INTERACTION OF PLANE STRESS WAVES WITH A SPHERICAL CAVITY IN ELASTIC AND VISCOELASTIC MEDIA

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THESIS

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1. INTRODUCTION

1.1. Statement of the Problem

With the advent of nuclear weapons, considerable attention has been given to deep underground shelters to provide protection against structural damage caused by the blast wave as well as heat and radioactive fallout. Stress waves in rock or soil may take the form of a directly transmitted ground shock (in the case of a surface or underground burst) or an air-induced shock. In the latter case (for a surface or air burst) the shock waves moving out in the air produce disturbances in the ground. If a cavity is present in the rock this causes an alteration of the incident stress wave and as a consequence a magnification of stress around the cavity. There is a possibility that this magnification of stress will cause failure in the rock by a spalling off of material into the cavity or by crushing.

This investigation treats the stress magnification around a cavity when it is engulfed by a stress wave. The shape of the cavity chosen for investigation is a sphere, this being the simplest boundary to consider from an analytical point of view. In the analysis the rock is represented by either an elastic or a viscoelastic material.

The methods developed in this study are by no means limited to the case of a spherical cavity in an elastic or viscoelastic medium. The methods may be adapted to cases of a lined cavity, with either a thin shell liner or a thick liner treated as a second continuum. In addition, a rigid or stiff inclusion or a fluid-filled cavity can also be treated.

In addition to the problem of stresses around a spherical cavity due to incident longitudinal (P) and transverse (S) waves, solutions to the simpler problem of a step pressure symmetrically applied to the cavity surface are presented briefly.
1.2. Basic Assumptions

The following assumptions relating to the configuration and the material are made:

(1) The body is assumed to be of infinite size so that the only waves that need to be considered in determining the stresses are the incident stress wave and the wave of disturbance caused by the cavity. In bedrock there may be additional waves which are reflected, say, from the ground surface. However, since the maximum stress occurs after the incident wave has traversed only a few cavity diameters past the front of the cavity, these additional waves may not affect the maximum stress.

(2) The cavity is assumed to be spherical in shape. An underground installation will generally have a more complicated shape and there will also be an entrance shaft. However, even the simplest three-dimensional shape is of interest in exploring the range of dynamic stress concentration factors.

(3) The deformations in the stress waves are assumed small enough to use the classical small strain, small rotation theory. For this problem, where the body is not thin in any direction, and where the strains are small, this assumption is justified.

(4) The material is assumed to be either linearly elastic or linearly viscoelastic, isotropic, and homogeneous. For wave propagation problems where the stresses are small compared to the stress at failure, rock is known to be nearly elastic. It does, however, exhibit internal dissipation which can, to a certain extent, be represented by linear viscoelasticity.* For larger stresses, greater departures from elastic behavior would be expected.

(5) The incident stress wave is assumed to be either a plane longitudinal (P) wave or a plane shear (S) wave. This assumption is acceptable if

* See Section 5.4.
the material is homogeneous and isotropic and if the distance from the source is large compared to the diameter of the cavity. A curved wave front could also be treated.

(6) The viscoelastic material is assumed to have a finite instantaneous elasticity. This ensures that all stress waves travel with a finite speed. Occasionally, viscoelastic models are chosen without this property. However, for wave propagation problems, this requirement appears to be essential if the wave front is to be represented reasonably.

1.3. Method of Solution

If it is possible to obtain a mathematical solution to certain dynamic problems in an elastic material, then the mathematical solution to the same problem in a viscoelastic material can be obtained by use of the following well-known correspondence principle:

If the dynamic solution for any boundary value problem, where the stress is specified on a known boundary, can be obtained for an elastic material, the solution to the same boundary value problem in a viscoelastic material is obtained by taking the Laplace (or Fourier) transform of the elastic solution. Here the elastic moduli are replaced by the corresponding viscoelastic transform "moduli" (or complex "moduli" in the case of the Fourier transform). The transform solution so obtained is then inverted to obtain the solution to the viscoelastic problem.

The solution of the problem of the interaction of plane stress waves with a spherical cavity in an elastic material (Chapter 2 and Appendix B) is obtained by expansion of the disturbing wave caused by the presence of the cavity in spherical harmonics. For each harmonic three diverging wave solutions, one longitudinal and two shear waves, are obtained. The three solutions are expressed in terms of generating functions (analogously to the d'Alembert solution for wave propagation in rods or strings) which are determined from the boundary conditions. For each harmonic, the boundary
conditions provide three ordinary linear differential equations in terms of
the three generating functions. These can easily be solved by a numerical
solution of the initial value problem, which is described in 2.5.

With the aid of the elastic solution and the correspondence principle just stated, the solution for the viscoelastic material expressed in
terms of the Laplace transform is given in Chapter 3. The inversion of the
transform to obtain the required stresses is not easily performed in terms
of known functions, so that a numerical method is given in 3.4 which involves
expressing the inverse by a series of orthogonal terms and calculating the
coefficients of the terms from discrete values of the transform.

It is possible, for some viscoelastic materials, to obtain the inverse
of the transforms of the generating functions in terms of expressions which,
for materials having instantaneous elasticity, contain convolution integrals.
The boundary conditions provide integro-differential equations of the Volterra
type in place of the ordinary differential equations obtained for the elastic
material. These equations are integrated numerically (as described in 4.3)
by methods similar to those used for the elastic material.

1.4. Previous Work

The theoretical basis of the solution for the elastic material can
be traced back to the middle of the nineteenth century. In 1862, Lord Kelvin (3)
obtained the general static solution to boundary value problems involving
spherical surfaces in an elastic material. In 1882, Lamb (4)(5) obtained the
solution to the vibrations of an elastic sphere; his theoretical development
forms a basis to the solution of dynamic problems associated with the spheric-
cal boundary.

Since then, Sato (6), Eringen (7), and others (8) have obtained solu-
tions by assuming that the wave motion is of the steady state harmonic type.
The solution to transient problems has been obtained in some cases from the steady state solution by use of the Fourier integral.\(^9\)

The solution presented herein is related to the solution given by Paul\(^{10}\) and by Yoshihara, Robinson and Merritt\(^{11}\) for the case of interaction of plane waves with a cylindrical cavity. The similarity lies in the fact that here also the solution is expressed in the form of generating functions analogously to the d'Alembert solution for wave propagation in rods and strings. One advantage of this method, besides its simplicity, is that in the numerical results, a sharp front of the diverging waves is apparent from the very form of the solution.

The mathematical theory of viscoelasticity has been the subject of extensive investigation since about 1940. The solution of the interaction of plane waves with a spherical cavity in a viscoelastic material is readily obtained from the elastic solution by use of the correspondence principle\(^1\) stated in 1.3. In this way, the solution to the viscoelastic problem is found by application of Laplace or Fourier transforms and subsequent inversion. Except for simple problems, the inversion is not easily obtained in terms of familiar functions. A numerical method of obtaining the inverse of the Laplace transform is given here which is similar to the method given by Papoulis\(^{12}\).

Sometimes the inverse of the transform of the generating functions can be obtained in terms of known functions. In the present investigation this inversion is given in Chapter 4 for the Maxwell and standard linear models, using the inversion given by Morrison\(^{13}\) for one-dimensional wave propagation.

The problem of determining the dissipative properties of rock has received considerable attention in the past few decades, the experimental
work of Born\textsuperscript{(14)} and McDonal et al\textsuperscript{(15)} being notable contributions. C. W. Horton\textsuperscript{(16)} has approximated the experimental observations of McDonal et al\textsuperscript{(15)} by a standard linear model.

1.5. Notation

Each symbol is explained when it is introduced in the text. The following list summarized the main uses of certain symbols which occur in more than one place in the text. In discussions of special topics other meanings may be ascribed to the symbols, at which time they will be redefined.

- \( c_p, c_s \) elastic P and S wave speeds, in cavity radii per second, defined by (2.4)
- \( c_p(s), c_s(s) \) viscoelastic P and S wave speeds, in cavity radii per second, defined by (3.9)
- \( c_{p0}, c_{s0} \) fastest viscoelastic P and S wave speeds, in cavity radii per second, defined by (3.10)
- \( c = c_p/c_s \)
- \( c(s) = c_p(s)/c_s(s) \)
- \( c_{st} \) static value of \( c(s) \) defined by (3.12) or (3.13)
- \( e_{ij} \) deviatoric strain tensor
- \( F_m^n, G_m^n, H_m^n \) generating functions (of one variable)
- \( f_m^n(r,t), g_m^n(r,t), h_m^n(r,t) \) functions expressed in terms of the generating functions, defined by (2.9)
- \( H(t) \) Heaviside step function
- \( I_n(z) \) modified Bessel function of the first kind
- \( J_{ij}(\beta, \tau) \) defined by (4.18)
- \( i, j \) Cartesian tensor subscripts
- \( K_m^n(x_i) \) spherical harmonics (Appendix A)
$L^n(r,x_i)$ spherical harmonic function defined by (2.25)

$m, n$ integer scripts indicating associated spherical harmonic defined by (2.16)

$p^m_n(\mu), P^m_n(\mu)$ spherical harmonic functions defined by (A.3) and (A.4)

$p_i, p_i^*, p$ viscoelastic parameters defined in Fig. 6, equation (3.8), and Section 3.5

$Q^n_m(r,x_i), R^n_m(r,x_i),$ $S^n_m(r,x_i), V^n_m(r,x_i)$ functions of spherical harmonics defined by (2.13)

$r$ radius in multiples of the cavity radius (spherical coordinate)

$r_0$ radius of the spherical cavity

$r_1 = r-1$

$s$ Laplace transform parameter

$s_{ij}$ deviatoric stress tensor

$t_{nm}^{ij}, T_0^p, T_p^p, T_s^s$ traction acting on the inside of a spherical surface

$t$ time, seconds

$t_p = tC_p$, time in radius transit times

$t_s = tC_s$, time in radius transit times

$t_{p0} = tC_{p0}$, time in radius transit times

$t_{s0} = tC_{s0}$, time in radius transit times

$u_i$ displacement

$x_i$ Cartesian coordinates

$x, y, z$ defined by (4.10)

$\alpha_i, \alpha^i, \alpha$ viscoelastic parameters defined in Fig. 6, equation (3.8), and Section 3.5

$\alpha(s), \beta(s)$ viscoelastic parameters defined by (3.14)
functions of the generating functions defined by (2.12) and (2.13)

\( \delta(t) \) Dirac delta function

\( \delta_{ij} = 1 \ (i=j), \quad 0 \ (i \neq j) \); Kronecker delta

\( \epsilon_{ijk} \)
- 1 if ijk are in cyclic order,
- 1 if ijk are not in cyclic order,
- 0 if \( i=j, j=k, \) or \( k=i \);
the Cartesian tensor alternator

\( \epsilon_{ij}, \epsilon_{nm} \) strain tensor

\( \eta \) dummy variable in (4.10)

\( \theta, \phi \) spherical angular coordinates (Fig. 2)

\( \lambda, \mu \) Lamé constants

\( \lambda(s), \mu(s) \) viscoelastic Lamé constants defined by (3.8)

\( \rho \) density

\( \sigma_{ij}, \sigma_{\theta\theta}, \sigma_{\phi\phi}, \sigma_{\theta\phi} \) stresses

\( \sigma_{ij}, \sigma_{nm}, \sigma_{\theta\theta}, \sigma_{\phi\phi}, \sigma_{\theta\phi} \) stresses associated with a spherical harmonic \( \chi_m^n \)

\( \sigma^p_r, \sigma^p_s, \sigma^p_{\theta\theta}, \sigma^p_{\phi\phi} \)

\( \sigma^{st}_{\theta\phi} \) incident step wave stresses

\( \sigma^{st} \) long-time static stress

\( \tau \) dummy time variable

\( \varphi, \omega, \psi \) displacement potentials defined by (2.9)

\( \omega \) frequency

- superior bar denotes the Laplace transform, i.e.

\[
\mathcal{F}(s) = \int e^{-st}f(t)\,dt
\]
2. SOLUTION FOR THE ELASTIC MATERIAL

To obtain the solution of any problem for a viscoelastic material by use of the correspondence principle, the elastic solution must first be known. In this chapter, the solution for the interaction of incident P and S waves with a spherical cavity in an elastic material is obtained by expansion of the disturbing wave in spherical harmonics. For each harmonic, the boundary condition provides ordinary linear differential equations in terms of generating functions. These equations are integrated numerically and the stresses corresponding to the disturbing wave are added to the stresses from the incident wave to obtain the total stress field.

2.1. Equations of Motion

If the displacements are sufficiently small, the following equations hold for an elastic body; (17)

\[ \sigma_{ij} = \delta_{ij} \Delta + 2\mu \epsilon_{ij} \]  \hspace{1cm} (2.1)

\[ \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]  \hspace{1cm} (2.2)

\[ \Delta = \frac{\partial u_k}{\partial x_k} \]

\[ (\lambda + \mu) \frac{\partial \Delta}{\partial x_i} + \mu \nabla^2 u_i = \rho \frac{\partial^2 u_i}{\partial t^2} \]  \hspace{1cm} (2.3)

Equations (2.1) are Hooke's Law for isotropic elasticity relating the stress components \( \sigma_{ij} \) and strain components \( \epsilon_{ij} \) in Cartesian coordinates \( x_i \).

* Cartesian tensor notation will be used throughout.  See also 1.5 for an explanation of the Kronecker delta, \( \delta_{ij} \).
Here $\lambda$ and $\mu$ are the Lamé elastic constants. Equations (2.2) relate the strain components $e_{ij}$ and dilatation $\Delta$ to the displacement $u_i$. Equations (2.3) are the equations of motion, in terms of displacements (the Navier equations), for a homogeneous elastic body with density $\rho$, where $\nabla^2$ is the Laplacian differential operator.

If the longitudinal and shear wave speeds, $C_p$ and $C_s$, respectively, and their ratio $c$, given by

$$
C_p^2 = \frac{\lambda + 2\mu}{\rho} \\
C_s^2 = \frac{\mu}{\rho} \\
c = \frac{C_p}{C_s}
$$

are used to eliminate the Lamé constants, the stress-strain relation (2.1) and the equations of motion (2.3) may be written as

$$
\sigma_{ij} = \mu \left[ \delta_{ij} (C_s^{-2} - 2) \Delta + 2e_{ij} \right] \tag{2.5}
$$

$$
\left[ \nabla^2 - \frac{1}{C_s^2} \frac{\partial^2}{\partial t^2} \right] u_i = \left[ 1 - \frac{C_p^2}{C_s^2} \right] \frac{\partial \Delta}{\partial x_i} \tag{2.6}
$$

The displacement field $u_i$ can be represented as the sum of the gradient of a scalar potential and the curl of a vector potential; (17)

$$
u_i = \frac{\partial \varphi}{\partial x_i} + \epsilon_{ijk} \frac{\partial \psi_k}{\partial x_j} \tag{2.7}
$$

where the scalar potential $\varphi$ gives the irrotational or dilatational component.

* See 1.5 for an explanation of the Cartesian alternator $\epsilon_{ijk}$
of \( u_i \) and the vector potential \( \psi_k \) gives the equivoluminal or rotational component of \( u_i \). On substitution of (2.7) into (2.6), the equations of motion are satisfied if

\[
\nabla^2 \phi = \frac{1}{c_p^2} \frac{\partial^2 \phi}{\partial t^2} \tag{2.8a}
\]

\[
\nabla^2 \psi_k = \frac{1}{c_s^2} \frac{\partial^2 \psi_k}{\partial t^2} \tag{2.8b}
\]

although these are not necessary as will be demonstrated by the solution applicable to a spherical boundary. Equation (2.8a) is the scalar wave equation and (2.8b) is the vector wave equation.

2.2. Expansion of the Disturbing Wave in Spherical Harmonics

When the incident wave envelops the cavity, a disturbance is transmitted into the medium, propagating away from the cavity boundary. This disturbance or disturbing wave*, when added to the incident wave which would exist were there no cavity, gives the complete displacement field. If the disturbing wave can be determined, the displacement and stresses from the incident and disturbing waves can be found anywhere in the medium and specifically at the boundary where the disturbing wave has its greatest effect.

The displacement of the disturbing wave which satisfies the equation of motion (2.6) can be expanded in solid spherical harmonics \( K_m^n(x_i) \)** (i.e.

* The disturbing wave contains both reflected and diffracted parts.

** The development of spherical harmonics is given in standard texts. (19) The main properties of these functions needed for this investigation are given in Appendix A.
\[ \varphi = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_m^n(r,t) K_m^n \]

where

\[ f_m^n(r,t) = \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^n \left[ \frac{1}{r} f_{m,0}^n \left( t - \frac{r}{C_p} \right) \right] \]

\[ \psi_i = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} g_m^n(r,t) \epsilon_{ijk} \frac{\partial K_m^n}{\partial x_k} \]

\[ \omega_i = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \omega_m^n(r,t) x_k K_m^n \]

\[ h_m^n(r,t) = \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^n \left[ \frac{1}{r} h_{m,0}^n \left( t - \frac{r}{C_s} \right) \right] \]

where \( r \) will be in multiples of the cavity radius and \( C_p \) and \( C_s \) are in cavity radii per second.

The displacement is obtained from

\[ u_i = \frac{\partial \varphi}{\partial x_i} + \epsilon_{ijk} \frac{\partial}{\partial x_j} (\psi_k + \omega_k) \]

\[ (2.10) \]
The displacement computed from each term of $\varphi$ in (2.9a) represents a longitudinal wave (P wave) and the displacement computed from each term of $\psi$ and $\omega_i$ represents a shear wave (S wave). It is noted that the solution $\omega_i$ (2.9c) does not satisfy the vector wave equation (2.8b).

This expansion in spherical harmonics gives a solution of the equation of motion (2.6) for the disturbing wave in terms of the independent generating functions $\tilde{F}_m(t-r/C_p)$, $\tilde{G}_m(t-r/C_s)$, and $\tilde{H}_m(t-r/C_s)$; the spherical harmonics separate out the parts depending on the angular spherical coordinates. The generating functions for the disturbing wave are obtained from the boundary condition on the spherical surface of the cavity where $r$ is a constant. It is then easy to show that the generating functions $\tilde{F}_m(t-r/C_p)$, $\tilde{G}_m(t-r/C_s)$, and $\tilde{H}_m(t-r/C_s)$ are analogous to the d'Alembert expression $f(t-x/C)$ which is a solution to wave propagation problems in an infinite rod or string. In this investigation, the initial displacement and velocity in the disturbing wave are zero.

The incident wave can also be expanded in spherical harmonics but the expansion includes converging as well as diverging components. The converging components have the same form as (2.9) except that the generating functions $\tilde{F}_m(t-r/C_p)$, $\tilde{G}_m(t-r/C_s)$, and $\tilde{H}_m(t-r/C_s)$ are replaced by $\tilde{F}_m(t+r/C_p)$, $\tilde{G}_m(t+r/C_s)$, and $\tilde{H}_m(t+r/C_s)$, respectively. Because all the components of the diverging and converging wave have singularities at the origin, whereas the incident wave has no singularities there, it is necessary that the following relations hold:

\[
\tilde{F}_m(t) = -\tilde{F}_m^*(t) \\
\tilde{G}_m(t) = -\tilde{G}_m^*(t) \\
\tilde{H}_m(t) = -\tilde{H}_m^*(t)
\]
This means that on reaching the origin, the converging components of the incident wave give rise to equal and opposite diverging components.

However, it is not necessary to carry out a harmonic analysis of the potentials of the incident wave. For this investigation, the disturbing wave components for each harmonic are determined from the boundary condition and the stress is found by adding the total stress from all the components of the disturbing wave to the stress from the incident wave.

2.3. Displacement and Stress

For the disturbing wave, the displacement \( u_{i}^{nm} \) associated with each harmonic \( K_{m}^{n}(x_{i}) \) is obtained in Cartesian coordinates by substituting (2.9) into (2.10):

\[
\begin{align*}
  u_{i}^{nm} &= \frac{x_{i}K_{m}^{n}}{r} \left[ \frac{\partial f_{m}^{n}}{\partial r} + n \frac{\partial g_{m}^{n}}{\partial r} \right] + r \frac{\partial K_{m}^{n}}{\partial x_{i}} \left[ \frac{f_{m}^{n}}{r} - \frac{\partial g_{m}^{n}}{\partial r} - \frac{(n+1)\partial g_{m}^{n}}{\partial r} \right] \\
  &\quad + \varepsilon_{i}^{m} K_{m}^{n} \frac{\partial K_{m}^{n}}{\partial x_{j}} \left[ h_{j}^{m} \right]
\end{align*}
\]

(2.11)

where use has been made of (A.5) of Appendix A.

The strain \( \varepsilon_{ij}^{nm} \) and stress \( \sigma_{ij}^{nm} \) associated with each harmonic can be obtained in Cartesian coordinates by substituting (2.11) into (2.2) and (2.5). However, the stresses of interest are the stresses in spherical coordinates. The traction \( T_{j}^{nm} \) acting on the spherical cavity wall (needed to express the boundary condition) is found by contracting the Cartesian stress \( \sigma_{ij}^{nm} \) (given by (2.11), (2.2), and (2.5)) with the normal out from the body, \(-x_{i}/r\):

* The superscripts \( m,n \) are, of course, not tensorial in character; they indicate the spherical harmonic under consideration.
where

$$\gamma_m = \mu \left\{ c^2 \frac{\partial^2 f_m}{\partial r^2} + \frac{2}{r} \left[ c^2 (n+1)-(n+2) \right] \frac{\partial f_m}{\partial r} + n \frac{\partial^2 g_m}{\partial r^2} - \frac{2n}{r} \frac{\partial g_m}{\partial r} \right\}$$

$$\Delta_n = \mu \left\{ \frac{2}{r} \frac{\partial f_m}{\partial r} + \frac{2(n-1)}{r^2} f_m - \frac{\partial^2 g_m}{\partial r^2} - \frac{2n}{r} \frac{\partial g_m}{\partial r} - \frac{2(n-1)}{r^2} g_m \right\}$$

$$\chi_n = \mu \left\{ \frac{1}{r} \frac{\partial h_m}{\partial r} + \frac{n-1}{r^2} h_n \right\}$$

The stresses $\sigma_{\theta\theta}^{nm}$ (meridional), $\sigma_{\phi\phi}^{nm}$ (circumferential or 'hoop'), $\sigma_{\theta\phi}^{nm}$ (Fig. 1), which at the cavity boundary are the nonvanishing stresses, are determined from the Cartesian stress (given by (2.11), (2.2), and (2.5)) as follows;

$$\sigma_{\theta\theta}^{nm} = \sigma_{ij}^{nn} \left( \frac{x_i x_j}{r^2} - \delta_{ij} \right) \left( \frac{x_i x_j}{r^2} - \delta_{ij} \right) \left( 1 - \frac{x_i^2}{r^2} \right)$$

$$= \alpha_m^{n} K_m^{n} + \beta_m^{n} Q_m^{n} + \frac{h_m}{r} S_m^{n}$$

$$\sigma_{\phi\phi}^{nm} = \sigma_{ij}^{nn} \left( \epsilon_{1ki} \frac{x_k}{r} \right) \left( \epsilon_{1kj} \frac{x_j}{r} \right) \left( 1 - \frac{x_i^2}{r^2} \right)$$

$$= \alpha_m^{n} K_m^{n} - \beta_m^{n} R_m^{n} - \frac{h_m}{r} S_m^{n}$$

$$\sigma_{\theta\phi}^{nm} = \sigma_{ij}^{nn} \left( \frac{x_i x_j}{r^2} - \delta_{ij} \right) \left( \epsilon_{1kj} \frac{x_j}{r} \right) \left( 1 - \frac{x_i^2}{r^2} \right)$$

$$= \beta_m^{n} S_m^{n} - \frac{h_m}{r} V_m^{n}$$
where 
\[ \alpha_m^n = \mu \left\{ (c^2-2) \frac{\partial^2 f_n^m}{\partial r^2} + \frac{2}{r} \left[ c^2 (n+1) - (2n+1) \right] \frac{\partial f_n^m}{\partial r} + \frac{2n}{r^2} \frac{\partial g_n^m}{\partial r} \right\} \]

\[ \beta_m^n = \mu \left\{ \frac{f_n^m}{r^2} - \frac{1}{r} \frac{\partial g_n^m}{\partial r} - \frac{(n+1)}{r^2} g_n^m \right\} \]

\[ Q_m^n = 2 \left[ \frac{n}{r^2} - 2(n-1) \frac{x_1}{r^2} + \frac{2}{r^2} \frac{\partial k_n^m}{\partial x_1} + r^2 \frac{\partial^2 k_n^m}{\partial x_1^2} \right] \left( 1 - \frac{x_1^2}{r^2} \right) \]

\[ R_m^n = 2 \left[ \frac{(n-1)}{r^2} k_n^m - 2(n-1) \frac{x_1}{r^2} \frac{\partial k_n^m}{\partial x_1} + r^2 \frac{\partial^2 k_n^m}{\partial x_1^2} \right] \left( 1 - \frac{x_1^2}{r^2} \right) \]

\[ S_m^n = 2 \varepsilon \frac{1}{r^2} \frac{x_i}{r^2} k_n^m \left[ \frac{(n-1)}{r^2} \frac{\partial k_n^m}{\partial x_j} - \frac{\partial^2 k_n^m}{\partial x_j \partial x_k} \right] \left( 1 - \frac{x_1^2}{r^2} \right) \]

\[ V_m^n = \frac{Q_m^n}{2} + \varepsilon \frac{1}{r} \frac{x_i}{r} \frac{x_k}{r} \frac{\partial^2 k_n^m}{\partial x_j \partial x_k} \left( 1 - \frac{x_1^2}{r^2} \right) \]

and where \( \left( \frac{x_i}{r^2} - \delta_{ij} \right) \sqrt{1 - \frac{x_1^2}{r^2}} \) and \( \varepsilon \frac{x_i}{r} \sqrt{1 - \frac{x_1^2}{r^2}} \) are unit vectors in the \( \theta \) and \( \phi \) directions, respectively.

2.4. \textbf{Boundary Equations}

The presence of the cavity requires that the traction on the spherical boundary, \( r = r_0 = 1 \), due to the combined incident and disturbing waves be zero, that is

\[ T_{j}^0 = \sum_{n=0}^{\infty} \sum_{m}^{2n} \left[ x_j k_p q^p + \frac{\partial k_q^p}{\partial x_j} \Delta^p + \varepsilon_{j,k} x_k \frac{\partial k_q^p}{\partial x_q} x_p \right] = 0 \quad (2.14) \]

where \( T_{j}^0 \) is the traction on the spherical boundary due to the incident wave.
The vectors \( n x^m \), \( \frac{\partial K_n^m}{\partial x_j} \), and \( \epsilon_{j k} x^k \frac{\partial K_n^m}{\partial x_j} \) associated with each spherical harmonic are orthogonal over the spherical surface to the same vectors associated with any other harmonic (see Ref. 20, p. 384). Therefore the boundary equations for each harmonic can be separated out by multiplying (2.14) by \( n x^m, \frac{\partial K_n^m}{\partial x_j} \) in turn and integrating over the spherical boundary of unit radius.

To carry out this process, we use the following identities (20) for integration over a unit sphere \( S_1 \):

\[
\int_0^\pi \int_0^{2\pi} \left( \frac{\partial K_n^m}{\partial x_i} \right) \left( \frac{\partial K_n^m}{\partial x_i} \right) \sin \theta \, d\theta \, d\phi = n(n+1) \int_0^\pi \int_0^{2\pi} (K_n^m)^2 \sin \theta \, d\theta \, d\phi
\]

\[
\int_0^\pi \int_0^{2\pi} \left( \epsilon_{j k} x^k \frac{\partial K_n^m}{\partial x_i} \right) \left( \epsilon_{j h} x^h \frac{\partial K_n^m}{\partial x_i} \right) \sin \theta \, d\theta \, d\phi = n(n+1) \int_0^\pi \int_0^{2\pi} (K_n^m)^2 \sin \theta \, d\theta \, d\phi \quad (2.15)
\]

\[
\int_0^\pi \int_0^{2\pi} (K_n^m)^2 \sin \theta \, d\theta \, d\phi = \frac{2\pi}{2n+1} \frac{(n-m)!(n+m)!}{(n!)^2} \quad \text{for } m \neq 0
\]

\[
= \frac{4\pi}{2n+1} \quad \text{for } m = 0
\]

The following three boundary equations are obtained for each spherical harmonic \( K_n^m(x) \):

\[
\left[ \gamma_n^m + n \Delta_n^m \right] N_n^m = \int_0^\pi \int_0^{2\pi} n x^m N_n^m \sin \theta \, d\theta \, d\phi
\]

\[
\left[ \gamma_n^m + n(2n+1) \Delta_n^m \right] N_n^m = \int_0^\pi \int_0^{2\pi} \frac{\partial K_n^m}{\partial x_j} r_j^0 \sin \theta \, d\theta \, d\phi
\]

\[
(2.16)
\]
From the boundary equations for each harmonic (2.16) and from (2.12) and (2.9) it is seen that the first two are coupled ordinary differential equations in the generating functions \( F_n(t-l/C_p) \) and \( G_n(t-l/C_s) \), while the third is an ordinary differential equation in the wave function \( H_n(t-l/C_s) \). This separation occurs because the vector \( \varepsilon_{j \delta k} \frac{\partial K_m^n}{\partial x_j} \) is orthogonal to the vectors \( x_j K_m^n \) and \( \frac{\partial K_m^n}{\partial x_j} \).

For problems with the spherical cavity, it is convenient to replace the variables \( t-r/C_p \) and \( t-r/C_s \) in (2.9) by either the variables \( t_p-(r-1) \) and \( t_p-c(r-1) \) or the variables \( t_s-(r-1)/c \) and \( t_s-(r-1) \), where \( t_p=tC_p^* \) is a measure of time in multiples of the time required for the incident \( P \) wave to traverse a cavity radius (i.e., a unit length), and \( t_s=tC_s^* \) is a measure of time in multiples of the time required for the incident \( S \) wave to traverse a cavity radius. By doing this, the generating functions on the boundary become \( F_m^n(t_p) \), \( G_m^n(t_p) \), and \( H_m^n(t_p) \) when \( t_p \) is the proper measure of time and \( F_m^n(t_s) \), \( G_m^n(t_s) \), and \( H_m^n(t_s) \) when \( t_s \) is indicated.

There is an analogy here to the reflection of plane waves from a plane boundary. The coupled components \( F_m^n \) (\( \Psi \) potential) and \( G_m^n \) (\( \Psi_i \) potential) for each harmonic are analogous to reflected \( P \) (longitudinal) and \( SV \) (shear)
plane waves (2) respectively. The component \( H^n_m \) (potential) whose displacement is perpendicular to the displacement of the components \( F^n_m \) and \( G^n_m \), is analogous to a reflected SH (shear) plane wave. (2) As in the spherical case, the first two reflected waves (P and SV) are not coupled to the third (SH).

In the interesting problem of the effect of an internal explosion, a step pressure, \( -\sigma H(t) \), where \( H(t) \) is the Heaviside unit step function, is symmetrically applied to the cavity boundary. Then only the harmonic \( K_0 = 1 \) is involved and the boundary equations (2.16) reduce to the following equation in \( F^0_0(t_p) \):

\[
c^2 F^0_0(t_p) + 4F^0_0 + 4F^0_0 = -\frac{\sigma}{\mu} H(t_p)
\]

(2.17)

where the dot indicates differentiation with respect to the argument.

2.4.1 Boundary Equations for the Incident P Wave

The incident plane P wave is taken as a step longitudinal wave traveling in the negative \( x_1 \) direction as shown in Fig. 2. The solution for any form of incident plane P wave can be found from the solution for the step wave by the Duhamel integral; (10)

\[
\sigma(t) = P(0)\sigma(s)(t) + \int_0^t \frac{dP(\tau)}{d\tau} \sigma(s)(t-\tau)d\tau
\]

where \( \sigma(s) \) is any stress due to a unit step wave, and \( \sigma \) is the corresponding stress due to an arbitrary plane stress wave \( P(t) \), beginning at \( t = 0 \).

If the longitudinal stress \( \sigma_{11} \) is \( -\sigma_p \) (compression) in the strained region, the lateral stress \( \sigma_{22} \) or \( \sigma_{33} \) is \( -(1-2/c^2)\sigma_p \) and the stress for the incident wave can be written.
\[ \sigma_{ij} = - \left[ \delta_{ij} \delta_{lj} \frac{2}{c^2} + \delta_{lj} \left( 1 - \frac{2}{c^2} \right) \right] \sigma_p H \left[ t_p + (x_1 - 1) \right] \]  

(2.18)

where \( H \) is the unit step function. The variable \( \left[ t_p + (x_1 - 1) \right] \) is chosen so that when the step front just reaches the cavity, the time \( t_p \) is taken as zero. The displacement potential which gives (2.18) is of the scalar type and satisfies the wave equation (2.8a).

The traction on the cavity boundary is

\[ T_j^p = \left[ x_1 \delta_{lj} \frac{2}{c^2} + x_j \left( 1 - \frac{2}{c^2} \right) \right] \sigma_p H \left[ t_p + (x_1 - 1) \right] \]  

(2.19)

and the meridional and hoop stresses are

\[ \sigma_{\theta\theta}^p = - \left[ 1 - \frac{x_1^2}{r^2} \frac{2}{c^2} \right] \sigma_p H \left[ t_p + (x_1 - 1) \right] \]

\[ \sigma_{\phi\phi}^p = - \left[ 1 - \frac{2}{c^2} \right] \sigma_p H \left[ t_p + (x_1 - 1) \right] \]  

(2.20)

\[ \sigma_{\phi\theta}^p = 0 \]

The boundary equations for each harmonic are found by substituting (2.19) into (2.16). The traction on the spherical boundary due to the incident \( P \) wave is symmetric about the polar axis \( x_1 \), and, since the boundary is also symmetric about \( x_1 \), this corresponds to a symmetric structure loaded symmetrically and therefore the reflected waves for each harmonic are symmetric about the polar axis and involve the spherical harmonics \( K_0^n \) only. The integrands on the right-hand side of (2.16) are therefore functions of \( x_1 \) only and simplify to single integrals along \( x_1 \). On a sphere of unit radius the boundary equations

* These correspond to zonal surface harmonics, not tesseral harmonics.
for the incident P wave are:

\[
\left[ \gamma_0^n + n\Delta_0^n \right] = \left( \frac{2n+1}{2} \right) \frac{\sigma_p}{c^2} \int_{1-t_p}^{1} \left[ (c^2 - 2) + 2x_1^2 \right] K^n_0 dx_1 \quad \text{for} \quad 0 \leq t_p \leq 2
\]

\[
= \left( \frac{2n+1}{2} \right) \frac{\sigma_p}{c^2} \int_{-1}^{1} \left[ (c^2 - 2) + 2x_1^2 \right] K^n_0 dx_1 \quad \text{for} \quad t_p \geq 2
\]

\[
\left[ n\gamma_0^n + n(2n+1)\Delta_0^n \right] = \left( \frac{2n+1}{2} \right) \frac{\sigma_p}{c^2} \int_{1-t_p}^{1} \left[ (c^2 - 2) nK^n_0 + 2x_1 \frac{\partial K^n_0}{\partial x_1} \right] dx_1 \quad \text{for} \quad 0 \leq t_p \leq 2
\]

\[
= \left( \frac{2n+1}{2} \right) \frac{\sigma_p}{c^2} \int_{-1}^{1} \left[ (c^2 - 2) nK^n_0 + 2x_1 \frac{\partial K^n_0}{\partial x_1} \right] dx_1 \quad \text{for} \quad t_p \geq 2
\]

(2.21)

where, from Appendix A,

\[
k^n_0 = \frac{(-1)^{n+1}}{n!} r^{2n+1} \frac{\partial^n}{\partial x_1^n} \left( \frac{1}{r} \right) \bigg|_{r=1}
\]

Equation (2.16) shows that the \( \omega_i \) potential is zero. The boundary equations (2.21) are coupled ordinary differential equations in the generating functions \( F^n_0(t_p) \) and \( G^n_0(t_p) \).

Once the generating functions are found as in Section 2.5, the stresses in the cavity surface \( \sigma^{n0}_{\theta\theta} \) and \( \sigma^{n0}_{\phi\phi} \) for each harmonic component of the reflected wave are given by (2.13) by putting \( r = 1 \). The stress in the interior can be found from (2.12) and (2.13) using the generating functions \( F^n_0 \left[ t_p - (r-1) \right] \) and \( G^n_0 \left[ t_p - c(r-1) \right] \).
2.4.2. Boundary Equations for the Incident S Wave

Just as for the incident P wave, the incident S wave is taken as a step shear stress traveling in the negative $x_1$ direction and with displacement in the $x_2$ direction. If the shear stress $\sigma_{12}$ is $\sigma_s$ in the stressed region, then the stress for the incident S wave can be written

$$\sigma_{ij} = [\delta_{i2}\delta_{j1} + \delta_{i1}\delta_{j2}] \sigma_s H \left[ t_s + (x_1 - 1) \right]$$

where the time $t_s$ is taken as zero just when the incident reaches the front of the cavity. The displacement potential which gives (2.22) is of the vector type $\psi_1$, and it satisfies the wave equation (2.8b).

The traction on the cavity boundary is given by

$$T^s_j = - \left[ x_2 \delta_{j1} + x_1 \delta_{j2} \right] \sigma_s H \left[ t_s + (x_1 - 1) \right]$$

and the other spherical stresses needed are given by

$$\sigma^s_{\theta \theta} = - 2 \frac{x_1 x_2}{r^2} \sigma_s H \left[ t_s + (x_1 - 1) \right]$$

$$\sigma^s_{\theta \phi} = \frac{x_3}{r} \sigma_s H \left[ t_s + (x_1 - 1) \right]$$

$$\sigma^s_{\phi \phi} = 0$$

For the boundary equations it is shown in Appendix C that the potentials $\phi$ and $\psi_1$ of the disturbing wave involve harmonics of the type $K^n_{1c}$ only, and the potential $\omega_1$ involves harmonics of the type $K^n_{1s}$ only, where
\[ k_{1c}^n = x_2 L_n(x_1, r) \]
\[ k_{1s}^n = x_s L_n(x_1, r) \]

When these harmonics are substituted into the integrals of the boundary equations (2.16), the surface integrals simplify to integrals along \( x_1 \). The boundary equations for the incident shear wave are:

\[ \gamma_1^n + n \Delta_1^n = -\frac{n(2n+1)}{2(n+1)} \sigma_s \int_{1-t_s}^{1} 2x_1(1-x_1^2) L_n dx_1 \quad \text{for} \quad 0 \leq t_s \leq 2 \]
\[ = -\frac{n(2n+1)}{2(n+1)} \sigma_s \int_{-1}^{+1} 2x_1(1-x_1^2) L_n dx_1 \quad \text{for} \quad t_s \geq 2 \]

\[ n \gamma_1^n + n(2n+1) \Delta_1^n = -\frac{n(2n+1)}{2(n+1)} \sigma_s \int_{1-t_s}^{1} \left[ 2x_1 L_n + (1-x_1^2) \frac{\partial L_n}{\partial x_1} + 2x_1 (1-x_1^2) \frac{\partial L_n}{\partial r} \right] dx_1 \]
\[ = -\frac{n(2n+1)}{2(n+1)} \sigma_s \int_{-1}^{+1} \left[ 2x_1 L_n + (1-x_1^2) \frac{\partial L_n}{\partial x_1} + 2x_1 (1-x_1^2) \frac{\partial L_n}{\partial r} \right] dx_1 \quad \text{for} \quad t_s \geq 2 \]

\[ n(n+1) \chi_1^n = \frac{n(2n+1)}{2(n+1)} \sigma_s \int_{1-t_s}^{1} \left[ (1-3x_1^2) L_n + x_1 (1-x_1^2) \frac{\partial L_n}{\partial x_1} \right] dx_1 \]
\[ = \frac{n(2n+1)}{2(n+1)} \sigma_s \int_{-1}^{+1} \left[ (1-3x_1^2) L_n + x_1 (1-x_1^2) \frac{\partial L_n}{\partial x_1} \right] dx_1 \quad \text{for} \quad t_s \geq 2 \]

where \( L_n = L_n(x_1, r) \)
When the generating functions $F^n(t_s)$, $G^n(t_s)$, and $H^n(t_s)$ are found from the boundary equations (2.26) as in Section 2.5, the stresses at the cavity surface are found from (2.13) by setting $r = 1$. The stresses in the interior are given by (2.12) and (2.13) using the generating functions $F^n[t_s-(r-1)/c]$, $G^n[t_s-(r-1)]$, and $H^n[t_s-(r-1)]$. The formulas (2.13) simplify to

$$Q^n_{1c} = x_2 q^n$$

$$Q^n_{1s} = x_3 q^n$$

$$R^n_{1c} = 2 \left[ n(n-1) L^n - 2(n-1) x_1 \frac{\partial L^n}{\partial x_1^2} \right] x_2 \left/ \left(1 - \frac{x_1^2}{r^2}\right) \right.$$

$$S^n_{1c} = -2 x_3 \left[ (n-1) \frac{x_1}{r} L^n - r \frac{\partial L^n}{\partial x_1} \right] \left/ \left(1 - \frac{x_1^2}{r^2}\right) \right.$$ (2.27)

$$S^n_{1s} = 2 x_2 \left[ (n-1) \frac{x_1}{r} L^n - r \frac{\partial L^n}{\partial x_1} \right] \left/ \left(1 - \frac{x_1^2}{r^2}\right) \right.$$ (2.27)

$$V^n_{1} = Q^n_{1c} - r x_3 \frac{\partial L^n}{\partial r}$$

where

$$q^n = 2 \left[ n(n-1) \left(\frac{x_1}{r}\right)^2 L^n - 2(n-1) x_1 \frac{\partial L^n}{\partial x_1} + r^2 \frac{\partial^2 L^n}{\partial x_1^2} \right] \left/ \left(1 - \frac{x_1^2}{r^2}\right) \right.$$ (2.27)

$$L^n = L^n \left[x_1, r(x_1, x_2, x_3)\right]$$ (2.27)

2.5. Numerical Solution of the Boundary Equations

All the boundary equations (2.17), (2.21), and (2.26) are ordinary linear differential equations for the generating functions $F^n_m$, $G^n_m$, and $H^n_m$ in the independent variable time $(t_p$ or $t_s)$. If the equations were uncoupled,
the order of the highest derivative would be $2n+4$, $n\neq 0$ (and 2 when $n=0$), where $n$ is the degree of the associated spherical harmonic. Since $n$ may be large, the solution by the classical method is very cumbersome. However, all the boundary equations are initial-value problems so that numerical integration starting with known conditions at zero time is easily carried out with the aid of a digital computer.

The method used to solve the boundary equations by numerical integration, called the trapezoidal rule, consists of the following procedure:

1. Divide the time up into equal small intervals generally $1/50$ to $1/500$ of a radius transit time. The greater number of divisions is required for boundary equations associated with the higher harmonics because they oscillate more rapidly.

2. Assuming the highest derivative as the latest value obtained, calculate the derivatives at a new point starting with the second highest by the trapezoidal integration rule

$$F_N^{(j)} = F_0^{(j)} + \frac{\Delta}{2} (F_0^{(j+1)} + F_N^{(j+1)}) \quad (2.28)$$

where $\Delta$ is the time interval, $F_N^{(j)}$ the $j$'th derivative at the new point and $F_0^{(j+1)}$ the $(j+1)$'st derivative at the old point.

3. Substitute the derivatives calculated in (2) into the boundary equation to recalculate the highest derivative.

4. Using the highest derivative calculated in (3), repeat (2) and (3) until successive highest derivatives agree to within a small test allowance. When the iteration is complete the last set of derivatives calculated in (3) are assumed to be sufficiently accurate.

The procedure is similar for the coupled equations except that in the computation of the highest derivatives, both unknown functions appear.
The coupled equations are of such a form that the highest derivative of $F_m^n$ appears in the first equation only and the highest derivative of $G_m^n$ appears in the second equation only, so that recalculation of the highest derivatives is straightforward, i.e., does not involve the solution of simultaneous algebraic equations.

For the initial conditions, no part of the incident wave has reached the cavity for $t<0$, and therefore all the wave functions $F_m^n$, $G_m^n$, and $H_m^n$ of the diverging disturbing wave and their derivatives appearing in the boundary equation are zero at the boundary and in the interior for $t<0$. Also, from the boundary equations (2.21) and (2.26) for the incident $P$ and $S$ waves, the right-hand side tends to zero as $t \to 0^+$, so that all the derivatives of the generating functions appearing in the boundary equations tend to zero as $t \to 0^+$. This is not so for a step pressure applied symmetrically to the cavity boundary.

For an incident $P$ (or $S$) wave, there is a cusp in the highest derivative for the $F_0^n$ (or $G_1^n$) generating function at the instant the incoming step wave becomes tangent to the cavity. This is taken care of in the numerical procedure because continuity of slope is not assumed in the highest derivative.

During the computation, errors occur due to truncation inherent in the trapezoidal rule, round off, and finiteness of the test allowance. These are discussed in many references, (see e.g. Ref. 20, pp. 174-187). These errors tend to be cumulative as the number of intervals is increased. Therefore, a good check is obtained by comparing the numerical solution as $t \to \infty$ with the static solution. This comparison is discussed in detail in Chapter 5.

The computation of the stresses $\sigma_{\rho\rho}^{nm}$, $\sigma_{\phi\phi}^{nm}$, and $\sigma_{\rho\phi}^{nm}$ at the cavity surface is straightforward. When the numerical integration has been completed for some value in time, the generating functions $F_m^n$, $G_m^n$, and $H_m^n$ and their
derivatives are known at the boundary and the stresses in the cavity surface can be calculated from (2.13) using (2.9) for \( f_m^n \), \( g_m^n \), and \( h_m^n \).

As the value of \( n \) increases, the polynomial expressions for the spherical harmonics and associated formulas for the tractions become poorly conditioned in that calculation involves subtracting a large number from an almost equal large number. Use of straightforward computation without double precision limits \( n \) to about 13 on a digital computer with eight decimal places.

Computations were carried out using the IBM 7094 system of the Department of Computer Science of the University of Illinois. Some unusual features of the program are discussed in Appendix E.
With the aid of the elastic solution given in Chapter 2, the Laplace transform of the viscoelastic solution is obtained in this chapter by means of the correspondence principle. The boundary equation for each spherical harmonic gives three algebraic equations in terms of the viscoelastic generating functions. The required stresses are determined from the stress transforms by a numerical scheme which involves expressing the inverse by a series of orthogonal terms. Two viscoelastic materials which are elastic in dilatation, represented by the Maxwell model and the standard linear model, are chosen for calculation.

3.1. Viscoelastic Stress-Strain Relation

The material is assumed to be linearly viscoelastic, isotropic, and homogeneous, and to have a finite instantaneous elasticity so that all waves propagate with a finite speed. For isotropic viscoelasticity, the stress-strain relation is completely described by one relation for the dilatational component of stress and strain and another relation which is common to the six deviatoric components of stress and strain. (1) The deviatoric components of stress $s_{ij}$ and strain $e_{ij}$ are defined as

$$s_{ij} = \sigma_{ij} - \delta_{ij} \frac{\sigma_{kk}}{3}$$

$$e_{ij} = e_{ij} - \delta_{ij} \frac{\Delta}{3}$$

where

$$\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$
The general linear viscoelastic stress-strain relation (for either the dilatation or deviatoric components) can be derived by considering a unidimensional model such as the one shown in Fig. 5 made up of a combination of spring and dashpot elements. The force applied to the model represents the stress $\sigma$ and the elongation represents the strain $\epsilon$. The spring elements have spring constants $E_j$ (that is, the elongation $\delta = E_j P$, where $P$ is the force) and the dashpot elements have coefficients of viscosity $\eta$ (that is, $\eta \frac{d\delta}{dt} = P$). The stress-strain relation is determined by summing the forces and elongations in the spring and dashpot elements and at the same time maintaining compatibility and equilibrium in the model.

One of the simplest viscoelastic models is the Maxwell model shown in Fig. 4, and the stress-strain relation is found by summing the elongation rates in the elastic and viscous elements;

$$\frac{d\epsilon}{dt} = \frac{d}{dt}(\delta_{\text{elastic}} + \delta_{\text{viscous}}) = \frac{1}{E} \frac{d\sigma}{dt} + \frac{p}{E} \sigma \tag{3.2}$$

where $E$ is the spring constant, $E/p$ the coefficient of viscosity of the dashpot, and $1/p$ is called the relaxation time of the Maxwell model. If the material is at rest for $t<0$, the Laplace transform of (3.2) is

$$\overline{\sigma} = \frac{E}{s + \frac{p}{E}} \overline{\epsilon} \tag{3.3}$$

where the bar denotes the Laplace transform* defined as

$$\overline{\sigma(x,t)} = \int_0^\infty e^{-st} \sigma(x,t) dt$$

* See, for example, Ref. 22.
A general viscoelastic model having instantaneous elasticity can be represented as in Fig. 5 by combining Maxwell elements with a spring in parallel. When a dashpot is added in parallel, this model is called the generalized Maxwell model. All the parallel elements have the same elongation $\epsilon$ and, for a material initially at rest, the stress-strain relation is found by summing up the forces in the elements and using (3.3);

$$\bar{\sigma} = E \left[ \alpha_0 + \sum_{j=1}^{n} \frac{\alpha_j s}{s+p_j} \right], \quad \text{where} \quad \sum_{j=0}^{n} \alpha_j = 1$$ (3.4)

The parameters $\alpha_j$, $p_j$, and $E$ are defined in Fig. 5. The ratio $\frac{\bar{\sigma}}{\bar{\epsilon}}$ will be called the $s$-varying modulus for the viscoelastic material.

If a step strain $\epsilon_0 H(t)$ is applied to the model the resulting stress variation with time is called the relaxation function $\sigma_R$;

$$\bar{\sigma}_R = E \left[ \alpha_0 + \sum_{j=1}^{n} \frac{\alpha_j s}{s+p_j} \right] \frac{\epsilon_0}{s}$$ (3.5)

The instantaneous elasticity $E_0$ is given by the instantaneous stress response $\sigma_0$ to the applied step strain, and, since $\sigma_0$ is determined from

$$\lim_{s \to \infty} \frac{\sigma}{\epsilon} = (22)$$

$$E_0 = \frac{\sigma_0}{\epsilon_0} = \frac{1}{\epsilon_0} \lim_{s \to \infty} \frac{\sigma}{\epsilon} = E$$ (3.6)

As $t \to \infty$, the relaxation function gives the static stress-strain relation or static modulus $E_{st}$. Since the static stress $\sigma_{st}$ is determined from

$$\lim_{s \to \infty} \frac{\sigma}{\epsilon} = (22)$$

$$\frac{\sigma_{st}}{\epsilon_0} = E_{st}$$
Direct examination of the model of Fig. 5 shows that the above instantaneous modulus $E$ and the static modulus $E\alpha_0$ are correct because, if a step elongation is applied, the dashpots initially do not elongate whereas after a long time the Maxwell elements carry no stress.

The viscoelastic stress-strain relation for the dilatation and for the deviatoric components can therefore be written

$$
\bar{\sigma}_{kk} = 3K \left[ \alpha^t_0 + \sum_{j=1}^{m} \frac{\alpha^s_j}{s+p_j} \right] \Delta, \quad \text{where} \quad \sum_{j=0}^{m} \alpha^t_j = 1
$$

$$
= [3\lambda(s) + 2\mu(s)]\Delta
$$

$$
\bar{\sigma}_{ij} = 2G \left[ \alpha^t_0 + \sum_{j=1}^{n} \frac{\alpha^s_j}{s+p_j} \right] \bar{\varepsilon}_{ij}, \quad \text{where} \quad \sum_{j=0}^{n} \alpha^t_j = 1
$$

$$
= 2\mu(s)\bar{\varepsilon}_{ij}
$$

where $K$ and $G$ are the instantaneous bulk and shearing moduli, respectively, $K\alpha^t_0$ and $G\alpha^t_0$ are the static bulk and shearing moduli respectively, and $\mu(s)$ and $\lambda(s)$ are the $s$-varying Lamé "constants."

The viscoelastic $s$-varying shear and longitudinal wave speeds, $C_s(s)$ and $C_p(s)$ respectively, and their ratio $c(s)$, are defined as

$$
C^2_s(s) = \frac{\mu(s)}{\rho} = \frac{G}{\rho} \left[ \alpha^t_0 + \sum_{j=1}^{n} \frac{\alpha^s_j}{s+p_j} \right]
$$

$$
C^2_p(s) = \frac{\lambda(s) + 2\mu(s)}{\rho} = \frac{K}{\rho} \left[ \alpha^t_0 + \sum_{j=1}^{m} \frac{\alpha^s_j}{s+p_j} \right] + \frac{4}{3} \frac{G}{\rho} \left[ \alpha^t_0 + \sum_{j=1}^{n} \frac{\alpha^s_j}{s+p_j} \right]
$$

(3.9)
c(s) = C_p(s)/C_s(s) \hspace{1cm} (3.9 \text{ cont.})

The instantaneous shear and longitudinal wave speeds, \(C_{s0}\) and \(C_{p0}\) respectively, are given by

\[
C_{s0}^2 = G/\rho \hspace{1cm} (3.10)
\]

\[
C_{p0}^2 = (3K + 4G)/3\rho
\]

The transform of the stress-strain relation is obtained by substituting (3.1) into (3.8);

\[
\bar{\sigma}_{ij} = \mu(s) \left[ \delta_{ij} (c_s^2(s) - 2\Delta + 2\bar{\varepsilon}_{ij}) \right] \hspace{1cm} (3.11)
\]

The static stress-strain relation, determined from the static moduli \(G\alpha_0\) and \(K\alpha_0\), is given by

\[
\sigma_{ij}^{st} = G\alpha_0 \left[ \delta_{ij} (c_s^2 - 2\Delta + 2\varepsilon_{ij}) \right] \hspace{1cm} (3.12)
\]

where

\[
c_s^2 = \frac{K\alpha_0}{G\alpha_0} + \frac{4}{3} = \lim_{s \to 0} c_s^2(s)
\]

If the material is represented by a Maxwell model, the static modulus as determined by (3.7) is zero, but if the strain \(\varepsilon_0\) is replaced by a step strain rate \(\frac{d\varepsilon_0}{dt}\), then there is a static modulus between the stress and the strain rate, given by

\[
\sigma^{st} = \frac{\varepsilon_0}{\rho} \frac{d\varepsilon_0}{dt}
\]

For the three dimensional case, the relation becomes
where

\[ c_{st}^2 = \frac{K_p}{G_p} + \frac{4}{3} \]  

(3.13)

3.2. **Disturbing Wave for Viscoelastic Material**

By the correspondence principle stated in 1.3 the solution for a traction specified on the spherical cavity in a viscoelastic material is obtained by taking the Laplace transform of the elastic solution given by (2.16), (2.13), (2.12), and (2.9) and replacing the elastic constants \( C_p, C_s, \) and \( c \) by the corresponding viscoelastic \( s \)-varying parameters \( C_p(s), C_s(s), \) and \( c(s) \).

The only terms in the elastic solution involving time are the traction \( T_{ij}^0 \) applied to the cavity boundary and the generating functions of the disturbing wave. For the viscoelastic material, it is convenient to specify the time either by \( t_{p0} = tCP_0 \), which is measured in multiples of the time required for the fastest incident \( p \) wave components to traverse a cavity radius, or by \( t_{s0} = tCs0 \), which is measured in multiples of the time required for the fastest incident \( s \) wave components to traverse a cavity radius. If the elastic generating functions \( F_n^{m}[tC_p-(r-l)], G_n^{m}[tC_p-c(r-l)], \) and \( H_n^{m}[tC_p-c(r-l)] \), when \( t \) is the proper measure of time, are replaced by new generating functions \( F_n^{m}[tC_p0-(r-l)], G_n^{m}[tC_p0-cC_p0(r-l)], \) and \( H_n^{m}[tC_p0-cC_p0(r-l)] \), then, because these functions for the disturbing wave are zero when their argument is negative, the Laplace transforms are, respectively, \( F_n^{m}(s)e^{-(r-l)C_p0s/C_p}, G_n^{m}e^{-(r-l)cC_p0s/C_p}, \) and \( H_n^{m}e^{-(r-l)cC_p0s/C_p} \). The transforms of the viscoelastic generating functions are the same except that

\* \( C_p0 \) or \( C_s0 \) is in cavity radii per second. The quantities \( t_{p0} \) or \( t_{s0} \) are measured in radius transit times.
$C_p$ is replaced by $C_p(s)$ and $c$ by $c(s)$. Introducing the new viscoelastic parameters

$$\alpha(s) = \frac{C_p\dot{0}^s}{C_p(s)} \quad (3.14)$$

$$\beta(s) = \frac{C_s\dot{0}^s}{C_s(s)}$$

we obtain the following expansion of the disturbing viscoelastic wave from (2.9);

$$\Phi = \sum_{n=0}^{\infty} \sum_{m} f^n_m(r,s)K_m^n$$

$$\Psi_i = \sum_{n=\Phi}^{\infty} \sum_{m} g^n_m(r,s)\epsilon_{ijk} k^{\alpha} \phi_{i,m}^{k,n}$$

$$\omega_i = \sum_{n=1}^{\infty} \sum_{m} h^n_m(r,s)x_iK_m^n$$

where, when $t_{p0}$ is the proper measure of time,

$$f^n_m = \left(\frac{1}{r} \frac{d}{dr}\right)^n \left[\frac{1}{r} F^n_m(s)e^{-(r-1)\alpha(s)}\right]$$

$$g^n_m = \left(\frac{1}{r} \frac{d}{dr}\right)^n \left[\frac{1}{r} G^n_m(s)e^{-(r-1)c(s)\alpha(s)}\right] \quad (3.15a)$$

$$h^n_m = \left(\frac{1}{r} \frac{d}{dr}\right)^n \left[\frac{1}{r} H^n_m(s)e^{-(r-1)c(s)\alpha(s)}\right]$$

and, when $t_{s0}$ is the proper measure of time,
\[
\begin{align*}
\tilde{f}_m^n &= \left(\frac{1}{r} \frac{d}{dr}\right)^n \left[\frac{1}{r} f_m^n(s) e^{-(r-1)\beta(s)c(s)}\right] \\
\tilde{g}_m^n &= \left(\frac{1}{r} \frac{d}{dr}\right)^n \left[\frac{1}{r} g_m^n(s) e^{-(r-1)\beta(s)c(s)}\right] \\
\tilde{h}_m^n &= \left(\frac{1}{r} \frac{d}{dr}\right)^n \left[\frac{1}{r} h_m^n(s) e^{-(r-1)\beta(s)c(s)}\right]
\end{align*}
\]

As in the case of the elastic material, the components of \( \tilde{\Phi} \) are P waves, and the components of \( \tilde{\Psi}_i \) and \( \tilde{\Omega}_i \) are S waves.

The formulas for the stresses of the disturbing wave given in 2.3 still apply except that \( \mu \) and \( c \) are replaced by \( \mu(s) \) and \( c(s) \) respectively and the terms involving the generating functions are replaced by the transforms of the viscoelastic generating functions given in (3.15). The traction on a spherical surface is given by (2.12) except that \( \gamma_m^n, \Delta_m^n, \) and \( \chi_m^n \) are obtained from their transforms;

\[
\begin{align*}
\tilde{\gamma}_m^n &= \mu(s) \left\{ c^2(s) \frac{d^2 f_m^n}{dr^2} + \frac{2}{r} \left[ c^2(s)(n+1)-(n+2) \right] \frac{df_m^n}{dr} + n \frac{d^2 g_m^n}{dr^2} - \frac{2n}{r} \frac{dg_m^n}{dr} \right\} \\
\tilde{\Delta}_m^n &= \mu(s) \left\{ \frac{2}{r} \frac{d^2 f_m^n}{dr^2} + \frac{2(n-1)}{r^2} \frac{df_m^n}{dr} - \frac{d^2 g_m^n}{dr^2} - \frac{2n}{r} \frac{dg_m^n}{dr} - \frac{2(n-1)}{r^2} \frac{h_m^n}{r} \right\} \\
\tilde{\chi}_m^n &= \mu(s) \left\{ \frac{1}{r} \frac{d h_m^n}{dr} + \frac{n-1}{r^2} \frac{h_m^n}{r} \right\}
\end{align*}
\]

The stresses \( \sigma_{nm}^{\eta\phi}, \sigma_{nm}^{\phi\phi}, \) and \( \sigma_{nm}^{\phi\theta} \) for a viscoelastic material are given by (2.13) except that \( \alpha_m^n, \beta_m^n, \) and \( \gamma_m^n \) are obtained from their transforms;

\[
\begin{align*}
\tilde{\alpha}_m^n &= \mu(s) \left\{ [c^2(s)-2] \frac{d^2 f_m^n}{dr^2} + \frac{2}{r} \left[ c^2(s)(n+1)-(2n+1) \right] \frac{df_m^n}{dr} + \frac{2n}{r} \frac{dg_m^n}{dr} \right\}
\end{align*}
\]

(3.16)
3.3. Boundary Equations

For any traction applied to the cavity surface, the boundary equations (2.16) give the solution for the viscoelastic material except that \( \mu \), \( \gamma^m_n \), \( \Delta^m_n \), and \( T_j^0 \) are replaced by \( \mu(s) \), \( \gamma^m_n \), \( \Delta^m_n \), \( \chi^m_n \), and \( T_j^0 \) respectively. When this is done, it is seen from (3.16) and (3.15) that the first two boundary equations are coupled algebraic equations in the transforms of the viscoelastic generating functions \( \bar{F}^n_m(s) \) and \( \bar{G}^n_m(s) \), whereas the third is an algebraic equation in \( \bar{H}^n_m(s) \) alone.

For the problem of a step pressure symmetrically applied to the cavity surface, the boundary equation for a viscoelastic material, in terms of time \( t_{p0} \), is given by

\[
[c^2(s)\alpha^2(s) + 4\alpha(s) + 4] \bar{F}_0^0(s) = -\frac{\sigma}{\mu(s)} \frac{1}{s}
\]  

(3.18)

3.3.1. Boundary Equations for the Incident P Wave

Because of dissipation, the traveling step wave taken as the incident wave for the elastic material is no longer a possible solution for the viscoelastic material. However, any incoming plane P wave can be treated in the following way:

(1) Assume that a plane viscoelastic wave traveling in the negative \( x_1 \) direction causes a step stress \( \sigma_{11} = -\sigma_p H(t_{p0}) \) on a plane at a specified position \( x_1 \), which is chosen in this investigation as the plane \( x_1 = 1 \) tangent to the cavity boundary.
(2) The incident viscoelastic wave for this case which satisfies the equations of motion is obtained by taking the Laplace transform of the incident elastic wave given by (2.18) and replacing \( C_p \) and \( c \) by \( C_p(s) \) and \( c(s) \), respectively. The viscoelastic solution for the incident P wave in terms of \( t_{p_0} \) becomes

\[
\tilde{\sigma}_{ij} = -\left[2\delta_{i1}\delta_{j1} + \delta_{ij}(c^2(s)-2)\right] \frac{\sigma_p(s)}{c^2(s)} \frac{e}{s} \tag{3.19}
\]

(3) If the form of the incident plane P wave is given at some position \( x_1 \) away from the spherical cavity, then an analysis can be made to determine the variation of stress \( \sigma_{11} \) with time at the position \( x_1 = 1 \). The stresses around the spherical cavity due to the stress variation \( \sigma_{11} \) can be determined from the stresses due to the wave given by (3.19) by the Duhamel integral.

For the viscoelastic wave (3.19), the traction on a spherical surface is given by

\[
\tilde{T}^p_{ij} = \left[2 \frac{x_1}{r} \delta_{ij} + \frac{1}{r} \left(c^2(s)-2\right)\right] \frac{\sigma_p(s)}{c^2(s)} \frac{e}{s} \tag{3.20}
\]

and the other stresses needed are given by

\[
\tilde{\sigma}^p_{\theta\theta} = -\left[1 - \frac{x_1^2}{r^2} \frac{2}{c^2(s)}\right] \frac{\sigma_p(s)}{s} \frac{e}{s} \tag{3.21}
\]

\[
\tilde{\sigma}^p_{\phi\phi} = -\left[1 - \frac{2}{c^2(s)}\right] \frac{\sigma_p(s)}{s} \frac{e}{s}
\]

The boundary equations are obtained as in 2.4.1 with \( \mu, \gamma^m_n, \) and \( \Delta^m_n \) in (2.21) replaced by \( \mu(s), \gamma^m_n, \) and \( \Delta^m_n \), and \( \tilde{T}^p \) given in (2.19) replaced by \( \tilde{T}^p_j \) given in (3.20);
\begin{equation}
\gamma^n_m + n\gamma^n_0 = \left( \frac{2n+1}{2} \right) \frac{\sigma_p e^{-\alpha(s)}}{s c^2(s)} \int_1^\infty \left[ (c^2(s) - 2) + 2x_1^2 \right] k_0^\alpha e^{x_1 \alpha(s)} dx_1
\end{equation}

\begin{equation}
\gamma^n_0 + n(2n+1)\alpha^n_0 = \left( \frac{2n+1}{2} \right) \frac{\sigma_p e^{-\alpha(s)}}{s c^2(s)} \int_1^\infty \left[ (c^2(s) - 2) n k_0^\alpha + 2x_1 \frac{\partial k_0^\alpha}{\partial x_1} \right] x_1 \alpha(s) dx_1
\end{equation}

When \( F^n_0 \) and \( G^n_0 \) are determined from (3.22), the stresses (2.13) are found by inverting \( \alpha^n_0 \) and \( \beta^n_0 \).

3.3.2. Boundary Equations for the Incident $S$ Wave

Just as in the method for the $P$ wave, the incident shear wave is taken as a wave traveling in the negative $x_1$ direction which causes a step shear stress $\sigma_{12} = \sigma_s H(t-t_0)$ on the plane $x_1 = 1$. The solution for the viscoelastic wave for this case is obtained by taking the Laplace transform of (2.22) with respect to $t-t_0$ and replacing $\zeta_s$ by $\zeta_s(s)$;

\begin{equation}
\bar{\sigma}_{ij} = \left[ \delta_{i2} \delta_j 1 + \delta_{i1} \delta_j 2 \right] \frac{\sigma_s}{s} (x_1 - 1)^\beta(s)
\end{equation}

The traction on a spherical surface is given by

\begin{equation}
\bar{t}^s_j = - \left[ \frac{x_2}{r} \delta_j 1 + \frac{x_1}{r} \delta_j 2 \right] \frac{\sigma_s}{s} (x_1 - 1)^\beta(s)
\end{equation}

and the other stresses needed are given by

\begin{equation}
\bar{\sigma}_\theta \theta = - \frac{2x_1 x_2}{r^2} \frac{\sigma_s}{s} (x_1 - 1)^\beta(s)
\end{equation}

\begin{equation}
\bar{\sigma}_\theta \phi = \frac{x_3}{r} \frac{\sigma_s}{s} (x_1 - 1)^\beta(s)
\end{equation}
The boundary equations are given by (2.26) with \( \mu, \gamma_1^n, \Delta_1^n, \) and \( \chi_1^n \) replaced by \( \mu(s), \gamma_1^n, \Delta_1^n, \) and \( \chi_1^n, \) respectively, and \( T_j^s \) in (2.23) replaced by \( T_j^s \) in (3.24):

\[
\gamma_1^n + n\Delta_1^n = -\frac{n(2n+1)}{2(n+1)} \frac{\sigma_s}{s} e^{-\beta(s)} \int_{-1}^{+1} 2x_1(1-x_1^2) L^n e x_1^\beta(s) \, dx_1
\]

\[
n\gamma_1^n + n(2n+1)\Delta_1^n = -\frac{n(2n+1)}{2(n+1)} \frac{\sigma_s}{s} e^{-\beta(s)} \int_{-1}^{+1} \left[ 2x_1 L^n(1-x_1^2) \frac{\partial L^n}{\partial x_1} + 2x_1(1-x_1^2) \frac{\partial L^n}{\partial r} \right] e x_1^\beta(s) \, dx_1
\]

\[
n(n+1)\chi_1^n = \frac{n(2n+1)}{2(n+1)} \frac{\sigma_s}{s} e^{-\beta(s)} \int_{-1}^{+1} \left[ (1-3x_1^2) L^n(1-x_1^2) \frac{\partial L^n}{\partial x_1} \right] e x_1^\beta(s) \, dx_1
\]

where \( L^n = L^n(x_1, r) \)

When \( F_1^n, G_1^n, \) and \( H_1^n \) have been determined from (3.26), the stresses (2.13) are found by inverting \( \alpha_1^n, \beta_1^n, \) and \( \chi_1^n. \)

### 3.4. Numerical Computation of the Stresses from the Stress Transforms

The stress functions \( \alpha^n_m, \beta^n_m, \) and \( \chi^n_m \) needed for the nonvanishing stresses in the cavity surface are given in (3.17) and (3.15) by the values of \( F^n_m, G^n_m, \) and \( H^n_m \) determined from the boundary equations (3.22) or (3.26). The inversion of \( \alpha^n_m, \beta^n_m, \) and \( \chi^n_m \) in terms of known functions or by contour integration (when this can be done) results in complicated formulas for the viscoelastic material (as, for example, in Chapter 4) so that a numerical scheme is desirable. The method here used involves making a transformation so that the Laplace transform becomes an integral over a finite interval,
representing the inverse in a half range sine series, and calculating the coefficients of this series from discrete values of the transform at equidistant points on the real line of the transform parameter $s$. This method, which is similar to the one given by Papoulis\textsuperscript{(12)} for other orthogonal sets, is developed as follows.

It is desired to obtain the function $f(t)$ in the Laplace transform

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

(3.27)

knowing the function $F(s)$. A substitution for $t$ is made by letting

$$e^{-\sigma t} = \sin r$$

(3.28)

where $\sigma$ is a real constant. The Laplace transform (3.27) becomes

$$F(s) = \frac{1}{\sigma} \int_0^{\pi/2} \cos r (\sin r)^{s-1} g(r) dr$$

(3.29)

where

$$g(r) = f \left[ \frac{1}{\sigma} \ln \left( \frac{1}{\sin r} \right) \right]$$

If $s/\sigma$ is assumed to take on even integral values $2k$, where $k$ is an integer, then (3.29) becomes

$$F(2k\sigma) = \frac{1}{\sigma} \int_0^{\pi/2} \cos r (\sin r)^{2k-1} g(r) dr$$

(3.30)

The inverse function $g(r)$ defined over the interval $(0, \frac{\pi}{2})$ is expanded in a half range sine series;}
This sine series representation assumes that the values of \( g(r) \) at its end points are zero, that is, \( f(t) \) is zero at \( t = 0 \) and \( f(t) \) approaches zero as \( t \to \infty \). If these values are not zero at the end points, but can be determined there, the method can still be used in a manner described below.

A simple way to obtain the coefficients \( c_n \) is to expand \( \cos r(\sin r)^{2k-1} \) in terms of the sine series (3.31) as follows:

\[
\cos r(\sin r)^{2k-1} = \left( e^{i r} + e^{-i r} \right) \left( e^{i r} - e^{-i r} \right)^{2k-1}
\]

\[
= \frac{1}{2^{2k-1}} \left\{ (-1)^{k-1} \sin 2 kr + (-1)^{k-2} \left[ \binom{2k-1}{1} - \binom{2k-1}{0} \right] \sin 2 (k-1) r \\
+ \ldots + (-1)^{k-j-1} \left[ \binom{2k-1}{j} - \binom{2k-1}{j-1} \right] \sin 2 (k-j) r \\
+ \ldots + \left[ \binom{2k-1}{k-1} - \binom{2k-1}{k-2} \right] \sin 2 r \right\} \tag{3.32}
\]

If (3.32) is substituted into (3.30), by orthogonality,

\[
2^{2k-1} \sigma F(2k\sigma) = \frac{\pi}{4} \left\{ (-1)^{k-1} + (-1)^{k-2} \left[ \binom{2k-1}{1} - \binom{2k-1}{0} \right] c_{k-1} \\
+ \ldots + (-1)^{k-j-1} \left[ \binom{2k-1}{j} - \binom{2k-1}{j-1} \right] c_{k-j} \\
+ \ldots + \left[ \binom{2k-1}{k-1} - \binom{2k-1}{k-2} \right] c_{1} \right\} \tag{3.33}
\]

When \( k = 1 \), (3.33) gives

\[
2\sigma F(2\sigma) = \frac{\pi}{4} c_{1}
\]
When \( k = 2 \), (3.33) gives

\[
2^3 \sigma F(4\sigma) = \frac{\pi}{4} \left[ -c_2 + (3-1)c_1 \right]
\]

so that the coefficients \( c_1, c_2, \ldots, c_n \) are obtained from the values of \( F(s) \) at the equidistant points \( 2\sigma, 4\sigma, \ldots, 2n\sigma \). From (3.33), the recursive formula for \( c_k \) is

\[
c_k = \left\{ \left[ \begin{array}{c} (2k-1) \\ 1 \\
0 \\
\end{array} \right] - \left[ \begin{array}{c} (2k-1) \\ j \\
0 \\
\end{array} \right] \right\} c_{k-1} + \ldots + (-1)^{j-1} \left\{ \left[ \begin{array}{c} (2k-1) \\ j \\
0 \\
\end{array} \right] - \left[ \begin{array}{c} (2k-1) \\ j-1 \\
0 \\
\end{array} \right] \right\} c_{k-j} + \ldots \\
+ (-1)^{k-2} \left\{ \left[ \begin{array}{c} (2k-1) \\ k-1 \\
0 \\
\end{array} \right] - \left[ \begin{array}{c} (2k-1) \\ k-2 \\
0 \\
\end{array} \right] \right\} c_1 + (-1)^{k-1} \frac{4}{\pi} 2^{2k-1} \sigma F(2k\sigma) \tag{3.34}
\]

When the values of the coefficients \( c_n \) are found, the inverse \( f(t) \) is given at any \( t \) by substituting into (3.31)

\[
T = \sin^{-1}(e^{-\sigma t})\tag{3.35}
\]

The choice of the constant \( \sigma \) is still left open, and for optimum convergence its choice depends on how the inverse is distributed along the \( t \) axis. If \( f(t) \) has its significant values near \( t = 0 \), then \( \sigma \) should be chosen fairly large (because \( F(s) \) will have its significant values for large \( s \)). However, if \( f(t) \) is spread out over the time axis, \( \sigma \) should be chosen smaller. In the numerical work, the value of \( \sigma \) was chosen by trial so that, for the elastic material, the solution converged best to the solution given by the method in 2.5. The convergence for the viscoelastic material is generally somewhat better than for the elastic material because the internal viscosity damps out the oscillations in the solution.

The numerical method has the disadvantage that very high accuracy of the transform is required in order to obtain enough significant figures for the
coefficients $c_n$ when $n$ gets large. It has been found that, as a rule, approximately $0.8n$ significant figures are required to compute $n$ sine waves. The number of sine waves needed for reasonable convergence depends on how much the inverse oscillates in the time interval $(0, \infty)$. With 28 figure accuracy (34 sine waves), it has been found that reasonable results for the incident P and S waves can be obtained for $\alpha_m^n$, $\beta_m^n$, and $h_m^n$ up to about $n=6$.

Another slight disadvantage is that as $t \to 0$, the slope of every sine wave becomes infinite. However, it has been found that the error in the inverse due to this is confined to a small interval $0 < t_p^0 \leq 0.1$, approximately.

The method gives a smooth inverse over the interval $(0, \infty)$ and assumes that at its end points the inverse is zero. A cusp or nonzero end value will cause a poor representation in the vicinity. This problem is overcome by removing the source of the difficulty and then adding it back separately as is now discussed.

As an example, the modal stress in the disturbing wave due to the incident P wave associated with the harmonic $K_0^0 = 1$, will be calculated. The boundary equation (3.22) is

$$[4 + 4\alpha + c^2 \alpha^2] \bar{F}_0^0 = \frac{\sigma}{2sc^2 \alpha^2} [4 - 4\alpha + c^2 \alpha^2 - e^{-2\alpha(4 + 4\alpha + c^2 \alpha^2)}]$$

where $\alpha$ is $\alpha(s)$ and $c$ is $c(s)$. The stress at the cavity surface is given by the function $\alpha_0^0$, where

$$\alpha_0^0 = [-2 - 2\alpha + (c^2 - 2)\alpha] \bar{F}_0^0$$

The inverse of $\alpha_0^0$ has both a cusp at $t_{p0}^0 = 2$ and a nonzero value as $t_{p0}^0 \to \infty$. For the elastic material, where $\alpha(s) = s$, the asymptotic value of $\alpha_0^0$
as \( t_p \to \infty \) is given by

\[
\alpha^0_{st} = \lim_{s \to 0} s \alpha^0_0 = -\frac{3c^2 - 4}{6c^2} \sigma_p \tag{3.38}
\]

This agrees with the static solution given in Appendix D. For viscoelastic materials, the static asymptote \( \alpha^0_{st} \) is determined by substituting \( c_{st} \) given by (3.12), or (3.13) for the Maxwell model, for \( c \) in (3.38). To use the numerical method, the Laplace transform of the term \( \alpha^0_{st} (1 - e^{-tp_0}) \), that is, \( \alpha^0_{st}/s(s+1) \), is subtracted from the transform \( \alpha^0_0 \), the transform inverted as above, and the term \( \alpha^0_{st} (1 - e^{-tp_0}) \) is added to the calculated inverse. By fixing the two end points in this way, it has been found that quite good convergence is obtained throughout the time interval \((0, \infty)\).

The cusp in the stress function \( \alpha^0_0 \) is due to the term in (3.36) containing the factor \( e^{-2\alpha} \). If only this part of the solution is considered, then \( \alpha^0_0 \) becomes

\[
\alpha^0_0 = \left[ (c^2 - 2) - \frac{2}{\alpha^2} \right] \frac{e^{-2\alpha}}{2sc^2} \sigma_p \tag{3.39}
\]

For the elastic material, \( \alpha = s \), and the inverse has the solution

\[
\alpha^0_0 = \left[ \frac{c^2 - 2}{2c^2} (t_p - 2) + O((t_p - 2)^2) \right] H(t_p - 2) \tag{3.40}
\]

near \( t_p = 2 \), where \( O((t_p - 2)^2) \) is a term of the order of \( (t_p - 2)^2 \). The cusp is therefore given by the angle change \( \theta = \frac{c^2 - 2}{2c^2} \). For the viscoelastic material it is possible from (3.9) and (3.14) to represent \( \alpha(s) \) as \( s + \gamma(s) \), where

\[
\gamma(s) = k_0 + \frac{k_1}{s} + \frac{k_2}{s^2} + \ldots \tag{3.41}
\]
and, since in this investigation, \( \lim_{s \to \infty} c^2(s) = c^2 \) (see 3.5), the inverse has the solution

\[
\alpha_0^0 = \left[ \frac{2-2}{2c^2} \right] e^{-2k_0(t_{p0}-2)} + 2 \left( (t_{p0}-2)^2 \right) H(t_{p0}-2) \tag{3.42}
\]

near \( t_{p0} = 2 \). Therefore the cusp for the viscoelastic material is given by the angle change \( \theta = \frac{c^2-2}{2c^2} e^{-2k_0} \).

To obtain the cusp in the inverse, the Laplace transform of the term \( \frac{2\theta}{\pi} \sin \frac{\pi t_{p0}}{2} H[(t_{p0})-H(t_{p0}-2)] \), that is, \( \theta(1-e^{-2s})/(s^2 + \frac{\pi^2}{4}) \), is subtracted from the transform \( \alpha_0^0 \), the transform inverted as above, and the term \( \frac{2\theta}{\pi} \sin \frac{\pi t_{p0}}{2} [H(t_{p0})-H(t_{p0}-2)] \) is added to the calculated inverse.

In summary, the transform \( F(s) \) used for the numerical scheme is

\[
F(s) = \alpha_0^0(s) - \frac{\alpha_0^0 st}{s(s+1)} - \frac{\theta(1-e^{-2s})}{s^2 + \frac{\pi^2}{4}} \tag{3.43}
\]

and the inverse \( \alpha_0^0 \), containing the required cusp and end value is given by

\[
\alpha_0^0 = \sum c_n \sin 2nr \alpha_0^0 st (1-e^{-t_{p0}})
+ 2\theta \sin \frac{\pi t_{p0}}{2} [H(t_{p0})-H(t_{p0}-2)] \tag{3.44}
\]

Although the transforms \( \alpha_m^0, \beta_m^0, \) and \( \pi_m^0 \) for the other harmonics are more complicated, the same method can still be used without difficulty.

3.5. Choice of Viscoelastic Materials

The viscoelastic materials chosen for the calculations will be assumed to act purely elastically in dilatation. This agrees with observations for materials like rock.\(^{(20)}\) It will also be assumed that the ratio
of instantaneous moduli $K/G$ is $5/3$ (corresponding to Poisson's ratio of $1/4$). This ensures that the fastest $P$ and $S$ waves travel with the same ratio of velocities as measured in most rocks. However, for the long-time static conditions, the material will not, in general, be representative of rock.

A viscoelastic material which has instantaneous elasticity and gives the simplest stress-strain relation is represented by the Maxwell model (Fig. 4). This material has the disadvantage of flowing unrestrictedly under constant stress, and is therefore not a realistic material for rock. However, the material takes account of internal dissipation, and the short-time stresses due to impulsive or step loadings may be representative for the material.

The required parameters for the Maxwell model are:

$$K = \frac{5G}{3}$$
$$\mu(s) = \frac{Gs}{s+p}$$
$$\beta^2(s) = s(s+p)$$
$$\alpha^2(s) = \frac{9s^2(s+p)}{9s+5p}$$
$$c^2(s) = 3 + \frac{5p}{3s}$$

where the viscoelastic constant $1/p$ has the same dimensions as the time variable (e.g. $t_{p0}$, time measured in terms of the number of cavity radii traversed by the fastest $P$-wave components).

A viscoelastic material which has instantaneous elasticity and does not flow unrestrictedly under constant stress may often be represented by the standard linear model. The model is composed of a spring and a Maxwell element in parallel (Fig. 5). The stress-strain relation is given by the first two terms in (3.4). If $\alpha_1$ is replaced by $1-\alpha$ for the spring in the Maxwell element and $\alpha/p_1$ by $1/p$ for the dashpot, the following parameters are obtained:
\[ K = \frac{5G}{3} \]

\[ \mu(s) = G \left[ \frac{s + \alpha(1-\alpha)p}{s + (1-\alpha)p} \right] \]

\[ \beta^2(s) = s^2 \left[ \frac{s + (1-\alpha)p}{s + \alpha(1-\alpha)p} \right] \]

\[ \alpha^2(s) = s^2 \left[ \frac{5}{9} + \frac{4}{9} \left( \frac{s + \alpha(1-\alpha)p}{s + (1-\alpha)p} \right) \right] \]

\[ c^2(s) = \frac{4}{3} + \frac{5}{3} \left[ \frac{s + (1-\alpha)p}{s + \alpha(1-\alpha)p} \right] \]

(3.46)

where the viscoelastic constant \( \frac{1}{p} \) is a characteristic time. When \( \alpha \rightarrow 0 \) the standard linear model approaches a Maxwell model, whereas when \( \alpha \rightarrow 1 \), the model becomes purely elastic.

The choice of the constants \( \alpha \) and \( p \) will be discussed in Chapter 5.
4. SOLUTION FOR THE VISCOELASTIC MATERIAL
BY THE GENERATING FUNCTIONS

The transforms of the generating functions which have been derived
in Chapter 3 can be inverted, for special viscoelastic materials having
instantaneous elasticity, to obtain expressions containing convolution
integrals. For these materials, the boundary condition gives integro-
differential equations of the Volterra type in place of the ordinary differential equations for the elastic material in Chapter 2. These equations are
integrated numerically by the same method as Chapter 2. Only the spherically
symmetric case is treated in this chapter for two materials, represented by
the Maxwell model and the standard linear model.

4.1. Inversion of the Generating Functions of the Disturbing Wave

For some viscoelastic materials having instantaneous elasticity,
the transforms of the generating functions, such as $F^n_m(s)e^{-(r-1)\alpha(s)}$, can be
inverted in terms of expressions containing convolution integrals, such as

$$\int_{r-1}^{\tau} F^n_m(t) f(\tau, r) d\tau$$

where $f(\tau, r)$ is a known function. Two materials, one
represented by the Maxwell model and the other by the standard linear model,
will be considered specifically for the case where the parameter $c(s)$, the
ratio of the $s$-varying $P$-wave speed to the $s$-varying $S$-wave speed, is independent of $s$. Also, because the formulas become involved, only the spherically
symmetric wave represented by the term $F^0_0$ will be considered. However, in
theory, all the terms $f^n_m$, $g^n_m$, and $h^n_m$ in (3.15) can be derived by following the
procedure presented.
4.1.1. Inversion for the Maxwell Model

The required transform parameters of the Maxwell model for the case where \( c(s) = c, \) a constant, are:

\[
\begin{align*}
\mathcal{C}_p^2(s) &= \frac{Gc^2}{\rho} \left( \frac{s}{s+p} \right) \\
\alpha^2(s) &= s(s+p)
\end{align*}
\]  

(4.1)

and the stress-strain relation (with time variable \( t_{p0} \)) is given by

\[
\sigma_{ij} = \left( \frac{s}{s+p} \right) \left[ \varepsilon_{ij} \left( \frac{c^2}{c^2-2} \right) \Delta + \frac{2}{c^2} \varepsilon_{ij} \right]
\]

This can be inverted to obtain

\[
\rho \sigma_{ij} + \frac{\partial \sigma_{ij}}{\partial t_{p0}} = \left[ \varepsilon_{ij} \left( \frac{c^2}{c^2-2} \right) \frac{\partial \Delta}{\partial t_{p0}} + \frac{2}{c^2} \frac{\partial \varepsilon_{ij}}{\partial t_{p0}} \right]
\]  

(4.2)

When (4.1) is substituted into (3.15a), the transform of the generating function containing \( \Phi_0^0 \) is given by

\[
r_f^0 = \Phi_0^0(s) e^{-r_1 \sqrt{s(s+p)}}
\]  

(4.3)

where

\[ r_1 = r - 1 \]

To invert (4.3) consider the transform

\[
\tilde{a} = \Phi_0^0(s) e^{-r_1 \sqrt{s(s+p)}}
\]  

(4.4)

where

\[ r_f^0 = -\frac{da}{dr} \]
The inverse of (4.4) can be obtained from the tables (23) as follows:

\[ a = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \left[ \int_{0}^{\infty} F_{0}^{0}(t') e^{-s t'} dt' \right] \frac{e^{-r_{1}\sqrt{s(s+p)}}}{\sqrt{s(s+p)}} ds \]

\[ = \int_{0}^{\infty} F_{0}^{0}(t') e^{\frac{-t^{2}}{2}} I_{0} \left( \frac{p}{2} \sqrt{\left( t_{p0}^{0} - t' \right)^{2} - r_{1}^{2}} \right) H[(t_{p0}^{0} - t') - r_{1}] dt' \]

\[ = \int_{0}^{t_{p0}^{0}} F_{0}^{0}(t_{p0}^{0} - t) e^{\frac{-pt^{2}}{2}} I_{0} \left( \frac{p}{2} \sqrt{\left( t_{0}^{0} - r_{1}^{2} \right)} \right) dt \cdot H(t_{p0}^{0} - r_{1}) \]

provided that interchanging the integrals \( \int_{\gamma - i\infty}^{\gamma + i\infty} \) and \( \int_{0}^{\infty} \) is legitimate. Here, \( I_{n} \) is the modified Bessel function of the first kind.

The generating function of the Maxwell model is given by (4.5) and (4.4);

\[ r_{f_{0}^{0}} = \left[ F_{0}^{0}(t_{p0}^{0} - r_{1}) e^{\frac{-pt_{1}^{2}}{2}} + \frac{p}{4} \int_{r_{1}}^{t_{p0}^{0}} F_{0}^{0}(t_{p0}^{0} - t) e^{\frac{-pt^{2}}{2}} I_{1}(z) dt \right] H(t_{p0}^{0} - r_{1}) \]

where \( z = \frac{p}{2} \sqrt{t_{0}^{2} - r_{1}^{2}} \)

* See, for example, Ref. 24. Use will be made of the identities

\[ \frac{d}{dz} \left( \frac{I_{n}(z)}{z^{n}} \right) = \frac{I_{n+1}(z)}{z^{n}} ; \quad \lim_{z \to 0} \frac{I_{n}(z)}{z^{n}} = \frac{1}{n!} 2^{n} \]

** The Dirac delta function arising from differentiation of the step function is multiplied by a factor equal to zero at the singularity.
The first term of the generating function on the right-hand side of (4.6) represents a 'decaying elastic' component and the second term, an integral which tends to zero as $t_{p0} \to r_1$, is a 'memory' component which includes all but the wave components traveling the fastest. These latter are accounted for in the first term.

When $F_0^n$ is replaced by $F_n^m$, the generating function (4.6) is applicable to all the terms of $\Phi$ in (2.9). However, the operation $(1/r \partial/r)^n$ required to determine the functions $f^m_n$ generates lengthy expressions, and this is one reason why the technique in Chapter 3 seems preferable for computation to the present one.

4.1.2. Inversion for the Standard Linear Model

The required transform parameters of the standard linear model for the case $c(s) = c$, a constant, are

$$C_p^2(s) = \frac{Gc^2}{\rho} \left[ \frac{s + c(1-c)p}{s + (1-c)p} \right]$$

$$\alpha^2(s) = s^2 \left[ \frac{s + (1-c)p}{s + c(1-c)p} \right]$$

(4.7)

and the stress-strain relation (with time variable $t_{p0}$) is given by

$$\bar{\sigma}_{ij} = \left[ \frac{s + c(1-c)p}{s + (1-c)p} \right] \delta_{ij} \left( \frac{c^2 - 2}{c^2} \right) \Delta + \frac{2}{c^2} \varepsilon_{ij}$$

This can be inverted to obtain

$$(1-c)p \sigma_{ij} + \frac{\partial \sigma_{ij}}{\partial t_{p0}} = \left[ \alpha(1-c)p + \frac{\partial}{\partial t_{p0}} \right] \delta_{ij} \left( \frac{c^2 - 2}{c^2} \right) \Delta + \frac{2}{c^2} \varepsilon_{ij}$$

(4.8)
When (4.7) is substituted into (3.15a), the transform of the generating function containing $F^0_0$ is given by

$$rF^0_0 = F^0_0(s)e^{-r_s^*}(s+(1-\alpha)p)s+\alpha(1-\alpha)p$$

(4.9)

The inversion of (4.9) is similar to the inversion for the Maxwell model but more complicated because the transform is not tabulated directly. The inversion for the case where $F^0_0(s) = 1/s$ has been obtained by Morrison, (13) and if his procedure is used, the generating function for the standard linear model becomes

$$rf^0_0 = \left\{e^{-\beta r}_1 F^0_0(t_p_0-r_1) + \beta^2 r 1 e^{-\lambda r}_1 \int^{t-p_0}_0 F^0_0(t_p_0-\tau)e^{-\gamma \tau} \left[ \frac{I_1(x)}{x} \right. \right.$$

$$+ 4B\lambda \int^{r_1+\eta}_0 (r_1+\eta)e^{-\beta \eta} \frac{I_1(y) I_1(z)}{y z} d\eta \left. \right] d\tau \right\} H(t_p_0-r_1)$$

(4.10)

where $\gamma = \frac{\beta^2}{2} (1-\alpha^2)$

$\beta = \frac{\beta^2}{2} (1-\alpha^2)$

$\lambda = p\alpha(1-\alpha)$

$x = \beta \sqrt{(\tau-r_1)(\tau+r_1)}$

$y = \beta \sqrt{(\tau-r_1)(\tau+r_1+2\eta)}$

$z = 2\sqrt{2B\lambda r_1}\eta$

The generating function of the standard linear model (4.10) is of the same form as the generating function of the Maxwell model (4.6). The first term on the right-hand side of (4.10) represents a "decaying elastic" component.
and the second a "memory" component. An additional term in the memory component, involving the integral in the brackets of (4.10) cancels out the unlimited flow effect which occurs with the Maxwell model.

When \( F^0 \) is replaced by \( f^n \) the generating function (4.10) is applicable to all terms of \( f^n \) in (2.9). Again, the operation \( \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^n \) to determine the functions \( f^n \) generates lengthy expressions.

4.2. Displacement, Stress, and Boundary Equations

4.2.1. Displacement, Stress, and Boundary Equation for the Maxwell Model

For the spherically symmetric case, the displacement is in the radial direction, and is given by

\[
  u^0_0 = \frac{x_i}{r} u^0_0 = \frac{\partial t^0_0}{\partial r} \quad (4.11)
\]

Substituting (4.6) for the Maxwell model, we have

\[
  u^0_0 = -e^{-\frac{p^2 r_1}{2 z}} \left[ \left( \frac{1}{r} + \frac{p^2}{2r} \right) F^0_0(t_0 - r_1) + \frac{1}{r} F^0_0(t_0 - r_1) \right] H(t_0 - r_1)
  - \frac{2}{4} \int_{r_1}^{t_0} F^0_0(t_0 - \tau) e^{-\frac{p^2 \tau}{2 z}} \left[ \frac{r_1 - r_0}{r^2} \frac{I_1(z)}{z} + \frac{p^2 r_1^2}{4r} \frac{I_2(z)}{z^2} \right] d\tau H(t_0 - r_1)
  \tag{4.12}
\]

where the dot indicates differentiation with respect to the argument.

The traction on a spherical surface is in the radial direction \( (\sigma_r) \), where, from (2.12) and (4.2)

\[
  p \sigma_r + \frac{\partial \sigma_r}{\partial t_{p0}} = -\frac{\partial \phi^0_0}{\partial t_{p0}} \quad (4.13)
\]

in which
The only other stress for the spherically symmetric case is the hoop stress $\sigma_h$. From (2.13) and (4.2),

$$p_0 \sigma_h + \frac{\partial \sigma_h}{\partial r_{p0}} = \frac{\partial \sigma_0^0}{\partial r_{p0}}$$

where

$$\sigma_0^0 = \left( \frac{c^2-2}{c^2} \right) \frac{\partial f_0^0}{\partial r^2} + \frac{2}{r} \left( \frac{c^2-1}{c^2} \right) \frac{\partial f_0^0}{\partial r}$$

For computational purposes, the differential equation (4.14) can be expressed in integral form;

$$\sigma_h = \sigma_0^0(t_{00}, r) - e^{-t_{00}} \int_{0}^{t_{00}} e^{\tau_0^0(t_{00}, \tau_{00}, r)} d\tau$$
If a step pressure, \(-\sigma H(t_{p0})\), is symmetrically applied to the cavity boundary \((r=1\) and \(r_1=0\)), from (4.13) the boundary equation for the Maxwell model becomes

\[
p\sigma H(t_{p0}) + \sigma \delta(t_{p0}) = c^2 \int_0^{t_{p0}} \bar{F}_0(t_{p0}) + (4+pc^2)\cdot \bar{F}_0(t_{p0}) + (4+2p)\bar{F}_0(t_{p0})
\]

\[
- p^2 \int_0^{t_{p0}} \bar{F}_0(t_{p0}-\tau) e^{-\frac{p\tau}{2}} \frac{i_1(z)}{z} d\tau \frac{H(t_{p0})}{c^2}
\]

(4.16)

where \(F_0(0)\) is zero and \(\delta(t_{p0})\) is the Dirac delta function.

The initial conditions, obtained from (4.16), are given by

\[
\bar{F}_0(t_{p0}) = \sigma \delta(t_{p0}) - \frac{4\sigma}{c} H(t_{p0})
\]

(4.17)

\[
\bar{F}_0(t_{p0}) = \sigma H(t_{p0})
\]

\[
\bar{F}_0(t_{p0}) = F_0(t_{p0}) = 0
\]

where \(t_{p0}\) is the interval \((0^-,0^+)\) of \(t_{p0}\).

When the function \(F_0(t_{p0})\) has been determined from (4.16) and (4.17) as in 4.3, the displacement (4.12) and stress (4.13) or (4.14) can be found anywhere in the medium. For the Maxwell model, the stress and displacement will be determined at the boundary and at \(r=2\).

4.2.2. Displacement, Stress, and Boundary Equation for the Standard Linear Model

The formulas for \(\frac{\partial \bar{f}_0}{\partial r}\) and \(\frac{\partial^2 \bar{f}_0}{\partial r^2}\) are quite lengthy but simplify considerably at the boundary. The formulas for the displacement and stress will therefore be given for \(r=1\) only.
The radial displacement corresponding to (4.11) is

\[ u_r^{00} = \left. \frac{\partial f_0}{\partial r} \right|_{r=1} = - \left[ (1+\beta)F_0^0(t_{p0}) + \tilde{F}_0^0(t_{p0}) \right] H(t_{p0}) \]

+ \beta^2 \int_0^{t_{p0}} F_0^0(t_{p0}-\tau) e^{-\gamma \tau} \left[ \frac{I_1(x)}{x} + 2\beta \lambda J_{11} \right] d\tau H(t_{p0})

(4.18)

where

\[ J_{ij} = \int_0^\infty \eta e^{-\beta \eta} \frac{I_j(y)}{y^j} d\eta \]

The traction on a spherical surface, \( \sigma_r \), is given by

\[ (\lambda + 2\beta)\sigma_r + \frac{\partial \sigma_r}{\partial t_{p0}} = - \left[ \lambda + \frac{\partial}{\partial t_{p0}} \right] \gamma_0^0 \]

(4.19)

where \( \gamma_0^0 \) is defined in (4.13) and

\[ \frac{\partial^2 f_0^0}{\partial r^2} \bigg|_{r=1} = \left[ 2(1+\beta-\lambda \beta)F_0^0(t_{p0}) + 2(1-\beta)\tilde{F}_0^0(t_{p0}) + \tilde{F}_0^0(t_{p0}) \right] H(t_{p0}) \]

- \beta^2 \int_0^{t_{p0}} F_0^0(t_{p0}-\tau) e^{-\gamma \tau} \left\{ (1+\lambda) \frac{I_1(x)}{x} - 2\beta \lambda J_{01} \right\} d\tau H(t_{p0})

+ 2\beta \lambda (1+\lambda) J_{11} - 2\beta^2 \lambda^2 J_{21} + 2\beta^3 \lambda J_{22} \]

(4.20)

Here, \( J_{ij} \) is defined in (4.18) and \( \beta \) and \( \lambda \) are defined in (4.10).

The hoop stress \( \sigma_r \) is given by

\[ (\lambda + 2\beta)\sigma_r + \frac{\partial \sigma_r}{\partial t_{p0}} = \left[ \lambda + \frac{\partial}{\partial t_{p0}} \right] \alpha_0^0 \]

(4.21)

where \( \alpha_0^0 \) is defined in (4.14). The integral form for (4.21) is
\[
\sigma_h = c_0(t_{p0}) - 2\beta \int_0^{t_{p0}} e^{(\lambda + 2\beta)(\tau - t_{p0})} c_0(\tau) \, d\tau
\]  
(4.22)

The boundary equation for an applied step pressure, \(-\sigma H(t_{p0})\), is

\[
(\lambda + 2\beta)\sigma H(t_{p0}) + \sigma(t_{p0})
\]

\[
= \left[ c^2 \tilde{F}_0(t_{p0}) + \left[ 4 + (\lambda + 2\beta)c^2 \right] \tilde{F}_0(t_{p0}) + 4(1 + \beta + \lambda)F_0(t_{p0}) \right] + 2\lambda(2 + 2\beta - \beta c^2)F_0(t_{p0}) - 2\beta^2 \int_0^{t_{p0}} \left[ \lambda F_0(t_{p0} - \tau) + \tilde{F}_0(t_{p0} - \tau) \right] \]

\[
e^{-\gamma\tau} \left[ \frac{I_1(x)}{x} - 2\beta\lambda c^2 J_{01} + 2\beta\lambda(2 + \lambda c^2) J_{11} - 2\beta^2\lambda c^2 J_{22} \right] \frac{H(t_{p0})}{\lambda c^2}
\]

(4.23)

and the initial conditions are the same as (4.17).

When the function \(F_0(t_{p0})\) has been determined from (4.23) and (4.17) as described in 4.3, the displacement (4.18) and stress (4.22) will be found at the boundary.

4.3. **Numerical Integration of the Boundary Equations**

The boundary equations (4.16) and (4.23) are linear integro-differential equations of the Volterra type in the function \(F_0(t_{p0})\). They can be integrated by the same numerical method as discussed in 2.5, that is by integrating the highest derivative (\(\tilde{F}_0\) in this case) through the boundary equation and using the trapezoidal approximation rule to obtain the lower derivatives from the highest.
The integral term, \( \int_0^{t_P} F_0(t_P - \tau) f(\tau) d\tau \) say, where \( f(\tau) \) is the known function in (4.16) or (4.23), is treated as follows. When a new time position \( t_{p0} \) is chosen, the integral \( \int_0^{t_{p0}} F_0(t_{p0} - \tau) f(\tau) d\tau \) is evaluated numerically by the trapezoidal rule at all points except the end point \( \tau = 0 \) because \( F_0(t_{p0}) \) is not yet known there. (See the graphical representation in Fig. 6.)

For each iteration only the increment \( \Delta \left[ \frac{f(0) + f(\Delta)}{2} \right] \left[ \frac{F_0(t_{p0} - \Delta) + F_0(t_{p0})}{2} \right] \), where \( \Delta \) is the time interval, is recalculated.

The functions, \( f(\tau) \), in the integral term of the boundary equations, such as \( e^{-\beta \tau} I_{\frac{\beta \tau}{2}} \) for the Maxwell model and \( e^{-\gamma \tau} J_{k_j} \), defined in (4.18) for the standard linear model, are evaluated numerically at discrete values of \( \tau \) prior to integration of the boundary equations. The exponential and Bessel functions are available in computer library subroutines, but the term \( J_{k_j} \) is evaluated as follows.

The integral \( J_{k_j}(\tau, \beta) \), defined as \( \int_0^\infty \eta e^{-\beta \eta} I_{\frac{\beta \eta}{2}} d\eta \), where \( \gamma = \beta \sqrt{\tau(\tau + 2\eta)} \), is broken into two parts, \( \int_0^N \) and \( \int_N^\infty \), where \( N \) is large enough so that the part \( \int_N^\infty \) is very small compared to \( \int_0^N \). It is also convenient to replace the variable \( \eta \) by \( \xi \tau \) when \( \tau > 1 \). The main part of the integral can be evaluated by Simpson's rule. A good upper bound of the integral \( \int_N^\infty \) is determined from the asymptotic formula

\[
\int_N^\infty \eta^k e^{-\beta \eta} I_{\frac{\beta \eta}{2}} d\eta < \frac{N^k e^{-N\beta} \left( 1 - \sqrt{\frac{2\tau}{N}} \right)}{\sqrt{2\pi} (2\beta^2 \tau)^{\frac{k-1+2i}{4}} \beta^{1-\sqrt{\frac{2\tau}{N}}}} \quad \text{for} \quad 0 \leq \tau \leq 1 \quad (4.24)
\]
where \( y = \beta \sqrt{\tau (\tau + 2\eta)} \)

and,

\[
\tau^{k+1} \int_0^\infty e^{-\frac{\beta \xi}{M}} \frac{I_j(y)}{y^j} \, d\xi < \frac{M}{e} \frac{\left(k - \frac{1+2j}{4}\right) - M\beta \tau (1 - \sqrt{\frac{2}{M}})}{\sqrt{\pi} \left(\sqrt{2 \beta \tau}\right)^2 \left(1 - \sqrt{\frac{2}{M}}\right)}
\]

for \( \tau > 1 \)

where \( y = \beta \tau \sqrt{1+2\xi} \)

When the function \( F_0(t_{p0}) \) is found in this way, to calculate the displacement and stress in (4.12), (4.15), (4.18), and (4.22), the integrals \( \int_0^{t_{p0}} \) can be evaluated by the trapezoidal rule.
5. DISCUSSION AND INTERPRETATION OF
THE RESULTS OF COMPUTATIONS

The maximum stresses at the spherical cavity due to incident P and
S waves, calculated for both elastic and viscoelastic materials, are dis-
cussed in this Chapter. Also a comparison is made between the method of
computation described in Chapters 2 and 4 and that described in Chapter 3.
Finally, a discussion is made of available information about the internal
dissipation in bedrock and the choice of viscoelastic parameters to approxi-
mate this dissipation.

5.1. Comparison of Results for the Two Numerical Methods

The two methods of obtaining numerical results of the stress for
each mode of the disturbing wave given in Chapters 2 to 4 are described briefly
as follows:

(1) Differential boundary equations in the case of the elastic
material (Chapter 2) and integro-differential boundary equations in the case
of viscoelastic materials (Chapter 4) in terms of the generating functions
are integrated numerically by a step-by-step procedure. The stresses for each
mode are then calculated directly from the numerical values of the generating
functions and their derivatives.

(2) The Laplace transform expressions for the stresses are obtained
directly from the Laplace transform of the boundary equations and are inverted
by a numerical scheme which involves expressing the inverse by a series of
orthogonal terms and obtaining the coefficients of these terms from discrete
values of the stress transforms (Chapter 3).

A comparison of results from both methods is desirable not only as a
check on the accuracy of results, but also it provides a means of determining
how many terms of the series used in method (2) are needed for adequate con-
vergence.
5.1.1. Comparison for the Elastic Material

A comparison of the two methods is made of the stress in the cavity boundary of the elastic solid for the following cases: (1) the symmetrically applied step pressure (Fig. 7); (2) the modal stresses, for \( n=0 \) to \( n=4 \), in the case of the disturbance due to the incident P wave (Figs. 15 to 19); and (3) the modal stresses, for \( n=1 \) to \( n=6 \), in the case of the disturbance due to the incident S wave (Figs. 27 to 33). In almost all cases the difference is so small that only one curve is shown. Where there is a difference the results from method (1) is shown as a solid line and those points obtained by method (2) which differ from method (1) are shown by small circles.

As seen from the results only the higher modes give differences and this can be attributed primarily to not having enough terms in method (2) to give sufficient convergence (maximum of 34 terms). Thus, for the elastic material, method (2) gives fairly accurate results for modes up to about \( n=4 \) for the disturbance due to the P wave and \( n=6 \) for the disturbance due to the S wave. Method (1) is therefore preferable for the elastic material and gives accurate results up to about \( n=13 \) (which is the limit discussed in 2.5).

Another check is that as \( t \to \infty \), the modal stresses must approach a value equal to the static stress obtained from the theory of elasticity. As shown in the figures, all the results confirm that the modal stresses by method (1) approach the static value as \( t \) becomes large.*

5.1.2. Comparison for Two Viscoelastic Materials

In Chapter 4, integro-differential equations are obtained in the case of a step pressure symmetrically applied to the cavity boundary (mode \( n=0 \)) for two viscoelastic materials, a Maxwell model and a standard linear model, which have a time-independent Poisson's ratio. A comparison of the results of the

* It should be recalled that for method (2) the static value is incorporated into the numerical scheme.
lateral boundary stress (Fig. 7) and boundary velocity (Fig. 8) show no difference between method (1) and method (2).

In summary, the results indicate that both methods are sufficiently accurate for most of the modal stresses calculated but that method (1) is more accurate when the stress response oscillates rapidly as it does for the higher modes. However, the boundary equations required for method (1) can be readily obtained only for special materials and then only for the first one or two modes, so that method (2) is preferable for the viscoelastic material. The results obtained of the modal stresses for the viscoelastic material by method (2) should be at least as accurate as the results for the elastic material by the same method. This should be expected because the viscoelastic modal response has smaller oscillations than the elastic response and the accuracy of method (2) depends primarily on having a sufficient number of terms to represent the oscillations.

5.2. Results for the Elastic Material

5.2.1. Step Pressure Symmetrically Applied to the Cavity Boundary

The lateral stress, velocity and displacement at the boundary for a step pressure symmetrically applied to the cavity are shown in Figs. 7 to 9 for Poisson's ratio 1/4. There is a dynamic oscillation in all quantities, which decays rapidly with time. This damping is, of course, due to spatial dispersion rather than to material anelasticity. The ratio of the maximum dynamic lateral stress at the boundary to the final static lateral stress is 1.43 for Poisson's ratio 1/4.

The radial stress and hoop stress at a radius of twice the boundary radius are shown in Fig. 10 for Poisson's ratio 1/4. As the step wave travels away from the boundary, the front (which is made up of the shortest wave lengths
in a harmonic analysis) dies out as \(1/r\) whereas the static radial stress (which is made up of the longest wavelength in a harmonic analysis) dies out as \(1/r^2\). A more complete discussion of symmetric waves is given in Ref. (25).

5.2.2. Stresses at the Cavity Boundary Due to the Incident P Wave

The maximum dynamic and static stress is always the meridional stress \(\sigma_{\theta\theta}\) at the equator \((\theta = 90^0)\) acting parallel to the direction of travel of the incident P wave. The total stress for this case (the stress due to the incident wave plus modes \(n=0\) to \(n=4\) of the disturbing wave) is shown in Fig. 11 for Poisson's ratios of 0, 0.25 and 0.4. The stress \(\sigma_{\theta\theta}^{n=0}\) at the equator for each mode of the disturbing wave for Poisson's ratio of 1/4 is shown in Figs. 15 to 19.

The ratio of maximum dynamic to static stress (Table 1) increases from 1.06 for Poisson's ratio of 0 to 1.18 for Poisson's ratio of 0.40 and the time when the dynamic stress reaches its maximum is 4 to 5 radius transit times after the incident wave reaches the cavity.

As a check on the number of modes required for adequate convergence, stresses were calculated for 11 modes and the results showed that for the maximum dynamic stress about 5 to 7 modes suffice.

In Fig. 13 the variation of meridional stress \(\sigma_{\theta\theta}\) along the meridian \((\theta = 0\) to \(\pi)\) is shown for various times \(t_p\). One interesting feature is that, when \(t_p = 2\), that is, when the incident step wave just reaches \(\theta = 180^0\), \(\sigma_{\theta\theta}\) (the total stress due to the incident wave plus 5 modes of the disturbing wave) is close to zero for \(\theta\) greater than 150°. For the case of a wave traveling around the sphere at the boundary, Huyghens' principle asserts that this wave can nowhere travel faster than \(C_p\), the speed of a P wave. For \(t_p = 2\), the maximum value of \(\theta\) so obtained is \((\pi/2 + 1)\) radians or 147°, so that if all
harmonic modes of the disturbing wave were included, the stress would be zero for $\theta > 147^0$.

5.2.3. **Stresses at the Cavity Boundary Due to the Incident S Wave**

The maximum tensile dynamic stress due to the incident S wave is the meridional stress $\sigma_{\theta\theta}$ for $\Phi = 0^0$ and $\theta$ somewhere between $120^0$ and $150^0$, usually $135^0$. The maximum static value is $\sigma_{\theta\theta}$ for $\Phi = 0^0$ and $\theta = 135^0$. The total stress $\sigma_{\theta\theta} (\Phi = 0^0, \theta = 135^0)$ due to the incident wave plus 11 modes of the disturbing wave is shown in Figs. 20 to 22 for Poisson's ratios of 0, 1/4 and 0.4 and the modal stresses for the disturbing wave are shown in Figs. 27 to 33 for Poisson's ratio of 1/4.

The ratio of maximum dynamic to static stress (Table 2) increases from 1.18 for Poisson's ratio of 0.4 to 1.44 for Poisson's ratio of 0, so that this amplification factor is larger for the incident S wave than for the incident P wave. The convergence was slower than for the P wave, which required that the total stress be based on 11 to 13 modes of the disturbing wave (which is the limit discussed in 2.5).

There are a few interesting features illustrated by Figs. 24 and 25 which show the variation of the stress $\sigma_{\theta\theta}$ along the meridian $\Phi = 0$ ($\theta = 90^0$ to $\theta = 180^0$) at different times. When $t_S = 2$, that is, when the incident S wave just reaches $\theta = 180^0$, $\sigma_{\theta\theta}$ is close to zero for $\theta$ greater than $150^0$, a similar result to the one already discussed for the P wave. Another feature is that the maximum dynamic stresses arise in the form of a disturbance which travels forward (from $\theta = 90^0$ to $\theta = 180^0$) and then backwards (from $\theta = 180^0$ to $\theta = 90^0$). The maximum stress can occur when the disturbance is traveling forward, as it does for Poisson's ratio of 0 (at $t_S = 1.8$), or when the disturbance is traveling backward, as it does for Poisson's ratio of 1/4 or 0.40 (at $t_S = 3.5$).
5.3. Results for the Viscoelastic Material

As discussed in 3.5, the viscoelastic materials chosen for calculation have an initial Poisson's ratio of 0.25 and a purely elastic dilatational stress-strain relation. There remains the parameter $p$, the viscous parameter in shear, and, for the standard linear model, $\alpha$, from which the ratio of stiffness of the two springs can be determined. When $\alpha \to 1$, the standard linear model approaches an elastic material whereas, when $\alpha \to 0$, it approaches a Maxwell model. A value of $\alpha = 1/2$ was chosen as a suitable value for calculation. The most important parameter is $p$, and, for both the Maxwell model and the standard linear model, if $p$ gets small (i.e., the dashpot becomes stiff), the material approaches an elastic material. If $p$ becomes large, the Maxwell model tends toward a viscous liquid, whereas the standard linear model tends toward an elastic model. The discussion in 5.4 indicates that the bedrock is very nearly elastic so that a value of $p$ was arbitrarily taken as 1 (in units of $1/t_{p0}$ or $1/t_{s0}$).

5.3.1. Step Pressure Symmetrically Applied to the Cavity Boundary

The results for hoop stress (Fig. 7), velocity (Fig. 8) and displacement (Fig. 9) at the boundary for the Maxwell model and the standard linear model where both models have parameters given in 5.3 are compared to the results for the elastic material ($\nu = 1/4$). The dynamic oscillation for the viscoelastic materials is less than the elastic material. This effect is due, of course, to the presence of internal material damping. For example, the ratio of maximum dynamic to static hoop stress is reduced from 1.43 for the elastic material to 1.32 for the standard linear model and 1.01 for the Maxwell model with the parameters used.
The displacement and velocity at the boundary shown in Figs. 8 and 9 are of less interest than the stresses because they reflect the undesirable peculiarities of the simplified viscoelastic models chosen. For instance, for the Maxwell model, the deflection approaches a constant rate, and for the standard linear model, the long-time static deflection is twice that of the elastic model, because $\alpha = 0.5$. However, the initial response is the same for all models because the viscoelastic models have the same instantaneous elasticity as the elastic material. Also the long-time static hoop stress (which does not depend on Poisson's ratio) is the same for all models.

The radial and hoop stresses in the medium at a radius twice the boundary radius are shown in Fig. 10 for the elastic and Maxwell models. The front of the step wave attenuates as $\frac{1}{r} e^{-p^* p_0 / 2}$ for the Maxwell model compared to $\frac{1}{r}$ for the elastic material, whereas for both materials the static stress attenuates as $\frac{1}{r^2}$.

5.3.2. Stresses at the Cavity Boundary Due to the Incident P Wave

Just as for the elastic material, the maximum stress is the meridional stress $\sigma_{\theta\theta}$ at the equator ($\theta = 90^\circ$). In Fig. 12 a comparison is made of the stress $\sigma_{\theta\theta}(t)$ ($\theta = 90^\circ$) for the elastic material, the Maxwell model and the standard linear model. The elastic material has Poisson's ratio of $1/4$ and the parameters of the Maxwell and standard linear models are given in 5.3. The stress $\sigma_{\theta\theta}$ is the sum of the stresses of the incident wave and 5 modes of the disturbing wave.

The maximum dynamic stress, and the ratio of the maximum dynamic to static stress (Table 1), are, of course, less for the two viscoelastic materials than for the elastic material. In fact, for the Maxwell model chosen ($p = 1$), the dynamic stress is always less than the static stress (due partly to the
time dependent Poisson's ratio). In Figs. 14 to 19 a comparison is made of the modal stresses $o^{n0}_{\theta\theta}(t) (\theta = 90^0)$ for the three materials. Unfortunately, the long-time static moduli are different for all three models, and a comparison can be useful only for small time. As seen in Figs. 15 to 19 all the stress responses start out the same because the instantaneous moduli of all three models are the same, but the dynamic oscillations are subsequently damped out by the internal viscosity.

5.3.3. Stresses at the Cavity Boundary Due to the Incident S Wave

Just as for the P wave, we compare the elastic material, the Maxwell model and the standard linear model whose parameters are given in 5.3. The maximum stress for this case is $\sigma_{\theta\theta} (\theta = 135^0; \phi = 0^0)$, on the leeward side of the cavity.

The total stress $\sigma_{\theta\theta} (\theta = 135^0, \phi = 0^0)$ is shown for the three materials in Fig. 23 (incident wave plus 8 modes of the disturbing wave) and the incident S-wave stresses $\sigma^S_{\theta\theta}$ and modal stresses $\sigma^{n1}_{\theta\theta}$ of the disturbing wave are shown in Figs. 26 to 33. Table 2 lists the maximum dynamic stresses for the three materials.

Just as for the P wave, the three materials have the same initial response (for $t_{po} = 0$ to 0.5), and subsequent dynamic oscillations are damped out in the viscoelastic materials. Because the long-time static moduli are different, the non-zero modal static stresses are not the same so that a comparison can be made only for small time.

For mode $n=4$, the stress $\sigma^{41}_{\theta\theta} (\theta = 135^0, \phi = 0^0)$ is very small so that $\sigma^{41}_{\theta\theta} (\theta = 150^0, \phi = 0^0)$ is shown instead.
The maximum dynamic stress is, of course, reduced for the viscoelastic materials (Table 2 and Fig. 23). The ratio of the maximum dynamic to static stress is reduced from 1.19 for the elastic material (for 8 modes of the disturbing wave compared to 1.23 for 11 modes of the disturbing wave) to 1.06 for the standard linear model \( p = 1, \alpha = 1/2 \).

5.4. Experimental Observations in Bedrock

The information which is available\(^{(15)}(26)\) indicates that, for small amplitude waves, the rock near the Earth's surface is dissipative to a small degree, but that wave attenuation measurements show it to be a material of the solid friction type rather than viscoelastic. According to field\(^{(15)}\) and laboratory\(^{(14)}\) measurements, a traveling sinusoidal plane P or S wave has the following characteristics:

1. the decrease of amplitude \( A \) of the traveling wave in a distance \( x \) is given by

\[
A = A_o e^{-\alpha x}
\]

where \( \alpha \) is called the attenuation. For bedrock, the attenuation is proportional to the frequency \( \omega \) of the wave;

\[
\alpha = c_1 \omega
\]

(2) The velocity of propagation of the wave is independent of the frequency, In other words, bedrock is non-dispersive.

These characteristics are satisfied by a material of the solid friction type and cannot be satisfied by any viscoelastic model\(^{(26)}\). Some attempt has been made\(^{(26)}\) to develop mathematical theories for a material of the solid friction type, but, since all such theories are nonlinear, only very simple problems can be solved.
However, a viscoelastic model can be made to represent the attenuation of the bedrock over a limited frequency range. Since the viscoelastic material is dispersive whereas the bedrock is not, the approximation is a rough one.

C. W. Horton (16) has fitted a standard linear model to the attenuation measurements of McDonal et al (15) in Pierre shale, a material which is isotropic. The results of measurements in various bedrocks (26) indicate that Pierre shale is more dissipative than most bedrocks. If a cavity is introduced in the bedrock, the rock near the cavity will become more dissipative because of loss of hydrostatic compaction and because of fragmentation due to excavation. Therefore, the results of Pierre shale may apply for the determination of stresses around cavities in less dissipative bedrocks.

A comparison of attenuation variation with frequency is shown in Fig. 34 between the measurements and the standard linear model fitted to the measurements. The viscoelastic parameters determined by Horton in terms of the parameters defined in 3.5 are $\alpha = 0.8$ and $p = 3470$ sec.$^{-1}$. The attenuation of the standard linear model fits the measurements over a frequency range of 0 to 150 cycles per second but, at higher frequencies, the viscoelastic material has much less damping than the bedrock. The dispersion for this standard linear model is small with a wave speed variation of only 10 per cent in the frequency range 0-150 cycles per second. Thus, if the input waveform does not contain large harmonic components with frequencies greater than 150 cycles per second, then the results for the standard linear model approximate those for the real bedrock. For example, the change in form of the front of a step wave (which is made up of the highest frequencies) is not well predicted by the viscoelastic model.

From the measurements of the P wave in Pierre shale in addition to the measurements of the S wave, Horton (16) determined the viscoelastic
parameters assuming that the material in volumetric stress and strain behaved as a standard linear model. The parameters obtained ($\alpha = 0.92$, $p' = 90800 \, \text{sec}^{-1}$) indicate that in comparison to the shear behavior, the material is very close to being elastic in bulk.

If we choose a cavity radius of 100 feet and, given that the instantaneous S-wave speed for Pierre shale is 2680 ft./sec. from measurements, $p = 3470/26.8 = 130$ (units of $1/t_{so}$). The measured P-wave speed is 7180 ft./sec. (which gives a high Poisson's ratio of 0.415 compared to 0.25 in most rocks), so that $p = 3470/71.0 = 49$ (units of $1/t_{po}$).

These parameters are used for two cases; (1) the hoop stress due to a step pressure symmetrically applied to the spherical cavity (Fig. 7), and, (2) the modal stress $\sigma_{\theta \theta}^{31} (\theta = 135^0)$ in the disturbing wave due to the incident step S wave (Fig. 28). It is seen that, although there is a time delay in the response of this model as compared to the elastic material, the magnitude of oscillation is very close to the magnitude of the oscillations of the elastic material. Therefore, the dynamic stresses are expected to be very close to the dynamic stresses of the elastic material although the time of occurrence will be a little later.

It may be concluded that, if the distance from the source of the stress wave is large compared to the diameter of the cavity, the material can be treated as purely elastic in determining the dynamic stress factors; the attenuation from the elastic dynamic stress factors would be of the order of 1 per cent or less. The explanation for this is twofold. First, when the wave travels from the source to the cavity, the bedrock acts as a filter by removing the harmonic wave components of high frequency. This, of course, assumes that failure does not take place around the cavity. Secondly,
attenuation of the resulting waveform passing the cavity, with the higher frequencies now removed, alters the wave only after traveling distances long compared to the size of the cavity.
6. SUMMARY AND CONCLUSIONS

6.1. Derivation of the Solution for the Spherical Cavity

When the incident stress wave envelops the cavity, a disturbance is transmitted into the medium, propagating away from the cavity boundary. For an elastic material, this disturbing wave can be expanded in spherical harmonics, where three diverging wave solutions, one longitudinal and two shear waves, are obtained for each spherical harmonic. The three solutions for each harmonic are expressed in terms of generating functions (analogously to the d'Alembert solution for wave propagation in rods or strings) which are determined from the boundary conditions. For each harmonic, the condition at the boundary of the spherical cavity provides three ordinary linear differential equations in terms of the three generating functions. Once these equations are solved the stresses in the disturbing wave can be found anywhere in the medium and added to the stress due to the incident wave to obtain the total stress field.

With the aid of the elastic solution and the correspondence principle stated in Section 1.3., the solution for the viscoelastic material is obtained by taking the Laplace transform of the elastic solution and replacing the elastic moduli by the viscoelastic transform "moduli". In this way, the boundary conditions provide, for each harmonic, three algebraic equations of the transforms of the viscoelastic generating functions in terms of the transform parameter. When these algebraic equations are solved, the transforms of the stresses can be obtained anywhere in the medium and added to the transform of the stress due to the incident wave.

It is possible, for some viscoelastic materials having instantaneous elasticity, to obtain the inverse of the transforms of the generating functions in terms of expressions which contain convolution integrals. For each harmonic,
the boundary conditions provide integro-differential equations of the Volterra type in place of the ordinary differential equations obtained for the elastic solution. Once these equations are solved, the stresses in the disturbing wave can be found anywhere in the medium and added to the stress due to the incident wave to obtain the total stress field.

6.2. Comparison of Results for the Two Numerical Methods

Method (1), described in Chapters 2 and 4, consists of the numerical integration of linear, ordinary differential or integro-differential boundary equations for each harmonic of the disturbing wave in terms of generating functions, and the calculation of the stresses from the calculated values of the generating functions and their derivatives. Method (2), described in Chapter 3, consists of obtaining the inverse of the stress transforms for each harmonic of the disturbing wave by a numerical scheme which expresses the inverse by a series of orthogonal terms.

The results indicate that both methods are sufficiently accurate for most of the modal stresses calculated but that method (1) is more accurate when the stress response oscillates rapidly as it does for the higher modes. However, although the boundary equations required for method (1) are obtained easily for the elastic material they are not easily obtained for the viscoelastic materials, so that method (2) seems preferable for viscoelastic materials.

6.3. Results for the Elastic and Viscoelastic Materials

For the problem of an incident step P wave enveloping a spherical cavity in an elastic material, the ratio of the maximum dynamic to maximum static stress around the cavity increases from 1.06 for Poisson's ratio of 0 to 1.18 for Poisson's ratio of 0.40 (Fig. 11). The modes for \( n \) greater than
4 contribute very little to the maximum dynamic stress. On the other hand, for
the incident step S wave enveloping the cavity in an elastic material, this
ratio varies from 1.18 for Poisson's ratio of 0.40 to 1.44 for Poisson's ratio
of 0. In this case the higher modes contribute considerably to the maximum
dynamic stresses, 13 modes being used for calculation. Also the time when the
dynamic stress reaches a maximum is 4 to 5 radius transit times for the inci-
dent P wave as compared to 1.8 to 3.5 radius transit times for the incident S
wave.

As expected, the dynamic oscillations obtained in the modal responses
for the elastic material are damped out for the viscoelastic materials and
therefore the ratios of the maximum dynamic to maximum static stress are
smaller for viscoelastic materials (Figs. 7, 12 and 23).

6.4. Implications of the Behavior of Bedrock for the Analysis

The results of measurements (26) show that although bedrock is very
nearly elastic, it is dissipative in such a way that, during wave propagation,
it acts as a filter by removing the harmonic components of high frequency but
leaving the harmonic components of low frequency unaltered. It follows that,
if the size of the cavity is small compared to the distance from the source,
the calculation of the dynamic stresses around the cavity can be made by
assuming that the material is perfectly elastic. The change of wave form from
the source to the cavity should, however, be calculated taking the dissipative
character of the rock into account.
REFERENCES


The solution of Laplace's equation
\[ \nabla^2 \psi = 0 \] 
(A.1)
suitable for spherical boundaries is given by
\[ \psi = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[ a_{nc} K_{mc}^{n} + a_{ms} K_{ms}^{n} \right] + \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[ b_{nc} K_{mc}^{n} + b_{ms} K_{ms}^{n} \right] r^{-(2n+1)} \] (A.2)
where the terms in the first series on the right hand side have singularities at \( r = \infty \), and the terms in the second series have singularities at \( r = 0 \). By separation of Laplace's equation in spherical coordinates,
\[ K_{mc}^{n} = r^{n} p_{n}^{m}(\mu) \cos m \Phi \] (A.3)
\[ K_{ms}^{n} = r^{n} p_{n}^{m}(\mu) \sin m \Phi \]
where,
\[ p_{n}^{m}(\mu) = (1-\mu^2)^{\frac{m}{2}} \frac{d^{m} p_{n}(\mu)}{d \mu^{m}} \]
\[ \mu = \frac{x_{1}}{r} = \cos \theta \]
and
\[ p_{n}^{m} \left( \frac{x_{1}}{r} \right) = (-1)^{n+1} r^{n+1} \frac{\partial^{n}}{\partial x_{1}^{n}} \left( \frac{1}{r} \right) \] (A.4)
On a sphere concentric with the origin, the functions \( \frac{K_{mc}^{n}}{r^{n}} \) and \( \frac{K_{ms}^{n}}{r^{n}} \) are defined as surface harmonics and these surface harmonics form a complete set of orthogonal functions over the surface of the sphere. There are \( 2n+1 \) independent surface harmonics (represented by the subscripts of \( K_{mc}^{n} \))
or \( K_n^m \) for each degree \( n \) of the spherical harmonic. The functions \( P_n(\mu) \) are Legendre polynomials and correspond to the axially symmetric surface harmonics \( K_n^0 \) (\( K_0^0 \) is zero and the subscript \( c \) is dropped).

The spherical harmonics \( K_n^m (K_n^m or K_n^m) \), when expressed in Cartesian coordinates \( (x_i) \) become polynomials in \( x_i \) which are homogeneous of degree \( n \), and therefore satisfy the Euler relation \( (17) \)

\[
\frac{\partial K_n^m}{\partial x_i} = nK_n^m
\]

(A.5)

Along with relation (A.5), the following relations already discussed are used in this investigation:

\[
\nabla^2 K_n^m = 0
\]

(A.6)

\[
\int (K_n^m K_p^q) d\sigma = 0 \quad \text{where} \quad n \neq P, \ m \neq q
\]

* The subscript \( s \) or \( c \) will be dropped except where needed.
APPENDIX B

EXPANSION OF THE SOLUTION IN SPHERICAL HARMONICS
FOR THE ELASTIC MATERIAL

In this appendix, three vector solutions in terms of generating functions will be obtained from the elasticity equations of motion (2.6) (the Navier equations) for every spherical harmonic \( K_{m}^{n}(x_{i}) \). It will be shown in Appendix D that the solution obtained is complete in the sense that any traction applied to a spherical boundary is completely represented by the solution. The three vector solutions to equation (2.6) for harmonic waves, replacing \( \frac{\partial^{2}u_{i}}{\partial t^{2}} \) by \( \omega^{2}_{\jmath} u_{i} \), have been given by Sato\(^{(6)}\), Eringen\(^{(7)}\) and others\(^{(8)}\) following the solution given by Lamb for the vibrating sphere.\(^{(4,5)}\)

The three solutions can be constructed by a method similar to the one given in Morse and Feshbach, Chapter 13.\(^{(8)}\) The first solution satisfies the scalar wave equation (2.8a) and gives the irrotational component of the displacement field. For the static case, the right hand side of (2.8a) is zero, and the solution appropriate for spherical boundaries is given by (A.2). The solution of (2.8a) is obtained by replacing the constants in (A.2) by functions of \( r \) and \( t \).

\[
\varphi = \sum_{n=0}^{\infty} \sum_{m=0}^{n} f_{m}^{n}(r, t) K_{m}^{n}(x_{i})^{*} \tag{B.1}
\]

where, by substitution of (B.1) into (2.8a), \( f_{m}^{n} \) must satisfy

\[
\frac{\partial^{2}f_{m}^{n}}{\partial r^{2}} + \frac{2(n+1)}{r} \frac{\partial f_{m}^{n}}{\partial r} = \frac{1}{c_{p}^{2}} \frac{\partial^{2}f_{m}^{n}}{\partial t^{2}} \tag{B.2}
\]

\( * \ K_{m}^{n} \) is either \( K_{mc}^{n} \) or \( K_{ms}^{n} \) (See Appendix A)
When \( n = 0 \), (B.2) becomes

\[
\frac{\partial^2}{\partial r^2} (rf_0^n) = \frac{1}{c_p^2} \frac{\partial^2}{\partial t^2} (rf_0^n)
\]

and the solution (2) is

\[
rf_0^n = F_0^n(t-r/c_p) + D_0^n(t+r/c_p)
\]

The generating function \( F_0^n(t-r/c_p) \) is an arbitrary longitudinal (P) wave diverging from the origin and the generating function \( D_0^n(t+r/c_p) \) is an arbitrary P wave converging towards the origin. The disturbing wave referred to in this investigation contains only diverging components so that only the diverging components will be discussed in what follows.

It can be shown by induction that the solution of (B.2) is

\[
f_{m}^{n}(r,t) = \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^n \left[ \frac{1}{r} F_{m}^{n}(t-r/c_p) \right] \quad (B.3)
\]

Thus (B.1) and (B.3) gives the first vector solution of (2.6) because if (2.8a) is satisfied, so is (2.6). Each term of (B.1) is a P wave diverging from the origin. The shape is given by the generating function \( F_{m}^{n}(t-r/c_p) \) which is determined from the boundary condition.

The displacement for each term of the first, or \( \varphi \), solution (the displacement in the P wave) is given by

\[
v_{i}^{nm} = \frac{\partial \varphi_{m}^{n}}{\partial x_{i}} = \frac{\partial K_{m}^{n}}{\partial x_{i}} f_{m}^{n} + \frac{x_{i}}{r} \frac{\partial f_{m}^{n}}{\partial r} \quad (B.4)
\]

and is therefore specified in direction by the plane containing the vectors \( x_{i} \) (the radius vector) and \( \frac{\partial K_{m}^{n}}{\partial x_{i}} \) (the gradient of the solid harmonic).

There remains the equivoluminal or rotational components of the displacement in (2.6). If we subtract out the longitudinal solution \( v_{i} \) which gives the irrotational component, the remaining component \( w_{i} \) must satisfy the
homogeneous part of (2.6):

\[
\left[ \nabla^2 - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \right] w_i = 0 \quad \text{where} \quad \frac{\partial w_i}{\partial x_i} = 0 \quad (B.5)
\]

From (2.6), it is found that the curl of \( w_i \) satisfies the same equation as \( w_i \):

\[
\left[ \nabla^2 - \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} \right] \varepsilon_{ijk} \frac{\partial w_k}{\partial x_j} = 0 \quad \text{where} \quad \frac{\partial}{\partial x_i} (\varepsilon_{ijk} \frac{\partial w_k}{\partial x_j}) = 0 \quad (B.6)
\]

By direct verification, one solution which satisfies (B.5) is:

\[
\mathbf{w}_i^{(1)} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} h_n^m(r,t) \varepsilon_{ijk} x_j \frac{\partial k_m^n}{\partial x_k} \quad (B.7)
\]

where

\[
h_n^m = \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^n \left[ \frac{1}{r} H_m^n(t-r/c_s) \right]
\]

The displacement of \( \mathbf{w}_i^{(1)} \) for each term in (B.7) is perpendicular to the corresponding term of \( \mathbf{v}_i \) in (B.4).

By virtue of (B.5) and (B.6) another solution is curl \( \mathbf{w}_i^{(1)} \):

\[
\mathbf{w}_i^{(2)} = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left[ \sum_{n=1}^{\infty} \sum_{m=0}^{n} g_m^n(r,t) \varepsilon_{k1p} x_1 \frac{\partial k_m^n}{\partial x_p} \right]
\]

where

\[
g_m^n(r,t) = \left( \frac{1}{r} \frac{\partial}{\partial r} \right)^n \left[ \frac{1}{r} G_m^n(t-r/c_s) \right] \quad (B.8)
\]

The displacement of \( \mathbf{w}_i^{(2)} \) for each term in (B.8) is at right angles to the corresponding term of \( \mathbf{w}_i^{(1)} \) in (B.7) and is therefore in a plane containing the vectors \( x_i \) and \( \frac{\partial k_m^n}{\partial x_i} \).
The two solutions $w_i^{(1)}$ and $w_i^{(2)}$ represent a series of diverging shear (S) waves and $G_m^n(t-r/C_s)$ and $H_m^n(t-r/C_s)$ are the generating functions which are determined from the boundary conditions.

The solutions $w_i^{(1)}$ and $w_i^{(2)}$ can be expressed as the curl of displacement potentials as in equation (2.7). The vector potential $\psi_i$ which gives $w_i^{(1)}$ is

$$\psi_i = \sum_{n=1}^{\infty} \sum_{m=0}^{n} h_m^n x_i k_m^n$$  \hspace{1cm} (B.9)

and from (B.8) the vector potential $\psi_i$ which gives $w_i^{(2)}$ is

$$\psi_i = \sum_{n=1}^{\infty} \sum_{m=0}^{n} g_m^n \epsilon_{ijk} x_j \frac{\partial k_m^n}{\partial x_k}$$  \hspace{1cm} (B.10)

where $h_m^n$ and $g_m^n$ are given by (8.7) and (8.8).

To summarize, a solution to the equations of motion (2.6) by expansion in spherical harmonics is given by

$$u_i = \frac{\partial \phi}{\partial x_i} + \epsilon_{ijk} \frac{\partial}{\partial x_j} (\psi_k + \omega_k)$$  \hspace{1cm} (B.11)

where $\phi$, $\psi_k$ and $\omega_k$ are three independent displacement potential solutions given in (B.1), (B.3), (B.9) and (B.10).
APPENDIX C

SPHERICAL HARMONICS ENTERING THE SOLUTION FOR THE INCIDENT SHEAR WAVE

Those harmonics $K^n_m(x_j)$ (that is, either $K^{nc}_m$ or $K^{ms}_m$) which enter into the solution for the incident shear wave can be determined by considering the integrals in the boundary equations (2.16) and using the value of $T^0_j$ given in (2.23). The integrands in (2.16) become

$$ x_j K^{n_j}_m = (2x_1 x_2 K^n_m) \sigma_s H[t_s + (x_1 - 1)] $$  \hspace{1cm} (C.1) \\

$$ \frac{\partial K^n_m}{\partial x_j} T^s_j = \left( x_2 \frac{\partial K^n_m}{\partial x_1} + x_1 \frac{\partial K^n_m}{\partial x_2} \right) \sigma_s H[t_s + (x_1 - 1)] $$  \hspace{1cm} (C.2) \\

$$ \varepsilon_{jkl} x_l K^n_k \frac{\partial K^n_m}{\partial x_j} T^s_j = \left[ x_2 \left( x_3 \frac{\partial K^n_m}{\partial x_2} - x_2 \frac{\partial K^n_m}{\partial x_3} \right) + x_1 \left( x_1 \frac{\partial K^n_m}{\partial x_3} - x_3 \frac{\partial K^n_m}{\partial x_1} \right) \right] $$  \hspace{1cm} (C.3) \\

If $x_1$, $x_2$, $x_3$, $\frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial x_3}$ are replaced by $r\mu$, $r\sqrt{1 - \mu^2} \cos \phi$, $r\sqrt{1 - \mu^2} \sin \phi$, $-\frac{\sin \phi}{r\sqrt{1 - \mu^2} \partial \phi}$ and $\frac{\cos \phi}{r\sqrt{1 - \mu^2} \partial \phi}$ respectively, the following expressions are obtained for the integrals (C.1), (C.2) and (C.3):

$$ x_j \left\{ K^{nc}_m \right\}_{m_s} T^s_j = 2r^2 \mu \sqrt{1 - \mu^2} \cos \phi \left\{ \frac{\cos m \Phi}{\sin m \Phi} \right\} \sigma_s H[t_s + (x_1 - 1)] $$  \hspace{1cm} (C.5) \\

$$ \frac{\partial}{\partial x_j} \left\{ K^{nc}_m \right\}_{m_s} T^s_j = \left[ r\sqrt{1 - \mu^2} \left( 2\mu \frac{\partial p^m_n}{\partial r} + \frac{\partial p^m_n}{\partial x_1} \right) \cos \phi \left\{ \frac{\cos m \Phi}{\sin m \Phi} \right\} - \frac{\mu}{\sqrt{1 - \mu^2}} \right. $$  \hspace{1cm} (C.6)
where \( m \) is \( P(\nu, x) \nabla \frac{\partial}{\partial x} \left\{ K_{mc}^{n} + \frac{\partial p_{m}^{n}}{\partial x} \right\} T_{j}^{s} = \left[ -r^{2} \nu \sqrt{1-\mu^{2}} \frac{\partial p_{m}^{n}}{\partial x} \sin \Phi \left\{ \cos m\Phi \right\} \right.
- \left. r \frac{2\mu - 1}{\nu} p_{n}^{m} \cos \Phi \left\{ -m \sin m\Phi \right\} \sigma H[t+(x_{1}-1)] \right]\]

(C.7)

where \( p_{n}^{m} = p_{n}^{m}(r, x_{1}) \)

and where corresponding terms in the brackets \( \left\{ \right\} \) are used, so that each of (C.5), (C.6) and (C.7) is two equations.

When (C.5), (C.6) and (C.7) are substituted in the integrals of (2.16), the part depending on \( \Phi \) is integrated from zero to \( 2\pi \). Therefore, because of orthogonality of the cosine and sine functions, (C.5) and (C.6) have non-zero integrals for the harmonic \( K_{lc}^{n} \) only and (C.7) has non-zero integrals for the harmonic \( K_{ls}^{n} \) only. Therefore the potentials \( \Phi \) and \( \psi_{1} \) of the disturbing wave involve the harmonics \( K_{lc}^{n} \) only whereas the potential \( \omega_{1} \) involves the harmonic \( K_{ls}^{n} \) only.
APPENDIX D

STATIC SOLUTION FOR THE SPHERICAL CAVITY

D.1. Expansion of the Disturbance in Spherical Harmonics

If there were no cavity, the stress due to the incident step $P$ and $S$ plane waves would eventually be constant and homogeneous throughout the body. The presence of the cavity causes a disturbance in this static state which, as in the dynamic problem, can be expanded in spherical harmonics. This expansion for the static disturbance is similar to the expansion for the dynamic case and three vector solutions (one dilatational and two equivoluminal) are obtained for each harmonic. One set of solutions associated with the spherical harmonics $r^{-(2n+1)}K^n_m$ or $K^{-n-1}_m$ dies out as $r \to \infty$. These are the ones which enter the solution for the spherical cavity. For each spherical harmonic, the solution can be expressed as (17)

$$u_i = A^n_m \left[ r^2 \frac{\partial K^n_m}{\partial x_i} + \alpha^n_n K^n_m \right] + B^n_m \frac{\partial K^n_m}{\partial x_i} + C^n_m \epsilon_{ijk} \frac{\partial K^n_m}{\partial x_k}$$  \hspace{1cm} (D.1)

where

$$\alpha^n_n = -2 \frac{c^2 n + (n+1)}{c^2 (n+3) - (n+1)}$$

The term in (D.1) containing the constant $A^n_m$ is the irrotational component, and the other two terms are equivoluminal components, the displacement from the one containing $C^n_m$ being orthogonal to the other two.

The traction on a spherical surface and the other required spherical stresses $\sigma_{\theta\theta}$, $\sigma_{\phi\phi}$, and $\sigma_{\phi\phi}$ are expressed by the same formulas as (2.12) and (2.13) except that $\gamma^n_m$, $\Delta^n_m$, $\chi^n_m$, $\alpha^n_m$, $\beta^n_m$, and $h^n_m$ are replaced by $\gamma^n_{m \text{ st}}$, $\Delta^n_{m \text{ st}}$, $\chi^n_{m \text{ st}}$, $\alpha^n_{m \text{ st}}$, $\beta^n_{m \text{ st}}$, and $h^n_{m \text{ st}}$, respectively, where
\[ \gamma_m^n = [2n(c^2-1) + \alpha_n c^2(n+3) - \alpha_n(n+4)] A_{m\nu}^n \]

\[ \Delta_m^n = [2n+\alpha_n] A_{m\nu}^n + 2(n-1)B_{m\nu}^n \]

\[ \chi_m^n = -(n-1)C_{m\nu}^n \]

\[ \alpha_m^n = [(c^2-2)(\alpha_n(n-3)+2n) + 2\alpha_n] A_{m\nu}^n \]

\[ \beta_m^n = [A_{m\nu}^n + B_{m\nu}^n] \mu \]

\[ h_m^n = -C_{m\nu}^n \]

For the static solution given in (D.1) and (D.2), \( n \) takes on the negative integers. Because the factor \( r^{-2n+1} \) in \( K_{m}^{-n-1} \) is absorbed by the functions \( f_m^n, g_m^n, \) and \( h_m^n \) in the dynamic solution, there is a direct correspondence between the asymptotic values \( \alpha_{m\nu}^n, \beta_{m\nu}^n, \) and \( h_{m\nu}^n \) in the dynamic solution and the values \( \alpha_{m\nu}^{-n-1}, \beta_{m\nu}^{-n-1}, \) and \( h_{m\nu}^{-n-1} \) obtained in the static solution, where \( n \) is now a positive integer. The correspondence, obtained from (2.13), is

\[ K_m^n \alpha_m^n + Q_m^n \beta_m^n + S_m^n h_m^n / r = K_m^{-n-1} \alpha_{m\nu}^{-n-1} + Q_m^{-n-1} \beta_{m\nu}^{-n-1} + S_m^{-n-1} h_{m\nu}^{-n-1} / r \]

\[ K_m^n \alpha_m^n - R_m^n \beta_m^n - S_m^n h_m^n / r = K_m^{-n-1} \alpha_{m\nu}^{-n-1} - R_m^{-n-1} \beta_{m\nu}^{-n-1} - S_m^{-n-1} h_{m\nu}^{-n-1} / r \]  

(D.3)

\[ S_m^n \beta_m^n - V_m^n h_m^n / r = S_m^{-n-1} \beta_{m\nu}^{-n-1} - V_m^{-n-1} h_{m\nu}^{-n-1} / r \]

where \( Q_m^n, R_m^n, S_m^n, \) and \( V_m^n \) are the terms derived from \( K_m^n \) in (2.13).

The boundary equations are the same as (2.16) except that the integer \( n \) in the formula for \( N_m^n \) in (2.15) must be replaced by the positive integer \(-n-1\).
D.2. Static Solution for the P Wave

The boundary equations from (2.21) are (for \( r_0 = 1 \)):

\[
\gamma^n_{0\ st} + n\Delta^n_{0\ st} = \left( \frac{2j+1}{2} \right) \frac{\sigma}{c^2} \int_{-1}^{+1} \left[ (c^2 - 2) + 2\gamma_1 \right] K^n_0 \, dx_1
\]

\[
\eta^n_{0\ st} + n(2n+1)\Delta^n_{0\ st} = \left( \frac{2j+1}{2} \right) \frac{\sigma}{c^2} \int_{-1}^{+1} \left[ (c^2 - 2)nK^n_0 + 2\gamma_1 \frac{\partial K^n_0}{\partial x_1} \right] \, dx_1
\]

where \( j = -n-1 \)

Nonzero solutions to (D.4) are obtained for \( n = -1 \) and \( n = -3 \) only.

For \( n = -1 \) and from (D.2),

\[
\alpha^{-1}_{0\ st} = 2\alpha_0^{-1}
\]

\[
\beta^{-1}_{0\ st} = \alpha_0^{-1} + \frac{(3c^2 - 4)}{12c^2}
\]

where \( \alpha_0^{-1} \) is an arbitrary constant which does not affect the boundary equation or the stresses.

For \( n = -3 \),

\[
\alpha^{-3}_{0\ st} = -\frac{160}{3c^2 (9c^2 - 4)} \frac{\sigma}{p}
\]

\[
\beta^{-3}_{0\ st} = -\frac{(6c^2 + 4)}{3c^2 (9c^2 - 4)} \frac{\sigma}{p}
\]

For \( n = -1 \), \( \alpha_0^{0\ st} \) and \( \beta_0^{0\ st} \) are obtained by substituting (D.5) into (D.3). In order to satisfy (D.3), it is necessary that

\[
A_0^{-1} = -\frac{3c^2 - 4}{12c^2} \frac{\sigma}{p}
\]
With this condition, (D.3) gives

\[ \alpha_0^{st} = - \frac{3c^2-4}{6c^2} \sigma_p \]  

and \( \beta_0^{st} \) as an arbitrary constant.

For \( n = -3 \), \( \alpha_0^{st} \) and \( \beta_0^{st} \) are obtained by substituting (D.6) into (D.3);

\[ \alpha_0^{st} = \frac{20(c^2-2)}{c^2(9c^2-4)} \sigma_p \]  

\[ \beta_0^{st} = -\frac{6c^2+4}{3c^2(9c^2+4)} \sigma_p \]  

The total solution is determined by adding to the solution given by (D.7) and (D.8) the homogeneous stress

\[ \sigma_{ij}^p = -[28_{11}i_j + (c^2-2)\delta_{ij}j_k] \frac{\sigma_p}{c^2} \]  

D.3. Static Solution for the S Wave

The boundary equations from (2.26) are (for \( r_0 = 1 \)):

\[ n \gamma_1^{st} + n \Delta_1^{st} = -\frac{i(2j+1)}{2(j+1)} \sigma_s \int_{-1}^{1} 2x_1(1-x_1^2)L_j^j \, dx_1 \]

\[ n \gamma_1^{st} + n(2n+1)\Delta_1^{st} = -\frac{i(2j+1)}{2(j+1)} \sigma_s \int_{-1}^{1} \left[ 2x_1L_j^j + (1-x_1^2) \frac{\partial L_j^j}{\partial x_1} + 2x_1(1-x_1^2) \frac{\partial L_j^j}{\partial x_1} \right] \, dx_1 \]

\[ n(n+1)x_1^{n} = \frac{i(2j+1)}{2(j+1)} \sigma_s \int_{-1}^{1} \left[ (1-3x_1^2)L_j^j + x_1(1-x_1^2) \frac{\partial L_j^j}{\partial x_1} \right] \, dx_1 \]  

(D.10)
where \( j = -n-1 \)

\[
L_j^j(r, x_1) = \frac{(-1)^{n+1}}{n!} \frac{\partial^{n-1}}{\partial x_1^{n-1}} \left( \frac{1}{r^3} \right)
\]

The only nonzero solution to (D.10) is for \( n = -3 \), where

\[
\alpha_{1, \text{st}}^{-3} = -\frac{160}{3(9c^2-4)} \sigma_s
\]

\[
\beta_{1, \text{st}}^{-3} = -\frac{6c^2+4}{3(9c^2-4)} \sigma_s
\]

\[
h_{1, \text{st}}^{-3} = 0
\]

If (D.11) is substituted into (D.3),

\[
\alpha_{1, \text{st}}^2 = -\frac{20(c^2-2)}{(9c^2-4)} \sigma_s
\]

\[
\beta_{1, \text{st}}^2 = \frac{(6c^2+4)}{3(9c^2-4)} \sigma_s
\]

The total solution is determined by adding to the solution given by (D.12), the homogeneous stress

\[
\sigma_{ij}^s = [\delta_{ij} \delta_{11} + 8 \delta_{ij} \delta_{11}] \sigma_s
\]

D.4. Completeness of the Dynamic Solution Given in Appendix B

An elastic solution for a spherical cavity of radius \( r_0 \) loaded by boundary tractions is given by (D.1). The boundary traction corresponding to this solution is given by (17).
\[
\begin{align*}
    r \frac{T_{n,m}^n}{\mu} &= A_n^m \left\{ (2n + \alpha_n) r^2 \frac{\partial K_{m}^n}{\partial x_i} + [2n(m-1) + \alpha_n(m(n+3) + n + 2)] x_i K_{m}^n \right\} \\
    &+ 2(n-1) B_n^m \frac{\partial K_{m}^n}{\partial x_i} + (n-1) C_n^m \frac{\partial K_{m}^n}{\partial x_j} \\
    &+ (n-1) C_n^m \frac{\partial K_{m}^n}{\partial x_k}
\end{align*}
\]  

(D.14)

It is shown in Ref. 20 that the solution (D.1) or (D.14) is the general solution so that its representation of the traction \( T_i \) at the boundary is complete. On the boundary surface (D.14) may be written as

\[
T_{n,m}^n = A_n^m x_i K_{m}^n + B_n^m \frac{\partial K_{m}^n}{\partial x_i} + C_n^m \frac{\partial K_{m}^n}{\partial x_j} \\
\]  

(D.15)

This equation is of the same form as the traction (2.12) derived from the dynamic solution given in Appendix B, where, at any instant of time,

\[
\begin{align*}
    \gamma &= A_n^m \\
    \Delta &= B_n^m \\
    \chi &= C_n^m
\end{align*}
\]  

(D.16)

Thus, since (D.15) is complete, then at any instant of time the dynamic solution (2.12) must also be complete. It only remains to verify that the coefficients \( \gamma_n^m \), \( \Delta_n^m \), and \( \chi_n^m \) can be determined for all time. This has been shown in 2.4 by the setting up and solving of the boundary equations (2.16).
APPENDIX E

SOME REMARKS CONCERNING THE COMPUTER PROGRAM

The boundary equations for each mode (equations (2.21) for the incident P wave and (2.26) for the incident S wave) when written out in full in terms of the generating functions on the left-hand side and the integrated polynomial expressions of time \( t_p \) or \( t_s \) on the right-hand side become very lengthy as \( n \) takes on larger values. For the elastic material, it was convenient to derive and store these expressions directly in the computer instead of by hand.

The technique of deriving all such formulas is based on the use of recurrence formulas for obtaining expressions for the spherical harmonics and expressions of functions such as \( f_m^n \), \( g_m^n \), etc. in terms of the generating functions \( F_m^n \), \( G_m^n \), and \( H_m^n \) and upon the simple rules for differentiation and integration of polynomials.

The recurrence formula for \( f^0_0(r,t) \) in terms of \( F^0_0(t-(r-1)) \) is given by

\[
f^n_0 = \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \left( f^{n-1}_0 \right)
\]

where

\[
f^0_0 = \frac{F^0_0}{r}
\]

For example,

\[
f^1_0 = \left( \frac{1}{r} \frac{\partial}{\partial r} \right) \left( \frac{1}{2} F^1_0 (t_p - (r-1)) \right) = - \frac{1}{r^3} F^1_0 - \frac{1}{r^2} \frac{\partial}{\partial r} F^1_0 \tag{E.1}
\]

The operations indicated can be performed by the computer by using the simple rules for differentiating polynomials. Expression (E.1) can be stored in the computer as two row vectors, one giving the coefficients of
the terms, another giving the exponent of \( r \) for each term, while the order of the derivative (e.g., \( r^1 \)) is determined by counting from left to right. In this way, two vectors represent the formula for \( f_0^n \). Similarly, a pair of vectors represents the formulas for \( g_m^n, h_m^n, \gamma_m^n, \Delta_m^n, \chi_m^n, \alpha_m^n, \) and \( \beta_m^n \) in terms of \( F_m^n, G_m^n, \) and \( H_m^n \). Finally, the left-hand side of the boundary equations can be set up and stored in this way.

The recurrence formula for the axially symmetric spherical harmonic \( K_0^n(r, x_1) \) is given by

\[
K_0^n = -\frac{r^{2n+1}}{n} \frac{\partial}{\partial x_1} \left( \frac{K_0^{n-1}}{r^{2n-1}} \right)
\]

where \( K_0^0 = 1 \)

For example, for \( K_0^2 \):

\[
K_0^2 = -\frac{r^5}{2} \frac{\partial}{\partial x_1} \left( \frac{K_0^1}{r^3} \right) = -\frac{r^5}{2} \frac{\partial}{\partial x_1} \left( \frac{x_1}{r^3} \right) = \frac{3}{2} x_1^2 - \frac{1}{2} r^2 \quad (E.2)
\]

This process of differentiation to obtain \( K_0^2 \) from \( K_0^1 \) can be performed by the computer using the simple rules for differentiating. Expression (E.2) can be stored in the computer as three row vectors, one giving the coefficients of the terms, another giving the exponents of \( x_1 \), and another giving the exponents of \( r \). In this way, three vectors represent the formula for \( K_0^n \).

Similar sets of vectors are easily derived for \( \frac{\partial K_0^n}{\partial x_1}, \quad \frac{\partial^2 K_0^n}{\partial x_1^2}, \quad Q_0^n, \quad R_0^n \). Also the integrals on the right-hand side of the boundary equations are obtained by the simple rule of integration of polynomials, and stored as row vectors.
Once all the formulas for the boundary equations are stored as vectors, the generating functions for each set of boundary equations are numerically integrated as in 2.5. When the generating functions and their derivatives are calculated for a position of time, the required modal stresses as given by (2.13) are calculated using the stored expressions for $K_m^n$, $Q_m^n$, etc.

For the higher modes, the equations were too lengthy to print out and have been solved numerically without ever being seen.
<table>
<thead>
<tr>
<th>Case</th>
<th>Maximum Static Principal Tensile Stress $\sigma_{st}$ ($\theta = 90^\circ$)</th>
<th>Maximum Dynamic Principal Tensile Stress $\sigma_d$ ($\theta = 90^\circ$)</th>
<th>Time for $\sigma_d$ ($t_p$ or $t_{p0}$)</th>
<th>Ratio $\frac{\sigma_d}{\sigma_{st}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic, $v = 0$</td>
<td>1.93</td>
<td>2.05</td>
<td>4.0</td>
<td>1.06</td>
</tr>
<tr>
<td>Elastic, $v = 1/4$</td>
<td>1.85</td>
<td>1.98</td>
<td>4.2</td>
<td>1.07</td>
</tr>
<tr>
<td>Elastic, $v = 0.40$</td>
<td>1.70</td>
<td>2.00</td>
<td>4.8</td>
<td>1.18</td>
</tr>
<tr>
<td>Maxwell Model, $^*$ ($p = 1$)</td>
<td>1.50</td>
<td>1.50</td>
<td>-</td>
<td>1.00</td>
</tr>
<tr>
<td>Standard Linear Model, $^*$ ($p = 1$, $\alpha = 1/2$)</td>
<td>1.75</td>
<td>1.86</td>
<td>4.2</td>
<td>1.01 $^{**}$</td>
</tr>
</tbody>
</table>

$^*$ Both viscoelastic models have instantaneous Poisson's ratio 1/4 and are elastic in dilatation.

$^{**}$ This is the ratio of the maximum dynamic stress to the maximum elastic static stress ($v = 1/4$)

**TABLE 1. MAXIMUM DYNAMIC STRESSES AROUND A SPHERICAL CAVITY DUE TO AN INCIDENT STEP P WAVE**
<table>
<thead>
<tr>
<th>Case</th>
<th>Maximum Static</th>
<th>Maximum Dynamic</th>
<th>Ratio σ_d/σ_st</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Principle Tensile Stress σ_st</td>
<td>Tensile Stress σ_d</td>
<td>Location for σ_st</td>
</tr>
<tr>
<td>Elastic, v = 0, 11 modes</td>
<td>2.15</td>
<td>2.99</td>
<td>θ=135°, φ=0°</td>
</tr>
<tr>
<td>Elastic, v = 0, 13 modes</td>
<td>2.15</td>
<td>3.09</td>
<td>θ=135°, φ=0°</td>
</tr>
<tr>
<td>Elastic, v = 1/4, 11 modes</td>
<td>2.61</td>
<td>3.22</td>
<td>θ=135°, φ=0°</td>
</tr>
<tr>
<td>Elastic, v = 0.40, 11 modes</td>
<td>3.00</td>
<td>3.55</td>
<td>θ=135°, φ=0°</td>
</tr>
<tr>
<td>Elastic, v = 1/4, 8 modes</td>
<td>2.61</td>
<td>3.09</td>
<td>θ=135°, φ=0°</td>
</tr>
<tr>
<td>Maxwell Model*, (p=1)</td>
<td>3.33</td>
<td>-</td>
<td>θ=135°, φ=0°</td>
</tr>
<tr>
<td>Standard Linear Model* (p=1, α=1/2), 8 modes</td>
<td>2.90</td>
<td>2.77</td>
<td>θ=135°, φ=0°</td>
</tr>
</tbody>
</table>

* Both viscoelastic models have instantaneous Poisson's ratio 1/4 and are elastic in dilatation.
** This is the ratio of the maximum dynamic stress to the maximum elastic static stress. (v=1/4).

TABLE 2. MAXIMUM DYNAMIC STRESSES AROUND A SPHERICAL CAVITY DUE TO AN INCIDENT STEP S WAVE
FIG. 1. STRESSES IN SPHERICAL COORDINATES

FIG. 2. RELATION BETWEEN SPHERICAL AND CARTESIAN COORDINATES
FIG. 3. DIRECTION OF TRAVEL OF THE INCIDENT STEP WAVE

\[ \sigma \rightarrow \frac{1}{\eta} \rightarrow \frac{E}{\rho} \rightarrow E \rightarrow \sigma \]

FIG. 4. MAXWELL MODEL
FIG. 5. GENERALIZED MAXWELL MODEL

FIG. 6. NUMERICAL INTEGRATION OF INTEGRO-DIFFERENTIAL BOUNDARY EQUATIONS; REPRESENTATION OF \[ \int_{0}^{t_{p0}} F_{0}(t_{p0} - \tau)f(\tau)\,d\tau \]
FIG. 7. HOOP STRESS AT THE BOUNDARY DUE TO STEP PRESSURE SYMMETRICALLY APPLIED TO THE CAVITY BOUNDARY
FIG. 8. BOUNDARY VELOCITY DUE TO STEP PRESSURE SYMMETRICALLY APPLIED TO THE CAVITY BOUNDARY
FIG. 9. BOUNDARY DISPLACEMENT DUE TO STEP PRESSURE SYMMETRICALLY APPLIED TO THE CAVITY BOUNDARY
FIG. 10. RADIAL AND HOOP STRESS (FOR \( r = 2 \times \text{CAVITY RADIUS} \)) DUE TO STEP PRESSURE APPLIED TO THE CAVITY BOUNDARY
FIG. 11. STRESS $\sigma_{\theta\theta}$ ($\theta = 90^\circ$) DUE TO INCIDENT STEP P WAVE: ELASTIC MATERIAL
FIG. 13. VARIATION OF STRESS $\sigma_{\theta\theta}$ ALONG THE MERIDIAN DUE TO THE INCIDENT STEP P WAVE: ELASTIC MATERIAL ($\nu = 1/4$)
FIG. 14. INCIDENT STEP P-WAVE STRESS $\sigma^{P}_{\theta\theta}$ ($\theta = 90^\circ$)
FIG. 15. MODAL STRESS $\sigma_{\theta\theta}$ ($\theta = 90^\circ$) IN DISTURRING WAVE (INCIDENT STEP P WAVE)
FIG. 16. MODAL STRESS $\sigma_{\theta \theta}^{10}$ ($\theta = 0^0$) IN DISTURBING WAVE (INCIDENT STEP P WAVE)
Fig. 18. Modal stress $\sigma_{\theta\theta}^{\lambda 0}$ ($\theta = 0^0$) in disturbing wave (incident step P wave)
FIG. 19. MODAL STRESS $\sigma_{\theta\theta}^{40} (\theta = 90^\circ)$ IN DISTURBING WAVE (INCIDENT STEP P WAVE)
FIG. 21. STRESS $\sigma_{\theta\theta}$ DUE TO INCIDENT STEP $S$ WAVE IN ELASTIC MATERIAL (INCIDENT WAVE PLUS 11 MODES OF THE DISTURBING WAVE)
FIG. 23. STRESS $\sigma_{\theta\theta}$ DUE TO INCIDENT STEP S WAVE (INCIDENT WAVE PLUS 7 MODES OF THE DISTURBING WAVE)
FIG. 24. VARIATION OF STRESS $\sigma_{\theta\theta}$ ALONG THE MERIDIAN (INCIDENT STEP S WAVE): ELASTIC MATERIAL ($\nu = 1/4$)

FIG. 25. VARIATION OF STRESS $\sigma_{\theta\theta}$ ALONG THE MERIDIAN (INCIDENT STEP S WAVE): ELASTIC MATERIAL ($\nu = 1/4$)
Fig. 26: Incident Step S-Wave Stresses

\[ \text{Elastic (v)} = \frac{1}{3} \text{p} \theta \theta \text{ and } \theta \theta \theta \theta \text{ at } a = 1/4 \]
FIG. 28. MODAL STRESS $\sigma_{\theta\theta}$ ($\theta = 135^\circ$) IN DISTURBING WAVE (INCIDENT STEP S WAVE).
FIG. 30. MODAL STRESS $\sigma_{\theta\theta}^{11}$ ($\theta = 135^\circ$) IN DISTURBING WAVE (INCIDENT STEP S WAVE)

FIG. 31. MODAL STRESS $\sigma_{\theta\theta}^{51}$ ($\theta = 135^\circ$) IN DISTURBING WAVE (INCIDENT STEP S WAVE)
FIG. 32. MODAL STRESS $\sigma_{\theta\theta}^{61} (\theta = 135^\circ)$ IN DISTURBING WAVE (INCIDENT STEP $S$ WAVE)

FIG. 33. MODAL STRESS $\sigma_{\theta\theta}^{71} (\theta = 135^\circ)$ IN DISTURBING WAVE (INCIDENT STEP $S$ WAVE)
The solid curve is the fitted standard linear model; the circles are McDonald's (1959) experimental data. (After C. W. Horton (15))

FIG. 34. ATTENUATION OF TRANSVERSE WAVES VERSUS FREQUENCY
VITA

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