(d) Elasticity method.

In the beam method it is assumed that no distortion of the cross-section of the folded plate structure takes place. As a result, the longitudinal stresses on a section cut transversely through the folded plate system are considered to vary linearly with the vertical distance from the centroidal axis just as they would in a beam. Furthermore, the in-plane shear forces are therefore parabolically distributed. Finally, transverse moments are computed from equilibrium using the incremental change in shear plus the vertical loads. This method is justifiably used frequently for long-span folded plate structures.
Method (b) differs from the Beam Method in that the linear variation of longitudinal stresses is assumed to exist over the individual plates rather than over the whole section. Method (b) disregards the transverse moment and longitudinal stresses caused by the relative displacement of the longitudinal joints. This effect is taken into consideration in method (c). In 1963, the Task Committee on Folded Plate Construction (Committee on Masonry and Reinforced Concrete) of the American Society of Civil Engineers recommended the use of method (c) for the design of simply-supported folded plate structures.\(^{(13)}\)

The elasticity solution as developed by Goldberg and Leve\(^{(7)}\) makes use of the classical theory of plates for loads perpendicular to the plane of the plate and the plane stress theory of elasticity for loads in the plane of the plate. Useful as it is, the elasticity method has some shortcomings, namely: (1) It applies only to folded plates simply-supported at the ends by flexible diaphragms. (2) It cannot handle folded plates with tie-rods or transverse stiffening ribs. (3) It cannot handle folded plates with intermediate column supports.

The advent of the high-speed electronic digital computer makes it possible to analyze numerically the cases listed above as being excluded from the Goldberg and Leve elasticity method.

1.2. Previous Work

First papers on folded plate theory appeared in the technical literature beginning in 1930. A theory based on a linear variation of longitudinal stresses in each plate was proposed by Ehlers\(^{(5)}\) in that
year. This was an improvement over the beam analysis which considers a linear variation of longitudinal stresses across the entire structure. Ehler's theory considered the structure to be hinged along the longitudinal joints. In the same year Craemer\textsuperscript{(3)} proposed a method that considered the transverse moments due to continuity at the joints. Winter and Pei\textsuperscript{(24)} introduced, for the first time in the United States, the folded plate theory neglecting relative joint displacements.

In 1932 Gruening\textsuperscript{(9)} proposed a theory that takes into account the relative displacement of the joints of the folded plate. Gaafar\textsuperscript{(6)} and Yitzhaki\textsuperscript{(25)} modified and further developed this theory. Werfel\textsuperscript{(22)} first proposed a theory which included all possible stress resultants. Later Goldberg and Leve\textsuperscript{(7)} developed the so-called elasticity solution. De Fries-Skene and Scordelis\textsuperscript{(4,18)} presented a direct stiffness method in matrix form for both the ordinary\textsuperscript{(18)} and the Goldberg and Leve elasticity\textsuperscript{(4)} methods. Reviews of several design theories of folded-plate structures have been given by Traum\textsuperscript{(21)} and more recently by Powell.\textsuperscript{(14)}

Experimental and analytical studies by Ali\textsuperscript{(1)} and Paulson\textsuperscript{(12)} showed that the number and spacing of transverse diaphragms has a significant effect on the transverse distribution of longitudinal stresses. Goldberg, Gutzwiller and Lee\textsuperscript{(8)} used a finite difference technique to study continuous folded plates. The effect of transverse stiffeners on the behavior of closed prismatic structures was studied by Craemer.\textsuperscript{(2)}
1.3. Object and Scope

The objective of this study is to develop a numerical method for the elastic analysis of transversely stiffened prismatic folded plates either simply-supported or continuous in the longitudinal direction. The method must be able to handle folded plates with tie-rods, supporting columns, concentrated loads and transverse stiffening ribs that have an eccentricity relative to the plane of the plate.

The representative problems shown are compared with existing solutions whenever possible.

1.4. Nomenclature

The symbols used in this work are defined where they first appear. For convenience they are summarized below:

- $A$: a square matrix of coefficients
- $B$: a column matrix
- $b$: width of stiffening beam
- $D = \frac{Eh^3}{12(1-\nu^2)}$: flexural stiffness of plate
- $d$: depth of stiffening beam
- $\overline{d} = \frac{d}{\sqrt{3}}$: separation of bars of equivalent beam
- $E$: Young's modulus
- $e$: eccentricity of beam relative to plane of the plate
- $F$: a column matrix of boundary forces
- $G$: shear modulus of elasticity
- $h$: thickness of plate
\( K \) = a stiffness matrix

\( L_x, L_y \) = grid lengths in x- and y-directions respectively

\( M_x(J,K), M_y(J,K) \) = plate moments about the y- and x-axes respectively at extensional node \( J,K \)

\( M_{xy}(m,n) \) = plate twisting moment at shear node \( m,n \)

\( M \) = an influence matrix (Section 2.8)

\( m \) = plate moment per unit length

\( N_x(J,K), N_y(J,K) \) = plate membrane forces in the x- and y-directions respectively at extensional node \( J,K \)

\( N_{xy}(m,n) \) = in-plane shear force at shear-node \( m,n \)

\( p \) = load per unit area in z-direction

\( Q_x(J,n), Q_y(m,K) \) = transverse shear in x- and y-directions at locations \( J,n \) and \( m,K \) respectively

\( q \) = load per unit area in x-direction

\( R_y(J,K) \) = reaction at edge at location \( J,K \)

\( r \) = load per unit area in y-direction

\( t = \frac{h}{\sqrt{3}} \) = separation of bars of framework

\( U_{J,n}, V_{m,K} \) = plate displacements in x- and y-directions at locations \( J,n \) and \( m,K \) respectively

\( W_{J,K} \) = plate displacement in z-direction at location \( J,K \)

\( \bar{X}_{J,n}, \bar{Y}_{m,K}, \bar{Z}_{J,K} \) = total external loads in the x-, y-, and z-directions at locations \( J,n \), \( m,K \), and \( J,K \) respectively

\( \alpha \) = rotation of bar of framework

\( Y_{xy}(m,n) \) = twist of surface at point \( m,n \)

\( \Delta \) = column matrix of boundary displacements
\[ \delta = \text{column matrix of interior displacements} \]

\[ \epsilon^T_{x(J,K)}, \epsilon^B_{x(J,K)} = \text{extensional strain in x-direction at level of top and bottom bars of framework respectively} \]

\[ \epsilon_{x(J,K)}, \epsilon_{y(J,K)} = \text{extensional strains at mid-depth of plate in x- and y-directions respectively} \]

\[ \eta = \text{plate displacement in the Y-direction} \]

\[ \nu = \text{Poisson's ratio} \]

\[ \xi = \text{plate displacement in the Z-direction} \]

\[ \sigma_{x(J,K)}, \sigma_{y(J,K)} = \text{extensional stresses in x- and y-directions respectively at extensional node J,K} \]

\[ \tau_{xy(m,n)} = \text{in-plane shear stress at mid-depth of plate at shear-node m,n} \]

\[ \varphi = \text{angle of inclination of the plate with the horizontal} \]

\[ \chi_{x(J,K)} = \text{curvature of plate in the x-direction at location J,K} \]

\[ \chi_{y(J,K)} = \text{curvature of plate in the y-direction at location J,K} \]

\[ \psi = \text{rotation of plate about X-axis} \]
2.1. *Introductory Remarks*

In the method of analysis for prismatic folded plates presented in this work the individual plates are solved by means of a discrete system model developed by Schnobrich. The details of the model are presented in Section 2.4.

A stiffness method of solution is used for the analysis of the structure as a whole. The description of this method as applied to folded plates is given in Section 2.8.

2.2. *Assumptions and Limitations*

The assumptions made relating the flexural action of the plate are those of the classical small-deflection theory of plates. For the in-plane or membrane action of the plate the assumptions are those of the plane-stress theory of elasticity.

The transverse stiffeners are assumed to resist bending about an axis parallel to the longitudinal span of the plate and to be completely flexible when bent about the weak axis. Torsional resistance of the stiffener is neglected. This omission of the torsional stiffness may be justified on the grounds that stiffeners are usually located at or near locations where the twist of the surface is zero. The stiffener and the plate are assumed to work monolithically, therefore they are assumed to have the same strain at their points of junction.
For bending of the plate, the following partial differential equation must be satisfied:

\[
\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} = \frac{p}{D}
\]  

where \( W \) is the deflection of the plate in the z-direction, \( D \) is the flexural stiffness of the plate and \( p \) is the load per unit area acting perpendicular to the plane of the plate.

The in-plane actions must satisfy simultaneously the following partial differential equations:

\[
\frac{Eh}{(1-v^2)} \left[ \frac{\partial^2 U}{\partial x^2} + \frac{(1-v)}{2} \cdot \frac{\partial^2 U}{\partial y^2} + \frac{(1+v)}{2} \cdot \frac{\partial^2 V}{\partial x \partial y} \right] + q = 0 \quad (2)
\]

\[
\frac{Eh}{(1-v^2)} \left[ \frac{\partial^2 V}{\partial y^2} + \frac{(1-v)}{2} \cdot \frac{\partial^2 V}{\partial x^2} + \frac{(1+v)}{2} \cdot \frac{\partial^2 U}{\partial x \partial y} \right] + r = 0 \quad (3)
\]

where \( U \) and \( V \) are displacements in the x- and y-directions respectively, \( E \) is Young's modulus of elasticity, \( v \) is Poisson's ratio, \( h \) is the thickness of the plate and \( q \) and \( r \) are the loads per unit area in the x- and y-directions, respectively.

Within the domain of the small-deflection theory, the normal deflection of the plate is governed by Eq. (1). This equation is independent of Eqs. (2) and (3) which are in turn coupled. Discretization of the plate results in one system of linear equations involving \( W \) displacements only and another system involving \( U \) and \( V \) displacements only. This situation is altered only when the plate is stiffened with eccentric ribs. In this latter case the entire system of linear equations is
coupled. Uncoupling of the equations implies that the equations may be solved as two separate sets of simultaneous equations. With a given storage capacity of the computer, much finer grids can be used if the uncoupled systems are solved separately.

2.3. Coordinate Systems

For the analysis of the structure a global orthogonal frame of reference \((X,Y,Z)\) is established as shown in Fig. 2. The \(X\) and \(Y\) axes lie in a horizontal plane, the \(X\) axis running parallel, and the \(Y\) axis perpendicular to the fold lines. The \(Z\) axis points downwards.

In addition to this global system, local systems of orthogonal coordinates are defined for each plate. The local frame of reference for plate \(n\) is denoted as \((x_n,y_n,z_n)\), Fig. 2. The \(x_n\) and \(y_n\) axes lie in the plane of the plate, the \(x_n\) axis being parallel and the \(y_n\) axis perpendicular to the fold lines. The \(z_n\) axis points in a direction such that the triad \((x_n,y_n,z_n)\) is a right-handed system.

2.4. Description of the Discrete System Model

A discrete system model\(^{(17)}\) will be used for the analysis of stresses and deformations of the structure. The model was originally developed for the analysis of cylindrical shells but it is equally applicable to plates provided the curvature is made zero. The model consists of a grid of straight, weightless, rigid bars arranged as shown in Fig. 3. The network is made of two bar-joint systems called the primary and secondary systems. They are arranged in two layers as in sandwich construction.
At the intersection of the bars of the primary system there are extensional elements capable of developing extensional forces and bending moments in both the longitudinal and transverse directions. The bars of the secondary system are connected to the midpoints of the bars of the primary system and are fastened to deformable shear nodes. These shear nodes provide the resultant in-plane shear forces and twisting moments.

The distance between the top and bottom rigid bars, thickness \( t \), of the grid is equal to \( h/\sqrt{3} \), where \( h \) is the actual thickness of the plate. This separation insures that the bending response of the model corresponds to that of the plate. The derivation of this relationship is given in Section 2.6.3.

The stress-strain properties of the material comprising the nodes of the model are the same as those of the material in the actual structure. Since the nodes are subjected to plane stress, the following stress-strain relations are valid for the nodes:

\[
\begin{align*}
\sigma_x &= \frac{E}{(1-\nu^2)} \left( \epsilon_x + \nu \epsilon_y \right) \\
\sigma_y &= \frac{E}{(1-\nu^2)} \left( \epsilon_y + \nu \epsilon_x \right) \\
\tau_{xy} &= \frac{E}{2(1+\nu)} \cdot \gamma_{xy}
\end{align*}
\]

where \( \sigma_x, \sigma_y \) and \( \tau_{xy} \) are the extensional stresses in the \( x \)- and \( y \)-directions and the shear stress in the \( x-y \) plane respectively. The corresponding unit strains are \( \epsilon_x, \epsilon_y \), and \( \gamma_{xy} \).
As shown in Fig. 4 the U and V displacements are defined at the center of the \(L_x\) and \(L_y\) bars of the primary system along the \(x\)- and \(y\)-directions respectively. These displacements are at mid-depth of the plate. The \(W\) displacements are defined at the intersections of the primary bars along the \(z\)-direction.

Uniformly distributed loads are replaced by concentrated loads equal in magnitude to the total loads acting in the respective contributing areas. Non-uniform loads are replaced by statically equivalent concentrated loads. The loads are called \(X\), \(Y\), and \(Z\); they act in the direction of the local coordinate axes at the points where the respective displacements are defined.

2.5. **Sign Convention**

In this section the sign convention for the following items is given: displacements, internal and external forces, eccentricity of stiffeners and angle of inclination of plate.

2.5.1. **Displacements**

As shown in Fig. 5, the \(U\), \(V\), and \(W\) displacements are positive in the positive \(x_n\), \(y_n\), and \(z_n\)-directions respectively. Rotations of the \(L_y\) bars, \(\alpha_x\), about the \(x_n\) axis are positive if they are clockwise when viewed from the origin in the positive \(x_n\) direction. Rotations of the \(L_x\) bars, \(\alpha_y\), about the \(y_n\) axis are positive if counterclockwise when viewed from the origin in the positive \(y_n\)-direction.
2.5.2. **External Forces**

The $X, Y,$ and $Z$ external forces are positive in the positive directions of the local coordinates.

2.5.3. **Internal Forces**

Bending moments $M_x$ and $M_y$ are positive if they produce tension in the bottom fibers, as shown in Fig. 6. In-plane forces $N_x$ and $N_y$ are positive if they produce tension. Twisting moments $M_{xy}$ acting on the positive face are positive if counterclockwise when viewed from the origin in the positive $x_n$ direction as shown in Fig. 7. Moments $M_{yx}$ on the positive face are positive if clockwise when viewed from the origin in the positive $y_n$-direction. In-plane shear forces $N_{yx}$ and $N_{xy}$ acting on the positive faces are positive if they act in the positive $x_n$ and $y_n$-directions respectively. Transverse shears $Q_x$ and $Q_y$ acting on the positive faces are positive when acting in the positive $z_n$-direction.

2.5.4. **Eccentricity of Stiffeners**

The eccentricity of the stiffener is positive if its center of gravity is located above the middle surface of the slab.

2.5.5. **Angle of Inclination of Plate**

The angle of inclination of a plate is positive when measured from the positive $Y$ axis counterclockwise to the middle plane of the plate.
2.6. **Equilibrium Equations**

In this section the equilibrium equations for typical interior points are derived. The analysis of any plate problem by means of the discrete system model requires the solution of a system of linear simultaneous equations. Each equation relates the appropriate external force applied at a point of displacement definition with the displacement at that point and at neighboring displacement points. These equations are called U, V, or W equations according to the point and direction for which equilibrium is established. Collectively they are called equilibrium equations.

For the derivation of the equilibrium equations reference is made to the identification scheme shown in Fig. 8. The equations are presented in operator form in Appendix A. The equilibrium equations for the stiffeners are given in Appendix B.

2.6.1. **U Equation**

Consideration of the equilibrium of the horizontal forces acting on the U bar, shown in Fig. 9, gives:

\[ N_x(J,K) - N_x(J,K-1) + N_{yx}(m+1,n) - N_{yx}(m,n) + \bar{X}(J,n) = 0 \]  

(5)

where the N's are the internal forces and \( \bar{X}(J,n) \) is the total external load acting upon that bar. The internal force resultants may be expressed in terms of the stresses by the following equations:

\[
N_x(J,K) = L_y \cdot h \cdot \sigma_x(J,K)
\]
\[
N_{yx}(m+1,n) = L_x \cdot h \cdot \tau_{yx}(m+1,n)
\]

(6)
Substitution of Eqs. (4) into Eqs. (6) yields:

\[ N_x(J,K) = \frac{EhL_y}{(1-\nu)^2} \left[ \varepsilon_x(J,K) + \nu \varepsilon_y(J,K) \right] \]  
\[ N_{yx(m+1,n)} = \frac{EhL_x}{2(1+\nu)} \gamma_{yx(m+1,n)} \]

The strains at points \((J,K)\) and \((m+1,n)\) are as follows:

\[ \varepsilon_x(J,K) = \frac{U_{J,n+1} - U_{J,n}}{L_x} \]
\[ \varepsilon_y(J,K) = \frac{V_{m+1,K} - V_{m,K}}{L_y} \]
\[ \gamma_{yx(m+1,n)} = \frac{U_{J+1,n} - U_{J,n}}{L_y} + \frac{V_{m+1,K} - V_{m+1,K-1}}{L_x} \]

Substitution of Eqs. (8) into Eqs. (7) yields the force-displacement equations:

\[ N_x(J,K) = \frac{EhL_y}{(1-\nu)^2} \left[ \frac{U_{J,n+1} - U_{J,n}}{L_x} + \frac{\nu(V_{m+1,K} - V_{m,K})}{L_y} \right] \]

and

\[ N_{yx(m+1,n)} = \frac{EhL_x}{2(1+\nu)} \left[ \frac{U_{J+1,n} - U_{J,n}}{L_y} + \frac{V_{m+1,K} - V_{m+1,K-1}}{L_x} \right] \]

Upon substitution of the force-displacement relations into Eq. (5), the "U equation" is obtained:
\[
\frac{L_y}{L_x} \left[ U_{j,n+1} - 2U_{j,n} + U_{j,n-1} \right] + \frac{(1-v)}{2} \cdot \frac{L_y}{L_x} \left[ U_{j+1,n} - 2U_{j,n} + U_{j-1,n} \right] \\
+ \frac{(1+v)}{2} \cdot \left[ V_{m+1,K} - V_{m,K} - V_{m+1,K-1} + V_{m,K-1} \right] + \frac{(1-v^2)}{Eh} \bar{Y}_{(J,n)} = 0
\]

(11)

2.6.2. \textbf{V Equation}

The derivation of the "V equation" is similar to the derivation of the "U equation." Equilibrium of the forces shown in Fig. 10 gives:

\[
N_y(J,K) - N_y(J-1,K) + N_{xy}(m,n+1) - N_{xy}(m,n) + \bar{V}(m,K) = 0
\]

(12)

Substitution of force-displacement relations into Eq. (12) results in the "V equation":

\[
\frac{L_x}{L_y} \left[ V_{m+1,K} - 2V_{m,K} + V_{m-1,K} \right] + \frac{(1-v)}{2} \cdot \frac{L_y}{L_x} \left[ V_{m,K+1} - 2V_{m,K} + V_{m,K-1} \right] \\
+ \frac{(1+v)}{2} \cdot \left[ U_{J,n+1} - U_{J,n} - U_{J-1,n+1} + U_{J-1,n} \right] + \frac{(1-v^2)}{Eh} \bar{V}(m,K) = 0
\]

(13)

2.6.3. \textbf{W Equation}

In the z-direction, equilibrium of the forces shown in Fig. 11 gives:

\[
Q_x(J,n+1) - Q_x(J,n) + Q_y(m+1,K) - Q_y(m,K) + \bar{Z}(J,K) = 0
\]

(14)

where the $Q$'s are the internal transverse shears and $\bar{Z}(J,K)$ represents the external load applied at point $(J,K)$. The transverse shear may be expressed in terms of the bending and twisting moments; e.g., $Q_x(J,n)$
may be obtained by taking moments about the right end of the U bar shown in Fig. 12:

\[ Q_x(J,n) = \frac{M_x(J,K) - M_x(J,K-1)}{L_x} + \frac{M_yx(m+1,n) - M_{yx}(m,n)}{L_x} \] (15)

Substitution of expressions for the \( Q \)'s similar to Eq. (15) into Eq. (14) gives:

\[
\begin{align*}
&\frac{M_x(J,K+1) - 2M_x(J,K) + M_x(J,K-1)}{L_x} + \frac{M_y(J+1,K) - 2M_y(J,K) + M_y(J-1,K)}{L_y} \\
&+ \frac{M_yx(m+1,n+1) - M_yx(m,n+1) - M_{yx}(m+1,n) + M_{yx}(m,n)}{L_x} \\
&+ \frac{M_{xy}(m+1,n+1) - M_{xy}(m,n+1) - M_{xy}(m+1,n) + M_{xy}(m,n)}{L_y} + Z(J,K) = 0 \\
&\text{(16)}
\end{align*}
\]

The bending and twisting moments in Eq. (16) may be expressed in terms of the displacements. Only the moments \( M_x(J,K) \) and \( M_{yx}(m+1,n) \) will be derived in this section. The derivation of moments \( M_y \) and \( M_{xy} \) proceeds in a similar fashion.

The moments may be expressed as follows:

\[
M_x(J,K) = \left(\frac{L}{2}\right)\left(\frac{h}{2}\right) \cdot L_y \cdot \left[ \sigma^B_x(J,K) - \sigma^T_x(J,K) \right] \] (17)

\[
M_{yx}(m+1,n) = -\left(\frac{L}{2}\right)\left(\frac{h}{2}\right) \cdot L_x \cdot \left[ \tau^B_{yx}(m+1,n) - \tau^T_{yx}(m+1,n) \right] \] (18)

where superscripts \( B \) and \( T \) indicate the top or bottom bars of the framework. Substitution of Eqs. (4) into Eqs. (17) and (18) gives:
If the strain quantities are replaced by the appropriate displacement terms given by Eqs. (8), Eqs. (19) and (20) become:

\[
M_x(J,K) = \frac{EhtL_y}{4(1-v^2)} \left[ \frac{u^B_{J,n+1} - u^T_{J,n+1}}{L_x} - \frac{u^B_{J,n} - u^T_{J,n}}{L_x} \right] + v \left( \frac{v^B_{m+1,K} - v^T_{m+1,K}}{L_y} - \frac{v^B_{m,K} - v^T_{m,K}}{L_y} \right)
\]

\[
M_{yx}(m+1,n) = -\frac{EhtL_x}{8(1+v)} \left[ \frac{u^B_{J+1,n} - u^T_{J+1,n}}{L_y} - \frac{u^B_{J,n} - u^T_{J,n}}{L_y} \right]
\]

From Fig. 13, it may be seen that:

\[
\lambda_1 = \lambda_2
\]

therefore, since the sines of both angles are equal,

\[
\frac{u^B_{J,n} - u^T_{J,n}}{L_x} = \frac{t}{L_x} (W_{J,K-1} - W_{J,K})
\]

Substitution of expressions for the difference between bottom and top
displacements, as in Eq. (24), into Eqs. (21) and (22) yields:

\[ M_x(J,K) = \frac{Eh t^2}{4(1-\nu^2)} \left[ \frac{L_y}{L_x} \left( -W_{J,K+1} + 2W_{J,K} - W_{J,K-1} \right) + \frac{\nu}{L_y} \left( -W_{J+1,K} + 2W_{J,K} - W_{J-1,K} \right) \right] \]  

\[ M_{yx}(m+1,n) = \frac{Eh t^2}{4(1+\nu)L_y} \left( W_{J+1,K-1} - W_{J+1,K-1} + W_{J,K} - W_{J-1,K} \right) \]  

The moments per unit length are obtained by dividing the moment expressions just derived by \( L_y \) and \( L_x \), respectively,

\[ m_x(J,K) = \frac{Eh t^2}{4(1-\nu^2)} \left[ \frac{L_y}{L_x} \left( -W_{J,K+1} + 2W_{J,K} - W_{J,K-1} \right) + \frac{\nu}{L_y} \left( -W_{J+1,K} + 2W_{J,K} - W_{J-1,K} \right) \right] \]  

\[ m_{yx}(m+1,n) = \frac{Eh t^2}{4(1+\nu)L_y} \left( W_{J+1,K-1} - W_{J+1,K-1} + W_{J,K} - W_{J-1,K} \right) \]  

Equations (27) and (28) are the equivalents in the model of the usual moment curvature relations of plate theory: (20)

\[ m_x = -D\left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \]  

\[ m_{yx} = -D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \]
where \( D = \frac{Eh^3}{12(1-\nu^2)} \).

Therefore, for the response of the model to be equal to that of the plate

\[
\frac{Eh^3}{12(1-\nu^2)} = \frac{Eht^2}{4(1-\nu^2)}
\]

from which

\[
t = \frac{h}{\sqrt{3}} \quad (30)
\]

Substitution of moment expressions similar to Eqs. (25) and (26) into Eq. (16) yields the "W equation":

\[
\frac{t^2 L_x}{4L_x^3} \left[ -W_{J+2,K} + \frac{4}{3} W_{J+1,K} - \frac{6}{3} W_{J,K} + \frac{4}{3} W_{J-1,K} - \frac{2}{3} W_{J-2,K} \right] \\
+ \frac{t^2 L_y}{4L_y^3} \left[ -W_{J,K+2} + \frac{4}{3} W_{J,K+1} - \frac{6}{3} W_{J,K} + \frac{4}{3} W_{J,K-1} - \frac{2}{3} W_{J,K-2} \right] \\
+ \frac{t^2 L_y}{2L_x L_y} \left[ -W_{J+1,K+2} + \frac{2}{3} W_{J+1,K+1} + \frac{2}{3} W_{J+1,K} + \frac{2}{3} W_{J+1,K-1} + \frac{2}{3} W_{J-1,K+1} \right] + \frac{(1-\nu^2)}{Eh} \overline{Z}(J,K) = 0
\]

A comparison of the equilibrium equations obtained from the model with finite difference expressions of Eqs. (1), (2), and (3) will show the mathematical equivalence of the two procedures. If the plate is transversely stiffened by beams, the \( V \) and \( W \) equilibrium equations for the stiffener must be added to the corresponding equations for the plate.
2.7. **Boundary Forces**

Because of the special conditions existing at the boundaries of a plate, the formulas for the boundary forces differ from those of a typical interior point. In this section, the formulas for the four boundary forces (see Fig. 14a) are derived. They are applicable whether the longitudinal edge is free or restrained by adjacent plates. The points and bars of the framework near a longitudinal edge are identified by means of the notation shown in Fig. 15.

Compatibility of deformations at the longitudinal joints of a folded plate requires that all plates meeting at that joint have the same rotation about the X axis and that the three displacements along the X, Y, and Z axes be equal. Therefore, for each panel length three linear displacements and one rotation must be defined at the edge. The U and W displacements at the edge are defined in the usual manner. As shown in Fig. 14b, the rotation and the V displacement at the edge are defined at the W points.

### 2.7.1. In-Plane Force $N_y$

The in-plane force $N_y$ at point $(J,K)$ on the edge is expressed in terms of the strains in the following manner:

$$N_y(J,K) = \sigma_y(J,K) \cdot h \cdot L_x = \frac{EhL}{(1-\nu^2)} \left[ \varepsilon_y(J,K) + \nu \varepsilon_x(J,K) \right] \quad (32)$$

The strains at $(J,K)$ are given in terms of the displacements by the expressions:
The \( y \)-strain at the edge is given in terms of the displacement at the edge and at an interior point. Substitution of Eqs. (33) into Eq. (32) gives \( N_y \) at the edge:

\[
N_y(J,K) = \frac{EhL_x}{(1-v^2)} \left[ \frac{2(V_{J,K} - V_{m,K})}{L_y} + v \frac{(U_{J,n+1} - U_{J,n})}{L_x} \right]
\]

(34)

2.7.2. Longitudinal Shear Force \( N_{yx} \)

Longitudinal equilibrium of the forces shown in Fig. 16 requires that:

\[
N_{yx}(J,n) = N_{yx}(m,n) + N_x(J,K-1) - N_x(J,K)
\]

(35)

The forces in the right-hand side of the equation above may be expressed in terms of the displacements as follows:

\[
N_{yx}(m,n) = \frac{EhL_x}{2(1+v)} \left[ \frac{U_{m,n} - U_{m,n-1}}{L_y} + \frac{V_{m,K} - V_{m,K-1}}{L_x} \right]
\]

(36)

\[
N_x(J,K-1) = \frac{EhL_y}{2(1-v^2)} \left[ \frac{U_{J,n} - U_{J,n-1}}{L_x} + \frac{2v(V_{J,K-1} - V_{m,K-1})}{L_y} \right]
\]

\[
N_x(J,K) = \frac{EhL_y}{2(1-v^2)} \left[ \frac{U_{J,n+1} - U_{J,n}}{L_x} + \frac{2v(V_{J,K} - V_{m,K})}{L_y} \right]
\]
Substitution of Eqs. (36) into Eq. (35) yields the longitudinal shear at the edge:

\[
N_{yx}(J,n) = \frac{EhL_x}{2(1+\nu)} \left[ \frac{U_{J,n} - U_{J-1,n}}{L_y} + \frac{V_{m,K} - V_{m,K-1}}{L_x} \right] \\
+ \frac{EhL_y}{2(1-\nu^2)} \left[ \frac{-U_{J,n-1} + 2U_{J,n} - U_{J,n+1}}{L_x} + \frac{2v(V_{J,K-1} - V_{J,K-1} - V_{J,K+1} + V_{J+1,K+1})}{L_y} \right]
\]

(37)

2.7.3. Transverse Moment \( M_y \)

The transverse moment \( M_y \) at a typical interior point is given by:

\[
M_y = -\frac{Eh^2L_x}{4(1-\nu^2)} \left[ \chi_x + \nu \chi_y \right]
\]

(38)

where \( \chi_x \) and \( \chi_y \) denote the curvatures along the \( x \)- and \( y \)-directions.

For a point \((J,K)\) at the edge the curvatures are as follows:

\[
\chi_x(J,K) = \frac{W_{J,K+1} - 2W_{J,K} + W_{J,K-1}}{L_x^2}
\]

(39)

\[
\chi_y(J,K) = \frac{\alpha(J,K) - \frac{W_{J,K} - W_{J-1,K}}{L_y}}{L_y^2}
\]

where \( \alpha(J,K) \) is the rotation of the plate at point \((J,K)\). Therefore the expression for the \( y \)-bending moment at the edge is:
The transverse forces acting at edge point \((J,K)\) are shown in Fig. 17. Equilibrium of the forces shown requires that the reaction at the edge of the plate be:

\[
R_y(J,K) = Q_x(J,n) - Q_x(J,n+1) + Q_y(m,K) \tag{41}
\]

The shears on the right-hand side of this equation may be expressed in terms of the moments as follows:

\[
Q_x(J,n) = \frac{M_x(J,K) - M_x(J,K-1)}{L_x} - \frac{M_{yx}(m,n)}{L_x}
\]

\[
Q_x(J,n+1) = \frac{M_x(J,K+1) - M_x(J,K)}{L_x} - \frac{M_{yx}(m,n+1)}{L_x} \tag{42}
\]

\[
Q_y(m,K) = \frac{M_y(J,K) - M_y(J-1,K)}{L_y} + \frac{M_{xy}(m,n+1) - M_{xy}(m,n)}{L_y}
\]

Substitution of Eqs. (42) into Eq. (41) yields:

\[
R_y(J,K) = \left[ \frac{M_y(J,K) - M_y(J-1,K)}{L_y} + \frac{M_{xy}(m,n+1) - M_{xy}(m,n)}{L_y} \right]
\]

\[
+ \left[ \frac{-M_x(J,K+1) + 2M_x(J,K) - M_x(J,K-1)}{L_x} \right. + \frac{M_{xy}(m,n+1) - M_{xy}(m,n)}{L_x} \right] \tag{43}
\]
Equation (43) corresponds to Kirchhoff's expression for the edge reaction. The second bracket is equal to the rate of change of the twisting moment along the edge of the plate.\(^{(20)}\) Substitution of the expressions for the moments into Eq. (43) yields the reaction at the edge:

\[
R_y(J,K) = \frac{Eht^2}{(1-\nu^2)} \left\{ \frac{L_x}{4L_y} \left[ \frac{3W_{J,K-1} W_{J-1,K} + W_{J-2,K}}{L_y} - 2\alpha(J,K) \right] \right. \\
+ \frac{1}{2L_y L_x} \left[ -W_{J,K+1} + 2W_{J,K} - W_{J,K-1} + W_{J-1,K+1} - 2W_{J-1,K} + W_{J-1,K-1} \right] \\
+ \frac{L_y}{8L_x} \left[ -W_{J,K+2} - W_{J,K+1} + 6W_{J,K} - 4W_{J,K-1} + W_{J,K-2} \right] \\
+ \frac{V}{L_x} \left[ \alpha(J,K+1) - 2\alpha(J,K) + \alpha(J,K+1) \right] \right\} 
\]

(44)

2.8. Description of Method of Solution

The method of solution used belongs to the class variously known as "stiffness," "displacement" or "equilibrium" methods. The structure is divided into segments corresponding to the individual plates; the boundaries of the segments therefore correspond to the longitudinal joints of the folded plate structure.

The displacements and rotations at the longitudinal joints of the structure are treated as unknowns. Each plate is first analyzed assuming that the longitudinal joints are completely fixed. Next, the displacements at the joints are each given unit values in turn and the resulting boundary forces computed. Thus, each boundary force may be expressed as a "fixed-edge" value plus a linear combination of the
unknown displacements at the boundaries. The actual boundary displace-
ments are obtained from the equations of equilibrium of forces and
moments at the joints. Once the boundary displacements are found, the
plate may be analyzed again under the action of external loads and the
known boundary displacements.

The forces at the boundaries of each segment are expressed
in matrix form as follows:

\[ \{F\} = [K] \{\Delta\} + \{F_b\} \quad (45) \]

Where \( F \) is the column matrix of the boundary forces,

\[ F = \begin{bmatrix} N_x^b \\ N_y^b \\ R_y^b \\ M_y^b \end{bmatrix} \quad (46) \]

and \( \Delta \) is the column matrix of boundary displacements,

\[ \Delta = \begin{bmatrix} U^b \\ V^b \\ W^b \\ \phi^b \end{bmatrix} \quad (47) \]

In Eqs. (46) and (47) above the superscript \( b \) indicates that the forces
and displacements are at the boundary. The number of elements in \( F \)
and \( \Delta \) is equal to the number of boundary-degrees-of-freedom of the
plate. Along the boundaries of a segment there are four forces and four degrees of freedom for each panel length, Fig. 14. Thus, for a plate with eight panel lengths in the longitudinal direction there are sixty-four boundary forces and sixty-four degrees of freedom (thirty-two on each boundary).

The square matrix \( K \) in Eq. (45) is a stiffness matrix. The column matrix \( F_f \) represents the values of the boundary forces when the boundaries are completely restrained. Thus, Eq. (45) is similar to the slope-deflection equations used in the theory of plane frames.

In order to obtain the \( F_f \) column matrix it is necessary to generate a matrix \( A \) that relates the interior displacements \( \delta_0 \) of the fixed plate with the external loads \( B_0 \) acting on the plate according to the relation:

\[
[A]\{\delta_0\} = \{B_0\} \tag{48}
\]

Equation (48) is in effect the matrix representation of the system of equilibrium equations derived in Section 2.6. The set of linear simultaneous algebraic equations given by Eq. (48) may be solved by means of the Gauss elimination procedure. Once the displacements \( \delta_0 \) of the fixed plate are known, the boundary forces \( F_f \) are easily computed by means of the equations presented in Section 2.7.

In order to obtain the matrix \( K \) it is necessary to give, in turn, a unit value to each boundary displacement keeping all others equal to zero. A unit displacement at a boundary affects the equilibrium equations of neighboring points only. For instance, referring to Table 1, a unit \( U \) displacement at point B will affect the equilibrium equations
for points 5, 6, and 8 only. The table shows the right-hand sides of the equilibrium equations for unit displacements at the boundaries for the plate alone. Table 2 shows the additional terms when there is a transverse stiffener present. To illustrate how the tables are formed the term corresponding to point (m,K) in Fig. 15 will be derived for the case when no stiffener is present. The only force entering the equilibrium equation of this point and involving the boundary displacement $V_{J,K}$ is $N_y(J,K)$, which is given by Eq. (34). The term involving $V_{J,K}$ is:

$$\frac{2EhL_x}{(1-\nu^2)L_y} \cdot V_{J,K}$$

The term is taken to the right-hand side and multiplied by $\frac{(1-\nu^2)}{Eh}$, and $V_{J,K}$ is given a unit value. The net result is:

$$\frac{2L_x}{L_y}$$

This term appears in Table 1 in the fifth row under $V_A$. Other terms in the tables are obtained in a similar manner.

When a boundary point is given a unit displacement the resulting set of linear simultaneous equations will have the form:

$$[A][\delta_1] = [B_1]$$

(49)

where $A$ is exactly the same as in Eq. (48) and $\delta_1$ represents the interior displacements of the plate when the $i$th boundary point is given a unit displacement.
Solution of the set of linear simultaneous equations given by Eq. (49) yields the column matrix \( \mathbf{\delta}_i \) from which the boundary forces that make up the \( i \)th row of stiffness matrix \( K \) are obtained.

Thus, the following sets of linear simultaneous equations have to be solved:

\[
\begin{align*}
[A][\mathbf{\delta}_0] &= [\mathbf{B}_o] \\
[A][\mathbf{\delta}_1] &= [\mathbf{B}_1] \quad i = 1, \ldots, l
\end{align*}
\]

(50)

where \( l \) is the number of boundary degrees of freedom of the plate. In the computer program developed advantage is taken of the fact that the \((l + 1)\) systems of equations represented by Eq. (50) have the same matrix of coefficients \( A \). In the Gauss elimination procedure the elimination part is performed only once on \( A \) keeping \((l + 1)\) right-hand vectors. The back substitution operation, however, must be done for each one of the \( B \) vectors, or a total of \((l + 1)\) times.

From the solution of Eqs. (50) the matrix \( M \) is obtained:

\[
M = [\mathbf{\delta}_1 \mathbf{\delta}_2 \ldots \mathbf{\delta}_i \ldots \mathbf{\delta}_l]
\]

(51)

where the \( \mathbf{\delta}'s \) are column vectors whose elements are the displacements of the interior points of the plate. Matrix \( M \) is an influence matrix. Its element \( m_{i,K} \) located in the \( i \)th row and \( k \)th column represents the displacement of the \( i \)th interior point due to a unit displacement at the \( k \)th boundary point.

Once matrix \( M \) is known, a set of boundary forces for each column of \( M \) is computed. These boundary forces make up the columns of the stiffness matrix \( K \).
In order to establish equilibrium at the edges of the plate the forces must be given in terms of displacements in the global system of coordinates. Referring to Fig. 18, the transformation for displacements from the local to the global system is given by the following relations:

\[ \begin{align*}
\lambda &= u \\
\eta &= V \cos \varphi + W \sin \varphi \\
\xi &= -V \sin \varphi + W \cos \varphi \\
\psi &= \alpha
\end{align*} \]  

(52)

where \( \lambda, \eta, \xi \) and \( \psi \) are the displacements along the \( X, Y, \) and \( Z \) axes and the rotation about the \( X \) axis respectively and \( \varphi \) is the angle of inclination of the particular plate relative to the horizontal. With the transformations given by Eq. (52) the boundary forces may be expressed now in terms of the displacements in the global system.

Equilibrium of forces at the joints of the folded plate may be established at this stage. Using the superscripts \( \text{L} \) and \( \text{R} \) to denote forces at the left and right of each joint, the equilibrium equations are as follows (Fig. 19):

\[ \sum F_x = 0: \quad N_x^L - N_x^R = 0 \]

\[ \sum F_y = 0: \quad N_y^R \cdot \cos \varphi^R - N_y^L \cdot \cos \varphi^L + \left( R_y^R + Z^R \right) \cdot \sin \varphi^R \\
- \left( R_y^L - Z^L \right) \cdot \sin \varphi^L = 0 \]
\[
\sum F_Z = 0: \quad -N^R_y \cdot \sin \phi^R + N^L_y \cdot \sin \phi^L + (R^R + Z^R) \cdot \cos \phi^R \\
- (R^L_y - Z^L) \cdot \cos \phi^L = 0
\]

\[
\sum M_x = 0: \quad M^L_y - M^R_y = 0 \tag{53}
\]

where $Z^L$ and $Z^R$ are external forces applied at the joints of the structure. External couples could also have been considered in the moment equilibrium equations. Wherever there is a transverse stiffener, the forces in Eq. (53) are to be taken as the sum of the forces in the plate and in the stiffener.

Solution of the system given by Eq. (53) yields the final boundary displacements $(\lambda, \eta, \xi, \psi)$ in the global system of coordinates. Fixity against rotation or translation of points along the fold lines may be obtained by elimination of rows and columns in Eqs. (53).

The final boundary displacements in the local system may be obtained from the displacements in the global system.

If the column vector of final boundary displacements in the local system is called $\Delta_{rb}$, then the final interior displacements $\delta$ in each plate are given by

\[
\{\delta\} = \{\delta_o\} + [M]^T [\Delta_{rb}] \tag{54}
\]

where $[M]^T$ is the transpose of $M$ (Eq. 51).

As the final step in the analysis, the internal forces and moments are computed from the known displacements of the plate.
NUMERICAL RESULTS

3.1. General Remarks

The method of analysis developed in this study is used for the solution of three example problems:

1. Simply-Supported Trough-Shaped Folded Plate
2. Two-Span Continuous V Folded Plate
3. Two-Span Continuous Saw-Tooth Folded Plate.

The numerical results obtained in examples 1 and 2 are compared with available solutions. All the structures analyzed are loaded symmetrically and have two lines of symmetry; therefore only one quarter of the structure need be analyzed. For the solution of the problems shown in this study a FORTRAN program for an IBM 7094 digital computer was written.

3.2. Simply-Supported Trough-Shaped Folded Plate

A three plate, trough-shaped, simply supported folded plate with uniformly distributed load is considered. The supports are assumed to be the usual diaphragm support, rigid in their own plane but flexible normal to that plane. The loading, material properties, cross-section, and dimensions of the structure are given in Table 3. Two grid sizes* were used: 8 x 5 and 10 x 8. The values of the elasticity solutions

* The two numbers denote the panels in the longitudinal and transverse directions respectively.
calculated to the 7th harmonic are used as a basis for comparison. The midspan values of longitudinal stresses at the free edge and at the fold line and the transverse moment at the fold line are given in Table 3. The transverse distribution of longitudinal stresses at midspan is shown in Fig. 20. Very good agreement is obtained between the elasticity solution of Goldberg and Leve and the 8 x 5 grid solution from the model. The 10 x 8 solution is not shown since it coincides with the solid line to the scale of the figure. The transverse variations of M and W displacements, both taken at midspan, are given in Figs. 21 and 22. Again good agreement is found with the elasticity solution. Within plotting accuracy, the same curve for W displacements was obtained for the elasticity method, the 8 x 5, and the 10 x 8 grid. Figure 23 shows the shear M at the simple support. The discontinuous line represents extrapolated values since the model defines the shears at points one half panel away from the support.

It is interesting to note that the longitudinal stresses obtained by the beam method come within 5% of the values given by the elasticity solution. The midspan moment M at the fold lines predicted by either of the engineering methods is 33% larger than that given by both the elasticity solution and the model. This discrepancy is probably due to the fact that the engineering analyses do not take into account the contribution of the twisting moments in resisting external loads.

3.3. Two-Span Continuous V Folded Plate

The properties and loading of the inverted V folded plate used in this example are shown in Table 4. As outlined in this table,
three sizes of stiffener supports are considered at the center column line of the structure. The three cases are designated 2-DB (with Deep Beam), 2-B (with Beam), and 2-NB (with No Beam). The two end supports are assumed to be normal simple support diaphragms.

The $N_x$ values at the free edge and at the fold line for structure 2-DB are shown in Figs. 24 and 25. The discontinuous lines indicate the solution obtained with the model using a 12 x 5 grid. The solid lines, presented for comparison, were obtained by Goldberg, Gutzwiller and Lee $^8$ by extrapolation from 12 x 5 and 24 x 10 finite difference solutions. At the center support their 12 x 5 solution deviates farther from the extrapolated values than does the 12 x 5 values obtained with the model. Near the center of each longitudinal span there is a slight reversal in the accuracy trend between the Goldberg solution and the model.

The values of $M_x$ along the free edge and of $M_y$ at the center of each longitudinal span for structure 2-DB are given in Figs. 26 and 27. The values obtained with the model are practically identical to the values given by Goldberg et al. Figure 28 shows the transverse distribution of $N_x$ values at several sections of folded plate 2-DB. As shown in the figure, the distribution of stresses is strongly non-linear. Moreover, this nonlinearity becomes more pronounced as the beam size is decreased, as shown in Fig. 29a. Thus, the beam method could in no way predict these stresses. Other methods of solution, such as those suggested for short span folded plate structures, $^{16}$ are not applicable because of the support conditions. Figures 29b and 29c show the in-plane shears at the simple support and at the center. The total in-plane
shear, as represented by the integrated or summed value at the simple end support increases with decreasing beam size. At the center, although the peak value increases with decreasing beam size, the total area under the curve decreases. The transverse shear reactions (not shown) tend to do exactly the opposite. The net result is that the vertical reaction at the center increases with increasing beam size, as shown in Table 5. The relative distribution of vertical loads transferred to the various supports approaches that of a continuous beam as the stiffener size is increased.

3.4. Two-Span Continuous Saw-Tooth Folded Plate

A saw-tooth folded plate continuous over central column supports is considered in this example. Five variations of the same problem are studied. The loading, material properties, and geometry of the structure are shown in Table 6. In addition, an interior plate with the same loading and dimensions of plate DE in Table 6 is considered. The structures are designated 3-NB (No Beam), 3-B (with Beam), 3-BO (with Beam and Poisson's ratio equal to zero), 3-DB (with Deep Beam), 3-BTR (with Beam and Tie-Rod), and 3-CP (Central Plate). Two sizes of central stiffeners and two values of Poisson's ratio were used. The two end supports are assumed to be normal simple support diaphragms. All structures are supported at the center on points B and D. These points are completely restrained against vertical and horizontal displacements for all structures except 3-BTR. Structure 3-BTR rests on roller supports at points B and D. The horizontal movement of these
rollers is assumed to be partially restrained by a steel tie-rod with an area of 0.44 square inches.

The changes of the significant forces in the folded plate structure for the different versions of the problem are discussed in the following sections.

3.4.1. Comparison of Transverse Moments $M_y$

Figures 30 and 31 show the longitudinal variation of the transverse moment $M_y$ in the plate along the first valley and the central ridge. A substantial reduction in the magnitude of the moment occurs in the vicinity of the stiffener. The magnitude and longitudinal variation of $M_y$ is practically unaffected, however, by a change in stiffener depth from 21 to 50 inches (an increase in stiffness of 12.5 times). Similar results were obtained for other longitudinal sections. The design moment for the central ridge is affected very little by the introduction of the stiffener. The major portion of the slab along the ridge must still be designed for a moment of about -0.35 k-ft/ft. The maximum moment along the first valley is decreased by about 25% by the introduction of the stiffener.

Figure 32 shows the moment taken by the stiffener. A change in Poisson's ratio from 0.18 to 0.0 for the material properties assumed for the plate elements causes almost no change in stiffener moment. Notice that a small moment is present at the free edge of the beam for the case $\nu = 0.18$. This moment is balanced by the moment in the central slab-strip and will shrink to zero as the mesh size is decreased. The
reason for the existence of this small moment is explained in the following paragraphs.

At a free edge of the slab the moment boundary condition for the slab-strip of width \( L_x \) is:

\[
\frac{Eh^3 L_x}{12(1-\nu^2)} \left[ \chi_y + \nu \chi_x \right] = 0
\]  

(55)

For the free end of a beam the condition is:

\[-EI \chi_y = 0 \]  

(56)

For non-zero values of \( \chi_x \) these conditions can be satisfied simultaneously only when \( \nu = 0 \). The condition imposed at the free end of the beam for the combined beam and slab-strip is:

\[-\chi_y \left[ \frac{Eh^3 L_x}{12(1-\nu^2)} + EI \right] - \frac{Eh^3 L_x}{12(1-\nu^2)} \cdot \nu \chi_x = 0 \]  

(57)

This condition satisfies the requirement of statics that the net moment be equal to zero and will converge to the condition given by Eq. (56) when the mesh length \( L_x \) is decreased. For the case \( \nu = 0 \) the moment at the free edge of the beam is zero, independent of the mesh size.

When the depth of the stiffener is increased from 21 to 50 inches the moment taken by the stiffener changes by a very small amount.

3.4.2. Comparison of In-Plane Shears \( N_{xy} \)

Figure 33 shows the values of the in-plane shear forces for structure 3-B0 at the end support and at the center column line. These
values are extrapolations of the shears from their points of definition at the center of each mesh panel to the edge line of support. The curves are essentially a series of parabolas. The curves for the other structures are similar. From the data presented in the figure, the load on the span is split between center and end supports in the ratio of 37% to 63% or in about the same ratio as would be obtained by a beam approach.

3.4.3. Comparison of Forces \( N_y \)

The axial forces in the stiffeners for structures 3-BO and 3-BTR are shown in Fig. 34. The difference between the ordinates of the solid line immediately to the left and right of points B and D is due to the presence of horizontal reactions at these points. The introduction of a roller support at points B and D eliminates any horizontal reaction, therefore bringing the values of the axial forces closer together. Small tensile forces are present at the ridge lines but the major portion of the stiffener is subjected to compression. As shown in Fig. 35 the value of \( N_y \) in the slab along the first valley stays at a constant value of about 0.5k/ft for most of the span and then dips to a large compressive value near the center support of the structure. The same variation occurs for fold line D. This transition of \( N_y \) from tension to compression is independent of the presence of stiffeners for the problem studied. The tensile force in the tie-rod between D and D', on the other side of the line of symmetry, was 1043 lbs. The tie-rod between points B and D carried a compressive force of 126 pounds. Provisions may be built into the computer program to disregard compressive
forces arising in tie rods. The resulting compressive force in the present case was deemed small enough not to warrant any modifications of results since the tie rod would not buckle at such a low stress. It must be pointed out that at the free end of a stiffener the same situation occurs for \( \mathbf{N}_{y} \) forces that occurs for transverse moments \( \mathbf{M}_{y} \). The condition established is that the sum of the axial force in the stiffener and the in-plane force in the central slab-strip must be zero at the free edge. For the case \( \nu = 0 \) both the axial force in the beam and \( \mathbf{N}_{y} \) in the slab are zero.

3.4.4. Comparison of Longitudinal Stresses \( \mathbf{N}_{x} \)

As shown in Fig. 36, the distribution of longitudinal stresses at the center support and at midspan is almost linear. Table 4 shows that the main effect of the stiffener is to bring the values of the tensions and compressions at the edges of the plates closer together. The transverse variations of longitudinal stresses in a central plate are plotted in Fig. 37. The stress at the line of symmetry remains unchanged whereas that at the support increases from \(-12.1\) to \(-14.2\) k/ft. The total compressive force, however, remains practically unchanged. Thus, there exists a stress concentration at the support that can not be predicted with the coarser grids. This concentration of stress is due to the fact that the support is assumed to provide a reaction concentrated at a point. Similar concentrations of stress are obtained if the supports are assumed to provide a reaction distributed uniformly over a short length, as in Ref. (15). A more realistic picture would
be obtained if a very fine grid was used around the column perimeter since actual columns are neither line nor point supports.
4.1. Conclusions

A stiffness method of analysis for folded plate structures using a discrete system model has been developed. By giving unit values to the boundary displacements of the fixed plate, stiffness matrices for the plates are generated. Equations of equilibrium yield the final boundary displacements at the fold lines. The method may be used for both simply-supported and continuous structures. Excellent agreement was found with other elasticity solutions, even when relatively coarse grids were used. Rapid variations of forces and moments were successfully predicted. Although roller-type diaphragms were used at the end supports, the method is not limited to this type of end condition. Rotational and translational fixity of points along the fold lines may be obtained by elimination of columns and rows in the equations of equilibrium at the longitudinal joints. Structures with tie-rods or with columns placed at points other than the center line can be successfully treated.

For the continuous cases studied, the introduction of transverse stiffeners affects the relative distribution of applied load between the supports. This effect is more pronounced for plates of low span-to-width ratio. The transverse distributions of longitudinal stresses is affected by the presence of transverse stiffeners. Transverse moments $M_y$ in the plates are reduced considerably in the immediate vicinity of the transverse stiffeners but the longitudinal distribution
of $M_y$ is relatively insensitive to large changes in stiffener size. The zone of reduced $M_y$ on both sides of the stiffener is wider for exterior plates than for interior ones.

4.2. **Recommendations for Future Study**

The analysis of folded plate structures supported at the ends by flexible stiffeners with torsional resistance should be undertaken in future studies. The study of folded plates with rectangular holes and of box girders should not present great difficulties. Simply-supported and continuous folded plates having more than two plates intersecting at a joint can also be studied.

The method of solution should be combined with a model similar to the one used by Mohraz and Schnobrich\(^{11}\) to study the elasto-plastic behavior of folded plates. Laboratory tests to failure should be conducted on small scale folded plates in order to verify experimentally the ultimate capacities predicted by the analytical models.

Finally, by using a model with non-orthogonal coordinates, the method may be extended to folded triangular plates and to umbrella and pyramidal roofs.
LIST OF REFERENCES


<table>
<thead>
<tr>
<th>R.H.S. of Point</th>
<th>Displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$U_B$</td>
</tr>
<tr>
<td>1</td>
<td>--</td>
</tr>
<tr>
<td>2</td>
<td>--</td>
</tr>
<tr>
<td>3</td>
<td>--</td>
</tr>
<tr>
<td>4</td>
<td>--</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{(1+v)}{2}$</td>
</tr>
<tr>
<td>6</td>
<td>$-\frac{(1-v) \cdot L_x}{2 \cdot L_y}$</td>
</tr>
<tr>
<td>7</td>
<td>--</td>
</tr>
<tr>
<td>8</td>
<td>$-\frac{(1+v)}{2}$</td>
</tr>
</tbody>
</table>

**TABLE 1.** RIGHT-HAND SIDE OF EQUILIBRIUM EQUATIONS FOR UNIT DISPLACEMENTS AT THE BOUNDARIES. NO STIFFENER PRESENT.
### Table 2: Terms to Be Added to Right-Hand Side of Equilibrium Equations for Unit Boundary Displacements When a Rectangular Stiffener Is Present.

<table>
<thead>
<tr>
<th>R.H.S. of Point</th>
<th>Displacement Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$V_A$</td>
</tr>
<tr>
<td>1</td>
<td>--</td>
</tr>
<tr>
<td>2</td>
<td>--</td>
</tr>
<tr>
<td>3</td>
<td>--</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{2bde(1-v^2)}{hL^2}$</td>
</tr>
<tr>
<td>5</td>
<td>$-\frac{2bd(1-v^2)}{hL^2}$</td>
</tr>
<tr>
<td>6</td>
<td>--</td>
</tr>
<tr>
<td>7</td>
<td>--</td>
</tr>
<tr>
<td>8</td>
<td>--</td>
</tr>
</tbody>
</table>
Forces at Midspan

<table>
<thead>
<tr>
<th>Forces at Midspan</th>
<th>8 x 5 Grid</th>
<th>10 x 8 Grid</th>
<th>Elasticity***</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_x$ at A (K/ft)*</td>
<td>20.55</td>
<td>21.08</td>
<td>21.52</td>
</tr>
<tr>
<td>$N_x$ at B (K/ft)**</td>
<td>-10.28</td>
<td>-10.54</td>
<td>-10.70</td>
</tr>
<tr>
<td>$M_y$ at B (K-ft/ft)***</td>
<td>-4.162</td>
<td>-4.184</td>
<td>-4.236</td>
</tr>
</tbody>
</table>

* Beam method: 20.48 K/ft.
** Beam method: -10.24 K/ft.
*** Engineering method: -5.625 K-ft/ft.
**** Highest harmonic taken is 7.

TABLE 3. SOLUTION OF SIMPLY SUPPORTED FOLDED PLATE. PROBLEM 1.
Dead load: 1.414 lb/ft²

\[ E = 10.6 \times 10^6 \text{ psi} \]
\[ v = 0.3 \]
\[ c = 0.0" \]

<table>
<thead>
<tr>
<th>Structure Designation</th>
<th>Beam Size</th>
<th>Conditions of Support at Center of Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-DB</td>
<td>8&quot; x 24&quot;</td>
<td>Supported at points A, B, and C</td>
</tr>
<tr>
<td>2-B</td>
<td>8&quot; x 3.6&quot;</td>
<td>Supported at A and C</td>
</tr>
<tr>
<td>2-NB</td>
<td>No Beam</td>
<td>Supported at A and C</td>
</tr>
</tbody>
</table>

**Table 4. Properties and Loading of Two-Span Continuous V Folded Plate. Problem 2.**
<table>
<thead>
<tr>
<th>Percentage of Total Load</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>At End Supports</td>
<td>At Center Supports</td>
</tr>
<tr>
<td>48.3</td>
<td>51.7</td>
</tr>
<tr>
<td>42.3</td>
<td>57.7</td>
</tr>
<tr>
<td>40.9</td>
<td>59.1</td>
</tr>
<tr>
<td>37.5</td>
<td>62.5</td>
</tr>
</tbody>
</table>

**TABLE 5. RELATIVE DISTRIBUTION OF TOTAL VERTICAL REACTION BETWEEN END AND CENTER SUPPORTS. PROBLEM 2.**
<table>
<thead>
<tr>
<th>Problem Designation</th>
<th>Grid Size</th>
<th>Beam Size</th>
<th>( v )</th>
<th>Tie-Rod</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-NB</td>
<td>12x6</td>
<td>No Beam</td>
<td>0.18</td>
<td>No</td>
<td>Vertical and horizontal displacements at points B and D are restrained.</td>
</tr>
<tr>
<td>3-B</td>
<td>12x6</td>
<td>8&quot;x21&quot;</td>
<td>0.18</td>
<td>No</td>
<td>&quot;</td>
</tr>
<tr>
<td>3-BO</td>
<td>12x6</td>
<td>8&quot;x21&quot;</td>
<td>0.00</td>
<td>No</td>
<td>&quot;</td>
</tr>
<tr>
<td>3-DB</td>
<td>12x6</td>
<td>8&quot;x50&quot;</td>
<td>0.18</td>
<td>No</td>
<td>&quot;</td>
</tr>
<tr>
<td>3-BTR</td>
<td>12x6</td>
<td>8&quot;x21&quot;</td>
<td>0.00</td>
<td>Yes</td>
<td>Roller supports at points B and D permit horizontal movements only.</td>
</tr>
<tr>
<td>3-CP</td>
<td>16x6</td>
<td>No Beam</td>
<td>0.18</td>
<td>No</td>
<td>Rotations and horizontal displacements restrained along fold lines D and E. Vertical displacements unrestrained everywhere except at point D.</td>
</tr>
</tbody>
</table>

**TABLE 6. SOLUTION OF TWO-SPAN CONTINUOUS SAW-TOOTH FOLDED PLATE. PROBLEM 3.**
TABLE 7. VALUES OF $N_x$ OVER CENTRAL SUPPORTS. PROBLEM 3.

<table>
<thead>
<tr>
<th>Structure</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-NB</td>
<td>10867</td>
<td>-13265</td>
<td>10697</td>
<td>-12142</td>
<td>10557</td>
</tr>
<tr>
<td>3-B</td>
<td>10602</td>
<td>-12376</td>
<td>11334</td>
<td>-11918</td>
<td>11024</td>
</tr>
<tr>
<td>3-DB</td>
<td>10709</td>
<td>-12140</td>
<td>11569</td>
<td>-11743</td>
<td>11325</td>
</tr>
</tbody>
</table>

$N_x$ in lb/ft at location.
FIG. 1 SOME COMMON FOLDED PLATE CROSS-SECTIONS.
FIG. 2. GLOBAL AND LOCAL SYSTEMS OF COORDINATES.

FIG. 3. DISCRETE SYSTEM MODEL.
FIG. 4. CONTRIBUTING AREAS AND LOCATION OF DISPLACEMENT POINTS.

Contribution Area for Perpendicular Loads

Contribution Area for Transverse Loads

Contribution Area for Longitudinal Loads
FIG. 5. SIGN CONVENTION FOR DISPLACEMENTS.

FIG. 6. POSITIVE BENDING MOMENTS AND NORMAL FORCES.

FIG. 7. POSITIVE TWISTING MOMENTS AND SHEAR FORCES.
FIG. 8. IDENTIFICATION OF INTERIOR POINTS ON PLATE.

FIG. 9. IN-PLANE FORCES ACTING ON U BAR \((J, n)\).
FIG. 10. IN-PLANE FORCES ACTING ON V BAR \((m, K)\)

FIG. 11. TRANSVERSE FORCES ACTING ON W POINT \((J, K)\).

FIG. 12. MOMENTS AND TRANSVERSE SHEARS ACTING ON U BAR \((J, n)\).
FIG. 13. DISPLACEMENTS AND ROTATIONS OF BAR \((J, n)\)
FIG. 14. FORCES AND DISPLACEMENTS AT BOUNDARY.
FIG. 15. IDENTIFICATION OF POINTS NEAR EDGE OF PLATE.

FIG. 16. IN-PLANE FORCES ACTING UPON LONGITUDINAL BAR (J, n) AT EDGE.
FIG. 17. TRANSVERSE FORCES ON W POINT AT EDGE.
FIG. 18. LOCAL AND GLOBAL DISPLACEMENTS AT EDGE OF PLATE.

FIG. 19. FORCES ACTING UPON LONGITUDINAL JOINTS OF PLATE.
FIG. 20. Nx Values at Midspan. Problem 1.

Elasticity Solution
8 x 5 Grid
FIG. 21. Variation of $M_y$ at Midspan. Problem 1.

FIG. 22. Transverse Displacements at Midspan. Problem 1.
Elasticity Solution

8 x 5 Grid

FIG. 23. $N_{xy}$ AT SIMPLE SUPPORT. PROBLEM 1.
FIG. 24. \( N_x \) AT FREE EDGE. PROBLEM 2-DB.

FIG. 25. \( N_x \) AT FOLD. PROBLEM 2-DB.
FIG. 26. $M_x$ ALONG FREE EDGE. PROBLEM 2-DB.

FIG. 27. $M_y$ AT CENTER OF EACH SPAN. PROBLEM 2-DB.
FIG. 28. VARIATION OF $N_x$ ACROSS SEVERAL SECTIONS OF FOLDED PLATE. PROBLEM 2-DB
FIG. 29. $N_x$ AND $N_{xy}$ VALUES. PROBLEM 2.
FIG. 30. VARIATION OF $M_y$ ALONG FIRST VALLEY.
PROBLEM 3.

FIG. 31. VARIATION OF $M_y$ ALONG CENTRAL RIDGE.
PROBLEM 3.
FIG. 32. MOMENT IN TRANSVERSE STIFFENER. PROBLEM 3.
FIG. 33. TRANSVERSE VARIATION OF $N_{xy}$. PROBLEM 3-BO.
FIG. 34. AXIAL FORCES IN STIFFENER. PROBLEM 3.

FIG. 35. VARIATION OF $N_y$ ALONG VALLEY LINE B.
FIG. 36. TRANSVERSE VARIATION OF $N_x$. PROBLEM 3-NB.
FIG. 37. DISTRIBUTION OF LONGITUDINAL STRESSES IN CENTRAL PLATE, PROBLEM 3-CP.
APPENDIX A. OPERATORS FOR INTERIOR POINTS

For convenience the operators for internal forces and the equilibrium equation patterns are summarized in this appendix.

Operators for forces:

- □ W point
- × U point
- △ V point

\[
N_x = \frac{Eh}{(1-v^2)}
\]

\[
N_y = \frac{Eh}{(1-v^2)}
\]
\[ N_{yx} = \frac{Eh}{2(1+v)} \]

\[ N_{xy} = \frac{Eh}{2(1+v)} \]

\[ M_x = \frac{Eh^3}{12(1-v^2)} \]

\[ M_y = \frac{Eh^3}{12(1-v^2)} \]
Equilibrium equations:

\[ M_{yx} = \frac{Eh^3}{12(1-v^2)} \]

\[ M_{xy} = \frac{Eh^3}{12(1-v^2)} \]

\[ W + \frac{(1-v^2)}{Eh} \bar{Z} = 0 \]
Where

\[ A_1 = \frac{L_y}{L_x} \]
\[ A_2 = v \]
\[ A_3 = \frac{L_x}{L_y} \]
\[ A_4 = 1.0 \]
\[ A_5 = \frac{1-v}{L_y} \]
\[ A_6 = \frac{(1-v)}{L_x} \]
\[ A_7 = \frac{L_y^2}{L_x} \]
\[ A_8 = \frac{v}{L_y} \]
\[ A_9 = 2(A_7 + A_8) \]
\[ A_{10} = \frac{L_x^2}{L_y} \]
\[ A_{11} = \frac{v}{L_x} \]
\[ A_{12} = 2(A_{10} + A_{11}) \]
\begin{align*}
    B_1 &= \frac{(1 + v)}{2} \\
    B_2 &= \frac{(1 - v)}{2} \cdot \frac{L_x}{L_y} \\
    B_3 &= 2(A_1 + B_2) \\
    B_4 &= \frac{(1 - v)}{2} \cdot \frac{L_y}{L_x} \\
    B_5 &= 2(A_3 + B_4) \\
    G_1 &= 6G_5 + 6G_6 + 4G_4 \\
    G_2 &= 2G_4 + 4G_5 \\
    G_3 &= 2G_4 + 4G_6 \\
    G_4 &= -\frac{t^2}{2L_x L_y} \\
    G_5 &= -\frac{t^2 L_x}{4L_y^3} \\
    G_6 &= -\frac{t^2 L_y}{4L_x^3}
\end{align*}
B.1. General Remarks

Force-displacement relations and the corresponding operator expressions associated with eccentric stiffeners are derived in this appendix. Because the stiffeners are assumed to be completely flexible in torsion, the operators will involve $V$ and $W$ displacements only. The $V$ and $W$ operators obtained for the stiffeners must be added to the appropriate operators for the plate itself in order to obtain the complete equilibrium expressions for the plate-stiffener structure.

Only stiffeners of rectangular cross-section are considered. The actual stiffener has a width $b$ and depth $d$. This stiffener has an eccentricity $e$ relative to the middle plane of the plate. The equivalent stiffener beam used in the model has dimensions width $b$, depth $\bar{d}$, and eccentricity $e'$ (Fig. B.1). For the equivalent and actual beams to have identical response it is necessary that $e'$ be equal to $e$ and $\bar{d}$ to be equal to $d/\sqrt{3}$.

In the following derivation the $g$ superscript placed to the left of the symbol denotes that the quantity to which it is appended applies to the stiffener. The right-hand superscript indicates whether the term applies to the top or the bottom of the equivalent beam, thus $S^T_\epsilon Y(J,K)$ means the strain in the top fiber of the equivalent beam at the point $(J,K)$. 
B.2. Strain-Displacement Relations

Referring to Fig. B.2, the top and bottom strains at location \((J,K)\) of the equivalent beam are:

\[
\varepsilon_y^{ST}(J,K) = \frac{S_{y}^{T}(m+1,K) - S_{y}^{T}(m,K)}{L_y}
\]

(B.1)

\[
\varepsilon_y^{SB}(J,K) = \frac{S_{y}^{B}(m+1,K) - S_{y}^{B}(m,K)}{L_y}
\]

The displacements at the top and bottom may be expressed as follows (Fig. B.3):

\[
S_{y}^{B}(m,K) = V(m,K) - \left(\frac{1}{2} - \frac{e'}{d}\right)(S_{y}^{T}(m,K) - S_{y}^{B}(m,K))
\]

(B.2)

\[
S_{y}^{T}(m,K) = V(m,K) + \left(\frac{1}{2} + \frac{e'}{d}\right)(S_{y}^{T}(m,K) - S_{y}^{B}(m,K))
\]

and since

\[
\frac{S_{y}^{T}(m,K) - S_{y}^{B}(m,K)}{\frac{d}{L_y}} = \frac{W(J,K) - W(J-1,K)}{L_y}
\]

(B.3)

then

\[
S_{y}^{B}(m,K) = V(m,K) - \left(\frac{1}{2} - \frac{e'}{d}\right) \cdot \frac{d}{L_y} \left(\frac{W(J,K) - W(J-1,K)}{L_y}\right)
\]

(B.4)

\[
S_{y}^{T}(m,K) = V(m,K) + \left(\frac{1}{2} + \frac{e'}{d}\right) \cdot \frac{d}{L_y} \left(\frac{W(J,K) - W(J-1,K)}{L_y}\right)
\]

Similarly,
Substitution of Eqs. B.4 and B.5 into Eqs. B.1 relates the strains at the nodes to the plate displacements by the following equations:

\[
S_{V}^{T}(m+1,K) = V(m+1,K) - \left(\frac{1}{2} - \frac{e'}{a}\right) \cdot \frac{d}{L_y} (W(J+1,K) - W(J,K))
\]

\[
S_{V}^{T}(m+1,K) = V(m+1,K) + \left(\frac{1}{2} + \frac{e'}{a}\right) \cdot \frac{d}{L_y} (W(J+1,K) - W(J,K))
\]

\(B.5\)

B.3. Axial Forces and \(V\) Operators

The force resultants at top and bottom of the equivalent beam are found from the strains by

\[
S_{N}^{T}(J,K) = \frac{Ebd}{2} \cdot S_{T}^{e}(J,K)
\]

\[
S_{N}^{B}(J,K) = \frac{Ebd}{2} \cdot S_{B}^{e}(J,K)
\]

therefore, the total axial force in the beam at location \((J,K)\) is:

\[
S_{N}(J,K) = \frac{Ebd}{2} \left[ S_{T}^{e}(J,K) + S_{B}^{e}(J,K) \right]
\]

\(B.8\)

Substitution of Eq. B.6 into Eq. B.8 gives:

\[
S_{N}(J,K) = \frac{Ebd}{L_y} \left[ (V_{m+1,K} - V_{m,K}) + \frac{e'}{L_y} (W_{J+1,K} - 2W_{J,K} + W_{J-1,K}) \right]
\]

\(B.9\)
Similarly,

\[ S_{N_y}(J-1,K) = \frac{Ebd}{L_y} \left[ (V_{m+1,K} - 2V_{m,K} + V_{m-1,K}) + \frac{e'}{L_y} (W_{J+1,K} - 3W_{J,K} + 3W_{J-1,K} - W_{J-2,K}) \right] \]  \hspace{1cm} (B.10)

With the contribution of the beam's own weight neglected, the equilibrium equation of axial forces acting on bar \((m,K)\) may be written as follows:

\[ S_{N_y}(J,K) - S_{N_y}(J-1,K) = 0 \]  \hspace{1cm} (B.11)

Upon substitution of Eqs. B.9 and B.10 into Eq. B.11, the equilibrium equation for the beam can be written as:

\[ \frac{Ebd}{L_y} \left[ (V_{m+1,K} - 2V_{m,K} + V_{m-1,K}) + \frac{e'}{L_y} (W_{J+1,K} - 3W_{J,K} + 3W_{J-1,K} - W_{J-2,K}) \right] = 0 \]  \hspace{1cm} (B.12)

Multiplying this expression by \(\frac{(1-v^2)}{Eh}\) and putting it in operator form, the following set of line-operators are obtained:

\[
\begin{bmatrix}
C_1 & -C_2 & & \\
& & C_1 & \\
& -C_3 & C_4 & -C_3 & C_4
\end{bmatrix}
\begin{bmatrix}
V \\
W
\end{bmatrix} = 0
\]  \hspace{1cm} (B.13)

where

\[ C_1 = \left( \frac{bd}{hL_y} \right) (1-v^2) \]
\[ C_2 = 2C_1 \]
\[ C_3 = \frac{e'bd}{gL_y^2} (1-v^2) \]
\[ C_4 = 3C_3 \]  \hspace{1cm} (B.14)
B.4. **Bending Moments and W Operators**

The moment at location \((J,K)\) in the stiffener, defined about the mid-depth of the slab, is as follows:

\[
S_y(J,K) = \left( \frac{bd}{2} \right) \left[ \frac{S_B}{\sigma_y(J,K)} \cdot \left( \frac{\alpha}{2} - \epsilon' \right) - \frac{S_T}{\sigma_y(J,K)} \cdot \left( \frac{\bar{\alpha}}{2} + \epsilon' \right) \right]
\]  

(B.15)

This moment may be expressed in terms of strains by substituting the appropriate strains for the stress terms within the brackets:

\[
S_y(J,K) = \frac{Ebd}{4} \left[ \frac{S_B}{\epsilon_y(J,K)} - \frac{S_T}{\epsilon_y(J,K)} \right] - \frac{Ebd}{2} \left[ \frac{S_B}{\epsilon_y(J,K)} + \frac{S_T}{\epsilon_y(J,K)} \right]
\]  

(B.16)

Finally with the strain-displacement relations of Eq. B.6, Eq. B.16 can be written as:

\[
S_y(J,K) = - \frac{Ebd}{2} \left[ \left( \frac{\alpha}{2} \right)^2 + (\epsilon')^2 \right] \left( W_{J+1,K} - W_{J,K} - 2W_{J,K} + W_{J-1,K} \right)
\]  

\[
- \frac{Ebd}{L_y} \left( V_{m+1,K} - V_{m,K} \right)
\]  

(B.17)

The transverse shear forces at points \((m,K)\) and \((m+1,K)\) are given by:

\[
S_{Sy}(m,K) = \frac{S_y(J,K) - S_y(J-1,K)}{L_y}
\]  

(B.18)

\[
S_{Sy}(m+1,K) = \frac{S_y(J+1,K) - S_y(J,K)}{L_y}
\]

Transverse equilibrium of forces at point \((J,K)\) requires that
Substitution of Eqs. B.18 into B.19 yields:

\[
- S_\varphi y(m,K) + S_\psi y(m+1,K) + S_\lambda y(j,K) = 0 \quad (B.19)
\]

The \( W \) equilibrium equation for the beam is obtained by further substitution of the expressions for moments into Eq. B.20:

\[
\frac{S_M y(J+1,K) - 2S_M y(J,K) + S_M y(J-1,K)}{L_y} + S_\lambda y(j,K) = 0 \quad (B.20)
\]

Multiplying this expression by \( \frac{(1-v^2)\varphi}{Eh} \) and converting it into operator form, the following set of line operators are obtained:

\[
\begin{bmatrix}
  C_3 & -C_4 & C_4 & -C_3 \\
  \Delta & \Delta & \Delta & \Delta
\end{bmatrix} V + \begin{bmatrix}
  C_5 & -4C_5 & 6C_5 & -4C_5 & C_5 \\
  \# & \# & \# & \# & \#
\end{bmatrix} W + \frac{(1-v^2)}{Eh} S_\lambda y(j,K) = 0 \quad (B.22)
\]

where \( C_3 \) and \( C_4 \) are given by Eq. B.14 and

\[
C_5 = -\frac{(1-v^2)bd}{hL_y^3} \left[ \frac{(d')^2 + (e')^2}{2} \right]. \quad (B.23)
\]
B.5. **Determination of Eccentricity and Depth of Equivalent Beam**

Referring to Fig. B.4, the strain at the level of fiber AB is:

\[ \varepsilon_{AB} = \frac{dV}{dy} \]  \hspace{1cm} (B.24)

because \( V \) displacements are defined at the level of this fiber. The strain at a point a distance \( \xi \) above fiber AB is related to the displacements by:

\[ \varepsilon_{\xi} = \frac{dV}{dy} + \frac{\xi}{R} \]  \hspace{1cm} (B.25)

Since \( \frac{1}{R} = \frac{d^2W}{dy^2} \),

\[ \varepsilon_{\xi} = \frac{dV}{dy} + \frac{d^2W}{dy^2} \cdot \xi \]  \hspace{1cm} (B.26)

The force in an elemental area \( b \cdot d\xi \) is:

\[ dN = E \cdot b \left( \frac{dV}{dy} \frac{d^2W}{dy^2} \cdot \xi \right) \cdot d\xi \]

The total force in the cross-section is given by

\[ N = \int \left( \frac{d}{2} + e \right) E \cdot b \left( \frac{dV}{dy} + \frac{d^2W}{dy^2} \cdot \xi \right) \cdot d\xi - \left( \frac{d}{2} - e \right) \]  \hspace{1cm} (B.27)

Integration of this expression yields
\[ N = Ebd \left( \frac{dV}{dy} + e \cdot \frac{d^2W}{dy^2} \right) \quad (B.28) \]

For Eq. B.9 to be the discrete representation of Eq. B.28 it is necessary that \( e' \) be equal to \( e \).

Similarly, the total moment about the mid-depth of the slab is:

\[ M = \int \frac{dV}{dy} + \frac{d^2W}{dy^2} \cdot (\xi) \cdot d\xi \quad (B.29) \]

Integration of this expression yields:

\[ M = -Ebde \cdot \frac{dV}{dy} - Ebd \cdot \frac{d^2W}{dy^2} \left[ \frac{d^2}{12} + e^2 \right] \quad (B.30) \]

For Eq. B.17 to be equivalent to Eq. B.30 it is necessary that \( \bar{d} = d/\sqrt{3} \).

B.6. Coupling of Linear Equations due to Eccentricity of Stiffeners.

The operators given by Eq. B.13 and Eq. B.22 must be added to the \( V \) and \( W \) operators obtained for the plate alone. Notice that in the general case the \( V \) operator involves \( W \) displacements and vice versa. This means that the simultaneous equations will be coupled. If the eccentricity \( e \) of the stiffener is zero Eq. B.14 will involve \( v \) displacements only and Eq. B.22 \( w \) displacements only. In this case the \( U \) and \( V \) equations will be coupled and the \( W \) equations will be independent.
FIG. B-1. DIMENSIONS OF ACTUAL AND EQUIVALENT STIFFENER.

FIG. B-2. IDENTIFICATION OF POINTS ALONG STIFFENER.
FIG. B-3. V DISPLACEMENTS AT POINT \((m, K)\).

FIG. B-4. ELEMENT OF BENT STIFFENER.
APPENDIX C. DESCRIPTION OF COMPUTER PROGRAM

A brief description of the FORTRAN program developed for this study is given in this appendix. The program uses the so-called "ping-pong technique" and consists of ten core-loads as follows:

1. Reads all information concerning the geometry, material properties and loading of the structure. Echo-prints all information.

2. Generates for each type of plate one force-displacement relation for each displacement point. Next, two column vectors for unit perpendicular and in-plane loads are generated. Finally, it generates one column vector for unit movements of each boundary degree-of-freedom.

3. Solves the equations generated in core-load 2 by Gauss elimination procedure.

4. Generates a force matrix for each type of plate. Forces are expressed as a "fixed" value plus a linear combination of the movements at the edges in the local system of coordinates.

5. Generates a force matrix for each plate. Forces are expressed in terms of edge-movements in the global system of coordinates.

6. Sets up equilibrium equations at the longitudinal edges of the folded plate.

7. Solves the equations generated in core-load 6.
8. Transforms the solution obtained in core-load 7 back to local coordinates.

9. Solves for final displacements in each plate.