RELIABILITY ANALYSIS OF REDUNDANT
DUCTILE STRUCTURAL SYSTEMS

By
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A practical method for determining the point estimate of the collapse probability of ductile structural systems, particularly, plastic frameworks and trusses is developed. In addition to the assumption that classical simple plastic method of analysis is applicable, the magnitudes of loads and the component resistance capacities of a structure are assumed to be stochastic with known distributions (e.g., normal).

The first step of the proposed method of analysis is to identify the major collapse mechanisms. This step is transformed into an unconstrained nonlinear minimization problem and Hooke and Jeeves pattern search was adopted as the solution technique. Once the major mechanisms are identified, their corresponding performance functions may be developed through the principle of virtual work, from which the individual mechanism collapse probabilities are calculated. The system collapse probability is then evaluated through the method of PNET.

Results of the proposed method showed good agreement with corresponding Monte Carlo calculations for a wide variety of example problems examined.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>INTRODUCTION</td>
</tr>
<tr>
<td>1.1</td>
<td>General Remarks</td>
</tr>
<tr>
<td>1.2</td>
<td>Objective and Scope of Study</td>
</tr>
<tr>
<td>1.3</td>
<td>Organization</td>
</tr>
<tr>
<td>2</td>
<td>TECHNICAL REVIEW OF PREVIOUS WORK</td>
</tr>
<tr>
<td>2.1</td>
<td>Determinate Structural Systems</td>
</tr>
<tr>
<td>2.2</td>
<td>Parallel Redundant Systems</td>
</tr>
<tr>
<td>2.2.1</td>
<td>Parallel Brittle Elements</td>
</tr>
<tr>
<td>2.2.2</td>
<td>Parallel Ductile Elements</td>
</tr>
<tr>
<td>2.3</td>
<td>General Brittle Systems</td>
</tr>
<tr>
<td>2.4</td>
<td>General Structural Systems</td>
</tr>
<tr>
<td>2.4.1</td>
<td>A Conceptual Model</td>
</tr>
<tr>
<td>2.4.2</td>
<td>Rosenblueth's Point Estimate</td>
</tr>
<tr>
<td>2.4.3</td>
<td>Monte Carlo Simulation</td>
</tr>
<tr>
<td>2.5</td>
<td>Redundant Ductile Systems</td>
</tr>
<tr>
<td>2.5.1</td>
<td>Analysis of Plastic Frameworks</td>
</tr>
<tr>
<td>2.5.2</td>
<td>Gorman and Moses's Method</td>
</tr>
<tr>
<td>2.5.3</td>
<td>Bounds on Collapse Probability</td>
</tr>
<tr>
<td>3</td>
<td>FORMULATIONS OF SYSTEM RELIABILITY</td>
</tr>
<tr>
<td>3.1</td>
<td>Introduction</td>
</tr>
<tr>
<td>3.2</td>
<td>Reliability Analysis of Individual Failure Mode</td>
</tr>
<tr>
<td>3.3</td>
<td>Point Estimate of System Collapse Probability</td>
</tr>
<tr>
<td>3.3.1</td>
<td>The PMET Method</td>
</tr>
<tr>
<td>3.3.2</td>
<td>Illustrative Example</td>
</tr>
<tr>
<td>3.4</td>
<td>Identification of Major Failure Modes</td>
</tr>
<tr>
<td>3.4.1</td>
<td>Exhaustive Enumeration</td>
</tr>
<tr>
<td>3.4.2</td>
<td>Simulation</td>
</tr>
<tr>
<td>3.4.3</td>
<td>Heuristic Search</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
</tr>
<tr>
<td>4.3.1</td>
<td>Description of Solution Technique</td>
</tr>
<tr>
<td>4.3.2</td>
<td>Illustrative Example</td>
</tr>
<tr>
<td>4.4</td>
<td>Computer Program</td>
</tr>
<tr>
<td>4.5</td>
<td>Monte Carlo Calculations</td>
</tr>
<tr>
<td>5.1.1</td>
<td>Problem 1</td>
</tr>
<tr>
<td>5.1.2</td>
<td>Problem 2</td>
</tr>
<tr>
<td>5.1.3</td>
<td>Problem 3</td>
</tr>
<tr>
<td>5.1.4</td>
<td>Problem 4</td>
</tr>
<tr>
<td>5.2.1</td>
<td>Problem 5</td>
</tr>
<tr>
<td>5.2.2</td>
<td>Problem 6</td>
</tr>
<tr>
<td>5.2.3</td>
<td>Problem 7</td>
</tr>
<tr>
<td>5.2.4</td>
<td>Problem 8</td>
</tr>
<tr>
<td>5.3</td>
<td>Summary of Results</td>
</tr>
<tr>
<td>6.1.1</td>
<td>Number of Significant Mechanisms</td>
</tr>
<tr>
<td>6.1.2</td>
<td>Minimization Algorithm</td>
</tr>
</tbody>
</table>
6.1.3 Effects of Formulations ...................... 109
6.2 On the Synthesis of System Collapse Probability .... 109
   6.2.1 The PNET Method .......................... 109
   6.2.2 Bounds on System Reliability ............... 114

7 SUMMARY AND CONCLUSIONS .......................... 125
   7.1 Summary ....................................... 125
   7.2 Concluding Remarks ............................ 125
       7.2.1 Suggestions for Further Studies .......... 126

REFERENCES .......................................... 128

APPENDIX
   A SYSTEMATIC GENERATION OF COMPATIBILITY EQUATIONS GOVERNING
       THE ANGLES OF ROTATIONS OF POTENTIAL PLASTIC HINGES .... 133
   B SYSTEMATIC GENERATION OF ELEMENTARY MECHANISMS .......... 140
# LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>SUMMARY OF POSSIBLE MECHANISMS</td>
</tr>
<tr>
<td>4.1</td>
<td>SUMMARY OF COMPUTATIONS</td>
</tr>
<tr>
<td>5.1</td>
<td>SUMMARY OF RESULTS FOR PROBLEM 1</td>
</tr>
<tr>
<td>5.2</td>
<td>SUMMARY OF RESULTS FOR PROBLEM 2</td>
</tr>
<tr>
<td>5.3</td>
<td>SUMMARY OF RESULTS FOR PROBLEM 2A</td>
</tr>
<tr>
<td>5.4</td>
<td>SUMMARY OF RESULTS FOR PROBLEM 2B</td>
</tr>
<tr>
<td>5.5</td>
<td>SUMMARY OF RESULTS FOR PROBLEM 3</td>
</tr>
<tr>
<td>5.6</td>
<td>SUMMARY OF RESULTS FOR PROBLEM 4</td>
</tr>
<tr>
<td>5.7</td>
<td>SUMMARY OF RESULTS FOR PROBLEM 5</td>
</tr>
<tr>
<td>5.8</td>
<td>SUMMARY OF RESULTS FOR PROBLEM 6</td>
</tr>
<tr>
<td>5.9</td>
<td>SUMMARY OF RESULTS FOR PROBLEM 6A</td>
</tr>
<tr>
<td>5.10</td>
<td>SUMMARY OF RESULTS FOR PROBLEM 7</td>
</tr>
<tr>
<td>5.11</td>
<td>MAJOR MECHANISMS IDENTIFIED IN PROBLEM 8</td>
</tr>
<tr>
<td>6.1</td>
<td>LOAD AND RESISTANCE STATISTICS FOR NORMAL AND LOG-NORMAL DISTRIBUTIONS</td>
</tr>
<tr>
<td>6.2</td>
<td>LOAD AND RESISTANCE STATISTICS FOR EXTREME VALUE DISTRIBUTIONS</td>
</tr>
<tr>
<td>6.3A</td>
<td>SUMMARY OF RESULTS (NORMAL DISTRIBUTION)</td>
</tr>
<tr>
<td>6.3B</td>
<td>SUMMARY OF RESULTS (LOG-NORMAL DISTRIBUTION)</td>
</tr>
<tr>
<td>6.3C</td>
<td>SUMMARY OF RESULTS EXTREME-VALUE DISTRIBUTION (WEIBULL M, GUMBEL L)</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>A &quot;Strictly&quot; Parallel Redundant System</td>
<td>25</td>
</tr>
<tr>
<td>2.2</td>
<td>A Multi-member Truss</td>
<td>25</td>
</tr>
<tr>
<td>3.1</td>
<td>One-story One-bay Frame</td>
<td>47</td>
</tr>
<tr>
<td>3.2</td>
<td>Potential Locations of Plastic Hinges</td>
<td>47</td>
</tr>
<tr>
<td>3.3</td>
<td>Coordinates of the Nodes of One-story Frame</td>
<td>47</td>
</tr>
<tr>
<td>3.4</td>
<td>Labelling of Variables of One-story Frame (Formulation 1F)</td>
<td>48</td>
</tr>
<tr>
<td>3.5</td>
<td>Labelling of Variables of One-story Frame (Formulation 2F)</td>
<td>48</td>
</tr>
<tr>
<td>3.6</td>
<td>&quot;Elementary&quot; Mechanisms of One-story Frame</td>
<td>48</td>
</tr>
<tr>
<td>3.7</td>
<td>A Truss Structure</td>
<td>49</td>
</tr>
<tr>
<td>3.8</td>
<td>Definition of Generalized and Global Coordinates</td>
<td>49</td>
</tr>
<tr>
<td>3.9</td>
<td>Definition of Local Coordinates</td>
<td>49</td>
</tr>
<tr>
<td>4.1</td>
<td>Cross-section of a Typical Solution Point</td>
<td>61</td>
</tr>
<tr>
<td>4.2</td>
<td>Descriptive Flow Diagram for Exploratory Move</td>
<td>61</td>
</tr>
<tr>
<td>4.3</td>
<td>Descriptive Flow Diagram for Pattern Search</td>
<td>62</td>
</tr>
<tr>
<td>4.4</td>
<td>Illustration of the method of Hooke &amp; Jeeves Pattern Search (starting point = (0.0, 3.0))</td>
<td>62</td>
</tr>
<tr>
<td>5.1</td>
<td>Some &quot;Combinatorially Possible&quot; Failure Modes</td>
<td>99</td>
</tr>
<tr>
<td>5.2</td>
<td>Frame Structure for Problems 2 &amp; 2B</td>
<td>99</td>
</tr>
<tr>
<td>5.3</td>
<td>Potential Plastic Hinges Locations for Problems 2, 2A &amp; 2B</td>
<td>100</td>
</tr>
<tr>
<td>5.4</td>
<td>Frame Structure for Problem 2A</td>
<td>100</td>
</tr>
<tr>
<td>5.5</td>
<td>Frame Structure for Problem 3</td>
<td>101</td>
</tr>
<tr>
<td>5.6</td>
<td>Potential Plastic Hinges Locations for Problem 3</td>
<td>101</td>
</tr>
<tr>
<td>5.7</td>
<td>Frame Structure for Problem 4</td>
<td>102</td>
</tr>
<tr>
<td>5.8</td>
<td>Potential Plastic Hinges Locations for Problem 4</td>
<td>102</td>
</tr>
<tr>
<td>5.9</td>
<td>Truss Structure for Problem 5</td>
<td>103</td>
</tr>
<tr>
<td>5.10</td>
<td>Truss Structure for Problems 6 &amp; 6A</td>
<td>103</td>
</tr>
<tr>
<td>5.11</td>
<td>Truss Structure for Problem 7</td>
<td>104</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>5.12</td>
<td>Truss Structure for Problem 8</td>
<td>104</td>
</tr>
<tr>
<td>5.13A</td>
<td>Summary of Results</td>
<td>105</td>
</tr>
<tr>
<td>5.13B</td>
<td>Summary of Results</td>
<td>106</td>
</tr>
<tr>
<td>5.13C</td>
<td>Summary of Results</td>
<td>107</td>
</tr>
<tr>
<td>6.1</td>
<td>Error Associated with Assumption of Perfect Correlation of Two Partially Correlated Failure Modes</td>
<td>120</td>
</tr>
<tr>
<td>6.2</td>
<td>Error Associated with Assumption of Statistically Independent of Two Partially Correlated Failure Modes</td>
<td>121</td>
</tr>
<tr>
<td>6.3</td>
<td>Structures Considered in Schueller &amp; Grimmelt (1981)</td>
<td>122</td>
</tr>
<tr>
<td>6.4A</td>
<td>Summary of PNET Results in Schueller &amp; Grimmelt</td>
<td>123</td>
</tr>
<tr>
<td>6.4B</td>
<td>Summary of PNET Results in Schueller &amp; Grimmelt</td>
<td>124</td>
</tr>
<tr>
<td>A.1</td>
<td>Network Representation of a Member</td>
<td>133</td>
</tr>
<tr>
<td>A.2</td>
<td>Numbering of Nodes, Branches and Circuit</td>
<td>136</td>
</tr>
<tr>
<td>B.1</td>
<td>Definition of Local Generalized Coordinates</td>
<td>140</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

1.1 General Remarks

The traditional means of assuring structural safety through prescribing conservative conditions in design may not be economical nor adequate. Because the information used in the development of a design invariably contains uncertainty, the safety of a structure may be measured realistically in terms of the probability of failure. In fact the probabilistic approach to structural safety has been the subject of research for more than 30 years (e.g. Freudenthal, 1947, 1956; Brown, 1960).

A structure or structural system, whether it is a bridge, a building or an offshore platform, is built up of many components or elements. Naturally, its load-resisting capacity will be a function of the capacities of the individual components. Thus, the probability of collapse of the system will also depend on those of its components. Civil engineering structures are invariably one-of-a-kind systems; as such, information (statistical or otherwise) may be available (at best) only for the components. For this reason, the probability of collapse of a structure must be determined through the synthesis of the probabilistic information on the corresponding components. A practical model for the reliability analysis of a structure should give a realistic, rather than conservative, estimate of the collapse probability of the structure.

Depending on whether or not the failed components can continue to carry loads, the collapse of a redundant system may or may not be a function of the sequence in which the components may fail (Ang & Amin, 1968). For a statically determinate system, failure of one component is tantamount to collapse of the system, whereas the failure of one or more components of a redundant system may not necessarily lead to the collapse of the system.
1.2 Objective and Scope of Study

The primary objective of this study is the development of a practical method for determining the point estimate (not just bounds) of the collapse probability of a class of ductile redundant structural systems. Specifically, systems that are made up of elastic-plastic materials, whose resistance functions can be idealized as rigid-plastic, are emphasized. Two common types of such ductile redundant structures, namely, plastic frameworks and trusses are examined.

It is assumed that the classical simple plastic method of analysis (Hodge, 1959; Neal, 1977) is applicable and the collapse of a truss is due to the yielding (tensile or compressive) of its members. It is further assumed that the magnitudes of the externally applied loads are stochastic with known distributions, whereas the points of applications of these loads are deterministic. Similarly, the resistance capacities of the components of a structure are stochastic with known or assumed distributions (e.g. normal).

1.3 Organization

Chapter 2 contains a technical review of previous work on the reliability analysis of structural systems.

In Chapter 3, the basic formulations for the analyses of redundant plastic frameworks and trusses are presented, with illustrative examples.

Chapter 4 examines the available algorithms that can be used to identify the significant failure modes. The solution technique of the algorithm selected in this study is then described, illustrated and implemented. Monte Carlo simulation procedures used to verify the accuracy of the results are also described.

In Chapter 5, the method of reliability analysis developed in this study is applied to various examples of frames and trusses. The results are compared with those of Monte Carlo simulations; available bounds are also evaluated.
Chapter 6 discusses the methods of analysis and observations obtained in this study.

Chapter 7 contains the summary and conclusions of this study, with suggestions for further studies.
CHAPTER 2

TECHNICAL REVIEW OF PREVIOUS WORK

2.1 Determinate Structural Systems

The work done prior to 1964 on the safety analysis of structures was summarized in a paper by Freudenthal, Garrelts, and Shinozuka (1966), in which the probability of failure, $P_f$, for a statically determinate structure consisting of $m$ members with statistically independent resistances under an external load $S$, was given as follows:

$$P_f = \int_0^\infty \left\{ 1 - \prod_{i=1}^m \left[ 1 - F_{R_i}(c_i S) \right] \right\} f_S(s) \, ds$$  \hspace{1cm} (2.1)

where $F_{R_i}$ denotes the CDF of the resistance of the $i$th member, and $c_i$ is a coefficient relating the load effect on the $i$th member induced by the external load $S$; and $f_S(s)$ denotes the PDF of the load $S$. In general, the member resistance or strength may also vary over two distinct ranges of strengths, such as tension and compression of an axial member. Therefore, the positive and negative ranges of the load should be considered separately in Eq. 2.1. A more complete and general expression of $P_f$ for a statically determinate structure, derived by Ang and Amin (1967), was given as:

$$P_f = 1 - \int_0^\infty \left[ \int_{-\infty}^{\alpha_1 S} \cdots \int_{-\infty}^{\alpha_m S} f_{P_1}(r_{p1}, \ldots, r_{pm}) \, dr_{p1} \cdots dr_{pm} \right] f_S(s) \, ds \int_{-\infty}^{\alpha_1 S} \cdots \int_{-\infty}^{\alpha_m S} f_{n1}(r_{n1}, \ldots, r_{nm}) \, dr_{n1} \cdots dr_{nm} \, ds$$  \hspace{1cm} (2.2)

where $\alpha_i S$ is the absolute value of the force induced in member $i$. $f_{P_1}, \ldots, f_{pm}(r_{p1}, \ldots, r_{pm})$ and $f_{n1}, \ldots, f_{nm}(r_{n1}, \ldots, r_{nm})$ are the
m-dimensional joint probability density functions for the resistances $r_1, \ldots, r_m$ in the positive and negative ranges of the member strengths, respectively.

2.2 Parallel Systems

2.2.1 Parallel Brittle Elements

The collapse probability of a statically indeterminate structure consisting of parallel elements such as the one shown in Fig. 2.1, was investigated by Shinozuka, et al (1965) with the following assumptions:

1. the structure consists of only brittle component members;

2. the load is divided equally among all the surviving members;

3. the resistances of the members are statistically independent with identical distribution;

4. collapse of the system requires the failure of all the members.

It is convenient to introduce the following symbols:

- $P'_{mk}(s)$ = the probability that $(m-k)$ among $m$ initially surviving members will fail due to one application of a (deterministic) load $s$.
- $p^i_{jk}(s)$ = the probability that failure will occur to $(j-k)$ members among the currently existing $j$ members with strength greater than $s/i$; thus reducing the number of remaining members from $j$ to $k$.
- $P_{mk}$ = the probability that $(m-k)$ among $m$ initially existing members will fail due to one application of the (statistical) load $S$.

From these definitions, the probability of failure of the structure is:

$$P_f = P_{mo}$$ (2.3)
and,

\[ p_{mk} = \int_0^\infty p'_m(s) f(s) ds \] \hspace{1cm} (2.4)

Also, from the definition,

\[ p'_m(s) = [1 - F_0(s)]^m \] \hspace{1cm} (2.5)

and,

\[ p'_mk(s) = \binom{m}{1} F_0(s) p_m^{m-1} k(s) + \binom{m}{2} [F_0(s)]^2 p_m^{m-2} k(s) + \ldots + \binom{m}{m-k} [F_0(s)]^{m-k} p_{kk}(s) \] \hspace{1cm} (2.6)

with

\[ p_{kk}(s) = [1 - F_0(s)]^k \] \hspace{1cm} (2.7)

For the probabilities \( p^i_{jk}(s) \), \( m \geq i > j > k \), the following recurrence formula applies:

\[ p^i_{jk}(s) = \binom{j}{i} [F_0(s) - F_0(s)]^i p^j_{(j-1)k}(s) \]

\[ + \binom{j}{2} [F_0(s) - F_0(s)]^2 p^j_{(j-2)k}(s) + \ldots \]

\[ + \binom{j}{j-k} [F_0(s) - F_0(s)]^{j-k} p^j_{kk}(s) \] \hspace{1cm} (2.8)

where,

\[ p^j_{kk}(x) = [1 - F_0(s)]^k \] \hspace{1cm} (2.9)
The expression of $P_{mm}(s)$ can be obtained by using the recurrence formulas of Eqs. 2.8 and 2.9 in Eqs. 2.5 through 2.7.

2.2.2 Parallel Ductile Elements

Shinozuka and Itagaki (1966) extended the results for brittle elements to materials that will fail through plastic yielding instead of brittle fracture. Again, the yield resistances of the members are identically distributed and statistically independent. In general, for $k \neq m$ and $k \neq 0$, $P'_{m(m-k)}$ is given by:

$$P'_{m(m-k)}(s) = m(m-1) \cdots (m-k+1) \int \cdots \int_{k-fold}^{f(y_i)} f'(y_i)$$

where $f'(y_i)$ is the PDF of $S$ and $y_i$ is the dummy resistance variable, and the $k$-fold integral is to be carried out in the domain

$$0 < y_1 < \frac{s}{m}, y_1 < y_2 < \frac{s-y_1}{m-1}, \ldots, y_{k-1} < y_k < \frac{s-\sum_{i=1}^{k} y_i}{m-k}$$

For $k=0$,

$$P'_{m0}(s) = m! \int \cdots \int_{(m-1)-fold}^{m-1} f_0(y_i)[F_0(s-\sum_{i=1}^{m-1} y_i)$$

$$- F_0(y_{m-1})] \, dy_1 \cdots dy_{m-1}$$
For \( k = m \), the expression for \( p'_{mn} \) is given by Eq. 2.5. Having \( p'_{m(m-k)} \), Eq. 2.4 is then used to evaluate \( p_{m(m-k)} \).

### 2.3 General Brittle Systems

Yao and Yeh (1967) presented an expression of \( p'_{mk}(s) \) for brittle structures in which the load need not be divided equally among the surviving members (e.g. Fig. 2.2). The expression basically includes all possible permutations of the sequences of member failures. Later, Yao and Yeh (1969) reformulated the reliability analysis of parallel brittle systems so that it is possible to count all the failure paths systematically.

Ishizawa (1968) modelled the brittle failure of a redundant structural system as a Markov process. For a structure of \( n \) degrees of indeterminacy, there are \( n+2 \) states in the state space of the process. Each state constitutes the set of all elements representing the stable configurations that have the same number of redundancy. The original state of the system is represented by the 0th state and the \((n+1)\)th state represents the collapse state, and is the only absorbing state of the state space. The transition probability matrix \([\gamma]\) with a typical element \( \gamma_{ik} \) representing the conditional probability that after the application of the external load, the system is in state \( k \) given that it was in state \( i \) before the load was applied; \([\gamma]\) is an upper triangular matrix. Ishizawa (1968) assumed that the CDF of the resistance of any member remains the same throughout the process and thus the process is a first-order Markov process.

Defining \( \alpha_{i}^{j}|k_{0}, \ldots, k_{j-1} \) S as the force in member \( i \) when the external load \( S \) acts on the modified structure obtained by removing \( k_{0}, \ldots, k_{j-1} \) from the original structure, and \( p_{i}^{j}|k_{0}, \ldots, k_{j-1} \) S as the failure probability of member \( i \) under the load effect \( \alpha_{i}^{j}|k_{0}, \ldots, k_{j-1} \) S, it may be shown that

\[
p_{i}^{j}|k_{0}, \ldots, k_{j-1} = F_{i}(\alpha_{i}^{j}|k_{0}, \ldots, k_{j-1} S)
\]

\[
= \int_{0}^{\alpha_{i}^{j}|k_{0}, \ldots, k_{j-1} S} f_{1}(r_{1})dr_{1} \quad (2.13)
\]
The probability that only members $i_o, \ldots, i_{\ell-1}$ will fail when the external load $S$ acts on the modified structure, with members $k_o, \ldots, k_{\ell-1}$ removed from the original structure, is given by:

$$p^j_{i_0, \ldots, i_{\ell-1}|k_o, \ldots, k_{\ell-1}} = \prod_{\alpha=0}^{m-j} p^j_{i_\alpha|k_o, \ldots, k_{\ell-1}}$$

and

$$(1 - p^j_{i_\beta|k_o, \ldots, k_{\ell-1}}) \quad (2.14)$$

The reliability of the modified structure, therefore, is:

$$Q^j_{i_0, \ldots, i_{m-j}|k_o, \ldots, k_{\ell-1}} = \prod_{\alpha=0}^{m-j} (1 - p^j_{i_\alpha|k_o, \ldots, k_{\ell-1}})$$

$$(2.15)$$

By considering all the possible failure modes, the conditional probability $\mu_{ij}$ that the system will be in state $j$ at time $t_{n+1}$, given that it is in state $i$ at time $t_n$, can be expressed in terms of $Q^j_{i_0, \ldots, i_{m-j}|k_o, \ldots, k_{\ell-1}}$ and $p^j_{i_0, \ldots, i_{\ell-1}|k_o, \ldots, k_{\ell-1}}$. $\mu_{ij}$ corresponds to the times at which a change in the system occurs. A one-step transition probability matrix, $T_i$, which corresponds to a transition from state $i$, $i=0,1,\ldots,n+1$, during the time interval required for a system change can be written as:

$$T_i = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \mu_{i,i} & \mu_{i,i+1} & \cdots & \mu_{i,n+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & 1
\end{bmatrix}$$

$$(2.16)$$
Since the system can only pass to a state of smaller number of redundancy, the Chapman-Kolmogorov equation yields,

\[ \gamma = T_0 \cdots T_n + 1 \]  

(2.17)

So far, the reliability analysis of brittle redundant structures is limited to only one load variable, such that upon the failure of a member, the load effects on the remaining members caused by the redistribution of the applied load can be expressed as a function of the original load. The practical shortcoming of these models is that all the possible sequence of failures of the members has to be identified; the number of such failure sequences can be very large for a practical structure.

2.4 General Structural Systems

2.4.1 A Conceptual Model

Based on the premise that the failure of a structural system occurs when the capacity of the system is less than the applied load, a general model for the reliability analysis of any redundant system (brittle or ductile), was proposed by Ang and Amin (1967). The probability distribution function \( F_R(r) \) of the ultimate capacity of a system \( R \) was given as follows:

\[
F_R(r) = \text{Prob} (R \leq r) = \sum_{i=1}^{k} \text{Prob} \left [(V_i \leq r) \cap (R \in G_i) \right ]
\]  

(2.18)

in which \( R \) is a random vector whose components \( R_1, \ldots, R_n \) are random variables representing the resistances of the individual members; \( G_i \) is a set of vectored sample values of \( R \) such that if \( R \) is in \( G_i \), failure will occur through the \( i \)th path of failure; \( V_i \) is the resistance of the system if failure occurs through path \( i \). The sets \( G_i, i=1,2,\ldots,k, \) are mutually exclusive. The determination of the sets \( G_i \) requires an analysis of the structural system which may best be described through examples.
Consider the system of three cables carrying a vertical load as a special case, as shown in Fig. 2.1. Let \( \alpha_i \), \( i=1,2,3 \), be the force in member \( i \) resulting from a unit load. Then,

\[ \alpha_1 + \alpha_2 + \alpha_3 = 1.0 \]

Assume that the individual members are brittle. Then member 1 will rupture first if

\[ \frac{R_2}{\alpha_2} > \frac{R_1}{\alpha_1} \quad \text{and} \quad \frac{R_3}{\alpha_3} > \frac{R_1}{\alpha_1} \]

(2.19)

and the rupture of member 2 will follow that of member 1, if in addition

\[ \frac{R_3}{\alpha_3} > \frac{R_2}{\alpha_2} \]

(2.20)

The failure path denoted by the above sequence of member ruptures will occur if values of \( R = \{R_1, R_2, R_3\} \) belongs to the sets \( G_1 = \left\{ \frac{R_1}{\alpha_1} < \frac{R_2}{\alpha_2} < \frac{R_3}{\alpha_3} \right\} \); and

\[
P[V_1 \leq r] \cap (R \in G_1)] = \int_{D} f(r_1, r_2, r_3) \, dr_3 \, dr_2 \, dr_1
\]

(2.21)

in which,

\[
D = (0 < r_1 \leq \alpha_1 r; \; \frac{\alpha_2}{\alpha_1} r_1 < r_2 \leq \frac{\alpha_2}{\alpha_1 + \alpha_3} r; \; \frac{\alpha_3}{\alpha_2} r_2 < r_3 \leq r)
\]

(2.22)

In this case, there are a total of \( 3! = 6 \) paths of failure corresponding to the number of permutations of the orders of rupture of the three members.
For the case where the cables are ductile (e.g., elastic—perfectly plastic material), the ultimate capacity of the system corresponds to the state in which all three members have yielded. The order in which the members yield is immaterial since the system capacity is determined when all the members have yielded and therefore there is only one failure path. Hence, for this ductile system,

\[
F_R(r) = P \left[ \left( \sum_{i=1}^{3} R_i \right) < r \right]
\]

(2.23)

and

\[
F_R(r) = \iiint_{\{0 < r_1 + r_2 + r_3 < r \}} f(r_1, r_2, r_3) \, dr_3 \, dr_2 \, dr_1
\]

(2.24)

The formulation represented by Eq. 2.18 applies to general structural systems; however, the model has practical limitations. The identification of the sets \( G_i \) is far from trivial and their physical meanings may depend on whether the structure exhibits brittle or ductile behavior. Another difficulty is associated with the evaluation of the multiple integrals for obtaining CDF of the system capacity \( F_R(r) \).

### 2.4.2 Rosenblueth's Point Estimate

For a function \( Y \) of \( N \) variables; i.e., \( Y = Y(X_1, \ldots, X_N) \), Rosenblueth (1975) suggested that the joint probability density function be discretized at \( 2^N \) points defined by the coordinates \( \mu_{X_i} \pm \sigma_{X_i} \), \( i = 1, 2, \ldots, N \). The probability concentration at each point is given by

\[
P = \frac{1}{2^N} \left[ 1 + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left( + \right) p_{ij} \right]
\]

(2.25)
where the + sign is used if the ith and jth components of the point are given by \((\mu_{x_i} + \sigma_{x_i} & \mu_{x_j} + \sigma_{x_j})\) or \((\mu_{x_i} - \sigma_{x_i} & \mu_{x_j} - \sigma_{x_j})\); whereas the sign is used if the ith and jth components of the point are given by \((\mu_{x_i} - \sigma_{x_i} & \mu_{x_j} + \sigma_{x_j})\) or \((\mu_{x_i} + \sigma_{x_i} & \mu_{x_j} - \sigma_{x_j})\). \(\rho_{ij}\) is the correlation coefficient between \(X_i\) and \(X_j\).

Then, by treating the external loads and component resistances as variables, the structure is evaluated at each of these \(2^N\) point using deterministic analysis, the mean and variance of the structural system resistance can be estimated from these results. The accuracy of the method is unknown and if N is large, \(2^N\) deterministic analyses are required, which could be computationally expensive.

2.4.3 Monte Carlo Simulation

With the availability of high speed digital computers, Monte Carlo simulation techniques have been used in reliability studies (e.g. Warner, 1968). This could be a computationally expensive procedure for structures of practical complexity; however, it is a useful method for checking the results of other analysis procedures.

Monte Carlo is used in this study (see Sect. 3.3.2 and Chapter 5) to verify and validate the PNET method developed herein for ductile systems.

2.5 Redundant Ductile Systems

2.5.1 Analysis of Plastic Frameworks

Stevenson and Moses (1970) developed a method of reliability analysis for plastic frameworks that could collapse through the formation of plastic hinge mechanisms. The performance function \(Z_i\) associated with a plastic mechanism (failure mode) \(i\) is the difference between the internal virtual work and the external virtual work for that mechanism, such that a negative value of \(Z_i\) implies the occurrence of collapse through mechanism \(i\).
Mathematically, $Z_i$ can be written as:

$$Z_i = \sum_j a_{ij} M_j + \sum_k b_{ik} S_k$$  \hspace{1cm} (2.26)

where $a_{ij}$ and $b_{ik}$ are the resistance and load coefficients, respectively; $M_j$ is the plastic moment capacity at the plastic hinge $j$; and $S_k$ is the load that is active in producing mechanism $i$.

The reliability index $\beta_i$ for mechanism $i$ may then be written as:

$$\beta_i = \frac{\mu_{Z_i}}{\sigma_{Z_i}}$$  \hspace{1cm} (2.27)

where $\mu_{Z_i}$ and $\sigma_{Z_i}$ are the mean-value and standard deviation, respectively, of the performance function $Z_i$; i.e.

$$\mu_{Z_i} = \sum_j a_{ij} \mu_{M_j} + \sum_k b_{ik} \mu_{S_k}$$  \hspace{1cm} (2.28)

where $\mu_{M_j}$ and $\mu_{S_k}$ are the mean values of $M_j$ and $S_k$.

Assuming there is no correlation among the variables, the standard deviation of $Z_i$ is,

$$\sigma_{Z_i} = \left[ \sum_j (a_{ij})^2 \sigma_{M_j}^2 + \sum_k (b_{ik})^2 \sigma_{S_k}^2 \right]^{1/2}$$  \hspace{1cm} (2.29)

If there are $n$ possible mechanisms of the system, the probability of collapse of the system is given by:

$$P(\text{collapse}) = P_f = P(Z_1 < 0 \cup Z_2 < 0 \cup \ldots \cup Z_n < 0)$$  \hspace{1cm} (2.30)

Treating the performance function of each mechanism $Z_i$ as a random variable, $P_f$ may be written as:
\[ P_f = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{Z_1, Z_2, \ldots, Z_n} (z_1, z_2, \ldots, z_n) \, dz_1 \, dz_2 \cdots dz_n \]  

(2.31)

where \( f_{Z_1, Z_2, \ldots, Z_n} (z_1, z_2, \ldots, z_n) \) is the \( n \)-dimensional joint PDF of \( Z_1, \ldots, Z_n \).

2.5.2 Gorman & Moses's Method

Gorman and Moses (1979) assumed that \( Z_i, i = 1, 2, \ldots, n \) are random variables with joint normal distribution and used Clark's approximation (1961) for the CDF of \( \max(Z_1, \ldots, Z_n) \) to estimate the system collapse probability \( P_f \).

Observe that

\[ Y_n = \min(Z_1, \ldots, Z_n) = -\max(-Z_1, \ldots, -Z_n) \]  

(2.32)

Therefore,

\[ P_f = P(\min[Z_1, \ldots, Z_n] < 0) = P(Y_n < 0) \]  

(2.33)

As a first order estimate,

\[ P_f = \phi\left(-\mu_{Y_n}/\sigma_{Y_n}\right) \]  

(2.34)

Clark's approximation for the CDF of \( \max(Z_1, \ldots, Z_n) \) may be summarized as follows:
Let
\[ x_1 = z_1 \]
\[ x_2 = \max (z_1, z_2) \]
\[ x_3 = \max (x_2, z_3) = \max [\max (z_1, z_2), z_3] \]
\[ = \max (z_1, z_2, z_3) \]
\[ \vdots \]
\[ \vdots \]
\[ x_n = \max (x_{n-1}, z_n) = \max (z_1, \ldots, z_n) \]

Then, for \( i = 3, 4, \ldots, n+1 \)

\[ \mu_{x_{i-1}} = \mu_{x_{i-2}} \phi(\alpha) + \mu_{z_{i-1}} \phi(-\alpha) + a \phi(\alpha) \quad (2.35a) \]

and

\[ E(x_{i-1}^2) = (\mu_{x_{i-2}}^2 + \sigma_{x_{i-2}}^2) \phi(\alpha) + (\mu_{z_{i-1}}^2 + \sigma_{z_{i-1}}^2) \phi(-\alpha) \]
\[ + (\mu_{x_{i-2}} + \mu_{z_{i-1}}) a \phi(\alpha) \]

From which,

\[ \sigma_{x_{i-1}}^2 = E(x_{i-1}^2) - \mu_{x_{i-1}}^2 \quad (2.35b) \]

where:

\[ a^2 = \sigma_{x_{i-2}}^2 + \sigma_{z_{i-1}}^2 - 2 \sigma_{x_{i-2}} \sigma_{z_{i-1}} \rho_{x_{i-2}z_{i-1}} \]
and,

\[ \alpha = \frac{(\mu_{X_{i-2}} - \mu_{Z_{i-1}})}{a} \]

For each \( i \), with \( j = i, \ldots, n \), the correlation is:

\[ \rho_{X_{i-1}, Z_j} = \frac{\sigma_{X_{i-2}, Z_{j-1}} \phi(\alpha)}{(\sigma_{X_{i-1}}^2 - \mu_{X_{i-1}}^2)^{1/2}} \]

(2.36)

For \( n = 2 \), the result is exact.

2.5.3 Bounds on Collapse Probability

General Bounds --- In practice, the joint probability density function, \( f_{Z_1, \ldots, Z_n}(z_1, \ldots, z_n) \) is generally not known. Furthermore, even if it were known, the computational effort required in the evaluation of the \( n \)-fold integral of Eq. 2.31 is often prohibitive. If all the mechanisms are perfectly correlated, Eq. 2.30 can be simplified as:

\[ P_f = \max_i P(Z_i < 0) = \max_i P_{f_i} \]

(2.37)

whereas, if all the mechanisms are assumed to be statistically independent, Eq. 2.30 becomes:

\[ P_f^* = 1 - \prod_{i=1}^{n} [1 - P(Z_i < 0)] = 1 - \prod_{i=1}^{n} [1 - P_{f_i}] \]

(2.38)

For small \( P(Z_i < 0) \), Eq. 2.38 can be approximated also by:

\[ P_f^* = \sum_{i=1}^{n} P(Z_i < 0) = \sum_{i=1}^{n} P_{f_i} \]

(2.39)
As pointed out, e.g. by Ang and Amin (1967), Eqs. 2.37 and 2.39 represent respectively the lower and upper bounds of the true collapse probability of a ductile system.

For a general determinate system subjected to an external load \( S \), the forces induced in the members are always perfectly correlated. If the member resistances are assumed to be statistically independent, the resulting collapse probability \( P_f^{nier} \) would be:

\[
P_f^{nier} = 1 - \int_0^\infty \sum_{i=1}^{m} \left[ 1 - F_{F_i i}(s) \right] f_S(s) ds
\]

\[
- \int_0^\infty \sum_{i=1}^{m} F_{F_i i}(s) f_S(s) ds
\]

(2.40)

Ang and Amin (1967) further proved that:

\[
P_f^{'} \leq P_f^* \leq P_f^{nier} \leq P_f^{\ast}
\]

(2.41)

Vanmarcke's Bounds --- Moses and Kinser (1967) have shown that Eq. 2.30 can be expressed as follows:

\[
P_f = P_{f_1} + \sum_{i=2}^{n} a_i P_{f_i}
\]

(2.42)

where \( a_i \) is the conditional probability that the first \( i-1 \) modes survive given that mode \( i \) has failed i.e.

\[
a_i = P[Z_1 > 0 \cap Z_2 > 0 \cap \cdots \cap Z_{i-1} > 0 | Z_i < 0]
\]

(2.43)

Vanmarcke (1971) suggested a set of upper bounds on \( a_i \) when all but one of the \( i-1 \) first modal survival events are eliminated in Eq. 2.43; i.e.
\[ a_i \leq \min_{k=1}^{i-1} P[Z_k > 0 | Z_i < 0] = a_i^* \] (2.44)

The upper bound on the system collapse probability suggested by Vanmarcke is then:

\[ P_f \leq \sum_{i=1}^{n} a_i^* P_{f_i} \] (2.45)

with \( a_i^* = 1 \).

The approximation of \( P(Z_k > 0 | Z_i < 0) \) may be expressed in terms of the correlation coefficient \( \rho_{ik} \) between the performance functions \( Z_i \) and \( Z_k \) as follows:

\[ P[Z_k > 0 | Z_i < 0] = 1 - \frac{g(\max(\beta_k', |\rho_{ik}|, \beta_i))}{g(\beta_i)} \] (2.46)

where \( \beta_i \) is the reliability index for mechanism \( i \) defined in Eq. 2.27 and \( P_{f_i} = g(\beta_i) \) (2.47)

Vanmarcke (1971) also used Eq. 2.37 as the lower bound.

Ditlevsen's Bounds --- Ditlevsen (1979) derived bounds for the structural collapse probability through "indicator function" algebra. However, the relevant bounds can also be derived using conventional set theory as follows:

Let: \( E = \) event that the system collapse; \( E_i = \) event that the system collapse through mode \( i \); \( \overline{E}, \overline{E}_i = \) complements of \( E \) and \( E_i \), respectively.

Assume that there are \( m \) possible failure modes.
Then, the event $E$ can be written as

$$E = E_1 \cup E_2 \cup \cdots \cup E_m$$

$$= E_1 \cup \overline{E}_1 E_2 \cup \overline{E}_1 \overline{E}_2 E_3 \cup \cdots \cup (\overline{E}_1 \overline{E}_2 \cdots \overline{E}_{m-1}) E_m \quad (2.48)$$

The events in Eq. 2.48 are mutually exclusive; therefore,

$$P(E) = P(E_1) + P(\overline{E}_1 E_2) + P(\overline{E}_1 \overline{E}_2 E_3) + \cdots$$

$$+ P(\overline{E}_1 \cdots \overline{E}_{m-1} E_m) \quad (2.49)$$

For any two events $A$ and $B$,

$$P(\overline{A}B) = P(B) - P(AB) \quad (2.50)$$

It follows that for any $i = 2, 3, \ldots, m-1$

$$P(\overline{E}_1 \cdots \overline{E}_{i-1} E_i) = P(E_i) - P(\overline{E}_1 \cdots \overline{E}_{i-1} E_i)$$

$$= P(E_i) - P[(E_1 \cup E_2 \cup \cdots \cup E_{i-1}) E_i]$$

$$= P(E_i) - P[(E_1 E_i) \cup (E_2 E_i) \cup \cdots (E_{i-1} E_i)] \quad (2.51)$$

Since the probabilities are nonnegative,
\[ P(\bar{E}_1 \cdots \bar{E}_{i-1} E_i) \geq \max\{[P(E_i) - \sum_{j=1}^{i-1} P(E_i E_j)], 0\} \]  

(2.52)

Using Eq. 2.52 in Eq. 2.49 yields:

\[ P_f = P(E) \geq P(E_1) + \sum_{i=2}^{m} \max\{[P(E_i) - \sum_{j=1}^{i-1} P(E_i E_j)], 0\} \]  

(2.53)

On the other hand, for \( j=1,2,\ldots,i-1 \), since

\[ P(\bar{E}_1 \bar{E}_2 \cdots \bar{E}_{i-1} E_i) \leq P(\bar{E}_j E_i) = P(E_i) - P(E_j E_i) \]  

(2.54)

\[ P(\bar{E}_1 \bar{E}_2 \cdots \bar{E}_{i-1} E_i) \leq P(E_i) - \max_{j<i} P(E_i E_j) \]  

(2.55)

Therefore, with Eq. 2.55, Eq. 2.49 also yields

\[ P_f = P(E) \leq P(E_1) + \sum_{i=2}^{m} \left[ P(E_i) - \max_{j<i} P(E_i E_j) \right] \]  

(2.56)

or

\[ P_f = P(E) \leq \sum_{i=1}^{m} P(E_i) - \sum_{i=2}^{m} \sum_{j<i} P(E_i E_j) \]  

(2.57)

Eqs. 2.53 and 2.57 are, respectively, the lower and upper bounds of \( P_f \). As pointed out by Ditlevsen (1979), the arrangement of the order of the failure modes will influence the right hand sides of Eq. 2.53 and Eq. 2.57.

Observe that to apply Eqs. 2.53 and 2.57 requires the evaluation of the probability of the joint event \( E_i E_j \) (i.e. \( Z_i < 0 \cap Z_j < 0 \)). In the case of the standardized \( m \)-dimensional normal distribution, the events
corresponding to any two half spaces with hyperplane boundaries orthogonal to each other are mutually independent. From the geometric interpretation of the failure mode reliability indices $\beta_i$ and $\beta_j$, and for $\rho_{ij} \geq 0$,

$$q_i + q_j \geq P(E_i \cap E_j) = P(Z_i < 0 \cap Z_j < 0) \geq \max(q_i, q_j)$$ \hspace{1cm} (2.58)

where:

$$q_i = \phi(-\beta_i) \phi\left(-\frac{\beta_i - \rho_{ij} \beta_i}{\sqrt{1 - \rho_{ij}^2}}\right)$$

$$q_j = \phi(-\beta_j) \phi\left(-\frac{\beta_j - \rho_{ij} \beta_j}{\sqrt{1 - \rho_{ij}^2}}\right)$$ \hspace{1cm} (2.59)

and $\phi$ is the standardized normal distribution function.

If $\rho_{ij} < 0$, then

$$q_i + q_j \geq P(E_i \cap E_j) \leq \min(q_i, q_j)$$ \hspace{1cm} (2.60)

Ditlevsen's bounds on $P_f$ are narrow when the system collapse probability is small (e.g. $P_f < 10^{-4}$).

**Augusti & Baratta's Bounds** — So far, the bounds of the system collapse probability discussed above are based on the knowledge of all the possible failure mechanisms. In reality, this may sometimes be difficult to obtain. For plastic frames under proportional loads $\lambda(S)$, where the load factor $\lambda$ is a random variable, Augusti and Baratta (1972) suggested a probabilistic equivalent of static and kinematic methods of limits analysis to determine the bounds on the collapse probability. For a given value of $\lambda$, a statically admissible bending moment diagram is found if the absolute
value of the moments $M_j$ in equilibrium with the given loads are nowhere exceeding the yield capacities $M_j$. Then, the static theorem of limit analysis ensures that the structure does not collapse. This implies

$$P_\psi(\lambda) \geq P_f(\lambda)$$

(2.61)

where,

$$P_\psi(\lambda) = 1 - \sum_{j=1}^{N} \left[ P(|\lambda M_j| < M_j) \right]$$

(2.62)

and $N$ is the total number of critical sections to be considered.

From the kinematic theorem of plasticity theory, a collapse mechanism will occur if $Z_i$ of Eq. 2.26 is less than or equal to zero. Since the plastic frame is restricted to proportional loading $\lambda(S)$, the external virtual work may be expressed in terms of a single variable $\lambda$, i.e.

$$- \sum_{k} b_{ik} S_k = C_0^\lambda$$

(2.63)

Substitute Eq. 2.63 into Eq. 2.26, the frame will collapse in mode $i$ if:

$$\lambda \geq C_0 + \sum_{j=1}^{N} C_j M_j$$

(2.64)

where

$$C_j = \frac{a_{ij}}{C_0}$$

In general, the computation of the lower bound curve $P_\psi(\lambda)$ involves the $n$-fold integration of the joint PDF of the $N$ variables $M_j$, i.e.

$$P_\psi(\lambda) = \int \cdots \int f_{M_1, \ldots, M_N}(m_1, \ldots, m_N) \, dm_1 \cdots dm_N$$

$$\lambda \geq C_0 + \sum_{i=1}^{N} C_i M_i$$

(2.65)
For a given $\lambda$, the collapse probability is bounded by:

$$P_\psi(\lambda) \geq P_f(\lambda) \geq P_\gamma(\lambda)$$  \hspace{1cm} (2.66)
Fig. 2.1 A "Strictly" Parallel Redundant System

Fig. 2.2 A Multi-member Truss
3.1 Introduction

For the class of structural systems that are built up of elements of elastic-plastic material such as mild steel, failure or collapse of a frame system may be assumed to be caused by the formation of plastic hinge mechanisms, whereas the collapse of a truss structure of similar material may be assumed to be caused by the compressive or tensile yielding of the elements. The order in which the plastic hinges are formed, or the order in which the elements yield, is immaterial.

A structural failure mode is a distinct combination of element failures that causes the structure to collapse (either partially or completely depending on the definition of collapse). In general, a structural system may collapse in different failure modes; and for a practical structure, the possible collapse modes may not be simple to identify. Nevertheless, if there are n potential collapse modes, $E_1, E_2, \ldots, E_n$, the occurrence of any one of them will constitute collapse of the system. Hence, if $E_i$ is the event that the system collapses through mode $i$, the probability of collapse of the system is:

$$P_f = P(E_1 \cup E_2 \cup \ldots \cup E_n)$$ (3.1)

For structural systems of plastic frameworks or trusses, the collapse of a system through mechanism $i$ is the event $(Z_i < 0)$, i.e. $E_i = (Z_i < 0)$, where $Z_i$ is defined by Eq. 2.26. Therefore, for such a system, Eq. 3.1 becomes Eq. 2.30. From Eq. 3.1, three essential steps may be observed in the analysis of collapse of structural systems.

1. Identification of all the significant failure modes;
2. Modeling and analysis of the individual failure modes;
3. Combining the individual failure mode probabilities to obtain the collapse probability of the system.
Previous works on the reliability analysis of structural systems using failure mode method (Stevenson & Moses 1970, Vanmarcke 1971, Gorman & Moses 1979, Ditlevsen 1979) implicitly assume that all the possible failure modes are known, and thus avoided the problem of identifying the significant failure modes. However, for structures of practical complexity, the number of potential failure modes under a given loading condition is too large to be enumerated, whereas the major failure modes may be difficult to identify. A method for the identification of the major modes of failure, therefore, is necessary. A method for this purpose is discussed in Sect. 3.4 through 3.6, and steps 2 and 3 will be discussed in Sect. 3.2 and 3.3, respectively.

3.2 Reliability Analysis of Individual Failure Mode

Following Stevenson and Moses (1971), the performance function $Z_i$ for a mechanism $i$ may be expressed by Eq. 2.26. For a frame having $p$ potential plastic hinges and $q$ external loads, this equation may be written in matrix form as:

$$Z_i = \{A_i\}^t \{M\}' + \{A_{II}\}_i \{S\}_{q \times 1}$$

where $\{A_i\}_I$ and $\{A_{II}\}_i$ are the column vectors of the resistance and load coefficients defining mechanism $i$; whereas $\{M\}$ and $\{S\}$ are the column vectors of the plastic moment capacities and applied loads, respectively. The components of $\{M'\}$ are always positive.

However, if the deformed shape of mechanism $i$ is also of interest, then an element $a_{ij}$ of vector $\{A_i\}_I$ may be interpreted as the virtual displacement at hinge $j$. A component of $\{A_{II}\}_i$ will be positive or negative depending on whether the rotation of hinge $j$ is counterclockwise or clockwise, whereas the sign of a component of $\{A_{II}\}_i$ will depend on whether the translation of joint $j$ is in the opposite or the same direction as that of the global coordinates directions. Also, the components of $\{M'\}$ may not always be positive. Each component of the vector $\{M\}$ will take on the
same sign as its corresponding component of \{A_i\}_i such that the internal virtual work done by each potential hinge is non-negative; whereas an element of \{S\} will have a positive or negative value according to whether the load is acting in the same or opposite direction of the global coordinates.

Eq. 3.2 is more general than Eq. 2.26; the vector \{M'\} includes all plastic resistances and loads regardless of whether the resistances or loads are active in mechanism i. If a particular load j is inactive in mechanism i, its corresponding \(a_{ij}\) in Eq. 3.2 will be zero.

Since the plastic moment capacities and the applied loads are assumed to be random variables with known distributions, the mean value of \(Z_i\) is;

\[
\mu_{Z_i} = \{A\}_i^t \{\mu_{M'}\} \tag{3.3}
\]

and the corresponding variance is;

\[
\sigma_{Z_i}^2 = \{A\}_i^t [V_{M'}] \{A\}_i \tag{3.4}
\]

where \([V_{M'}]\) is the covariance matrix of resistances and loads. If all the components of \{M'\} are mutually statistically independent, then \([V_{M'}]\) will be reduced to a diagonal matrix of variances.

If all the components of \{M'\} are normal variates, \(Z_i\) will also be normal. Moreover, if some of the components of \{M'\} are non-normal variates, the normal distribution may still be prescribed for \(Z_i\) by virtue of the Central Limit Theorem, as \(Z_i\) is the sum and differences of many random variables. In the worst case, when the number of loads and resistances active in mechanism i is small, the method of equivalent normal distribution suggested by Paloheimo (1974) and Rackwitz (1976) may be used.

Prescribing the normal distribution for \(Z_i\), the probability of collapse of the system through the ith mechanism is given by,

\[
P_{F_i} = P(Z_i < 0) = 1 - \phi\left(\frac{\mu_{Z_i}}{\sigma_{Z_i}}\right) = 1 - \phi(\beta_i) \tag{3.5}
\]
Similar results may be obtained for trussed structures.

3.3 Point Estimate of System Collapse Probability

3.3.1 The PNET Method

The performance functions $Z_i$, $i=1,2,...,m$, are invariably positively correlated, as these correlations are the results of the plastic moments or loads that are common to two or more collapse mechanisms of a system. In terms of Eq. 3.2, it can be shown that the correlation coefficient $\rho_{ij}$ between mechanisms $i$ and $j$ ($i \neq j$) is,

$$\rho_{ij} = \frac{(A)_i^t[V']M'[V']_j}{\sigma Z_i \sigma Z_j}$$

(3.6)

From Eqs. 3.1 and 2.26, it is seen that these correlations between the mechanisms are important in the evaluation of the system collapse probability.

The PNET (Probabilistic Network Evaluation Technique) method previously developed for the analysis of activity networks (Ang, et al, 1975) appears to be appropriate also for calculating the point estimate of the collapse probability of a structural system, taking into account the effects of the correlations between potential collapse modes. Applied to the collapse probability of redundant plastic frameworks or trusses, the PNET method is based on the premise that those plastic mechanisms that are highly correlated (e.g. with $\rho_{ij} > \rho_o$) may be assumed to be perfectly correlated; whereas those with low correlations (with $\rho_{ij} < \rho_o$) may be assumed to be statistically independent. $\rho_o$ is some specified "demarcating" correlation coefficient. On this basis, the possible plastic mechanisms can be divided into several groups, such that within each group the mechanisms are highly correlated with a single "representative" mechanism; the "representative" mechanism within each group is the mechanism having the highest probability of failure in the group. The "representative"
mechanisms between the different groups may be assumed to be statistically independent. Thus, the collapse probability of the system is approximated by,

\[ P_f = \sum_r P(Z_r < 0) \] (3.7)

where \( r \) stands for the "representative" mechanisms. Observe that only individual collapse mode probabilities are required.

Central to the PNET approach is the value of the demarcating correlation \( \rho_0 \), which defines the transition between high and low correlations. A value of \( \rho_0 = 0.5 \) has been shown to be appropriate for activity networks (Ang, et al., 1975); however, a value of \( \rho_0 = 0.7 \) or 0.8 appears to be more appropriate for the collapse probability analysis of plastic frameworks and trusses.

3.3.2 Illustrative Example

For the purpose of illustrating the PNET method, consider the example of a simple one-story one-bay frame as shown in Fig. 3.1. The structure is subjected to the loads \( S_1 \) and \( S_2 \). The load and resistance statistics are as follows:

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>C.O.V.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1 )</td>
<td>360 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>480 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>( S_1 )</td>
<td>100 K</td>
<td>0.10</td>
</tr>
<tr>
<td>( S_2 )</td>
<td>50 K</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Assume that the bars are prismatic and resistance capacities along a member are perfectly correlated; moreover, the resistance capacities of the members with the same labels (e.g., \( M_1 \)) are perfectly correlated; whereas variables of different labels (e.g., \( M_1, M_2, S_1, S_2 \)) are statistically independent normal variates. Under these assumptions, the potential locations of
plastic hinges are limited to the points of high applied moments, which are those marked in Fig. 3.2. The corresponding collapse mechanisms are those summarized in Table 3.1.

**TABLE 3.1 SUMMARY OF POSSIBLE MECHANISMS**

<table>
<thead>
<tr>
<th>Mechanism i</th>
<th>Hinges Involved In Mechanism</th>
<th>$Z_i$</th>
<th>$\beta_i = \frac{\mu_{Z_i}}{\sigma_{Z_i}}$</th>
<th>$\Phi(-\beta_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1, 2, 3, 6</td>
<td>$4M_1 + 2M_2 - 10S_1 - 15S_2$</td>
<td>1.82</td>
<td>0.0346</td>
</tr>
<tr>
<td>2</td>
<td>1, 2, 3, 4</td>
<td>$4M_1 - 15S_2$</td>
<td>2.21</td>
<td>0.0135</td>
</tr>
<tr>
<td>3</td>
<td>1, 2, 5, 6</td>
<td>$2M_1 + 4M_2 - 10S_1 - 15S_2$</td>
<td>2.26</td>
<td>0.0120</td>
</tr>
<tr>
<td>4</td>
<td>5, 6, 7</td>
<td>$4M_2 - 10S_1$</td>
<td>3.02</td>
<td>0.0013</td>
</tr>
<tr>
<td>5</td>
<td>1, 2, 5, 7</td>
<td>$2M_1 + 2M_2 - 15S_2$</td>
<td>3.23</td>
<td>0.0006</td>
</tr>
<tr>
<td>6</td>
<td>3, 4, 6</td>
<td>$2M_1 + 2M_2 - 10S_1$</td>
<td>3.30</td>
<td>0.0005</td>
</tr>
</tbody>
</table>

In Table 3.1, the mechanisms are arranged in the order of decreasing probabilities of collapse, i.e. in decreasing $P(Z_i < 0)$. The correlation coefficients between any pair of mechanisms are as follows:

<table>
<thead>
<tr>
<th>Modes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>0.87</td>
<td>0.89</td>
<td>0.47</td>
<td>0.82</td>
<td>0.73</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>0.60</td>
<td>0.0</td>
<td>0.82</td>
<td>0.36</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.0</td>
<td>0.77</td>
<td>0.91</td>
<td>0.78</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>symmetric</td>
<td>1.0</td>
<td>0.47</td>
<td>0.82</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.0</td>
<td>0.55</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.0</td>
</tr>
</tbody>
</table>

$P_i = P_6 + P_4 + P_5$
Using a demarcating correlation of $\rho_0 = 0.7$, all the mechanisms, except mechanism 4, are "represented" by mechanism 1. Therefore, there are only two "representative" mechanisms, namely mechanisms 1 and 4, in this example. Thus, in accordance with Eq. 3.7, the collapse probability of the frame is estimated as

$$P_f = P(Z_1 < 0) + P(Z_4 < 0)$$

$$= 0.0346 + 0.0013 = 0.036$$

Observe that if the mechanisms are arranged in order of decreasing $P(Z_i < 0)$, as in this example, it is not necessary to evaluate all the $\rho_{ij}$ in order to estimate $P_f$ (e.g. $\rho_{2,3}, \rho_{2,4}$ etc.). The correlation of those mechanisms that are already "represented" by an earlier mechanism need not be evaluated subsequently.

According to Eqs. 2.37 and 2.39, the correct probability for this example is bounded as follows:

$$0.0346 \leq P_f \leq 0.0625$$

whereas, the corresponding bounds of Eqs. 2.53 and 2.57 are:

$$0.0367 \leq P_f \leq 0.0447$$

The accuracy and validity of the PNET method may be evaluated through large-sample Monte Carlo calculations. For this example, a total of 20,000 samples were performed, among which 802 failures were observed giving a collapse probability of the frame of 0.04. The procedure involved in the Monte Carlo calculations is described in Sect. 4.5.
3.4 Identification of Major Failure Modes

The major failure modes are those collapse mechanisms whose contribution to the system collapse probability are significant. One of the main problems in the practical reliability analysis of general structural systems using failure mode analysis lies in the identification of the major failure modes. There are three general approaches that may be used for this purpose:

1. Exhaustive enumeration
2. Simulation
3. Heuristic search

3.4.1 Exhaustive Enumeration

The theory of plastic analysis may be used for an exhaustive enumeration of all possible mechanisms (including those that are not kinematically admissible) through the superposition of certain independent mechanisms (Hodge, 1959; Neal, 1979). Then, by comparing the reliability index $\beta_i$ of the failure modes, the mechanisms can be arranged in order of significance. Unfortunately, the total number of possible failure modes will increase rapidly with the complexity of a structure. Exhaustive enumeration of all possible failure modes will easily become prohibitive for most structures of practical configurations.

3.4.2 Simulation

Through simulations, random loads and random element resistances are generated according to their respective distributions, and a structure is then analyzed to determine the corresponding failure mode. The process is repeated until enough failure modes are discovered. Obviously, this approach requires large computational efforts; in addition, it is difficult to guarantee that all the significant modes have been found.
3.4.3 Heuristic Search

Heuristic search schemes, if designed properly, may be efficient for identifying the major failure modes. An ideal heuristic method should be able to identify failure modes in the order of decreasing failure probabilities, and with computational effort proportional (linearly) to the number of basic random variables, e.g., external loads and element resistance capacities. Based on the observation that major failure modes (especially those exhibiting different structural behavior at collapse) will correspond mathematically to local minima, the heuristic method proposed here is to identify the major failure modes through mathematical programming; the failure modes, however, may not necessarily be in decreasing order of failure probabilities.

Use of Optimization Techniques --- It can be seen from Eq. 3.2 that the mechanism having the smallest reliability index $\beta_1$ is the most significant collapse mechanism of the system. The reliability index of a system $\beta$ may be expressed in terms of a vector of variables $\{X\}$, in which the components may be the amount of rotations and translations of the potential hinges under the loads (Sect. 3.5.1 & 3.6.1) or the weights of certain independent mechanisms (Sect. 3.5.2 & 3.6.2). Then, the problem of identification of the most significant failure mode can be transformed into finding the values of the components of $\{X\}$ such that the resulting $\beta$ value is the minimum. The objective function for minimization is the general expression of $\beta(\{X\})$.

Of course, there may be more than one significant failure mode for a system; consequently, the minimization problem involves the determination of the local minima. Thus, in addition to the "absolute" minimum, identification of the local minima is also essential.

For a major mechanism $i$ with its corresponding point $X_i$ that is not a local minimum in the $n$-dimensional space, it is assumed that $X_i$ will be close to a local minimum $X_j$ which corresponds to the major mechanism $j$. The implication of the closeness of two points in the $n$-dimensional space is that the two mechanisms will involve many common loads and plastic hinges
and hence are highly correlated. Thus, failure to identify mechanism i will not affect the accuracy of the estimate of the system collapse probability $P_f$ because according to the PNET method, mechanism i, even if identified, will be "represented" by mechanism j. Therefore, mechanism i will not have significant contribution to $P_f$.

The formulation of the general safety index $\beta(\{X\})$ is described below.

3.5 Formulations For Frames

3.5.1 Formulation F1

Objective Function —— The reliability index $\beta(\{X\})$ of a structural system may be formulated in more than one way. One is to consider the rotations and the translations of the potential hinges under the loads as the set of variables $\{X\}$. With this formulation, the general expression of the performance function $Z$ for a mechanism may be written in matrix form as:

$$Z = \{X\}^t \{M\}' = \{-----\} \{-----\} \{\text{S} \}$$

where $\{X_I\}$ = vector of rotations of the potential hinges; a component $x_{II}$ of potential hinge i will be positive for counterclockwise rotation, and negative for clockwise rotation.

$\{X_{II}\}$ = vector of translations of the hinges; a component $x_{II}$ is positive or negative depending on whether the translation of hinge i is in the opposite or the same direction as the global coordinates.

$\{M\}'$ = as defined in Eq. 3.2 when the deformed shape of the mechanism is also required.

Observe that for a particular mechanism i (under a unit deformation), the corresponding vector $\{X\}$ is unique and a component $x_j$ has the physical
meaning of the virtual displacement of hinge j. Denote it as $\{X\}_j$; then it is clear that $\{X\}_j$ is identical to the vector $\{A\}_j$ in Eq. 3.2 provided that the components of $\{M\}$ are arranged in the same order.

From Eq. 3.8, the general expression for the reliability index becomes:

$$\beta(\{X\}) = \frac{\{X\}_j^t \{\mu_M\}}{\sqrt{\{X\}_j^t [V_M'] \{X\}_j}}$$

(3.9)

where $\{\mu_M\}$ and $[V_M']$ are the mean-value vector and the covariance matrix of $\{M\}$, respectively.

Since each of the components of $\{\mu_M\}$ takes on the same sign as its corresponding components of $\{X\}_j$ such that the internal virtual work done by the individual potential hinge is nonnegative, it is convenient to assign $\mu_{\{M\}}$ to be always positive. Under this assignment of the sign convention, the translation of a hinge under load $S_i$ is $x_{II_i}$, which is positive or negative depending on whether the translation is in the opposite or the same direction as the load $S_i$. $\beta$ can be written as:

$$\beta(\{X\}) = \frac{|\{X\}_I^t \{\mu_M\} + \{X\}_II^t \{\mu_S\}|}{\sqrt{\{X\}_I^t [V_M'] \{X\}_I}}$$

(3.10)

**Constraints** --- Not all the variables in $\{X\}$ are independent of each other. They must satisfy the conditions of kinematic admissibility, which may be divided into two parts:

1. The external virtual work done by the external loads must be positive, i.e.

$$-\{X\}_I^t \{S\} > 0$$

(3.11)
2. Continuity of the structure must be preserved, i.e. compatibility conditions must be maintained.

The inequality constraint of the external virtual work, Eq. 3.11 is usually automatically satisfied since negative virtual work \((\{X_{II}\}^t \{S\} > 0)\) will not give a minimum value in Eq. 3.10. Therefore, the inequality constraint can be released in the course of finding the minimum \(\beta\). The second condition of kinematic admissibility can be further divided into two parts as follows:

1. Relationship among the angles of rotations of potential hinges \(\{X_I\}\).

2. Relationship between the rotations and translations of the hinges under external loads. (i.e.\(\{X_I\} \& \{X_{II}\}\))

Given the geometric configuration of a structure, the second set of compatibility conditions can be easily written down by inspection. Generally, all the components of \(\{X_{II}\}\) can be expressed as linear functions of the components of \(\{X_I\}\). The first set of compatibility requirements may not be obvious; however, they can be generated systematically, e.g. by the method suggested by Fenves & Gonzalez-Carlo (1971). Appendix A describes the Fenves & Gonzalez-Carlo method in detail.

In general the number of independent variables for a plane frame structure is always equal to the number of potential plastic hinges minus the degree of redundancy of the structure.

**Illustrative Example** --- Consider the one-story one-bay rectangular frame discussed previously in Sect. 3.3.2 (see Fig. 3.1) with the same loads and resistance statistics. The coordinates of the nodes are shown in Fig. 3.3 and the components of the vector \(\{X\}\) are shown in Fig. 3.4.

From Eq. 3.9, the objective function \(\beta\) is given as:

\[
\beta = \frac{360(|x_1| + |x_2| + |x_7| + |x_8|) + 480(|x_3| + |x_4| + |x_5| + |x_6|) - 50x_9 - 10x_{10}}{[(|x_1| + |x_2| + |x_7| + |x_8|)^2 54^2 + (|x_3| + |x_4| + |x_5| + |x_6|)^2 72^2 + x_9 215^2 + x_{10} 10^2]^{1/2}}
\]
The relationship between the rotations and translations of the hinges can be obtained by inspection, which are:

\[ x_4 = x_5 \]

\[ 15x_1 + x_9 = 0 \]

\[ 10(x_1 + x_2 + x_3) + x_{10} = 0 \]

The compatibility equations governing the rotations of the potential hinges can be obtained from the following observations:

1. If the frame is going to sway, both columns must sway in an equal amount; i.e.
   \[ x_7 + x_8 = 0 \]

2. The algebraic sum of the rotations of the potential hinges in a closed loop must be zero; i.e.
   \[ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 0 \]

3. Because of the symmetry of the frame, the following compatibility equation must be satisfied:
   \[ x_7 + x_2 + x_3 + x_4 = 0 \]

However, the above equations may not be obvious. Therefore, a method for the systematic generation of the compatibility equations is necessary (Appendix A).

Observe that the compatibility constraints are all linear equations; thus these equations may be "substituted" into the objective function to reduce the number of variables (from 10 to 4 in this example) and the problem becomes essentially an unconstrained minimization problem.

In general, the number of independent variables for a plane frame structure is equal to the number of potential plastic hinges minus the degree of redundancy of the structure.
3.5.2 **Formulation F2**

In plastic-limit analysis, it is well known that every collapse mechanism can be regarded as a linear combination of certain independent mechanisms (Hodge, 1959; Neal, 1977). Following Hodge's terminology, these independent mechanisms will be called "elementary" mechanisms. The number of "elementary" mechanisms, $N_e$, is equal to the number of potential plastic hinges minus the number of degrees of redundancy of the system. Actually, the set of "elementary" mechanisms is not unique; any $N_e$ mutually independent mechanisms would constitute a set of "elementary" mechanisms. Each "elementary" mechanism satisfies the compatibility equations but not necessarily the positive virtual external work requirement. Even though the "elementary" mechanisms of simple systems can be easily identified by inspection, the "elementary" mechanisms of a complex system (such as gable frames) may have to be generated systematically, e.g. by the procedure proposed by Watwood (1979). The details of Watwood's method is described in Appendix B.

**Objective Function** --- Once a set of "elementary" mechanisms is generated, the performance functions of the "elementary" mechanisms may be written as:

$$\{Z^e\} = [A^e] \{M'\} \tag{3.12}$$

where

$[A^e] = \text{matrix of virtual displacements corresponding to the elementary mechanisms; an element } a^e_{ij} \text{ is the virtual displacement at hinge } j \text{ of mechanism } i.$

$\{M'\} = \text{vector of plastic resistances and applied loads.}$

If every "elementary" mechanism is a one-degree-of-freedom mechanism, then
it is convenient to define a normalized matrix \([A']\), such that its elements are;

\[
\hat{a}_{ij} = \frac{a_{ij}}{\sqrt{\sum_{j} a_{ij}^2}}
\]  

(3.13)

By treating the "weight" of each "elementary" mechanism as a component of \(\{X\}\), the reliability index of a system, \(\beta\), can be expressed as a function of the vector \(\{X\}\). Denoting,

\[
\{Y\} = [A']^t \{X\}
\]  

(3.14)

the system reliability index is,

\[
\beta(\{X\}) = \frac{\{Y\}^t \{W\}'}{[\{Y\}^t [V_M'] \{Y\}]^{1/2}}
\]  

(3.15)

Since each of the "elementary" mechanisms already satisfies the compatibility requirements, a collapse mechanism formed by the linear combination of such elementary mechanisms will also satisfy compatibility. Therefore, with this formulation, there is no constraint equations to be considered.

**Numerical Example ---** Consider the same example problem discussed earlier in Sect. 3.3.2 (see Fig. 3.1). Relabel the potential plastic hinges as shown in Fig. 3.5. Fig. 3.6 shows the "elementary" mechanisms generated by the method suggested by Watwood(1979). For this example, it can be seen that the first mechanism generated is a "panel" mechanism; the second one is a "beam" mechanism and the third and fourth are "joint" mechanisms; all are of one-degree-of-freedom. Strictly speaking, the "joint" mechanisms are not possible mechanisms, as there is no applied
couple at the joints. However, they are clearly mutually independent of each other, and are also independent of the other two mechanisms and, therefore are "elementary" mechanisms. For rectangular frames, Watwood's method always generates "joint" mechanisms, "beam" mechanisms and "panel" mechanisms as part of the set of "elementary" mechanisms.

For this example, Eq. 3.12 becomes;

\[
\begin{aligned}
\begin{bmatrix}
\mathbf{Z}_1 \\
\mathbf{Z}_2 \\
\mathbf{Z}_3 \\
\mathbf{Z}_4 \\
\end{bmatrix}
&= 
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{m}_1 \\
\mathbf{m}_2 \\
\mathbf{m}_3 \\
\mathbf{m}_4 \\
\end{bmatrix}
\end{aligned}
\]

and Eq. 3.14 gives

\[
\begin{aligned}
\begin{bmatrix}
\mathbf{y}_1 \\
\mathbf{y}_2 \\
\mathbf{y}_3 \\
\mathbf{y}_4 \\
\mathbf{y}_5 \\
\mathbf{y}_6 \\
\mathbf{y}_7 \\
\mathbf{y}_8 \\
\mathbf{y}_9 \\
\end{bmatrix}
&= 
\begin{bmatrix}
-\frac{1}{\sqrt{5}} & 0 & 0 & 0 \\
\frac{1}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & -\frac{1}{\sqrt{7}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & \frac{2}{\sqrt{7}} & 0 & 0 \\
0 & -\frac{1}{\sqrt{7}} & 0 & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{5}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{5}} & 0 & 0 & 0 \\
-\frac{1}{\sqrt{5}} & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{7}} & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_1 \\
\mathbf{x}_2 \\
\mathbf{x}_3 \\
\mathbf{x}_4 \\
\mathbf{x}_5 \\
\mathbf{x}_6 \\
\mathbf{x}_7 \\
\mathbf{x}_8 \\
\mathbf{x}_9 \\
\end{bmatrix}
\end{aligned}
\]
3.6 Formulations For Trusses

3.6.1 Formulation T1

The identification of the major failure modes through optimization can similarly be applied to a ductile truss system, in which the collapse is through the tensile and/or compressive yielding of its members. In this case, every member is a potential yielding element.

Objective Function --- Treating the amount of yielding for each bar as a variable \( x \) in \( \{X_I\} \) of Eq. 3.8, \( x \) being positive or negative depending on whether the bar yields in tension or compression, the reliability index for the system can also be described by Eq. 3.9. A variable \( x \) in \( \{X_{II}\} \) represents a translation of a joint under an applied load; in this case, \( x \) is positive or negative depending on whether the translation is in the opposite or the same direction as the applied load.

For the special case in which the compressive yield strength of each bar is identical to its tensile yield strength, Eq. 3.10 remains valid, and serves as the objective function in the minimization process.

Constraints --- Again, the kinematic admissibility conditions (consisting of the compatibility requirement and the external work done) must be satisfied. As in the case of frame structures, the requirement of positive external virtual work is usually automatically satisfied.

Although all the components of \( \{X_{II}\} \) can be expressed in terms of \( \{X_I\} \), it is not as easily done as in the case of frame structures. Furthermore, for a redundant truss, there are more bars than the degrees of freedom and hence the components of \( \{X_I\} \) are not mutually independent of each other. A systematic method to generate both sets of compatibility equations will be required. Although such a method cannot be described in a compact matrix form as in Eq. A.1, it can easily be programmed for computer calculations. The method is explained and illustrated as follows.

Illustrative Example --- Consider a truss as shown in Fig. 3.7 and
define the global coordinates, generalized coordinates \( \{D\} \), and local coordinates \( \{d\} \) as shown in Figs. 3.8 and 3.9. The local coordinates \( \{d\} \) are related to the displacements of the joints, in generalized coordinates, as follows:

\[
\begin{align*}
\begin{bmatrix}
  d_1 \\
  d_2 \\
  d_3 \\
  d_4 \\
\end{bmatrix} &= \begin{bmatrix}
  \cos \theta & \sin \theta & 0 & 0 \\
  -\sin \theta & \cos \theta & 0 & 0 \\
  0 & 0 & \cos \theta & \sin \theta \\
  0 & 0 & -\sin \theta & \cos \theta \\
\end{bmatrix} \begin{bmatrix}
  D_1 \\
  D_2 \\
  D_3 \\
  D_4 \\
\end{bmatrix} \\
\end{align*}
\]

Define the "destination" array \( DA \) for member \( i \) as a 4x1 column vector with the following components:

- \( DA_{11} \) = Label denoting the generalized coordinate in global X direction in the positive (or A) end of member \( i \);
- \( DA_{12} \) = Label denoting the generalized coordinate in global Y direction in the positive (or A) end of member \( i \);
- \( DA_{13} \) = Label denoting the generalized coordinate in global X direction in the negative (or B) end of member \( i \);
- \( DA_{14} \) = Label denoting the generalized coordinate in global Y direction in the negative (or B) end of member \( i \).

Thus, for this example (see Figs. 3.7 and 3.8), \( DA_i \), \( i=1, \ldots, 5 \) are as follows:

\[
\begin{align*}
DA_1 &= \begin{bmatrix} 1 \\ 2 \\ 7 \\ 8 \end{bmatrix} ; \\
DA_2 &= \begin{bmatrix} 3 \\ 4 \\ 7 \\ 8 \end{bmatrix} ; \\
DA_3 &= \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} ; \\
DA_4 &= \begin{bmatrix} 7 \\ 8 \\ 5 \\ 6 \end{bmatrix} ; \\
DA_5 &= \begin{bmatrix} 1 \\ 2 \\ 5 \\ 6 \end{bmatrix}
\end{align*}
\]

In general, the change in length of member \( i \) can be written as:

\[
x_i = (d_3 - d_1)_i = \cos \theta_i (D_{DA_{13}} - D_{DA_{11}}) + \sin \theta_i (D_{DA_{14}} - D_{DA_{12}})
\]

\[(3.17)\]
where \( x \) is positive if the member is under tensile yielding and negative if the member is under compressive yielding. For this example, \( \theta_1 = 0^\circ; \theta_2 = -45^\circ; \theta_3 = 0^\circ; \theta_4 = 90^\circ \) and \( \theta_5 = 45^\circ \). Hence,

\[
\begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5
\end{bmatrix} =
\begin{bmatrix}
 -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\
 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
 -1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0
\end{bmatrix}\begin{bmatrix}
 D_1 \\
 D_2 \\
 D_3 \\
 D_4 \\
 D_5 \\
 D_6 \\
 D_7 \\
 D_8
\end{bmatrix}
\]

Since \( D_1 = D_2 = D_3 = D_4 = 0 \) in this example, the above matrix equation reduces to:

\[
\begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5
\end{bmatrix} =
\begin{bmatrix}
 0 & 0 & 1 & 0 \\
 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \\
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & -1 \\
 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0
\end{bmatrix}\begin{bmatrix}
 D_5 \\
 D_6 \\
 D_7 \\
 D_8
\end{bmatrix}
\]

In general, for a plane truss having \( m \) members (bars) and \( n \) external degrees of freedom (2 degrees for a joint unless restrained),

\[
\{X\}_{mx1} = [A]_{mxn} \{D\}_{nx1}
\]

Since the amount of yielding in each bar is the independent variable, it is necessary to express \( \{D\} \) in terms of \( \{X\} \).
Suppose there is a matrix \( [B]_{mxm} \) such that \( [B] [A] = [-I \, O]_{mxn} \); then using this in Eq. 3.18 yields,

\[
[B] [A] \{D\} = [-I \, O]_{mxn} \{D\} = [B] \{X\}
\]

or

\[
\{D\}_{nx1} = [\text{Rows of}] \{X\}_{mx1} ; \quad [\text{Last m-n Rows of } B] \{X\}_{mx1} = \{0\}
\]

The matrix \( [B]_{mxm} \) can be obtained by performing elementary row operations on \( [A]_{mxn} \) such that \( [A] \) becomes \( [I \, O]_{mxn} \) and at the same time perform exactly the same set of elementary row operations on \( [I]_{mxm} \) (Watwood, 1979). Given the geometric configuration and the labelling of the joints and members, the generating procedure described above can be easily programmed for computer calculations.

For the present example, \( [B] \) is given as:

\[
[B] = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
1 & -\sqrt{2} & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & -\sqrt{2} & 0 & 0 & 0 \\
-1 & \sqrt{2} & -1 & -1 & \sqrt{2}
\end{bmatrix}
\]

Since the external loads are applied at \( D_7 \) and \( -D_8 \) only, Eq. 3.9 (assuming all the loads and resistances are mutually independent) yields

\[
\beta = \frac{\sum_i^5 x_i \mu_{m_i} - x_3 \mu_{s_i} + (x_1 - \sqrt{2} x_2 + x_4) \mu_{s_2}}{[\sum_i x_i^2 \sigma_{m_i}^2 + x_3^2 \sigma_{s_i}^2 + (x_1 - \sqrt{2} x_2 + x_4)^2 \sigma_{s_2}^2]^{1/2}}
\]

where \( \mu_{m_i} \) and \( \sigma_{m_i} \) are the mean and standard deviation, respectively, of
the yield strength of member \(i\); \(\mu_{s_i}\) and \(\sigma_{s_i}\) are the mean and standard deviation, respectively, for load \(s_i\).

The constraint equation in this example is

\[-x_1 + \sqrt{2}x_2 - x_3 - x_4 + \sqrt{2}x_5 = 0\]

In general, for an indeterminate truss of \(b\) bars and \(n\) degrees of freedom (\(b > n\)), there are \(b - n\) linear constraints which can be "substituted" into the objective function such that in effect the problem is an unconstrained nonlinear optimization problem in \(n\) variables.

3.6.2 Formulation T 2

The identification of the major failure modes through the linear combination of "elementary" mechanisms can also be applied to a truss system whose collapse is due to yielding of its members. Eqs. 3.12 through Eq. 3.15 are applicable except that an element \(a_{e}^{ij}\) of \([A^e]\) now represents the amount of yielding (positive if tensile; negative if compressive) in member \(j\) of mechanism \(i\). Again, with this formulation, the resulting problem is independent of the constraints.
Fig. 3.1 One-story One-bay Frame

Fig. 3.2 Potential Locations of Plastic Hinges

Fig. 3.3 Coordinates of the Nodes of One-story Frame
Fig. 3.4 Labelling of Variables of One-story Frame (Formulation 1F)

Fig. 3.5 Labelling of Variables of One-story Frame (Formulation 2F)

Fig. 3.6 "Elementary" Mechanisms of One-story Frame
Fig. 3.7 A Truss Structure

Fig. 3.8 Definition of Generalized and Global Coordinates

Fig. 3.9 Definition of Local Coordinates
CHAPTER 4

CALCULATIONAL METHODS

4.1 Remarks on Optimization Technique

Mathematical programming is a branch of applied mathematics concerned with solving the following problem: minimize (or maximize) the objective function \( f(x_1, \ldots, x_n) \) such that a set of constraints \( g_i(x_1, \ldots, x_n) \leq 0, i = 1, 2, \ldots, m \), are satisfied. It is a rapidly growing field of study and has a broad range of applications.

Mathematical programming problems vary according to the nature of the objective functions and constraint functions. If all the functions are linear, it is called a linear programming problem; otherwise, they may be referred to collectively as nonlinear programming problems. In general, there is no unique solution technique that is suitable for all nonlinear optimization problems.

Depending on the nature of the constraints, mathematical programming problems may also be divided into three broad classes:

1. unconstrained optimization problems;
2. problems with equality constraints;
3. problems with inequality constraints (with or without equality constraints as well).

From Chapter 3, it may be seen that the problem of identification of the major failure modes may be formulated as a nonlinear optimization problem, of either type 1 or type 2; theoretically, a type 2 problem may be reduced to type 1. Therefore, methods appropriate for solving type 1 problems are emphasized.
4.2 Selection of Algorithm

4.2.1 Introduction

Solution techniques for unconstrained optimization problems are of two classes: direct and indirect methods. Direct methods start at an arbitrary point in the solution space and proceed, stepwise, towards the optimum by successive improvements; whereas indirect methods involve the solution of a set of simultaneous equations to extract the optimum.

From Chapter 3, it may be observed that the objective function in the present problem contains discontinuities in the gradients, since absolute signs appear in the objective function. The presence of the absolute signs is because the internal virtual work at a hinge (in the case of frames) or in a member (in the case of trusses) is always nonnegative; whereas + and − signs of x must be maintained to indicate the proper direction of the displacement at the hinge (or member).

For those solution points corresponding to the major mechanisms, discontinuities in the gradients in some directions (though not necessarily all directions) are usually present (e.g. see Fig. 4.1). Thus, the necessary condition that all the first derivatives must be zero for a local extremum would not apply. Approximate gradients obtained through forward or backward finite differences are always positive or negative; whereas the approximate gradients obtained through central differences are the average values of those obtained from the forward and backward finite differences and can be positive in some directions and negative in others. Therefore, to test a point suspected to be a local minimum, the functional values of points in the neighborhood of the pertinent point are evaluated and compared. This is not a rigorous way to test local optimality; however for the present purpose, a point in the neighborhood of the local minimum is sufficient. The precise point of local optimality is not important as the objective is the determination of the major mechanisms. Example results show that the solution point obtained in this manner usually correspond to the major mechanisms.
4.2.2 Examination of Available Algorithms

Because of the nature of the objective function, a solution technique using direct search involving the evaluations of the objective function only appears to be appropriate.

Multidimensional search strategy, like chess strategy, generally have three phases: opening game, middle game and ending game. The opening moves set the stage; the middle one push for advantages and the final phase strive to reach the goal. Various search procedures differ from one another mostly in the middle phase. The strategic approach of changing tactics as the search progresses seems to work well in practice (Beightler et al, 1979). For problems of relatively low dimensionality (e.g. n < 20), the Hooke and Jeeves (1961) pattern search algorithm is chosen as the solution technique for this study because it can be easily programmed and its performance for the example problems (Chapter 5) is better than the other algorithms examined. Other multidimensional search methods, that do not require derivatives, such as Rosenbrock's rotating coordinate method (1960), Powell's method (1964), and the modified Hooke and Jeeves search procedure proposed by Pappas (1980), were also examined. Results show that Powell's method is unsatisfactory for this type of problems; whereas the performance of Rosenbrock's method and Pappas's method are comparable in efficiency to that of the Hooke and Jeeves algorithm.

For direct methods requiring first or second derivatives, the derivatives are estimated with the corresponding central finite differences. Among the algorithms tested include the "One-at-a-time Golden Section Method", "Steepest Descent Method", "Conjugate Gradient Method" (see Bazaraa et al, 1979), "Quasi-Newton Method" proposed by Davidon, Fletcher and Powell (1963), "Nelder & Mead's Flexible Polyhedron" (1964) and the "Generalized Reduced Gradient" method. Because of the peculiar nature of the objective function (presence of discontinuities in gradients and singularities of the Hessian matrices), the algorithms mentioned above do not perform well for one or more of the following reasons:

1. excessive computation time;
2. premature termination of the solution seeking process;
3. inability to determine multiple local minima regardless of starting points.

4.3 The Hooke and Jeeves Pattern Search

4.3.1 Description of Solution Technique

The direct search method of Hooke and Jeeves (1961) contains two types of moves; namely, exploratory and pattern moves. The procedure of going from a given point to the following point is called a move and is termed a "success" if the new value of the objective function is more favorable; otherwise it is a "failure". The aim of an exploratory move is to acquire knowledge concerning the behavior of the objective function and the information of whether the move is a success or failure is then used to determine a probable direction for a "pattern" move; the "pattern" move aims at moving quickly towards the optimal point. The point from which a "pattern" move is made is designated a base point. Procedures involved in an exploratory move and a "pattern" search can be summarized with the flow diagrams shown in Figs. 4.2 and 4.3. Computationally the Hooke and Jeeves algorithm may be described as follows:

1. Choose a starting point \( \{x\} \) as the initial base point \( b_1 \) and incremental step size \( \{\Delta x\} \) whose \( i \)th component is \( \Delta x_i \) (\( i =1,2,\ldots,n \)).

2. The objective function value at the initial base point \( b_1 \), i.e. \( f(b_1) \), and the value \( f(b_1 + \Delta x_1) \) is compared. If the new point is better than the base point, i.e. \( f(b_1 + \Delta x_1) < f(b_1) \) for minimization problem, the point \( b_1 + \Delta x_1 \) is called the temporary head \( t_{11} \), where the first subscript shows that the first "pattern" is being developed and the second subscript indicates that the first variable \( x \) has been perturbed. If the point \( b_1 + \Delta x_1 \) is not better than \( b_1 \), discard \( b_1 + \Delta x_1 \) and try \( b_1 - \Delta x_1 \). If the new point
is better than $b_1$, it is used as the temporary $t_{11}$; otherwise $b_1$ is designated the temporary head $t_{11}$. Perturbation of $x_2$, the next variable, is then performed in a similar manner; this time about the temporary head $t_{11}$, instead of the original base point $b_1$. In general, the $j$th temporary head $t_{1j}$ is obtained from the preceding one $t_{1,j-1}$ as follows:

$$
t_{1,j} = \begin{cases} 
    t_{1,j-1} + \Delta x_j & \text{if } f(t_{1,j-1} + \Delta x_j) < f(t_{1,j-1}) \\
    t_{1,j-1} - \Delta x_j & \text{if } f(t_{1,j-1} - \Delta x_j) < f(t_{1,j-1}) \leq f(t_{1,j-1} + \Delta x_j) \\
    t_{1,j-1} & \text{if } f(t_{1,j-1}) < \min[f(t_{1,j-1} + \Delta x_j), f(t_{1,j-1} - \Delta x_j)] 
\end{cases}
$$

(4.1)

This expression covers all $1 \leq j \leq n$ if the convention

$$
t_{10} = b_1
$$

(4.2)

is adopted. When all the variables have been perturbed, the last temporary head $t_{1n}$ is designated the second base point $b_2$, i.e. $t_{1n} = b_2$, and the exploratory move is completed.

3. The original base point $b_1$ and the newly determined base $b$ together establish the first "pattern" move ($b_2 - b_1$). The initial temporary head is given by:

$$
t_{20} = b_2 + \alpha(b_2 - b_1)
$$

(4.3)
where \( \alpha \) is the acceleration factor \( ( \geq 1.0 ) \) and is used to speed up convergence; \( \alpha \) is a preassigned value. The double subscript \( 20 \) shows that a second "pattern" is developing and the variables have not been perturbed. A local exploration about \( t_{20} \) is now carried out to establish the new temporary heads \( t_{21}, \ldots, t_{2n} \) similar to Eq. 4.1; the only difference being that the first subscript will be 2 instead of 1. When all the variables have been perturbed, the last temporary head \( t_{2n} \) is designated the base point \( b_3 \). If \( b_3 \) is better than \( b_2 \), as before, a new temporary head \( t_{30} \) is established similarly with Eq. 4.3. In general, if \( b \) is better than \( b_{n-1} \) and for \( n \) greater than than 1, \( t_{n0} \) is given by:

\[
t_{n0} = b_n + \alpha(b_n - b_{n-1})
\]  

(4.4)

4. Steps 2 and 3 are repeated until the new last temporary head \( t_{mn} \) is not better than the old base point \( b_m \); in this case, let \( b_{m+1} = b_m \) and establish a new "pattern" by making exploratory moves of shorter step sizes. The step size vector is now changed to \( \gamma(\Delta x) \), where \( 0 < \gamma < 1 \) and \( \gamma \) is called the reduction factor. Treating \( b_{m+1} \) as \( b_1 \), Steps 2, 3, and 4 are continued until one of the following criteria is satisfied.

1. The new base point fails to improve the objective function value of the old base point by a prescribed value \( \varepsilon \) (\( \varepsilon \) is called the convergence criterion) even after a double reduction in step size.

2. The step sizes are smaller than \( \gamma^m(\Delta x) \), where \( m \) is a prescribed integer.
4.3.2 Illustrative Example Using Hooke & Jeeves Pattern Search

Consider the following problem:

Minimize \( f(x) = (x_1 - 2)^4 + (x_1 - 2x_2)^2 \)

Choose initial step sizes \( \Delta x_1 = \Delta x_2 = 0.2 \), the acceleration factor \( \alpha = 1.0 \), the reduction factor \( \gamma = 0.5 \), the convergence criterion \( \varepsilon = 0.0001 \), and the initial starting point \((0.0, 3.0)\). Then, the base point is \( b_1 = (0.0, 3.0) \) and \( f(b_1) = 52.0 \).

Next, compute \( f(x + \Delta x_1) = f(0.2, 3.0) = 44.1376 \) which is better than \( f(b_1) \) and hence \( t_{11} = (0.2, 3.0) \). Now, perturb variable \( x_2 \) about \( t_{11} \); giving \( t_{11} + \Delta x_2 = (0.2, 3.2) \). Since \( f(0.2, 3.2) = 48.9376 \) which is greater than \( 44.1376 \), the move is a "failure". Next try the point \( t_{11} - \Delta x_2 = (0.2, 2.8) \) which gives \( f(0.2, 2.8) = 39.6576 \) which is smaller than \( 44.1376 \), and thus is a "success". At this point, all the variables have been perturbed and the first exploratory moves are completed. From Eq. 4.3,

\[
t_{20} = (0.2, 2.8) + 1.0x((0.2, 2.8) - (0.0, 3.0)) = (0.4, 2.6)
\]

The iteration procedure is repeated (after five iterations) when \( f(t_{52}) > f(b_5) = f(2.0, 1.0) = 0 \), at which the pattern move is abandoned. Now let \( b_6 = b_5 \) and reduce step size by \( \gamma = 0.5 \) with \( t_{60} = b_6 = (2.0, 1.0) \). Since \( f(b_6) = f(t_{62}) \), at iteration 7, the step size is again reduced by half (i.e. \( \Delta x_1 = \Delta x_2 = 0.05 \)). Again, no improvement is observed; at this stage, since the step size has been reduced twice without improvement, the algorithm is terminated. The solution is then found to be \((2.0, 1.0)\) with \( f(2.0, 1.0) = 0.0 \). Table 4.1 summarizes the computations and Fig. 4.4 shows the path taken by the algorithm starting from \((0.0, 3.0)\). The points generated are numbered sequentially and the acceleration step that is rejected is shown by dotted lines in Fig. 4.4.
<table>
<thead>
<tr>
<th>Iteration $k$</th>
<th>Step Size $\Delta x_1 = \Delta x_2$</th>
<th>Base Point $b_k$, $f(b_k)$</th>
<th>$t_{k,j-1}$</th>
<th>$t_{k,j-1} + \Delta x_j$</th>
<th>$t_{k,j-1} - \Delta x_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2 (0.0, 3.0) 52</td>
<td>1 (0.0, 3.0)</td>
<td>(0.2, 3.0) 44.1376</td>
<td>(0.2, 3.0) 39.6576</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 44.1376</td>
<td>48.9376</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.2 (0.2, 2.8) 39.6576</td>
<td>1 (0.4, 2.6) 29.5936</td>
<td>(0.6, 2.6) 25.0016</td>
<td>(0.6, 2.4) 21.4816</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 25.0016</td>
<td>28.8416</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.2 (0.6, 2.4) 21.4816</td>
<td>1 (1.2, 2.0) 10.0</td>
<td>(1.2, 2.0) 8.2496</td>
<td>(1.2, 1.8) 6.1696</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 8.2496</td>
<td>10.6496</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.2 (1.2, 1.8) 6.1696</td>
<td>1 (1.8, 1.2) .3616</td>
<td>(2.0, 1.2) .16</td>
<td>(2.0, 1.0) 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 (2.0, 1.2) .16</td>
<td>(2.0, 1.4) .64</td>
<td>(2.0, 1.0) 0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.2 (2.0, 1.0) 0</td>
<td>1 (2.8, 0.2) 6.1696</td>
<td>(3, 0.2) 7.76</td>
<td>(2.6, 0.2) 4.9696</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 6.1696</td>
<td>3.3696</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.1 (2.0, 1.0) 0</td>
<td>1 (2.0, 1.0) 0</td>
<td>(2.1, 1.0) .011</td>
<td>(1.9, 1.0) .011</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 (2.0, 1.0) 0</td>
<td>(2.0, 1.1) .04</td>
<td>(2.0, 0.9) .04</td>
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</tr>
<tr>
<td>7</td>
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<td>(1.95, 1.0) .0025</td>
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<td></td>
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<td>(2.0, 1.05) .01</td>
<td>(2.0, 0.95) .01</td>
<td></td>
</tr>
</tbody>
</table>
4.4 Computer Programs

A computer subroutine was written in FORTRAN to implement the Hooke and Jeeves pattern search algorithm described in Sect. 4.3.1. The required input data include the maximum allowable number of objective function evaluations, the initial starting point, the initial step sizes, the convergence criterion \( \varepsilon \), the acceleration factor \( \alpha \) and the reduction factor \( \gamma \). In theory, the choice of these input parameters (except the number of maximum allowable objective function evaluations) will affect the performance of the algorithm. Experience from this study indicates that the following appear to give satisfactory results: convergence criterion \( \varepsilon = 0.0001 \), reduction factor \( \gamma = 0.5 \), acceleration factor \( \alpha = 1.0 \), initial step size of each component = 0.2, the absolute value of each component of the initial starting point not more than 1.0, and the maximum allowable number of step size reduction \( m = 20 \). The algorithm is terminated when any one of the following conditions is met:

1. the maximum allowable objective function evaluation is reached;
2. the maximum allowable number of step size reduction \( m \) is reached;
3. the convergence criterion is met.

The computer program will print out all the components of the point at which the algorithm terminates, its corresponding objective function value, the final incremental step size, and the number of objective function evaluations performed.

There is more than one way to formulate the mathematical programming problem. In the formulation using the hinge rotations as variables (Sect. 3.5.1), a computer program is written such that given the data defining the geometry of the structure, the program generates the linear constraint equations (say, \( N_c \) of them) automatically. The user may then arbitrarily select \( N_c \) hinge rotations as the dependent variables and express them in terms of the independent variables. The objective function is then assembled in terms of the independent variables only and input as part of the Hooke and Jeeves pattern search subroutine. The user is allowed to try different starting points in one run of the program. The program applies also to truss structures using the yielding of the members as variables (Sect. 3.6.1).
For the formulation using the "weights" of elementary mechanisms as variables (Sect. 3.5.2 & 3.6.2), a different computer program has been developed that generates the set of "elementary" mechanisms using Watwood's method; the objective function is then generated automatically given the geometry of the structure and the statistics of the loads and resistances. The Hooke and Jeeves algorithm subroutine is then called to search for the optimal solution. In addition to the output printed by the Hooke and Jeeves subroutine, the computer program also prints the coefficient matrix \([A]\) of Eq. 3.2 and the matrix \([A^e]\) of Eq. 3.12. Again, different starting points are allowed in one run of the program.

4.5 Monte Carlo Calculations

Problems of structural system reliability may also be evaluated through large-sample Monte Carlo calculations. The Monte Carlo calculations require that all the potential failure modes be known and enumerated; consequently, the identification of the failure modes as described in Sect. 3.5 & 3.6 will still be necessary. For a particular problem, assume that all the possible mechanisms have been identified and listed; the steps involved in the Monte Carlo calculations may then be summarized as follows:

1. For each trial, a pseudo random number is automatically generated for each of the independent variables according to their respective prescribed distributions. Using these generated values of the loads and resistance capacities, the values of the performance functions for all the possible failure modes are computed. If any one of the performance functions thus calculated is nonpositive, the structure has collapse.

2. Step 1 is repeated \(n\) times, and the number of trials in which the structure collapsed is recorded.
3. The number of trials in which the structure collapsed divided by \( n \) is the estimated probability of collapse.

The number of trials, or sample size \( n \), must be sufficiently large to minimize the sampling error. Depending on the theoretical collapse probability, the number of trials \( n \) used in this study is greater than 100 times the average number of trials required for a collapse to occur; i.e. \( n > 100/\text{estimated } p_f \).

Monte Carlo calculations were performed for all the example problems presented in Chapter 5.

\[
\begin{align*}
  n &> \frac{100}{1 - P_{\text{ref}}} \\
  n &> \frac{100}{0.01} \\
  n &> 10,000 \quad \text{OK}
\end{align*}
\]
EXPLORATORY MOVE

ENTER

INCREASE COORDINATE

(Is Move a Success?) Yes → Retain New

No → DECREASE COORDINATE

(Is Move a Success?) Yes → New Functional

No → RESET COORDINATE

EXIT

Routine carried out for each coordinates separately.

Fig. 4.2 Descriptive Flow Diagram for Exploratory Move
START: Evaluate function at initial base point

1. Start at base point → Make exploratory move → Is present functional value below that at base point? 
   - Yes → \( \text{Fig. 4.3 Descriptive Flow Diagram for Pattern Search} \)
   - No → 

2. Set new base point → Make pattern move → Make exploratory move → Is present functional value below that at base point? 
   - Yes → \( \text{Fig. 4.3 Descriptive Flow Diagram for Pattern Search} \)
   - No → 

3. Is step size small enough? 
   - No → Decrease step size 
   - Yes → 

STOP

Fig. 4.3 Descriptive Flow Diagram for Pattern Search

The Numbers Denote the Order in Which Points Are Generated

Fig. 4.4 Illustration of the method of Hooke & Jeeves Pattern Search (starting point = (0.0, 3.0))
CHAPTER 5

ILLUSTRATIVE APPLICATIONS

This chapter presents a number of redundant plastic frames and trusses; the probability of collapse for each structure is calculated using the method described in Chapter 3. For each example problem, the major collapse modes were determined using the two formulations. For formulations 1T, 2F and 2T (Sect. 3.5.2 & 3.6), the initial starting points were \{(+1,0,...,0), (0,+1,0,...,0),..., (0,...,0,+1)\}; whereas in formulation F1 (Sect. 3.5.1), the initial points for frames included points corresponding to all possible "beam" mechanisms as well.

In order to examine the accuracy of the results, Monte Carlo simulations were performed for all the examples. In general, the computer time roughly increases linearly with the sample size and a typical sample of 100,000 trials requires approximately 40 seconds of execution time on Cyber 175. The simple bounds (e.g. Ang & Amin, 1967) and the narrow bounds (Ditlevsen, 1979) on the system collapse probabilities were also calculated.

In all the example problems discussed in this chapter, unless otherwise stated, the variables having the same label (e.g. \(M_1\)) and all the sections along a member are assumed to be perfectly correlated; whereas variables having different labels (e.g. \(M_1\) and \(M_2\)) are assumed to be statistically independent. Furthermore, all variables are assumed to be normal variates. The methods discussed herein, however, are equally applicable for non-normal variates; in particular, equivalent normal distributions may be used (Rackwitz, 1976).

5.1 Frame Structures

For a frame having \(N_e\) "elementary" mechanisms, a collapse mechanism can be considered as the linear combination of the \(N_e\) elementary mechanisms. The number of possible mechanisms may be determined as follows.

For each elementary mechanism \(i, i=1,...,N_e\), two cases are involved in the linear combination, namely, (1) its weight \(x_i\) is zero, or (2) its
weight \( x_i \) is non-zero. If the collapse mechanism is of one-degree-of-freedom, the ratio of non-zero \( x_i \), \( i=1, \ldots, N_e \) will define the mechanism. Given that exactly \( r \) non-zero weights are involved, there could be as many as \( \binom{N_e}{r} \) possible mechanisms. Thus, by considering all the possible cases of \( r \), i.e. \( r = 1, 2, \ldots N_e \), the total number of one-degree-of-freedom failure modes can be shown to be \( 2^{N_e} - 1 \).

5.1.1 Problem 1

The first example considered is a simple one-story one-bay portal frame under concentrated vertical and lateral loads as shown in Fig. 3.1. The same problem was used earlier in Sect. 3.3.2 to illustrate the PNET method. The statistics of the loads and component resistances are the same as those stated in Sect. 3.3.2, namely,

\[
\begin{align*}
\text{Mean} \\
M_1 & \quad 360 \text{ ft-K} \\
M_2 & \quad 480 \text{ ft-K} \\
S_1 & \quad 100 \text{ K} \\
S_2 & \quad 50 \text{ K}
\end{align*}
\]

For the present example, there are 7 critical sections as shown in Fig. 3.2; the structure has three degrees of redundancy. Therefore, there are \( 7-3 = 4 \) elementary mechanisms (as shown in Fig. 3.6). Combinatorially, there may be a maximum of \( 2^4 - 1 = 15 \) mechanisms. However, some of these 15 "possible" modes violate the kinematic admissibility constraints; i.e. the external virtual work done must be positive (e.g. "joint mechanisms of Fig. 3.6"); whereas some of the other modes contradict the assumption that the two column capacities are perfectly correlated. An example of this is shown in Fig. 5.1a where equilibrium of the left hand corner implies \( M_2 \) is greater than \( M_1 \); whereas the opposite is implied when equilibrium of the right hand
corner is considered. Such failure modes, therefore, should also be deleted from the list of physically admissible mechanisms. Also, some of the "apparent" failure modes may be impossible to occur because the internal virtual works are identical to some other mechanisms, but the external works are less (e.g. compare Fig. 5.1b and c). The list of physically admissible mechanisms should be limited to one-degree-of-freedom mechanisms. Two (or more)-degree-of-freedom mechanisms, such as the one shown in Fig. 5.1d, are not considered because in practice it is highly unlikely that two mechanisms can occur simultaneously. Deleting all such physically inadmissible mechanisms, six mechanisms remain for this problem, as listed in Table 3.1. Fig. 3.6 shows the elementary mechanisms generated by Watwood's method. The results for this example are summarized in Table 5.1, which shows the $\beta$ values of all 6 physically admissible mechanisms arranged in the order of decreasing $P_{\tilde{f}}$. For those mechanisms that are identified by the Hooke and Jeeves algorithm, they are labelled "I" in the table; otherwise they are labelled "N". Based on Formulation F1 (Sect. 3.5.1), 4 mechanisms are identified, whereas only 3 mechanisms are identified in Formulation F2 (Sect. 3.6.1). Table 5.1 also contains the calculated $P_{\tilde{f}}$ base on the PNET method using 3 different values of $\rho_o$, i.e. 0.6, 0.7 and 0.8. For each $\rho_o$ value, the cases of (1) using all 6 mechanisms; (2) using only those mechanisms identified in Formulation F1; (3) using only those mechanisms identified in Formulation F2, are considered. The results of Monte Carlo simulation as well as the simple and narrow bounds on $P_{\tilde{f}}$ are also listed in the table.

Although example 1 has three dominant mechanisms, the second and third modes are highly correlated with the first (and most significant) mechanism. Therefore, the structure has essentially one dominant mechanism and, as expected, the system collapse probability is very close to its lower bound probability.

5.1.2 Problem 2

The second problem is a two-story one-bay rectangular frame subjected to concentrated loads, as shown in Fig. 5.2. The vertical loads represent
TABLE 5.1 SUMMARY OF RESULTS FOR PROBLEM 1

Number of physically admissible mechanisms = 6

<table>
<thead>
<tr>
<th>Mechanism i (ordered)</th>
<th>$\beta_i$</th>
<th>$P(Z_i &lt; 0)$</th>
<th>Based on Formulation F1</th>
<th>Based on Formulation F2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.82</td>
<td>$34.6 \times 10^{-3}$</td>
<td>I*</td>
<td>I</td>
</tr>
<tr>
<td>2</td>
<td>2.21</td>
<td>$13.5 \times 10^{-3}$</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>3</td>
<td>2.26</td>
<td>$12.0 \times 10^{-3}$</td>
<td>N*</td>
<td>N</td>
</tr>
<tr>
<td>4</td>
<td>3.02</td>
<td>$1.3 \times 10^{-3}$</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>5</td>
<td>3.23</td>
<td>$0.6 \times 10^{-3}$</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>6</td>
<td>3.30</td>
<td>$0.5 \times 10^{-3}$</td>
<td>I</td>
<td>N</td>
</tr>
</tbody>
</table>

PNET Calculations of System Collapse Probability, $P_f$

<table>
<thead>
<tr>
<th>$\rho_0$</th>
<th>Using All 6 Mechanisms</th>
<th>Using Mechanisms Identified in Formulation F1</th>
<th>Using Mechanisms Identified in Formulation F2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.036</td>
<td>0.036</td>
<td>0.036</td>
</tr>
<tr>
<td>0.7</td>
<td>0.036</td>
<td>0.036</td>
<td>0.036</td>
</tr>
<tr>
<td>0.8</td>
<td>0.036</td>
<td>0.036</td>
<td>0.036</td>
</tr>
</tbody>
</table>

$P_f$ by Monte Carlo Simulations

$P_f = 0.040$
Sample Size = 20,000

Bounds on $P_f$ (Based on All 6 Mechanisms)

Simple Bounds = 0.0346 - 0.0625
Narrow Bounds = 0.0367 - 0.0448

*I = Mechanisms identified; N = Mechanisms not identified
the equivalent uniformly distributed loads, such that the external virtual works are the same. The potential locations of the plastic hinges in the frame are those shown in Fig. 5.3. The statistics of the applied loads and the moment capacities of the members are as follows:

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>C.O.V.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>80 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>$M_2$</td>
<td>200 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>$F_1$</td>
<td>20 K</td>
<td>0.25</td>
</tr>
<tr>
<td>$F_2$</td>
<td>40 K</td>
<td>0.15</td>
</tr>
<tr>
<td>$P$</td>
<td>3.5 K</td>
<td>0.25</td>
</tr>
</tbody>
</table>

There are, therefore, 14 potential plastic hinges and 6 degrees of redundancy; thus, there are $(14-6) = 8$ elementary mechanisms. Combinatorially, there could be as many as $2^8 - 1 = 255$ possible failure mechanisms. By exhaustively inspecting all the 255 possible mechanisms, it is found that, for one reason or another, only 25 of them are admissible. The first 8 major mechanisms are those described below:

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Hinges Involved in Mechanism</th>
<th>Failure Mode Equation $Z_i$</th>
<th>$\beta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5,6,7</td>
<td>$4M_2 - F_2L_2/2$</td>
<td>2.98</td>
</tr>
<tr>
<td>2</td>
<td>1,2,4,6,8,9</td>
<td>$6M_1 + 2M_2 - 3pL_1 - F_2L_2/2$</td>
<td>3.06</td>
</tr>
<tr>
<td>3</td>
<td>1,2,4,6,7,8</td>
<td>$4M_1 + 3M_2 - 3pL_1 - F_2L_2/2$</td>
<td>3.22</td>
</tr>
<tr>
<td>4</td>
<td>3,4,6,8,9</td>
<td>$4M_1 + 2M_2 - F_2L_2/2$</td>
<td>3.28</td>
</tr>
<tr>
<td>5</td>
<td>1,2,3,4</td>
<td>$4M_1 - 3pL_1$</td>
<td>3.38</td>
</tr>
<tr>
<td>6</td>
<td>1,2,4,6,9,10,11</td>
<td>$8M_1 + 2M_2 - 4pL_1 - F_2L_2/2$</td>
<td>3.50</td>
</tr>
<tr>
<td>7</td>
<td>1,2,6,7,11,13</td>
<td>$4M_1 + 6M_2 - 4pL_1 - F_1L_1/2 - F_2L_2/2$</td>
<td>3.64</td>
</tr>
<tr>
<td>8</td>
<td>1,2,6,7,10,11</td>
<td>$4M_1 + 4M_2 - 4pL_1 - F_2L_2/2$</td>
<td>3.72</td>
</tr>
</tbody>
</table>
The correlation coefficients between any pair of the first 8 major mechanisms are as follows:

<table>
<thead>
<tr>
<th>Nodes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td></td>
<td>0.69</td>
<td>0.88</td>
<td>0.83</td>
<td>0.00</td>
<td>0.50</td>
<td>0.91</td>
</tr>
<tr>
<td>2</td>
<td>1.00</td>
<td>0.95</td>
<td>0.95</td>
<td>0.67</td>
<td>0.99</td>
<td>0.80</td>
<td>0.91</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.00</td>
<td>0.95</td>
<td>0.49</td>
<td>0.90</td>
<td>0.93</td>
<td>0.99</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.00</td>
<td></td>
<td>0.41</td>
<td>0.90</td>
<td>0.83</td>
<td>0.90</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.00</td>
<td></td>
<td></td>
<td>0.77</td>
<td>0.31</td>
<td>0.43</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td>Symmetric</td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td>0.75</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td>0.95</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
</tr>
</tbody>
</table>

The representative mechanisms for the respective $\rho_0$ are as follows:

- $\rho_0 = 0.6$: representative mechanisms = 1, 5, 20
- $\rho_0 = 0.7$ or $0.8$: representative mechanisms = 1, 2, 5, 17.

The results of analysis are summarized in Table 5.2, which shows that only the first, second and the fifth mechanisms are identified by both formulations, and the third mechanism is identified in Formulation F1 but not in Formulation F2. Nevertheless, based on the 3 or 4 identified mechanisms, the calculated $P_f$ values using the PNET method are the same as those obtained using all 25 physically admissible mechanisms for all 3 values of $\rho_0$ (0.6, 0.7 and 0.8) used. The $P_f$ obtained using $\rho_0 = 0.7$ or 0.8 is close to that obtained by Monte Carlo simulation, and lies within the simple and narrow bounds.
TABLE 5.2  SUMMARY OF RESULTS FOR PROBLEM 2

Number of physically admissible mechanisms = 25

<table>
<thead>
<tr>
<th>Mechanism i (ordered)</th>
<th>$\beta_i$</th>
<th>$P(Z_i &lt; 0)$</th>
<th>Based on Formulation F1</th>
<th>Based on Formulation F2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.98</td>
<td>$1.44 \times 10^{-3}$</td>
<td>$I^*$</td>
<td>I</td>
</tr>
<tr>
<td>2</td>
<td>3.06</td>
<td>$1.11 \times 10^{-3}$</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>3</td>
<td>3.22</td>
<td>$0.64 \times 10^{-3}$</td>
<td>I</td>
<td>N</td>
</tr>
<tr>
<td>4</td>
<td>3.28</td>
<td>$0.52 \times 10^{-3}$</td>
<td>$N^*$</td>
<td>N</td>
</tr>
<tr>
<td>5</td>
<td>3.38</td>
<td>$0.36 \times 10^{-3}$</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>6</td>
<td>3.50</td>
<td>$0.23 \times 10^{-3}$</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>7</td>
<td>3.64</td>
<td>$0.14 \times 10^{-3}$</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>8</td>
<td>3.72</td>
<td>$0.10 \times 10^{-3}$</td>
<td>N</td>
<td>N</td>
</tr>
</tbody>
</table>

PNET Calculations of System Collapse Probability, $P_f$

<table>
<thead>
<tr>
<th>$\rho_0$</th>
<th>Using All 25 Mechanisms</th>
<th>Using Mechanisms Identified in Formulation F1</th>
<th>Using Mechanisms Identified in Formulation F2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.0018</td>
<td>0.0018</td>
<td>0.0018</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0029</td>
<td>0.0029</td>
<td>0.0029</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0029</td>
<td>0.0029</td>
<td>0.0029</td>
</tr>
</tbody>
</table>

$P_f$ by Monte Carlo Simulations

$P_f = 0.0025$

Sample Size = 160,000

Bounds on $P_f$ (Based on All 25 Mechanisms)

Simple Bounds = 0.00144 - 0.00445

Narrow Bounds = 0.00252 - 0.00371

*I = Mechanisms identified;  N = Mechanisms not identified*
Problem 2A — This example differs from Problem 2 only by the assumption of correlations between the resistance capacities of the columns and the beams. As shown in Fig. 5.4, the resistance capacities of the columns for the same story are assumed to be perfectly correlated; whereas, the columns of different stories are statistically independent. The same distributions, however, are prescribed throughout; i.e. they are all normal. The same assumptions apply for the beams in the same floor and between different floors. The statistics of the applied loads and resistance capacities of the members are as follows:

<table>
<thead>
<tr>
<th>Mean</th>
<th>C.O.V.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>80 ft-K</td>
</tr>
<tr>
<td>$M_2$</td>
<td>200 ft-K</td>
</tr>
<tr>
<td>$M_3$</td>
<td>80 ft-K</td>
</tr>
<tr>
<td>$M_4$</td>
<td>200 ft-K</td>
</tr>
<tr>
<td>$F_1$</td>
<td>20 K</td>
</tr>
<tr>
<td>$F_2$</td>
<td>40 K</td>
</tr>
<tr>
<td>$P$</td>
<td>3.5 K</td>
</tr>
</tbody>
</table>

In this case, there are 44 physically admissible mechanisms; the first 6 major modes are summarized below:

<table>
<thead>
<tr>
<th>Mechanism $i$</th>
<th>Hinges Involved in Mechanism $i$</th>
<th>Failure Mode Equation $Z_i$</th>
<th>$\beta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5,6,7</td>
<td>$4M_2 - F_2L_2/2$</td>
<td>2.98</td>
</tr>
<tr>
<td>2</td>
<td>1,2,4,6,7,8</td>
<td>$3M_1 + 3M_2 + M_3 - 3pL_1 - F_2L_2/2$</td>
<td>3.31</td>
</tr>
<tr>
<td>3</td>
<td>1,2,4,6,8,9</td>
<td>$4M_1 + 2M_2 + 2M_3 - 3pL_1 - F_2L_2/2$</td>
<td>3.36</td>
</tr>
<tr>
<td>4</td>
<td>1,2,3,4</td>
<td>$4M_1 - 3pL_1$</td>
<td>3.38</td>
</tr>
<tr>
<td>5</td>
<td>3,4,6,8,9</td>
<td>$2M_1 + 2M_2 + 2M_3 - F_2L_2/2$</td>
<td>3.50</td>
</tr>
<tr>
<td>6</td>
<td>3,6,7,9,10,11</td>
<td>$M_1 + 3M_2 + 3M_3 - PL_1 - F_2L_2/2$</td>
<td>4.15</td>
</tr>
</tbody>
</table>
The correlation coefficients between any pair of the first 6 modes are as follows:

<table>
<thead>
<tr>
<th>Modes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>0.90</td>
<td>0.77</td>
<td>0.00</td>
<td>0.88</td>
<td>0.93</td>
</tr>
<tr>
<td>2</td>
<td>1.00</td>
<td>0.96</td>
<td>0.40</td>
<td>0.93</td>
<td>0.94</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.00</td>
<td>0.55</td>
<td>0.93</td>
<td>0.89</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.00</td>
<td>0.22</td>
<td>0.14</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Symmetric</td>
<td>1.00</td>
<td>0.97</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
</tr>
</tbody>
</table>

The representative mechanisms for $\rho_o = 0.6$ or 0.7 are 1 and 4; whereas for $\rho_o = 0.8$, the representative mechanisms are 1, 3 and 4. The results of analysis for this example are summarized in Table 5.3. From Table 5.3, it can be seen that the assumption of statistical independence among the columns of the two different has increased the number of physically admissible mechanisms from 25 to 44. Nevertheless, there are only 5 mechanisms that are significant. The sixth mechanism has a collapse probability which is less than 1% of the probability of the first mode. For a given $\rho_o$, the PNET results are identical regardless of whether it is using all 44 mechanisms or only those mechanisms that are identified in Formulation F1 or F2. The calculated $P_f$ using $\rho_o = 0.6$ or 0.7 is close to that obtained from the Monte Carlo simulation. Comparing the results of this problem with the previous one (Problem 2), it is observed that the relative order of significance of the individual mechanisms has been changed. For example, the mechanism involving hinges 1,2,4,6,8,9 is the second most important mechanism in problem 2 ($\beta = 3.06$), whereas it is ranked third in this example ($\beta = 3.36$). For the same mechanism $i$, the assumption of perfect (or high) correlations among loads and among resistances will lead to a higher $\sigma^2_i$ (and hence lower $\beta_i$), resulting in a higher $P_f$ than the assumption of statistical independence (or low correlations). On the other hand, the extreme case of perfect correlations among all the variables will
TABLE 5.3 SUMMARY OF RESULTS FOR PROBLEM 2A

Number of physically admissible mechanisms = 44

<table>
<thead>
<tr>
<th>Mechanism i (ordered)</th>
<th>$\beta_i$</th>
<th>$P(Z_i &lt; 0)$</th>
<th>Based on Formulation F1</th>
<th>Based on Formulation F2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.98</td>
<td>$1.44 \times 10^{-3}$</td>
<td>I*</td>
<td>I</td>
</tr>
<tr>
<td>2</td>
<td>3.31</td>
<td>$0.43 \times 10^{-3}$</td>
<td>I</td>
<td>N</td>
</tr>
<tr>
<td>3</td>
<td>3.36</td>
<td>$0.38 \times 10^{-3}$</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>4</td>
<td>3.38</td>
<td>$0.36 \times 10^{-3}$</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>5</td>
<td>3.50</td>
<td>$0.23 \times 10^{-3}$</td>
<td>N*</td>
<td>N</td>
</tr>
<tr>
<td>6</td>
<td>4.15</td>
<td>$0.01 \times 10^{-3}$</td>
<td>N</td>
<td>N</td>
</tr>
</tbody>
</table>

PNET Calculations of System Collapse Probability $P_f$

<table>
<thead>
<tr>
<th>$P_0$</th>
<th>Using All 44 Mechanisms</th>
<th>Using Mechanisms Identified in Formulation F1</th>
<th>Using Mechanisms Identified in Formulation F2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.0018</td>
<td>0.0018</td>
<td>0.0018</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0018</td>
<td>0.0018</td>
<td>0.0018</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0022</td>
<td>0.0022</td>
<td>0.0022</td>
</tr>
</tbody>
</table>

$P_f$ by Monte Carlo Simulations

$P_f = 0.0017$

Sample Size = 160,000

Bounds on $P_f$ (Based on All 44 Mechanisms)

Simple Bounds = 0.00144 - 0.00284
Narrow Bounds = 0.00188 - 0.00235

* I = Mechanisms identified; N = Mechanisms not identified
lead to a lower bound $P_f$ because all the possible mechanisms will be perfectly correlated.

**Problem 2B** — The purpose of this problem is to examine the effect of $p_o$ in the PNET method, particularly when the system collapse probability is of the order of $10^{-4}$. The problem is essentially the same as problem 2 (Fig. 5.2) except that the load and resistance statistics have been changed to the following:

<table>
<thead>
<tr>
<th>Mean</th>
<th>C.O.V.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>70 ft-K</td>
</tr>
<tr>
<td>$M_2$</td>
<td>230 ft-K</td>
</tr>
<tr>
<td>$F_1$</td>
<td>22 K</td>
</tr>
<tr>
<td>$F_2$</td>
<td>35 K</td>
</tr>
<tr>
<td>$P$</td>
<td>2.5 K</td>
</tr>
</tbody>
</table>

The first 6 major mechanisms are as follows:

<table>
<thead>
<tr>
<th>Mechanism i</th>
<th>Order of Mech. in Ex. 2</th>
<th>Hinges Involved in Mechanism i</th>
<th>Failure Mode Equation $Z_i$</th>
<th>$B_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>5,6,7</td>
<td>$4M_2 - F_2L_2/2$</td>
<td>3.86</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>1,2,3,4</td>
<td>$4M_1 - 3pL_1$</td>
<td>3.99</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1,2,4,6,8,9</td>
<td>$6M_1 + 2M_2 - 3pL_1 - F_2L_2/2$</td>
<td>4.02</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3,4,6,8,9</td>
<td>$4M_1 + 2M_2 - F_2L_2/2$</td>
<td>4.05</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
<td>10,11,13</td>
<td>$2M_1 + 2M_2 - F_1L_2/2$</td>
<td>4.19</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>1,2,4,6,7,8</td>
<td>$4M_1 + 3M_2 - 3pL_1 - F_2L_2/2$</td>
<td>4.26</td>
</tr>
</tbody>
</table>
Observe that with a change in the loads and resistance capacities, there is a change in the order of the major mechanisms (compare columns 1 and 2 in the above table), in addition to changes in $\beta$.

The correlation coefficients between pairs of the first 6 major mechanisms are as follows:

$$
\begin{array}{cccccc}
\text{Modes} & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1.00 & 0.00 & 0.76 & 0.86 & 0.71 & 0.92 \\
2 & 1.00 & 0.60 & 0.38 & 0.20 & 0.38 & \\
3 & 1.00 & 0.96 & 0.61 & 0.95 & & \\
4 & & 1.00 & 0.65 & 0.96 & & \\
5 & & & 1.00 & & 0.71 & \\
6 & & & & & 1.00 & \\
\end{array}
$$

The representative mechanisms for the respective $\rho_o$ are as follows:

- $\rho_o = 0.6$: representative mechanisms = 1, 2.
- $\rho_o = 0.7$: representative mechanisms = 1, 2, 18.
- $\rho_o = 0.8$: representative mechanisms = 1, 2, 3, 5, 18.

The results of analysis are summarized in Table 5.4. In this case (with $P_f = 10^{-4}$), the results show the collapse probability obtained with $\rho_o = 0.6$ or 0.7 (0.000091) is lower than the narrow bound $P_f$, indicating that a higher $\rho_o$ value (e.g. 0.8) is required, which gives a result of $P_f = 0.000135$. 

TABLE 5.4 SUMMARY OF RESULTS FOR PROBLEM 2B

Number of physically admissible mechanisms = 25

<table>
<thead>
<tr>
<th>Mechanism (ordered)</th>
<th>$\beta_i$</th>
<th>$P(Z_i &lt; 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.86</td>
<td>$5.7 \times 10^{-5}$</td>
</tr>
<tr>
<td>2</td>
<td>3.99</td>
<td>$3.4 \times 10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>4.02</td>
<td>$3.0 \times 10^{-5}$</td>
</tr>
<tr>
<td>4</td>
<td>4.05</td>
<td>$2.6 \times 10^{-5}$</td>
</tr>
<tr>
<td>5</td>
<td>4.19</td>
<td>$1.4 \times 10^{-5}$</td>
</tr>
<tr>
<td>6</td>
<td>4.26</td>
<td>$1.1 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

PNET Calculations of System Collapse Probability, $P_f$

Using All 25 Mechanisms

<table>
<thead>
<tr>
<th>$\rho_0$</th>
<th>$P_f$ Using All 25 Mechanisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.000091</td>
</tr>
<tr>
<td>0.7</td>
<td>0.000091</td>
</tr>
<tr>
<td>0.8</td>
<td>0.000135</td>
</tr>
</tbody>
</table>

$P_f$ by Monte Carlo Simulations

$P_f = 0.000114$

Sample Size = 800,000

Bounds on $P_f$ (Based on All 25 Mechanisms)

Simple Bounds = 0.000057 - 0.000214

Narrow Bounds = 0.000112 - 0.000152
5.1.3 Problem 3

An unsymmetrical two-story two-bay rectangular frame subjected to concentrated loads is shown in Fig. 5.5.

The potential plastic hinge locations, 19 of them, are shown in Fig. 5.6. Since the frame has 9 degrees of redundancy, there are 19 - 9 = 10 elementary mechanisms. Exhaustive inspection revealed at least 111 physically admissible mechanisms, out of the combinatorially maximum of \(2^{10} - 1 = 1023\). The statistics of the applied loads and moment capacities of the members are as follows:

<table>
<thead>
<tr>
<th>Member</th>
<th>Mean</th>
<th>C.O.V.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M_1)</td>
<td>70 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>(M_2)</td>
<td>150 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>(M_3)</td>
<td>70 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>(M_4)</td>
<td>90 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>(M_5)</td>
<td>120 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>(F_1)</td>
<td>38 K</td>
<td>0.15</td>
</tr>
<tr>
<td>(F_2)</td>
<td>20 K</td>
<td>0.25</td>
</tr>
<tr>
<td>(F_3)</td>
<td>26 K</td>
<td>0.25</td>
</tr>
<tr>
<td>(P)</td>
<td>7 K</td>
<td>0.25</td>
</tr>
</tbody>
</table>

The first 8 most major mechanisms are listed as follows:

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Hinges Involved in Mechanism</th>
<th>Failure Mode Equation (Z_i)</th>
<th>(\beta_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,2,4,6,7,9,11,13,15,16</td>
<td>(5M_1 + 3M_2 + 3M_3 + 2M_4 - F_1L_2/2 - F_2L_2/2 - 4pL_1)</td>
<td>1.85</td>
</tr>
<tr>
<td>2</td>
<td>1,2,3,4,15,16</td>
<td>(6M_1 - 3pL_1)</td>
<td>1.89</td>
</tr>
<tr>
<td>3</td>
<td>1,2,6,7,11,13,15,16,18</td>
<td>(5M_1 + 4M_2 + 2(M_3 + M_4 + M_5) - (F_1 + F_2 + F_3)L_2/2 - 4pL_1)</td>
<td>1.92</td>
</tr>
<tr>
<td>4</td>
<td>1,2,4,6,7,8,15,16</td>
<td>(5M_1 + 3M_2 + M_3 - F_1L_2/2 - 3pL_1)</td>
<td>1.97</td>
</tr>
<tr>
<td>5</td>
<td>10,11,13</td>
<td>(2M_2 + M_3 - F_2L_2/2)</td>
<td>1.98</td>
</tr>
<tr>
<td>6</td>
<td>16,17,18</td>
<td>(M_1 + 3M_5 - F_3L_2/2)</td>
<td>2.00</td>
</tr>
<tr>
<td>7</td>
<td>1,2,4,6,7,9,13,14,15,16</td>
<td>(5M_1 + 3M_2 + M_3 + 4M_4 - F_1L_2/2 - F_2L_2/2 - 4pL_1)</td>
<td>2.06</td>
</tr>
<tr>
<td>8</td>
<td>5,6,7</td>
<td>(4M_2 - F_2L_2/2)</td>
<td>2.07</td>
</tr>
</tbody>
</table>
The correlation coefficients between pairs of the 8 major mechanisms are summarized below:

<table>
<thead>
<tr>
<th>Modes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>0.65</td>
<td>0.89</td>
<td>0.91</td>
<td>0.44</td>
<td>0.04</td>
<td>0.97</td>
<td>0.59</td>
</tr>
<tr>
<td>2</td>
<td>1.00</td>
<td>0.55</td>
<td>0.68</td>
<td>0.00</td>
<td>0.09</td>
<td>0.63</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.00</td>
<td>0.83</td>
<td>0.35</td>
<td>0.45</td>
<td>0.88</td>
<td>0.61</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.00</td>
<td>0.03</td>
<td>0.05</td>
<td>0.87</td>
<td>0.72</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.00</td>
<td>0.00</td>
<td>0.45</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Symmetric</td>
<td>1.00</td>
<td>0.04</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td>1.00</td>
<td>0.58</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The representative mechanisms for the respective $\rho_o$ are as follows:

- $\rho_o = 0.6$: representative mechanisms = 1, 5, 6, 8.
- $\rho_o = 0.7$: representative mechanisms = 1, 2, 5, 6, 8.
- $\rho_o = 0.8$: representative mechanisms = 1, 2, 5, 6, 8, 27, 40, 44.

The results of analysis are summarized in Table 5.5. Although there are many significant mechanisms in this problem, Table 5.3 only listed the first 8 of them. It can be observed that some of the major mechanisms are identified in Formulation F1 but not in Formulation F2, and vice versa. Formulation F2 even misses the first and the most significant mechanism. This shows that initial points in addition to the "routine" ones may be necessary for more complex problems. For this example, regardless of formulations, $P_f$ calculated using $\rho_o = 0.7$ or 0.8 is close to the Monte Carlo result. This example also illustrates that the simple (or even the narrow) bounds may be too wide to be useful.
TABLE 5.5 SUMMARY OF RESULTS FOR PROBLEM 3

Number of physically admissible mechanisms = 111

<table>
<thead>
<tr>
<th>Mechanism i (ordered)</th>
<th>$\beta_0$</th>
<th>$P(Z_i &lt; 0)$</th>
<th>Based on Formulation F1</th>
<th>Based on Formulation F2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.854</td>
<td>$31.9 \times 10^{-3}$</td>
<td>I*</td>
<td>N</td>
</tr>
<tr>
<td>2</td>
<td>1.886</td>
<td>$29.7 \times 10^{-3}$</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>3</td>
<td>1.917</td>
<td>$27.8 \times 10^{-3}$</td>
<td>N*</td>
<td>I</td>
</tr>
<tr>
<td>4</td>
<td>1.967</td>
<td>$24.6 \times 10^{-3}$</td>
<td>N</td>
<td>I</td>
</tr>
<tr>
<td>5</td>
<td>1.981</td>
<td>$23.8 \times 10^{-3}$</td>
<td>I</td>
<td>N</td>
</tr>
<tr>
<td>6</td>
<td>1.996</td>
<td>$23.0 \times 10^{-3}$</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>7</td>
<td>2.064</td>
<td>$19.6 \times 10^{-3}$</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>8</td>
<td>2.065</td>
<td>$19.5 \times 10^{-3}$</td>
<td>I</td>
<td>I</td>
</tr>
</tbody>
</table>

PNET Calculations of System Collapse Probability, $P_f$

<table>
<thead>
<tr>
<th>$\beta_0$</th>
<th>Using All 111 Mechanisms</th>
<th>Using Mechanisms Identified in Formulation F1</th>
<th>Using Mechanism Identified in Formulation F2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.098</td>
<td>0.098</td>
<td>0.092</td>
</tr>
<tr>
<td>0.7</td>
<td>0.128</td>
<td>0.128</td>
<td>0.115</td>
</tr>
<tr>
<td>0.8</td>
<td>0.134</td>
<td>0.128</td>
<td>0.115</td>
</tr>
</tbody>
</table>

$P_f$ by Monte Carlo Simulations

$P_f = 0.116$
Sample Size = 5,000

Bounds on $P_f$ (Based on All 111 Mechanisms)

Simple Bounds = 0.034 - 0.412
Narrow Bounds = 0.071 - 0.298

*I* = Mechanisms identified; *N* = Mechanisms not identified
5.1.4 Problem 4

Consider the two-story two-bay symmetrical frame subjected to unsymmetrical concentrated loads as shown in Fig. 5.7. The load and resistance statistics for this example are summarized as follows:

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>C.O.V.</th>
</tr>
</thead>
<tbody>
<tr>
<td>M_1</td>
<td>70 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>M_2</td>
<td>150 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>M_3</td>
<td>70 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>M_4</td>
<td>90 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>M_5</td>
<td>150 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>M_6</td>
<td>90 ft-K</td>
<td>0.15</td>
</tr>
<tr>
<td>F_1</td>
<td>38 K</td>
<td>0.15</td>
</tr>
<tr>
<td>F_2</td>
<td>20 K</td>
<td>0.25</td>
</tr>
<tr>
<td>F_3</td>
<td>36 K</td>
<td>0.15</td>
</tr>
<tr>
<td>F_4</td>
<td>20 K</td>
<td>0.25</td>
</tr>
<tr>
<td>P</td>
<td>7 K</td>
<td>0.25</td>
</tr>
</tbody>
</table>

The potential plastic hinge locations are shown in Fig. 5.8. The structure has 24 potential plastic hinges and 12 degrees of redundancy and therefore has 24 - 12 = 12 elementary mechanisms. There are at least 247 physically admissible mechanisms, out of a combinatorial number of 4095. It may be emphasized that for such a complex frame, exhaustive inspection would become very tedious; nevertheless, it is necessary to have all the physically admissible mechanisms, particularly when performing Monte Carlo calculations. In this example, only the first 48 major mechanisms were used in the Monte Carlo calculations because the contributions of the remaining mechanisms are small (e.g. the 49th mechanism has a collapse probability which is less than 1% of the probability of the first mechanism).

The first 8 major mechanisms are as follows:
<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Hinges Involved in Mechanism</th>
<th>Failure Mode Equation $Z_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,2,4,6,7,8,15,16</td>
<td>$5M_1 + 3M_2 + M_3 - F_1L_2/2 - 3pL_1$</td>
<td>1.80</td>
</tr>
<tr>
<td>2</td>
<td>21,22,23</td>
<td>$M_3 + 3M_6 - F_4L_2/2$</td>
<td>1.85</td>
</tr>
<tr>
<td>3</td>
<td>10,13,14</td>
<td>$M_3 + 3M_4 - F_2L_2/2$</td>
<td>1.85</td>
</tr>
<tr>
<td>4</td>
<td>1,2,3,4,15,16</td>
<td>$6M_1 - 3pL_1$</td>
<td>1.89</td>
</tr>
<tr>
<td>5</td>
<td>5,6,7</td>
<td>$4M_2 - F_1L_2/2$</td>
<td>2.07</td>
</tr>
<tr>
<td>6</td>
<td>3,6,7,8</td>
<td>$M_1 + 3M_2 + M_3 - F_1L_2/2$</td>
<td>2.13</td>
</tr>
<tr>
<td>7</td>
<td>1,2,5,7,13,14,15,17,19,21,23</td>
<td>$3M_1+2M_2+2M_3+4M_4+2M_5+2M_6-(F_2+F_4)L_2/2-4pL_1$</td>
<td>2.15</td>
</tr>
<tr>
<td>8</td>
<td>12,13,14</td>
<td>$4M_4 - F_2L_2/2$</td>
<td>2.17</td>
</tr>
</tbody>
</table>

The correlation coefficients among any of these 8 mechanisms are:

**Correlation Coefficients, $\rho_{ij}$**

<table>
<thead>
<tr>
<th>Modes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>0.01</td>
<td>0.68</td>
<td>0.73</td>
<td>0.78</td>
<td>0.71</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.00</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>0.37</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.48</td>
<td>0.98</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.53</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.00</td>
<td>0.98</td>
<td>0.44</td>
<td>0.44</td>
<td>0.44</td>
<td>0.44</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Symmetric</td>
<td>1.00</td>
<td>0.48</td>
<td>0.00</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td>0.47</td>
<td>1.00</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

The representative mechanisms for the respective $\rho_o$ are as follows:

- $\rho_o = 0.6$: representative mechanisms = 1, 2, 3, 10.
- $\rho_o = 0.7$: representative mechanisms = 1, 2, 3, 4, 10, 26.
- $\rho_o = 0.8$: representative mechanisms = 1, 2, 3, 4, 5, 7, 10, 21, 23.
### TABLE 5.6 SUMMARY OF RESULTS FOR PROBLEM 4

Number of physically admissible mechanisms = 247

<table>
<thead>
<tr>
<th>Mechanism i (ordered)</th>
<th>( \beta_i )</th>
<th>( P(Z_i &lt; 0) )</th>
<th>Based on Formulation F1</th>
<th>Based on Formulation F2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.805</td>
<td>35.5 ( \times 10^{-3} )</td>
<td>I*</td>
<td>I</td>
</tr>
<tr>
<td>2</td>
<td>1.852</td>
<td>32.0 ( \times 10^{-3} )</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>3</td>
<td>1.852</td>
<td>32.0 ( \times 10^{-3} )</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>4</td>
<td>1.886</td>
<td>29.6 ( \times 10^{-3} )</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>5</td>
<td>2.065</td>
<td>19.5 ( \times 10^{-3} )</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>6</td>
<td>2.128</td>
<td>16.7 ( \times 10^{-3} )</td>
<td>N*</td>
<td>N</td>
</tr>
<tr>
<td>7</td>
<td>2.146</td>
<td>15.9 ( \times 10^{-3} )</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>8</td>
<td>2.174</td>
<td>14.9 ( \times 10^{-3} )</td>
<td>I</td>
<td>N</td>
</tr>
<tr>
<td>9</td>
<td>2.174</td>
<td>14.9 ( \times 10^{-3} )</td>
<td>I</td>
<td>N</td>
</tr>
<tr>
<td>10</td>
<td>2.287</td>
<td>11.1 ( \times 10^{-3} )</td>
<td>I</td>
<td>I</td>
</tr>
</tbody>
</table>

**PNET Calculations of System Collapse Probability, \( P_f \)**

<table>
<thead>
<tr>
<th>( P_0 )</th>
<th>Using First 48 Mechanisms</th>
<th>Using Mechanisms Identified in Formulation F1</th>
<th>Using Mechanisms Identified in Formulation F2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.1106</td>
<td>0.1106</td>
<td>0.1106</td>
</tr>
<tr>
<td>0.7</td>
<td>0.1440</td>
<td>0.1402</td>
<td>0.1402</td>
</tr>
<tr>
<td>0.8</td>
<td>0.1808</td>
<td>0.1597</td>
<td>0.1597</td>
</tr>
</tbody>
</table>

**\( P_f \) by Monte Carlo Simulations**

\[ P_f = 0.135 \]

Sample Size = 5,000

**Bounds on \( P_f \) (Based on first 48 Mechanisms)**

Simple Bounds = 0.0355 - 0.3170

Narrow Bounds = 0.1153 - 0.2197

*I* = Mechanisms identified; \( N \) = Mechanisms not identified
The results of analysis for this example is summarized in Table 5.6. Table 5.6 shows that the first five mechanisms plus the tenth mechanism are identified in both Formulations F1 and F2. In addition, mechanisms 8 and 9 are also identified in Formulation F1. However, both formulations yielded the same $P_f$ using three values of $\rho_o$, namely, 0.6, 0.7 and 0.8. The $P_f$ calculated using $\rho_o = 0.7$ agrees best with the Monte Carlo result. Again, the simple bound probabilities in this problem is too wide to be useful.

5.2 Truss Structures

In all the problems of truss structures discussed in this Chapter, it is assumed that the collapse of the structure is due to the yielding of the members, either in tension or compression; post-yield behavior is assumed to be perfectly plastic. Also the joints are considered as frictionless hinges.

For a truss structure consisting of $n$ members having $r$ degrees of redundancy, yielding in any $(r+1)$ members will render the structure a mechanism. Therefore, there could be $\binom{n}{r+1}$ possible mechanisms. In general, given a set of $(r+1)$ yielded members forming a mechanism, the corresponding failure mode equation is not easy to express by inspection. The coefficients of the failure mode equation are obtained by solving a set of simultaneous linear compatibility equations. For this purpose, a method of systematic generation of failure mode equations proposed by Gorman (1981) is applicable.

5.2.1 Problem 5

Consider a one-tier truss structure under the concentrated external loads as shown in Fig. 5.9, with the statistics of the applied loads and the yield capacities for the members as follows (tensile and compressive yield capacities are assumed to be equal):

<table>
<thead>
<tr>
<th></th>
<th>Mean (Kips)</th>
<th>C.O.V.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>14</td>
<td>0.15</td>
</tr>
<tr>
<td>$M_2$</td>
<td>10</td>
<td>0.15</td>
</tr>
<tr>
<td>$M_3$</td>
<td>20</td>
<td>0.15</td>
</tr>
<tr>
<td>F</td>
<td>3</td>
<td>0.15</td>
</tr>
<tr>
<td>P</td>
<td>13</td>
<td>0.30</td>
</tr>
</tbody>
</table>
Since the structure has 5 members and 1 degree of redundancy, there are \( \binom{5}{2} = 10 \) possible mechanisms. The first 5 major failure modes are as follows:

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Members Yielding</th>
<th>Failure Mode Equation</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 3^-, 5^- )</td>
<td>( M_2 + 0.7071 M_3 - P )</td>
<td>2.38</td>
</tr>
<tr>
<td>2</td>
<td>( 2^-, 5^- )</td>
<td>( M_1 + 0.7071 M_3 - P - F )</td>
<td>2.46</td>
</tr>
<tr>
<td>3</td>
<td>( 1^+, 2^- )</td>
<td>( 2M_1 - P )</td>
<td>2.62</td>
</tr>
<tr>
<td>4</td>
<td>( 4^+, 5^- )</td>
<td>( 1.4142 M_3 - P )</td>
<td>2.65</td>
</tr>
<tr>
<td>5</td>
<td>( 1^+, 3^- )</td>
<td>( M_1 + M_2 - P + 1.414 F )</td>
<td>3.23</td>
</tr>
</tbody>
</table>

The correlation coefficients between any pair of the first 5 major mechanisms are listed below:

<table>
<thead>
<tr>
<th>Modes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>0.85</td>
<td>0.57</td>
<td>0.90</td>
<td>0.79</td>
</tr>
<tr>
<td>2</td>
<td>1.00</td>
<td>1.00</td>
<td>0.85</td>
<td>0.85</td>
<td>0.83</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>1.00</td>
<td>0.46</td>
<td>0.89</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>1.00</td>
<td>0.56</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td></td>
</tr>
</tbody>
</table>

The representative mechanisms for all three values of \( \rho \) (i.e. 0.6, 0.7 and 0.8) are 1 and 3. Results of the analysis are shown in Table 5.7, which shows that for \( 0.6 \leq \rho \leq 0.8 \), the \( P_f \) obtained using the mechanisms identified in Formulation T2 is identical to that obtained by using all 10 possible mechanisms, and the PNET results are in good agreement with the Monte Carlo simulation \( P_f \). Unfortunately, Formulation T1 missed the important mechanism 3 and the resulting \( P_f \) is equal to the lower bound probability.
### TABLE 5.7 SUMMARY OF RESULTS FOR PROBLEM 5

Number of physically admissible mechanisms = 10

<table>
<thead>
<tr>
<th>Mechanism (ordered)</th>
<th>$\beta_i$</th>
<th>$P(Z_i &lt; 0)$</th>
<th>Based on Formulation T1</th>
<th>Based on Formulation T2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.38</td>
<td>$8.70 \times 10^{-3}$</td>
<td>$I^*$</td>
<td>$I$</td>
</tr>
<tr>
<td>2</td>
<td>2.46</td>
<td>$6.90 \times 10^{-3}$</td>
<td>$I$</td>
<td>$I$</td>
</tr>
<tr>
<td>3</td>
<td>2.62</td>
<td>$4.44 \times 10^{-3}$</td>
<td>$N^*$</td>
<td>$I$</td>
</tr>
<tr>
<td>4</td>
<td>2.65</td>
<td>$4.00 \times 10^{-3}$</td>
<td>$I$</td>
<td>$N$</td>
</tr>
<tr>
<td>5</td>
<td>3.23</td>
<td>$0.62 \times 10^{-3}$</td>
<td>$N$</td>
<td>$N$</td>
</tr>
</tbody>
</table>

PNET Calculations of System Collapse Probability, $P_f$

<table>
<thead>
<tr>
<th>$\rho_0$</th>
<th>Using All 10 Mechanisms</th>
<th>Using Mechanisms Identified in Formulation T1</th>
<th>Using Mechanisms Identified in Formulation T2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.0131</td>
<td>0.0087</td>
<td>0.0131</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0131</td>
<td>0.0087</td>
<td>0.0131</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0131</td>
<td>0.0087</td>
<td>0.0131</td>
</tr>
</tbody>
</table>

$P_f$ by Monte Carlo Simulations

$P_f = 0.0149$

Sample Size = 28,000

Bounds on $P_f$ (Based on All 10 Mechanisms)

Simple Bounds = 0.0087 - 0.0248

Narrow Bounds = 0.0113 - 0.0188

*I = Mechanisms identified; N = Mechanisms not identified*
5.2.2 Problem 6

This example is a two-tier truss as shown in Fig. 5.10, which could be a simplified model of an offshore structure. Again, for convenience, it is assumed that the members have the same yield limits in tension and compression. The applied loads and member capacities are as follows:

<table>
<thead>
<tr>
<th>Mean (Kips)</th>
<th>C.O.V.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>60</td>
</tr>
<tr>
<td>$M_2$</td>
<td>6</td>
</tr>
<tr>
<td>$M_3$</td>
<td>32</td>
</tr>
<tr>
<td>$M_4$</td>
<td>14</td>
</tr>
<tr>
<td>$M_5$</td>
<td>10</td>
</tr>
<tr>
<td>$M_6$</td>
<td>20</td>
</tr>
<tr>
<td>$F$</td>
<td>4</td>
</tr>
<tr>
<td>$P$</td>
<td>12</td>
</tr>
</tbody>
</table>

The structure is statically indeterminate to the second degree; hence, in general, yielding in three members are required to cause the collapse or the structure. However, in many cases, yielding in two members are sufficient to cause partial collapse of the truss. Since the structure has 10 members, combinatorially there could be as many as $\binom{10}{3} = 120$ possible collapse modes; however, under the given loading conditions and the assumptions of correlations among the member resistances, there are only 20 physically admissible mechanisms. The first 8 of these may be described as follows:
The correlation coefficients among the 8 major mechanisms are as follows:

<table>
<thead>
<tr>
<th>Modes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>0.62</td>
<td>0.92</td>
<td>0.65</td>
<td>0.54</td>
<td>0.92</td>
<td>0.54</td>
<td>0.74</td>
</tr>
<tr>
<td>2</td>
<td>1.00</td>
<td>0.82</td>
<td>0.87</td>
<td>0.86</td>
<td>0.80</td>
<td>0.86</td>
<td>0.86</td>
<td>0.65</td>
</tr>
<tr>
<td>3</td>
<td>1.00</td>
<td>0.86</td>
<td>0.64</td>
<td>0.94</td>
<td>0.80</td>
<td>0.80</td>
<td>0.78</td>
<td>0.78</td>
</tr>
<tr>
<td>4</td>
<td>1.00</td>
<td>0.61</td>
<td>0.76</td>
<td>0.91</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
<td>0.67</td>
</tr>
<tr>
<td>5</td>
<td>1.00</td>
<td>0.80</td>
<td>0.51</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
</tr>
<tr>
<td>6</td>
<td>Symmetric</td>
<td>1.00</td>
<td>0.63</td>
<td>0.77</td>
<td>0.77</td>
<td>0.77</td>
<td>0.77</td>
<td>0.77</td>
</tr>
<tr>
<td>7</td>
<td>1.00</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
</tr>
<tr>
<td>8</td>
<td>1.00</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
<td>0.56</td>
</tr>
</tbody>
</table>

The representative mechanisms for the respective \( \rho_o \) are as follows:

- \( \rho_o = 0.6 \): representative mechanisms = 1, 5, 7, 10.
- \( \rho_o = 0.7 \): representative mechanisms = 1, 2, 10.
- \( \rho_o = 0.8 \): representative mechanisms = 1, 2, 8.

The main results of the analysis are summarized in Table 5.8. In general, Table 5.8 shows similarly good agreements between the probabilities obtained by the PNET method using \( \rho_o = 0.7 \) or 0.8 and the results of Monte Carlo simulation. Higher \( \rho_o \) values tend to lead to higher \( P_f \).
TABLE 5.8 SUMMARY OF RESULTS FOR PROBLEM 6

Number of physically admissible mechanisms = 20

<table>
<thead>
<tr>
<th>Mechanism i (ordered)</th>
<th>( \beta_i )</th>
<th>( P(Z_i &lt; 0) )</th>
<th>Based on Formulation T1</th>
<th>Based on Formulation T2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.808</td>
<td>( 35.3 \times 10^{-3} )</td>
<td>*I</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.834</td>
<td>( 33.3 \times 10^{-3} )</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.835</td>
<td>( 33.2 \times 10^{-3} )</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.933</td>
<td>( 26.6 \times 10^{-3} )</td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>2.257</td>
<td>( 12.0 \times 10^{-3} )</td>
<td>N*</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2.260</td>
<td>( 11.9 \times 10^{-3} )</td>
<td>N</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>2.293</td>
<td>( 10.9 \times 10^{-3} )</td>
<td>N</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>2.427</td>
<td>( 7.7 \times 10^{-3} )</td>
<td>I</td>
<td></td>
</tr>
</tbody>
</table>

PNET Calculations of System Collapse Probability, \( P_f \)

\[
\begin{array}{ccc}
\rho_0 & \text{Using All 20 Mechanisms} & \text{Using Mechanisms Identified in Formulation T1} & \text{Using Mechanisms Identified in Formulation T2} \\
0.6 & 0.0622 & 0.0353 & 0.0353 \\
0.7 & 0.0726 & 0.0686 & 0.0686 \\
0.8 & 0.0763 & 0.0763 & 0.0686 \\
\end{array}
\]

\( P_f \) by Monte Carlo Simulations

\( P_f = 0.0764 \)

Sample Size = 5,000

Bounds on \( P_f \) (Based on All 20 Mechanisms)

Simple Bounds = 0.0353 - 0.220

Narrow Bounds = 0.0558 - 0.133

\*I = Mechanisms identified; N = Mechanisms not identified
Problem 6A --- In order to examine the effect of \( \rho_0 \), particularly for small \( P_f \) (of the order of \( 10^{-4} \)), the load and resistance statistics of Problem 6 are changed to the following:

<table>
<thead>
<tr>
<th>Mean (Kips)</th>
<th>C.O.V.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1 )</td>
<td>90</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>6</td>
</tr>
<tr>
<td>( M_3 )</td>
<td>32</td>
</tr>
<tr>
<td>( M_4 )</td>
<td>14</td>
</tr>
<tr>
<td>( M_5 )</td>
<td>10</td>
</tr>
<tr>
<td>( M_6 )</td>
<td>20</td>
</tr>
<tr>
<td>( F )</td>
<td>4</td>
</tr>
<tr>
<td>( P )</td>
<td>12</td>
</tr>
</tbody>
</table>

The first 8 major mechanisms may be described as follows:

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Member yielding</th>
<th>Failure Mode</th>
<th>Equation ( Z_i )</th>
<th>( \beta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4(^+), 5(^-)</td>
<td>( \sqrt{2} M_3 - 2.2P )</td>
<td>( Z_1 )</td>
<td>3.899</td>
</tr>
<tr>
<td>2</td>
<td>1(^+), 2(^-)</td>
<td>( 2M_1 - 4.6P )</td>
<td>( Z_2 )</td>
<td>3.940</td>
</tr>
<tr>
<td>3</td>
<td>6(^+), 7(^-)</td>
<td>( \sqrt{2} M_4 - 0.8485P )</td>
<td>( Z_3 )</td>
<td>4.002</td>
</tr>
<tr>
<td>4</td>
<td>9(^+), 10(^-)</td>
<td>( \sqrt{2} M_6 - 1.2P )</td>
<td>( Z_4 )</td>
<td>4.006</td>
</tr>
<tr>
<td>5</td>
<td>8(^-), 9(^-)</td>
<td>( M_5 + \frac{M_6}{\sqrt{2}} - 1.2P )</td>
<td>( Z_5 )</td>
<td>4.066</td>
</tr>
<tr>
<td>6</td>
<td>3(^-), 5(^-), 10(^-)</td>
<td>( M_2 + \frac{M_3}{\sqrt{2}} + \frac{M_6}{\sqrt{2}} - 2.2P )</td>
<td>( Z_6 )</td>
<td>4.170</td>
</tr>
<tr>
<td>7</td>
<td>7(^-), 10(^-)</td>
<td>( M_4 + \frac{M_6}{\sqrt{2}} - F - 1.2P )</td>
<td>( Z_7 )</td>
<td>4.406</td>
</tr>
<tr>
<td>8</td>
<td>1(^+), 3(^-), 10(^-)</td>
<td>( M_1 + \frac{M_2}{\sqrt{2}} + \frac{M_6}{2} - \frac{F}{\sqrt{2}} - 2.404P )</td>
<td>( Z_8 )</td>
<td>4.419</td>
</tr>
</tbody>
</table>

whereas the correlation coefficients among the first 8 major mechanisms are tabulated below:
TABLE 5.9  SUMMARY OF RESULTS FOR PROBLEM 6A

Number of physically admissible mechanisms = 20

<table>
<thead>
<tr>
<th>Mechanism i (ordered)</th>
<th>$\beta_i$</th>
<th>$P(Z_i &lt; 0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.899</td>
<td>$4.8 \times 10^{-5}$</td>
</tr>
<tr>
<td>2</td>
<td>3.940</td>
<td>$4.1 \times 10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>4.002</td>
<td>$3.2 \times 10^{-5}$</td>
</tr>
<tr>
<td>4</td>
<td>4.006</td>
<td>$3.1 \times 10^{-5}$</td>
</tr>
<tr>
<td>5</td>
<td>4.066</td>
<td>$2.4 \times 10^{-5}$</td>
</tr>
<tr>
<td>6</td>
<td>4.170</td>
<td>$1.5 \times 10^{-5}$</td>
</tr>
<tr>
<td>7</td>
<td>4.406</td>
<td>$0.5 \times 10^{-5}$</td>
</tr>
<tr>
<td>8</td>
<td>4.419</td>
<td>$0.4 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

PNET Calculations of System Collapse Probability, $P_f$

Using All 25 Mechanisms

<table>
<thead>
<tr>
<th>$\rho_0$</th>
<th>$P_f$ Using All 25 Mechanisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.000152</td>
</tr>
<tr>
<td>0.7</td>
<td>0.000152</td>
</tr>
<tr>
<td>0.8</td>
<td>0.000157</td>
</tr>
<tr>
<td>0.9</td>
<td>0.000200</td>
</tr>
</tbody>
</table>

$P_f$ by Monte Carlo Simulations

$P_f = 0.00019$

Sample Size = 500,000

Bounds on $P_f$ (Based on All Mechanisms)

Simple Bounds = 0.000048 - 0.000212

Narrow Bounds = 0.000169 - 0.000199
### Correlation Coefficients, $\rho_{ij}$

<table>
<thead>
<tr>
<th>Modes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>0.28</td>
<td>0.27</td>
<td>0.27</td>
<td>0.38</td>
<td>0.89</td>
<td>0.34</td>
<td>0.35</td>
</tr>
<tr>
<td>2</td>
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<td>0.27</td>
<td>0.27</td>
<td>0.37</td>
<td>0.38</td>
<td>0.34</td>
<td>0.96</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.00</td>
<td>0.26</td>
<td>0.37</td>
<td>0.38</td>
<td>0.79</td>
<td>0.34</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.00</td>
<td>0.88</td>
<td>0.66</td>
<td>0.79</td>
<td></td>
<td>0.79</td>
<td>0.50</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Symmetric</td>
<td>1.00</td>
<td>0.72</td>
<td>0.78</td>
<td>0.58</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1.00</td>
<td></td>
<td>0.65</td>
<td>0.55</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1.00</td>
<td></td>
<td></td>
<td>0.52</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The representative mechanisms for the respective $\rho_o$ are as follows:

- $\rho_o = 0.6$ or $0.7$ representative mechanisms = 1, 2, 3, 4.
- $\rho_o = 0.8$ representative mechanisms = 1, 2, 3, 4, 7.
- $\rho_o = 0.9$ representative mechanisms = 1, 2, 3, 4, 5, 6, 9.

The results are summarized in Table 5.9, which tend to support the observation that a higher value of $\rho_o$ may be more appropriate for small $P_f$. It appears that a value of 0.9 is more appropriate in the present case where $P_f = 10^{-4}$.

#### 5.2.3 Problem 7

This problem is a simple truss commonly used in bridge structures. The structure is shown in Fig. 5.11, with the following applied loads and member resistance statistics:
<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Members Yielding</th>
<th>Failure Mode Equation $Z_i$</th>
<th>$\beta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1^+$</td>
<td>$M_1 - 1/2 F_1 - 1/4 F_2$</td>
<td>2.265</td>
</tr>
<tr>
<td>2</td>
<td>$4^-, 5^+$</td>
<td>$M_4 + 3/4 M_5 - F_1 - 1/2 F_2$</td>
<td>2.351</td>
</tr>
<tr>
<td>3</td>
<td>$7^-$</td>
<td>$M_7 - 5/6 F_1 - 5/12 F_2$</td>
<td>2.373</td>
</tr>
<tr>
<td>4</td>
<td>$5^+, 8^+$</td>
<td>$5/4 M_5 + M_6 - 5/4 F_1$</td>
<td>2.441</td>
</tr>
<tr>
<td>5</td>
<td>$3^+$</td>
<td>$M_3 - 1/4 F_1 - 1/2 F_2$</td>
<td>2.764</td>
</tr>
<tr>
<td>6</td>
<td>$10^-$</td>
<td>$M_{10} - 5/12 F_1 - 5/6 F_2$</td>
<td>2.866</td>
</tr>
<tr>
<td>7</td>
<td>$4^-, 6^+$</td>
<td>$M_4 + 3/4 M_6 - 1/2 F_1 - F_2$</td>
<td>3.032</td>
</tr>
<tr>
<td>8</td>
<td>$2^+, 4^-$</td>
<td>$M_2 + M_4 - 3/4 F_1 - 3/4 F_2$</td>
<td>3.142</td>
</tr>
</tbody>
</table>

The structure has 10 members and 1 degree of redundancy; combinatorially there could be as many as $\binom{10}{2} = 45$ possible mechanisms. However, only 19 are physically admissible; the first 8 major mechanisms can be described as follows:
The correlation coefficients between any pair of the first 8 modes are as follows:

\[
\begin{array}{cccccccc}
\text{Modes} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 1.00 & 0.46 & 0.37 & 0.43 & 0.28 & 0.27 & 0.35 & 0.41 \\
2 & 1.00 & 0.46 & 0.80 & 0.34 & 0.34 & 0.70 & 0.76 \\
3 & 1.00 & 0.42 & 0.27 & 0.26 & 0.35 & 0.40 \\
4 & 1.00 & 0.23 & 0.22 & 0.30 & 0.41 \\
5 & 1.00 & 0.31 & 0.40 & 0.37 \\
6 & \text{Symmetric} & 1.00 & 0.39 & 0.36 \\
7 & 1.00 & 0.74 \\
8 & 1.00 \\
\end{array}
\]

The representative mechanisms for the respective \( \rho_o \) are as follows:

- \( \rho_o = 0.6 \) representative mechanisms = 1, 2, 3, 5, 6.
- \( \rho_o = 0.7 \) representative mechanisms = 1, 2, 3, 5, 6, 11.
- \( \rho_o = 0.8 \) representative mechanisms = 1, 2, 3, 4, 5, 6, 7, 8, 11.

The results of analysis are summarized in Table 5.10. Based on the mechanisms identified in Formulation T1 with \( \rho_o = 0.6 \) or 0.7, \( P_f \) is found to be 0.0349 which is very close to the value of 0.0346 obtained with Monte Carlo simulations. However, because some of the major mechanisms are not identified in Formulation T2, the resulting \( P_f \) is not satisfactory; additional initial points should improve the results.
TABLE 5.10 SUMMARY OF RESULTS FOR PROBLEM 7

Number of physically admissible mechanisms = 19

<table>
<thead>
<tr>
<th>Mechanism i (ordered)</th>
<th>$\beta_i$</th>
<th>$P(Z_i &lt; 0)$</th>
<th>Based on Formulation T1</th>
<th>Based on Formulation T2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.265</td>
<td>$1.18 \times 10^{-3}$</td>
<td>I*</td>
<td>I*</td>
</tr>
<tr>
<td>2</td>
<td>2.351</td>
<td>$9.4 \times 10^{-3}$</td>
<td>I</td>
<td>N</td>
</tr>
<tr>
<td>3</td>
<td>2.373</td>
<td>$8.8 \times 10^{-3}$</td>
<td>I</td>
<td>N</td>
</tr>
<tr>
<td>4</td>
<td>2.441</td>
<td>$7.3 \times 10^{-3}$</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>5</td>
<td>2.764</td>
<td>$2.8 \times 10^{-3}$</td>
<td>I</td>
<td>N</td>
</tr>
<tr>
<td>6</td>
<td>2.866</td>
<td>$2.1 \times 10^{-3}$</td>
<td>I</td>
<td>N</td>
</tr>
<tr>
<td>7</td>
<td>3.032</td>
<td>$1.2 \times 10^{-3}$</td>
<td>N*</td>
<td>N</td>
</tr>
<tr>
<td>8</td>
<td>3.142</td>
<td>$0.8 \times 10^{-3}$</td>
<td>N</td>
<td>N</td>
</tr>
</tbody>
</table>

PNET Calculations of System Collapse Probability, $P_f$

<table>
<thead>
<tr>
<th>$\rho_o$</th>
<th>Using All 19 Mechanisms</th>
<th>Using Mechanisms Identified in Formulation T1</th>
<th>Using Mechanisms Identified in Formulation T2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.0349</td>
<td>0.0349</td>
<td>0.0193</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0350</td>
<td>0.0349</td>
<td>0.0193</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0444</td>
<td>0.0422</td>
<td>0.0193</td>
</tr>
</tbody>
</table>

$P_f$ by Monte Carlo Simulations

$P_f = 0.0346$

Sample Size = 15,000

Bounds on $P_f$ (Based on All 19 Mechanisms)

Simple Bounds = 0.0118 - 0.0450

Narrow Bounds = 0.0302 - 0.0407

*I = Mechanisms identified; N = Mechanisms not identified*
5.2.4 Problem 8

This example is a 25-member, 6-span truss structure as shown in Fig. 5.12; the applied loads and resistance capacities of the members are as follows:

<table>
<thead>
<tr>
<th>Member</th>
<th>Mean (Kips)</th>
<th>C.O.V.</th>
</tr>
</thead>
<tbody>
<tr>
<td>M_1 to M_{10}</td>
<td>10</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{11}</td>
<td>24</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{12}</td>
<td>12</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{13}</td>
<td>7</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{14}</td>
<td>12</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{15}</td>
<td>24</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{16}</td>
<td>28</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{17}</td>
<td>7</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{18}</td>
<td>5</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{19}</td>
<td>7</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{20}</td>
<td>10</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{21}</td>
<td>10</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{22}</td>
<td>7</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{23}</td>
<td>5</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{24}</td>
<td>7</td>
<td>0.15</td>
</tr>
<tr>
<td>M_{25}</td>
<td>28</td>
<td>0.15</td>
</tr>
<tr>
<td>F_1</td>
<td>10</td>
<td>0.25</td>
</tr>
<tr>
<td>F_2</td>
<td>8</td>
<td>0.25</td>
</tr>
<tr>
<td>F_3</td>
<td>10</td>
<td>0.25</td>
</tr>
<tr>
<td>F_4</td>
<td>8</td>
<td>0.25</td>
</tr>
</tbody>
</table>
TABLE 5.11 MAJOR MECHANISMS IDENTIFIED IN PROBLEM 8

<table>
<thead>
<tr>
<th>Mechanism</th>
<th>Failure Mode Equation (Z_1) (+ = tension, - = compression)</th>
<th>(\beta_i)</th>
<th>(P_{F_i})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(M_1^+ + .375M_{13}^- + .3M_{20}^- + .3M_{21}^- .625F_1 - .5F_2 - .25F_3 - .125F_4)</td>
<td>1.883</td>
<td>2.98x10^{-2}</td>
</tr>
<tr>
<td>2</td>
<td>(M_6^+ + .375M_{13}^- + .3M_{20}^- + .3M_{21}^- - .125F_1 - .25F_2 - .5F_3 - .625F_4)</td>
<td>2.144</td>
<td>1.60x10^{-2}</td>
</tr>
<tr>
<td>3</td>
<td>(M_5^+ + M_{10}^- + 1.125M_{13}^- + .9M_{20}^- + .9M_{21}^- .375F_1 - .75F_2 - 1.5F_3 - 1.125F_4)</td>
<td>2.150</td>
<td>1.58x10^{-2}</td>
</tr>
<tr>
<td>4</td>
<td>(M_5^+ + M_{10}^- + 1.8M_{21}^- + 1.8M_{22}^- .125F_1 - .125F_3 - 1.5F_4)</td>
<td>2.151</td>
<td>1.57x10^{-2}</td>
</tr>
<tr>
<td>5</td>
<td>(M_2^+ + M_7^- + 1.125M_{13}^- + .9M_{20}^- + .9M_{21}^- - 1.125F_1 - 1.5F_3 - 1.5F_2 - .75 - F_3 - .375F_4)</td>
<td>2.194</td>
<td>1.41x10^{-2}</td>
</tr>
<tr>
<td>6</td>
<td>(M_1^+ .6M_{19}^- + .6M_{20}^- .75F_1 - .75F_2)</td>
<td>2.206</td>
<td>1.37x10^{-2}</td>
</tr>
<tr>
<td>7</td>
<td>(M_6^+ .6M_{21}^- + .6M_{22}^- .75F_3 - .75F_4)</td>
<td>2.206</td>
<td>1.37x10^{-2}</td>
</tr>
<tr>
<td>8</td>
<td>(M_1^+ .6M_{21}^- + .6M_{22}^- .5F_1 - .25F_2 - .5F_3 - .25F_4)</td>
<td>2.330</td>
<td>0.99x10^{-2}</td>
</tr>
<tr>
<td>9</td>
<td>(M_2^+ + M_7^- + 1.8M_{19}^- + 1.8M_{20}^- .125F_1 - 1.5F_3 - 2.25F_2)</td>
<td>2.497</td>
<td>0.63x10^{-2}</td>
</tr>
<tr>
<td>10</td>
<td>(M_{10}^- + .75M_{13}^- + .75M_{14}^+ + .6M_{20}^- .25F_1 - .5F_2 - F_3 - .5F_4)</td>
<td>2.593</td>
<td>0.47x10^{-2}</td>
</tr>
<tr>
<td>11</td>
<td>(M_4^+ 1.2M_{21}^- + 1.8M_{22}^- .125F_3 - .75F_4)</td>
<td>2.700</td>
<td>0.35x10^{-2}</td>
</tr>
<tr>
<td>12</td>
<td>(M_7^- + .75M_{12}^- + .75M_{13}^- + .6M_{21}^- .5F_1 - F_2 - .5F_3 - .25F_4)</td>
<td>2.856</td>
<td>0.21x10^{-2}</td>
</tr>
</tbody>
</table>
The structure has 25 members and 5 degrees of redundancy; in general, yielding in six members are required to cause the collapse of the structure. Combinatorially, there could conceivably be of \( \binom{25}{6} = 177,100 \) mechanisms. In reality, the actual number of physically admissible mechanisms will be much less than this number. It is obviously impractical to exhaustively examine all of these possible failure modes. In this case, the major failure modes are identified through the Hooke and Jeeves pattern search algorithm. Additional initial points were used with both formulations described in Sect. 3.5.2 and 3.6.2 were used. The first 12 major failure modes identified are those shown in Table 5.11. The corresponding correlation coefficients between any pairs of the 12 mechanisms can be tabulated as follows:

<table>
<thead>
<tr>
<th>Modes</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>.41</td>
<td>.52</td>
<td>.28</td>
<td>.73</td>
<td>.90</td>
<td>.25</td>
<td>.87</td>
<td>.63</td>
<td>.46</td>
<td>.27</td>
<td>.59</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>.72</td>
<td>.66</td>
<td>.64</td>
<td>.53</td>
<td>.23</td>
<td>.90</td>
<td>.48</td>
<td>.26</td>
<td>.61</td>
<td>.58</td>
<td>.48</td>
</tr>
<tr>
<td>3</td>
<td>1.0</td>
<td>.85</td>
<td>.67</td>
<td>.31</td>
<td>.68</td>
<td>.59</td>
<td>.35</td>
<td>.85</td>
<td>.69</td>
<td>.58</td>
<td>.48</td>
<td>.48</td>
</tr>
<tr>
<td>4</td>
<td>1.0</td>
<td>.40</td>
<td>0.0</td>
<td>.82</td>
<td>.61</td>
<td>0.0</td>
<td>.69</td>
<td>.90</td>
<td>.41</td>
<td>.41</td>
<td>.41</td>
<td>.41</td>
</tr>
<tr>
<td>5</td>
<td>1.0</td>
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<td>.35</td>
<td>.61</td>
<td>.83</td>
<td>.59</td>
<td>.38</td>
<td>.84</td>
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<td>.49</td>
<td>.49</td>
<td>.49</td>
</tr>
<tr>
<td>6</td>
<td>1.0</td>
<td>0.0</td>
<td>.66</td>
<td>.81</td>
<td>.30</td>
<td>.00</td>
<td>.49</td>
<td>.49</td>
<td>.49</td>
<td>.49</td>
<td>.49</td>
<td>.49</td>
</tr>
<tr>
<td>7</td>
<td>Symmetric</td>
<td>1.0</td>
<td>.53</td>
<td>0.0</td>
<td>.54</td>
<td>.79</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
<td>.54</td>
</tr>
<tr>
<td>8</td>
<td>1.0</td>
<td>.37</td>
<td>.49</td>
<td>.62</td>
<td>.54</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
</tr>
<tr>
<td>9</td>
<td>1.0</td>
<td>.35</td>
<td>0.0</td>
<td>.63</td>
<td>.52</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>.57</td>
<td>.52</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
</tr>
<tr>
<td>11</td>
<td>1.0</td>
<td>.39</td>
<td>.52</td>
<td>.36</td>
<td>.54</td>
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<td>.36</td>
<td>.54</td>
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<td>.54</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
<td>.54</td>
<td>.36</td>
</tr>
</tbody>
</table>

The representative mechanisms for the respective \( \rho_o \) are as follows:

\[ \rho_o = 0.6 \quad \text{representative mechanisms} = 1, 2, 11, 12. \]
\[ \rho_o = 0.7 \quad \text{representative mechanisms} = 1, 2, 4, 9, 10, 12. \]
\[ \rho_o = 0.8 \quad \text{representative mechanisms} = 1, 2, 3, 5, 11. \]

It should be emphasized that Monte Carlo simulation and the methods
for calculating the simple and narrow bounds require the knowledge of all the possible failure modes, which is not available in this case. Based on the first 12 major mechanisms, the results of Monte Carlo calculations and PNET using different values of $\rho_o$ are as follows:

### PNET Calculations of System Collapse Probability, $P_f$

<table>
<thead>
<tr>
<th>$\rho_o$</th>
<th>Based on First 12 Mechanisms Identified</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.0514</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0746</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0792</td>
</tr>
</tbody>
</table>

**$P_f$ by Monte Carlo Simulations**

(Based on First 12 Mechanisms Identified)

$P_f = 0.0736$

Sample Size = 5,000

### Bounds on $P_f$ (Based on First 12 Mechanisms Identified)

Simple Bounds = 0.0298 - 0.1453

Narrow Bounds = 0.0459 - 0.1110

In this example, the PNET result using $\rho_o = 0.7$ is closest to that obtained from the Monte Carlo simulation.

### 5.3 Summary of Results

In order to validate the accuracy of the results obtained by the proposed PNET method, the calculated collapse probabilities $P_f$ are expressed in terms (ratios) of the Monte Carlo results. These ratios, for the structural systems considered in this chapter, are summarized in Figs. 5.13A, B and C corresponding to $\rho_o = 0.6$, 0.7 and 0.8 respectively.
Comparing the results in these three figures, it may be observed that the calculated failure probabilities clearly tend to be increasingly conservative as $p_0$ increases. The bounds for the corresponding collapse probabilities are also indicated in these figures.

In general, the results of the example problems obtained by the approximate PNET method of analysis are in close agreements with those obtained with Monte Carlo calculations, and lie well within the narrow bounds, and hence also in the simple bounds.
Fig. 5.1 Some "Combinatorially Possible" Failure Modes

Fig. 5.2 Frame Structure for Problems 2 & 2B
Fig. 5.3 Potential Plastic Hinges Locations for Problems 2, 2A & 2B

Fig. 5.4 Frame Structure for Problem 2A
Fig. 5.5 Frame Structure for Problem 3

Fig. 5.6 Potential Plastic Hinges Locations for Problem 3
Fig. 5.7 Frame Structure for Problem 4

Fig. 5.8 Potential Plastic Hinges Locations for Problem 4
Fig. 5.9 Truss Structure for Problem 5

Fig. 5.10 Truss Structure for Problems 6 & 6A
Fig. 5.11 Truss Structure for Problem 7

Fig. 5.12 Truss Structure for Problem 8
Fig. 5.13A Summary of Results
Fig. 5.13B Summary of Results
Fig. 5.13C Summary of Results
6.1 On the Identification of Major Mechanisms

It may be emphasized that the analysis of the reliability of structural systems requires the knowledge of all the potential failure or collapse modes of a system. The identification of such potential collapse modes, for structures of practical significance, is not a trivial problem. Regardless of the calculational method, including Monte Carlo, the collapse modes (particularly the major ones) must first be identified.

6.1.1 Number of Significant Mechanisms

For most structures used in practice, the number of possible mechanisms are generally quite large; invariably, however, only a few of them will contribute significantly to the system collapse probability. Moreover, the major contributions to the system collapse probability will come from those mechanisms that have low mutual correlations. This observation confirms the basis of the PNET method.

6.1.2 Minimization Algorithm

For a given structure, the major mechanisms may be identified as those with small reliability indices, $\beta_i$, and thus may be determined numerically using available algorithms for optimization. The relevant objective function invariably contain discontinuities in the gradients and possibly singularities in the Hessian matrices at the solution points; thus local optimality may be obtained by comparing the values of the objective functions around the optimal point. In general, the local minimum point $\{X\}$ will correspond to a mechanism. However, regardless of the types of formulations, there may be cases when the algorithm solution does not converge to a point which corresponds to a one-degree-of-freedom mechanism; experience shows that this may occur for problems with relatively large number of variables (e.g. $n \geq 20$). Nevertheless, one can rely on checking
the physical meanings of all the solutions to confirm that those solutions are indeed one-degree of freedom mechanisms. In fact, because of the availability of the physical interpretations of the solution points, the convergence criterion of the algorithm may be relaxed to reduce computations. In cases when the solution corresponds to a two-degree of freedom mechanism, interpretation of its physical meaning may provide information about which 2 (or more) mechanisms are involved. Examination of these 2 (or more) mechanisms may then lead to initial starting points for the subsequent trials.

Because of the vastness of the multidimensional hyperspace, good initial points (in the sense that it is close to a local minimum) are essential for the rapid convergence of solutions to the local minima. Good engineering judgement of the likely significant mechanisms will form a set of good initial starting points.

6.1.3 Effects of Formulations

For a given structure, regardless of the type of formulation, the problem of identification of significant mechanisms will always end up with an unconstrained optimization problem with the same number of independent variables $N$. Experience shows that because of the nature of the "routine starting points", rapid convergence to solution points may be achieved if the solution points contain many zeroes (i.e. very little interaction among the variables). Formulations 1F (Sect. 3.5.1) and 1T (Sect. 3.6.1) provide the flexibility for the user to select the independent variables such that the major mechanisms will likely correspond to points having many zeroes. In some truss problems, this advantage may increase the rate of convergence significantly. On the other hand, formulations 2F & 2T (Sect. 3.5.2 & 3.6.2 respectively) have the important advantage that it can be completely automated for computer implementation; for this reason, it is preferable.

6.2 On the Synthesis of System Collapse Probability

6.2.1 The PNET Method
The PNET method is used in this study to combine the individual collapse modes to obtain the system collapse probability $P_f$. Central to the PNET method is the appropriate value of the demarcating correlation coefficient $\rho_o$, which may depend on the level of the collapse probability.

**The Demarcating Correlation** --- From the results of the example problems examined in this study, it appears that a value of $\rho_o = 0.7$ (or 0.8 for conservative results) is suitable for calculating the collapse probability of structural systems in the order of $10^{-3}$. However, for $P_f$ in the order of $10^{-4}$, a value of $\rho_o = 0.8$ (or 0.9 for conservative results) may be necessary.

It is difficult to determine the precise effect of $\rho_o$. It appears, nevertheless, that the value of $\rho_o$ tends to increase with the reliability of the system (i.e., larger values of $\rho_o$ should be used for smaller $P_f$).

Recall from Sect. 3.3.1 that the PNET method is based on the premise that mechanisms with correlations higher than $\rho_o$ can be assumed to be perfectly correlated, whereas those with correlations less than $\rho_o$ can be assumed to be statistically independent. The following may provide an approximate basis for examining the effect of $\rho_o$.

Consider the case when there are two mechanisms, with $\rho_1 = \rho_2 = \beta$, and $0 < \rho_{12} < 1$. From Eq. 2.59,

$$q_1 = q_2 = q = \phi(-\beta) \left( -\frac{\beta \sqrt{1 - \rho_{12}}}{\sqrt{1 + \rho_{12}}} \right)$$

Eq. 2.58, then gives

$$2q \geq P(E_1E_2) \geq q$$

Substituting Eq. 6.2 into Eqs. 2.53 and 2.57 yields the narrow bounds,

$$2P_{f_1} - q \geq P_f \geq 2P_{f_1} - 2q$$

(6.3)
where $P_f$ is the single mode failure probability.

For a system with two mechanisms that are perfectly correlated, the failure probability of the system is

$$P_f' = P_{f_1}$$  \hspace{1cm} (6.4)

Therefore, if the two mechanisms are partially correlated, Eq. 6.4 will underestimate $P_f'$; the error would be:

$$\frac{P_f - P_f'}{P_f} = \frac{P_f - P_{f_1}}{P_f}$$  \hspace{1cm} (6.5)

Since the exact value of $P_f$ is unknown, but is bounded as in Eq. 6.3, the error would also be bounded by:

$$\frac{P_{f_1} - q}{2P_{f_1} - q} > \text{error} > \frac{P_{f_1} - 2q}{2P_{f_1} - 2q}$$  \hspace{1cm} (6.6)

Fig. 6.1 shows the above error bounds plotted against $\beta$ for different values of $\rho_{12}$. The errors associated with the perfect correlation assumption are large; however, the case with $\beta_1 = \beta_2 = \beta$ would represent the situation with the largest error. From Fig. 6.1, it can be observed that to maintain a given level of error (e.g., 40%), the value of $\rho_{12}$ increases with $\beta$.

Furthermore, for high $\rho_{12}$, the error is closer to the lower bound error than the upper bound. This can be observed from the extreme case of $\rho_{12} = 1$ when Eq. 6.1 would give:

$$q = \phi(-\beta) \phi(0) = 0.5 P_{f_1}$$  \hspace{1cm} (6.7)

Substituting this into Eq. 6.3 yields:

$$1.5P_{f_1} > P_f > P_{f_1}$$  \hspace{1cm} (6.8)
With Eq. 6.4, $P_f$ therefore lies on the lower bound of Eq. 6.3; the error, therefore, will tend towards the lower bound of Eq. 6.6 as $\rho_{12}$ approaches 1.0.

On the other hand, if the two partially correlated mechanisms are assumed to be statistically independent (i.e. $\rho_{12} = 0$), the system failure probability is,

$$P_f = 1 - (1-P_{f_1})^2 \quad (6.9)$$

For large $\beta$ or small $P_{f_1}$,

$$P_f \approx 2P_{f_1} \quad (6.10)$$

Using the average of the narrow bounds (Eq. 6.3) for the correct probability $P_f$; i.e.

$$P_f \approx 2P_{f_1} - 1.5q \quad (6.11)$$

the error that would be incurred in assuming two partially correlated mechanisms to be statistically independent would be (from Eqs. 6.10 and 6.11)

$$\frac{2P_{f_1} - (2P_{f_1} - 1.5q)}{2P_{f_1} - 1.5q} = \frac{1.5q}{2P_{f_1} - 1.5q} \quad (6.12)$$

With $\beta_1 = \beta_2 = \beta$, the error for the two-mechanism case would be maximum; Eq. 6.12 therefore represents the maximum possible error. Fig. 6.2 shows this error plotted against $\beta$ for different values of $\rho_{12}$. From Fig. 6.2, it may be observed that for a given value of $\rho_{12}$, the error associated with the statistically independent assumption improves with increasing values of $\beta$. This implies that assuming the representative mechanisms are statistically independent (for a given $\rho_0$) will improve with $\beta$. 
Non-normal Variables --- For variables that have non-normal distributions, the method of equivalent normal distributions may be adopted (Paloheimo, 1974; Rackwitz, 1976). Once the equivalent normal distributions of the variables are obtained, the PNET method may then be applied as described herein for normal variates. Results obtained for a separate study (Schueller and Grimmelt, 1981) indicate that the PNET method performs equally well for problems involving non-normal variates. These results are summarized in Tables 6.3A, B and C, which include verifications by Monte Carlo simulations.

In the study by Schueller & Grimmelt (1981), the loads and plastic resistances are considered to be random variables modeled by normal, lognormal, and extreme value distributions. The various combinations of distribution, for the loads and resistances are as follows: normal/normal, lognormal/lognormal, and Gumbel/Weibull. In addition, three ratios of load and resistance C.O.V.'s were used; namely, \( \sigma_L / \sigma_M = 0.3/0.05, 0.3/0.1, \) and \( 0.1/0.1 \). Fig. 6.3 describes the five example problems, and Table 6.1 summarizes the statistics of loads and resistances, for normal and lognormal distributions, whereas Table 6.2 tabulates the load and resistance statistics for the extreme value distributions. The frames are assumed to collapse through the formation of plastic-hinge mechanisms; at a beam-column joint there is only one possible hinge. Mutual independence is assumed between the vertical and horizontal loads, as well as between the loads and moment capacities.

The plastic moment capacities are assumed to be either perfectly correlated \( (\rho_M = 1) \) or statistically independent \( (\rho_M = 0) \). The results of the analysis with the PNET method using \( \rho_o = 0.7 \) and 0.8 for normal, lognormal and extreme value distributions are summarized in Tables 6.3A, B and C respectively. Most of the results using \( \rho_o = 0.8 \) are identical to those obtained with \( \rho_o = 0.7 \), otherwise, the values obtained with \( \rho_o = 0.8 \) are shown in brackets in Tables 6.3A, B and C. Figs. 6.4A and B also summarize the PNET results (with \( \rho_o = 0.7 \)) expressed as ratios of the Monte Carlo-calculated \( P_f \) reported by Schueller and Grimmelt (1981). In those cases where Monte Carlo \( P_f \) are not available, the average value of the upper and lower narrow bounds is used. From the results shown in Figs. 6.4A and
6.4B, it is difficult to discern any effect of the distributions on $\rho_0$; on this basis, $\rho_0$ may be assumed to be independent of the distributions.

The performance of the PNET method relative to other methods may be seen from the forthcoming report by Schueller and Grimmel (1981).

6.2.2 Bounds on System Reliability

For the narrow bounds of the failure probability, the ordering of the failure modes may influence the values of the upper and lower bound values. It is possible, for a particular ordering of the failure modes, to produce a lower bound value less than the general lower bound. The closest narrow bounds may be obtained only if all the possible orderings of the failure modes (there are $m!$ such orders for a structure having $m$ possible mechanisms) are examined, which is obviously impractical. The narrow bounds for the example problems described in Chapter 5 were obtained by arranging the failure modes in the order of decreasing failure probabilities $P_{fi}$. The bounds obtained in this way compared favorably with those obtained from random ordering of the mechanisms; this should insure that the narrow lower bound is not less than the general lower bound value of the system failure probability.
TABLE 6.1 LOAD AND RESISTANCE STATISTICS FOR NORMAL AND LOG-NORMAL DISTRIBUTIONS

I, Loads (kN)

<table>
<thead>
<tr>
<th>Type of Distribution</th>
<th>Normal or Log-normal</th>
<th>Normal or Log-normal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Standard Deviation $\sigma$</td>
</tr>
<tr>
<td>H</td>
<td>50</td>
<td>5.0</td>
</tr>
<tr>
<td>V</td>
<td>40</td>
<td>4.0</td>
</tr>
</tbody>
</table>

II, Plastic Moment Capacities (kNm)

<table>
<thead>
<tr>
<th>Type of Distribution</th>
<th>Normal or Log-normal</th>
<th>Normal or Log-normal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Standard Deviation $\sigma$</td>
</tr>
<tr>
<td>Problem 1A</td>
<td>134.9</td>
<td>6.745</td>
</tr>
<tr>
<td>Problem 1B</td>
<td>61.95</td>
<td>3.098</td>
</tr>
<tr>
<td>Problem 2A</td>
<td>101.0</td>
<td>5.05</td>
</tr>
<tr>
<td>Problem 3</td>
<td>176.8</td>
<td>8.84</td>
</tr>
</tbody>
</table>
TABLE 6.2 LOAD AND RESISTANCE STATISTICS FOR EXTREME VALUE DISTRIBUTIONS

I, Loads (KN)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\alpha$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\alpha$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>61.35</td>
<td>2.15</td>
<td>0.5965</td>
<td>60.382</td>
<td>84.05</td>
<td>6.45</td>
<td>0.1988</td>
<td>81.15</td>
<td></td>
</tr>
<tr>
<td>$V$</td>
<td>49.08</td>
<td>1.72</td>
<td>0.7457</td>
<td>48.31</td>
<td>67.25</td>
<td>5.16</td>
<td>0.2486</td>
<td>64.92</td>
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II, Plastic Moment Capacities (KNm)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$k$</th>
<th>$\omega$</th>
<th>$\epsilon$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$k$</th>
<th>$\omega$</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1A</td>
<td>121.2</td>
<td>2.38</td>
<td>48.96</td>
<td>119.2</td>
<td>18.2</td>
<td>108.6</td>
<td>4.31</td>
<td>24.53</td>
<td>105.2</td>
<td>22.3</td>
</tr>
<tr>
<td>Problem 1B</td>
<td>55.65</td>
<td>1.1</td>
<td>48.96</td>
<td>54.75</td>
<td>8.1</td>
<td>49.9</td>
<td>1.97</td>
<td>24.53</td>
<td>48.29</td>
<td>10.3</td>
</tr>
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<td>1.82</td>
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<td>12.1</td>
<td>81.3</td>
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<td>23.5</td>
<td>142.5</td>
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<td>24.53</td>
<td>137.8</td>
<td>29.5</td>
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<td>48.96</td>
<td>119.2</td>
<td>18.2</td>
<td>108.6</td>
<td>4.31</td>
<td>24.53</td>
<td>105.2</td>
<td>22.3</td>
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</table>
### TABLE 6.3A SUMMARY OF RESULTS (NORMAL DISTRIBUTION)

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<thead>
<tr>
<th>Correlation between member resistances</th>
<th>H →</th>
<th>H → v</th>
<th>H → v v</th>
<th>H → v v v</th>
<th>H → v v v v</th>
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</thead>
<tbody>
<tr>
<td>( \rho_0 ) or ( \rho_0 )</td>
<td>PNET</td>
<td>MONTE CARLO</td>
<td>PNET</td>
<td>MONTE CARLO</td>
<td>PNET</td>
</tr>
<tr>
<td>( \rho_0 = 0.7 ) or 0.8</td>
<td>( \phi = 0.7 ) or 0.8</td>
<td>( \phi = 0.7 ) or 0.8</td>
<td>( \phi = 0.7 ) or 0.8</td>
<td>( \phi = 0.7 ) or 0.8</td>
<td></td>
</tr>
<tr>
<td>( \phi = 0.7 ) or 0.8</td>
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<td>( \phi = 0.7 ) or 0.8</td>
<td>( \phi = 0.7 ) or 0.8</td>
<td>( \phi = 0.7 ) or 0.8</td>
<td></td>
</tr>
<tr>
<td>( \phi = 0.7 ) or 0.8</td>
<td>( \phi = 0.7 ) or 0.8</td>
<td>( \phi = 0.7 ) or 0.8</td>
<td>( \phi = 0.7 ) or 0.8</td>
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<td>( \phi = 0.7 ) or 0.8</td>
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<td>( \phi = 0.7 ) or 0.8</td>
<td>( \phi = 0.7 ) or 0.8</td>
<td>( \phi = 0.7 ) or 0.8</td>
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</tr>
</tbody>
</table>

* Values Obtained from Graphical Results Reported by Scheuller & Grimmelt.  
† Average of Upper and Lower Bounds Used in place of Monte Carlo.
<table>
<thead>
<tr>
<th>( p_M )</th>
<th>( r )</th>
<th>( \rho_0 = 0.7 ) or 0.8</th>
<th>( \rho_0 = 0.7 ) or 0.8</th>
<th>( \rho_0 = 0.7 ) or 0.8</th>
<th>( \rho_0 = 0.7 ) or 0.8</th>
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<td>5.9x10^{-3}</td>
<td>1.6x10^{-2}</td>
</tr>
<tr>
<td>0.1</td>
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<td>7.6x10^{-3}</td>
<td>1.1x10^{-2}</td>
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</tr>
<tr>
<td>0.1</td>
<td>1</td>
<td>1.5x10^{-11}</td>
<td>5x10^{-3}</td>
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<tr>
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<td>2x10^{-12}</td>
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<td>4x10^{-7}</td>
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<td>7x10^{-7}</td>
<td>4x10^{-7}</td>
<td>3.4x10^{-6}</td>
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* Values Obtained from Graphical Results Reported by Scheuller & Grimmelt. \( \text{\&} \) Average of upper and Lower Bounds Used in place of Monte Carlo.
### TABLE 6.3C SUMMARY OF RESULTS EXTREME-VALUE DISTRIBUTION (WEIBULL M, GUMBEL L)

<table>
<thead>
<tr>
<th>Correlation between number resistances</th>
<th>$\rho_H$</th>
<th>$\rho_M$</th>
<th>PNET $\rho_o = 0.7$ or 0.8</th>
<th>MONTE CARLO $\rho_o = 0.7$ or 0.8</th>
<th>PNET $\rho_o = 0.7$ or 0.8</th>
<th>MONTE CARLO $\rho_o = 0.7$ or 0.8</th>
<th>PNET $\rho_o = 0.7$ or 0.8</th>
<th>MONTE CARLO $\rho_o = 0.7$ or 0.8</th>
<th>PNET $\rho_o = 0.7$ or 0.8</th>
<th>MONTE CARLO $\rho_o = 0.7$ or 0.8</th>
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</thead>
<tbody>
<tr>
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<td>.3 / .05</td>
<td>6.7 x 10^{-2}</td>
<td>7 x 10^{-2}</td>
<td>.86</td>
<td>.87</td>
<td>.27</td>
<td>.3</td>
<td>.46</td>
<td>.6</td>
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<tr>
<td></td>
<td>.3 / .1</td>
<td></td>
<td>.49</td>
<td>.5</td>
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<td>1</td>
<td>.92</td>
<td>.9</td>
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<td>1</td>
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<td>.1 / .1</td>
<td></td>
<td>2 x 10^{-6}</td>
<td>2 x 10^{-7}</td>
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<td>2 x 10^{-2}</td>
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<td>4 x 10^{-3}</td>
<td>5.2 x 10^{-3}</td>
<td>7 x 10^{-3}</td>
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<td>.8</td>
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<td>.9</td>
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<td>3.1 x 10^{-3}</td>
<td>4 x 10^{-3}</td>
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</tbody>
</table>

* Values Obtained from Graphical Results Reported by Scheuller & Grimmelt. 

** Average of Upper and Lower Bounds Used in place of Monte Carlo.
Fig. 5.1 Error Associated with Assumption of Perfect Correlation of Two Partially Correlated Failure Modes
Fig. 6.2 Error Associated with Assumption of Statistically Independent of Two Partially Correlated Failure Modes
Problem No.

1A Horizontal Load only

1B Both Horizontal and Vertical Loads

2A Horizontal Load only

2B Both Horizontal and Vertical Loads

3 Horizontal Load only

$h = 5.0 \text{ m}, \ell = 10.0 \text{ m}$ for all systems and load cases

Fig. 6.3 Structures Considered in Schueller & Grimmel (1981)
Fig. 6.4A Summary of PNET Results in Schueller & Grimmelt
Fig. 6.4B Summary of PNET Results in Schueller & Grimmelt
CHAPTER 7

SUMMARY AND CONCLUSIONS

7.1 Summary

An approximate method has been developed for the reliability analysis of ductile redundant frames and trusses. Frames are assumed to fail through the formation of plastic hinge mechanisms whereas trusses are assumed to collapse through the yielding of the members. The applied loads and resistance capacities of the elements are prescribed to be of normal distributions; however, non-normal distributions may be used with the same method. The problem of the identification of the significant failure modes is formulated in two ways; both formulations will result in an unconstrained nonlinear programming problem and the Hooke and Jeeves algorithm is employed as the solution technique. Once the major mechanisms are identified, their corresponding performance functions can be developed through the principle of virtual work, from which the individual mode collapse probabilities are calculated. The system collapse probability is then evaluated through the method of PNET.

With the PNET method, information on the correlations between different mechanisms are accounted for in the evaluation of the system collapse probability. Results of the proposed PNET method show consistently good agreement with corresponding Monte Carlo calculations for a wide variety of structural systems examined.

7.2 Concluding Remarks

On the basis of the results of the present study, the following conclusions may be observed:

1. The PNET method appears to be an effective tool for the reliability analysis of ductile redundant frames and trusses. The effects of correlations between mechanisms (particularly the major ones) are
properly included in the evaluation of the system collapse probability. Although the appropriate value of the demarcating correlation \( \rho_o \) may depend on the level of the system collapse probability, a value of 0.7 appears to be suitable for \( P_f \geq 10^{-3} \). Generally, a higher value of \( \rho_o \) would be required for smaller \( P_f \); e.g. for \( P_f < 10^{-3} \), a value of 0.8 (or even 0.9 for \( P_f \ll 10^{-4} \)) may be necessary. Also, \( \rho_o \) seems to be independent of the form of distributions.

2. There are invariably a large number of potential collapse mechanisms for practical structures; however, only a small number of such mechanisms are significant, and need to be considered in the evaluation of system reliability. Moreover, among the major mechanisms, only those that have low mutual correlations will contribute significantly to the system collapse probability.

7.2.1 Suggestions For Further Studies

1. Effective algorithms for identifying the major failure modes through the unconstrained optimization formulations need further investigation, especially when the number of variables becomes large (e.g. \( n > 20 \)). The algorithm selected should be efficient and at the same time can guarantee that all major mechanisms have been identified. The problem of identification of the significant failure modes through mathematical programming need not be restricted to linear constraint functions; e.g. the effect of axial forces on the moment capacities of potential plastic hinges and the possibility of buckling may be included in the formulation, although the resulting mathematical problem may be more complicated. Since it is not unusual for a structure of practical complexity to have a large number of variables (e.g. \( n > 50 \)), it is unlikely that presently available algorithms will be able to find a local minimum efficiently. It is suggested that the approach of systematic subdivision of a given structure into
substructures be investigated. The significant failure modes of each substructure would be identified individually and the failure mode probability combined according to the PNET method to estimate the collapse probability of the system.

2. The appropriate value of the demarcrating correlation \( \rho_o \) for systems with small collapse probabilities, \( P_f < 10^{-4} \) should be investigated more thoroughly. For such cases, the results may be validated using the average (e.g. geometric mean) value of the narrow bounds in place of Monte Carlo calculations, as the latter would be costly.

3. The method of analysis can be easily extended to cases when there are more than one dominant (and mutually exclusive) loading conditions. In this case, one can apply the Theorem of Total Probability to express the system collapse probability as:

\[
P_f = \sum_{i=1}^{n} P_f^L_i P(L_i)
\]  \hspace{1cm} (7.1)

where \( P_f^L_i \) is the system collapse probability under the \( i \)th loading condition, and \( P(L_i) \) is the probability that the \( i \)th loading condition will occur, and \( n \) is the number of loading conditions.

4. The application of the PNET method for the reliability analysis of brittle redundant structural systems needs to be studied. The identification of the major failure modes of a brittle system will be more difficult than in ductile systems because the failure modes will be dependent on the sequence in which the components may fail.
REFERENCES


31. Schueller, G.I., and Grimmelt, M., "Benchmark Study On Methods to Determine Collapse Probabilities of Redundant Structures", Techn. University of Munchen; (To be published.)


APPENDIX A

SYSTEMATIC GENERATION OF COMPATIBILITY EQUATIONS GOVERNING
THE ANGLES OF ROTATIONS OF POTENTIAL PLASTIC HINGES

Following Fenves and Gonzalez-Carlo (1971), the geometrical
configuration of a frame structure is treated as a flow network. A member
is represented as an oriented branch going from its positive (or A) end to
the negative (or B) end as shown below.

Positive (A) end

Negative (B) end

Fig. A.1 Network Representation of a Member

Using network terminology, the set of compatibility requirements
governing the angles of rotations of potential plastic hinges \( \{ \mathbf{X}_1 \} \), is
written in matrix notation as follows:

\[
\mathbf{C} \mathbf{t} \wedge \mathbf{t} \Delta \{ \mathbf{X}_1 \} = \{ 0 \}
\] (A.1)

where:

\( \mathbf{C} \) = the branch-circuit matrix. The typical submatrix \( \mathbf{C}_{ir} \) of \( \mathbf{C} \) is
\( (T_i^T H_{is}', -T_i^T H_{is}', 0) \) depending on whether the member \( i \) is
(positively, negatively, not) included in the circuit defined by the
rth link. In the definition of \( H_{is}' \), \( s \) is the joint incident on the
negative end of link \( r \).

\( T_i \) = a square rotation matrix which transforms the force vector at the
negative end of member \( i \) from local to global coordinates.

\( H_{is}' \) = a translation matrix representing the effect of the load vector \( \mathbf{P}_s' \) at
joint \( s \) on the force vector at the negative end of member \( i \), both in
global coordinates.
Γ = a diagonal matrix which transforms the vector of negative end member forces \( R_F \) to the vector of forces at both ends of the member \( R_F \).

\( \Delta = "extractor" \) matrix which selects only the force components that are of interest \( R_F \).

Hence, in matrix form, \( R_F \) can be written as:

\[
\bar{R}_F = \Delta R_F = \Delta^\top R_F \quad (A.2)
\]

For a typical member \( i \) of a plane frame, \( R_F \) is:

\[
R_{F_i} = \begin{bmatrix}
p_{xA_i} \\
p_{yA_i} \\
m_{zA_i} \\
p_{xB_i} \\
p_{yB_i} \\
m_{zB_i}
\end{bmatrix}, \quad \bar{R}_{F_i} = \begin{bmatrix}
m_{zA_i} \\
m_{zB_i}
\end{bmatrix} \quad (A.3)
\]

Γ has diagonal submatrices \( \Gamma_i \) and

\[
\Gamma_i = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-\text{Ly} & \text{Lx} & 1 \\
0 & 0 & 0
\end{bmatrix} \quad (A.4)
\]
For structure consists of only straight members, \( L_y = 0 \) and \( L_x = L \), the length of the member; therefore,

\[
\mathbf{T}_i = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

(A.5)

\( \Delta \) is a diagonal matrix of submatrices \( \Delta_i \)

\[
\Delta_i = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

(A.6)

Also

\[
\mathbf{T}_i = \begin{bmatrix}
\cos \theta_i & \sin \theta_i & 0 \\
-\sin \theta_i & \cos \theta_i & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

(A.7)

\[
\mathbf{H}_{is} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-L_{ys} & L_{xs} & 0 \\
\end{bmatrix}
\]

(A.8)

where

\( L_{ys} \) = (y coordinate of s) - (y coordinate of the negative end of member i)
\( L_{xs} \) = (x coordinate of s) - (x coordinate of the negative end of member i)

For clarity, the method of systematic generation of compatibility constraints discussed above may be illustrated through an example.
Illustrative Example

Consider the one-story one-bay rectangular portal frame discussed previously in Sect. 3.3.2 (see Fig. 3.1), with the same load and resistance statistics. Arbitrarily label the node number, branch number and the definition of circuit 1 as shown in Fig. A.2. The coordinates of the nodes are shown in Fig. 3.3 and the components of the variables \{X\} are shown in Fig. 3.4.

Fig. A.2 Numbering of Nodes, Branches and Circuit

From Eq. A.6, \( \Delta \) is,

\[
\Delta = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Using Eq. A.5, matrix \( \Gamma \) is written as follows:
and therefore,

\[
\Delta \mathbf{v} = \begin{bmatrix}
0 & 15 & 1 \\
0 & 0 & 1 \\
0 & 10 & 1 \\
0 & 0 & 1 \\
0 & 15 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\mathbf{v} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 15 & 1 \\
--- & I \\
0 \\
--- & I \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 15 & 1 \\
--- & I
\end{bmatrix}
\]

\[
\mathbf{v} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 10 & 1 \\
--- & I \\
0 \\
--- & I \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 15 & 1 \\
--- & I
\end{bmatrix}
\]

24 x 12

8 x 12
According to the labels of Fig. A.2, node 5 is the node incident on the negative end of circuit 1, therefore $s = 5$ and 

$$H_{i5} = H'_{i5}$$

For member 1, joint 2 is the negative end of the member and

$$-T^t_1 H'_{15} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 15 & 20 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 15 & 20 & 1 \end{bmatrix}$$

Similarly, for members 2, 3 and 4

$$T^t_2 H'_{25} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 15 & 10 & 1 \end{bmatrix} ; \quad T^t_3 H'_{35} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 15 & 0 & 1 \end{bmatrix} ; \quad T^t_4 H'_{45} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$c^t = \begin{bmatrix} 0 & -1 & 15 & 1 & 0 & 15 & 1 & 0 & 15 & 0 & 1 & 0 \\ 1 & 0 & 20 & 0 & 1 & 10 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}_{3 \times 12}$$

and

$$c^t \cdot r^t \Delta^t = \begin{bmatrix} 0 & 15 & 15 & 15 & 15 & 15 & 0 \\ 20 & 20 & 20 & 10 & 10 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}_{3 \times 8}$$

The set of compatibility equations governing the hinge rotations for this example, therefore, are:
The above set of compatibility equations is equivalent to those obtained by inspection in Sect. 3.5.1. However, the method discussed herein can be easily programmed for computer calculations and thus the generation of the compatibility equations become automatic.

Observe that the compatibility constraints are all linear equations; these may be "substituted" into the objective function to reduce the number of variables (from 10 to 4 in this example) and the problem becomes essentially an unconstrained minimization problem.

In general the number of independent variables for a plane frame structure is always equal to the number of potential plastic hinges minus the degree of redundancy of the structure.
For many structural systems with non-standard configurations, it is necessary to have a procedure to generate all the "elementary" mechanisms systematically. The procedure discussed below (Watwood, 1979) requires only information on the joints and members of a structure.

B.1 Restriant Equations

Consider the planar beam element as shown in Fig. B.1.

![Diagram of a planar beam element](image)

**Fig. B.1 Definition of Local Generalized Coordinates**

A coordinate transformation is introduced on the local generalized coordinates \( \{S\} \), Fig. B.1, in order to separate the rigid body motion from the member deformation. There are three deformation coordinates \( S_1', S_2', S_3' \) and three rigid body coordinates \( S_4', S_5', S_6' \); together they form the transformed coordinate vector \( \{S'\} \), and can be chosen as:
### Components

<table>
<thead>
<tr>
<th>Components</th>
<th>Physical Meaning</th>
<th>Vector {S}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_1' )</td>
<td>End 2 axial displacement</td>
<td>( (0 \ 0 \ 0 \ 1 \ 0 \ 0) )</td>
</tr>
<tr>
<td>( S_2' )</td>
<td>End 1 rotation</td>
<td>( (0 \ 0 \ 1 \ 0 \ 0) )</td>
</tr>
<tr>
<td>( S_3' )</td>
<td>End 2 rotation</td>
<td>( (0 \ 0 \ 0 \ 0 \ 1) )</td>
</tr>
<tr>
<td>( S_4' )</td>
<td>Rigid-body translation in x direction</td>
<td>( (1 \ 0 \ 0 \ 1 \ 0) )</td>
</tr>
<tr>
<td>( S_5' )</td>
<td>Rigid-body translation in y direction</td>
<td>( (0 \ 1 \ 0 \ 0 \ 1) )</td>
</tr>
<tr>
<td>( S_6' )</td>
<td>Rigid-body rotation about end 1</td>
<td>( (0 \ 0 \ 1 \ 0 \ -L) )</td>
</tr>
</tbody>
</table>

Thus the full transformation is assembled as,

\[
\{S\} = [T] \{S'\} \tag{B.1}
\]

where,

\[
\{S\} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -L \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad \{S'\} = \begin{bmatrix} S_1' \\ S_2' \\ S_3' \\ S_4' \\ S_5' \\ S_6' \end{bmatrix} \tag{B.2}
\]

The inverse relation is,

\[
\{S'\} = [T]^{-1} \{S\} \tag{B.3}
\]

in which
For any mechanism, all members except those containing hinges or axial yielding must move as rigid bodies. This implies that the deformation coordinates $S_1$, $S_2$, and $S_3$ for each member must remain zero during the mechanism. To enforce those constraints formally, a constraint matrix $[C]_{3\times6m}$ composed of diagonal submatrices $[C_j]$ is introduced. A typical diagonal submatrix $[C_j]$ for the jth member is made up of the first three rows of $[T]^{-1}$. Defining $\{S_d\}'$ as:

$$\begin{bmatrix}
S_1' \\
S_2' \\
S_3' \\
\vdots
\end{bmatrix}$$

- first element

$$\begin{bmatrix}
S_1' \\
S_2' \\
S_3' \\
\vdots
\end{bmatrix}$$

- second element

and $\{S\}$ is redefined as:
Then

$$[C]{S} = {S'}$$  \hspace{1cm} (B.7)

The components of \(\{S\}\) are not independent, but are restrained to move such that the compatibility of the assembled structure is preserved. This can be enforced by introducing the usual structural compatibility matrix \([A]\); i.e.

$$\{S^*\} = [A] \{r\}$$  \hspace{1cm} (B.8)

where the components of \(\{S^*\}\) are the generalized coordinates of all the elements expressed in the global coordinate system; \(\{r\}\) is the column matrix of the external degrees of freedom (in general, 3 per joint unless constraints are imposed) expressed in the global coordinate system, and \([A]\) is the compatibility matrix, which depends on the numbering of the elements and the external degrees of freedom.

The relationship between \(\{S\}\) and \(\{S^*\}\) can be expressed as:

$$\{S\} = [Q] \{S^*\}$$  \hspace{1cm} (B.9)

The matrix \([Q]\) for the structure consists of diagonal submatrices \([q^j]\), where the superscript \(j\) stands for the \(j\)th member, and is given by:
in which \( \lambda_1 \) and \( \lambda_2 \) are the direction cosines of the local x-axis and y-axis relative to the global coordinate system. Combining Eqs. B.7 through B.9 leads to:

\[
[C] [Q] [A] \{r\} = [C] \{r\} = \{S_d\}
\]  \hspace{1cm} (B.11)

To find a mechanism, one must find a solution to Eq. B.11 such that \( \{S_d\} = \{0\} \). However, unless the given structure is already a mechanism, no such solution exists. At this point, one introduces releases that will form a mechanism. For planar structures, three releases per element will be inserted. The required three releases may be a hinge at each end of the element and an axial release at end 2. Insertion of a release is equivalent to adding an external degree of freedom. Mathematically, this is done with replacing a row of the matrix \([Q][A]\) with zeroes where they correspond to \( S_3, S_4 \), and \( S_6 \) for each element. For each row that is "zeroed out", a zero column is added to \([Q][A]\) except that a "1" is placed in the position which correspond to the row that was "zeroed out" earlier. Of course, \( \{r\} \) must be expanded by one component for each column added to \([Q][A]\). This modified form of \([Q][A]\) and \( \{r\} \) is substituted into Eq. B.11, yielding

\[
[C_2] \{r_t\} = \{S_d'\}
\]  \hspace{1cm} (B.12)

in which \( [C_2] = [C] \) multiplied by the modified form of \([Q][A]\), and \( \{r_t\} \) is the matrix \( \{r\} \) augmented by the member releases. Then, specifying \( \{S_d\} = \{0\} \) should yield the possibility of solutions; i.e.

\[
[C_2] \{r_t\} = \{0\}
\]  \hspace{1cm} (B.13)
The order of \([C_2]\) is \(3mN_t\), in which \(m\) is the number of elements (members) and \(N_t\) is the number of external degrees of freedom plus 3m.

**B.2 Solution of Equations**

Consider the case in which \([C_2]\) is of rank 3m. For this situation, there exist a nonsingular matrix \([B]\) of order \(N_t\) by \(N_t\) such that:

\[ [C_2] [B] = [I 0] \]  \hspace{1cm} (B.14)

The identity matrix \([I]\) is of order \(3m \times 3m\) and the null matrix \([0]\) is of order \(3m \times (N_t - 3m)\). The last \((N_t - 3m)\) columns of \([B]\) is the desired solution since the column is orthogonal to all rows of \([C_2]\).

To generate \([B]\), one performs a Gaussian reduction using column operations. The first row of \([C_2]\) is scanned to find the component with the largest absolute value. This column is designated as the pivot column, and after normalization, it is transferred to the first column. It is then used to reduce all other non-zero entries in the first row. The second row of \([C_2]\) is then scanned from the second column on, and the procedure is repeated until all rows have been exhausted. Identical operations are performed on the identity matrix \([I]\) (\(N_t\) by \(N_t\)) and generate the matrix \([B]\).

In the event that \([C_2]\) is of rank less than 3m (say, \(=3m-1\)), a pivot column will fail on one row as all row components are zero. One may simply move down to the next row without shifting over one column to begin the search and continue as usual. This will result in an addition of a column of zeroes on the right hand side of Eq. B.14, and an additional column of \([B]\) is orthogonal to all rows of \([C_2]\); this means that there is an additional solution.

The original external degrees of freedom \([r]\) is a subset of the extended degrees of freedom \([r_t]\). Therefore, the corresponding rows of \([B]\) can be extracted to express the set of "elementary" mechanisms in terms of the original external degrees of freedom only.