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TWO RULES FOR DEDUCING VALID INEQUALITIES FOR 0-1 PROBLEMS*

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Abstract. We present two rules, one of which is equivalent to linear programming, for obtaining consequence inequalities from systems of linear inequalities in which each variable is restricted to being zero or one.

A *zero-one problem* is a system of inequalities

$$\text{(S)} \qquad \begin{aligned} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n &\geqq A_1,\\ &\cdots\\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n &\geqq A_m \end{aligned}$$

in which each x_i is restricted to being either zero or one. An inequality

$$\text{(I)} \qquad \sum t_i x_i \geqq R$$

is said to be *valid for* (S) if every $(x_1, \cdots, x_n) \in \{0, 1\}^n$ satisfying (S) satisfies (I). In particular, if there are no $(x_1, \cdots, x_n)$ satisfying (S) any inequality is valid.

(I) can be *linearly deduced* from (S) iff there are $\theta_1, \cdots, \theta_m \geqq 0$ and $\gamma_1, \cdots, \gamma_n \geqq 0$ such that

$$\sum_{j=1}^{m} a_{ji}\theta_j - \gamma_i \leqq t_i, \qquad 1 \leqq i \leqq n,$$

and

$$\sum_1^m A_j\theta_j - \sum_1^n \gamma_j \geqq R.$$

Any inequality linearly deduced from (S) is valid for (S). If (I) is linearly deduced from (S) we can obtain (I) by adding together nonnegative multiples of the inequalities of (S), the inequalities $-x_i \geqq -1$, and the inequalities $x_i \geqq 0$ using multipliers θ_i, γ_i, and $t_i - (\sum_{j=1}^m a_{ji}\theta_j - \gamma_i)$ respectively. The converse does not hold: for example if (S) consists of the single inequality $x_1 \geqq 1/2$, $x_1 \geqq 1$ is valid but cannot be linearly deduced from (S).

We give a second method of deducing valid inequalities from previous ones. With appropriate apologies, we call it the nameless rule.

Nameless rule. For $1 \leqq M \leqq n$, if

$$(\text{P}_1) \qquad t_1x_1+t_2x_2+\cdots+sx_M+\cdots+t_nx_n \geqq P,$$

$$(\text{P}_2) \qquad t_1x_1+t_2x_2+\cdots+rx_M+\cdots+t_nx_n \geqq T,$$

are both valid for (S) then so is

$$\text{(C)} \qquad t_1x_1+t_2x_2+\cdots+(r+P-T)x_M+\cdots+t_nx_n \geqq P.$$

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The nameless rule is clearly correct. If $(x_1, \cdots, x_n) \in \{0; 1\}^n$ satisfies (S) and $x_M = 0$, then $(x_1, \cdots, x_n)$ satisfies (C) because (P_1) is valid. If $x_M = 1$, then the correctness of (C) follows from the validity of (P_2). If $s - r > P - T > 0$, (C) cannot be linearly deduced from (P_1) and (P_2).

THEOREM. *Every valid inequality for* (S) *can be deduced by a finite sequence of linear deductions and applications of the nameless rule. In other words, if* (I) *is valid there is a sequence* $I_1, I_2, \cdots, I_L$ *such that*: (i) $I_1, \cdots, I_m$ *are the inequalities of* (S); (ii) $I_L = (I)$; (iii) *for* $m < j \leqq L$ *either* I_j *can be linearly deduced from* $I_1, \cdots, I_{j-1}$ *or* I_j *can be deduced from* I_h, I_k *by the nameless rule* $(k, h < j)$.

Proof. We induct on the number of variables. For $n = 1$ we may assume (S) consists of the two inequalities $x_1 \geqq \alpha$ and $-x \geqq \beta$. If $\alpha > 0$, $x_1 \geqq 1$ is valid for (S). We first obtain $0x_1 \geqq 0$ from (S) by linear deduction and then use the nameless rule with $s = 1, P = \alpha, r = T = 0$ to obtain $\alpha x_1 \geqq \alpha$. Similarly, if $\beta > -1$, we can use the inequality $0x_1 \geqq 0$ and the nameless rule $(s = P = 0,\ r = -1,\ T = \beta)$ to deduce $(-1-\beta)x_1 \geqq 0$. If $\alpha > 1$ or $\beta > 0$ the system (S) has no zero-one solutions and we can linearly deduce any inequality.

We now assume that our theorem has been established for systems with $n - 1$ variables and that (I) is valid for (S). It follows that

$$(\mathrm{I}_1) \qquad t_2x_2 + \cdots + t_nx_n \geqq R$$

is valid for

$$(\mathrm{S}_1) \qquad \begin{aligned} a_{12}x_2 + \cdots + a_{1n}x_n &\geqq A_1, \\ &\cdots \\ a_{m2}x_2 + \cdots + a_{mn}x_n &\geqq A_m, \end{aligned}$$

and that

$$(\mathrm{I}_2) \qquad t_2x_2 + \cdots + t_nx_n \geqq R - t_1$$

is valid for

$$(\mathrm{S}_2) \qquad \begin{aligned} a_{12}x_2 + \cdots + a_{1n}x_n &\geqq A_1 - a_{11}, \\ &\cdots \\ a_{m2}x_2 + \cdots + a_{mn}x_n &\geqq A_m - a_{m1} \end{aligned}$$

(if either of these assertions failed we could construct $(x_1, \cdots, x_n)$ satisfying (S) which did not satisfy (I)). By induction hypothesis, (I_1) can be deduced from (S_1) and (I_2) from (S_2).

LEMMA 1. *For some* $\Gamma \geqq 0$,

$$(\mathrm{I}_1') \qquad \Gamma x_1 + t_2x_2 + \cdots + t_nx_n \geqq R$$

can be deduced from (S).

Proof of lemma. We induct on the number of steps (linear deductions and nameless rule applications) in the deduction of (I_1) from (S_1). If (I_1) is obtained from (S_1) by a single linear deduction we use the same multipliers on the inequalities of (S) to obtain (I_1') for some $\Gamma \geqq 0$. If (I_1) is deduced from (S_1) by an application of the nameless rule to rows h, k of (S_1), then (S) contains two inequalities

$$(\mathrm{P}_1) \qquad a_h x_1 + \cdots + a_{hn} x_n \geqq A_h,$$

$$(\mathrm{P}_2) \qquad a_{k1} x_1 + \cdots + a_{kn} x_n \geqq A_k,$$

such that $a_{h2} = a_{k2}, \cdots, a_{h(M-1)} = a_{k(M-1)}, \quad a_{h(M+1)} = a_{k(M+1)}, \cdots, a_{hn} = a_{kn}$. Form ($\mathrm{P}'_1$), ($\mathrm{P}'_2$) by replacing a_{h1}, a_{k1} by $\max(a_{h1}, a_{k1}, 0)$. (P'_1) can be linearly deduced from (P_1), and (P'_2) can be linearly deduced from (P_2). Application of the nameless rule to (P'_1), (P'_2) yields (I'_1) with $\Gamma = \max(a_{h1}, a_{k1}, 0)$.

If (I_1) is deduced from (S_1) in several steps we consider the final step in the deduction.

Case 1. If the final step is a linear deduction, (I_1) is linearly deduced from inequalities $J_1, \cdots, J_Q$, each of which is deduced from (S_1). Since the deductions of $J_1, \cdots, J_Q$ from (S_1) are each shorter than the deduction of (I_1), the induction hypothesis implies that $J'_1, \cdots, J'_Q$ may be deduced from (S), where J'_k differs from J_k in that the left-hand side of J'_k has an additional term $\Gamma_k x_1$, for some $\Gamma_k \geqq 0$. (I'_1) can be linearly deduced from $J'_1, \cdots, J'_Q$.

Case 2. If the final step is an application of the nameless rule to inequalities J_1, J_2, then the induction hypothesis implies that J'_1, J'_2 can be deduced from (S), where J'_k is formed from J_k by addition of the term $\Gamma_k x_1$ to the left-hand side. We form J''_1, J''_2 by replacing $\Gamma_1 x_1, \Gamma_2 x_1$ by $(\max \Gamma_1, \Gamma_2) x_1$. J''_1 can be linearly deduced from J'_1 and J''_2 from J'_2. Application of the nameless rule to J''_1, J''_2 yields (I'_1). Q.E.D.

LEMMA 2. *For some* $\Delta \geqq 0$,

$$(\mathrm{I}'_2) \qquad -\Delta x_1 + t_2 x_2 + \cdots + t_n x_n \geqq R - t_1 - \Delta$$

can be deduced from (S).

Proof of lemma. This proof is isomorphic to the proof of Lemma 1. We induct on the length of the deduction of (I_2) from (S_2). If (I_2) is obtained from (S_2) by a single linear deduction using multipliers $\theta_1, \cdots, \theta_m, \gamma_2, \cdots, \gamma_n$ (see the definition of linear deduction) we perform the same linear deduction on (S) using $\gamma_1 = \max(0, \sum \theta_j a_{j1})$ to obtain (I'_2) with $\Delta = \max(0, -\sum \theta_j a_{j1})$. If ($\mathrm{I}_2$) is obtained from rows h, k of (S_2) by the nameless rule, let $-\Delta = \min(a_{h1}, a_{k1}, 0)$. By adding multiples of $-x_1 \geqq -1$ to rows h, k we obtain two inequalities which have first term $-\Delta x_1$, to which we may apply the nameless rule to obtain (I'_2) (linear deduction allows us to add multiples of $-x_1 \geqq -1$ to an inequality).

If the deduction of (I_2) from (S_2) takes several steps we consider the final step, as in Lemma 1.

Case 1. If the final step is a linear deduction of (I_2) from inequalities $J_1, \cdots, J_Q$ using multipliers $\theta_1, \cdots, \theta_Q$, the induction hypothesis yields inequalities $J'_1, \cdots, J'_Q$ each of which may be deduced from (S). J'_k differs from J_k in that $-\Delta_k x_1$ has been added to the left-hand side and $-\Delta_k$ has been added to the right-hand side. We can linearly deduce (I'_2) from $J'_1, \cdots, J'_Q$ with $\Delta = \sum \theta_j \Delta_j$.

Case 2. If the final step is an application of the nameless rule to inequalities J_1, J_2, the induction hypothesis implies there are J'_1, J'_2 as above, each of which can be deduced from (S). We add a multiple of $-x_1 \geqq -1$ to one of the inequalities to insure that the coefficient of x_1 is the same in both. The nameless rule then yields (I'_2) with $\Delta = \max(\Delta_1, \Delta_2)$. Q.E.D.

Lemmas 1 and 2 show that (I_1') and (I_2') can be deduced from (S). We may deduce (I) from (I_1) and (I_2) by the nameless rule, taking $M = 1$, $s = \Gamma$, $r = -\Delta$, $P = R$, and $T = R - t_1 - \Delta$. Q.E.D.

Remark. E. Balas has informed me that he had experimented with the nameless rule in practical zero-one algorithms several years ago, but did not investigate its theoretical power.