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A SIMPLE APPROXIMATION FOR THE FUNDAMENTAL FREQUENCIES OF TWO-SPAN AND THREE-SPAN CONTINUOUS BEAMS

By

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and

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Technical Report

to

OFFICE OF NAVAL RESEARCH

Contract N6ori-071(06), Task Order VI

Project NR-064-183

UNIVERSITY OF ILLINOIS

URBANA, ILLINOIS

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SYNOPSIS

A rapid approximate method is presented for calculating the fundamental frequencies of flexural vibration of two-span beams and of particular arrangements of three-span beams which are continuous over non-deflecting supports and are elastically restrained against rotation at their end supports. The end restraints may be provided by actual coil springs or they may represent the effect of adjoining members, but in all cases the stiffnesses of these restraints are assumed to be positive. The mass per unit of length and the flexural rigidity of the beams may vary from one span to the next, but in any one span these quantities are considered constant. Two numerical examples are included to illustrate the application of the method.

SIGN CONVENTION

The following sign convention is used. Clockwise rotations are taken as positive. Bending moments at the ends of a span are considered positive when acting in a clockwise direction on the beam.

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BASIS AND GENERAL DESCRIPTION OF METHOD

When a continuous beam is in a state of free oscillations, each of the spans is elastically restrained against rotation at its ends by the rigidity of the contiguous spans and vibrates with the same frequency as that of the continuous system. Therefore, the problem of determining the natural frequencies of a continuous beam is basically the same as that of determining the corresponding frequencies of one of its spans only, with proper consideration of the actual restraints existing at its ends. The stiffnesses of these restraints depend on the properties of all the spans and on the order of the desired natural frequency.

Consider a continuous beam oscillating in its fundamental mode of free vibration. Let the supports be numbered consecutively starting with 1 at one end and terminating with z at the other end. Let θ_j be the rotation of the beam at an interior support j, and $M_{j,j-1}$ and $M_{j,j+1}$ be the internal bending moments at end j of the span between (j-1) and j, and that between j and (j+1), respectively. The relationship between these quantities may be expressed by the equations

$$M_{j,j-1} = -K_{j,j-1}\theta_j \quad \text{and} \quad M_{j,j+1} = -K_{j,j+1}\theta_j, \tag{1}$$

in which $K_{j,j-1}$ and $K_{j,j+1}$ are the stiffnesses of the internal restraints at support j. For a hinged condition K = 0, whereas for a fixed condition K = infinity. The negative signs in the foregoing expressions denote that for a positive restraint (positive value of K), the moment exerted by the restraint on the span acts in a direction opposite to the direction of rotation of the span. The end moments $M_{1,2}$ and $M_{z,z-1}$ are related to the end rotations θ_1 and θ_z by expressions similar to

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those given in Eq. (1). It will be assumed that the stiffnesses of the end restraints, $K_{1,2}$ and $K_{z,z-1}$, are positive and known.

For a natural mode of free vibration, no external moment acts on the system; therefore,

$$M_{j,j-1} + M_{j,j+1} = 0 \; ; \tag{2}$$

whence

$$K_{j,j-1} + K_{j,j+1} = 0 \; . \tag{3}$$

Expressed in words, Eq. (3) states that the sum of the stiffnesses at a joint is equal to zero. It should be pointed out that this relationship holds true not only for the fundamental mode, but for the higher natural modes as well.

The procedure to be presented consists of: (a) isolating from the continuous beam the span from j to (j+1) subjected to positive end restraints: (b) determining the stiffnesses of these restraints, $K_{j,j+1}$ and $K_{j+1,j}$; and (c) evaluating the fundamental frequency of the continuous beam from the approximation[1]

$$f \cong \left[1 + \frac{1}{2} \frac{\beta_{j,j+1}}{5 + \beta_{j,j+1}} \right] \left[1 + \frac{1}{2} \frac{\beta_{j+1,j}}{5 + \beta_{j+1,j}} \right] \frac{\pi}{2L_j^2} \sqrt{\frac{E_j I_j}{m_j}} \; , \tag{4}$$

in which L_j, $E_j I_j$, and m_j are, respectively, the length, the flexural rigidity of the cross section, and the mass per unit of length of the span between j and (j+1), and $\beta_{j,j+1}$ and $\beta_{j+1,j}$ are dimensionless quantities related to the stiffnesses of the end restraints by the equations

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1. "A Simple Approximation for the Natural Frequencies of Partly Restrained Bars," by N. M. Newmark and A. S. Veletsos, Journal of Applied Mechanics, Vol. 19, 1952, p. 563.

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$$\beta_{j,j+1} = K_{j,j+1} \frac{L_j}{E_j I_j} \quad \text{and} \quad \beta_{j+1,j} = K_{j+1,j} \frac{L_j}{E_j I_j} \; . \tag{5}$$

The frequency f is expressed in cycles per second. Eq. (4) is applicable to positive restraints only; it is for this reason that the isolated span must be positively restrained.

TWO-SPAN BEAMS

For a two-span beam, such as that shown in Fig. 1, it is only necessary to determine the stiffness of the restraint exerted by one span upon the other. Let f_1 and f_2 be the fundamental frequencies of spans (1,2) and (2,3), assuming that the beam is hinged at support 2 ($\beta_{2,1} = \beta_{2,3} = 0$). These frequencies may readily be evaluated from Eq. (4).

If the supports are numbered so that $f_2 \leq f_1$, the stiffness $K_{2,3}$ of the restraint exerted by the dynamically stiffer span (1,2) on the dynamically weaker span (2,3) will be greater than or equal to zero, and the fundamental frequency $\bar{f}$ of the continuous beam will lie between f_2 and f_1

From the results of numerical calculations based on exact solutions, the following empirical approximation has been found for $K_{2,3}$,

$$K_{2,3} \cong (K_{2,3})_s \left[1 - \left(\frac{f_2}{f_1} \right)^2 \right], \tag{6}$$

in which $(K_{2,3})_s$ is the stiffness of the restraint provided by span (1,2) under static conditions. It can readily be shown[2] that

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2. See for example "A Direct Method of Moment Distribution," by T. Y. Lin, Transactions A.S.C.E., Vol. 102, 1937, p. 565.

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$$(K_{2,3})_s = 4 \frac{E_1 I_1}{L_1} \left[1 - \frac{1}{4 + \beta_{1,2}} \right] \qquad (7)$$

With $K_{3,2}$ known and $K_{2,3}$ determined from Eq. (6), span (2,3) may now be treated as a bar subjected to positive end restraints, and its fundamental frequency, which is also the desired frequency of the continuous beam, may be evaluated from Eq. (4) In this case $j = 2$ and $j+1 = z = 3$.

The accuracy of Eq. (6) and that of the natural frequencies determined by the foregoing procedure have been checked for over three hundred representative beams having end restraints in the range between hinged and fixed conditions and spans with ratios of lengths, ratios of flexural rigidities of cross section, and ratios of masses per unit of length in the range between zero and one. The greatest error was found to occur in the case of beams which have a ratio of flexural rigidities of cross section from about 0.2 to 0.4 and have the extreme end of the dynamically stiffer span hinged or practically unrestrained and the end of the other span clamped or very nearly fixed

As an indication of the accuracy of Eq. (6) some representative results, including those for which the error is maximum, are given in Fig. 2 In this figure, the abscissas $K_{2,3} / (K_{2,3})_s$ were determined from the exact solution, whereas the quantities f_1 and f_2 for the ordinates were computed from Eq. (4). The vertical distances between the various points in this figure and the diagonal line represent the error involved in Eq. (6). These particular results are applicable to two-span beams simply supported at one end and elastically restrained at the other. It should be noted that for the limiting values of $f_2/f_1 = 0$ and

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$f_2/f_1 = 1.00$ Eq. (6) is exact.

Figure 2 indicates that, when the ratio of the flexural rigidities is the variable, the error in Eq. (6) is appreciable. However, because the natural frequencies of elastically restrained bars are not very sensitive to the stiffnesses of the end restraints, the error in the natural frequencies determined by using Eq. (6) is for all practical purposes insignificant. By comparing the exact natural frequencies of the more than three hundred beams referred to previously with those determined by the foregoing procedure, it was found that the maximum error in the frequencies determined by the approximate method is within $\pm$ 5 percent.

Example. - As an illustration, consider a beam having the following characteristics:

$$L_1 = 0.80L_2, \quad E_1I_1 = E_2I_2, \quad m_1 = 0.81m_2,$$

$$K_{1,2} = 1.0E_1I_1/L_1 \quad \text{and} \quad K_{3,2} = 5.0E_2I_2/L_2.$$

The frequencies f_2 and f_1, determined from Eq. (4), are

$$f_2 = 1.00 \times 1.25\, f_o = 1.25\, f_o,$$

$$f_1 = 1.083 \times 1.00 \times \frac{1}{0.576}\, f_o = 1.88\, f_o,$$

where

$$f_o = \frac{\pi}{2L_2^2}\sqrt{E_2I_2/m_2}.$$

The static stiffness of the restraint exerted by span (1,2) on span (2,3) is

$$(K_{2,3})_s = 0.80 \times 4.0\, E_1I_1/L_1 = 3.2\, E_1I_1/L_1,$$

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and the corresponding dynamic stiffness, computed from Eq. (6), is

$$K_{2,3} = 0.558 \times 3.2\, E_1 I_1 / L_1 = 1.79\, E_1 I_1 / L_1 .$$

Then,

$$\beta_{2,3} = 1.79 \times \frac{1}{0.80} = 2.24, \quad \beta_{3,2} = 5.0,$$

and

$$\bar{f} = 1.155 \times 1.25\, f_o = 1.44\, f_o .$$

The exact value of $\bar{f}$, neglecting the effects of damping, rotatory inertia, and shearing deformation, is $1.43\, f_o$.

THREE-SPAN BEAMS

Consider the three-span beam shown in Fig. 3. Let f_1^o, f_2^o, and f_3^o be, respectively, the fundamental frequencies of spans (1,2), (2,3), and (3,4), assuming that the beam is hinged over its interior supports ($\beta_{2,1} = \beta_{2,3} = \beta_{3,2} = \beta_{3,4} = 0$). These frequencies are determined from Eq. (4) Only those cases will here be considered for which f_1^o and f_3^o are sufficiently larger than f_2^o so that, when the beam vibrates in its fundamental mode, the restraints exerted on the central span are positive.

The stiffnesses $K_{2,3}$ and $K_{3,2}$ are determined by successive approximations as follows: One assumes a value for, say, $K_{3,2}$ and, by treating the portion of the beam between supports 1 and 3 as a two-span continuous beam in the manner described previously, calculates an approximate value for $K_{2,3}$. Using this value of $K_{2,3}$ and working with the portion of the beam between supports 2 and 4, one then computes a new value for $K_{3,2}$. From this revised value of $K_{3,2}$, one then obtains

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a new value of $K_{2,3}$. This procedure is repeated until the values of both $K_{2,3}$ and $K_{3,2}$ converge. Reasonable convergence is generally obtained in two or three cycles.

Having $K_{2,3}$ and $K_{3,2}$, the fundamental frequency $\bar{f}$ of the continuous beam may be calculated from Eq. (4) by considering the central span as an elastically restrained bar. As before, by comparing the exact and the approximate natural frequencies for a number of representative beams covering the possible range of variables, it has been concluded that the maximum error in the value of $\bar{f}$ determined by the foregoing procedure is of the order of $\pm$ 5 percent.

Example. - As an illustration, consider a beam having the following characteristics.

$$E_1 I_1 = 0.80 E_2 I_2, \quad L_1 = 0.85 L_2, \quad m_1 = 0.80 m_2,$$

$$E_3 I_3 = 0.80 E_2 I_2, \quad L_3 = 0.90 L_2, \quad m_3 = 0.70 m_2,$$

$$K_{1,2} = 4.0 E_1 I_1 / L_1 \quad \text{and} \quad K_{4,3} = 1.6 E_3 I_3 / L_3.$$

$$\text{The frequency } f_2^o = \frac{\pi}{2L_2^2} \sqrt{E_2 I_2 / m_2} = f_o.$$

The frequencies f_1^o and f_3^o, evaluated from Eq. (4), are

$$f_1^o = 1.22 \times 1.00 \times 1.384 \, f_o = 1.69 \, f_o,$$

$$f_3^o = 1.00 \times 1.12 \times 1.320 \, f_o = 1.48 \, f_o.$$

In this particular case, the successive approximation procedure is started by taking for the dynamic stiffness $K_{3,2}$ a value equal to one-half the corresponding static stiffness $(K_{3,2})_s$. The value of the latter is determined from Eq. (7) by replacing the

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quantities $E_1 I_1$ and L_1 by $E_3 I_3$ and L_3, and $\beta_{1,2}$ by $\beta_{4,3}$,

$$(K_{3,2})_s = 0.8214 \times 4.0\, E_3 I_3/L_3 = 3.286\, E_3 I_3/L_3 = 2.92\, E_2 I_2/L_2.$$

Hence,

$$_1K_{3,2} = 0.5 \times 2.92\, E_2 I_2/L_2 = 1.46\, E_2 I_2/L_2.$$

In this expression $_1K_{3,2}$ denotes the first approximation to $K_{3,2}$. In general, $_nK_{j,j+1}$ will designate the value of $K_{j,j+1}$ at the beginning of the n-th cycle of the procedure.

The portion of the beam between supports 1 and 3 is now treated as a two-span continuous beam with $K_{3,2}$ equal to $1.46E_2 I_2/L_2$. The frequencies f_1 and f_2 of the individual spans (assuming the beam hinged at support 2) are

$$f_1 = f_1^o = 1.69\, f_o,$$

and

$$f_2 = 1.11\, f_o.$$

The static stiffness of the restraint provided by span (1,2) on span (2,3) is determined from Eq. (7) as

$$(K_{2,3})_s = 0.875 \times 4.0\, E_1 I_1/L_1 = 3.50\, E_1 I_1/L_1 = 3.29\, E_2 I_2/L_2.$$

The first approximation to the corresponding dynamic stiffness is obtained from Eq. (6) as

$$_1K_{2,3} = 0.569 \times 3.29\, E_2 I_2/L_2 = 1.87\, E_2 I_2/L_2.$$

Next, the portion of the beam between supports 2 and 4 is considered, with $K_{2,3}$ taken equal to $_1K_{2,3}$. On the assumption that the beam is hinged at support 3, the fundamental frequencies of the individual

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spans are

$$f_2 = 1.136\, f_o \quad \text{and} \quad f_3 = f^o_3 = 1.48\, f_o .$$

The dynamic stiffness $K_{3,2}$ is obtained from Eq. (6) by substituting $(K_{3,2})_s$ for $(K_{2,3})_s$ and f_3 for f_1,

$$K_{2\,3,2} = 0.411 \times 2.92\, E_2 I_2 / L_2 = 1.20\, E_2 I_2 / L_2 .$$

This newly computed value of $K_{3,2}$ leads to $K_{2\,2,3} = 1.91\, E_2 I_2 / L_2$ which, in turn, leads to $K_{3\,3,2} = 1.19\, E_2 I_2 / L_2$. It should be observed that, for all practical purposes, $K_{2\,2,3}$ is equal to $K_{1\,2,3}$ and $K_{3\,3,2}$ is equal to $K_{2\,3,2}$. Therefore, the β values for the central span may be taken as

$$\beta_{2,3} = 1.91 \quad \text{and} \quad \beta_{3,2} = 1.19.$$

The fundamental frequency $\bar{f}$ of the continuous beam is finally evaluated from Eq. (4), where $j = 2$, as follows:

$$\bar{f} = 1.138 \times 1.096\, f_o = 1.25\, f_o .$$

The exact value of $\bar{f}$, neglecting the effects of damping, rotatory inertia, and shearing distortion, is also equal to $1.25\, f_o$.

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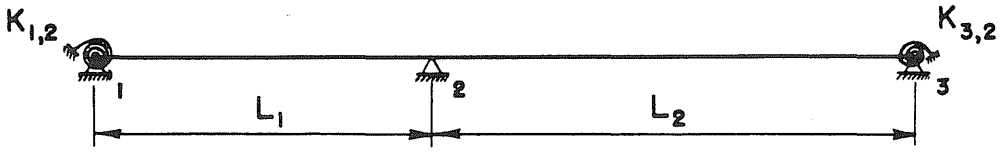

FIG. 1

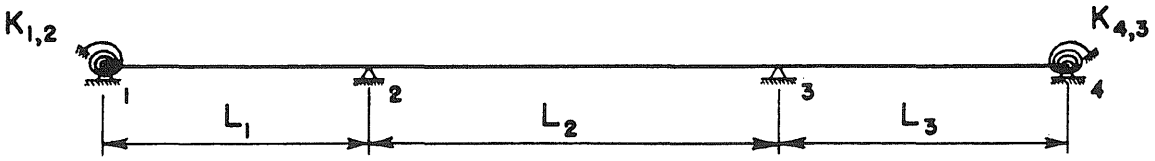

FIG. 3

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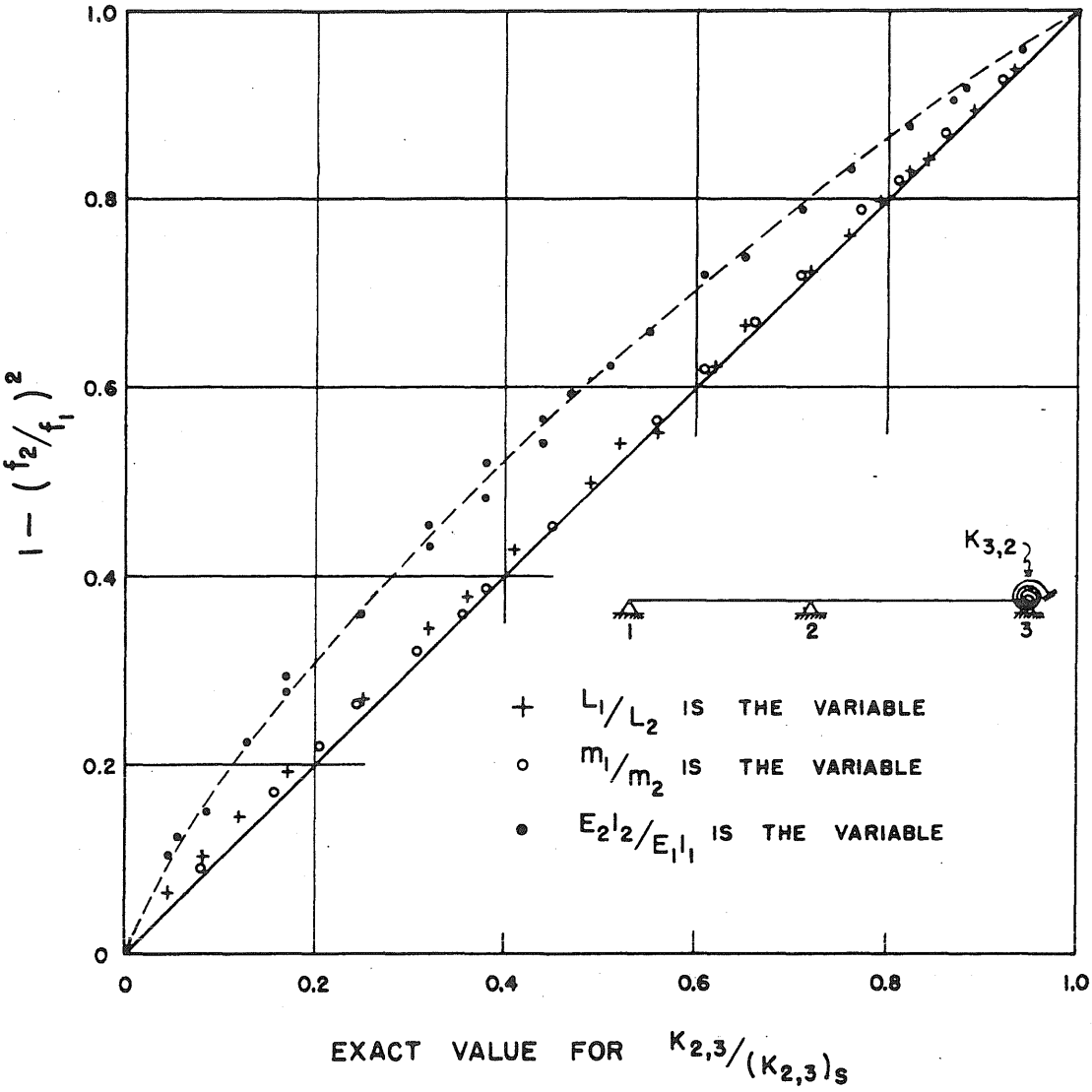

FIG. 2