A NOTE ON THE LIMIT ANALYSIS OF GRID FRAMEWORKS

Metz Reference Room
Civil Engineering Department
E106 C. E. Building
University of Illinois
Urbana, Illinois 61801

By
F. S. SHAW

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OF GRID FRAMEWORKS

By
F. S. Shaw
Professor of Civil Engineering
University of New South Wales
Visiting Professor of Civil Engineering
University of Illinois, 1961-62

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1. Introduction

When analysing the plastic behavior of rigid-jointed plane frames frequently the process used is that of obtaining the "believed" lowest upper bound to the collapse load by some mechanism consideration, and then of attempting to show that this bound is also the solution by obtaining an associated statically admissible moment field. For the latter purpose the "moment-balancing" process is probably the simplest tool, and if the chosen mechanism is not the correct collapse mechanism its use enables a lower bound to be found and also in general indicates the manner in which the mechanism can be modified in order to produce an improved upper bound. All of this is, of course, well known.

For plane grid-frameworks, for which it is a reasonable approximation to neglect the torsional stiffness of the constituent beam elements of the structure, the same situation holds. However, whilst in this case the mechanism technique is well known, this does not seem to be so for the associated moment-balancing technique. Discussion of the latter is the subject of this note. Here, only panel point or "joint" loading is considered in any detail.

2. The General Unit Problem

Consider a system of grid-beam elements of unequal length, rigidly connected together at their ends. Apart from constituting a plane system, loaded normal to that plane at the join of the individual elements, no regularity of beam layout is necessary. Let Fig. 1 represent

*Usually--and erroneously--called plastic moment distribution. The process is not one of distributing moments, but of balancing them.
(a perspective of) a portion of such a grid-frame. For convenience only, parallel systems of elements are shown, but these systems are not necessarily perpendicular to each other.

Fig. 1

Let (any) single joint 0 be given a displacement of arbitrary magnitude \( d \) in the direction and sense of the force \( F_0 \) acting at that joint. Then, with notation shown in the figure, and moment convention of

\[
M_{0,1} = \text{moment at joint 0 in member 0,1,}
\]

where a positive moment at a joint produces tension at the "bottom" of a beam (and so a negative, at the top), and a positive rotation \( \Theta \) of a beam at a joint accompanies a positive moment, application of the virtual displacement principle results in the "unit" moment equilibrium equation

\[
\sum_{i=1}^{n} (M_{0,i}/l_i) - \sum_{i=1}^{n} (M_{i,0}/l_i) = F_0
\]  

(1)

Together with this exists the requirement of moment equilibrium at joint 0 itself. For perpendicular grid systems, with the above notation this amounts to the two requirements.

\[
M_{0,1} = M_{0,3} ; M_{0,2} = M_{0,4} .
\]  

(2)

3. Regular grid-frameworks

Consider, now, any regular grid-framework, i.e., a grid pattern formed by parallel systems of beam elements of equal length \( l \). This requirement of regularity is no real restriction on the usefulness of the moment balancing process, but makes simple discussion possible. Examples of such patterns are shown in Fig. 2.

Fig. 2

In discussing the unit problem for such grids it is convenient to consider three different locations of the central unit joint 0. These are (a) interior joints, (b) boundary joints, and (c) joints adjacent to boundary.
(a) **Interior Joints**

Typical interior joints are shown in Fig. 2. For these the unit problem equilibrium equation, (1), reduces to

$$\sum_{i=1}^{n} M_{o,i} - \sum_{i=1}^{n} M_{i,o} = F_i$$

(3)

where \( n = 4, 6 \) and 4 respectively for the three configurations shown.

For equilibrium of joint 0 itself, for these three systems it is necessary that

$$M_{o,4} = M_{o,2}, \quad M_{o,1} = M_{o,3}$$

(4a)

$$\begin{cases} M_{o,6} + \frac{(M_{o,1} + M_{o,5})}{2} = M_{o,3} + \frac{(M_{o,2} + M_{o,4})}{2}, \\ M_{o,1} + M_{o,2} = M_{o,4} + M_{o,5} \end{cases}$$

(4b)

$$\begin{cases} M_{o,3} + M_{o,4} = M_{o,1} + M_{o,2}, \\ M_{o,4} + M_{o,1} = M_{o,2} + M_{o,3} \end{cases}$$

(4c)

respectively.

(b) **Boundary Joints**

For points 0 on supported boundaries, whether simply-supported or fully clamped, Figs. 3a, 3b, no moment equilibrium equations are required.

Figs. 3a, 3b, Fig. 4

For joints 0 on free boundaries (i.e., boundary joints not supported externally), see Fig. 4, the unit equation (3) applies. However, the joint equilibrium equations (4), whilst still holding, impose a restriction on (3) which should not be overlooked. Thus, since the angle 2\(\alpha\), see Fig. 4, is non-zero, it follows that here equilibrium requires

$$M_{o,2} = 0.$$
This is a consequence of the assumption of zero torsional stiffness.

Hence, for this case equation (3) reduces to

\[ M_{0,1} + M_{0,3} - (M_{1,0} + M_{2,0} + M_{3,0}) = aF_0, \]

together with

\[ M_{0,1} = M_{0,3}. \]

For the framework configuration in which more than three beam elements meet at a free boundary joint, equation (3) again holds together with two joint equilibrium equations after the style of (4b).

(c) Joints Adjacent to Boundary.

For joints adjacent to either a free or a simply supported boundary equation (3) together with joint equilibrium equations like (4) apply. It is, of course, again necessary to keep in mind that at that joint \( i \) which is the boundary joint connected directly to the joint 0 in question, Fig. 5, it is possible that \( M_{i,0} = 0 \)

Fig. 5.

If the boundary joints \( i \) are built in or clamped, then (3) and the appropriate equations (4) apply without modification.

4. Two Simple Examples

Two simple examples will illustrate an application of the foregoing. (a) In the first the problem to be solved is the three bay by three bay square grid structure shown in Fig. 6a. The plastic hinge moment for each beam element is of magnitude \( M_c \), \( L \) is the length of each beam element, the grid is simply-supported at the boundaries, and a force \( F \) acts normally to the grid at joint A.

Figs. 6a, 6b.
The suspected failure mechanism is shown in Fig. 6b. For this mechanism, writing \( m = M/M_c \), \( f = FL/M_c \), the non-dimensional collapse load upper bound is given by \( f^+ = 4.5 \). To show that this is in fact the collapse load itself it is necessary to obtain a corresponding statically admissible moment field.

The field, unique in this instance, is given in Fig. 6c.

In the first line of the computation is shown the information obtained from the mechanism, i.e., plastic hinges of magnitudes \( m = 1 \) have formed in the positions indicated.

To balance the unit beam system centered at A we have

\[
\Sigma m_{A,i} = +4, \quad f = 4.5,
\]

hence

\[
\Sigma m_{i,A} = -0.5.
\]

We satisfy this by choosing \( m_{DA} = m_{BA} = -0.25 \).

Joint equilibrium is satisfied automatically at B and D by noting that \( m_{BE} = m_{BA}, \ m_{DA} = m_{DF} \); however, it is not necessary to record these.

Next, consider equilibrium of the unit beam system centered at B. We have

\[
\Sigma m_{B_i} = +1.5, \ \Sigma m_{iB} = 1.0 + m_{CB}.
\]

Hence

\[
m_{CB} (\Sigma m_{iB}) = 0.5.
\]

From symmetry, \( m_{CD} (= m_{CH}) = 0.5 \) also.

Finally, checking equilibrium at joint C, we have

\[
\Sigma m_{C1} = 2.0, \ \Sigma m_{1C} = 2.0
\]

as required.
Inspection shows that the resulting equilibrium moment field nowhere exceeds the magnitude of the hinge moment and so is a statically admissible field as desired.

(b) The second problem is concerned with a square four bay by four bay grid structure. All internal joints are loaded by a normal force of magnitude $F$. Pertinent frame properties are the same as those in the first example.

The frame is shown in Fig. 7a, and the suspected collapse mechanism, for which $f = 1$, is shown in Fig. 7b.

Figs. 7a, 7b.

Development of an admissible moment field follows in Fig. 7c. Because of symmetry only one quarter of the frame need be considered, in fact only one eighth is necessary.

Fig. 7c.

The first line of figures at each joint is the information obtained from the mechanism shown in Fig. 7b.

To balance joint A it is necessary that

$$4.0 - (2m_{CA} + 2m_{BA}) = 1.0$$

i.e.

$$m_{CA} = m_{BA} = 0.75$$

To balance joint B we require

$$2 (1.0 + 0.75) - (1 + 2m_{DB}) = 1$$

i.e.

$$m_{DB} = 0.75$$

$= m_{DC}$ by symmetry.

Checking joint D, we have

$$\Sigma m_{DB} = 3.0, \Sigma m_{ID} = 2.0$$
Hence

\[ \sum m_{D1} - \sum m_{iD} = 1 \]

as required for equilibrium.

As can be seen, the solution constitutes a statically admissible moment field.

5. Remarks

As for the usual rigid-frame moment-balancing process, the above technique can also be used in a direct manner to obtain a lower bound and upper bound solution. It is, however, more convenient to commence the process with some possible mechanism configuration and associated upper bound to the collapse load. The process will then be found useful in indicating where hinges are incorrectly located, and so in suggesting more appropriate hinge locations.

The technique can easily cope with structures in which different beam arrangements form separate adjacent portions of the total configuration. Cut-outs in the grid-beam system, e.g., stairwell openings, present no special difficulty.

6. A More Difficult Example

(a) The structures considered in the foregoing two examples are extremely simple and not particularly realistic. As a consequence it was not necessary to be very systematic in the balancing process. If, however, a more complex structure is considered, it will be found that such an approach is not simple. It is frequently by no means obvious how to proceed. For this reason, it is useful to systematise the process a little.
To do so, it is convenient to adopt one or two of the thoughts used in the numerical "relaxation" process. This will be illustrated by another square grid problem.

In particular, let $R_o$ be the error, or amount by which a joint of a unit grid is not in equilibrium. Then from (3), in non-dimensional form we have

$$\Sigma m_{0,1} - \Sigma m_{1,0} - f_o = R_o$$  \hspace{1cm} (5)

and it is desired to reduce $R_o$ to zero, or to a negligibly small magnitude at each joint. Attention will be concentrated on this aspect.

Fig. 8

From equation (5), and referring to Fig. 8 and keeping in mind (since this is a perpendicular grid system) that $m_{03} = m_{01}$ for example, it follows that two unit balancing operators are available. Thus with

$$\Delta m_{02} \quad (\Delta m_{04}) = +1$$

it follows that

$$\Delta R_o = +2, \quad \Delta R_2 = -1, \quad \Delta R_4 = -1.$$ \hspace{1cm} (6a)

Also, with

$$\Delta m_{03} \quad (\Delta m_{01}) = +1$$

it follows that

$$\Delta R_o = +2, \quad \Delta R_1 = -1, \quad \Delta R_3 = -1.$$ \hspace{1cm} (6b)

These two operators make possible a systematic unit balancing process not dissimilar to that used in "relaxation." Either one, or a combination of both, can be used to reduce $R_o$ at any one joint. Alternatively, either can also be used for reducing an $R_i$ at an appropriate neighboring joint $i$. This latter situation exists where one or both of the moments at joint $i$ is specified a priori because of the believed existence there
of a plastic hinge. In such a circumstance, should the unit problem centered at that joint not be in equilibrium \( R_i \neq 0 \) then, with moments there already recorded as having magnitude \( M_c \) (or the non-dimensional equivalent), the balancing can only be accomplished by systematic modification to moments at adjacent joints. For instance, a residual or error \( R_1 \) could be transferred away from joint 1 to joints 2, 3, and 4 without altering \( R_0 \) by using equal and opposite multiples of the two unit operators at joint 0. Such moves are occasionally desirable.

In practice, although the balancing process can be carried out by commencing with any guessed set of moment values, it will usually be found to be easier to commence with some possible failure mechanism. From this the associated \( f^+ \) can be computed and so, keeping this and the relevant plastic hinge moments constant, the \( R_i \)'s can be calculated everywhere and then balancing attempted by means of the unit operators (6a), (6b). If the chosen mechanism is not the correct one it will be found impossible to produce a set of balanced joint moments which everywhere satisfy the yield condition \( -M_c \leq M_{i,j} \leq M_c \). In this circumstance, the balancing process will in general indicate, via the larger moment values, possibilities of more appropriate hinge locations.

(b) A more realistic structure embodying some of the above features is shown in Fig. 9. The grid is again built up of square bays, all beam elements being of length \( L \) and possessing a plastic hinge magnitude \( M_c \). Joints are loaded normal to the grid plane with forces:

\[
\begin{align*}
F/2 & \text{ at joints } 2, 9, 13, 21, 22, 30, 32; \\
F & \text{ at all others.}
\end{align*}
\]

Fig. 9

Without discussing in detail all steps of the process, the solution proceeds as follows:
(i) After a few doodling trials the two mechanisms shown in Figs. 10a, 10b seem reasonable. For ease of visualisation the normal displacement of each joint, as multiples of some arbitrary distance $d$, are recorded on each figure.*

Figs. 10a, 10b

A virtual displacement calculation produces the results:

For mechanism of Fig. 10a, $f_1^+ = 23/14$;
for that of Fig. 10b, $f_2^+ = 48/37 < f_1^+$.

$m_c = \pm 1$

An attempt to produce a statically admissible moment field for the latter fails. With all joints balanced, three moments are greater than the permissible $m_c$. However the moment solution suggests the mechanism shown in Fig. 11a.

Figs. 11a, 11b, 11c.

(ii) For this mechanism computation produces $f_3^+ = 44/35$, which is less than the value of $48/37$ just tried. Unfortunately, it is quickly shown that this likewise is not the correct solution, but one is led to the mechanism of Fig. 11b as a possibility.

(iii) For this latter the upper bound is $f_4^+ = 36/33$, $< f_3^+$. Again, it becomes apparent during the moment balancing that this is not the correct solution; however, in turn, the mechanism shown in Fig. 11c is suggested.

(iv) For this, the upper bound to the collapse load is

$$f_5^+ = 46/45, < f_4^+,$$

and this is, in fact, the correct solution.

*By noting such joint displacements it is a simple matter to verify that sufficient hinges are provided, and at the correct locations, to form a mechanism. The requirement is that for a mechanism there must exist a hinge, real or plastic, wherever there is a change in slope-relative to the plane of the grid- in any straight line of grid elements. At such a slope change two hinges can be inserted instead of one if so preferred; the magnitude of the associated collapse load is not affected in any way by so doing.
A resulting statically admissible moment field is shown in Fig. 12. In the actual computation of this, with the result

\[ F_r = \frac{46}{45} \cdot \frac{M_c}{L}, \]

i.e.,

\[ r = \frac{FL}{M_c} = \frac{46}{45}, \]

it seems more convenient, for the numerical process, to keep "r" at value unity and so to use a non-dimensional hinge moment of magnitude

\[ m_c = \frac{45}{46} = 0.978. \]

This is the magnitude recorded on Fig. 12.

7. Loads on Individual Beam Elements

Although making for simplification, it is not essential that structure loads be applied only at joints of the grid. If, however, the beam elements themselves are subjected to systems of normal forces, for each such element an additional amount equilibrium equation is required. This one is the familiar beam equation used in the limit analysis of rigid framed structures. For completeness it is included here.

Referring to Fig. 13, let A and B be the joints at the ends of the beam element A, B. Then, with the moment convention already adopted, and for any point P along the span of the beam, use of the virtual displacement principle results in the equilibrium relation between the three moments \( M_{A,B}, M_{B,A'}, M_P \) expressed by

\[ -\beta m_{A,B} + m_P - \alpha m_{B,A} = \alpha B_F \]  (external work \( W_e \))  \( (7) \)

where \( \alpha = a/L, \beta = b/L, m = M/L. \)
Reference:

FIG. 1 (Perspective)

FIG. 2 (Plan Views)
(a) PLAN

(b) MECHANISM

(c) SOLUTION

FIG. 6
(a) PLAN  (b) MECHANISM

(c) SOLUTION

FIG. 7
FIG. 8
UNIT OPERATORS

FIG. 9 (Plan)
(a) $f_1^+ = \frac{23}{14}$

(b) $f_2^+ = \frac{48}{37}$

**FIG. 10**

(a) $f_5^+ = \frac{44}{35}$

(b) $f_4^+ = \frac{36}{33}$

(c) $f_8^+ = \frac{46}{45}$

**FIG. 11**
FIG. 12 (Perspective)

SOLUTION
FIG. 13
TYPICAL BEAM ELEMENT