EFFECTS OF COLUMN STIFFNESS ON THE MOMENTS IN TWO-WAY FLOOR SLABS

By
S. H. SIMMONDS
and
C. P. SIESS

Metz Reference Room
Civil Engineering Department
B106 C. E. Building
University of Illinois
Urbana, Illinois 61801

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1. INTRODUCTION

1.1 Introductory Remarks

Probably no structural system incapable of an easy, accurate theoretical analysis is as widely used as the two-way floor slab. This form of floor construction has proved its efficiency and economy in carrying load in two directions to the supporting members. Since the type and stiffness of these supporting members have a significant influence on the behavior of this slab system it is imperative that a better understanding of these influences be obtained. While several types of two-way slabs are used extensively, this investigation is restricted to slabs supported along all edges by beams spanning between columns.

1.2 Object

It was the object of this investigation to determine the effect of variations in the stiffnesses of the supporting beams and columns on the bending moments in a continuous two-way floor slab system. This was accomplished by determining the elastic moments at specific points on the slab for different values of these stiffnesses.

Detailed objectives considered can be summarized as follows:

(a) To determine the magnitude and distribution of the slab bending moments for different combinations of the flexural and torsional stiffnesses of the supporting beams.

(b) To determine the effect of column stiffness on the magnitude of the slab moments for a given combination of beam stiffnesses.

(c) To determine the effect of column stiffness on the flexural and torsional moments in the beams.

(d) To consider the effects of partial loading.
1.3 Scope

The slab system investigated consists of nine square panels as shown in Fig. 1. The panels are supported on all four sides by elastic beams which frame into columns located at each beam intersection. Three values of the flexural stiffness of the supporting beams were considered in all combinations with three values of the torsional stiffness. For each combination of beam stiffnesses, five values of the column stiffness, ranging from zero to infinity, were used.

A reduced stiffness was considered for exterior beams as compared to interior beams. Similarly, corner and edge columns were considered to be less stiff than interior columns. However only one ratio of these stiffness values was used.

For each combination of beam and column stiffness a solution was obtained for a uniformly distributed load on all panels. A limited number of solutions were obtained for the condition of loading alternate panels with a uniformly distributed load (checkerboard loading).

1.4 Historical Review

Several attempts have been made to present a reasonable analysis for determining the moments and shears developed in continuous two-way slabs. In each instance, in order to obtain a solution, certain assumptions were required which limit the range of applicability of the results. A detailed historical review of existing classical solutions for single panel plates with various edge conditions has been given by H. M. Westergaard. (1)*

* Numbers in parentheses refer to entries in the list of references.
By combining these solutions with empirical observations, he obtained a set of coefficients which could be used in the design of continuous rectangular floor slabs.

A distribution procedure for determining the moments in a slab continuous over flexible beams was presented by Newmark. However solutions by this procedure are restricted to slabs which are continuous in one direction only and which must be simply supported along the other two edges. Siess and Newmark presented an approximate analysis for determining the moments in two-way slabs. Their method, based on a distribution procedure similar to the Cross method, allows the average positive and negative moments for any continuous rectangular floor to be obtained readily. Although the effect of the torsional resistance of the supporting beams could be taken into account, the beams were assumed to be non-deflecting.

With the advent of high-speed digital computing machines, new numerical techniques have become available. An elastic distribution procedure for determining the moments in rectangular slabs continuous in both directions over flexible beams was developed by Ang. Using this procedure, Morrison obtained moments at distinct points in a nine-panel continuous slab for various combinations of beam flexural and torsional stiffnesses. However the assumption that the action of the beams were concentrated along lines while the columns had finite dimensions resulted in unreasonably high moments in the region of the column corners.

Little, if any, account has been taken of the effect of column stiffness on the slab moments. In general the columns have been considered
infinitely stiff. In experimental studies of nine-panel two-way floor slabs (6)(7) the beams and columns were proportioned to have stiffnesses representative of those used in practice. In interpreting the results of these tests it appeared that rotation of the columns may have taken place. Therefore, it seemed desirable to investigate the effect of variable column stiffness on the magnitude and distribution of the slab moments.

1.5 **Acknowledgment**

This thesis was prepared under the direction of Dr. C. P. Siess, Professor of Civil Engineering. The author expresses his appreciation for the guidance and helpful advice received during the progress of this investigation.

1.6 **Notation**

\[
\begin{align*}
\text{b} &= \text{span length of one panel, center to center distance between columns} \\
\text{C} &= \text{a measure of the torsional rigidity of a beam cross-section} \\
\text{E}_b &= \text{modulus of elasticity of the beam material} \\
\text{E}_c &= \text{modulus of elasticity of the column material} \\
\text{E}_s &= \text{modulus of elasticity of the slab material} \\
G &= \frac{E_b}{2(1+\mu)} \quad \text{shear modulus of elasticity of the beam material} \\
h &= \text{distance between joints or node points of grid} \\
H &= \frac{E_b I}{N_b} \quad \text{ratio of beam flexural stiffness to slab stiffness} \\
H' &= \frac{E_b I}{N_h} = \frac{b}{h} H \quad \text{a measure of the flexural stiffness of a beam}
\end{align*}
\]
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tr>
<td>$I_b$</td>
<td>moment of inertia of the cross-section of a beam</td>
</tr>
<tr>
<td>$I_c$</td>
<td>moment of inertia of the cross-section of a column</td>
</tr>
<tr>
<td>$I_s$</td>
<td>$\frac{t^3}{12}$ moment of inertia per unit width of slab</td>
</tr>
<tr>
<td>$J$</td>
<td>$\frac{CG}{Nb}$ ratio of beam torsional stiffness to plate stiffness</td>
</tr>
<tr>
<td>$J'$</td>
<td>$\frac{CG}{Nh} = \frac{b}{h} J$ a measure of the torsional stiffness of a beam</td>
</tr>
<tr>
<td>$K$</td>
<td>$\sum \frac{k}{N} = \frac{4E I_c}{L_c N}$ a measure of the total flexural stiffness of a column</td>
</tr>
<tr>
<td>$L_c$</td>
<td>effective length of column</td>
</tr>
<tr>
<td>$m$</td>
<td>bending moment or twisting moment per unit width of slab</td>
</tr>
<tr>
<td>$\bar{m}$</td>
<td>bending moment or twisting moment acting on a section of slab of width $h$</td>
</tr>
<tr>
<td>$M$</td>
<td>flexural bending moment in a beam</td>
</tr>
<tr>
<td>$M_c$</td>
<td>end moment applied to column</td>
</tr>
<tr>
<td>$M'_s$</td>
<td>bending moment applied to slab due to presence of columns</td>
</tr>
<tr>
<td>$N$</td>
<td>$\frac{E I_s}{(1-\mu^2)}$ measure of slab stiffness</td>
</tr>
<tr>
<td>$q$</td>
<td>uniformly distributed load per unit of area</td>
</tr>
<tr>
<td>$t$</td>
<td>thickness of slab</td>
</tr>
<tr>
<td>$T$</td>
<td>torsional moment in a beam</td>
</tr>
<tr>
<td>$w$</td>
<td>vertical deflection of the slab, positive downward</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>rectangular reference coordinates</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Poisson's ratio</td>
</tr>
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</table>
2. METHOD OF ANALYSIS

2.1 Introductory Remarks

The analysis used in this investigation is an application of the finite difference equations. The difference operators were derived by the method presented by Ang and Prescott (8) which approximates the slab-beam system by a mathematical model based on Newmark's plate analog. By this method, the operators are formed from considerations of equilibrium of the model rather than from the governing differential equation. Internal discontinuities and boundary conditions are considered by introducing appropriate modifications to the model. A complete description of the procedure applied to slabs and the supporting beams has been given by Prescott (9). The modifications required to consider the effects of column stiffness are outlined in the following sections.

2.2 Basic Assumptions

The basic assumptions for the usual theories of flexure of beams and plates have been listed in a number of places in the literature (10)(11). The added assumptions required to account for the effects of the beams and columns are as follows:

(a) The behavior of the beams in both bending and torsion is linearly elastic.

(b) The neutral axes of the beams are assumed to coincide with the middle plane of the slab.

(c) The forces and moments acting on a beam are assumed to be distributed along a line and not over a finite width.
(d) Torsion in the beam is assumed to be uniform between node points and the effects of warping are neglected.

(e) The action of the column is assumed to be concentrated at a point, located at the intersection of the column center line and the middle plane of the slab.

2.3 Review of Fundamental Equations

The fundamental equations used in obtaining the operators are derived in the literature (10) and are listed below for convenience.

The deflection, $w$, is related to the loading by the differential equation

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{N}$$

The bending moments and twisting moments are related to the deflections as follows:

$$m_x = -N(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2})$$

$$m_y = -N(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2})$$

$$m_{xy} = -N(1-\mu) \frac{\partial^2 w}{\partial x \partial y}$$

Similarly the shears and reactions can be expressed as:

$$v_x = \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y}$$

$$v_y = \frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x}$$

$$R_x = v_x + \frac{\partial m_{xy}}{\partial y} = \frac{\partial m_x}{\partial x} + 2 \frac{\partial m_{xy}}{\partial y}$$

$$R_y = v_y + \frac{\partial m_{xy}}{\partial x} = \frac{\partial m_y}{\partial y} + 2 \frac{\partial m_{xy}}{\partial y}$$
The equations governing the flexure and twisting of beams are

\[ M_x = - E_b I_b \frac{\partial^2 w}{\partial x^2} \]
for a beam with its longitudinal axis parallel to the x axis

\[ M_y = - E_b I_b \frac{\partial^2 w}{\partial y^2} \]
for a beam with its longitudinal axis parallel to the y axis

\[ T = - CG \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial y} \right) \]
for a beam with its longitudinal axis parallel to the x axis

\[ T = - CG \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial x} \right) \]
for a beam with its longitudinal axis parallel to the y axis

The expressions for the end moments induced in the columns are

\[ M_c = - k \frac{\partial w}{\partial y} \]
for bending about the x axis

\[ M_c = - k \frac{\partial w}{\partial x} \]
for bending about the y axis

The resultant moments in the slab due to the column action are

\[ M_s = \Sigma M_c = - K N \frac{\partial w}{\partial y} \]
for bending about the x axis

\[ M_s = \Sigma M_c = - K N \frac{\partial w}{\partial x} \]
for bending about the y axis

2.4 Derivation of Difference Operators

Points are designated by the system shown in Fig. 3. In deriving the difference operators, the following relationships are used repeatedly and are listed for ready reference. By the use of finite differences, the derivatives at point \( o \) can be expressed in terms of the deflections as follows:
\( \begin{align*}
\frac{\partial w}{\partial x} \bigg|_0 &= \frac{1}{2h} (-w_w + w_e) \\
\frac{\partial w}{\partial y} \bigg|_0 &= \frac{1}{2h} (-w_s + w_n) \\
\frac{\partial^2 w}{\partial x^2} \bigg|_0 &= \frac{1}{2h} (w_w - 2w_o + w_e) \\
\frac{\partial^2 w}{\partial y^2} \bigg|_0 &= \frac{1}{2h} (w_s - 2w_o + w_n) \\
\frac{\partial^2 w}{\partial x \partial y} \bigg|_0 &= \frac{1}{4h^2} (w_{sw} - w_{nw} - w_{se} + w_{ne})
\end{align*} \)

By defining point \( o \) at the locations designated by the black dots, these expressions, when substituted into the fundamental equations, will produce the following operators for the moments and shears acting on the elements of the plate analog.
\[ M_{ox} = \frac{NH'}{h} \left( -w_w + 2w_o - w_e \right) \]
\[ T_{oe} = \frac{NJ'}{2h} \left( -w_s + w_n + w_{se} - w_{ne} \right) \]
\[ m_s = -\frac{NK}{2h} \left( -w_s + w_n \right) \]

Similar expressions can be written for the other moment and shear force quantities.

The operators required for points on the slab-beam system, have been previously derived, (8)(9) and are not repeated here. Due to discontinuities at the column, operators for points adjacent to the column will introduce fictitious deflections. The operators required to solve for these fictitious deflections involve consideration of the column effects and are derived below.

(a) Operators for an Interior Column

A portion of the plate analog in the vicinity of an interior column is shown in Fig. 4. The equilibrium of moments about the x-axis at point \( \mathbf{0} \) results in the equation

\[ + M_s - T_{oe} - \bar{m}_{\text{on}} - M_{on} + T_{ow} + \bar{m}_{os} + M_{os} = 0 \quad (1) \]

Each term in equation 1 can be expressed in terms of the deflections by the relationships given in Section 2.4 as follows:

\[ M_s = -\frac{NK}{2h} \left( -w_s + w_n \right) \]
\[ T_{oe} = \frac{NJ'}{2h} \left( -w_s + w_n + w_{se} - w_{ne} \right) \]
\[ m_{\text{on}} = \frac{N}{h} \left( -\mu_w - w_n + (2 + 2\mu)w_o - \mu w_e - w_s \right) \]
These expressions involve the fictitious deflections $w_s^-, w_s^+ w_n^-$ and $w_{ne}^-$. The fictitious deflection $w_n^-$ can be eliminated by considering the continuity of the slab. Thus by equating slopes we obtain

$$-w_s + w_n^- = -w_s + w_n$$

Solving for $w_n^-$ and substituting into the above moment equations, equation 1 can be written in the notation

$$w = 0 \quad (2)$$

The coefficients of the fictitious deflections $w_s^-, w_{ne}^- w_{nw}^-$ are contained in the dotted boxes. The operators required for determining the fictitious deflections $w_{ne}^-$ and $w_{nw}^-$ are obtained from consideration of moment equilibrium at points e and w respectively and have been derived by Prescott (9). Equation 2 provides the relationship necessary to solve for the fictitious deflection $w_s^-$. 

\[
\begin{align*}
M_{on} &= \frac{NH}{h} (-w_s + 2w_o - w_n) \\
T_{ow} &= \frac{NJ}{2h} (-w_{sw} + \frac{w_{nw}}{h} + w_s - w_n) \\
T_{os} &= \frac{N}{h} (-\mu w_o - w_n^+ + (2 + 2\mu)w_o - \mu w_e - w_s) \\
M_{os} &= \frac{NH}{h} (-w_s + 2w_o - w_n) \\
\end{align*}
\]
Similarly, by considering the equilibrium of moments about the y-axis, the following equation is obtained

\[ + M_s + T_{os} - \overline{m} - M_{oe} - T_{on} + \overline{m} + M_{ow} = 0 \]  

(3)

This equation is used to solve for the fictitious deflection \( w_e \), and can be expressed in operator notation as

\[
\begin{align*}
\begin{bmatrix}
-\frac{3}{2} & -\frac{1}{2} \\
2J' + K & J'
\end{bmatrix}
\begin{bmatrix}
J\\
0
\end{bmatrix}
= w = 0
\end{align*}
\]

(4)

It must be noted that equations 3 and 4 each contain a fictitious deflection at point designated \( \overline{nw} \). In general, these fictitious deflections will not be equal as \( w_{\overline{nw}} \) in equation 3 results from consideration of torsion in the beam parallel to the x-axis, whereas \( w_{\overline{nw}} \) in equation 4 results from consideration of torsion in the beam parallel to the y-axis.

(b) Operator for an Edge Column

As in the case for an interior column, the application of the difference patterns to points adjacent to an edge column involves the use of fictitious points. The necessary equations to solve for these fictitious deflections are obtained by conditions of moment equilibrium. Referring to Fig. 5 and considering the equilibrium of moments at point \( o \) about the x-axis we obtain

\[ + M_s + T_{ow} - T_{oe} + \overline{m}_{os} + M_{os} = 0 \]  

(5)
Expressing each term in equation 5 in terms of the deflections yields

\[
\begin{align*}
M_s &= -\frac{NK}{2h}(-w_s + \bar{w}_n) \\
T_{oe} &= \frac{NJ'}{2h}(-w_s + \bar{w}_n + \bar{w}_e - \bar{w}_n) \\
T_{ow} &= \frac{NJ'}{2h}(-w_{sw} + \bar{w}_n + w_s - \bar{w}_n) \\
\bar{m}_{os} &= \frac{N}{h}(-\mu w_n - \bar{w}_n + (2 + 2\mu)w_o - \mu w_e - \bar{w}_s) \\
M_{os} &= \frac{NH'}{h}(-w_s + 2w_o - \bar{w}_n)
\end{align*}
\]

When these expressions are substituted into equation 5 we obtain

in operator notation

\[
\begin{bmatrix}
\frac{J'}{J'} & \frac{K-2J'}{2H'} & \frac{J'}{J'} \\
\frac{K}{J'} & \frac{4+4H'+4\mu}{4H'+4\mu} & -2\mu \\
-J' & K-2H'+2J' & -J'
\end{bmatrix}
\begin{bmatrix}
w \\
0 \\
0
\end{bmatrix}
= 0
\]

Equation 6 provides the operator for determining the fictitious deflection \(\bar{w}_n\).

To obtain the value of the fictitious deflection \(\bar{w}_n\), the moments acting at point \(o\) about the \(y\)-axis are considered. Thus,

\[
+ M_s + T_{os} + M_{ow} - M_{oe} + \frac{\bar{m}_{ow}}{2} - \frac{\bar{m}_{oe}}{2} = 0
\]
Eliminating the fictitious deflection $w_w$ by considering continuity of the slab, equation 7 can be expressed in operator notation as

$$w = 0 \quad (8)$$

(c) Operator for a Corner Column

The required modification to the analog is shown in Fig. 6. Equilibrium of moments about the $x$-axis yields

$$+ M_s + T_{ow} + \frac{\bar{m}_{os}}{2} + M_{os} = 0 \quad (9)$$

In operator notation this can be represented as

$$w = 0 \quad (10)$$

Similarly considering moments about the $y$-axis we obtain

$$w = 0 \quad (11)$$
Equations 10 and 11 provide the operators required to determine the fictitious deflections $w_n$ and $w_e$. These operators are perfectly general for square grids and the beam and column constants in the $x$ and $y$ directions can be varied independently if so desired. However, where an axis of symmetry bisects the corner, as is the case in this analysis, equations 10 and 11 can be combined as fictitious points $n$ and $e$ become equal as do real points $s$ and $w$. Thus, for diagonal symmetry, equations 10 and 11 can be expressed as

$$\begin{pmatrix}
J' & 0 \\
0 & J'
\end{pmatrix} \begin{pmatrix}
\lambda - KJ' - 2H' - 1 - \mu \\
J' - 2H' + K - 1 - \mu
\end{pmatrix} \begin{pmatrix}
w_n \\
w_e
\end{pmatrix} = 0$$

(12)

(d) **Operator for Infinitely Stiff Column**

An infinitely stiff column is a column sufficiently rigid that no rotation of the slab directly adjacent to the column face is permitted. This definition provides the boundary condition for determining an operator for this case, namely that the slope of the slab over the column is zero at all times. Thus for bending about the $x$-axis we obtain

$$- w_s + w_n = 0$$

(13)

or for bending about the $y$-axis

$$- w_w + w_e = 0$$

(14)
It is obvious that equations 13 and 14 are valid for any column sufficiently rigid to be considered infinitely stiff and are independent of the column location in the slab system.

In this analysis the deflection of the slab immediately over the column has been taken as zero; that is, the columns have been considered non-deflecting. Therefore, when applying an operator to points adjacent to the column, the term containing the deflection of the slab at a point coincident with the column centerline can be neglected. However, if it is desired to consider the deflection or settlement of the columns, the numerical value of this deflection would be inserted in the appropriate operators. In either case, the number of simultaneous equations that must be solved is unchanged.

2.5 Correctness of Operators

Due to the use of fictitious deflections at points of discontinuity, the matrix developed is not symmetrical. However, since it represents the solution of a physical model, Betti's reciprocal theorem must apply. Deflections were obtained for a load of 100 units placed successively at five locations on the slab system. These deflections are tabulated below and inspection shows that the solutions obey the reciprocal theorem.

<table>
<thead>
<tr>
<th>Load Applied at Locations</th>
<th>(8,12)</th>
<th>(4,9)</th>
<th>(7,7)</th>
<th>(0,5)</th>
<th>(1,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deflection at Location</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(8,12)</td>
<td>+49.02504</td>
<td>+2.45238</td>
<td>+16.33311</td>
<td>-1.26985</td>
<td>-0.81820</td>
</tr>
<tr>
<td>(4,9)</td>
<td>+2.45238</td>
<td>+25.98918</td>
<td>+21.72452</td>
<td>+2.96430</td>
<td>-4.86964</td>
</tr>
<tr>
<td>(7,7)</td>
<td>+16.33311</td>
<td>+21.72452</td>
<td>+111.62684</td>
<td>-3.32998</td>
<td>-8.31917</td>
</tr>
<tr>
<td>(0,5)</td>
<td>-1.26985</td>
<td>+2.96430</td>
<td>-3.32998</td>
<td>+64.58309</td>
<td>+30.01472</td>
</tr>
<tr>
<td>(1,2)</td>
<td>-0.81820</td>
<td>-4.86964</td>
<td>-8.31917</td>
<td>+30.01472</td>
<td>+167.53193</td>
</tr>
</tbody>
</table>
3. OUTLINE OF ANALYSIS

3.1 General Remarks

Using the procedure presented in Chapter 2, it is possible to solve any two-way floor slab system with no special limitation as to size or pattern of the beams and columns. However, the accuracy of the solution is dependent on the interval of the discrete points considered, and for each point there is an independent equation which must be solved simultaneously with the equations developed from the other points. Thus there is a practical limit to the extent of problems that can be considered, beyond which the number of simultaneous equations to be solved becomes so great that their solution exceeds the capacity of the available computing facilities. In this investigation the digital computer used, the ILLIAC, has the capacity to solve 143 simultaneous equations which thus sets an upper limit to the size of problem which could be considered.

Since the theory is based on elastic principles, the results are at best applicable only to the elastic range. Although the operators derived permit any value of Poisson's ratio to be considered, the only value used in this investigation was zero, which is the value generally accepted in reinforced concrete slab analysis. Since the slab is supported at the boundaries on deflecting beams, a simple conversion of the moments to other values of Poisson's ratio is not possible.

It was the object of the investigation to determine the effects on the slab moments of a range of beam and column variables. A discussion of these variables and the slab system considered is contained in the following sections.
3.2 Description of Slab System

The slab system chosen (Fig. 1) consists of nine square panels arranged in three rows of three panels each. Beams are located along the border of each panel and columns are positioned at each intersection of the beams. The slab is considered to have the same thickness in each panel.

There are several distinct advantages to this system. It allows the consideration of interior, edge and corner panels, and consequently the effects of the three corresponding types of columns. In a general sense the results are applicable to a slab with any number of square panels. The geometry of this system corresponds to that of experimental slabs tested at the University of Illinois, (6)(7) which permits direct comparisons to be made. This arrangement of slab has both axial and diagonal symmetry which greatly reduces the portion of slab that must be considered and so allows the use of a finer grid spacing without exceeding the limiting number of equations.

3.3 Beam Variables

Although the theory permits using any combination of beam sizes, the beam variables were chosen to maintain the maximum symmetry of the system, and were constant along the length of the beam in any span.

The flexural stiffness of the beam is represented by a dimensionless ratio, \( H \), which is the flexural stiffness of the beam divided by the flexural stiffness of one panel width of slab; \( H = E_b I_b / N_b \). It is obvious that a greater value of \( H \) represents a greater beam flexural stiffness.
Similarly the torsional stiffness of the beam is represented by a dimensionless ratio, \( J \), which is the torsional stiffness of the beam divided by the flexural stiffness of one panel width of slab; \( J = \frac{C_G}{N_b} \).

The flexural and torsional stiffnesses of the beams were varied separately. In order to obtain a range of results, three values of each stiffness were chosen to approximate a rather flexible beam, a beam of medium flexibility, and a rather stiff beam. The numerical values of \( H \) and \( J \) were 0.25, 1.0, and 2.5. The three values of \( H \) were combined with the three values of \( J \) in all combinations.

In order to correspond to stiffnesses used in the test slab (6) and the conditions usually met in practice, the edge beams were made less stiff than the interior beams. Using the dimensions of the prototype for the test slab, the flexural and torsional stiffnesses of the edge beams were computed to be five-eighths of those for the interior beams. Therefore, values of the stiffness ratios \( H \) and \( J \), given for a particular slab system, refer to the stiffnesses of the interior beams and the flexural and torsional stiffnesses of the edge beams were in every case taken as five-eighths of these values.

The above definitions of the ratios representing the flexural and torsional stiffnesses of the beams are independent of the grid spacing, as they are referred to the width of panel, \( b \). However, in deriving and using the operators, it is more convenient to express the stiffness ratios in terms of the width of a grid strip, \( h \). Therefore, we define modified stiffness ratios \( H' = \frac{E_b I_b}{N h} = \frac{bH}{h} \) and \( J' = \frac{C_G}{Nh} = \frac{bJ}{h} \). These modified stiffness ratios appear in the derived operators and were the values used in writing the equations.
3.4 Column Variables

The magnitude of column stiffness is represented by the dimensionless ratio $K$, which is defined as the sum of the column stiffnesses both above and below the slab divided by the flexural stiffness of a unit width of slab. To obtain the effect on the slab moments for a full range of the column stiffness, $K$ was taken equal to 0, 10, 30, 90, and infinity for the interior columns. These values were chosen after initial studies showed they gave a reasonable distribution of effects between the two extreme values. It should be remembered that an infinite value of $K$ cannot be represented in the operators but, as shown in Section 2.4, this case is solved by resorting to the boundary conditions implied by an infinitely stiff column.

The above values of $K$ refer to a typical interior column. The values of stiffness for the corresponding edge and corner columns were reduced in proportion to the dimensions given for the prototype of the slab tested in Reference 6. These reductions were made by direct ratio of the moments of inertia of the uncracked cross-section about the corresponding axes. Thus values of stiffness for the corner columns were taken as $0.2K$ about both axes, and those for an edge column were taken as $0.3K$ about an axis parallel to the edge and $0.7K$ about an axis perpendicular to the edge. These ratios of corner and edge column stiffness to interior column stiffness were maintained for each analysis.

3.5 Loading Conditions

Analyses were made with a uniformly distributed load over all panels for all combinations of beam and column stiffness. In order to
gain some insight into the effect of partial loadings, a limited number of solutions with checker-board loadings were made. These consisted of loading the odd numbered panels in Fig. 1 with a uniformly distributed load for the complete range of column stiffness, but only for two cases of beam stiffness, namely $H = J = 0.25$ and $H = J = 2.5$. 
4. PRESENTATION OF RESULTS

4.1 General Remarks

For clarity and for the convenience of the reader, the results are presented in the form of plots. Although deflections and moments were obtained at each grid point shown in Fig. 2, values are presented only for points along grid lines 0-X, 1'-X, 8-X and 12-X, which represent positions along the beam lines and along the midspan of each panel. For each of these grid lines, figures are included which show the slab deflections along the grid line, the slab moments \( m_x \) along the grid line, and the slab moments \( m_y \) across the grid line. Additional figures show the flexural and torsional moments in both the interior and exterior beams for each case considered.

On each figure, values of the beam and column stiffness parameters are specified and the reader's attention is called to the interpretation of these values. The values of \( H, J \) and \( K \) given on any figure are those for the interior beams and the interior columns. It must be remembered that the flexural and torsional stiffness ratios for the exterior beams are five-eighths of those given in the figure title for the interior beams. Similarly the values of the corresponding effective stiffness ratios for the corner and edge columns are less than those given for an interior column; for the corner column this effective stiffness ratio is 0.2\( K \) about either axis; for the edge column it is 0.3\( K \) for bending about an axis parallel to the edge and 0.7\( K \) for bending about an axis perpendicular to the edge.
4.2 Presentation of Material

The figures are divided into two main groups, one relating to the presentation of the results, and the other to the interpretation of the results. The presentation of the results is organized as follows:

The slab deflections and bending moments, for all combinations of $H$ and $J$ and for the condition of all panels loaded, are plotted in Figs. 7-42. To facilitate reference to these figures during discussion, they are arranged according to location in the slab system, showing the results along line $0-X$, $4'-X$, $8-X$ and $12-X$, in that order. For each line, the figures are arranged in order of ascending values of $H$, and for each value of $H$ in order of ascending values of $J$.

The slab deflections and bending moments for the checkerboard loading are presented in Figs. 43-50. The arrangement of these figures is the same as for the condition of all panels loaded. To facilitate comparisons of the two patterns of loading, the deflections and moments for all panels loaded for the extreme values of column stiffness are also shown in Figs. 43-50 as dashed lines.

When interpreting these figures the reader should keep in mind the method of analysis used to determine the slab moments. The moments $m_x$ shown are the average unit moments along a strip of slab one grid interval in width, centered about the grid line indicated. The moments $m_y$ are the average unit moments across this grid line over one grid interval centered at the point considered.

The flexural moments in the beams for the condition of all panels loaded are given in Figs. 51-59. These are again arranged in order of
ascending values of \( H \), and for each value of \( H \) in order of ascending values of \( J \). The flexural moments for the condition of partial loading are given in Figs. 60 and 61. Similarly, the torsional moments in the beams for the condition of all panels loaded are presented in Figs. 62-70, and for the condition of partial loading in Figs. 71 and 72.

To assist in interpreting the results, plots were prepared showing the slab moments at certain points as a function of the parameter \( K \), for particular values of \( H \) and \( J \). These plots are contained in Figs. 73-107. Plots showing the effect of \( K \) on the beam torsional moments are given in Figs. 108-115. In order to show the relationship between moment and column stiffness more clearly, values of \( K \) are plotted in these figures to an arithmetic scale. Thus this relationship cannot be represented continuously to include very large values of \( K \). Therefore, the values of the moments corresponding to \( K = \infty \) are shown as short, horizontal lines to the right of the plots. The curves approach these values asymptotically.
5. EFFECTS OF STIFFNESS PARAMETERS

5.1 Introduction

In this chapter the effects of the stiffness parameters on the magnitudes and distributions of the deflections and moments in the slab and its supporting beams are discussed. The stiffness parameters varied were the beam flexural stiffness ratio, \( H \), the beam torsional stiffness ratio, \( J \) and the column stiffness ratio, \( K \).

The discussion is organized under the main divisions of deflections, slab moments, beam flexural and beam torsional moments. In each division, the effects of stiffness parameters are discussed followed by an explanation of these effects. This discussion is made first for the condition of all panels loaded and then for the condition of partial loading.

5.2 Deflections

The slab deflections are presented in Figs. 7-42 for all panels loaded, and in Figs. 43-50 for the checkerboard loading.

The slab deflections were determined from the difference operators in terms of \( q_h / N \) but are reported in terms of \( q_b / N \) so that the units would be independent of the grid spacing. Downward deflections are considered positive.

The deflections are the result of the loading and the interaction between the beam and column parameters. The influence of these factors on the slab deflections at specified locations of the slab system is presented first. Following this discussion is an explanation of this behavior.
(a) **Discussion of Deflections**

An examination of the Figs. 7-42 shows the beam flexural stiffness, $H$, has a marked influence on the magnitude of the deflections but little influence on the configuration of the deflections. Along both the interior and exterior beam lines, the deflections are doubled as the flexural stiffness ratio, $H$, is reduced from 2.5 to 1.0, and are doubled again as $H$ is reduced from 1.0 to 0.25. The effect of $H$ on the magnitude is not as pronounced along the midspan lines 0-X and 8-X. Here the magnitude of the deflection is doubled as $H$ is reduced from 2.5 to 0.25, and the deflection corresponding to $H = 1.0$ is about midway between these extreme values. For corresponding values of the column stiffness, $K$, and the beam torsional stiffness $J$, the configuration of the deflected slab is similar for all values of $H$.

The same trend is noted for the condition of checkerboard loading. Along the beam lines, the deflections for $H = J = 0.25$ are from three to four times those for $H = J = 2.5$. Along the midspan grid lines 0-X and 8-X the deflections for the more flexible beams are about two times the deflections for the stiffer beams.

The beam torsional stiffness ratio, $J$, has little effect on the magnitude or configuration of the deflections except adjacent to the edge beams. In this region, an increased $J$ will cause a reduction in the slope of the slab adjacent to the beam. This effect is most noticeable with high values of $K$.

For given values of $H$ and $J$, the column stiffness ratio, $K$, has considerable effect on both the magnitude and configuration of the
deflection. An examination of the figures shows that for \( K = \infty \) the maximum deflections at the center of each panel are nearly equal and that a decrease in \( K \) results in a marked increase in the deflections in the corner panels and a decrease in the deflections in the interior panel.

Along the edge beams adjacent to the corner panels the deflections for a value of \( K = 0 \) are about two times the deflections for \( K = \infty \), for all values of \( H \) and \( J \). However, along the beam adjacent to the edge panel, the deflections for \( K = \infty \) are about two times the deflections at \( K = 0 \) for \( H = 0.25 \) and about four times greater for \( H = 2.5 \) (Figs. 16-24 and 34-42).

Along line 8-X (Figs. 25-33), the deflections corresponding to large values of \( K \) are always less than those for small values. The deflection at the center of the corner panel for \( K = \infty \) is about one-half that for \( K = 0 \), and the deflection at the center of the edge panel for \( K = \infty \) is three-quarters of that for \( K = 0 \).

The effects of \( K \) along line 0-X (Figs. 7-15) are similar to those along the beam lines except that the effects are not as pronounced. The deflection at the center of the edge panel for \( K = \infty \) is about three-quarters of that for \( K = 0 \) and the deflection at the center of the interior panel for \( K = \infty \) is about twice that for \( K = 0 \).

For the checkerboard loading and small values of the beam stiffnesses, values of \( K \) greater than 10 have little effect on the magnitude of the deflections. In Fig. 49, for example, with \( H = J = 0.25 \) there is a greater difference in deflection for \( K = 0 \) and \( K = 10 \) than exists for \( K = 10 \) and \( K = \infty \). However, for \( H = J = 2.5 \), Fig. 50, the deflections for the \( K \) values shown are about equally spaced between \( K = 0 \) and \( K = \infty \).
(b) **Explanation of Deflections**

The changes in the magnitudes and configurations of the deflections might be best explained by comparing the results with the deflections obtained for a strip of slab considered as a uniformly loaded beam, continuous over three equal spans. In considering such a strip through the slab for all panels loaded, first consider all supports fixed with the analogous beam in a horizontal position. For this case, each span will appear as a beam with both ends built in and the maximum deflection in each span will be equal. This condition is closely approximated along the beam lines with $K = \infty$ and for strips at midspan such as $8-K$ if $J$ is high (Figs. 27, 30 and 33). The equality of deflections is reasonably satisfied even for small values of $J$ (Figs. 25, 28, and 31), although the fixed end conditions do not exist at the edge.

As the supports of the beam are relaxed, corresponding to a decrease in $K$, the deflections in the end spans increase and those in the middle span decrease. This is clearly shown in all figures by a marked increase in the deflections of the corner panels and a decrease in the deflections of the interior panel. As would be expected, there is a conflict of effects in the edge panel. For strips parallel to the edge, points in the edge panel deflect as points in the middle span of the analogous beam; for strips perpendicular to the edge, they deflect as points in the end span. Points along both beam lines adjacent to the edge panel behave as points on a middle span; that is, a decreasing $K$ value results in a decreasing deflection. Points in the interior of the edge panel begin to behave as points in the end span of a beam; that is, an increasing deflection with decreasing $K$ but with a reduced
rate of increase for further decreases in K. Points near the beam lines will behave as points on the end span for K large but, as K decreases, will reverse direction and behave like points on a middle span. For low values of H and J, this reversal of direction with reduced K extends about b/8 into the panel and for high values of H and J about b/4 into the panel from each beam line. Along a line through midspan, such as 8-X, the slower rate of increased deflection with reduced K is shown by the bunching of the deflection curves in the edge panel as shown in Figs. 25-33.

The analogy can be extended to show qualitative effects for the case of partial loading. Here, strips of the slab behave as continuous beams which are loaded in corresponding alternate spans. However, the deflections along the line of an interior beam, see Figs. 45 and 46, indicate that the behavior along this line is similar to that of a three-span beam with all panels loaded.

5.3 Slab Moments

The slab moments are presented in Figs. 7-42 for all panels loaded and in Figs. 43-50 for the checkerboard loading.

The moments are presented in terms of qb^2 and so represent the moments across a unit width of slab. Moments are considered positive when they produce compressive stresses in the top face of the slab.

The slab moments are divided into two main groups; moments m_x along a grid line parallel to the x-axis, and moments m_y across this grid line. Since the magnitude and distribution of the slab moments depend on the combined effects of the beam and column stiffness ratios, it is not
practical to discuss the effects of each stiffness ratio separately. Therefore, the effects of these ratios are discussed for each moment along or across a particular grid line.

(a) Slab Moments Along Line 12-X

The slab moments along the edge, line 12-X, are plotted in Figs. 34-42 as slab bending moments $m_x$.

An examination of these figures indicates that the beam flexural stiffness, $H$, has a great influence on the magnitude of the slab moments but a relatively insignificant effect on their distribution. For all panels loaded, the moments in the slab are doubled as $H$ is reduced from 2.5 to 1.0 and are doubled again as $H$ is reduced from 1.0 to 0.25, for constant values of $J$ and $K$. A similar reduction occurs with checkerboard loading, the moments increasing about four times as $H$ (and $J$) are reduced from 2.5 to 0.25. This increase in slab moment with reduced $H$ is explained by examining the moment-curvature expression for the beam. A reduced beam stiffness for constant curvature would cause a decrease in the moment capacity of the beams and thus result in the transfer of part of the total moment to the slab. Therefore, the curvature of the system does not remain constant but increases with decrease in beam stiffness; thus the increase in the slab moment is not linear with the decrease in beam stiffness.

The torsional stiffness of the edge beam, $J$, has relatively little effect on the magnitude of the slab moments along the edge, as shown in Figs. 40-42. As $J$ is increased from 0.25 to 2.5, the negative and positive moments are increased by about 10 percent due to the more torsionally stiff beam attracting additional load to the edge.
For given values of H and J and all panels loaded, the magnitude and distribution of the moments along the edge, particularly in the corner panel, are sensitive to changes in the column stiffness ratio K. At the face of the corner column, the moment can vary from a small positive to a negative value comparable to the negative moment at the face of the edge column. At midspan of the corner panel, location (8,12), the positive moment is approximately doubled as K is reduced from infinity to zero. The distribution of these moments are shown in Figs. 34-42 and the variation of the moments with K in Figs. 101-106. The distribution of the moments along line 12-X in the region of the edge panel is independent of K but the magnitude of the positive moment at midspan is about halved as K is reduced from infinity to zero. The changes in the moment in this region due to variation of K are shown in Figs. 87-99.

The effect of K on the moments along the edge of the slab can be explained by comparing the action of a strip of the slab adjacent to the edge to that of a three-span continuous beam. If all of the supports could be considered as hinges, corresponding to K = 0, the moment at the end would be zero. However, for higher values of H the moment in the slab at this point is positive. This is due to the fact that the loaded corner panel, as it deflects, tends to rotate the edge beam perpendicular to line 12-X about an axis parallel to the edge. This rotation develops a torsional moment in the edge beam which is transmitted to the corner column. Since the corner column cannot resist this moment, a positive end moment is produced in the edge beam along line 12-X. As the stiffness of the edge beam is increased, the torsional moment developed becomes
greater and so induces a higher positive moment at the column face. This effect is shown in Figs. 40-42. As the stiffness of the corner column increases, the torsional moment developed is resisted directly by the column and the moment at the face of the corner column becomes negative and increases rapidly with increased column stiffness. This effect is shown in Figs. 101-106. It must be remembered when examining these figures that the value of $K$ given refers to an interior column and the effective stiffness ratio is $0.2K$ for the corner column.

In accordance with the behavior of the three-span beams, as moments are applied to the ends, the positive moments in the end spans and the negative moments at the faces of interior supports are decreased. Figures 40-42 show about a 50 percent reduction in these positive moments and about a 25 percent reduction in the negative moments as $K$ increases from zero to infinity. A decrease in the negative end moments at the face of the edge column causes a constant positive increment to be applied to the moments in the middle span. This is clearly shown in Figs. 34-42.

A different behavior was noted for the checkerboard loading. As the stiffness of the corner column is increased, the negative moment at the face of the corner column increases and the positive moment at location $(8,12)$ decreases, as shown in Fig. 107. However at the face of the edge beam, location $(4',12)$ the moment continues to increase with increase in $K$, rather than decrease as was the case for all panels loaded. This increase in negative moment is shown in Figs. 49 and 50. The moment along the edge of the edge panel is nearly uniform in magnitude for $H = J = 2.5$ and becomes positive at the ends for high values of $K$ and low values of $H$ and $J$. 
The explanation for the apparently peculiar effects in the vicinity of the edge column with checkerboard loading is as follows: From loading the corner and interior panels but not the edge panels, there is a strong tendency to rotate the interior beam and the edge column about an axis perpendicular to the edge. When the column has zero stiffness, there is no resistance to this rotation and the behavior of the edge is similar to that of a three-span beam on knife supports and load only on the end spans. A nearly constant negative moment is induced in the middle span. However, as the column stiffness is increased, this rotation is decreased. This has the effect of increasing the negative moment at the column face adjacent to the loaded edge panel and decreasing the negative moment at the column face adjacent to the unloaded panel. For high values of $K$ and low values of $H$ and $J$, the rotation of the interior beam is sufficient to produce positive moments in the edge slab in the region adjacent to the edge column. As $H$ and $J$ are increased, the forces causing this rotation are resisted in part by the increased stiffness of the beams and the effect is thus reduced. Also, since there is less leakage of moment into the unloaded edge panel when the beams are stiff, the moments induced are more nearly uniformly distributed across this panel.

(b) Slab Moments Along Line 8-X

The slab moments $m_x$ along grid line 8-X are plotted in Figs. 25-33, 47 and 48.

The moments along this interior line are not as sensitive to changes in $H$ and $K$, but are more sensitive to $J$, than were the moments along the edge. Effects of the parameters are discussed at specific points.
At the exterior beam, location (12, 8), the magnitudes of the moments vary appreciably with changes in the stiffness ratios. For the condition of all panels loaded and for small values of the beam stiffness ratios, H and J, the moments at this point are also small, being positive for small values of K and negative for larger values of K, as shown in Fig. 25. When H is kept at 0.25 and J is increased to 2.5, Figs. 25-27, the positive moment for K = 0 increases from +0.0060 to +0.0165 $q b^2$ or an increase of 1.75 times; and the negative moment for $K = \infty$ increases from -0.0037 to -0.0257 $q b^2$ or an increase of 5.4 times. However, for constant J, the change in moments from increasing H is small for $K = \infty$ compared to the change for $K = 0$. Although the trend is the same for all values of J, consider the effect for $J = 2.5$ as H is increased from 0.25 to 2.5.

Referring to Figs. 27, 30 and 33, the moment for $K = 0$ is changed from +0.0165 to -0.0195 $q b^2$, a change of 0.0360 $q b^2$, while the moment for $K = \infty$ is changed from -0.0257 to -0.0330 $q b^2$, a much smaller change of 0.0073 $q b^2$. The difference between moments for different K values is thus seen to be much less for high values of H than for low values of H. An examination of Figs. 31-33 shows a reverse trend for J, the effect of K being less for low values of J than for high values of J.

These effects can be explained by considering the behavior of the slab system in the region of the corner panel. For very flexible beams, and for columns with no rotational resistance, the deflections in the beams adjacent to the corner panel are large. As the beams parallel to the x-axis deflect, they cause a rotation of the edge and corner columns, which in turn rotate the edge beam between them. With flexible edge beams, the deflection
of the slab along line 8-X is nearly uniform and thus the slab does not tend to rotate the edge beam as much as do the columns. This results in a positive moment being induced in the slab. As the torsional stiffness of the beams is increased, more of the column rotational effect is transferred to midspan of the beam and the positive moment is increased. However, increasing the flexural stiffness of the beams tends to decrease the slab deflections and prevent rotation of the columns. This effect is the same as adding stiffness to the columns. As the columns become stiffer, the edge beam is restricted from rotating with the slab and so a couple is developed along the edge which produces negative moments in the slab. When the columns are infinitely stiff, the rotation of the edge beam is a minimum and the negative moments in the slab are a maximum. Similarly, as J is increased for stiff columns, the rotation of the edge beam at midspan is reduced which results in higher negative slab moments.

The positive moments at the center of the corner panel vary considerably with changes in H, J and K. For all panels loaded and K = 0, a reduction in H from 2.5 to 0.25 will increase the moments about 60 percent for J = 0.25 and 70 percent for J = 2.5, and the effect of increasing J is small. On the other hand, for K = ∞, increasing J from 0.25 to 2.5 decreases the moments about 30 percent for H = 0.25 and about 15 percent for H = 2.5. However, the effect of H is reduced; as H is decreased from 2.5 to 0.25, the moment is increased about 30 percent for all values of J. From Figs. 87-93, the effect of K is seen to be greater for small values of H and large values of J. When H = 2.5 and J = 0.25 there is a 20 percent increase in the moment as K is reduced from infinity to zero. This increase is 40 percent when H = J = 0.25 or when H = J = 2.5 and is a maximum of 85 percent when H = 0.25 and J = 2.5.
This behavior is due to the following action of the corner panel. As the stiffness of the beams is increased, they carry a greater portion of the load and so reduce the moments in the slab. At the center of the slab this reduction is much less than in the slab adjacent to the beams. If the columns have zero stiffness, increasing J will cause the rotation of the edge to be more uniform but will have little effect on the amount of rotation and therefore little effect of the moment in the center of the panel. However if the columns are stiff, increasing J will decrease the amount of rotation at midspan of the edge beam and thus apply a negative couple along the edge of the slab which reduces the positive moments in the slab. The effects of K are more apparent for low values of H since stiffer beams tend to prevent rotation of the columns and so mask the effects of the column stiffness.

The negative moments at location (4', 8) for all panels loaded are reduced about 25 percent as H is reduced from 2.5 to 0.25, which is apparent from Figs. 25, 28 and 31. The effect of J on these moments is small. As J is increased from 0.25 to 2.5, the negative moments, for H = 2.5, are reduced about 6 percent, but, for H = 0.25, are increased about 3 percent. The effects of K are shown in Figs. 80-86, and are small for all values of H and J, generally being less than 10 percent except for a maximum effect of 15 percent for H = 0.25 and J = 2.5. Although the variation in magnitude is small with change in K, the pattern of variation is not uniform. For all values of H and J, and for K = 0, the moment at beam face 4 is greater than at face 4'. As K is increased the negative moment at face 4 is reduced and the negative moment at face 4' is increased, such that at K = 10 the moment at face 4 is less than that at 4'. For further increase in K, the moment at
both faces tends to reduce such that at \( K = \infty \) the moment at face 4' is nearly the same as at \( K = 0 \) and the moment at face 4 is less than the moment at face 4'. A detailed explanation of this distribution is given in subsection (g), slab moments across line 4'-X.

For the condition of checkerboard loading, the negative moments at face 4' are decreased by 0.014 \( qb^2 \) and at face 4 are increased by 0.011 \( qb^2 \) as \( H \) and \( J \) are decreased from 2.5 to 0.25. The difference results in increased torsional moment in the beam. This change in moment at the interior beam is due to the beam exerting an increased resisting couple on the slab as the torsional stiffness of the beam is increased. With the stiffer beams there is less effect from the loaded panels and the moments in the edge panel are thus distributed almost linearly across the panel. For more flexible beams this distribution is less uniform.

The positive moments at the center of the edge panel, location (0,8), are increased as \( H \) is decreased; this increase depends on the values of \( J \) and \( K \). As \( H \) is decreased from 2.5 to 0.25 but \( J \) kept constant at 0.25, this increase is 40 percent for \( K = 0 \) but only 10 percent for \( K = \infty \). For the same decrease in \( H \) but with \( J = 2.5 \), this increase is 30 percent for \( K = 0 \) and 20 percent for \( K = \infty \). Similarly the effect of \( J \), although small, is dependent on \( H \) and \( K \). For \( H = 0.25 \), increasing \( J \) from 0.25 to 2.5 increases the moment about 6 percent for \( K = 0 \) but decreases the moment slightly for \( K = \infty \). For \( H = 2.5 \), increasing \( J \) from 0.25 to 2.5 decreases the moment about 10 percent for all values of \( K \). The effect of increasing \( K \) from zero to infinity is greater for low values of \( J \). For \( H = 0.25 \) the moments are increased by 150 percent as \( K \) is increased from zero to infinity but for \( H = 2.5 \), this increase is only 35 percent.
This change in moment with $H$ and $K$ is due to the following behavior of the slab system. When the columns have no stiffness and the beams are flexible, the system tends to rotate uniformly about the edge causing a large, almost uniform deflection along $\delta$-$X$ as shown in Fig. 25. The edge panel tends to behave as a wide beam which is simply supported along the beam lines 4-$X$ and 12-$X$, and thus carries most of the load by one-way action. This results in a small moment $m_x$ along line $\delta$-$X$. As the stiffness of the beams or of the columns is increased, this rotation is reduced and more of the load is carried in the $x$-direction, causing $m_x$ to increase. Also, a resisting couple is generated along the interior beam perpendicular to line $\delta$-$X$ as the column stiffness is increased which results in a further increase in the moment at midspan of the edge panel. The lower the values of $H$, the greater the effect of increasing the column stiffness has on reducing the rotations of the interior beams and thus increasing $m_x$ at the center of the edge panel.

The moments across the edge panel along line $\delta$-$X$ are nearly uniform for the checkerboard loading. This is due to the panel behaving as if it had a couple applied along the edges due to this loading. The negative moments are greater for $K = 0$ and low values of $H$ and $J$, since the smaller beam stiffness allows a greater couple to be applied along the interior beams. For larger $K$ values this couple is resisted in part by the columns.

(c) Slab Moments Along Line 4'-X

The slab moments $m_x$ along the interior beam line 4'-X are presented in Figs. 16-24 and 45-46.

In general the moments in the slab adjacent to the interior beam are distributed similarly to those of a uniformly loaded three-span beam.
For the condition of all panels loaded, the moments, except in the region of the edge column, are approximately doubled as \( H \) is reduced from 2.5 to 1.0 and are doubled again as \( H \) is reduced from 1.0 to 0.25.

The moments at the face of the edge column, location (12,4'), vary greatly with changes in all the parameters. When \( K = 0 \), reducing \( H \) from 2.5 to 0.25, for \( J = 0.25 \), changes the moment from +0.0009 to -0.0030 \( qb^2 \), but for \( J = 2.5 \) changes the moment from +0.0030 to -0.0237 \( qb^2 \). However when \( K = \infty \) the effect of reducing \( H \) from 2.5 to 0.25 is increased; for \( J = 0.25 \) from -0.0241 to -0.1274 \( qb^2 \) and for \( J = 2.5 \) from -0.0207 to -0.1038 \( qb^2 \). An examination of these moment values will show that increasing \( J \) from 0.25 to 2.5, for \( K = 0 \), will increase the negative moment when \( H = 0.25 \) and increase the positive moment when \( H = 2.5 \) and for \( K = \infty \) will decrease the negative moments for all values of \( H \). A further examination shows that the greatest effect on the moments at the internal face of the edge column is caused by changing \( K \). Because of symmetry these changes in the slab moments with change in \( K \) are the same as slab moments \( m_y \) at location (4',12) and are shown as such in Figs. 94-100. As \( K \) is increased there is a major increase in the negative moment, particularly for small values of \( H \).

This behavior is due to the following action of the slab system. As the slab is loaded, the beams perpendicular to the edge deflect and tend to rotate the columns about an axis parallel to the edge. These in turn rotate the edge beam. However for flexible beams, say \( H = 0.25 \), the deflection of the edge beam and the slab at midspan are approximately equal and there is little tendency for the slab to rotate the edge beam. Therefore, where the column stiffness is small, say \( K = 0 \), the column is unable to resist rotation and the resisting couple is generated in the middle of the
edge spans by the action of the slab on the beam. This resisting couple is transferred through torsion in the beam to the column causing a negative moment at the interior face of the edge column. As the torsional stiffness of the edge beam is increased, this resisting couple is increased and thus the negative moment at the column face is increased. As the flexural stiffness of the beams is increased, there is less tendency for the slab to rotate the edge beam. For a value of $H = 1.0$, the tendency for rotation at both locations is almost equal with the result that the edge rotates as a unit similar to a hinged edge. This results in almost zero moment at the column face as shown in Figs. 19 to 21. As $H$ is increased further, say to 2.5, the beams perpendicular to the edge offer a greater resistance to rotation of the edge about an axis parallel to the edge than does the slab and thus causes a positive moment to be induced at the column face. Adding torsional stiffness to the edge beam will cause this positive moment to increase.

However, as the column stiffness is increased the behavior changes since the resistance to rotation is now provided by the column. Again, for low values of $H$ there is a greater tendency for the interior beam to rotate the edge column but since this rotation is resisted directly by the column a larger negative moment is developed in the immediate vicinity of the column face. This moment can be very large and has a value of $-0.1274 q b^2$ for $K = \infty$ and $H = J = 0.25$. As $J$ is increased, this moment is spread out by the edge beam over a greater width of slab and thus causes the moment at the column face to decrease. This decrease is about 20 percent as $J$ increases from 0.25 to 2.5. When $H$ is increased to 2.5 there is less tendency for the beams to rotate the column, and so the negative moment at the column face is
further decreased. For $H = 2.5$ and $J = 0.25$ this moment for $K = \infty$ is reduced to $-0.0241 \text{ qb}^2$.

The positive moments $m_x$ at location $(\delta, \theta')$ are dependent mainly on the beam flexural stiffness $H$ and the negative moment existing at the face of the edge column. As $H$ is decreased from a value of 2.5 to 1.0, the moments are essentially doubled and are doubled again as $H$ is further decreased from a value of 1.0 to 0.25. This increase in moment is due to the decreased moment capacity of the beam as its flexural stiffness is reduced.

The changes in moments $m_x$ at location $(\delta, \theta')$ with variation of $K$ are shown in Figs. 80-86 as moments $m_y$ at location $(\theta', \delta)$. For any value of $H$ and $J$, the moments are about twice as large for $K = 0$ as for $K = \infty$. The larger positive moment for $K = 0$ is due to the smaller value of the negative end moment applied at the face of the edge column for this column stiffness. As the column stiffness is increased, the magnitude of the negative end moment is increased, and the positive midspan moment is reduced. An increase in the beam torsional stiffness $J$ has almost no effect on the magnitude of the moment for small values of $K$, but causes a small reduction in magnitude for large values of $K$.

The negative slab moments at the interior column are mainly dependent on the beam flexural stiffness $H$. As $H$ is decreased from 2.5 to 1.0, these slab moments are slightly more than doubled, and as $H$ is reduced from 1.0 to 0.25 are slightly more than doubled again. This is again due to the decreased moment capacity of the beams with decreased stiffness.
The moments at the exterior face of the interior column are independent of $J$ when the column has no stiffness; however, for stiffer columns, these moments are reduced as $J$ increases. For $K = \infty$, an increase in $J$ from 0.25 to 2.5 will reduce the negative moment at location $(4',4')$ about 10 percent. This reduction is due to the moment in the slab being distributed over a larger width from the column which reduces the maximum value at the column face. The moments at the interior face, location $(4,4)$ are essentially independent of $J$ for all values of $K$.

At the interior face of the interior columns the negative moment is a maximum for a column stiffness of $K = 0$ and is reduced as $K$ increases. This is the behavior which would be expected for this location if the moments along line $4'-X$ were compared with those for a three-span beam and the supports had an increasing resistance to rotation. The distribution and magnitude of the moments in the interior span of line $4'-X$ are consistent with this beam analogy; that is, the distribution of moment is the same as that for a beam with both ends fixed, and the magnitude is increased by a positive increment constant over the span as the end moments are decreased.

However, the negative moments at the exterior face of the interior column do not decrease regularly with increase in the column stiffness. As $K$ is increased from $K = 0$, the negative moment increases rapidly at first and then decreases for larger values of $K$. This trend can be seen more clearly in Figs. 73-78. The explanation of this behavior is given in detail in subsection (g), slab moments across line $4'-X$.

The behavior of the slab along line $4'-X$ for the checkerboard loading is very similar to that for all panels loaded. The effect of a
reversal of direction in the moment at the exterior face of an interior column as the column stiffness is increased is even more pronounced, and the reversal occurs at a higher value of the column stiffness as shown in Fig. 79.

(d) Slab Moments Along Line O-X

The slab moments \( m_x \) along line O-X are given in Figs. 7-5 and 43-44.

An examination of these figures for the condition of all panels loaded indicates that the slab moments along line O-X are similar in magnitude and distribution to the moments along line 8-X. In general, the slab moments along the midspans of the panels are much less sensitive to changes in the beam flexural stiffness, \( H \), than are the slab moments along the beam lines. A detailed effect of the parameters on the slab moments is given for specific locations along line O-X in the following paragraphs.

Moments at the edge of the edge panel, location (12,0), are dependent on the degree of rotation of the edge beam; the smaller the rotation, the greater the moment developed. For low values of \( H \), small values of the column stiffness will cause a positive moment at the edge. Comparing Figs. 7-9, this positive moment is seen to increase with an increase in the beam torsional stiffness \( J \). For \( H = 0.25 \) and \( K = 0 \), an increase in \( J \) from 0.25 to 2.5 will increase the moment from \(-0.0071\) to \(-0.0171\) \( q b^2 \), an increase of 140 percent. As the column stiffness is increased, the moment becomes negative and increases greatly with increase in beam torsional stiffness. Again for \( H = 0.25 \), but \( K = \infty \), an increase in
J from 0.25 to 2.5 will increase the moment from -0.0031 to -0.0255 \( gb^2 \), an increase of 750 percent.

A similar behavior occurs for changes in \( K \) and \( J \) for large values of \( H \), except that the moment is negative for all values of \( K \). When \( H = 2.5 \), the increase in the negative edge moment as \( J \) is increased from 0.25 to 2.5 is 230 percent for \( K = 0 \) and 290 percent for \( K = \infty \).

The changes in moments \( m_x \) at location \((12,0)\) as a result of changes in the column stiffness are shown in Figs. 87-93. Due to symmetry these changes are the same as those for moments \( m_y \) at location \((0,12)\). For all values of the beam stiffness parameters, the moments are increased in a negative sense as the column stiffness is increased. This effect is greatest for low values of \( H \) and high values of \( J \).

This behavior is explained by considering the rotation of the edge beam. For low values of beam and column stiffness, the beams perpendicular to the edge cause the edge columns to rotate about an axis along the edge more than the slab rotates the edge beam at midspan. This produces a positive moment at midspan of the edge. As the torsional stiffness of the edge beam increases, more of the rotating couple is transferred to the midspan of the edge and this moment is increased. For higher values of \( H \) and \( K \) the rotation of the edge columns is less than the rotation of the edge beam at midspan. This causes a negative moment to be developed in the slab at the midspan of the edge because the slab tries to rotate the edge beam. This negative moment increases with increased stiffness of the beams or columns since both of these effects will decrease the rotation of the columns and hence the rotation of the edge beam.
At the center of the edge panel, location \((8,0)\), the positive moments are much less sensitive to variation of \(K\) and \(J\) than were the moments at the edge. The effect of increasing \(J\) is to reduce slightly the positive moments. This reduction is greater for low values of \(H\) and high values of \(K\), but the maximum for \(H = 0.25\) and \(K = \infty\) is only 12 percent less. The effect of decreasing \(H\) is to increase the positive moment, the amount of this increase being dependent on the column stiffness. As \(H\) is decreased from 2.5 to 0.25, the positive moments are increased by 85 percent for \(K = 0\) but only about half this much for \(K = \infty\). Similarly, the effects on the slab moments as a result of changes in column stiffness are dependent on the flexural stiffness as shown in Figs. 80-86. Due to the symmetry of the slab system these changes are the same as for the moments \(m_y\) at location \((0,8)\). An increase in the column stiffness \(K\) from zero to infinity will decrease the positive moment about 45 percent for \(H = 0.25\) and about 30 percent for \(H = 2.5\).

The positive moments are increased as the flexural stiffness of the beams is decreased owing to the more flexible beams carrying less of the load. This increase is also a function of the column stiffness since the effective beam stiffness in the end spans is a function of the end moment applied by the edge column.

The negative moments across the interior beam, location \((4',0)\), are relatively insensitive to changes in the beam and column parameters for the condition of all panels loaded. A decrease in \(H\) from 2.5 to 0.25 reduces the negative slab moment by less than 5 percent for \(K = 0\) and about 15 percent for \(K = \infty\), while a variation of the beam torsional stiffness has
a negligible effect for all values of K. An increase in K, (Figs. 73-79) from zero to infinity will reduce the negative slab moment by about 20 percent for $H = 0.25$ and by less than 10 percent for $H = 2.5$.

The positive moments at the center of the interior panel are influenced primarily by changes in the column stiffness. An increase in the column stiffness from zero to infinity will increase the positive moment by 85 percent for $H = 0.25$ and 30 percent for $H = 2.5$. The effect of varying $H$ is dependent on the column stiffness. For $K = 0$, decreasing $H$ from 2.5 to 0.25 decreases the moment by 12 percent, but for $K = \infty$ this decrease in $H$ increases the moment by 25 percent. This behavior is due to the fact that small values of $H$ and $K$ allow the greatest rotations of the interior beams and columns, which cause a maximum amount of the load to be carried by negative moments along the edge of the interior panel. Thus the positive moments at the center of the panel are a minimum for these stiffness values. However, for large values of $H$ and $K$, these rotations are reduced and more of the load on the interior panel is carried by the positive moments at midspan. For the case of stiff columns, a reduction in the beam stiffness increases the portion of load carried by the slab and thus increases the positive slab moments.

For the condition of checkerboard loading, the behavior of the slab along line 0-X is quite different from that described above. Referring to Figs. 43 and 44, it can be seen that the moments across the loaded interior panel are distributed in a manner similar to those for a fixed ended beam subjected to a uniform load. However, the distribution of the moments across the unloaded edge panel, especially for small values of the column and beam stiffnesses are unique.
The behavior of this edge panel can be understood by considering the effect of the checkerboard loading on the slab system. The edge beam behaves like a continuous three-span beam loaded only on the end spans. This causes a large upward deflection at midspan of the middle span as shown by the deflections in Figs. 43 and 50. Strips of the slab parallel to the edge also behave in this fashion, as shown in Fig. 47, but since the interior beams, which constitute the supports for the three-span beam analogy, deflect appreciably the resulting deflections along line 0-X in the edge panel are downward. This variation in deflection is such as to cause positive curvature which results in an increase in positive moment along line 0-X as one proceeds from the edge towards the middle of the edge panel. For small values of the column and beam stiffness, the edge columns tend to rotate and thus rotate the edge beam producing a positive moment along the edge of the slab. These effects, particularly for low values of the column stiffness, are apparent in the slab moments m_x shown in Fig. 43.

As one proceeds further from the edge, the increase in positive moment decreases and then becomes negative. The reason for this negative moment can be seen by considering a strip of slab along line 0-X. This strip will tend to behave as a three-span beam loaded only on the middle span and will thus have a linear negative moment in the end spans. The effect of this negative moment is greater nearer the interior beam and so causes a net negative moment in this region. However, considering such a strip along line 0-X requires a rotation of the interior beam between the interior and exterior panels. When the columns and beams are less stiff, this rotation is resisted by the rotation of the columns in the opposite
direction owing to the influence of the loading and the interior beams along line 4'-X. For stiff columns, the rotation is resisted by the columns themselves. In either case this resistance to rotation causes a small couple to be applied, through torsion in the beam, which tends to reduce the negative moment in the exterior panel and increase the negative moment in the interior panel. This effect is clearly shown in Fig. 43. For stiffer beams, as in Fig. 44, this behavior, although still apparent is much reduced in magnitude since more of the load is carried by the beams and the distribution across the edge panel is almost linear.

A comment on the difference in negative moment on each side of the beams along line 0-X as shown in Figs. 43 and 44, may be useful. The difference in these moments should be resisted by torsion in the interior beam. However reference to Figs. 71 and 72 will show that the torsional moment in the beam is zero at this point. The explanation for this apparent discrepancy comes from a review of the procedure used to determine the moments. The moments were determined from consideration of equilibrium for an analogous model over a finite width and thus the moments shown are the average moments over this width. An evaluation of the total torsional moment in the beam over a grid interval will show that it is equal to the difference in the unit moments over this same interval.

In the above discussion of slab moments along the grid lines, nothing has been said about the location of the points of inflection or points of zero moment. From the figures it can be observed that, except near the discontinuous edge, the positions of the points of inflection are almost identical for all values of the beam and column stiffness parameters. At
the discontinuous edge, the point of inflection can lie anywhere between the edge and about $3b/16$ from the edge, depending on the degree of rotation permitted at the edge.

(e) **Slab Moments Across Line 12-X**

The slab moments $m_y$ across the edge beam, line 12-X, are given in Figs. 34-42 and 49-50.

For the condition of all panels loaded, the magnitude and distribution of the slab moments across the edge are seen to vary considerably for changes in the values of the beam and column stiffness parameters. When the beam torsional stiffness is low ($J = 0.25$), the moment is concentrated in the vicinity of the columns. The magnitude of this moment is a function of both the beam flexural stiffness, $H$, and the column stiffness, $K$. For $K = 0$, a reduction of $H$ from 2.5 to 1.0 increases the negative slab moment at the column face about two times and a further reduction from 1.0 to 0.25 will cause a further increase of about two times. For $K = 30$, these increases, as $H$ is decreased, are about three times. For $K = 0$ the moments at the columns are small but change from a positive value for $H = 2.5$ to a negative value for $H = 0.25$. At locations along the edge not in the vicinity of the columns, the moments are almost uniform and relatively small. Figures 34-36 show the effect of increasing $J$ for a value of $H = 0.25$. The effect of increasing $J$ is to decrease the magnitude of the negative moment at the face of the column and to spread this moment over a greater width of slab. When $H = 1.0$, an increase in $J$ will cause an almost uniform distribution of moment as shown in Figs. 38 and 39. For high values of $H$, an increase in $J$ will increase the negative moment at midspan and decrease the negative
moment at the column. The effects on the moment at the face of the edge column produced by varying $K$ with given values of $H$ and $J$ are shown in Figs. 94-98, and at the face of the corner column in Figs. 101-105. The reduced effective stiffness of the edge column about an axis parallel to the edge should be considered when interpreting these figures.

The distribution of the slab moments across the edge is explained by the following behavior of the slab system. For low values of the beam stiffness ratios $H$ and $J$, an examination of the deflections and slopes in Figs. 16, 25 and 34 shows a greater tendency for the beams perpendicular to the edge to rotate the corner and edge columns about an axis parallel to the edge than for the slab to rotate the edge beam. For low values of the column stiffness, say $K = 0$, the column is unable to develop a couple capable of resisting this rotation. Therefore the columns rotate and so rotate the edge beam. However, since there is less tendency for the beam to rotate at midspan than at the column, this rotation is resisted by the slab at midspan which thus develops a positive moment in the slab in this region and a negative moment in the region of the column. This distribution is clearly shown in Figs. 34-36 for $K = 0$. As $J$ is increased, the magnitude of these moments is increased.

Increasing the flexural stiffness of the beams reduces the deflection along the beam lines and so reduces their tendency to rotate the columns, but increases the tendency of the slab to rotate the edge beam. This reduces the negative moment at midspan, as shown in Figs. 34, 37 and 40. Again, increasing $J$ causes the moments to be more uniformly distributed across the edge. If $H$ is increased to 1.0 the tendency for the beams to
rotate the columns and the slab to rotate the edge beam are almost equal.
Figure 39 shows an almost linear distribution of the moment along the edge for all values of K. A further increase in H will cause the tendency of the slab to rotate the beam to be greater than that of the beams to rotate the columns and thus a larger negative moment is developed in the slab at midspan of the edge beam, as shown in Fig. 42.

An identical behavior is seen to exist for the condition of the checkerboard loading. In Fig. 49, for low values of the beam stiffness, the negative moment at the column face is seen to increase rapidly as the column stiffness is increased, whereas in Fig. 50, for high values of the beam stiffness, the negative moments are greatest at midspan of the loaded panel. Along the unloaded panel the slab moments are almost zero for all values of K.

(f) Slab Moments Across Line 8-X

The slab moments $m_y$ across line 8-X at midspan are given in Figs. 25-33 and 47-48.

An examination of Figs. 25-33 for the condition of all panels loaded shows that the distribution of the slab moments across line 8-X is almost independent of J and K but is a function of H. For all values of H, the moment is reduced by a constant increment over the length of the slab as the column stiffness is increased. This is shown by the moment curves, for different values of K, being almost parallel across the slab.

For low values of H, the positive moments across line 8-X are greater at the beam lines than at midspan (Figs. 25-27), but for large values of H, the moments are greater at midspan (Figs. 31-33). The
explanation of this is closely tied to the distribution of moments across line 12-X. For low values of \( H \) the difference in deflection between the midspan and the ends for an interior beam is greater than the difference in deflection between the center of the edge or corner panels and the corresponding midspans of the edge beams. This causes a greater curvature along the beam lines and hence a greater moment across line 8-X in the vicinity of the beams. As \( H \) is increased to 1.0, the curvature of the beams is decreased to a value about equal to the curvature for a strip of slab between the beams. This causes an almost constant moment across line 8-X as shown in Figs. 28-30. A further increase in \( H \), say to 2.5, will decrease the curvature along the beams at a rate greater than along a strip between the beams, resulting in a greater slab moment between beam lines than adjacent to beam lines. This effect is shown in Figs. 31-33.

A comparison of Figs. 25, 28 and 31 shows a decrease in the slab moments as the beam flexural stiffness is increased as a result of more of the load being carried by the beams.

A similar behavior is noted for the condition of checkerboard loading, as shown in Figs. 47 and 48. For small values of the beam stiffness, the moments are reasonably linear over the loaded panel, being about 50 percent greater at the discontinuous edge than at the interior beam. Again greater deflections and hence greater curvatures exist with more flexible columns, resulting in higher positive moments for low values of the column stiffness. The moments across the unloaded edge panel show this same trend with changes in the column stiffness but are smaller in magnitude, being a minimum at midspan. For large values of the beam stiffness ratios (Fig. 48),
the slab moments are reduced, particularly in the vicinity adjacent to the beams. The maximum moment occurs at midspan of the loaded corner panel which is consistent with the behavior for all panels loaded. The moments are small and fairly uniformly distributed across the unloaded edge panel. Again there is an almost constant increment in moment across the entire slab system corresponding to changes in the column stiffness \( K \).

\[ (g) \quad \text{Slab Moments Across Line} \ 4'-X \]

The slab moments \( m_y \) across the interior beam at line \( 4'-X \) are given in Figs. 16-24 and 45-46.

The reader's attention is called to the distinction between lines \( 4-X \) and \( 4'-X \). The figures presented show the moments across line \( 4'-X \) which, as shown in Fig. 2, is the line passing along the exterior face of the columns and interior beam. The difference between these moments and the moments across line \( 4-X \) is small and nowhere greater than 10 percent for all panels loaded. The difference in these moments is also a measure of the change in torsional moment along the beam.

For the condition of all panels loaded, the distribution of slab moments across line \( 4'-X \) is similar to the distribution of moments across the edge beam. Due to the continuity of the slab across the interior beam, the effects of varying the column stiffness, \( K \), and the beam torsional stiffness, \( J \), are less. Low values of the beam flexural stiffness result in a greater curvature along the beams perpendicular to line \( 4'-X \) than exists at midspan and thus results in a larger moment adjacent to the columns. This is shown in Figs. 16-18. An increase in the beam flexural stiffness reduces the curvature at the column face more rapidly than at
midspan such that at $H = 1.0$ the moments are nearly constant across the interior beam. A further increase in the beam flexural stiffness so reduces the curvature at the column face that it is less than that at midspan, resulting in slab moments adjacent to the column being less than at midspan, as shown in Figs. 22-24.

The variation in moment across line $4'-X$ for different values of the column stiffness requires some explanation. An examination of the figures shows a crossing of the moment lines for different values of $K$ in the region of the interior columns and across the edge span. This behavior is particularly noticeable in Figs. 19-21 where $H = 1.0$. For example, the negative moment at the exterior face of the interior column, location $(4', 4')$, increases as the column stiffness is increased from $K = 0$. However, after the column stiffness reaches a particular value, usually between $K = 10$ and $K = 30$, a further increase in the column stiffness will decrease the negative moment at this location. This trend is shown in Figs. 73-78. A similar behavior is exhibited across line $4'-X$ over the width of the edge panel and at the face of the edge column.

The reason for this behavior lies in the action of the slab system. Under the influence of the loading there is a tendency for the interior column to rotate outward which reduces the negative moment at its outside face. For a column of zero stiffness, this rotation, and hence this reduction, is a maximum. As the column stiffness is increased, this rotation is restricted, resulting in an increase of the negative moments at the outside face. However, with larger values of the column stiffness, the negative moment at the inside face of the edge column (location $4', 12$), is increased.
which tends to decrease the moment at the outside face of the interior column (location \(4',4'\)). An increase in the column stiffness will increase the effective stiffness of the beams, which will also decrease the negative slab moments at the interior column. Therefore, increases in the column stiffness, \(K\), for larger values of \(K\) will cause a net decrease in the negative moments at the outside face of the interior columns.

Rotation of the edge column about an axis perpendicular to the edge causes a similar reduction in the negative moment \(m_y\) at the face of this column, location \((12,4')\). The changes in these moments as a result of changes in \(K\) are from symmetry the same as the changes for moments \(m_x\) at location \(4',12\) which are shown in Figs. 94-99. The reason for the apparent greater change in moment with change in \(K\) at this location, as \(K\) approaches infinity, especially for low values of \(H\) (Figs. 16-18), is the relatively small value of the effective stiffness of the corner column compared to the corresponding stiffness of the edge column about an axis perpendicular to the edge. The effect is less noticeable with high values of \(H\) since the stiffer beams resist rotation of the columns and thus mask the effects of \(K\).

The behavior of the slab moments across line \(4'-X\) between the interior and edge columns is a function of the amount of rotation of these columns about axes perpendicular to this line. Changes in the moments due to changes in the column stiffness are greater nearer the edge column, because continuity of the slab over the interior column makes rotation of this column less sensitive to changes in \(K\). For columns with zero stiffness, the rotation of the edge perpendicular to line \(4-X\) is a maximum which
causes the slab to try to deflect into the pattern of a cylindrical trough with its axis parallel to the edge. This results in a reduced curvature across the interior beam (line 4'-X) and hence reduced moment. As the column stiffness is increased, the deflection of the interior beam (line 4'-X) is decreased at a greater rate than the deflection of a strip of slab at midspan along line 3-X. This results in more of the load being carried parallel to the edge and thus an increase in the negative slab moment across line 4'-X. Although this effect is most noticeable for \( H = 1.0 \), Figs. 19-21, since for this case the moments are distributed almost linearly across this line, actually it occurs for all values of \( H \).

Figures 19-21 also show the effect of the beam torsional stiffness \( J \) on the distribution of the moments. For small \( J \) (Fig. 19) the effect of the stiff edge column is to limit the deflection of the interior beam in the vicinity of the column, but the small torsional stiffness in the edge beam does not limit the rotation, and hence the deflection, of the slab at midspan to the same degree. Therefore, more load is carried in the direction perpendicular to line 4'-X which results in a greater negative moment across the line as shown. However, for large \( J \), (Fig. 21) the deflection of the slab adjacent to the edge beam at midspan is reduced to a greater extent, and more of the load is transferred to the edge beam. This reduces the moment across line 4'-X between the columns but increases the negative moment at the face of the edge beam.

For the condition of checkerboard loading, the negative slab moments at the outside face of the interior column vary with changes in \( K \) in a fashion similar to that for all panels loaded as shown in Fig. 79.
However, the behavior at the face of the edge column for partial loading is slightly different, since the negative slab moment is a minimum for the condition of zero column stiffness, as shown in Fig. 45. This is due to the greater rotation of the edge column under the action of the checkerboard loading which reduces the negative moment at the column face.

(h) Slab Moments Across Line O-X

The slab moments $m_y$ across line O-X at midspan are given in Figs. 7-15 and 45-44.

For the condition of all panels loaded the distribution of the moments across line O-X is very similar to the distribution of the moments across line 8-X. As was the case along the line 8-X, the positive moments are greater adjacent to the beams for low values of $H$, are almost uniformly distributed over the entire slab width for $H = 1.0$, and are greater at midspan for high values of $H$. However, whereas across line 8-X the maximum moments occur with a column stiffness of zero, across line O-X the maximum moments occur with a column stiffness of infinity. This difference would be expected if a strip of slab parallel to the y-axis were compared to a continuous three-span beam. For knife-edge supports, corresponding to $K = 0$, the positive moments in the middle span (line 0-X) are smaller than those in the end span (line 8-X), whereas for fixed supports, corresponding to $K = \infty$, the magnitude and distribution of the moments in each span are almost identical.

The change in the slab moments across line O-X due to varying $K$, are given in Figs. 73-78 for locations at the center of the interior panel and at the interior beam line. An examination of these figures shows that
at the center of the interior panel, location (0,0), the moments are independent of J and are about 25 percent larger for H = 0.25 than for H = 2.5. For values of K greater than 30, there is almost no change in moment with change in K; but for lower values of K, a decrease in K results in a decrease in moment. This decrease is much less for higher values of H.

The slab moments at the interior beam, location (4'-0), are independent of J but are sensitive to changes in H. The moments are doubled as H is reduced from 2.5 to 1.0 and are doubled again as H is further reduced from 1.0 to 0.25. There is little change in moment due to varying K for values of K greater than 30, and a small decrease in moment with decrease in K for values of K less than 30.

The effects of K on the slab moments \( m_y \) at the center of the edge panel, location (8,0), are shown in Figs. 80-85. For small H, these moments are almost independent of J but for large H are about 10 percent greater for J = 0.25 than for J = 2.5. For values of K larger than 10, these moments are nearly constant for a given H, but are approximately 20 percent greater for H = 0.25 than for H = 2.5. However, for values of K less than 10, the moments are reduced as K is reduced. This reduction is greater for small H, such that at K = 0 the moment is greater for H = 2.5 than for H = 0.25. The reason for this greater decrease in moment for small values of K and H is the greater rotation of the edge, which results in the panels adjacent to the edge assuming a cylindrical configuration with the axis of the cylinder parallel to the edge. This reduces the curvature across line 0-X and increases it along line 0-X, resulting in a
reduced moment $m_y$ and an increased moment $m_x$. These changes are clearly shown in Figs. 7-9. For larger values of $H$ the rotation of the edge is more restrained and consequently this reduction in moment for smaller $K$ values is reduced.

Changes in the moments across line $O-X$ at the edge beam, with change in $K$, are shown in Figs. 87-92. For $K$ greater than 30 the moments are almost constant and for smaller $K$ the moments decrease with decrease in $K$. Because a more torsionally stiff edge beam will attract load to the edge, the moments are about 15 percent greater for $J = 2.5$ than for $J = 0.25$.

For the checkerboard loading, the moments $m_y$ across line $O-X$ are given in Figs. 43 and 44. Generally, the moments are positive in the loaded panel, and negative in the unloaded panel. Maximum positive moments in the loaded interior panel are about 50 percent greater for $H = J = 0.25$ than for $H = J = 2.5$, but the maximum negative moments in the unloaded edge panel are approximately three times greater for the smaller beam stiffness. The changes in these moments due to changes in $K$ are given in Figs. 79, 86 and 93. As was the case for all panels loaded, the moments are practically constant for values of $K$ greater than 30, and the moments decrease with a decrease in $K$ for values of $K$ less than 30.

5.4 Beam Flexural Moments

The flexural and torsional moments in the supporting beams are discussed separately since they vary independently, both in magnitude and distribution, as functions of the loading and stiffness parameters.

The beam flexural moments, $M$, for both the exterior and interior beams are plotted in Figs. 51-59 for all panels loaded and in Figs. 60-61.
for the checkerboard loading. The flexural moments presented are the total flexural moments in the beams and have units of \( q_b^3 \).

Owing to the nature of the analog used in this study to approximate the slab system, the deflections and curvatures of the beams are identical to those of the adjacent slab. Therefore, the flexural moments along the exterior beams are proportional to the slab moments \( m_x \) along line 12-X and the flexural moments along the interior beams are proportional to the slab moments \( m_x \) along line 4'-X. Thus the behavior of the beam flexural moments as a result of variations in the beam torsional stiffness, \( J \), and the column stiffness, \( K \), will be the same as the behavior of the slab moments along the corresponding grid lines, as discussed in detail in Section 5.3. In general, changes in \( J \) have a negligible effect on either the magnitude or the distribution of the beam flexural moments. For all panels loaded, the effect of changes in \( K \), especially for values of \( K \) greater than 10, on the beam moments in the middle spans is small for all values of \( H \) and \( J \). In the end spans, however, the positive moments for \( K = 0 \) are about 50 percent greater for low values of \( H \) and about 100 percent greater for high values of \( H \) than the corresponding moments for \( K = \infty \). The moments at the discontinuous ends are practically zero for \( K = 0 \) but, for \( K = \infty \), are as large as the negative moments at the continuous end. For checkerboard loading, the moments in the spans adjacent to loaded panels are similar in distribution to those for all panels loaded, but in the spans adjacent to unloaded panels the moments are small and nearly linear over the span.

An increase in the beam flexural stiffness, \( H \), decreases the curvature along the beam lines. This decrease in curvature has the effect...
of reducing the moments in the adjacent slab, (Section 5.3). However, since the flexural stiffness of the beam is increased at a rate greater than the rate of decrease in curvature, the moments in the beams increase as \( H \) increases. For the condition of all panels loaded, this increase in moments is about 100 percent as \( H \) is increased from 0.25 to 1.0, and about 25 percent as \( H \) is increased from 1.0 to 2.5, for all values of \( J \) and \( K \). For the checkerboard loading, the maximum moments for \( H = J = 2.5 \) are about two and one-half times greater than those for \( H = J = 0.25 \).

5.5 Beam Torsional Moments

The beam torsional moments, \( T \), for both the exterior and interior beams are plotted in Figs. 62-70 for all panels loaded and in Figs. 71 and 72 for the checkerboard loading.

The torsional moments are presented in terms of \( qb^3 \). The sign convention used is such that a positive torsional moment causes a decrease in slope across the beam as one proceeds in the positive direction of the axes. The directions of \( T \) shown in Figs. 4-6 are positive.

The torsional moments were obtained by determining the rotations of the beams between adjacent grid lines and multiplying these rotations by the torsional stiffness of the beam. Since the rotations obtained were the average rotations over the grid spacing, the moments obtained are the average torsional moments over this spacing. In plotting Figs. 62-72 a smooth curve was drawn such that the average ordinate to the curve over a grid interval was equal to the value computed for the middle of the interval. The beam torsional moment is zero at the lines of symmetry bisecting the nine-panel structure; that is, at locations \((0,4)\) and \((0,12)\).
In the following discussion a distinction is made between exterior and interior beams, and in each case, between the middle span and the end spans of the three-span beam. The positive end of the beam is the end in the positive coordinate direction when referring to the quadrant shown in Fig. 2.

(a) **Exterior Beam**

For the condition of all panels loaded, the torsional moments in the exterior beams are much greater than those in the interior beams for corresponding values of the stiffness parameters. The effect of the beam flexural stiffness ratio, $H$ on the torsional moments in the exterior beams depends on the values of $J$ and $K$. For $K = 0$, values of $H$ less than 1.0 will cause positive torsional moments at the positive end of the beam spans, a value of $H = 1.0$ will cause almost no torsional moment in the edge beam, and values of $H$ greater than 1.0 will cause negative torsional moment at the positive end of the beam spans. For larger values of $K$, the torsional moments at the positive end of the beam spans are negative for all values of $H$.

At the positive end of the middle span (grid interval 3-4) for $K = 0$, the torsional moments go from positive to negative as $H$ is increased from 0.25 to 2.5, but for $K = \infty$ this change in $H$ causes the negative moments to decrease. This decrease for $K = \infty$ is such that the moments for $H = 2.5$ are about one-half of those for $H = 0.25$ for all values of $J$. The effect of increasing $J$ from 0.25 to 2.5, for all values of $K$, is to increase the moments about two times for $H = 0.25$ and four times for $H = 2.5$.

A similar behavior occurs at the negative end of the end span (grid interval 4'-5). Here, for $K = 0$, the moments are small and change
from negative to positive values as H is increased from 0.25 to 2.5. For K = \infty the moments are positive for all H, and an increase in H from 0.25 to 2.5 will reduce the moments by approximately one-half. An increase in J from 0.25 to 2.5 will increase the moments about two times for H = 0.25 and three times for H = 2.5.

At the positive end of the end span (grid interval 11-12), for K = \infty, the moments are almost numerically equal to those at the negative end of the span (grid interval 4'-5) except, of course, the signs are reversed. For K = 0, the moments for H = 0.25 are almost independent of J and have small positive values, but for H = 2.5 the moments are negative at the corner column, or positive end, and are about twice as large as those at the interior column, or negative end. These negative moments are tripled as J is increased from 0.25 to 2.5.

An examination of Figs. 62-70 shows that H has an important influence on the configuration of the torsional moment curves for the exterior beams. For low values of H (Figs. 62-64), the torsional moments increase at an increasing rate as one approaches the columns for all values of K except zero. For H = 1.0 (Figs. 65-67), the torsional moment curves are almost straight lines in any span for all values of K. For H = 2.5 (Figs. 68-70), the moments increase with decreasing rate as one approaches the columns for all values of K.

For the exterior beam the effect of an increase in the column stiffness K is to increase the magnitude of the torsional moments at each end of the beam spans. For higher values of H, an increase in K from zero to infinity increases the magnitude of the torsional moments at the end of
the beam spans about four times. For low values of \( H \) the moments reverse
sign between \( K = 0 \) and \( K = 10 \) and the moments at \( K = \infty \) are about three
times the magnitude of those for \( K = 10 \).

The above behavior of the torsional moments can be explained as
follows: For \( K = 0 \), and \( H \) less than 1.0, the cross-beams tend to rotate
the edge more than does the slab at midspan, which results in positive
moments at the positive ends of the edge beam spans. At \( H = 1 \), the tend­
ency of the cross-beams and slab to rotate the edge is almost equal, which
results in almost no torsional moment being developed in the edge beam.
For values of \( H \) greater than 1.0, the slab has a greater tendency to rotate
the edge beam, which results in negative moments at the positive ends of
the beam spans. For \( K = \infty \), the rotation of the edge caused by the cross­
beams is resisted directly by the columns which causes the same effect as
a greater rotation at midspan, namely negative moments at the positive
ends of the beam spans. For larger values of \( H \), the beams carry more of
the load, which reduces the tendency of the slab to rotate the edge beam.
Since the stiffer beams also restrain the column from rotating, the torsional
moments are reduced as \( H \) is increased.

An increase in the torsional stiffness, for the same angle of
twist, would cause a linear increase in the torsional moment. However,
with greater values of \( J \), more load is carried by the cross-beams, which
reduces the angle of twist in the edge beam caused by the deflecting slab.
This results in the increase in torsional moment being not linear with
increase in \( J \).

The effect of \( H \) on the configuration of the torsional moment
curves can be explained by considering the rotation of the edge. For \( H \)
less than 1.0, the beam is subjected to a couple from the slab which is
distributed over most of the middle portion of the span. This couple is
resisted by a couple generated by the column. Since this resisting couple
is concentrated at the column, the rate of change of the angle of twist
in the beam is greater in this region, resulting in an increased rate of
moment increase. For small H and J (Fig. 62), the torsional moments over
the middle third of the spans are almost constant and equal to zero for
all K. However, for H greater than 1.0, the resisting couple is provided
in part by the cross-beams and adjacent slab and is therefore more evenly
distributed in the region of the column. This causes the rate of change
in the angle of twist to be less in the region adjacent to the columns,
which results in the rate of increase of the torsional moments to be less
at the columns than at midspan.

For all values of beam stiffness there is a rotation of the edge
at midspan. As the column stiffness is increased, the column offers
increased resistance to rotation of the edge at the column, which causes
an increase in the angle of twist in the beam. Therefore, an increase in
K will increase the torsional moments in the exterior beam.

The torsional moments in the exterior beam for checkerboard loading
are presented in Figs. 71 and 72. As expected, the moments are small in
the middle span adjacent to the unloaded edge panel. In the end span adjacent
to the loaded corner panel, the maximum torsional moments are about equal in
magnitude to those for all panels loaded, although for large values of the
beam stiffness the distribution for partial loadings is more sensitive to
changes in the stiffness of the columns.
For low values of $H$ and $J$ (Fig. 71), the torsional moments in the middle span of the exterior beam are positive near midspan but, for values of $K$ greater than zero, become negative near the positive end of the span. However, for high values of $H$ and $J$ (Fig. 72), the torsional moments in the positive half of the middle span are positive throughout, except adjacent to the edge column for very high values of $K$.

This behavior can be explained in terms of the action of the checkerboard loading on the slab. Consider a strip of slab perpendicular to the edge at midspan and compare it to a three-span continuous beam loaded only on the central span. The end spans of this analogous beam will deflect upward and thus tend to rotate the edge beam at the center of the middle span outward. For low values of $K$ the edge column rotates inward. These opposite rotations cause positive moments in the positive half of the middle span. However, as $K$ is increased, this inward rotation in the region of the column is restricted. Also the deflecting cross-beam allows the slab to deflect which rotates the edge beam in the vicinity of the column inward. This develops a negative torsional moment near the column. This negative moment is greater for small $H$ and $J$ as the more flexible beams permit the slab to cause a greater rotation of the edge beam.

In the end spans adjacent to the loaded corner panel, the torsional moments, for low values of $H$ and $J$ and all values of $K$ greater than about 10, are very similar to those for the case of all panels loaded. For $K = 0$, the inward rotation at the corner column is greater than at the edge column, which produces a positive moment throughout the span. For high values of $H$ and $J$, the torsional moment for $K = \infty$ is almost identical
for both types of loading. Comparison of Figs. 70 and 72 shows that the points of zero torsional moment for smaller values of $K$, compared to $K = \infty$, are closer to the edge column for all panels loaded and are closer to the corner column for the checkerboard loading. A further comparison shows that the configurations of the torsional moment curves in this span are similar for both loadings for corresponding values of $K$. The difference is that the partial loading case has an almost constant positive increment applied over the length of the span, which is greater for small values of $K$. This increment is due to the increased rotation of the edge column about an axis perpendicular to the edge, because of the unloaded edge panel. This in turn increases the inward rotation of the edge beams some amount at mid-span and a greater amount at the corner column. The inward rotation of the edge at the edge column is not increased significantly because the checkerboard loading does not cause as great a rotation of the interior column. With infinitely stiff columns, there is no column rotation, hence no transfer of moment to the edge, resulting in the torsional moments being almost equal for both loadings.

(b) Interior Beam

The torsional moments for the interior beam are presented in Figs. 62-70 for all panels loaded and in Figs. 71-72 for the checkerboard loading.

(1) All Panels Loaded

For the condition of all panels loaded, the torsional moments in the interior beams are small for all values of beam and column stiffness. In each case, the maximum values of the torsional moments occur adjacent
to the columns, which is the condition for the exterior beam. However, compared to the exterior beam, the sign of the moment is reversed; a positive moment is induced at the positive ends of the spans for all values of $K$ except zero.

The torsional moments are increased as $H$ is reduced. A reduction in $H$ from 2.5 to 1.0 approximately doubles the magnitude of the maximum torsional moments. A further reduction in $H$ from 1.0 to 0.25 again doubles these moments. On the other hand, the opposite effect is produced by changes in $J$; the torsional moments are increased as $J$ is increased. An increase in $J$ from 0.25 to 1.0 doubles the maximum moments and a further increase from 1.0 to 2.5 increases these moments by 50 percent.

The most notable characteristic of the curves plotted in Figs. 62-70 for the torsional moments in the interior beam is the apparently random distribution with changes in the column stiffness. For the edge beam, there is a steady increase in the torsional moment adjacent to the columns as the column stiffness was increased, but for the interior beam, the maximum moment occurs at intermediate values of the column stiffness. The changes in the average torsional moment with change in $K$ are shown in Figs. 108-110 for grid interval 3-4 and in Figs. 112-114 for grid interval 4'-5. These figures clearly show that the value of $K$ for which the average torsional moment is a maximum is a function of $H$. An increase in $H$ causes the maximum moment adjacent to the interior columns to occur with higher values of $K$.

Figures 108 and 112 show that for $H = 0.25$ the maximum torsional moments adjacent to the interior column occur for a value of $K$ less than 10.
The moment for values of K of zero and infinity are shown as short horizontal lines. The curve is not drawn continuously to K = 0 since intermediate values of the moment for values of K between 0 and 10 were not determined. The curves approach the values for K = ∞ asymptotically.

The maximum torsional moment in the interior beam over grid interval 3-4 occurs for H = 1.0 (Fig. 109) when K is between 10 and 20, and for H = 2.5 (Fig. 110) when K is between 40 and 50. The maximum moment over grid interval 4'-5 occurs for H = 1.0 (Fig. 113) when K is between 30 and 50, and for H = 2.5 (Fig. 114) when K is between 50 and 60. The value of K for which the beam torsional moment is a maximum is essentially independent of J.

This behavior of the beam torsional moments can be explained by the action of the slab system. Consider two strips of slab perpendicular to the interior beam, one through the middle of the edge and corner panels and the other through the interior columns. These strips can be compared to three-span continuous beams loaded on all spans. When K = 0, there is no resistance to rotation of these strips about their supports. The action of strips at midspan is to rotate the interior beam outward. Similarly the action of strips adjacent to the interior columns is to rotate the interior beam in this region outward. Since all strips rotate the interior beam in the same direction, the torsional moment developed is small since it is a function of the difference in amount of rotation. For K = 0, the beam adjacent to the columns is rotated slightly more than the beam at midspan, which causes a positive moment at the negative end of the end span (grid interval 4'-5).
As the column stiffness is increased from zero, there is a definite resistance to rotation of the beam at the columns. However, the rotation of the strip at midspan is only slightly affected. This difference in amount of rotation causes an angle of twist in the beam which in turn causes a torsional moment. As \( K \) is further increased, the rotation of the interior beam at the columns is even more restrained but the increased stiffness of the columns decreases the deflections of the corner panels and thus decreases the rotation of the interior beam at midspan. After a particular increase in \( K \), the decrease in rotation at midspan is greater than at the columns and the torsional moments generated by the difference in these rotations is reduced. At \( K = \infty \), there is no rotation of the interior beams at the columns, but the difference in deflections between the edge and corner panels is smallest for this value of \( K \), which means that the tendency for the interior beam at midspan to rotate is smallest. Therefore the difference in rotation along the beam is not great and the torsional moments for \( K = \infty \) are also small.

For low \( H \), the restraint to rotation of the interior beam at midspan due to an increase in \( K \) is small; thus a small increase in \( K \), which will restrict rotation at the columns, can cause a large torsional moment. For high \( H \), the stiffer beams have already restrained rotation of the columns and so mask the effect of small increases in \( K \). Therefore, higher values of \( H \) cause the maximum torsional moment to occur at higher values of \( K \).

(2) Checkerboard Loading

The torsional moments for the interior beam for the checkerboard loading are plotted in Figs. 71 and 72. The change in these moments due to variations in \( K \) are plotted in Figs. 111 and 115 for grid intervals adjacent to the interior column.
The interior beam torsional moments are substantially increased for the checkerboard loading as compared to the condition of all panels loaded. The effect of $h$ for partial loading is reversed from the effect for all panels loaded. Values of the torsional moment for $h = j = 2.5$ are from two to three times larger than for $h = j = 0.25$ in the regions adjacent to the interior column.

An examination of the figures for the middle span of the interior beam (Figs. 71, 72 and 111) shows that for small values of $h$ and $j$ the torsional moment is greatest for $k = 0$, is reduced about one-third as $k$ is increased to 10, and is practically constant for any further increase in $k$. The same trend holds for large values of $h$ and $j$; the moments are greatest for $k = 0$, are reduced about 20 percent as $k$ is increased to 10, and are practically constant for further increases in $k$.

For the end span of the interior beam (Figs. 71, 72 and 115), the distribution is completely different for low and high values of $h$. For $h = j = 0.25$, the moments for $k = 0$ are negative at the end adjacent to the interior column and decrease to approximately zero at the edge column, but for $k = 10$ they are negative at the end adjacent to the interior column and become positive at an increasing rate as the edge column is approached. Further increases in $k$ do not significantly change either the magnitude or distribution of the torsional moments from those for $k = 10$. However, for $h = j = 2.5$, the distribution is identical for all values of $k$ and the magnitudes have a constant negative increment for each decrease in $k$.

The explanation for the torsional moments in the interior beam due to partial loading is as follows: In the middle span, the loaded
interior span rotates the middle portion of the interior beam inward. For
K = 0, the loading causes the interior beam at the interior column to rotate
slightly outward. This induces a twist in the beam causing a negative
moment in the positive half of the middle span. As K is increased, the
outward rotation of the columns is restricted, causing a decrease in this
negative moment. For higher values of H and J (H = J = 2.5), this difference
in rotation between midspan and the ends adjacent to the interior columns is
reduced to about one-third of that for H = J = 0.25, but the ten fold increase
in the torsional stiffness ratio causes the torsional moments to be about
three times greater for the stiffer beams.

The torsional moments in the end span of the interior beam are
also sensitive to the loading. For low values of H and J, and for K = 0,
the interior column rotates slightly outward. However, the rotation of
the slab at midspan is also outward, but by an amount greater than that
at the interior column, which causes a negative moment in the end span.
Since, for K = 0 the edge column rotates about an axis perpendicular to the
edge nearly the same amount as the slab at the positive end of the end span,
the angle of twist, and thus the torsional moment, is almost zero (Fig. 71).
Again, for low values of H and J but higher values of K, the rotation of
the interior column and the rotation of the beam at midspan are only slightly
reduced. However, an increase in K greatly restricts the rotation of the
edge column, and produces a rapid increase in the positive moment in the
end span adjacent to the edge column.

For higher values of H and J the effects are similar. With K = ∞
there is no rotation of the columns and, since there is little leakage of
load from other panels, the beam in the end span is subjected to a rather uniform twist outward owing to the edge panel being unloaded. This results in negative moments adjacent to the interior column and almost equal positive moments adjacent to the edge column (Fig. 72). As $K$ is reduced, the rotation of the interior column is only slightly affected but the rotation of the edge column about an axis perpendicular to the edge is greatly increased.

In a similar fashion, a decrease in $K$ causes an increased rotation of the interior beam over the span because of the smaller restraint to the slab from the more flexible columns. This increase in rotation from the interior column to the edge column is almost linear which causes an almost constant negative increment of the torsional moments over the span.
6. EFFECTS OF PARTIAL LOADING

6.1 Types of Partial Loading

In many cases the maximum deflections and moments in continuous slab-beam systems are produced by partial loadings; that is, a loading in which only some of the panels are loaded. Two patterns of partial loading are of primary interest in analyses, strip loadings and checkerboard loadings. For strip loading on the nine-panel slab considered, the load would be applied to rows of three panels in various combinations depending on whether maximum positive or negative moments were desired. For checkerboard loading, the load would be applied either to the even numbered or odd numbered panels shown in Fig. 1.

In this study, the partial loading considered was restricted to a checkerboard pattern since this type of loading retains the degree of symmetry used previously. Load was applied to the interior and corner panels; that is, panels 1, 3, 5, 7 and 9 in Fig. 1. The analyses were restricted to two combinations of beam flexural and torsional stiffness; namely, flexible beams \( H = J = 0.25 \) and stiff beams \( H = J = 2.5 \). A comparison of the deflection and moments for the checkerboard loading with those for all panels loaded is made in the following sections.

6.2 Comparison of Deflections

The deflections for sections at midspan of the panels and along the beam lines for the checkerboard loading are given in Figs. 43-50. To aid in comparison, the deflections for all panels loaded for the extreme values of column stiffness, zero and infinity, have been added to the figures as dashed lines.
The effect of the checkerboard loading is to cause positive (downward) deflections in the loaded panels, and small or negative deflections in the unloaded panels. In general, the positive deflections at midspan of the loaded panels and in the end spans of the exterior beam are approximately equal for both patterns of loading. On the other hand, the positive deflections of the interior beam for checkerboard loading are approximately one-half of these deflections for all panels loaded.

At the center of the loaded interior panel, location (0,0) in Figs. 43-44, the difference in deflection between the two patterns of loading is nowhere greater than 14 percent. For $K = 0$, the deflections for the checkerboard loading are 5 percent greater for flexible beams and 14 percent greater for stiff beams, but for $K = \infty$, the deflections are 12 percent less for flexible beams and 4 percent less for stiff beams as compared to the deflections for all panels loaded. However, the deflections at the midspan of the adjacent beams, location $(0,1/4)$ in Figs. 45-46, are significantly less for the checkerboard loading than for all panels loaded. For $K = \infty$ they are 50 percent less and for $K = 0$, they are 80 percent less for $H = J = 0.25$ and have reverse directions for $H = J = 2.5$. The deflections for $K = 0$ are small; therefore, these differences, although large percentagewise, are small numerically and thus not important.

The deflections at the center of the loaded corner panel, location $(8,8)$ in Figs. 47-48, are only slightly less for checkerboard loading than for all panels loaded. For the flexible beams the difference is less than 5 percent and for the stiff beams it is less than 2 percent.
The deflections of the interior beam adjacent to the corner panel, location (8,4) in Figs. 45-46, are for K = 0, 40 percent less and for K = \infty, 50 percent less for the checkerboard loading than for all panels loaded. However, the deflections for the exterior beam adjacent to this panel, location (8,12) in Figs. 49-50, for checkerboard loading are 21-26 percent greater for K = 0 and less than 7 percent greater for K = \infty than for all panels loaded.

For all values of the stiffness parameters, the deflections for the checkerboard loading at the center of the unloaded edge panel, location (0,8) in Figs. 48-49, are essentially zero, and at the midspan of the adjacent edge beam, location (0,12) in Figs. 49-50, the deflections are negative.

The differences in deflections for the two patterns of loading can be explained by considering the deflections and rotations of the beams. For the loaded interior panel, the deflection at the center relative to the deflection at midspan of the adjacent beams is greater for the checkerboard loading, because of the reduced restraint to rotation at the panel edge. However, since the deflections of the interior beams are reduced, owing to the edge panels being unloaded, the net deflections at the center of the panel are approximately equal for both loadings. A similar action takes place in the corner panel. Again, owing to the reduced restraint to rotation of the interior beams, the difference in deflections between the interior beams and the center of the panel is increased; but, owing to the reduced loaded area, the deflection of these beams is substantially reduced. The deflections of the edge beam adjacent to the corner panel are
greater for checkerboard loading than for all panels loaded, particularly for flexible columns, because of the increased rotation of the edge column about an axis perpendicular to the edge. For infinitely stiff columns, this rotation is zero and the difference is negligible for stiff beams but about 7 percent for flexible beams owing to the greater rotation of the interior beam in the vicinity of the edge column.

6.3 **Comparison of Slab Moments**

The slab moments are presented in Figs. 7-42 for the condition of all panels loaded and in Figs. 43-50 for the condition of partial loading. To aid comparison, moments for all panels loaded for K equal to zero and infinity are shown in the latter figures as dashed lines.

In general the maximum moments in continuous slabs will be produced by some pattern of partial loading rather than by loading all panels simultaneously, and the particular pattern of partial loading will depend on the location of the moment to be made a maximum, on the geometry of the structure, and on the relative stiffness of the slab, beams, and columns. In this study, analyses have been made for only one pattern of partial loading, the checkerboard loading in the odd-numbered panels described previously. Consequently, the moments obtained for this loading are not necessarily the maximums which could be obtained. For the structure with flexible beams (H = J = 0.25) the checkerboard loading used probably will not produce maximum moments at any point in the slab; since the slab tends to act like a beam three panels wide, strip loadings would produce moments greater than those given by the checkerboard pattern used. Nevertheless, the checkerboard loading does produce moments different from and sometimes
greater than those caused by loading all panels. For the structure with the stiffer beams \((H = J = 2.5)\) the checkerboard loading used should produce maximum positive moments in the slab and also maximum negative moments at the discontinuous edges. Nevertheless, it must be kept in mind that the comparisons which follow are very limited in scope and are valuable chiefly as a qualitative indication of the effect of partial loadings on the particular slabs and relative stiffness considered.

(a) Slab Moment Along Line 12-X

The slab moments \(m_x\) along the edge for the two patterns of loading are given in Figs. 49-50. At the face of the corner column, location \((12,12)\) for \(H = J = 2.5\), the small positive moments for \(K = 0\) are even smaller for checkerboard loading, but for \(K = \infty\) there is no significant difference. For \(H = J = 0.25\), the negative moments at this face for \(K = 0\) are very small for both loadings. For \(K = \infty\), the moments for checkerboard loading are only 5 percent larger than those for all panels loaded. At location \((8,12)\), for \(H = J = 2.5\), the positive moments for checkerboard loading are 20 percent greater for \(K = 0\) but about the same for \(K = \infty\). For \(H = J = 0.25\), these moments are 15 percent greater for \(K = 0\) and 12 percent greater for \(K = \infty\). At the outside face of the interior column, location \((4',12)\), for stiff beams, the negative moments for checkerboard loading are approximately 50 percent less for \(K = 0\) but are slightly greater for \(K = \infty\). For the flexible beams, the moments at this location are again approximately 50 percent less for \(K = 0\) but 12 percent greater for \(K = \infty\).

At location \((0,12)\) in the unloaded panel, for checkerboard loading the moments are negative whereas they were positive for all panels.
loaded. This difference is greatest for low values of $H$, $J$ and $K$; for $H = J = 0.25$ they are $-0.0255 \, q^2 b^2$ as compared to $+0.0249 \, q^2 b^2$ for $K = 0$, and $-0.0053 \, q^2 b^2$ as compared to $+0.0323 \, q^2 b^2$ for $K = \infty$. For the stiff beams these differences are smaller, since the initial positive moments for all panels loaded are considerably less.

This change in behavior is easily understood by considering the rotation of the edge column. With the checkerboard loading, a strip along the edge behaves as a three-span beam loaded only on the end spans and for small column stiffness the rotation of the edge column about an axis perpendicular to the edge is large. Such rotation in the loaded corner panel will cause a much greater negative moment at the corner column, a greater positive moment at the middle of the end span and a much smaller negative moment at the face of the edge column. In the unloaded edge panel, the moments are less, because of the removal of the load, and sizeable negative moments are induced because of the edge column rotation. For $K = \infty$, the columns do not rotate for either loading and there is thus almost no change in the loaded corner panel and practically zero moments in the unloaded edge panel, particularly for the system with stiff beams.

(b) Slab Moments Along $8-X$

The slab moments $m_x$ along line $8-X$ for both patterns of loading are given in Figs. 47-48. At the center of the loaded corner panel, location $(8,8)$, and at the edge, location $(12,8)$, the slab moments $m_x$ for checkerboard loading are not significantly different from those for all panels loaded for corresponding beam and column stiffness. For checkerboard loading, the negative moments at the outside face of the interior
beam, location (4',8), are 30 percent less for \( H = J = 0.25 \) and 10 percent less for \( H = J = 2.5 \), but at the inside face, location (4,8), are 50 percent less for \( H = J = 0.25 \) and 80-90 percent less for \( H = J = 2.5 \) for all values of \( K \). At the center of the unloaded edge panel, the moments for all values of the stiffnesses are negative for partial loading and positive for all panels loaded. For \( K \) large, these negative moments produced by the checkerboard loading are very much smaller than the corresponding positive moments for all panels loaded; however, for flexible beams and zero column stiffness, the negative moment is larger than the positive moment. For \( K = 0 \), the moments for checkerboard loading are \(-0.0282 \, qb^2\) compared to \(+0.0099 \, qb^2\) for \( H = J = 0.25 \) and \(-0.0087 \, qb^2\) compared to \(+0.0154 \, qb^2\) for \( H = J = 2.5 \).

For checkerboard loading, the negative moments at location (4',8) are less for the more flexible beams because the reduced beam stiffness and the greater twist from the unbalanced loading causes a greater rotation of the beam, which in turn reduces the negative moment at the outside face. However, this rotation causes an increase in the negative moment at the inside face, location (4,8), such that a sizeable negative moment exists across the unloaded panel. As the stiffness of the beams is increased, this rotation is decreased which results in less reduction of moment at the outside face and a greatly reduced moment at the inside face. The rotation of the interior beams is also responsible for the large negative moments at the center of the edge panel for the case of flexible beams and columns. The magnitude of this moment is substantially reduced when the column is made rigid, which restricts beam rotation, and very much reduced when the interior beams are also made stiffer.
(c) Slab Moments Along Line 4'-X

The slab moments $m_x$ along the interior beam line for the two patterns of loading are given in Figs. 45-46. Although the distribution of these moments is approximately the same for both patterns of loading, there is a substantial difference in their magnitudes. For $H = J = 2.5$, the moments are 50 percent less for the checkerboard loading than for all panels loaded. For $H = J = 0.25$ the moments for checkerboard loading are again 50 percent less except at location $(0,4)$ where, for $K = 0$, they are 70 percent less.

This behavior is readily explained by the action of the slab subjected to the checkerboard loading. Each span of the interior beam receives load from one-half of a panel, which would suggest behavior identical to that for all panels loaded except that the moments would be only one-half as large. When the beams and columns are stiff, there is little leakage of load from loaded to unloaded panels and this behavior is closely realized. It is also realized for columns of low stiffness, particularly near the interior column which has little tendency to rotate due to the checkerboard loading.

(d) Slab Moments Along Line 0-X

The slab moments $m_x$ along line 0-X are given in Figs. 43-44. For checkerboard loading these moments are greater at midspan and less at the edge of the loaded interior panel than for all panels loaded. For checkerboard loading, the positive moments at location $(0,0)$ are 50 percent greater for $H = J = 0.25$ and $K = 0$ and 5 percent greater for $H = J = 2.5$ and $K = \infty$, while the negative moments at the inside face of
the interior beam, location (4,0), are 40 percent less for the flexible beams and columns and 10 percent less for the stiffer beams and columns. The negative moments at the outside face of the interior beam, location (4',0) are less since the edge panel is not loaded. At the center of the unloaded edge panel, location (8,0), the moments for partial loading are practically zero for stiff beams and columns and the positive moments for flexible beams and columns are much less than those for all panels loaded. The moments at the edge, location (12,0), are about one-half as great for checkerboard loading as for all panels loaded for K = 0 and are essentially zero for K = ∞.

This behavior is explained by considering the amount of rotation of the interior beam between the interior and edge panels. For partial loading, there is a greater tendency for the loading to rotate the interior beam, which will increase the positive moments in the middle of the interior panel and reduce the negative moments at the inside face of the beam. These increases and reductions are greater for the more flexible beams and columns since the rotations are greater for these conditions. For infinitely stiff beams and columns, the moments in the unloaded edge panel should be zero, a condition very nearly attained with H = J = 2.5 and K = ∞. For more flexible beams and K = 0, the rotation of the interior beam induces some negative moment at the outside face of the interior beam.

(e) Slab Moments Across Line 12-X

The slab moments m across the edge are given in Figs. 34-42 and 49-50 for both patterns of loading. Except at the face of the corner column, these moments are less for the checkerboard loading than for all panels.
loaded. At the face of the corner column, location (12,12), the moments are similar for K = \infty but are larger for K = 0, particularly for flexible beams. Since the moments are small at this location for K = 0, this difference, although large percentagewise, is not important. At midspan of the edge of the loaded corner panel, location (8,12), there is no change in moment between the two loadings.

For K = \infty, the moments m_y at the face of the edge column are large and for the checkerboard loading are about 50 percent less than for all panels loaded. For K = 0, these moments are much smaller and are even less for partial loading. At the midspan of the edge of the unloaded edge panel, the moments are practically zero for K = \infty and have only small positive values for K = 0.

The explanation for this behavior is as follows: Except in the region adjacent to the edge column, the curvatures, and hence the moments m_y across the edge are not appreciably affected by unloading the edge panels. The load on the beam along line \( \frac{a}{2} \) is reduced by approximately one-half because of the reduced contributing loaded area for the checkerboard loading, which reduces the moments to about one-half the corresponding values for all panels loaded.

(f) Slab Moments Across Line 8-X

The slab moments m_y across the midspan line 8-X are shown in Figs. 47-48 as continuous lines for checkerboard loading and dashed lines for all panels loaded.

The moments at the center of the loaded edge panel, location (8,8), are slightly greater for the checkerboard loading than for all panels
loaded; however, the difference is less than 5 percent. From this location to the edge, grid intervals 8 to 12 in Figs. 47-48, the moments for checkerboard loading are greater. At the edge, location (8,12), they are from 14-19 percent greater for \( K = 0 \) and from 3-6 percent greater for \( K = \infty \) than for all panels loaded.

At the interior beam, location \((4',8)\) the moments for the checkerboard loading are 42 percent less for \( K = 0 \) and 50 percent less for \( K = \infty \), than for all panels loaded. Across the unloaded panel, the moments for checkerboard loading are significantly less than for all panels loaded, particularly for the stiff beams and columns.

The reason for the greater positive moment at the edge for the more flexible beams and columns is the greater rotation of the edge column about an axis perpendicular to the edge for partial loading, which reduces the negative moment at the outside column face and increases the moment at midspan. Also, for the more flexible elements, the deflections of the end span of the interior beams are greater, which tends to cause a more uniform deflection along line 8-X. This causes the moments at the interior beams for the checkerboard loading to be less than one-half those for all panels loaded because the moment is resisted in part by the slab in the unloaded edge panel. However, for stiff beams and columns, this deflection of the interior beams is greatly reduced, which all but prevents the unloaded edge panel from deflecting and thus developing moment.

(g) Slab Moments Across Line 4'-X

The slab moments \( m_y \) across the outside face of the interior beam are given in Figs. 16-24 and 45-46. As expected, these moments are generally
less for checkerboard loading than for all panels loaded. At the edge of
the unloaded edge panel, location \((0, 4')\), the moments for checkerboard
loading for \(H = J = 0.25\) are about one-fourth of those for all panels
loaded; for \(H = J = 2.5\), the moments are essentially zero for partial
loading. At the outside face of the interior column, location \((\frac{1}{4}, 4')\),
the moments are one-half of those for all panels loaded for all values of
beam and column stiffness. At location \((8, 4')\), the moments for partial
loading are 30 percent less for \(H = J = 0.25\) and 10 percent less for
\(H = J = 2.5\). At the face of the edge column, the moments are reduced by
one-half for \(K = 0\) but are increased about 10 percent for \(K = \infty\).

This reduction in negative moment at the face of the beam adjacent
to the loaded panels is due to the rotation of the beam. This reduction is
small at the interior column since there is little rotation of this column
under the partial loading. However, since the rotation of the edge column
about an axis perpendicular to the edge is appreciable for \(K = 0\), the
reduction in moment at the face of the edge column is greatly reduced.
This reduction does not occur for \(K = \infty\) as the column does not rotate.
Furthermore, rotation of the interior beam causes a slight increase in the
negative moments at this location.

(h) Slab Moments Across Line 4-X

For the condition of all panels loaded, the moments across the
inside face of the interior beam, line 4-X, do not vary from those across
the outside face, line 4'-X, by more than 10 percent and generally con-
siderably less. However, for checkerboard loading, there is loading on
one side of the interior beam only which, especially for stiffer beams,
causes a considerable unbalance in moments from one side to the other. The moments across \( y = \frac{1}{4} \times \) for the checkerboard loading are everywhere less than the corresponding moments for all panels loaded. This reduction is greater for flexible beams and columns since the rotation of the supporting beams helps to reduce the maximum negative moments across the face of the beam.

(i) **Slab Moments Across Line \( O-X \)**

The slab moments \( m_y \) across the midspan line \( O-X \) are given in Figs. 7-15 and 43-44. These moments are considerably different for partial loading as compared to those for all panels loaded, particularly for the more flexible beams and columns. At the center of the loaded interior panel, location \((0,0)\), the positive moments for checkerboard loading are 50 percent greater for \( H = J = 0.25 \) and \( K = 0 \) but only 5 percent greater for \( H = J = 2.5 \) and \( K = \infty \). At the interior beam, location \((\frac{1}{4},0)\), the positive moments are about 70 percent less for \( K = 0 \) and 50 percent less for \( K = \infty \).

A more noticeable difference occurs in the unloaded edge panel where the moments \( m_y \) are negative for checkerboard loading but were positive for all panels loaded. At location \((0,8)\), for \( H = J = 0.25 \) and \( K = 0 \), the moment \( m_y \) is \(+0.0099 \, qb^2\) for all panels loaded compared to \(-0.0282 \, qb^2\) for checkerboard loading. As the beam and column stiffness is increased, the magnitude of this negative moment for checkerboard loading is decreased.

The same trend holds for the edge at location \((0,12)\). For the flexible beams and \( K = 0 \), the moment \( m_y \) is \(+0.0249 \, qb^2\) for all panels loaded compared to \(-0.0255 \, qb^2\) for checkerboard loading. However, for the stiffer beams this negative moment for partial loading decreases with increase in column stiffness such that at \( K = \infty \) this moment is essentially zero.
6.4 Comparison of Beam Flexural Moments

The beam flexural moments for all panels loaded are given in Figs. 51-59 and for the checkerboard loading in Figs. 60 and 61.

The pattern of checkerboard loading considered in this investigation will not give the maximum beam flexural moments for the interior beams since both panels adjacent to the beam are not loaded. However, the positive moments for the edge beams may be reasonably close to the maximum moments, especially for the stiffer beams. For convenience, the comparison of the beam flexural moments for partial loading to those for all panels loaded is made separately, first for the exterior beams and then for the interior beams.

(a) Exterior Beams

In general, the magnitudes and distributions of the beam flexural moments in the end span of the exterior beam adjacent to the loaded corner panels are similar for both patterns of loading. The moments at the face of the corner column, location (12,12), for \( K = \infty \), are about the same for checkerboard loading as for all panels loaded; for \( K = 0 \), the difference in these moments, although large percentagewise, are not important since the moments are small. The positive moments at the middle of the end span, location (8,12), are about 20 percent greater for \( K = 0 \) and about 5 percent greater for \( K = \infty \) for checkerboard loading than for all panels loaded. However, for \( K = 0 \) the maximum moment for all panels loaded occurs not at the midspan of the end span but at about one-eighth the span closer to the corner column. Thus the difference in the maximum positive moments in the end span for \( K = 0 \) is only about 10 percent. This shift in position of maximum
positive moments does not occur for \( K = \infty \). The negative moments in the end span at the face of the edge column, location \((\frac{4}{4}',12)\), for \( K = 0 \) are about one-half as great for checkerboard loading as for all panels loaded. However, for \( K = \infty \), these moments are the same for both patterns of loading.

At the center of the middle span adjacent to the unloaded panel, location \((0,12)\), the moments are positive for all panels loaded and negative for checkerboard loading. The negative moments for \( K = \infty \) are practically zero but those for \( K = 0 \) are sometimes larger than the corresponding positive moments for all panels loaded. For example, for \( H = J = 2.5 \) and \( K = 0 \), the moments for partial loading are \(-0.0145 \text{ lb}^2\) as compared to \(+0.0065 \text{ lb}^2\) for all panels loaded.

The magnitudes and distributions of the flexural moments in the exterior beams are dependent on the amount of rotation of the edge and corner columns about axes perpendicular to the edge. Since the edge panel is not loaded for checkerboard loading, the sum of the negative moments at faces 4 and 4' of the edge column is approximately one-half of this sum for all panels loaded. When the column has zero stiffness, the negative moment for partial loading is divided essentially evenly between the two faces as shown in Fig. 60. However, when \( K = \infty \), no rotation of the edge column occurs and the moment at the loaded face 4' is not reduced from its value for all panels loaded. In fact, for \( K = \infty \) and flexible beams the negative moment at the face adjacent to the loaded panel is slightly greater for the checkerboard loading than for all panels loaded owing to the torsional effects of the interior beam framing into the edge column.

The greater negative moment at face 4' of the edge column for \( K = 0 \) for all panels loaded shifts the position of maximum positive moment
in the end span further from the edge column for checkerboard loading. Since this negative moment is not significantly different for the two patterns of loading when \(K = \infty\), no such shift in the position of the maximum positive moments occurs.

(b) Interior Beams

The effect of the checkerboard loading on the interior beam is to reduce the amount of load carried on each span to about one-half the load carried for the condition of all panels loaded. Since the checkerboard loading causes little rotation of the interior column, irrespective of the column stiffness, the distribution will be similar to that for all panels loaded. Owing to the above behavior, the flexural moments in the interior beam for the checkerboard loading are approximately one-half the corresponding flexural moments for all panels loaded.

6.5 Comparison of Beam Torsional Moments

The beam torsional moments, \(T\), for all panels loaded are given in Figs. 62-70 and for the checkerboard loading in Figs. 71 and 72.

In general, the torsional moments in the end span of the exterior beams, adjacent to a loaded panel, are approximately the same for both patterns of loading. However, the maximum torsional moments in the interior beam for partial loading are approximately five times greater for \(H = J = 0.25\) and ten times greater for \(H = J = 2.5\) than for all panels loaded. The torsional moments for the two patterns of loading are compared in more detail in the following sub-sections:

(a) Exterior Beams

The maximum torsional moments in the end spans of the exterior beams occur for \(K = \infty\) and are about equal for both patterns of loading.
These moments are reduced for both loadings as $K$ is reduced. The torsional moments for partial loading for $K = 0$ have a positive increment over the span as compared to the moments for all panels loaded because of the increased rotation of the corner column for the partial loading.

In the middle span adjacent to the unloaded edge panel, for $H = J = 0.25$, the torsional moments for partial loading are about one-third those for all panels loaded. For $H = J = 2.5$ these torsional moments for partial loading have insignificantly small positive values, especially for stiff columns. The reason for these smaller torsional moments for the stiffer beams and columns is that the added restraint of these stiffer elements restricts the transfer of effects from the loaded to the unloaded panels.

(b) **Interior Beams**

When load is applied to all panels there is little tendency for the interior beams and columns to rotate. This results in very small torsional moments in the interior beams as compared to those in the exterior beam. However, for the pattern of checkerboard loading, each span of the interior beam has load on only one side, which causes maximum rotations and thus maximum angles of twist. This results in torsional moments of the same order of magnitude as in the exterior beams.

Since the alternate spans of the interior beam are loaded on alternate sides for the checkerboard loading, only small moments are produced in the interior columns. At the positive end of the middle span the moment is negative and at the positive end of the end span it is positive.

For the interior beam, the maximum torsional moments for checkerboard loading are about ten times greater, for $H = J = 2.5$, and about five
times greater, for \( H = J = 0.25 \), than the maximum moments for all panels loaded. There is also a considerable difference between the distribution of the torsional moments for the two types of loading, particularly for low values of the column stiffness. Except for small variations near the interior column, the moments for checkerboard loading vary consistently with changes in column stiffness. Near the interior column the negative torsional moments are greatest for \( K = 0 \) but near the edge column the positive moments are greatest for \( K = \infty \).

The increased torsional moments for the checkerboard loading are due to the increased rotation of the edge column and adjacent slab about an axis perpendicular to the edge. Since this rotation is greatest for small values of the column stiffness, the effects of this increase are greatest for \( K = 0 \).

6.6 Summary

The slab deflections and bending moments for the two patterns of loading considered are presented in Figs. 43-50. As discussed in Section 6.3, the differences in these quantities shown are not the maximum differences that can occur.

The deflections at the center of the interior panel are greater for checkerboard loading than for all panels loaded by 14 percent for \( K = 0 \) and \( H = J = 2.5 \). At the center of the end span of the exterior beam, the deflection for checkerboard loading is about 25 percent greater for \( K = 0 \) but less than 7 percent greater for \( K = \infty \). At other locations the deflections for checkerboard loading are the same as or less than those for all panels loaded. Deflections along the interior beams for checkerboard loading are approximately one-half of those for all panels loaded.
The slab moments at the center of the interior panel are greater for checkerboard loading than for all panels loaded by 50 percent for flexible beams ($H = J = 0.25$) and $K = 0$, but by only 5 percent for stiff beams ($H = J = 2.5$) and $K = \infty$. The slab moments at the center of the end span adjacent to the exterior beam are greater for checkerboard loading by 20 percent for $K = 0$ but are essentially the same for $K = \infty$. Elsewhere, the slab moments for checkerboard loading are similar to or less than those for all panels loaded in the loaded panels and are very small or negative in the unloaded panels.

The beam flexural moments along the interior beams for checkerboard loading are similar to those for all panels loaded, but, owing to only one of the adjacent panels being loaded for each span, they are only one-half as great. In the exterior beam the flexural moments in the end span for checkerboard loading are greater than those for all panels loaded by about 20 percent for $K = 0$ and about 5 percent for $K = \infty$.

The torsional moments in the exterior beam for checkerboard loading are similar to those for all panels loaded in the end spans adjacent to the loaded panels but are almost zero in the middle span adjacent to the unloaded panel. However, with the checkerboard loading, the torsional moments in the interior beams are as large as those for the exterior beam. For small $H$ and $J$, these moments are not sensitive to changes in $K$, especially for values of $K$ greater than 10, but for large values of $H$ and $J$, the moments at the interior column are 50 percent greater for $K = 0$ than for $K = \infty$ but at the edge column are greater for $K = \infty$ by roughly the same amount.
7. SUMMARY AND GENERAL CONCLUSIONS

7.1 Outline of Investigation

The analytical study presented in this report is concerned with the influence of variable column stiffness on the deflections and moments in continuous, two-way floor slabs and their supporting beams.

Analyses were made for continuous structures composed of nine square panels arranged three by three and supported along all edges by flexible beams supported in turn on non-deflecting but flexible columns at the beam intersections. The primary variable was the column stiffness ratio, \( K \). Five values of \( K \), ranging from zero to infinity, were used for each combination of beam stiffness and loading. For a uniformly distributed load on all panels, three values of the beam flexural stiffness ratio, \( H \), were considered in all combinations with the same three values of the beam torsional stiffness ratio, \( J \). These values were 0.25, 1.0 and 2.5, corresponding to flexible, medium, and stiff beams, respectively. Stiffness of the edge columns and beams were less than those for the interior beams by a fixed ratio.

In order to obtain some insight into the effects of partial loading, a limited number of solutions were made for a checkerboard pattern of loading in which only the interior and corner panels were loaded with a uniformly distributed load. These solutions included all five values of the column stiffness but only two combinations of beam stiffness, \( H = J = 0.25 \) and \( H = J = 2.5 \).

The method of analysis employed is based on an application of the finite difference equations. The difference operators were obtained by
considerations of equilibrium of a mathematical model based on the concept of Newmark's plate analog which was modified to account for the presence of the beams and columns. The method of analysis is described in Chapter 2. The solutions of the simultaneous equations were obtained from the ILLIAC (the University of Illinois Digital Computer). This computer was also used to calculate slab and beam moments from the deflections.

7.2 Numerical Results

Numerical results were obtained for the deflections, slab bending moments, and beam flexural and torsional moments.

Solutions for the slab deflections and moments have been presented graphically only for the sections along the beam lines and at midspan of the panels. Beam flexural and torsional moments for both interior and exterior beams have also been presented graphically. To aid in the interpretation of the results, plots have been presented showing the effect of variations in column stiffness on the slab bending moments and the beam torsional moments.

The major changes in the deflections and moments for all panels loaded as the column stiffness ratio $K$ is reduced from infinity are summarized below. The changes for the checkerboard loading are similar in the loaded panels and are discussed in detail in Chapter 6.

(a) Deflections

For all panels loaded and for infinitely stiff columns, the configuration of the deflections is similar in all panels although the maximum deflections in the corner panel are greater than those in the interior panel by 23 percent for $H = J = 0.25$ and 8 percent for $H = J = 2.5$. The changes in the deflections as $K$ is reduced are summarized in the following table.
As K is reduced, the deflections at locations in "end spans" increase, whereas deflections at locations in "middle spans" decrease. The increase in the deflections at the center of the edge panel are smaller since, parallel to the edge, this panel acts as a middle span.

The change in deflections as K is reduced from infinity down to about 10 is less for flexible beams than for stiff beams, whereas, from K = 10 to zero, the change in deflection is greater for the flexible beams. This trend is shown in the table by the greater values of K at which the values of the deflections for the stiffer beams are changed by 10 percent from the values for K = \( \infty \). These tabulated values of K are representative of those for the H and J values given but are generally not sensitive to changes in J.

(b) Slab Moments

For all panels loaded, the slab moments along the beam lines and at the columns are sensitive to changes in \( H \). These moments are doubled as \( H \) decreases from 2.5 to 1.0 and are more than doubled again as \( H \) decreases from 1.0 to 0.25. The moments across the beam lines and at the centers of the panels are less sensitive to \( H \), and increase only about 25 percent as \( H \) decreases from 2.5 to 0.25.
Except across the exterior edge, the slab moments are essentially independent of changes in $J$. For $K = \infty$, the moments across the edge are negative and are reduced about 50 percent as $J$ decreases from 2.5 to 1.0 and are reduced another 50 percent as $J$ decreases from 1.0 to 0.25. For $K = 0$, these moments are small for all values of $J$ and for large $J$ can become positive.

The effects of $K$ on the slab moments are best seen by considering "column strips" and "middle strips" in the slab as shown in Fig. 116. These effects, which are summarized in the following table, are not the average effects across these strips but are the average effects across a width of slab of one grid interval centered in each strip.

### CHANGES IN SLAB MOMENTS WITH CHANGES IN $K$

<table>
<thead>
<tr>
<th>Location</th>
<th>Change in Moment As $K$ is Decreased From $\infty$ to 0</th>
<th>% Change as $K$ is Decreased From $\infty$ to 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strip</td>
<td>Moment At</td>
<td>$K$ is Decreased From $\infty$ to 0</td>
</tr>
<tr>
<td>Column</td>
<td>Corner column</td>
<td>decrease</td>
</tr>
<tr>
<td>Strip in</td>
<td>Center of end span</td>
<td>increase</td>
</tr>
<tr>
<td>Outer</td>
<td>Edge column</td>
<td>increase</td>
</tr>
<tr>
<td>Row</td>
<td>Center of middle span</td>
<td>decrease</td>
</tr>
<tr>
<td>Column</td>
<td>Edge column</td>
<td>decrease</td>
</tr>
<tr>
<td>Strip in</td>
<td>Center of end span</td>
<td>increase</td>
</tr>
<tr>
<td>Inner</td>
<td>Interior column</td>
<td>increase</td>
</tr>
<tr>
<td>Row</td>
<td>Center of middle span</td>
<td>decrease</td>
</tr>
<tr>
<td>Middle</td>
<td>Over edge beam</td>
<td>decrease</td>
</tr>
<tr>
<td>Strip in</td>
<td>Center of end span</td>
<td>increase</td>
</tr>
<tr>
<td>Outer</td>
<td>Over interior beam</td>
<td>decrease</td>
</tr>
<tr>
<td>Row</td>
<td>Center of middle span</td>
<td>decrease</td>
</tr>
<tr>
<td>Middle</td>
<td>Over edge beam</td>
<td>decrease</td>
</tr>
<tr>
<td>Strip in</td>
<td>Center of end span</td>
<td>increase</td>
</tr>
<tr>
<td>Inner</td>
<td>Over interior beam</td>
<td>increase</td>
</tr>
<tr>
<td>Row</td>
<td>Center of middle span</td>
<td>decrease</td>
</tr>
</tbody>
</table>

* become positive.
In general, as K decreases, the moments in the end spans increase and those in the middle spans decrease. Thus, as K decreases, the moments in the outer column strip decrease at the corner column and at the center of the middle span, and increase at the edge column and at the center of the end span. For the inner column strip, the moments at the interior column and at the center of the end span increase, and those at the edge column and at the center of the middle span decrease. For the middle strip in the outer row of panels, the moments over the beams and at the center of the middle span decrease, and those at the center of the end span increase. For the middle strip in the middle row of panels, the moments over the exterior beam and at the center of the middle span decrease, and those over the interior beam and at the center of the end span increase.

Moments in the middle strips at the exterior beam become positive for small values of beam stiffness, as K decreases from infinity to zero. For the increase in moments in the end spans of these strips, two values for the percentage change as K is decreased from infinity to zero are given for the flexible beams; the larger values apply for larger values of J. The moments at the edge for the column strips approach a value of zero as the column stiffness is decreased to zero.

In general, the rate of change in the moments for high values of K is greater for stiff beams than for flexible beams. This is indicated by the generally greater values of K for stiff beams for which the moments are changed by 10 percent.

(c) **Beam Flexural Moments**

For all panels loaded, the beam flexural moments are reduced about 20 percent as H decreases from 2.5 to 1.0 and about 50 percent as H decreases
from 1.0 to 0.25, however, changes in \( J \) have only a negligible effect on these moments. Moments in the middle spans are relatively independent of \( K \), especially for \( K \) greater than 10, but moments in the end spans, as \( K \) decreases from infinity to zero are increased about 50 percent for low values of \( H \) and 100 percent for high values of \( H \). The beam flexural moments at the discontinuous end decrease rapidly as \( K \) decreases, and for high values of \( H \) become positive as \( K \) approaches zero.

(d) **Beam Torsional Moments**

In general, the beam torsional moments are approximately zero at the center of each span and increase to maximum values at the ends of the spans.

For all panels loaded, the torsional moments in the exterior beam are small for \( K = 0 \) but increase substantially and reach maximum values at \( K = \infty \). These maximum moments increase from two to three times as \( J \) increases from 0.25 to 2.5 but decrease about 25 percent as \( H \) increases from 0.25 to 2.5. The torsional moments in the interior beams are small, being only about one-tenth of those in the exterior beam for corresponding values of \( H, J \) and \( K \).

For checkerboard loading, the torsional moments in the exterior beam are similar to those for all panels loaded in the end spans adjacent to the loaded panels but are almost zero in the middle span adjacent to the unloaded panel. However, with the checkerboard loading, the torsional moments in the interior beams are as large as those in the exterior beams. For small values of \( H \) and \( J \), these moments are not sensitive to changes in \( K \), especially for values of \( K \) greater than 10, but for large values of \( H \) and \( J \), these moments at the interior column are 50 percent greater for \( K = 0 \) than for \( K = \infty \) but at the edge column are greater for \( K = \infty \) as shown in Fig. 72.
7.3 General Conclusions

The following general conclusions from the results of this investigation are believed to be applicable to all continuous two-way floor slabs supported on flexible beams.

The slab deflections and moments are sensitive to changes in the beam flexural stiffness ratio, $H$, and increase at an increasing rate as $H$ is decreased. However, except across the edge, the slab deflections and moments for all panels loaded are not sensitive to changes in the beam torsional stiffness, $J$. Slab moments across a discontinuous edge decrease at an increasing rate as $J$ is decreased.

The effect of the column stiffness, $K$, on the slab moments for all panels loaded is great in the edge and corner panels but is relatively small in the interior panels. The negative moments across a discontinuous edge decrease at an increasing rate as $K$ is decreased, and can become positive as $K$ approaches zero. The positive moments in the end spans of the column and middle strips increase at an increasing rate as $K$ decreases; this increase is from 50-100 percent as $K$ decreases from infinity to zero. The negative moments at the face of the first interior column, for the column strips, and at the face of the first interior beam, for the inner middle strips, increase at an increasing rate as $K$ decreases; this increase is from 10-25 percent as $K$ decreases from infinity to zero. The slab moments in the loaded panels are essentially the same for checkerboard loading as for all panels loaded.

The beam flexural moments are primarily influenced by $H$ and decrease at an increasing rate as $H$ is decreased. These moments are approximately one-half as great for checkerboard loading as for all panels loaded since only one adjacent panel is loaded for any beam span.
The beam torsional moments decrease at an increasing rate as \( J \) is decreased. For all panels loaded the torsional moments in the interior beams are small, whereas those in the exterior beams are small at midspan but increase to much larger values at the ends. In the exterior beams, these moments decrease at an increasing rate as \( K \) is decreased. For the checkerboard loading, the torsional moments in the interior beams are similar in magnitude to those in the exterior beams and both vary with \( K \) in the same manner as the moments in the exterior beams for all panels loaded.
8. REFERENCES


FIGURE 1 - ARRANGEMENT OF NINE-PANEL SLAB
FIGURE 2 - CO-ORDINATE SYSTEM FOR LOCATING POINTS ON SLAB

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MOMENTS AT POINT O ACTING ABOUT Y-AXIS

FIGURE 4 - ANALOG MOMENTS FOR AN INTERIOR COLUMN
MOMENTS AT POINT O ACTING ABOUT X-AXIS

MOMENTS AT POINT O ACTING ABOUT Y-AXIS

FIGURE 5 - ANALOG MOMENTS FOR AN EDGE COLUMN
Figure 6 - Analog moments for a corner column.
LOADING - ALL PANELS

$H = 0.25$
$J = 0.25$

DEFLECTIONS IN TERMS OF $\frac{q b^4}{N}$

SLAB BENDING MOMENTS $\frac{m_y}{q b^2}$

SLAB BENDING MOMENTS $\frac{m_x}{q b^2}$

FIGURE 7 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 0-X,
LOADING - ALL PANELS, $H=0.25$, $J=0.25$
LOADING - ALL PANELS

$H = 0.25$

$J = 1.0$

DEFLECTIONS IN TERMS OF $\frac{q b^4}{N}$

SLAB BENDING MOMENTS $\frac{m_x}{cb^2}$

SLAB BENDING MOMENTS $\frac{m_y}{cb^2}$

FIGURE 8 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE O - X, LOADING - ALL PANELS. $H = 0.25, J = 1.0$
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LOADING - ALL PANELS

$H = 1.0$
$J = 2.5$

DEFLECTIONS in TERMS OF $\frac{qL^4}{EI}$

SLAB BENDING MOMENTS $\frac{M_x}{ab^2}$

SLAB BENDING MOMENTS $\frac{M_y}{ab^2}$

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LOADING - ALL PANELS
\[ H = 0.25 \]
\[ J = 0.25 \]

DEFLECTIONS IN TERMS OF \( \frac{q b^4}{N} \)

SLAB BENDING MOMENTS \( \frac{m_x}{q b^2} \)

SLAB BENDING MOMENTS \( \frac{m_y}{q b^2} \)

FIGURE 16 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE \( 4' - X \), LOADING - ALL PANELS, \( H = 0.25, J = 0.25 \)
LOADING - ALL PANELS

\[ H = 0.25 \]
\[ J = 1.0 \]

DEFLECTIONS IN TERMS OF \( \frac{qb^4}{N} \)

SLAB BENDING MOMENTS \( \frac{m_x}{qb^2} \)

SLAB BENDING MOMENTS \( \frac{m_y}{qb^2} \)

FIGURE 17 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 4'-X,
LOADING - ALL PANELS, H = 0.25, J = 1.0
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LOADING - ALL PANELS, H = 0.25, J = 2.5
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LOADING - ALL PANELS

$H = 2.5$
$J = 2.5$

DEFLECTIONS IN TERMS OF $\frac{q b^4}{N}$

SLAB BENDING MOMENTS $\frac{m_x}{qb^2}$

SLAB BENDING MOMENTS $\frac{m_y}{qb^2}$

FIGURE 24 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE $4'-X$, LOADING - ALL PANELS, $H = 2.5$, $J = 2.5$
LOADING - ALL PANELS

H = 0.25
J = 0.25

DEFLECTIONS IN TERMS OF \( \frac{qb^4}{N} \)

SLAB BENDING MOMENTS \( \frac{m_x}{qb^2} \)

SLAB BENDING MOMENTS \( \frac{m_y}{qb^2} \)

FIGURE 25 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 8 - X,
LOADING - ALL PANELS, H = 0.25, J = 0.25
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LOADING - ALL PANELS
H = 0.25
J = 2.5

DEFLECTIONS IN TERMS OF $\frac{qb^4}{N}$

SLAB BENDING MOMENTS $\frac{m_x}{qb^2}$

SLAB BENDING MOMENTS $\frac{m_y}{qb^2}$

FIGURE 27 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 8 - X,
LOADING - ALL PANELS, H = 0.25, J = 2.5
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LOADING - ALL PANELS

H = 1.0
J = 1.0

DEFLECTIONS IN TERMS OF \( \frac{qb^4}{N} \)

SLAB BENDING MOMENTS \( \frac{m_x}{qb^2} \)

SLAB BENDING MOMENTS \( \frac{m_y}{qb^2} \)

FIGURE 29 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 8-X, LOADING - ALL PANELS, H = 1.0, J = 1.0
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LOADING - ALL PANELS

\[ H = 2.5 \]
\[ J = 2.5 \]

DEFLECTIONS IN TERMS OF \( \frac{qb^4}{N} \)

SLAB BENDING MOMENTS \( \frac{m_x}{qb^2} \)

SLAB BENDING MOMENTS \( \frac{m_y}{qb^2} \)

FIGURE 33 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 8 - X, LOADING - ALL PANELS, \( H = 2.5 \), \( J = 2.5 \)
Figure 34 - Slab deflections and bending moments along line 12 - X, loading - all panels, H = 0.25, J = 0.25

Deflections in terms of \( \frac{q b^4}{N} \)

Slab bending moments \( \frac{m_x}{ab^2} \)

Slab bending moments \( \frac{m_y}{ab^2} \)
FIGURE 35 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 12 - X,
LOADING - ALL PANELS, H = 0.25, J = 1.0
FIGURE 36 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 12 - X, LOADING - ALL PANELS, $H = 0.25, J = 2.5$
LOADING - ALL PANELS
H = 1.0
J = 0.25

DEFLECTIONS IN TERMS OF $\frac{q b^4}{N}$

SLAB BENDING MOMENTS $\frac{m_x}{ab^2}$

SLAB BENDING MOMENTS $\frac{m_y}{ab^2}$

FIGURE 37 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 12-X,
LOADING - ALL PANELS, H = 1.0, J = 0.25
LOADING - ALL PANELS

H = 1.0
J = 1.0

DEFLECTIONS IN TERMS OF $\frac{q b^4}{N}$

SLAB BENDING MOMENTS $\frac{m_x}{q b^2}$

SLAB BENDING MOMENTS $\frac{m_y}{q b^2}$

FIGURE 38 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 12 - X,
LOADING - ALL PANELS, H = 1.0, J = 1.0
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LOADING - ALL PANELS

$H = 2.5$

$J = 1.0$

DEFLECTIONS IN TERMS OF $\frac{qb^4}{N}$

SLAB BENDING MOMENTS $\frac{m_x}{ab^2}$

SLAB BENDING MOMENTS $\frac{m_y}{ab^2}$

FIGURE 41 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 12-X,
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LOADING - CHECKERBOARD

$H = 2.5$
$J = 2.5$

DEFLECTIONS IN TERMS OF $\frac{q b^4}{N}$

SLAB BENDING MOMENTS $\frac{m_x}{q b^2}$

SLAB BENDING MOMENTS $\frac{m_y}{q b^2}$

FIGURE 44 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE O-X, LOADING - CHECKERBOARD, $H = 2.5$, $J = 2.5$
SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 4'-X
LOADING - CHECKERBOARD, H = 0.25, J = 0.25
Figure 46 - Slab deflections and bending moments along line 4'-X, loading - checkerboard, H = 2.5, J = 2.5
LOADING - CHECKERBOARD

H = 0.25
J = 0.25

DEFLECTIONS IN TERMS OF $\frac{qb^4}{N}$

SLAB BENDING MOMENTS $\frac{m_x}{qb^2}$

SLAB BENDING MOMENTS $\frac{m_y}{qb^2}$

FIGURE 47 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE B - X, LOADING - CHECKERBOARD, H = 0.25, J = 0.25
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LOADING - CHECKERBOARD

H = 0.25
J = 0.25

DEFLECTIONS IN TERMS OF \( \frac{q_b^4}{N} \)

SLAB BENDING MOMENTS \( \frac{m_x}{q_b^2} \)

SLAB BENDING MOMENTS \( \frac{m_y}{q_b^2} \)

FIGURE 49 - SLAB DEFLECTIONS AND BENDING MOMENTS ALONG LINE 12 - X, LOADING - CHECKERBOARD, H = 0.25, J = 0.25
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H = 0.25, J = 1.0

EXTERIOR BEAM FLEXURAL MOMENTS $\frac{M}{qb^3}$

INTERIOR BEAM FLEXURAL MOMENTS $\frac{M}{qb^3}$
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H=1.0, J=1.0
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H = 1.0, J = 2.5
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H = 0.25, J = 0.25
FIGURE 63 - BEAM TORSIONAL MOMENTS, LOADING - ALL PANELS,
H = 0.25, J = 1.0
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H = 1.0, J = 1.0
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J = 1.0, AT LOCATIONS (0,0), (0,4), (4,4), LOADING - ALL PANELS
FIGURE 78 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H, J = 2.5, AT LOCATIONS (0,0), (0,4), (4,4), LOADING - ALL PANELS
FIGURE 79-CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H = J, AT LOCATIONS (0,0),(0.41),(4/41),LOADING-CHECKERBOARD
FIGURE 80 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND J, H = 0.25, AT LOCATIONS (0,8),(4',8), LOADING - ALL PANELS
FIGURE 81 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND J, $H = 1.0$, AT LOCATIONS (0,8), (4',8), LOADING - ALL PANELS
FIGURE 82 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND J, H = 2.5, AT LOCATIONS (0,8), (4', 8), LOADING - ALL PANELS
FIGURE 83 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H,
J = 0.25, AT LOCATIONS (0,8), (4',B). LOADING - ALL PANELS
FIGURE 84 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H, J = 1.0, AT LOCATIONS (0,8), (4',8). LOADING - ALL PANELS
Figure 85 - Change in slab moment due to variation of $K$ and $H$, $J = 2.5$, at locations (0,8), (4',8), loading - all panels.
FIGURE 86 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H = J, AT LOCATIONS (0, B), (4', B). LOADING - CHECKERBOARD
FIGURE 87 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND J, H = 0.25, AT LOCATIONS (8,8), (0,12), LOADING - ALL PANELS
FIGURE 88 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND J, 
H = 1.0, AT LOCATIONS (8,8), (0,12), LOADING - ALL PANELS
FIGURE 89 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND J, H = 2.5, AT LOCATIONS (8,8), (0,12), LOADING - ALL PANELS
Figure 90 - Change in slab moment due to variation of $K$ and $H$, $J=0.25$, at locations (8,8), (0,12), loading - all panels.
FIGURE 91 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H, J = 1.0, AT LOCATIONS (8,8),(0,12), LOADING - ALL PANELS
FIGURE 92 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H, J = 2.5, AT LOCATIONS (8,8), (0,12), LOADING - ALL PANELS
FIGURE 93 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF $K$ AND $H = J$, AT LOCATIONS (8,8),(0,12), LOADING - CHECKERBOARD
FIGURE 94 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND J,
H=0.25, AT LOCATION (4',12'), LOADING - ALL PANELS
FIGURE 95 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND J,
H = 1.0, AT LOCATION (4', 12'), LOADING - ALL PANELS
FIGURE 96 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND J,  
H = 2.5, AT LOCATION (4',12), LOADING - ALL PANELS
FIGURE 97 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H, 
J=0.25, AT LOCATION (4',12'), LOADING - ALL PANELS
FIGURE 98 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF $K$ AND $H$, $J=1.0$, AT LOCATION $(4',12')$, LOADING - ALL PANELS
FIGURE 99 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H
J = 2.5 AT LOCATION (4', 12'), LOADING - ALL PANELS
FIGURE 100 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H = J, AT LOCATION (4',12'), LOADING - CHECKERBOARD
FIGURE 101 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND J,
H = 0.25, AT LOCATIONS (8,12), (12,12), LOADING - ALL PANELS
FIGURE 102 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND J, H = 1.0, AT LOCATIONS (8,12), (12,12), LOADING - ALL PANELS
FIGURE 103 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND J, H = 2.5, AT LOCATIONS (8,12), (12,12). LOADING - ALL PANELS
FIGURE 104 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H, J=0.25, AT LOCATIONS (8,12),(12,12), LOADING - ALL PANELS
FIGURE 105 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H,

J = 1.0, AT LOCATIONS (8,12), (12,12), LOADING - ALL PANELS
FIGURE 106 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H,
J = 2.5, AT LOCATIONS (8,12), (12,12), LOADING - ALL PANELS
FIGURE 107 - CHANGE IN SLAB MOMENT DUE TO VARIATION OF K AND H = J, AT LOCATIONS (8,12), (12,12), LOADING - CHECKERBOARD
FIGURE 108 - CHANGE IN AVERAGE BEAM TORSIONAL MOMENT OVER GRID INTERVAL 3-4 OF INTERIOR BEAM DUE TO VARIATION OF K, LOADING - ALL PANELS, H = 0.25
FIGURE 109 - CHANGE IN AVERAGE BEAM TORSIONAL MOMENT OVER GRID INTERVAL 3 - 4 OF INTERIOR BEAM DUE TO VARIATION OF K, LOADING - ALL PANELS, H = 1.0

LEGEND
A - H=1.0, J=0.25
B - H=1.0, J=1.0
C - H=1.0, J=2.5
FIGURE 110 - CHANGE IN AVERAGE BEAM TORSIONAL MOMENT OVER GRID INTERVAL 3-4 OF INTERIOR BEAM DUE TO VARIATION OF K, LOADING - ALL PANELS
H = 2.5

LEGEND
A - H = 2.5, J = 0.25
B - H = 2.5, J = 1.0
C - H = 2.5, J = 2.5
FIGURE 111 - CHANGE IN AVERAGE BEAM TORSIONAL MOMENT OVER GRID INTERVAL 3-4 OF INTERIOR BEAM DUE TO VARIATION OF K, LOADING - CHECKERBOARD

LEGEND
A - H=0.25, J=0.25
B - H=2.5, J=2.5
FIGURE 112 - CHANGE IN AVERAGE BEAM TORSIONAL MOMENT OVER GRID INTERVAL 4'-5' OF INTERIOR BEAM DUE TO VARIATION OF K, LOADING - ALL PANELS, H = 0.25
FIGURE 113 - CHANGE IN AVERAGE BEAM TORSIONAL MOMENT OVER GRID INTERVAL 4'-5' OF INTERIOR BEAM DUE TO VARIATION OF K, LOADING - ALL PANELS, H = 1.0

LEGEND
A - H = 1.0, J = 0.25
B - H = 1.0, J = 1.0
C - H = 1.0, J = 2.5
FIGURE 114 - CHANGE IN AVERAGE BEAM TORSIONAL MOMENT OVER GRID INTERVAL 4' - 5 OF INTERIOR BEAM DUE TO VARIATION OF K, LOADING - ALL PANELS, H = 2.5
LEGEND
A - \( H=0.25, J=0.25 \)
B - \( H=2.5, J=2.5 \)

FIGURE 115 - CHANGE IN AVERAGE BEAM TORSIONAL MOMENT OVER GRID INTERVAL 4' - 5'
OF INTERIOR BEAM DUE TO VARIATION OF K, LOADING - CHECKERBOARD
FIGURE 116 - POSITION OF COLUMN AND MIDDLE STRIPS