STOCHASTIC SEISMIC PERFORMANCE
EVALUATION OF BUILDINGS

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A method is presented for determining the probabilities that a structure will sustain various levels of damage or become unsafe due to earthquake loading during its lifetime. Uncertainties in the dynamic analysis associated with both the loading and the prediction of the structural response are considered.

The method is based on a nonlinear random vibration analysis and an analytical technique for evaluating the sensitivity of the response to various structural and load parameters. Recently obtained data are used to formulate the earthquake random process model. The structural modeling is grounded on a shear beam idealization, with equivalent story parameters systematically evaluated in order to reasonably represent actual structural behavior, and an analytical hysteretic degrading model is used to represent the structural restoring force characteristics. A seismic hazard model is used to evaluate the probabilities associated with all significant ground motion intensities, and various damage and safety criteria are examined for the purpose of evaluating the lifetime damage probabilities.
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CHAPTER 1

INTRODUCTION AND GENERAL PROCEDURE

1.1 Objective and Scope

Ensuring the safety of a structure while maintaining economy is the primary technical objective in structural design. To meet this objective it is necessary to be able to assess the safety of a design as well as estimate the costs of damage and failure that may be incurred over the life of a structure. Ideally, this would require precise knowledge of the lifetime loadings on the structure and knowledge of how the structure behaves under particular loadings. For buildings located in seismically active regions, however, the lifetime loadings cannot be ascertained with certainty and the structural response under a given seismic loading (especially in the inelastic range) is a complex phenomenon and difficult to predict. Although the state-of-the-art of earthquake engineering has progressed rapidly over the past few decades, there remain uncertainties associated with the analysis of multistory structures, in addition to inherent uncertainties and variabilities in the structural properties and loadings. Thus, it is difficult to speak of safety or to estimate the lifetime damage costs in deterministic terms.

For a realistic assessment of the adequacy of a structure to resist earthquake loadings, it is necessary not only to analyze the structure, but also to quantify all uncertainties associated with the analysis or performance predictions. Safety may then be defined in terms of the
probability that the building response will not violate a set of performance criteria during its lifetime. Decisions concerning the need for structural improvements may also be based on the probabilities that the structure will sustain various levels of damage over its lifetime.

The objective of this study is to present and demonstrate the use of an analytical method for evaluating the required probabilities. The method consists of a nonlinear random vibration solution for the structural response and the assessment of the uncertainties associated with modeling the structural system and ground excitation, as well as the method of analysis.

1.2 Outline of Approach

The general procedure for calculating the desired probabilities may be outlined as follows:

1. Evaluate the seismic hazard (i.e., the probabilities associated with all significant ground motion intensities) at a site for the duration of the life of a structure.
2. Construct a ground motion model in which the loading is modeled as a random process containing the variability of the ensemble of possible ground motions.
3. Mathematically model a structure and the structural material (i.e., formulate the hysteretic restoring force relations and the equations of motion), and assess the error of the model and the uncertainties in the model parameters.
4. Perform the random vibration analysis to determine the response statistics based on the structural and ground motion models for various load intensities, and evaluate the effect of modeling and model parameter uncertainties on these results.
5. Evaluate the final response statistics, accounting for all response uncertainties.
6. Determine a damage model and calculate the probabilities associated with the various damage levels based on the final response statistics, for various load intensities.
7. Combine the damage probabilities with all possible hazard levels to obtain the lifetime damage probabilities.

1.3 Notation

\( A, n, \alpha, \beta, \gamma \) = parameters controlling the hysteresis loop shape and yielding level

\( B \) = random process excitation matrix

\( c \) = coefficient of viscous damping

\( C, K \) = equivalent linear coefficients for hysteresis model

\( E[ \cdot ] \) = expected value

\( F_G \) = scale factor for computing RMS of Kanai-Tajimi power spectral density function

\( G \) = structural system coefficient matrix

\( k \) = story stiffness

\( m \) = story mass

\( N \) = Bayesian correction variable for prediction errors

\( p \) = general ground motion model or structural system parameter

\( q \) = total restoring force

\( s_o \) = white noise power spectral density

\( S \) = response covariance matrix

\( t_d \) = earthquake strong motion duration

\( \text{Var}[ \cdot ] \) = variance

\( z \) = hysteretic displacement (the hysteretic restoring force is given by \( k z \))

\( \delta \) = coefficient of variation measuring basic variability

\( \delta_A, \delta_\eta, \delta_\nu, \eta, \nu \) = parameters controlling the hysteresis loop deterioration

\( \Delta \) = coefficient of variation measuring prediction error

\( \varepsilon \) = hysteretic energy dissipated

\( \lambda_o \) = zero crossing rate

\( \rho \) = correlation coefficient

\( \sigma \) = standard deviation
$\omega, \beta' = \text{Kanai-Tajimi filter parameters}$

$\Omega = \text{total coefficient of variation equal to } \sqrt{\sigma^2 + \Delta^2}$
2.1 Seismic Hazard

2.1.1 Introduction and Review of Related Work

In order to assess the seismic adequacy of a structure, for a specific lifetime, it is necessary to determine the seismic hazard at the particular site. This may be done through the use of seismic risk maps, in which isoseismal contours for expected maximum earthquake motions are given. This approach is based on estimates of the maximum ground shaking experienced during the recorded historical period, but does not account for the occurrence frequency of such motions. Thus, the probability of exceeding a specified design ground motion may vary from region to region. More recently, contour maps (ATC, 1978) were introduced based on the work of Algermisson and Perkins (1976), such that the indicated design ground motions have relatively consistent exceedance probabilities. This information, however, is still incomplete because the occurrence frequencies of different motion intensities are not indicated. Therefore, the actual risk implied in a design based on this procedure is unknown. For this reason, methods for evaluating the exceedance probabilities associated with all significant ground motion intensities at a building site have been developed (Cornell, 1968; Ang, 1974; Der Kiureghian and Ang, 1977b).
The model presented by Der Kiureghian and Ang (1977b) is based on the assumption that an earthquake originates as an intermittent series of fault ruptures in the earth's crust, and the intensity of motion at a site is mainly contributed by the segment of the ruptured fault closest to the site. This is in contrast to the "point source" model of Cornell (1968) in which the total energy released during an earthquake is assumed to radiate from the focus.

The fault-rupture model of Der Kiureghian and Ang (1977b) will be used as it is believed that this approach is more appropriate for tectonic earthquakes. This is especially true for large earthquakes in which the total energy released may be distributed along a rupture zone of several hundred kilometers. The procedure to evaluate the probabilities of exceeding all significant ground shaking intensities, measured by one or more ground motion variables (e.g., maximum acceleration, velocity, etc.) at a particular site over a specific time duration is summarized in the following.

2.1.2 Fault-Rupture Model

Assuming an earthquake has occurred in the region around a site, the maximum intensity expected at the site will depend on the magnitude of the earthquake and the closest distance between the site and the fault rupture. Thus, the intensity may be represented as a function of two variables, i.e.,

\[ y = g(m,r) \]  \hspace{1cm} (2.1)

where \( r \) is the relevant distance, and \( m \) is the magnitude (e.g., in Richter scale). The magnitude of a given earthquake is highly variable, and thus may be described by a random variable, with a truncated exponential probability distribution,

\[ F_M(m) = \frac{1 - \exp[-\beta(m - m_0)]}{1 - \exp[-\beta(m_u - m_0)]} \quad m_0 < m \leq m_u \]  \hspace{1cm} (2.2)
in which \( m \) denotes the smallest magnitude of concern to engineers (e.g., \( m = 4 \)), \( m \) denotes the upper bound magnitude potentially possible in the region, and \( \beta \) is the the slope of the "magnitude recurrence" curve. The truncated exponential distribution follows directly from Richter's well known law of magnitudes.

To evaluate the length of the rupture zone for an earthquake of magnitude \( m \) (which determines the distance between the site and the nearest point of rupture, represented by the variable \( r \)), the relation

\[ \lambda = \exp(am - b) \]  

(2.3)

suggested by several authors (e.g., Krinitzsky, 1974), is used. The parameters \( a \) and \( b \) are constants appropriate for a given region. Additionally, the potential sources in the region surrounding the site are idealized as known faults, faults with known orientation only, or as completely unknown fault systems, depending on the information available.

From the preceding information the probability that the intensity at the site will exceed some value \( y \), given that an earthquake of magnitude bounded by \( m \) and \( m \) occurs in source \( i \), that is, \( P(Y > y | E_i) \), can be evaluated. Finally, the future occurrence of earthquakes in the region is modeled probabilistically as a homogeneous Poisson process. On these bases, Der Kiureghian and Ang (1977b) give the annual probability of exceeding a given ground shaking intensity, \( y \), as

\[ P(Y_1 > y) = 1 - \exp\left\{- \sum_{i=1}^{n} P(Y > y | E_i \nu_i) \right\} \]  

(2.4a)

\[ \approx \sum_{i=1}^{n} P(Y > y | E_i \nu_i) \]  

(2.4b)

where \( \nu_i \) is the mean occurrence rate of \( E_i \). The approximation of Eq. 2.4b is valid for small exceedance probabilities.
Assuming that the maximum annual intensities at the site between years are statistically independent, the corresponding exceedance probability in $T$ years becomes

$$P(Y_T > y) = 1 - [1 - P(Y_1 > y)]^T$$ (2.5)

**Uncertainties** — The probabilities obtained with the above model will be dependent on the physical relations assumed in the model (e.g., slip length - magnitude relation, intensity attenuation equation) as well as the values of the parameters (e.g., $m$, $m_0$ and $\beta$). These relations and parameter values may contain significant uncertainty which may be systematically quantified on the basis of Eqs. 2.1 and 2.3. In general, the uncertainty underlying the attenuation equation will tend to dominate.

### 2.2 Ground Motion Models

The seismic hazard model described above may be used to evaluate the probability associated with a given maximum ground motion intensity. The actual ground shaking, however, is generated by processes that are essentially random (Clough and Penzien, 1975). Thus, the detailed characteristics of the ground motion (and therefore its damage-producing potential) can vary widely from one event to another. In fact, it has been demonstrated (Biggs, et al., 1979) that two different earthquake time histories, with the same intensity, can give rise to very different structural responses. It is clear then, that knowledge of the occurrence probability of an earthquake characterized by a single intensity parameter or a set of parameters, is not sufficient to estimate structural damage probabilities. Ground motion modeling is necessary such that the random nature of the excitation and the randomness of the resulting structural response is captured. To this end, various approaches have been developed.

The best known and most widely used approach is that of the response spectrum method, in which the ground motion is characterized by
the maximum response of a single degree of freedom oscillator of varying natural frequency and specified damping ratio. Response spectra may be developed based on the normalized response of the oscillator subjected to a number of earthquakes, thereby modeling the ground motion based on the average of past recorded earthquakes (Housner, 1959). For design, response spectra have been constructed for peak values of the ground acceleration, velocity and displacement (Newmark, Blume and Kapur, 1973). More recently, response spectra have been developed in which the probability of exceeding given spectral ordinates is consistent for the entire range of structural frequencies (e.g., Der Kiureghian and Ang, 1977a; Anderson and Trifunac, 1978).

Although modeling ground motions using response spectra provides a definitive loading characterization, it does not allow for a detailed analysis of the structural response. Thus, response spectra are generally applicable for preliminary design (Clough and Penzien, 1975). A more detailed (and more costly) approach to modeling ground motions is to develop a set of loading time histories based on recorded earthquake accelerograms, adjusted for amplitude, frequency content, and duration. This approach, however, does not systematically consider the random nature of earthquake time histories.

An alternative approach is to assume that the ground motion is a random sequence of impulses generated at some distance and propagated to the site through the basement rock structure. This allows description of the loading as a random process; in fact, it has been shown (Amin and Ang, 1968) that an earthquake of a particular intensity may be modeled as a zero mean filtered Gaussian shot noise random process. That is, the acceleration at any instant is a zero mean Gaussian random variable whose variance is dependent on the particular point in time.

This approach to ground motion modeling takes the random nature of earthquake time histories into consideration and allows an efficient determination of the response statistics using random vibration theory (e.g., Lin, 1976). Recently, Wen (1976, 1980) and Baber and Wen (1980), showed that random vibration response statistics may also be obtained for degrading, hysteretic structures. For these reasons, the ground motion is modeled in this study as a random process.
2.2.1 Random Process Representation of Earthquake Motions

As indicated above, earthquake ground motions may be modeled as zero mean filtered Gaussian shot noise random processes. Since Gaussian random variables are completely described by their first and second moments, the ground motion model can be represented or specified by the autocorrelation function of the random process. For the unfiltered shot noise process, the autocorrelation is given as

\[ R_{xx}(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)] \]

\[ = I(t_1)\delta(t_2-t_1) \]

where \( I(t) \) governs the intensity variation of the process in time, and \( \delta \) is the Dirac delta function.

The above autocorrelation function implies a process with no statistical correlation between the variables at different time instants (and therefore has an infinitely wide mean square frequency content) and an infinite variance. Although it is possible to obtain meaningful results with such a representation (Bycroft, 1960), it is more realistic to use the autocorrelation function (or power spectral density function) of a filtered process.

Kanai (1957) and Tajimi (1960) suggested that a second order linear damped oscillator could serve as an appropriate filter. It remains then to examine actual earthquake records to determine the proper values of the filter fundamental frequency, \( \omega \), and damping ratio, \( \beta \), such that the resulting filtered motion has the statistically correct frequency content and correlation properties. In general, the parameters \( \omega \) and \( \beta \) will be affected by the epicentral distance, earthquake magnitude, and the ground layer rigidity.

The filter parameters, \( \omega \) and \( \beta \), may be evaluated on the basis of fitting an empirically obtained autocorrelation function, or equivalently fitting the corresponding empirical power spectral density. The latter approach is followed herein.
The power spectral density function of a Kanai-Tajimi filtered stationary shot noise (the Kanai-Tajimi spectrum) takes the form

\[ S_{aa}(\omega) = s_0 \frac{1 + 4 \frac{\beta^2}{\omega g} \left(\frac{\omega}{\omega_g}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_g}\right)^2\right]^2 + 4 \frac{\beta^2}{\omega g} \left(\frac{\omega}{\omega_g}\right)^2} \]  

(2.7)

where \( s_0 \) is the constant power spectral density of the stationary shot noise. Assuming that the frequency content of earthquakes does not vary significantly with time, this spectrum may be fitted to the average squared Fourier amplitude spectra obtained from normalized sample records. Housner and Jennings (1964) first used this approach and proposed values of \( \omega = 15.6 \) rad/sec and \( \beta = 0.64 \) for firm ground condition, based on the shape of an average pseudo-velocity response spectra (which is shown to be related to the Fourier amplitude spectra), for eight accelerograms (two components each of four earthquakes).

Strictly speaking, the frequency content of earthquakes varies with time, and the power spectral representation should, therefore, be in terms of evolutionary spectra (Priestley, 1967). Using this approach, Liu (1970) showed that the frequency content may be significantly different during the early and middle portions of the record. In a recent study by Moayyad and Mohraz (1982), however, it was found that the frequency content of earthquake accelerograms is approximately constant during the strong motion phase. Power spectra were then obtained on the basis of the Fourier amplitude spectra for the strong motion phase. The results are summarized in the form of spectral plots for soft, intermediate, and hard grounds, as shown in Fig. 2.1. The soft ground spectrum was based on the Fourier analysis of 161 records, the intermediate ground spectrum on 60 records and the hard ground spectrum on 26 records. The ordinates have been normalized so that the curves enclose the same area as the Kanai-Tajimi spectrum with the Housner and Jennings (1964) values, \( \omega = 15.6, \beta = 0.64 \) and \( s = 1.0 \) (the "Housner and Jennings spectrum"). This spectrum is also shown for comparison.
Fig. 2.1 Earthquake Power Spectra (after Moayyad and Mohraz, 1982)
A least squares procedure was used to evaluate the appropriate Kanai-Tajimi parameters for each of the three ground conditions. The results are listed in Table 2.1 and the curves obtained are plotted in Fig. 2.2 along with the Moayyad and Mohraz empirical curves.

Observe that the Kanai-Tajimi spectrum at $\omega=0$ is $s_0$, whereas in the low frequency region the ordinates of the empirical curves tend to zero. Also, in the high frequency region the Kanai-Tajimi spectrum approaches zero more slowly than the empirical curves. This is due to the large values of $\beta$ obtained from the least squares fit of the experimental spectra in the central frequency range. These anomalies should not affect the response of most structures, whose dominant frequencies are within the central range. However, in order to ensure that the total area of the Kanai-Tajimi Spectrum (the mean square value of the process), which defines the intensity of the process, is not unduly amplified by the high frequencies, "scale factors" may be applied.

The mean square, $\sigma_a^2$, of the process characterized by the two-sided Kanai-Tajimi power spectral density function may, therefore, be evaluated as

$$\sigma_a^2 = F_G \int_{-\infty}^{\infty} S_{aa}(\omega) d\omega = F_G \frac{s \omega \pi}{2\beta_g}(1 + 4 \beta_g^2)$$  \hspace{1cm} (2.8)

where $F_G$ is the scale factor for ground condition $G$. Based on the area

<table>
<thead>
<tr>
<th>Ground Condition</th>
<th>$\omega_g$</th>
<th>$\beta_g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soft</td>
<td>10.9</td>
<td>0.96</td>
</tr>
<tr>
<td>Intermediate</td>
<td>16.5</td>
<td>0.80</td>
</tr>
<tr>
<td>Hard</td>
<td>16.9</td>
<td>0.94</td>
</tr>
</tbody>
</table>
Fig. 2.2 Kanai-Tajimi Spectra (---) and Empirical Power Spectra (-----)
under the empirical spectra of Moayyad and Mohraz, the required values of \( F \) were evaluated. The results and the corresponding mean square values of \( \sigma^2 \) as a function of \( s \) are summarized in Table 2.2.

Lai (1982) used a different procedure to investigate the spectral characteristics of earthquake ground motions. Fourier amplitude spectra were calculated for the two horizontal components of 70 earthquake records and the method of spectral moments (Binder, 1978) was used to determine the Kanai-Tajimi parameters for each individual record. This work provides a basis for evaluating the degree of variability in the Kanai-Tajimi parameters between specific ground motion events. The coefficients of variation of \( \omega \) and \( \beta \) obtained by Lai (1982) for two ground conditions, classified as "rock" and "soil", are summarized in Table 2.3. For the purposes of the present study, the "soil" site in Lai (1982) will be assumed to correspond to the soft and intermediate ground conditions of Moayyad and Mohraz (1982), whereas the "rock" site will be assumed to correspond to the hard ground condition.

**Nonstationarity** — It is well known that the frequency content and intensity of earthquakes vary with time and as such are really nonstationary processes. Due to the limited results available for evolutionary spectra, however, the Kanai-Tajimi parameters presented for the strong-motion phase will be assumed to be valid for the entire loading history.

<table>
<thead>
<tr>
<th>Ground Condition (G)</th>
<th>Scale Factor ( (F_G) )</th>
<th>Mean Square ( \sigma^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soft</td>
<td>0.81</td>
<td>67.7 ( s_o )</td>
</tr>
<tr>
<td>Intermediate</td>
<td>0.83</td>
<td>95.7 ( s_o )</td>
</tr>
<tr>
<td>Hard</td>
<td>0.79</td>
<td>101.2 ( s_o )</td>
</tr>
</tbody>
</table>
Table 2.3. Coefficients of Variation for $\omega_g$ and $\beta_g$

<table>
<thead>
<tr>
<th>Ground Condition</th>
<th>c.o.v. [$\omega_g$]</th>
<th>c.o.v. [$\beta_g$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soil</td>
<td>0.425</td>
<td>0.426</td>
</tr>
<tr>
<td>Rock</td>
<td>0.398</td>
<td>0.391</td>
</tr>
</tbody>
</table>

Nonstationarity of the loading intensity (measured by $s$ or equivalently $\sigma^2$) varying in time is accounted for through the use of a semi-deterministic temporal multiplier. That is, the temporal variation of the root-mean-square or mean square, of the process is governed by a specific function of time whose parameters are random. A function intended for modeling the variation of the mean square acceleration as a function of time was proposed by Amin and Ang (1968), given by

$$\psi(t) = \begin{cases} 
(t/t_1)^2 & ; 0 \leq t \leq t_1 \\
1 & ; t_1 \leq t \leq t_2 \\
-\frac{1}{c}(t - t_2) & ; t_2 < t 
\end{cases}$$

(2.9)

A plot of the function is shown in Fig. 2.3.

Based on two components each of four earthquake records (the same set used by Housner, 1959, 1964), Amin and Ang (1968) recommended the mean parameter values, $t = 1.5$ sec., $t = 15$ sec., and $c = 0.18$ sec.~$^{-1}$.

Rather than introduce undue complexity at this time, the envelope parameters $t_1$ and $c$ will be assumed to be deterministic. The statistics of $t_2$ may then be defined in terms of the statistics of the strong motion duration (see Sect. 2.2.2), that is, the mean of $t_2$ may be taken as the mean duration plus the fixed value of $t_1$, whereas the coefficient of variation of $t_2$ is the same as that of the duration.
2.2.2 Duration of Strong Motion

Recently a number of studies have been carried out to define and estimate the duration of strong motion earthquakes (Bolt, 1974; Trifunac and Brady, 1975; McCann and Shah, 1979; Vanmarcke and Lai, 1980; Moayyad and Mohraz, 1982). Modeling the temporal variation of the earthquake as described in Fig. 2.3, requires specification of the mean and standard deviation of the duration of the strong motion phase, i.e., $t_1 - t$. For this purpose, the definitions proposed by Vanmarcke and Lai (1980) and Moayyad and Mohraz (1982) appear to be the most appropriate.

Based on the results of Moayyad and Mohraz (1982), the mean and coefficient of variation of the strong motion duration were evaluated for three soil conditions, as summarized in Table 2.4. Vanmarcke and Lai (1980) evaluated the strong motion durations for ground conditions classified as either "rock" or "soil". The corresponding means and coefficients of variation are also presented in Table 2.4.

It can be observed that there is significant difference in the coefficients of variation obtained from the two sets of data. This is due to the fact that the Moayyad and Mohraz (1982) data contains a large number of samples from the 1971 San Fernando earthquake, whereas the data used by Vanmarcke and Lai (1980) includes a broad range of earthquake magnitudes, epicentral distances, and motion intensities. For this reason, the coefficients of variation obtained with the
Table 2.4 Strong-Motion Duration

<table>
<thead>
<tr>
<th>Ground Condition</th>
<th>Based on Moayyad and Mohraz Data (1982)</th>
<th>Based on Vanmarcke and Lai Data (1980)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean (sec.)</td>
<td>Coefficient of Variation</td>
</tr>
<tr>
<td>Soft Ground</td>
<td>10.0</td>
<td>0.44</td>
</tr>
<tr>
<td>Intermediate Ground</td>
<td>6.9</td>
<td>0.42</td>
</tr>
<tr>
<td>Hard Ground</td>
<td>5.6</td>
<td>0.65</td>
</tr>
<tr>
<td>Sample Size</td>
<td>161</td>
<td></td>
</tr>
</tbody>
</table>

Vanmarcke and Lai (1980) data are more appropriate. The differences in the mean durations obtained do not appear to be significant and may be due primarily to differences in the duration definitions. For the present study the values shown in Table 2.5 are used.

Table 2.5 Proposed Strong-Motion Duration Statistics

<table>
<thead>
<tr>
<th>Ground Condition</th>
<th>Mean (sec.)</th>
<th>Coefficient of Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soft</td>
<td>10.0</td>
<td>0.9</td>
</tr>
<tr>
<td>Intermediate</td>
<td>7.0</td>
<td>0.9</td>
</tr>
<tr>
<td>Hard</td>
<td>5.5</td>
<td>1.0</td>
</tr>
</tbody>
</table>

2.2.3 Specification of Ground Motion Intensity

In order to evaluate the lifetime reliability of a structure, the probability of ground motions of particular intensities being exceeded during the life of the structure are evaluated (Sect. 2.1). Since the
ground motion is modeled as a random process, the intensity of the
loading is measured by the root mean square of the process, \( \sigma_a \), or
equivalently by the intensity of the power spectral density function,
\( s \). Results of the seismic hazard analysis, however, are in terms of
the probabilities of exceeding given maximum accelerations. This is
necessitated by a lack of available attenuation equations for \( \sigma \) or \( s \).
It is, therefore, necessary to relate the expected maximum acceleration,
\( \bar{a} \), to \( \sigma \) or \( s \). The relation given by Vanmarcke and Lai (1980) may
be used for this purpose; namely,

\[
r = \frac{\bar{a}_{\text{max}}}{\sigma_a} = \begin{cases} \\
\sqrt{2} \ln(2 \frac{f_o t_d}{\sigma_a}) & t_d > \frac{1.36}{f_o} \\
\sqrt{2} & t_d \leq \frac{1.36}{f_o} 
\end{cases}
\]

(2.10)

where,

- \( r \) = peak factor
- \( f_o = 1/2 \) expected rate of zero crossings
- \( t_d \) = duration of stationary strong motion phase
- \( d \)

As the frequency structure of the earthquake motion is modeled by
the Kanai-Tajimi spectrum, the value of \( f_o \) depends on \( \omega \) and \( \beta \). For
zero mean stationary Gaussian excitations, \( f_o \) may be calculated as

\[
f_o = \frac{1}{2\pi} \frac{\sigma_a}{\bar{a}}
\]

(2.11)

Noting that \( \sigma \) and \( \sigma_a \) are the zeroth and second spectral moments,
respectively, the procedure used by Lai (1980) for calculating the
moments of the Kanai-Tajimi spectrum (theoretically, all higher order
moments are infinite for the Kanai spectrum), may be used to determine
the effective zero crossing rate for the three ground conditions. Using
these values and with the mean durations previously calculated, the peak factors, $r$, are evaluated for the three ground conditions, as presented in Table 2.6.

Observe that the peak factor, $r$, is insensitive to changes in the zero-crossing rate and duration. Therefore, in the reliability analysis the constant values of $r$ given in Table 2.6 will be used.

Table 2.6 Peak Factors

<table>
<thead>
<tr>
<th>Ground Condition</th>
<th>$2f_0$ (sec.$^{-1}$)</th>
<th>$t_d$ (sec.)</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soft</td>
<td>10.2</td>
<td>10.0</td>
<td>3.0</td>
</tr>
<tr>
<td>Intermediate</td>
<td>10.5</td>
<td>7.0</td>
<td>2.9</td>
</tr>
<tr>
<td>Hard</td>
<td>12.9</td>
<td>5.5</td>
<td>2.9</td>
</tr>
</tbody>
</table>
CHAPTER 3

STRUCTURAL MODELING, HYSTERETIC RELATIONS
AND ASSOCIATED UNCERTAINTY

3.1 Structural Modeling

Models of varying complexity are available for estimating the response of a structure subjected to ground shaking. The structure may be modeled as a simple shear beam system with a single translational degree of freedom at each story, or with more detailed models such as those based on finite element idealizations.

Although the method to be used for the random-structural response analysis (Chapter 4) is general and not restricted to any particular class of structural models, for the purposes of this study, i.e., prediction of structural damage and safety, the shear beam model was found to be adequate and is used for reasons of simplicity. To ensure that response predictions compare reasonably well with those obtained by more detailed models, techniques are presented to determine the proper shear beam model parameters.

3.1.1 Shear-Beam Modeling of Buildings

In order to develop a simple shear beam idealization of a structure and maintain reasonable accuracy, it is necessary to determine for each story, an equivalent lateral stiffness and equivalent strength.
Equivalent Story Stiffness -- A method for evaluating the equivalent lateral stiffness of framed structures has been proposed by Anagnostopoulos (1972). With the assumptions that the shears above and below a joint in the columns are the same, the inflection points in the columns above and below a joint are located symmetrically with respect to the joint, and that the rotations of all joints in one floor are the same, the lateral stiffness is given as

\[ k = \frac{24E}{h^2} \cdot \frac{1}{\frac{2}{\sum k_c} + \frac{1}{\sum k_{gt}} + \frac{1}{\sum k_{gb}}} \]

(3.1)

where,

- \( E \) = modulus of elasticity
- \( h \) = story height
- \( \ell \) = girder length
- \( I \) = moment of inertia
- \( k_{c} = \frac{\sum I}{h} \) for all columns in a story
- \( k_{gt},k_{gb} = \frac{\sum I}{\ell} \) for all girders in the adjacent top and bottom floors

In the case of reinforced concrete frames, the moment of inertia, \( I \), must include the effects of cracking. Based on the work of Medland and Taylor (1971), Anagnostopoulos (1972) proposed a relation for determining the effective moment of inertia, as follows:

\[ \frac{I_{eff}}{I_{gross}} = \begin{cases} 0.40 & \text{for beams} \\ 0.80 & \text{for columns} \end{cases} \]

(3.2)

The larger coefficient for columns is a direct result of axial compression counteracting the effect of cracking.

An alternative approach to calculate the equivalent story lateral stiffness is possible if translational frequencies and mode shapes are
available from either actual response data or from an eigen-analysis of the "complete" stiffness matrix (generally available from the static analysis of the structure). In this instance, the shear beam simplification used in the dynamic analysis may be formulated by solving the inverse eigenvalue problem.

When the mass matrix is diagonal, but not necessarily uniform, the solution for the mode shapes and frequencies reduces to the problem of determining the eigenvalues and eigenvectors of the symmetric matrix \([B]\), given by

\[
[B] = [M]^{-1/2}[K][M]^{-1/2}
\]  

(3.3)

where \([M]\) and \([K]\) are, respectively, the mass and stiffness matrices. The natural frequencies and mode shapes are then

\[
\omega_i^2 = \lambda_i
\]  

(3.4)

\[
\{X_i\} = [M]^{-1/2} \{\phi_i\}
\]  

(3.5)

where:

\[
\omega_i = \text{\(i^{th}\) natural frequency}
\]

\[
\lambda_i = \text{\(i^{th}\) eigenvalue of \([B]\)}
\]

\[
\{X_i\} = \text{\(i^{th}\) mode shape}
\]

\[
\{\phi_i\} = \text{\(i^{th}\) eigenvector of \([B]\)}
\]
For the shear beam system, the stiffness matrix \([K]\) has the form

\[
\begin{bmatrix}
 k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\
-k_2 & k_2 + k_3 & -k_3 & \cdots & 0 \\
0 & -k_3 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & k_{n-1} + k_n & -k_n \\
0 & \cdots & 0 & -k_n & k_n
\end{bmatrix}
\]

where \(k_i\) is the equivalent lateral stiffness for story \(i\).

From this it is clear that if a mode shape and natural frequency are known, the corresponding eigenvalue and eigenvector may be determined using Eqs. 3.4 and 3.5, and the shear beam \([K]\) matrix then calculated by a simple back substitution in the eigenproblem equation,

\[
([M]^{-1/2} [K][M]^{-1/2} - \lambda I)\{\phi\} = 0
\]

(3.6)

This results in an equivalent shear beam stiffness matrix which has a mode shape and frequency exactly matching the chosen one. Of course, since the shear beam matrix is an idealization, all lateral mode shapes and frequencies cannot be matched simultaneously. It is possible to conceive of an optimization scheme that will minimize the error in all the modes; however, as one particular mode will generally tend to dominate the structural response, it is more desirable to obtain the stiffness matrix that matches only the dominant mode. From a simple study, it was found that using this procedure the chosen mode shape and frequency are matched exactly, as expected, and that the other mode shapes and frequencies close to this one are not significantly in error. This is illustrated in the case studies presented in Chapter 6.
Equivalent Story Strength -- Based on the assumption that a story will reach its maximum strength when hinges form in either the columns or girders, Anagnostopoulos (1972) proposed for framed structures, the maximum story strength

\[
F_{y_{\text{max}}} = \min \left\{ \frac{2\Sigma M_{yc}}{h}, \frac{2\Sigma M_{yc}}{h} \right\}
\]

(3.7)

where:

- \(\Sigma M_{yc}\) = sum of the plastic moment capacities for all girders in the floor;
- \(\Sigma M_{yc}\) = sum of the plastic moment capacities for all columns in the floor, reduced for the effect of axial load;
- \(h\) = story height.

Equation 3.7 is valid only if shear failure does not occur in the columns. For frames that are designed according to standard codes and are well detailed, premature shear failure should not occur (Anagnostopoulos, 1972). Another possible source of error in Eq. 3.7, as pointed out by Lai (1980), exists because the expression is based on the assumption that the column stiffnesses will be approximately equal to the girder stiffnesses for a particular story. If this is not the case, story strength may be underestimated. This error is further discussed and quantified, along with a complete discussion of the validity of the shear beam model, in Sect. 3.3.

3.2 Hysteretic Restoring Force Model

Buildings subjected to strong earthquake motions are often likely to undergo response in the inelastic range. In addition, the response of the building will be oscillatory and in some cases the stiffness and/or strength of the structural components will deteriorate. Thus, to
obtain an accurate analysis of a system subjected to earthquake loading, particularly at significant levels of damage, the restoring force model should be capable of reproducing inelastic, hysteretic, degrading behavior. Although numerous models have been proposed to describe the hysteretic behavior of both reinforced concrete and steel structures, they generally involve complicated sets of rules and require different mathematical functions for different phases of the loading.

A versatile hysteretic model with the desired attributes mentioned above was developed by Wen (1976, 1980) and Baber and Wen (1980). The hysteretic relation is given in terms of a nonlinear differential equation; hence, it is characterized by a single mathematical form. It allows analytical solutions (see Chapter 4) for the random vibration response statistics, whereas most empirical rule-based models require costly repeated time-history analyses (i.e., Monte Carlo simulation) to obtain the necessary statistics and probabilities.

The essence of the model may be described by considering the single-degree-of-freedom case in which the equation of motion is

\[ m \ddot{u} + c \dot{u} + q(u, t) = -ma \]  

(3.8)

where the restoring force \( q \) is given by

\[ q = \alpha ku + (1 - \alpha)kz \]  

(3.9)

The nonlinear differential equation governing \( z \) is given by

\[ \ddot{z} = A\dot{u} - \beta |\dot{u}|z^{n-1}z - \gamma |\dot{u}|z^n \]  

(3.10)

where \( A, \beta, \gamma \) and \( n \) are parameters that control the shape of the hysteresis loop.

Dividing Eq. 3.10 by \( \dot{u} \) yields

\[ \frac{dz}{du} = \frac{A - \beta |\dot{u}|}{\dot{u}} |z|^{n-1}z - \gamma |z|^n \]  

(3.11)

Solving Eq. 3.11 for \( z \) as a function of \( u \) reveals the nature and
versatility of the model. For the case $n=1$, the resulting hysteretic shapes for constant amplitude cyclic motion of $u$ and different combinations of $\beta$ and $\gamma$ are shown in Fig. 3.1. From Fig. 3.1, it is seen that $z$ is a transformed displacement variable, such that the restoring force given by Eq. 3.9 exhibits smooth hysteretic behavior.

For softening systems, $z$ attains a maximum value which is obtained by setting Eq. 3.11 to zero. For positive $u$ and $z$, this gives

$$z_{\text{max}} = \left[ \frac{A}{\beta + \gamma} \right]^{1/n}$$

(3.12)

The yield level $f_y$ is thus given by

$$f_y = (1 - \alpha)k_z_{\text{max}}$$

(3.13)

Other important physical properties of softening systems worth noting are that the initial stiffness, $k_i$, and the post-yield stiffness, $k_f$, are as follows:

$$k_i = \alpha k + (1 - \alpha)kA$$

(3.14)

$$k_f = \alpha k$$

(3.15)

Note that the ratio of post-yield to initial stiffness reduces to the value $\alpha$ when $A=1$, revealing the physical significance of $\alpha$. Finally, the sharpness of the transition from the linear to nonlinear range is governed by the parameter $n$, with the hysteresis approaching bilinear behavior as $n$ approaches $\infty$.

The model is also capable of reproducing degrading material behavior. This is obtained by introducing two additional parameters in Eq. 3.10, giving

$$\dot{z} = [A\dot{u} - \nu(\beta|\dot{u}|z|^{n-1} z + \gamma\dot{u}|z|^n)]/n$$

(3.16)
Fig. 3.1 Hysteretic Shapes for Different $\beta$ and $\gamma$
Prescribing $\eta$ and $\nu$ to be increasing functions of time will induce stiffness and strength degradation, respectively. This may be seen by examining Eqs. 3.12 and 3.14 and noting that increasing $\eta$ is equivalent to reducing $A$, $\beta$ and $\gamma$ in proportion and increasing $\nu$ is equivalent to increasing $\beta$ and $\gamma$ without affecting $A$. Furthermore, both strength and stiffness degradation are obtained by prescribing the parameter $A$ to be a decreasing function of time.

An in-depth study of the model, and the extension to the multi-degree-of-freedom case may be found in Baber and Wen (1980).

3.2.1 Degradation Laws - Energy and Displacement-Based

Baber and Wen (1980) obtained degrading material behavior by defining the parameters $A$, $\eta$ and $\nu$ as functions of the dissipated hysteretic energy, $\varepsilon$, given by

$$
\varepsilon = \int_{u_0}^{u_f} F \cdot du = (1-\alpha)k \int_{u_0}^{u_f} zdudt
$$

where $F=(1-\alpha)kz$ is the hysteretic restoring force. $A$, $\eta$ and $\nu$ may then be written as

$$
A(\varepsilon) = A_0 - \delta_A \varepsilon
$$
$$
\eta(\varepsilon) = 1.0 + \delta_\eta \varepsilon
$$
$$
\nu(\varepsilon) = 1.0 + \delta_\nu \varepsilon
$$

where the $\delta$'s are constants specified for the desired rate of degradation. A value of $\delta=0$ implies no degradation. Figure 3.2 illustrates the effects of the three $\delta$'s on the degradation of the hysteresis loop.
Fig. 3.2 Energy-Based Material Degradation
(reproduced from Baber and Wen, 1980)
In order to properly model the hysteretic behavior of reinforced concrete elements, modifications to the basic model described above are necessary. In these modifications, degradation in stiffness during loading in each cycle is dependent on the maximum deformation incurred in the previous cycle. This is based on the stiffness degrading model proposed by Clough (1966), illustrated in Fig. 3.3.

![Fig. 3.3 Clough's Stiffness Degrading Model](image)

The desired displacement-based stiffness degradation can be obtained by defining the stiffness degradation parameter \( \eta \), such that the value of \( A \) (which controls the initial loading slope of each hysteresis cycle) reflects the maximum deformation reached in the previous cycle. This is achieved by

\[
\eta_i = A_0 \left( \frac{u_i}{z_i} - \frac{u_{i-1}}{z_{i-1}} \right)
\]

where:

- \( \eta_i \) = value of \( \eta \) during the \( i \)th half cycle;
- \( A_0 \) = initial value of \( A \) prior to loading;
- \( u_i \) = peak displacement in \( i \)th half cycle;
- \( z_i \) = peak \( z \) value in \( i \)th half cycle.
Figure 3.4 shows the total lateral load versus the top story deflection for a model reinforced concrete frame (Park and Paulay, 1975; Wilby, 1974). The frame was modeled with the proposed modified degrading model and subjected to the same displacement history; the resulting hysteresis curves are shown in Fig. 3.5 which may be compared with those of Fig. 3.4.

Figure 3.6 shows the experimental load deflection curves of a single-story scaled reinforced concrete frame (Gulkan and Sozen, 1971); whereas Fig. 3.7 shows the load-deflection curves of the corresponding frame modeled with the proposed modified degrading behavior subjected to the same displacement history for comparison.

Figure 3.8 shows the behavior of the modified model under cycles of large constant amplitude displacements. This illustrates the model's ability to simulate the large stiffness loss in the first cycle due to initial cracking and the subsequent reduced rate of deterioration as generally observed in experimental work (e.g., Hwang, 1982). Figure 3.9 shows the behavior of the model during an earthquake type loading in which the displacements are initially small, build up during the strong-motion phase and finally decrease.

It is important to point out that the modified model is valid only when the frames under consideration are provided with adequate shear reinforcement to prevent significant shear cracking; otherwise deterioration may continue even under decreasing load (Park and Paulay, 1975; Hwang, 1982). In such a case, it may be necessary to consider degradation based on combined peak displacement and dissipated hysteretic energy, and to also include strength deterioration. Moreover, for members having significantly different tensile and compressive steels, the well-known "pinching" effect will be more marked (Park and Paulay, 1975).

3.2.2 Hysteresis Model Parameters by System Identification

In order to properly model the restoring force behavior for a real structure, it is necessary to determine appropriate values for the
Fig. 3.4 Load-Deflection Curves for Reinforced Concrete Building Model (after Park and Paulay, 1975)

Fig. 3.5 Proposed Displacement Based Stiffness Degrading Model
Fig. 3.6 Load-Deflection Curves for Reinforced Concrete Frame Model (after Gulkan and Sozen, 1971)

Fig. 3.7 Proposed Displacement Based Stiffness Degrading Model
Fig. 3.8 Behavior of Proposed Model under Cycles of Large Constant Amplitude Displacement

Fig. 3.9 Behavior of Proposed Model under Earthquake Type Displacement History
hysteresis loop shape parameters $A$, $B$ and $\gamma$. For this purpose, a systems identification technique based on the invariant imbedding filter (e.g., Distefano and Rath, 1974; Distefano and Pena-Pardo, 1976), was investigated. This approach, however, required the solution of a large number of first-order nonlinear differential equations (e.g., twenty for nondegrading systems). In addition, unsatisfactory results were observed when the response history was highly irregular, as in the case of earthquake response data. For these reasons, an alternative technique based on a least squares error minimization was developed. The method requires the solution of linear algebraic equations (three for nondegrading systems) and gave satisfactory results in all cases. Applications to some degrading systems is also possible.

Non-degrading Case -- For the non-deteriorating case, integration of Eq. 3.11 yields

$$z(u_i) = z_0 + A \int_{u_0}^{u_i} du - B \int_{u_0}^{u_i} \frac{|\dot{u}|}{u} |z|^{n-1} zdu - \gamma \int_{u_0}^{u_i} |z|^n du$$ (3.20)

Introducing the notations

$$I_{11} = \int_{u_0}^{u_i} du$$

$$I_{21} = \int_{u_0}^{u_i} \frac{|\dot{u}|}{u} |z|^{n-1} zdu$$ (3.21)

$$I_{31} = \int_{u_0}^{u_i} |z|^n du$$

Eq. 3.20 becomes

$$z(u_i) = z_0 + AI_{11} - B I_{21} - \gamma I_{31}$$ (3.22)
Given experimental force-displacement data points \((z_i, u_i)\), the integrals \(I_{1i}, I_{2i}\) and \(I_{3i}\) can be calculated directly and may be considered as observable quantities (assuming \(n\) is known and noting that \(\dot{u}/u\) is only a function of the displacement direction, taking on the values \(+1\) or \(-1\)), whereas \(z(u_i)\) is a theoretical value dependent on the parameters \(A, \beta\) and \(\gamma\). The values of \(A, \beta\) and \(\gamma\) should then be chosen so that the calculated value of \(z(u_i)\) is the same as the experimental value of \(z\) at displacement \(u_i\), for all data points. Of course, as the model is not perfect this is not possible. Instead, the values of \(A, \beta\) and \(\gamma\) are determined such that the sum of the squared errors over all the data points is minimized. The total squared error is

\[
E = \sum_{i=1}^{N} (z_i - z_0 - A I_{1i} + \beta I_{2i} + \gamma I_{3i})^2
\]  

(3.23)

where \(N\) is the number of observed data points and \(z, I_{1i}, I_{2i}\), and \(I_{3i}\) are the observed quantities defined earlier. The values of \(A, \beta\) and \(\gamma\) that minimize the error are then obtained by differentiating Eq. 3.23 with respect to the three parameters, setting each equation to zero, and solving the system of equations. The resulting set of equations is

\[
\begin{bmatrix}
\sum I_{1i}^2 & -\sum I_{1i} I_{2i} & -\sum I_{1i} I_{3i} \\
\sum I_{2i}^2 & \sum I_{2i} I_{3i} \\
\text{symmetric} & \sum I_{3i}^2 \\
\end{bmatrix}
\begin{bmatrix}
A \\
\beta \\
\gamma \\
\end{bmatrix}
=
\begin{bmatrix}
\sum I_{1i} (z_i - z_0) \\
-\sum I_{2i} (z_i - z_0) \\
-\sum I_{3i} (z_i - z_0) \\
\end{bmatrix}
\]  

(3.24)

Degrading Case -- For degrading hysteretic behavior, some modification is necessary. If the degrading behavior is represented by Eq. 3.19, the optimal values of \(A, \beta\) and \(\gamma\) are obtained by solving the set of equations (Eq. 3.24) with \(I_{1i}, I_{2i}\), and \(I_{3i}\) redefined as
\[ I_{ij} = \int_{u_0}^{u_i} \frac{1}{\eta} \, du \]

\[ I_{21} = \int_{u_0}^{u_i} \frac{1}{\eta} \, \left| \frac{\dot{u}}{u} \right| \, z \, |z|^{n-1} \, z \, du \]  
\( (3.25) \)

\[ I_{3i} = \int_{u_0}^{u_i} \frac{1}{\eta} \, |z|^n \, du \]

where \( \eta \) of Eq. 3.19 is determined from the observed data. For this case, the determination of \( A \) (see Eq. 3.19) may require iteration. In the test cases examined, convergence was invariably achieved in a few cycles.

If \( A \) is degraded as in Eq. 3.18, an additional equation is needed with a fourth quantity \( I \) defined as

\[ I_{4i} = \int_{u_0}^{u_i} \varepsilon \, du \]  
\( (3.26) \)

where \( \varepsilon \) is the observed dissipated hysteretic energy as a function of \( u \) (Eq. 3.17). Thus, a fourth row and column (observing symmetry) are added to the matrix of Eq. 3.24 whose elements are \( -\Sigma I_1, \Sigma I_1, \Sigma I_1, \Sigma I_1, \Sigma I_1, \Sigma I_1 \). The fourth unknown in the parameter vector is \( \delta \) and the fourth row of the vector on the right hand side of Eq. 3.24 is \( -\Sigma I_1 (z - z) \).

If \( \eta \) and \( \nu \) degrade on the basis of dissipated hysteretic energy as defined in Eq. 3.18, identification of the unknown parameters becomes more difficult. In this case, the error function becomes non-quadratic. Substituting Eqs. 3.18 into 3.16 and dividing through by \( \dot{u} \),

\[ \frac{dz}{du} = \frac{A - \delta A \varepsilon - (1.0 + \delta \nu \varepsilon)(\beta \frac{\dot{u}}{u} \, |z|^{n-1} \, z + \gamma |z|^n)}{1.0 + \delta \eta \varepsilon} \]  
\( (3.27) \)

Examination of the above expression reveals that if \( \beta, \gamma \) and \( \delta \) are known (\( \beta \) and \( \gamma \), at least, may be obtained using the preceding procedure
for the nondeteriorating case on the first nonlinear response cycle),
the parameters $A$, $\delta$, and $\delta_v$ may be identified assuming that the exponent $n$ is known as in the previous cases. This is more easily seen by rewriting Eq. 3.27 as

$$
\frac{dz}{du} = A \frac{1}{1 + \delta \varepsilon} - \delta_A \frac{\varepsilon}{1 + \delta \varepsilon} - \frac{\delta_v}{1 + \delta \varepsilon} \left( \epsilon \frac{|\dot{u}|}{u} |z|^{n-1} z + \gamma |z|^n \right)
$$

then,

$$
z(u_1) = z_0 + A 5_1 - A 16_1 - A (\delta 7_1 + \gamma 8_1) - \beta 9_1 - \gamma 10_1
$$

where in this case:

$$
I_{51} = \int_{u_0}^{u_1} \frac{du}{1 + \delta \varepsilon}
$$

$$
I_{61} = \int_{u_0}^{u_1} \frac{\varepsilon}{1 + \delta \varepsilon} \, du
$$

$$
I_{71} = \int_{u_0}^{u_1} \varepsilon \frac{|\dot{u}|}{u} \frac{|z|^{n-1} z}{1 + \delta \varepsilon} \, du
$$

$$
I_{81} = \int_{u_0}^{u_1} \varepsilon \frac{|z|^n}{1 + \delta \varepsilon} \, du
$$

$$
I_{91} = \int_{u_0}^{u_1} \frac{|\dot{u}|}{u} \frac{|z|^{n-1} z}{1 + \delta \varepsilon} \, du
$$

$$
I_{101} = \int_{u_0}^{u_1} \frac{|z|^n}{1 + \delta \varepsilon} \, du
$$
Thus, the system of equations for the solution of the unknowns $A$, $A_0$, and $A_0$ becomes

$$
\begin{bmatrix}
\Sigma I_{51}^2 & -\Sigma I_{51} I_{61} & -\Sigma I_{51} (\beta I_{71} + \gamma I_{81}) \\
\Sigma I_{61}^2 & \Sigma I_{61} (\beta I_{71} + \gamma I_{81}) \\
\text{symmetric} & \Sigma (\beta I_{71} + \gamma I_{81})^2
\end{bmatrix}
\begin{bmatrix}
A \\
\delta_A \\
\delta_V
\end{bmatrix} =
\begin{bmatrix}
\Sigma I_{51} (z_i - z_o + \beta I_{91} + \gamma I_{101}) \\
-\Sigma I_{61} (z_i - z_o + \beta I_{91} + \gamma I_{101}) \\
-\Sigma (\beta I_{71} + \gamma I_{81}) (z_i - z_o + \beta I_{91} + \gamma I_{101})
\end{bmatrix}
$$

(3.30)

The identification procedure requires knowledge of the parameter $n$. This does not present a major problem, since solution for the unknown parameter values requires little computation time. Results may be obtained for a number of trial values of $n$, and the best value selected. It should be mentioned that in using the method it was generally found necessary to include two data points close to either side of the peak of a hysteresis loop, in order to assure satisfactory results. This is necessary to define the discontinuity, at the peak, of the term $|\ddot{u}|/\dot{u}$ that appears in the integrals $I_2$, $I_7$, and $I_9$.

Results and Recommendations -- In order to validate the method, a single-degree-of-freedom hysteretic element was subjected to a variety of excitations and its responses recorded. These response records were then used to identify the hysteretic parameter values as discussed in the previous section. Ideally, the identified values should be the same.
as the parameter values of the original element used to generate the response data.

Generally, it was found that identification with a few cycles of nonlinear response gave the best results. Table 3.1 shows some results obtained when a nondegrading element with $A=B=\gamma=n=1.0$ (implying $z=0.5$) was used to generate the response. Note that when the system is linear, the values for $\beta$ and $\gamma$ are not correctly identified since they have almost no effect on the loop shape at this level. The value of $A$, however, that controls the initial stiffness is accurate. In almost all cases, the identified values of $A$, $\beta$ and $\gamma$ yield the exact value of $z$.

The procedure was also appraised for the degrading case by setting $\Delta=1.0$ ($A$, $\beta$ and $\gamma$ were kept equal to 1) and generating the corresponding response data. The results obtained using 3.5 cycles of response data are summarized in Table 3.2.

### Table 3.1 Nondegrading Case

<table>
<thead>
<tr>
<th>Excitation</th>
<th>No. of Response Cycles</th>
<th>Identified Parameters $\ z = \left( \frac{A}{\beta + \gamma} \right)^{1/n}$</th>
<th>Response Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin 5\pi t$</td>
<td>25</td>
<td>$A \</td>
<td>\beta \</td>
</tr>
<tr>
<td>$200\sin 5\pi t$</td>
<td>2.5</td>
<td>$A \</td>
<td>\beta \</td>
</tr>
<tr>
<td>$200\sin 5\pi t$</td>
<td>1</td>
<td>$A \</td>
<td>\beta \</td>
</tr>
<tr>
<td>$600\sin 5\pi t$</td>
<td>2.5</td>
<td>$A \</td>
<td>\beta \</td>
</tr>
<tr>
<td>$600\sin 5\pi t$</td>
<td>1</td>
<td>$A \</td>
<td>\beta \</td>
</tr>
<tr>
<td>$200t\sin 5\pi t$</td>
<td>9.5</td>
<td>$A \</td>
<td>\beta \</td>
</tr>
<tr>
<td>$200t\sin 5\pi t$</td>
<td>3.75</td>
<td>$A \</td>
<td>\beta \</td>
</tr>
</tbody>
</table>
In addition, a degrading case with $\delta = \delta = 0.5$ was investigated. Using the known values of $\beta = 1.0$ and $\gamma = 1.0$, and 3.5 cycles of response data, the results shown in Table 3.3 were obtained.

Table 3.3 Degrading Case – $\delta = \delta = 0.5$

<table>
<thead>
<tr>
<th>Excitation</th>
<th>Identified Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_o$</td>
</tr>
<tr>
<td>200$\sin 5\pi t$</td>
<td>1.00</td>
</tr>
<tr>
<td>El Centro Earthquake</td>
<td>1.01</td>
</tr>
</tbody>
</table>

Finally, the case in which the degradation is controlled by maximum cyclic displacements was considered. Four cycles of response data were generated with $A = 1.0$, $\beta = 3.06$, $\gamma = -1.02$ and $n = 2$ (values which are used to simulate reinforced concrete behavior). The identified values of $A$, $\beta$ and $\gamma$ obtained were 1.02, 2.54 and -0.47, respectively. Although the identified values of $\beta$ and $\gamma$ differ from those used to generate the response record, the value of $z$ implied is almost exact (7.02 versus 7.0). For further confirmation, the identified values of the parameters
were used to regenerate the response. This response data was found to be almost identical with that generated with the original set of parameters. Hence, the identification was assumed to be satisfactory.

With the method verified as described above, the required hysteresis parameters were identified using several sets of actual experimental force-displacement data. Based on these results, certain general rules for determining the proper hysteresis parameters for structural modeling purposes were developed. For steel structures it was found that $\beta = \gamma$ should be used; whereas for reinforced concrete $\beta = -3\gamma$ is appropriate, with $\gamma < 0$. Also, the values for $\alpha$ and $n$ should be 0.04 and 1, respectively, for steel, and 0.02 and 2 for reinforced concrete. The value of $A$ may be taken as 1, in which case the value of the parameter, $k$, is simply the initial stiffness of the restoring force (Eq. 3.14), calculated from elastic theory. Assuming that the yield level of the material is known, the absolute values of $\beta$ and $\gamma$ may be determined through Eqs. 3.12 and 3.13 (which express the yield strength as a function of the model parameters). Finally, steel structures may be modeled using the non-deteriorating model, whereas reinforced concrete behavior is well simulated by the displacement-based stiffness degrading model (assuming that adequate shear reinforcement is provided). This procedure was used to determine the parameter values for the hysteresis models used to obtain the results shown in Figs. 3.5 and 3.7.

3.3 Structural Response Uncertainty

The uncertainties associated with estimating the dynamic response of a structure are of two fundamentally different sources, those due to incomplete knowledge or information of the physical phenomena, and those associated with inherent variabilities. The former source of uncertainty results from the imperfect nature of the mathematical relations used to predict the structural response, whereas the latter results from the variabilities of the material properties.
In evaluating the uncertainty in the structural response estimate (for a given loading), it is necessary to consider the uncertainty associated with the parameters of the model (e.g., stiffness, damping, etc.) as well as the uncertainty underlying the mathematical idealization of the real structure. In evaluating the uncertainty of the model parameters, the inherent variability in the material or structural properties (measured by the c.o.v. \( \delta \)) as well as the estimation or modeling errors (measured by the c.o.v. \( \Delta \)), must be included.

The uncertainty of the structural response may be obtained by first-order analysis (Ang and Tang, 1975) as

\[
\text{Var}[X] \approx (E[\hat{X}])^2 \cdot \text{Var}[N] + (E[N])^2 \cdot \text{Var}[\hat{X}]
\]  

(3.31)

where \( E[\hat{X}] \) is the mean response obtained from the model using mean parameter values, \( E[N] \) and \( \text{Var}[N] \) represent the expected bias and variance of the dispersion error, respectively, in the response associated with the mathematical idealization of the structure, and \( \text{Var}[\hat{X}] \) is the variance of the response associated with the parameter uncertainties and the randomness of the loading history.

The variance of the response due to the model parameter uncertainties is also obtained by first-order approximation as

\[
\text{Var}_p[\hat{X}] \approx \sum_{i,j} \left( \frac{\partial \hat{X}}{\partial P_i} \right)^2 \rho_{ij} \frac{\sigma_i}{\sigma_P} \frac{\sigma_j}{\sigma_P}
\]  

(3.32)

where \( \overline{P} \) is the set of mean model parameters, \( \rho_{ij} \) is the correlation coefficient of the \( i^{th} \) and \( j^{th} \) parameter, and \( \sigma_i \) is the standard deviation of the \( i^{th} \) parameter. The variance of the response due to the random nature of the loading history, is obtained through a random vibration analysis and is presented in Chapter 4. It remains, therefore, to evaluate the statistics of \( N \) and the standard deviations of the model parameters.
3.3.1 Mathematical Idealization Error

The error in the response estimates associated with an idealized model of a structure is evaluated through the Bayesian correction variable, \( N \), as discussed above. For the purpose of evaluating the mean and variance of \( N \), comparisons between actual building responses under earthquake loads, and the corresponding responses predicted with a mathematical model would be needed.

Following the 1971 San Fernando earthquake, a study was published by the U.S. Department of Commerce (1973) in which nine structures, equipped with strong motion instruments were analyzed, making it possible to perform the required comparisons. Although the analyses were performed with linear and elastic models, the comparisons are valid as the majority of the structures exhibited only slight nonlinearity. The structural models were also quite detailed and accounted for most aspects of the structural behavior. From these studies it was found that the coefficient of variation (c.o.v.) of the maximum response, due to mathematical idealization, is about 15%. This value was, as mentioned, based on comparing linear models with observed linear response. For nonlinear behavior, the errors can be expected to be larger.

The models used in this study do not account for the effects of accidental torsion and simultaneous excitation in three directions. These effects are difficult to assess at this time; hence, to include these errors, a c.o.v. of 20\% will be assumed for the response prediction error associated with the mathematical idealization of the structure. This is the error, then, that would be expected if the physical parameters in the model were exactly known.

It is also necessary to assess the prediction error caused by any inadequacy of the hysteresis model. As was shown in Sect. 3.2, the model is quite flexible and with proper choice of the parameters can reproduce a variety of hysteretic behaviors accurately, including material degradation. There are, however, certain aspects of hysteretic behavior that the model in its present form cannot represent, e.g., the loop "pinching" phenomenon observed in reinforced concrete. With the
proper parameters, the error associated with the hysteretic force model is believed to be small. Therefore, for steel structures, $\Delta_H = 0.05$ is used and for reinforced concrete structures, owing to the more complicated nature of the behavior, $\Delta_H = 0.10$.

The total idealization error in the response estimates, therefore, is $\Delta^N = \sqrt{0.20^2 + 0.05^2} = 0.21$ for steel structures, and $\Delta^N = \sqrt{0.20^2 + 0.10^2} = 0.22$ for reinforced concrete structures.

3.3.2 Model Parameter Uncertainties

For the shear beam model, the basic parameters are the story stiffness, mass, damping, and yield strength. Uncertainties in these parameters would include the inherent variability of the material properties (such as the modulus of elasticity and the yield strength of the base materials), measured by the c.o.v. $\delta$, as well as the approximations used to evaluate the parameters, measured by the c.o.v. $\Delta$.

**Story Stiffness** -- The uncertainty in the estimated story stiffness, $k$ (Sect. 3.1.1) lies, primarily, in the calculation of the flexural stiffness of the structural members. For reinforced concrete beams and columns, Portillo Gallo and Ang (1976) determined that $\delta_{EI} = 0.2$ and $\Delta = 0.2$. For steel, assuming $\delta = 0.06$ and $\delta_{EI} = 0.05$ (Galambos and Ravindra, 1978; Yura, Galambos and Ravindra, 1978; Rojiani, 1978) yields $\delta_{EI} = 0.08$; the prediction error, $\Delta_{EI}$, may be neglected.

If the method of Eq. 3.1, is used for calculating the equivalent story stiffness, the uncertainty depends on the flexural stiffness uncertainty of the girders and columns framing into the story, and the uncertainty of the material modulus of elasticity. In addition, the error in the form of Eq. 3.1 must be included in the prediction error of $k$; this has been evaluated by Lai (1980) and is given as $\Delta^B = 0.08$. Lai (1980) also found that there is an expected bias (ratio of actual stiffness, calculated by detailed methods, to predicted stiffness) of 0.93.
Assuming that the stiffness of all the members framing into the story are perfectly correlated (due to common construction and workmanship) and neglecting the uncertainty in the member lengths and the story height, the coefficient of variation of the equivalent lateral story stiffness (by Eq. 3.1) can be shown to be (by first-order approximation) \( \sqrt{[\text{c.o.v.}(E)]^2 + [\text{c.o.v.}(EI)]^2} \). Therefore, the inherent variability of \( k \) is given as \( \delta_k = \sqrt{0.11^2 + 0.2^2} = 0.23 \) for reinforced concrete structures and \( \delta_k = \sqrt{0.06^2 + 0.08^2} = 0.10 \) for steel structures; the corresponding prediction error in \( k \) is \( \Delta_k = \sqrt{0.22^2 + 0.08^2} = 0.22 \) for reinforced concrete structures, and \( \Delta = 0.08 \) for steel structures. The total uncertainties (c.o.v.) are thus \( \Omega_k = \sqrt{0.23^2 + 0.22^2} = 0.32 \) for reinforced concrete, and \( \Omega = \sqrt{0.10^2 + 0.08^2} = 0.13 \) for steel structures.

If the equivalent story stiffnesses are computed using the mode matching technique, the prediction error associated with Eq. 3.1 should be removed.

**Damping** -- A number of studies have been performed in which damping coefficients were experimentally determined for both steel and reinforced concrete structures. Using small amplitude vibration data (so as not to confuse hysteretic energy dissipation with viscous damping energy dissipation), Portillo Gallo and Ang (1976) estimated the mean of \( \zeta \) (fraction of critical damping) for reinforced concrete structures as 0.04 with c.o.v.'s \( \delta_\zeta = 0.50 \), \( \Delta_\zeta = 0.25 \) and \( \Omega_\zeta = 0.56 \). Haviland (1976) estimated the mean and total coefficient of variation (without differentiating between basic variability and prediction error), again using small amplitude data, for concrete structures to be 0.043 and 0.76, respectively; whereas, the mean and total c.o.v. for steel structures were estimated to be 0.02 and 0.65, respectively. For reinforced concrete, the average from the two studies yields a c.o.v. of \( \Omega_\zeta = 0.65 \). The mean value of \( \zeta \) for reinforced concrete may be taken as 0.04. For steel structures, the values proposed by Haviland (1976) will be used, assuming that the total c.o.v. includes a prediction error of 0.25. Although the above results are for the damping of the fundamental mode, they are assumed valid for the other modes.

**Mass** -- For reinforced concrete structures Portillo Gallo and Ang (1976) recommended a value for the total c.o.v. of the story mass of
\( \Omega = 0.12 \). This is based on the consideration of the uncertainty in the unit weight of concrete, member dimensions, the weight of nonstructural elements, and in the estimation of the live loads. This value was found to be reasonable for a wide range of floor areas and dead loads, making it generally applicable. For steel structures, much of the results of Portillo Gallo and Ang (1976) are also applicable; on this basis and using the same approach, the total c.o.v. of the story mass for steel structures was determined to be \( \Omega^M = 0.11 \). For the purposes of this study, it is assumed that \( \delta = 0.05 \), therefore, \( \Delta^M = 0.11 \) for reinforced concrete structures, and \( \Delta^M = 0.10 \) for steel structures.

**Story Strength** -- The equivalent story strength can be determined by Eq. 3.7; on this basis the corresponding uncertainty will be a function of the uncertainty in the plastic moment capacity of either the girders or columns, depending on which controls the story strength. In addition, the uncertainty in the equation itself must be included in the prediction error.

For reinforced concrete beams, Portillo Gallo and Ang (1976) evaluated the uncertainty in the estimated flexural capacity, and determined that for beams of a wide range of reinforcing ratios, the coefficients of variation are essentially constant as follows: \( \delta^F = 0.12 \), \( \Delta^F = 0.12 \) and \( \Omega^F = 0.17 \). For steel beams, the coefficients of variation for the plastic moment capacity are given by Yura, Galambos and Ravindra (1978), as \( \delta^F = 0.11 \), \( \Delta^F = 0.07 \) and \( \Omega^F = 0.13 \). Also, the expected bias (ratio of experimental to predicted values) was found to be 1.1.

For columns, the statistics become more difficult to evaluate since the capacity is affected by the axial load. Thus, the uncertainty will depend on the uncertainty in the axial load and the ratio of \( P/P_B \) (axial load in the column to the column balanced load). For most structures designed to resist earthquakes, however, the axial load will be below \( P_B \), except in cases of extremely tall buildings, (Portillo Gallo and Ang, 1976; Blume, Newmark and Corning, 1961), and the axial load effect may be assumed to be statistically independent of the bending moment induced by the lateral earthquake loadings. Furthermore, it is reasonable to assume that the loading between two
stories is perfectly correlated implying that the c.o.v. of the axial column load at any story is the same as the c.o.v. of the weight of an individual story (0.12 for reinforced concrete structures, 0.11 for steel structures). With these considerations, and using the results of Portillo Gallo and Ang (1976), it was determined that the c.o.v. for the flexural capacity of the columns may be taken to be the same as that for the beams. The flexural capacities between all the beams and columns in a story may also be assumed to be perfectly correlated.

Finally, the bias and prediction error of Eq. 3.7 must be evaluated. On the basis of a detailed, inelastic frame analysis, Lai (1980) showed that the equation tends to underestimate the equivalent story strength (as expected, see Sect. 3.1.1). The bias was calculated as 1.1 and the c.o.v. of the prediction error was found to be 0.19.

Thus, for the equivalent story strength calculated using Eq. 3.7, \( \delta_F = 0.12, \Delta_F = \sqrt{0.12^2 + 0.19^2} = 0.22, \) and \( \Omega_F = 0.25 \) for reinforced concrete structures; whereas, \( \delta_F = 0.11, \Delta_F = \sqrt{0.07^2 + 0.19^2} = 0.20, \) and \( \Omega_F = 0.23 \) for steel structures.

When the equivalent story strength is calculated by a detailed inelastic frame analysis the prediction error associated with Eq. 3.7 should be remove.

Summary -- The results of this section are summarized in Table 3.4.
Table 3.4 Parameter Coefficients of Variation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Coefficient of Variation</th>
<th>Reinforced Concrete Structures</th>
<th>Steel Structures</th>
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<td>0.22</td>
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<td>Story Strength:</td>
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</tr>
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CHAPTER 4

RANDOM RESPONSE STATISTICS
AND SENSITIVITY COEFFICIENTS

4.1 Random Vibration Response

As the earthquake loading is modeled with a random process, the resulting structural response is defined by the statistics obtained from a random vibration analysis. Exact solution for the random vibration response statistics of nonlinear hysteretic degrading structures, however, is generally not possible. By replacing the nonlinear system with an equivalent linear one, Wen (1980) and Baber and Wen (1980) were able to obtain accurate results (verified by Monte-Carlo simulation) without resorting to the commonly used Krylov-Bogoliubov approximation, which has been shown to underestimate the root-mean-square response for nearly elastoplastic systems (Iwan and Lutes, 1968).

4.1.1 Stochastic Equivalent Linearization of Hysteretic Systems

Examining the equations of motion of a single-degree-of freedom system, Eqs. 3.8 through 3.10, reveals that the differential equation for \( z \), Eq. 3.10, is the source of nonlinearity. As \( \dot{z} \) is a function of \( \dot{u} \) and \( z \), the equivalent linear form is written as
Here, C and K are equivalent linear coefficients, chosen such that the resulting solution for $\dot{z}$ is as "close" as possible to that obtained with the original nonlinear equation. The values for C and K will, of course, be dependent on the response; hence, the solution for the equivalent linear system will require iterative procedures.

On the basis of minimizing the mean square residual of Eq. 3.8 (i.e., the residual obtained by substituting the solution of the equivalent linear system into Eq. 3.8), the optimal values for $C_e$ and $K_e$, for zero mean Gaussian excitations are (Atalik and Utku, 1976; Baber and Wen, 1980),

$$C_e = E[\frac{\partial^2 z}{\partial u^2}]$$

$$K_e = E[\frac{\partial^2 z}{\partial z}]$$

The general expressions for $C$ and $K$ in terms of the response statistics, as obtained by Baber and Wen (1980), are given in Appendix A. The nonlinear random vibration problem is, therefore, reduced to a linear one in which the coefficients of the linear system are response-dependent; the final solution being obtained by iteration.

4.1.2 Solution of the Equivalent Linear System

The random vibration solution of a linear system is relatively straightforward. The equations of motion are first decomposed into a system of first order differential equations, resulting in the general system equations, written in matrix form, as follows:

$$\dot{\mathbf{y}} + G\mathbf{y} = f(t)$$

For the current model, there will be three first-order equations for each degree of freedom - two replacing the second order dynamic
equilibrium equation (Eq. 3.8 for the single degree of freedom case) and one for the linearized hysteresis equation (Eq. 4.1).

Postmultiplying Eq. 4.3 by $\tilde{\gamma}$, and then taking expected values and adding the resulting equation to its transpose, gives the classical result

$$\dot{S} + GS + SG^T = B$$

(4.4)

where,

$$S = E[yy^T] \quad \text{i.e.,} \quad s_{ij} = E[y_i y_j]$$

(4.5a)

and

$$B = E[f_x^T] + E[\tilde{y}f^T]$$

(4.5b)

The desired response statistics are obtained by solving Eq. 4.4 for the zero time lag covariance matrix $S$.

For the case of earthquake loading, the vector $f(t)$ represents the ground acceleration. As discussed in Chapter 2, this motion is modeled by a filtered Gaussian shot noise random process. In implementation, however, an uncoupled "basement story" is added to the structure to act as the filter. The natural frequency and damping ratio of the filter are as specified in Chapter 2 for the specific soil condition. Furthermore, the equations of motion are written such that the base excitation appears only in the dynamic equilibrium equation of the filter (Baber and Wen, 1980). Thus, the vector $\tilde{\gamma}(t)$ has only one nonzero component which is a Gaussian shot noise random process. This term is written as the product of a time envelope function $\sqrt{\psi(t)}$ (see Eq. 2.9) and a stationary Gaussian shot noise process $\xi(t)$. It may be shown then, that the $B$ matrix has only one nonzero term, the diagonal term in the row associated with the filter dynamic equilibrium equation, $b$, given by

$$b = 2\pi\psi(t)s_0$$

(4.6)
in which $s$ is the constant power spectral intensity of the stationary shot noise process.

In the stationary case (i.e., stationary load, $\psi(t) = 1$, and nondegrading system) the covariance matrix $S$, is constant in time, and Eq. 4.4 becomes

$$GS + SG^T = B$$

The problem is, thus, reduced to solving a set of algebraic equations. This equation is a special case of the matrix equation $AX + XB = C$ (where $A$, $B$, and $C$ are matrices of dimensions $m \times m$, $n \times n$, and $m \times n$, respectively, and $X$ is the $m \times n$, matrix of unknowns). An efficient algorithm has been devised by Bartels and Stewart (1972) for solving this system when the matrices are of order up to 100. The algorithm is based on a series of transformations that reduce the matrix $A$ to lower triangular form and the matrix $B$ to upper triangular form. The transformed equation, which has the form $A'X' + X'B' = C'$ ($A'$ and $B'$ are the transformed lower and upper triangular matrices) is now easily solved for $X'$. The solution of the original equation is then obtained by transforming $X'$ back to $X$.

Bartels and Stewart also consider the special symmetric form of the equation which is of interest here. Significant reduction in the computation time is possible by taking advantage of the symmetry of the $S$ matrix.

Since the $G$ matrix contains the response dependent equivalent linear coefficients, the iterative solution process requires guessing initial values for the equivalent linear coefficients, solving the matrix equation for $S$, and then using these response statistics to recompute the equivalent linear coefficients. The procedure is continued until convergence of the equivalent linear coefficients or the response statistics is achieved.

For the nonstationary case, solution of Eq. 4.4 requires numerical integration. Many efficient algorithms, however, are readily available for solving systems of first order nonlinear ordinary differential equations.
For systems with deterioration governed by energy dissipation, one differential equation of the form

\[
\dot{E}_i = (1 - \alpha)kE[\dot{u}_i z_i]
\]  

(4.8)

representing the expected rate of energy dissipation for the \(i\)th degree of freedom (see Eq. A.3), is added for every degree of freedom. Note that \(E[\dot{u}_i z_i]\) is an element of the covariance matrix. The solutions of Eqs. 4.4 and 4.8, are the response statistics contained in the covariance matrix along with the expected energy dissipation, as functions of time.

4.1.3 Displacement-Dependent Stiffness Degradation

For systems with displacement dependent stiffness degradation additional considerations are needed in the random vibration analysis. For deterministic excitations, the stiffness degradation parameter \(\eta\) (governed by Eq. 3.19), is constant over each \(1/2\)-cycle of oscillation and then updated at the occurrence of a displacement peak. However, for random excitations, the amplitude of the response (peaks) as well as the time between the occurrence of the peaks are random quantities. Rigorously, determination of the time-varying expected value of \(\eta\) (required for evaluating the equivalent linear coefficients) would require an analysis of the time varying statistics of a complex nonhomogeneous pulse process.

An approximate solution can be obtained, however, by idealizing \(\eta(t)\) as a random variable whose density function remains constant over the expected period between peaks. Thus, the expected value of \(\eta(t)\) needs only to be calculated (or updated) at discrete time points. In implementing the method, the value of \(E[\eta(t)]\) is held constant for the current expected zero crossing period, assuming that the time between peaks is approximately the same as the zero crossing period. Any error caused by this approximation would be small as the value of \(\eta\) would not
be expected to change significantly if successive peaks occurred without a zero crossing.

For nonstationary response, the expected zero crossing rate is given as

\[ \lambda_o(t) = \frac{\sigma_u(t) \sqrt{1 - \rho_{uu}^2(t)}}{\pi \sigma_u(t)} \]  

(4.9)

if \( \dot{u} \) and \( u \) are assumed to be jointly Gaussian. The expected zero crossing period (i.e., the time between zero crossings) is then

\[ T_o(t) = \frac{1}{\lambda_o(t)} \]  

(4.10)

Following the deterministic definition of \( \eta \), its expected value at time \( t \) is

\[ E[\eta(t)] = \int_0^\infty A_o \frac{u_{pk}(t) + \bar{u}_{pk}(t - T_o)}{z[u_{pk}(t), - \bar{u}_{pk}(t - T_o)] + z[\bar{u}_{pk}(t - T_o)]} \]

\[ \cdot f_{pk}(u_{pk}, \sigma_u, \sigma_{\dot{u}}, \rho_{\dot{u}u}, t) du_{pk} \]  

(4.11)

where \( \bar{u}(t - T_o) \) and \( z[\bar{u}(t - T_o)] \) represent the expected peak displacement and the corresponding value of \( z \), respectively, at the previous update, and \( f_{pk}(u_{pk}, \sigma_u, \sigma_{\dot{u}}, \rho_{\dot{u}u}, t) \) is the probability density function (at time \( t \)) of the peak amplitude for nonstationary random response (Kobori and Minai, 1967),

\[ f_{pk}(u_{pk}, \sigma_u, \sigma_{\dot{u}}, \rho_{\dot{u}u}, t) = \exp\left(-\frac{u_{pk}^2}{2\sigma_u^2}\right) \exp\left(-\frac{\rho_{\dot{u}u}^2 u_{pk}^2}{2(1 - \rho_{\dot{u}u}^2)\sigma_u^2}\right) \]

\[ + \frac{\rho_{\dot{u}u}}{\sqrt{1 - \rho_{\dot{u}u}^2}} \frac{\pi}{2\sigma_u^2} \left(\frac{u_{pk}^2}{2\sigma_u^2} - 1\right) \text{erf}\left\{ \frac{u_{pk}}{\sqrt{2(1 - \rho_{\dot{u}u}^2)\sigma_u^2}} \right\} \]  

(4.12)
where

\[ \rho_{uu} = \frac{E[\dot{u}\dot{u}]}{\sigma_u \sigma_u} \quad (4.13) \]

Solution of Eq. 4.11 is extremely tedious, as the evaluation of \( z[u(t), -\dot{u}(t-T)] \) requires solution of a nonlinear differential equation (Eq. 3.11 with the right hand side divided by \( \eta \)), at each step of the numerical integration. Moreover, numerical integration for \( \ddot{u}(t) \) is required after \( E[\eta(t)] \) is calculated. Considerable simplification is obtained by using a first order approximation, giving

\[ E[\eta(t)] \approx A \frac{\ddot{u}_{pk}(t) + \ddot{u}_{pk}(t-T_o)}{z[\ddot{u}_{pk}(t), -\ddot{u}_{pk}(t-T_o)] + z[\ddot{u}_{pk}(t-T_o)]} \quad (4.14) \]

where

\[ \ddot{u}_{pk}(t) = \int_0^\infty u_{pk}(t)f_{pk}(u, \sigma_u, \sigma_u, \rho_{uu}, t)du \quad (4.15) \]

On this basis, one needs to solve the differential equation and perform the numerical integration for \( \ddot{u}(t) \) only once for each update. The above approximation has been compared with Eq. 4.11 for several cases and found to be accurate.

Finally, it is assumed that the expressions derived by Baber and Wen (1980) for the equivalent linear coefficients (Eq. A.4) remain valid for the present case (i.e., displacement-dependent degradation). This is based on the consideration that the assumption used in deriving Eq. A.4, namely, that \( E[\eta(t)] \) is a slowly varying function of time, still holds. Thus, for the case of displacement-dependent stiffness degradation the equivalent linear coefficients are evaluated by using Eq. 4.14 instead of Eq. A.2, to determine \( \ddot{\eta}(t) \) in Eq. A.4.
4.2 Additional Statistics

For assessing the safety of a structure, additional statistics of the response, besides the elements of the covariance matrix, are necessary. The statistics of the maximum response and the energy dissipation are of particular interest since these are useful or necessary for describing structural damage and failure.

4.2.1 Maximum Displacement Statistics

The maximum displacement statistics may be obtained using an approach developed by Yang and Liu (1981). The method is based on the simulation results of Shinozuka and Yang (1971), which indicate that the distribution, \( F_\mu (u,T_1,T_2) \), of the nonstationary peaks, \( u_p \), of \( u(t) \) in the time interval \( (T_1,T_2) \) can be represented by the Weibull distribution; i.e.,

\[
F_{u_p} (u,T_1,T_2) = 1 - \exp\left(-\frac{u}{\alpha\sigma}\right)^\alpha
\]

(4.16)

where the parameters \( \alpha \) and \( \sigma \) are dependent on \( T_1 \) and \( T_2 \). These parameters may be evaluated by assuming that the mean and mean square values of the peak at time \( t \), given that it has occurred, are (Yang; 1972, 1973)

\[
E[u_p(t)] = \sqrt{\frac{\pi}{2}} \sigma_u(t)
\]

(4.17)

and

\[
E[u_p^2(t)] = 2\sigma_u^2(t)
\]

(4.18)

Taking the time averages of Eqs. 4.17 and 4.18 over the interval \( (T_1,T_2) \) gives the mean and mean square of the Weibull-distributed peaks, from which the Weibull parameters \( \alpha \) and \( \sigma \) are evaluated by solving,
\[
\delta_{U_p}(T_1, T_2) = \frac{[\Gamma \left( \frac{2}{\alpha} + 1 \right) - \Gamma \left( \frac{1}{\alpha} + 1 \right)]^{1/2}}{\Gamma \left( \frac{1}{\alpha} + 1 \right)}
\] (4.19)

\[
\mathbb{E}[\bar{u}_p(T_1, T_2)] = \sigma^{1/\alpha} \Gamma \left( \frac{1}{\alpha} + 1 \right)
\] (4.20)

in which \(\delta\) is the coefficient of variation.

Assuming that the peaks occurring in \((T_1, T_2)\) are statistically independent and the total number of peaks, \(n\), is large, Yang and Liu (1981) obtained the distribution of the maximum response, \(U_m\), as

\[
F_{U_m}(u; T_1, T_2) = F_{U_p}(u; T_1, T_2) \approx \exp\left\{-\exp\left[-K^{\alpha-1}\frac{u}{\sigma} - K\right]\right\}
\] (4.21)

where

\[
K = (\alpha \mu n)^{1/\alpha} \left[ \alpha \mu \int_{T_1}^{T_2} 2^{\lambda^{+}_o(t)} dt \right]^{1/\alpha}
\] (4.22)

and \(\lambda^{+}_o\) is the time-varying zero upcrossing rate obtained using the method due to Rice (1941, 1945) assuming that the displacement, \(u\), and velocity, \(\dot{u}\), are jointly Gaussian. That is,

\[
\lambda^{+}_o(t) = \frac{\sigma_u(t) \sqrt{1 - \rho_{uu}^2(t)}}{2\pi \sigma_u(t)}
\] (4.23)

The mean value and standard deviation of the maximum response are then

\[
\mathbb{E}[U_m] = (K + 0.5772K^{1-\alpha})\sigma
\] (4.24)

\[
\sigma_{U_m} = \frac{\pi}{\sqrt{6}} K^{\alpha-1} \sigma
\] (4.25)
In the stationary case, $\alpha=2.0$ and $\sigma=\sigma(t)$, a constant, and Eqs. 4.24 and 4.25 reduce to the classical result (see Davenport, 1964).

Because of the assumption that the response process is Gaussian, the value of $\sigma_u$ may be underestimated (Baber and Wen, 1980). For the present study, the variance of the expected maximum is taken as $\sigma_u^2 + [(0.15)\langle U \rangle]$.

For the reliability analysis, the derivative of the expected maximum with respect to a general structural or ground motion parameter, $p$, is required. As the Weibull parameter $\alpha$, was found to be comparatively insensitive to parametric changes, its derivative may be neglected. Then,

$$\frac{\partial E[U_m]}{\partial p} = (K + 0.5772K^{1-\alpha}) \frac{\partial \sigma}{\partial p} + (1 + 0.5772(1-\alpha)K^{-\alpha}) \sigma \frac{\partial K}{\partial p}$$  \hspace{1cm} (4.26)

where,

$$\frac{\partial \sigma}{\partial p} = \frac{1}{\alpha^{1/\alpha} (\frac{1}{\alpha} + 1)} \frac{\partial E[u_p(T_1,T_2)]}{\partial p}$$  \hspace{1cm} (4.27)

and

$$\frac{\partial K}{\partial p} = \frac{1}{n} (\alpha \ell \ln n) 1/\alpha - 1 \int_{T_1}^{T_2} \frac{\partial \lambda^+}{\partial p} (t) dt$$  \hspace{1cm} (4.28)

in which

$$n = \int_{T_1}^{T_2} 2\lambda^+(t) dt$$

Evaluation of the derivatives appearing on the right hand sides of Eq. 4.27 and 4.28 are straightforward once the derivative of the covariance matrix is obtained (see Sect. 4.3). For the stationary case, $\alpha=2.0$, $\partial \sigma/\partial p = \partial \sigma/\partial p$ and $\partial \lambda^+/\partial p$ is a constant.

It may be mentioned that a more direct approach for evaluating the maximum response statistics has been developed by Suzuki and Minai (1980). This approach involves augmenting the equations of motion with
one additional nonlinear equation per degree of freedom, the solution of which is a nondecreasing function describing the maximum response that has occurred. The equation is given as

\[ \ddot{u}_m = |\dot{u}|H(u\dot{u})H(|u| - u_m) \]  

(4.29)

where \( H(x) = 1 \) for \( x \geq 0 \), and \( H(x) = 0 \) for \( x < 0 \).

This approach is compatible with the random vibration framework used herein. The maximum response statistics may be obtained by linearizing Eq. 4.29 and adding one such equation per degree of freedom. However, the linearization is complicated by the fact that the variable of interest, i.e., the maximum response, has nonzero mean. Furthermore, the determination of the equivalent linear coefficients requires the evaluation of triple integrals, for which closed form expressions may not be available.

4.2.2 Standard Deviation of Energy Dissipation

In Sect. 4.1.2, it is shown that the mean dissipated energy can be obtained directly from the zero time lag covariance matrix. In addition, the standard deviation is required for probabilistic assessment of structural failure in terms of energy dissipation. The standard deviation of the dissipated energy may be obtained from the two-time response covariance matrix (Pires, 1983) as summarized below. On the basis of Eq. 3.17, the mean square energy dissipated at time \( t \) is

\[ E[c^2(t)] = E[(1 - \alpha)^2k^2 \int_0^t z(\tau)\dot{u}(\tau)d\tau \cdot \int_0^t z(\tau)\dot{u}(\tau)d\tau] \]

\[ = (1 - \alpha)^2k^2 \int_0^t \int_0^t E[z(t_1)\dot{u}(t_1)z(t_2)\dot{u}(t_2)]dt_1dt_2 \]  

(4.30)

In general, the two-time joint probability density function of \( z \) and \( \dot{u} \) is necessary to evaluate the above integral. However, if \( z \) and \( \dot{u} \) are
zero mean and jointly Gaussian,

\[ E[z(t_1)\dot{u}(t_1)z(t_2)\dot{u}(t_2)] = E[z(t_1)\dot{u}(t_1)]E[z(t_2)\dot{u}(t_2)] + E[z(t_1)z(t_2)]E[\dot{u}(t_1)\dot{u}(t_2)] + E[z(t_1)\dot{u}(t_2)]E[z(t_2)\dot{u}(t_1)] \]  \tag{4.31}

Observing that

\[ E^2[\varepsilon(t)] = (1 - \alpha)^2 \int_0^t \int_0^t E[z(t_1)\dot{u}(t_1)]E[z(t_2)\dot{u}(t_2)]dt_1 dt_2 \]  \tag{4.32}

the variance of the energy dissipation is

\[ \sigma^2_\varepsilon(t) = E[\varepsilon^2(t)] - E^2[\varepsilon(t)] = (1 - \alpha)^2 \int_0^t \int_0^t \{E[z(t_1)z(t_2)]E[\dot{u}(t_1)\dot{u}(t_2)] + E[z(t_1)\dot{u}(t_2)]E[z(t_2)\dot{u}(t_1)]\} dt_1 dt_2 \]  \tag{4.33}

The necessary expectations may then be obtained from the two-time covariance matrix.

The differential equation that governs the two-time covariance matrix is obtained in a similar manner as Eq. 4.4 (the zero time lag equation). Again, writing the equations of motion as a system of first order equations gives

\[ \frac{\partial}{\partial t_1} y(t_1) = G(t_1)y(t_1) + f(t_1) \]  \tag{4.34}

Postmultiplying by \( y(t_2)^T \), taking expected values, and then interchanging expectation and differentiation, yields

\[ \frac{\partial}{\partial t_1} S(t_1, t_2) = G(t_1)S(t_1, t_2) + E[f(t_1)y(t_2)^T] \]  \tag{4.35}
where

\[ S(t_1, t_2) = E[y(t_1)y(t_2)^T] \]

is the desired two-time response covariance matrix. A simplification of Eq. 4.35 is obtained by recognizing that for \( t \geq t_1 \), \( E[f(t_1)y(t_2)^T] = 0 \), since the loading at time \( t \) is uncorrelated with the response prior to \( t \); thus

\[ \frac{\partial}{\partial t_1} S(t_1, t_2) = G(t_1)S(t_1, t_2) \quad , \quad t_1 \geq t_2 \quad (4.36) \]

The \( S(t, t) \) matrix is obtained by specifying \( t \) and solving Eq. 4.36 with the initial value of \( t = t_2 \). The complete solution is obtained by repeating the procedure for different values of \( t \), and observing that the \( ij \) element of the \( S \) matrix evaluated for \( t \geq t_2 \) is the \( ji \) element of the \( S \) matrix for \( t < t_2 \).

Details of the evaluation of the energy dissipation statistics may be found in Pires (1983).

### 4.3 Sensitivity Coefficients

From Eq. 3.32 it is seen that the reliability analysis requires evaluation of the derivatives of the response with respect to specific parameters, that is, the "sensitivity coefficients", with which the contribution of parameter uncertainty to the total response uncertainty may be evaluated.

Two approaches for evaluating the sensitivity coefficients are possible. The coefficients may be evaluated numerically by central finite difference using the response corresponding to parameter values above and below the mean value. Alternatively, an equation of the sensitivity coefficients can be derived by differentiating the equations of motion with respect to the parameter of interest; solutions of the resulting system are then the desired sensitivity coefficients. This method gives exact solutions and can be obtained directly. Also, the
equations are always linear and, therefore, easily solved. This direct approach is used in this study.

4.3.1 Derivation and Solution of the Derivative Equations

The pertinent equation is obtained by differentiating Eq. 4.4 with respect to the parameter of interest and interchanging the time and parameter derivatives. Letting \( p \) represent the parameter of interest, the derivative equation is

\[
\frac{\partial}{\partial t} \frac{\partial S}{\partial p} + \frac{\partial G}{\partial p} S + G \frac{\partial S}{\partial p} + \frac{\partial S}{\partial p} G^T + S \frac{\partial G^T}{\partial p} = \frac{\partial B}{\partial p}
\]  

(4.37)

where

\[
\frac{\partial S}{\partial p} = \frac{\partial}{\partial p} E[y^T y]
\]

(4.38)

Thus, the \( n \) th element of \( \partial S / \partial p \) is \( \frac{\partial}{\partial p} E[y_{ij} y_{ij}] \), which are the necessary response statistic derivatives (i.e., sensitivity coefficients).

Examination of Eq. 4.37 reveals that the sensitivity coefficients depend on the response level, as the equation contains the \( S \) matrix. This, of course, is characteristic of nonlinear systems. Thus, the solution for the \( \partial S / \partial p \) matrix will first require knowledge of the response covariance matrix \( S \), which is part of the solution of the random vibration analysis.

It may be pointed out that since the \( \partial G / \partial p \) matrix (containing the derivatives of the equivalent linear coefficients, which are functions of the unknown sensitivity coefficients) in Eq. 4.37 appears only as a multiplier of the \( S \) matrix (known), the equation is linear in terms of the unknown response derivatives. This is in clear contrast to the nonlinear equations (Eqs. 4.4 and 4.7) that govern the structural response.

For the stationary case, the response derivative matrix \( \partial S / \partial p \) is constant in time. Thus, after rearranging terms and taking the time
derivatives to be zero, Eq. 4.37 leads to the corresponding stationary derivative equation

\[
G \frac{\partial S}{\partial p} + \frac{\partial S}{\partial p} G^T = \frac{\partial B}{\partial p} - \frac{\partial G}{\partial p} S - S \frac{\partial G}{\partial p}
\]

(4.39)

Observe that the equation takes the form \( AX + XB = C \), discussed in Sect. 4.1.2, where the unknown matrix \( X \) represents the unknown response derivative matrix \( \partial S/\partial p \). However, the matrix \( \partial G/\partial p \) appearing on the right-hand side of Eq. 4.39 contains the unknown derivatives of the equivalent linear coefficients (functions of the unknown response derivatives). To solve the equations in this form, it is necessary to assume initial values for the derivatives of the equivalent linear coefficients, solve Eq. 4.39 for the unknown matrix \( \partial S/\partial p \) using the algorithm of Bartels and Stewart (discussed in Sect. 4.1.2), and then use these results to recalculate the derivatives of the equivalent linear coefficients. This procedure is continued until convergence of the solution is achieved.

Theoretically, the solution may be obtained without iteration (since the equation is linear), by expanding and factoring the equation into the form \( Ax = b \), such that \( x \) is a vector containing the unknown elements of the matrix \( \partial S/\partial p \) and \( A \) and \( b \) contain only known terms. However, the order of the matrix \( A \) increases rapidly with the number of degrees of freedom.

Because Eq. 4.39 is linear, the iterative procedure converged, in all test cases, after only a few cycles, even when the assumed initial values were off by orders of magnitude. Furthermore, the calculational time for each iteration is quite short since only the right-hand side of the equation is changed in each cycle. Thus, in each iteration only a simple back substitution is required. For these reasons, the iterative approach is used.

For the nonstationary case, the equation is solved numerically for the response statistics (see Sect. 4.1.2).
4.3.2 Evaluation of the Matrices $\partial B/\partial p$ and $\partial G/\partial p$

The matrix $\partial B/\partial p$ is easily evaluated for the case of base excitation since the matrix $B$ contains only one nonzero term, given by Eq. 4.6. If the parameter, $p$, is a structural parameter (e.g., a particular story stiffness or mass), the derivative of this term and, therefore, the entire $\partial B/\partial p$ matrix is zero. However, it may be necessary to calculate the derivatives with respect to the parameters $t_1$, $t_2$, or $c$, that define the temporal variation of the mean square intensity $ψ(t)$, given by Eq. 2.9. Specifically, the derivatives with respect to $t_1$ and $t_2$ would be necessary to evaluate the effect of uncertainty in the earthquake duration on the response. In these instances, the matrix $\partial B/\partial p$ has one nonzero term. The required expressions are given in Appendix B.

As the $G$ matrix contains primarily system parameters (i.e., story mass, stiffness, etc.), evaluation of the matrix $\partial G/\partial p$ is, for the most part, straightforward. However, the matrix $G$ also contains the equivalent linear coefficients. Differentiating Eq. A.4 with respect to a general parameter (represented by $p$) gives

\[
\frac{\partial C}{\partial p} = \left( \frac{\partial A}{\partial p} - \nu (B \frac{\partial F_1}{\partial p} + \frac{\partial B}{\partial p} F_1 + \gamma \frac{\partial F_2}{\partial p} + \frac{\partial F}{\partial p} F_2) \right)
\]

\[-\frac{\partial \bar{u}}{\partial p} (\beta F_1 + \gamma F_2) - \frac{\partial \bar{u}}{\partial p} [\bar{A} - \bar{u}(\beta F_1 + \gamma F_2)]/\bar{n}^2 \quad (4.40a)\]

\[
\frac{\partial K}{\partial p} = \left( - \bar{u} (B \frac{\partial F_3}{\partial p} + \frac{\partial B}{\partial p} F_3 + \gamma \frac{\partial F_4}{\partial p} + \frac{\partial F}{\partial p} F_4) \right)
\]

\[+\frac{\partial \bar{u}}{\partial p} (\beta F_3 + \gamma F_4) + \frac{\partial \bar{u}}{\partial p} [\bar{u}(\beta F_3 + \gamma F_4)]/\bar{n}^2 \quad (4.40b)\]
where
\[
\frac{\partial A}{\partial \rho} = \frac{\partial A}{\partial \rho} - \frac{\partial A}{\partial \rho} - \frac{\partial A}{\partial \rho} + \frac{\partial A}{\partial \rho} \varepsilon
\]
and
\[
\frac{\partial \nu}{\partial \rho} = \frac{\partial \nu}{\partial \rho} + \frac{\partial \nu}{\partial \rho} \varepsilon
\]
(4.41)

\[
\frac{\partial \eta}{\partial \rho} = \frac{\partial \nu}{\partial \rho} + \frac{\partial \eta}{\partial \rho} \varepsilon
\]
(4.42)

when \( \eta \) is governed by Eq. A.2 for energy-based stiffness degradation.

When \( \eta \) is governed by Eq. 4.14 (displacement-dependent stiffness degradation) the expression for \( \frac{\partial \eta}{\partial \rho} \), after substituting \( u \) for \( \bar{u}(t) \), \( u \) for \( \bar{u}(t-T) \), \( z \) for \( z(\bar{u}(t), -\bar{u}(t-T)) \), and \( z \) for \( z(\bar{u}(t-T)) \) becomes,

\[
\frac{\partial \eta}{\partial \rho} = A_0 \left( \frac{u_f + u_o}{z_f + z_o} \right) \left( \frac{\partial u_f}{\partial \rho} + \frac{\partial u_o}{\partial \rho} - (u_f + u_o) \left( \frac{\partial z_f}{\partial \rho} + \frac{\partial z_o}{\partial \rho} \right) \right) + \frac{\partial A_0}{\partial \rho} \left( \frac{u_f + u_o}{z_f + z_o} \right)
\]
(4.43)

As in the case of \( \eta(t) \), \( \frac{\partial \eta}{\partial \rho}(t) \) is a step function in time, updated whenever \( \eta(t) \) is updated. The derivative of the peak displacement \( u \) is obtained by differentiating Eq. 4.15, giving

\[
\frac{\partial \bar{u}_{pk}}{\partial \rho} = \int_0^\infty u_{pk} \frac{\partial f_{pk}(u_{pk}, \sigma', \sigma, \rho, t)}{\partial \rho} \, du_{pk}
\]
(4.44)

The derivative of the peak amplitude density function, \( f_{pk}(u_{pk}, \sigma', \sigma, \rho) \), is obtained by differentiating Eq. 4.12, noting that the response quantities \( \sigma', \sigma \) and \( \rho \) are functions of the parameter \( \rho \). Evaluation
of Eq. 4.43 also requires calculation of \( \frac{\partial z(u, u)}{\partial p} \). Although a closed form expression for \( z(u, u) \) is generally not available (as it is the solution of a nonlinear differential equation, Eq. 3.11) the required derivative expression may be evaluated as shown in Appendix B.

The derivation of the terms \( \frac{\partial F}{\partial p} \), \( \frac{\partial F}{\partial p} \), \( \frac{\partial F}{\partial p} \), and \( \frac{\partial F}{\partial p} \) (Eq. 4.40) is also presented in Appendix B. Since \( F, F, F \) and \( F \) (see Eqs. A.6) are functions of the response statistics \( \sigma, \sigma, \sigma, \) and \( \beta \), their derivatives will also depend on these statistics and their derivatives.

When derivatives with respect to story strength are required, some modifications are necessary as there is no single parameter in the basic model that defines story strength. This is seen by examining Eqs. 3.12 and 3.13. Examining Fig. 3.2 (for the case \( \delta_v = 0.20 \)), it is seen that varying the parameter \( \nu \) results in a change in strength without affecting other aspects of the model behavior. Thus, \( \nu \) can be considered as a strength parameter and variation in the yield strength may be assumed to be proportional to variation in \( \nu \). Development of the necessary expressions for evaluating response derivatives with respect to strength are given in Appendix B.

### 4.3.3 Additional Observations

From extensive testing, the direct, analytical approach for evaluating the derivatives appears to give accurate results (verified with finite difference solutions), and required approximately one half the computation time of the finite difference method.

Since the ground motion is filtered with an additional "basement story" (see Sect. 4.1.2), the response derivatives with respect to the filter parameters \( \omega \) and \( \beta \) are easily obtained. As formulated herein, these parameters appear as elements of the matrix \( G \) (defined by Eq. 4.3); thus, it is only necessary to calculate the proper \( \frac{\partial G}{\partial p} \) matrix in Eq. 4.37 or 4.39, to obtain the response derivatives with respect to the filter parameters.
Although the derivatives discussed above are for the single degree of freedom case, extension to the multi-degree of freedom problem is straightforward. The procedure is almost identical except that the derivatives of the equivalent linear coefficients for the \( i \)-th degree of freedom are now dependent on the \( i \)-th degree of freedom response statistics and the response statistics' derivatives. From this, it can be seen that the general derivative expressions given in Sect. 4.3.2 will frequently appear in much simpler form. For example, the term \( \beta \beta / \beta p \) in Eq. 4.40 will only be nonzero if the parameter \( p \) is \( \beta \) (\( \beta \) of the \( i \)-th degree of freedom) and the derivative of the equivalent linear coefficient for degree of freedom \( i \) is being calculated.
CHAPTER 5

LIFETIME DAMAGE AND SAFETY EVALUATION

5.1 Definition of Damage and Failure

5.1.1 Structural and Nonstructural Damage

Assessing building damage and safety in terms of monetary cost, or hazard to occupants, as a function of structural response variables (e.g., displacement, acceleration, dissipated energy, etc.), is a complex task fraught with uncertainty (Scawthorn, et al., 1981; Algan, 1982). The subject has recently attracted a number of studies (see review by Yao, 1979), including the use of fuzzy set theory (Yao, 1980; Ishizuka, Fu and Yao, 1980), which attempts to quantitatively measure imprecise but meaningful descriptors of damage.

Based on data obtained from buildings damaged during the 1971 San Fernando earthquake, Whitman, et al. (1974) determined that among the dynamic response quantities, damage repair cost correlated best with the interstory drift. This same conclusion was derived also by Scawthorn, et al., (1981) in an extensive study of buildings damaged in the June 12, 1978 Miyagiken-oki earthquake in Japan; specific relations between the damage states and interstory drifts were also presented, although these are not generally applicable for buildings in the U.S. A useful result of the study, however, is that the coefficient of variation in the drift-damage relation is approximately 45%. This large coefficient
of variation includes the inherent differences in the types of structures covered in the data.

In a recent study, Algan (1982) used the interstory drift to evaluate damage intensity covering both the structural system and the nonstructural walls of reinforced concrete buildings. Figure 5.1 shows damage intensity as a function of a "damage index", a measure of the story drift accounting for the relative contributions of frame-and-wall type behavior in the structural response. For frame structures, the damage index is the interstory drift, whereas for frame-wall structures it is the drift reduced by the effect of floor rotations. In Fig. 5.1, the elliptical form of the damage function shown is appropriate for nonstructural masonry partitions (determined from test data); the exponential form applies to nonstructural partitions detailed to accommodate small distortions by clearance or sliding, and also represents the results of an opinion survey on tolerable values of the damage index; whereas the linear form is for damage to the structural system (developed from an analysis of typical reinforced concrete members). The damage intensity may be approximately interpreted as follows: (0.0)=no damage, (0.1-0.3)=minor, (0.4-0.5)=moderate, (0.6-0.7)=substantial, and (0.8-0.9)=major damage.

![Damage Functions](image-url)
For steel structures, in lieu of a specific structural response-damage relation, structural performance may be evaluated in terms of the probability of exceeding specified ductility ratios.

Whitman, et al. (1974) attempted to relate the descriptions of damage states to actual repair costs (given in % of building replacement cost); their results are presented in Table 5.1. It should be cautioned that the damage ratios reflect only repair costs and do not include associated costs such as loss of income, cost of injury, etc. Using these results it is possible to estimate the expected repair cost, based on the damage intensities evaluated in the structural response analysis.

In order to account for prediction uncertainty in a general response-damage relation, the damage state indicator, D, (e.g., Algan's damage intensity) is written as

\[ D = N_D h(X) \]  

(5.1)

where \( N_D \) is a Bayesian correction variable used to account for bias and dispersion error in the assumed relation \( h(\xi) \), which is the function that relates the general response quantities, \( \xi \), to the damage state. The probability that damage will exceed some specific value, \( d \), is then

\[ P(D > d) = \int P(N_D h(X) > d | N_D = \xi) f_{N_D}(\xi) d\xi \]  

(5.2)

where \( f_{N_D}(\xi) \) is the probability density function of \( N_D \), obtained by examining the scatter of data in the response-damage relationship.

5.1.2 Structural Failure and Safety

Failure of a structure, as discussed by Takizawa and Jennings (1980), can be the result of localized failures of individual members and/or global structural instability caused by vertical loads combined with large lateral drifts. Evaluation of failure probabilities, therefore, must consider the possible modes of failure. Furthermore,
Table 5.1 Description of Damage States (after Whitman, et al., 1974)

<table>
<thead>
<tr>
<th>Damage State</th>
<th>Repair Cost/Replacement Cost (in %)</th>
<th>Central Value</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 No Damage</td>
<td></td>
<td>0</td>
<td>0 - 0.05</td>
</tr>
<tr>
<td>1 Minor non-structural damage—a few walls and partitions cracked, incidental mechanical and electrical damage</td>
<td>0.1</td>
<td>0.05 - 0.3</td>
<td></td>
</tr>
<tr>
<td>2 Localized non-structural damage—more extensive cracking (but still not widespread); possibly damage to elevators and/or other mechanical/electrical components</td>
<td>0.5</td>
<td>0.3 - 1.25</td>
<td></td>
</tr>
<tr>
<td>3 Widespread non-structural damage—possibly a few beams and columns cracked, although not noticeable</td>
<td>2</td>
<td>1.25 - 3.5</td>
<td></td>
</tr>
<tr>
<td>4 Minor structural damage—obvious cracking or yielding in a few structural members; substantial non-structural damage with widespread cracking</td>
<td>5</td>
<td>3.5 - 7.5</td>
<td></td>
</tr>
<tr>
<td>5 Substantial structural damage requiring repair or replacement of some structural members; associated extensive non-structural damage</td>
<td>10</td>
<td>7.5 - 20</td>
<td></td>
</tr>
<tr>
<td>6 Major structural damage requiring repair or replacement of many structural members; associated non-structural damage requiring repairs to major portion of interior; building vacated during repairs</td>
<td>30</td>
<td>20 - 65</td>
<td></td>
</tr>
<tr>
<td>7 Building condemned</td>
<td>100</td>
<td>65 - 100</td>
<td></td>
</tr>
<tr>
<td>8 Collapse</td>
<td>100</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
Criteria are needed for determining the likelihood of failure of individual members, given their response history.

Various attempts have been made to develop member failure prediction models. Difficulties arise, however, due to a lack of understanding of the mechanisms that lead to failure. In a study by Banon and Veneziano (1982) both energy dissipation (normalized to one half the maximum elastic energy stored in the member in anti-symmetric bending) and the flexural damage ratio (a measure of the maximum displacement that also accounts for strength degradation) were investigated as possible failure indicators for reinforced concrete members (see Fig. 5.2). Based on the failure points and the actual paths to failure of 29 test specimens of various configurations, subjected to various types of cyclic loading, contours of equal failure probability for given amounts of dissipated energy and flexural damage ratio were derived. In another study of reinforced concrete members, Hwang (1982) proposed an "energy index", which is a measure of the energy dissipated weighted by the maximum force and displacement in each cycle, as a means for predicting failure. The energy index evaluated at failure for a number of test specimens showed relatively little scatter (Fig. 5.3), for members with similar reinforcing ratios and subjected to

\[
D_1 = \text{flexural damage ratio}
\]

\[
D_2 = \text{normalized dissipated energy}
\]

Fig. 5.2 Experimental Failure Points and Contours of Equal Failure Probability (Banon and Veneziano, 1982)
Fig. 5.3 Experimental Failure Points (Hwang, 1982)
various types of cyclic loadings producing similar maximum shear stress. Prediction of a particular member's energy index at failure, therefore, requires consideration of the expected maximum shear stress level and the reinforcement ratio.

For steel members, failure is generally caused by the local buckling of the plate elements of a section or by flexural torsional buckling (Kato and Akiyama, 1982). Based on a large number of tests, Kato and Akiyama related local buckling to cumulative rotational ductility (essentially a measure of energy dissipation) and determined the allowable design ductilities based on column-and-beam section width to thickness ratios. For flexural torsional buckling, allowable limits on beam slenderness ratios were recommended. The data used by Kato and Akiyama (1982) for evaluating these limits may be useful in constructing probabilistic failure models for steel members, based on maximum displacements and dissipated energy.

Since the maximum displacement and energy dissipation statistics are available from the random vibration analysis (as discussed in Chapter 4), the models described above may, theoretically, be used to evaluate member failure probabilities. At the present time, however, computational limitations have restricted the response analysis to lumped mass models based on one degree-of-freedom per story. Therefore, only the general story response statistics, and not the detailed member response information, may be practically obtained.

In light of the above, structural safety may be expressed in terms of the probability of exceeding some critical level of damage based on the story drifts. For reinforced concrete structures, the linear function for structural damage given in Fig. 5.1 may be used in this regard. Safety will be defined as "damage intensity < 0.9". For steel structures, in lieu of a specific damage function for the structural system, safety may be evaluated in terms of the probability of exceeding some critical ductility ratio, e.g., as specified by a seismic code.
5.2 Lifetime Probability Evaluation

In the foregoing, methods for determining the response statistics (up to the second moment), for a load of given intensity have been presented. However, the probabilities that particular response or damage levels will be exceeded during the life of the structure are pertinent to its seismic safety or performance. Using the seismic hazard model presented in Chapter 2 to evaluate the exceedance probabilities of specific ground motion intensities (measured by the expected maximum acceleration) over a particular time duration, the desired lifetime exceedance probabilities may be evaluated as

\[ P(X_T > x) = \int_0^\infty P(X > x | A = a)f_T(a)da \]

where \( f(a) \) is the probability density function of the expected maximum acceleration in \( T \) years, and \( X \) represents the response quantity for which exceedance probabilities are needed (e.g., maximum drift, dissipated energy, etc.). The conditional cumulative distribution function, \( F_{X|a}(x) \), is obtained by fitting an appropriate probability distribution to the mean and variance of the response, when the random process loading has an expected maximum acceleration \( a \). For example, the extreme value Type I distribution has been found to fit simulation results for the maximum interstory drift reasonably well (Baber and Wen, 1980).

5.2.1 Error in Calculated Probability

The probability estimated with Eq. 5.3 represents a central value of the real exceedance probability. Because of the various uncertainties in the modeling and estimation (as represented by the c.o.v. \( \Delta \) ) underlying the calculation, the probability is also subject to
error. This error in the calculated probability may be represented by the standard deviation of the calculated probability. With this information, error bounds (or "confidence" limits) on the calculated probability may be constructed (Ang and Tang, 1975).

The variance of the probability may be evaluated with Eq. 3.32, where \( \hat{X} \) is replaced by \( P(X > x) \). The uncertainties, however, are strictly those associated with prediction errors only; i.e., should not include the inherent variabilities.

The required derivative of \( P(X > x) \) (see Eq. 3.32) is evaluated by differentiating Eq. 5.3, i.e.,

\[
\frac{\partial P(X_T > x)}{\partial p_i} = \int_0^\infty \left( \frac{\partial P(X > x \mid A = a)}{\partial p_i} f_1(a) + P(X > x \mid A = a) \frac{\partial f_1(a)}{\partial p_i} \right) \, da \quad (5.4)
\]

where,

\[
\frac{\partial P(X > x \mid A = a)}{\partial p_i} = \frac{\partial [1 - F_X|_a(x) \mid]}{\partial p_i} \quad (5.5)
\]

in which \( F_X|_a(x) \) is the cumulative distribution function obtained earlier. In general, \( F_X|_a(x) \) will be an analytic function with parameters that depend on the moments (e.g., the mean and mean square) of the response \( X \) for the given excitation level \( a \). The derivative of \( F_X|_a(x) \) may, therefore, be expressed as a function of the derivatives of the moments of \( X \) (although the derivatives of the higher order moments may, generally, be neglected); thus required derivatives may be obtained using the procedures outlined in Sect. 4.3. For the case in which \( F_X|_a(x) \) has the extreme value Type I asymptotic form, the derivative is obtained as

\[
\frac{\partial F_X|_a(x)}{\partial p_i} = \frac{\partial F_X|_a(x)}{\partial u} \frac{\partial u}{\partial p_i} + \frac{\partial F_X|_a(x)}{\partial \alpha} \frac{\partial \alpha}{\partial p_i} \quad (5.6)
\]

where \( u \) and \( \alpha \) are the parameters of the Type I distribution, which are functions of the mean and standard deviation of \( X \) for the given excitation level \( a \). The derivatives of \( u \) and \( \alpha \) are evaluated using the chain rule, i.e.,
\[
\frac{\partial u}{\partial p_i} = \frac{\partial u}{\partial \bar{X}} \frac{\partial \bar{X}}{\partial p_i} + \frac{\partial u}{\partial \sigma_X} \frac{\partial \sigma_X}{\partial p_i} \tag{5.7a}
\]

and

\[
\frac{\partial \alpha}{\partial p_i} = \frac{\partial \alpha}{\partial \bar{X}} \frac{\partial \bar{X}}{\partial p_i} + \frac{\partial \alpha}{\partial \sigma_X} \frac{\partial \sigma_X}{\partial p_i} \tag{5.7b}
\]

For the Type I distribution, the derivatives of \( u \) and \( \alpha \) with respect to \( \bar{X} \) and \( \sigma \) are easily obtained; whereas the derivatives of \( \bar{X} \) and \( \alpha \) with respect to a parameter \( p_i \) are obtained from the derivative analysis (Sect. 4.3).

The derivatives of the Type I distribution with respect to \( u \) and \( \alpha \) are,

\[
\frac{\partial F_{X|\alpha}(x)}{\partial u} = \{F_{X|\alpha}(x)\}{-\exp[-\alpha(x-u)]}\{\alpha\} \tag{5.8a}
\]

and

\[
\frac{\partial F_{X|\alpha}(x)}{\partial \alpha} = \{F_{X|\alpha}(x)\}{-\exp[-\alpha(x-u)]}\{u-x\} \tag{5.8b}
\]

Finally, the derivative of \( f_T(a) \) as required in Eq. 5.4 will be nonzero with respect to a parameter or the Bayesian correction factor in the seismic hazard model. In these cases, the derivatives may be evaluated by finite difference.
CHAPTER 6

ILLUSTRATIVE APPLICATIONS

6.1 Introduction

To illustrate the proposed methodology, two example structures are analyzed. The first is a four-story steel frame building, and the second a seven-story reinforced concrete building; the latter structure sustained damage during the 1971 San Fernando earthquake. Both structures have been analyzed by other researchers, thus, comparisons between the expected maximum response predicted using the random vibration formulation presented herein, and the response predicted by other approaches is possible. The four-story frame structure serves to illustrate the application of the method to steel structures and to investigate the sensitivity of the structural response to the various parameters. The seven-story building illustrates the modeling of reinforced concrete structures, and includes a lifetime damage and safety analysis.
6.2 Four-Story Steel Frame Building

6.2.1 Structure Description and Modeling

The building being examined was designed and analyzed in a study by Lai (1980). Results of a simulation analysis are also reported in the study and will be compared with the random vibration results obtained herein. The frame dimensions and story masses are shown in Fig. 6.1; the corresponding shear-beam model parameters are given in Table 6.1.

<table>
<thead>
<tr>
<th>Story</th>
<th>Lateral Stiffness (kips/in)</th>
<th>Strength (kips)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>107.4</td>
<td>65.43</td>
</tr>
<tr>
<td>2</td>
<td>74.8</td>
<td>56.62</td>
</tr>
<tr>
<td>3</td>
<td>65.9</td>
<td>45.16</td>
</tr>
<tr>
<td>4</td>
<td>60.9</td>
<td>32.08</td>
</tr>
</tbody>
</table>

The story stiffnesses were obtained using the mode matching technique discussed in Sect. 3.1.1., based on the first mode shape given by Lai (1980). Figure 6.2 shows a comparison of the mode shapes for the equivalent shear-beam model and the corresponding mode shapes obtained from a direct stiffness frame analysis (Lai, 1980). As expected, the first mode is matched exactly whereas the higher modes are only negligibly different, even though the structure is not particularly well suited for shear beam modeling (i.e., it has relatively strong columns and weak girders, except at the top story). Table 6.2 lists the modal periods for the frame and shear-beam systems. The story strengths shown in Table 6.1 were obtained from an inelastic frame analysis (Lai, 1980).
Fig. 6.1 Four Story Steel Frame Analyzed (after Lai, 1980)
Fig. 6.2 Comparison of Exact and Equivalent Shear-Beam Mode Shapes
Table 6.2 Comparison of "Exact" and Shear-Beam System Modal Periods (secs.)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frame Model</td>
<td>0.967</td>
<td>0.320</td>
<td>0.186</td>
<td>0.134</td>
</tr>
<tr>
<td>Equivalent Shear Beam</td>
<td>0.967</td>
<td>0.353</td>
<td>0.234</td>
<td>0.191</td>
</tr>
</tbody>
</table>

Based on the recommendations for steel structures presented in Sect. 3.2.2, the values of the hysteretic parameters $A$, $\alpha$ and $n$ are taken as 1, 0.04 and 1, respectively, and $\beta$ is taken equal to $\gamma$. The specific values of $\beta$ (and $\gamma$) are obtained using Eqs. 3.12 and 3.13 yielding values of 0.788, 0.634, 0.700 and 0.911, respectively, for the four stories.

Finally, viscous damping was taken to be approximately 1%, 2%, and 3% of critical for the first three modes, respectively, and the restoring force behavior was assumed to be nondegrading.

6.2.2 Load Description

The ground motion is modeled as discussed in Chapter 2, assuming an intermediate soil condition. Thus, the power spectral density function is given by the Kanai-Tajimi spectrum with mean parameter values $\bar{\omega} = 16.5$ and $\bar{\beta} = 0.80$, (Table 2.1) and the mean strong motion duration is 7.0 seconds (Table 2.5). Based on the results presented in Table 2.2 and the peak factors of Table 2.6, the relationship between the power spectral intensity, $S_o$, and the expected maximum acceleration for this soil condition is given by

$$E[\text{a}_{\text{max}}] = 2.9 \sigma_a = 28.4 \sqrt{S_o} \quad (6.1)$$

The power spectral density function used by Lai (1980) for generating the artificial earthquake motions in the simulation study was
similar to the Kanai-Tajimi spectrum used here for intermediate soil conditions. Furthermore, Lai (1980) used a total excitation time of 10.0 seconds with 1.0 second linear rise and decay times, thus implying a strong motion duration approximately equal to that used here. Thus, comparison of the random vibration and available simulation results should be valid.

The seismic hazard prescribed for the reliability analysis is that of the Boston area, as this is the location for which the frame was originally designed. The risk curves were evaluated using the method of Der Kiureghian and Ang (1977), as described in Chapter 2, based on the geologic source data given by Cornell and Merz (1975) and Taleb-Agha (1977). The parameter $\beta=1.65$ (slope of the magnitude recurrence curve) was used as recommended for the Boston area by Taleb-Agha (1977). Also, the parameters of the magnitude – slip length equation were taken as $a=1.596$ and $b=7.560$, which are based on worldwide data (Ambraseys and Tchalenko, 1968). Finally, the attenuation equation used is that given by Taleb-Agha (1977),

$$a = 0.00121e^{1.15m/R}$$

(6.2)

where $a$ is the maximum ground acceleration expressed as a fraction of gravity, $m$ is the earthquake magnitude in Richter scale, and $R$ is the shortest distance between the site and the slipped area in km. Figure 6.3 shows the annual and fifty-year hazard curves.

6.2.3 Discussion of Results

The structure was subjected to six levels of excitation with maximum acceleration ranging from $1/6g$ to $1.0g$. As the structural restoring force is nondegrading, a stationary response analysis was performed. The maximum response statistics and their derivatives (with respect to the various model parameters), are calculated for an equivalent stationary response duration equal to the strong motion duration.
Probabilty of Exceeding Given Acceleration

Fig. 6.3 Seismic Hazard Curves for Boston Area
Expected Drift -- Figure 6.4 shows the expected maximum drift for each story, for the six levels of excitation; the corresponding 5-sample simulation results obtained by Lai (1980) for three of the six excitation levels are also shown. It can be seen that the agreement with the simulation results is reasonable. Since only five samples were used in performing the simulation, the reported values are accurate only to within approximately 10-15%. Also, according to Lai (1980), the structural modeling used in the simulation tends to overestimate the first-story drifts. Thus, the apparent error in the first-story random vibration results at the 2/3g and 1.0g excitation levels may be exaggerated.

Observe that the displaced shape changes as the excitation level increases. The first mode shape, which initially dominates the lateral displacements becomes less significant at the higher excitation levels. This is caused by the yielding in the lower stories, and is accounted for by the nonlinear model used herein. Conventional modal analysis would have difficulty reproducing this behavior unless iterative techniques were used (Scawthorn, Iemura and Yamada, 1981).

Response Variance -- To evaluate the variance of the maximum drift, the derivatives (i.e., sensitivity coefficients) of the drift with respect to the stiffness, damping, mass and strength of each story (using the methods discussed in Chapter 4) were calculated. The derivatives of the maximum drift of each story with respect to the filter parameters, $\omega$ and $\beta$, and to the strong motion duration were also evaluated. The variance of the maximum drift is evaluated using the coefficients of variation (c.o.v.'s) given in Table 3.4; whereas for the filter parameters and duration, the c.o.v.'s given in Tables 2.3 and 2.5 were used.

On the basis of Eq. 3.32, the significance of each parameter uncertainty may be represented by the product of the absolute value of the sensitivity coefficient and the pertinent parameter standard deviation. Figure 6.5 presents these results as functions of the excitation level, for the uncertainty in the first-story maximum drift. It may be observed that the uncertainties in the stiffness, strength (particularly at the higher response levels) and load duration tend to
Fig. 6.4 Expected Maximum Story Drifts for Four-Story Steel Frame (---, simulation, after Lai, 1980)
Fig. 6.5 Relative Significance of Individual Model Parameter Uncertainties on the First-Story Response Uncertainty
dominate the response uncertainty. For the structural parameters, the response uncertainty of a particular story is primarily controlled by the uncertainty of the parameters of that story, and to a lesser degree by those of the adjacent stories. Because the ground motion spectrum has a fairly wide band, the structural response is not sensitive to changes in the filter parameters.

To complete the determination of the response variance due to parameter uncertainty, the correlation coefficients for the various parameters must be evaluated. For this purpose, it is assumed that due to common construction and workmanship, a given parameter (e.g., the mass of story 1), is perfectly correlated with that same parameter (i.e., mass) of all the other stories. Also, the stiffness and strength of all the stories are assumed to be perfectly correlated. All other parameters, including the load duration and filter parameters, are assumed to be independent of each other.

The total response uncertainty must also include, as discussed earlier, the uncertainty underlying the random nature of the earthquake time history and the uncertainty in the mathematical idealization of the structure. The dispersive error in the mathematical idealization of the structure has a coefficient of variation of 0.21, for steel structures (see Sect. 3.3.1), and the response variance associated with the random nature of the earthquake time history is evaluated from the random vibration response statistics.

Table 6.3 illustrates the relative contributions to the total variance of the first-story maximum drift, from each of the three sources of uncertainty. The contribution of the parameter uncertainty increases with the response level, since the response becomes much more sensitive to changes in the initial stiffness and the prescribed yield level as it reaches the nonlinear range. This behavior may be observed in Fig. 6.5. Table 6.3 also shows that the contribution from the randomness of the loading history remains fairly uniform throughout all response levels. Actually, the coefficient of variation of the maximum response, associated with the ground motion uncertainty increases continuously with load level; however, the effects of the parameter uncertainties simply increase faster. The increase of the coefficient
Table 6.3 Percentage Contribution from Uncertainty Sources to Total Variance of First Story Maximum Drift

<table>
<thead>
<tr>
<th>E[a_{max}] (g)</th>
<th>Structural Modeling</th>
<th>Parameter Uncertainty</th>
<th>Randomness of Loading History</th>
<th>Total Response c.o.v.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/6</td>
<td>35</td>
<td>19</td>
<td>46</td>
<td>0.40</td>
</tr>
<tr>
<td>1/3</td>
<td>29</td>
<td>25</td>
<td>46</td>
<td>0.44</td>
</tr>
<tr>
<td>1/2</td>
<td>25</td>
<td>30</td>
<td>45</td>
<td>0.48</td>
</tr>
<tr>
<td>2/3</td>
<td>21</td>
<td>35</td>
<td>44</td>
<td>0.53</td>
</tr>
<tr>
<td>5/6</td>
<td>18</td>
<td>38</td>
<td>44</td>
<td>0.57</td>
</tr>
<tr>
<td>1.0</td>
<td>17</td>
<td>38</td>
<td>45</td>
<td>0.58</td>
</tr>
</tbody>
</table>

of variation of the maximum response due to the randomness of the earthquake time history, with increasing load, results from the fact that it is inversely proportional to the number of response zero crossings (see Eqs. 4.24 and 4.25). As the load intensity increases, the structure softens and its apparent frequency decreases due to yielding. Thus, there are less zero crossings and the coefficient of variation increases. Physically, this is reasonable; with fewer zero crossings and fewer response peaks in a given time duration, prediction of the maximum response becomes (in terms of a statistical sampling) more uncertain. Since the apparent structural frequency eventually approaches a limiting value, controlled by the prescribed post-yield slope of the hysteresis, the coefficient of variation also approaches a limiting value. In addition, the stiffness and strength parameter uncertainty effects also cease to increase at the very high load levels for this same reason. This, along with consideration of the fact that the coefficient of variation of response due to structural modeling uncertainty was taken as a constant for all load levels, explains why the increase of the total response coefficient of variation levels off at the high load levels.
Ductility Exceedance Probabilities -- The steel frame was designed for a limiting ductility ratio of 4 under a peak ground acceleration of 1/3g. Thus, it is of interest to evaluate the lifetime probabilities of exceeding specified ductility ratios. The hysteresis model used herein exhibits smooth yielding and, therefore, does not have a clearly established yield point; however, the yield displacement may be approximately defined as the story strength divided by the initial stiffness. In any case, this is the definition of story yield displacement used by Lai (1980) in designing the frame. Thus, based on the story stiffnesses and strengths given in Table 6.1, the yield displacements for the first four stories are, respectively, 0.61, 0.76, 0.69 and 0.53 inches.

Figure 6.6 shows the expected maximum ductility, \( \mu \), and the corresponding standard deviation (considering all three sources of uncertainty) plotted as a function of the maximum ground acceleration for all four stories. It should be noted that the design ductility ratio of 4 was intended as the ductility ratio of the individual members. For the frame under investigation, the weak girders (i.e., in relation to the columns) control the yielding of the story (this is in fact the basis of Eq. 3.7 for evaluating equivalent story strength); thus, the story ductilities may be roughly interpreted as the girder ductilities. This interpretation is valid as long as the column and girder stiffnesses do not differ greatly.

Using Eq. 5.3 (where \( X \) represents the ductility ratio) the lifetime exceedance probabilities may be calculated. The seismic hazard curve is shown in Fig. 6.3, and the conditional cumulative distribution function for the ductility ratio is obtained by fitting a Type I extremal distribution to the maximum ductility ratio statistics (see Sect. 5.2).

The annual and 50-year exceedance probability curves are shown in Fig. 6.7. The dashed lines in these figures are the exceedance probabilities corresponding to the mean maximum-response; i.e., assuming there is no uncertainty in the calculated response. Comparison of the two sets of curves show the significance of considering response uncertainties. This points out that for critical structures in which
Fig. 6.6 Maximum Ductility Ratio Statistics
Fig. 6.7 Ductility Ratio Exceedance Probability Curves
Fig. 6.7 (Cont'd.)

(c) Story 3

(d) Story 4
exceedance probabilities for large response levels must be calculated, proper consideration of the response uncertainties is important.

Of particular interest is the probability of the first-story drift exceeding the design ductility ratio of 4. Given the occurrence of the design earthquake of 1/3g, the probability of exceeding the design ductility ratio is approximately 12%. However, the probabilities associated with the occurrence of such earthquakes is very low, resulting in a very low 50-year lifetime probability of exceeding the design ductility ratio (approximately $6 \times 10^{-7}$). This illustrates the need to evaluate response exceedance probabilities under all significant earthquake intensities as well as the occurrence probabilities associated with the particular earthquake intensities.

As discussed in Chapter 5, the exceedance probabilities obtained (Fig. 6.7) represent central values of the true probabilities, due to the various modeling assumptions made in the analysis. To quantify this error the variance of the calculated probability associated with these prediction errors is calculated (Sect. 5.2.1). Assuming that the probabilities obtained are the mean probabilities, and that the error in the probability is lognormally distributed, the error bounds (equivalent to 90% "confidence limits") were constructed for the first story 50-year exceedance probabilities, and are shown in Fig. 6.8.

![Fig. 6.8 First Story 50-Year Exceedance Probability with Error Bounds](image-url)
6.3 Orion Avenue Holiday Inn Building

6.3.1 Structure Description and Modeling

The structure is a seven-story reinforced concrete frame building with three bays in the short direction and eight bays in the long direction, covering approximately 62 ft by 160 ft in plan area. Figure 6.9 shows a typical floor framing plan and a typical transverse section, taken from the U.S. Dep't. of Commerce (1973) study of buildings damaged in the 1971 San Fernando Earthquake. The transverse direction was chosen for analysis since this is the direction in which the structure experienced the most severe shaking.

The structural framing consists of columns spaced approximately 20 feet apart and a spandrel beam around the perimeter of the building at each floor level. Reinforced concrete flat slabs comprise the rest of the structural system.

For the random vibration analysis, the structure is modeled with a shear-beam system using one degree of freedom per story. Table 6.4a gives the equivalent lateral story stiffnesses and strengths, and story masses; Table 6.4b shows the corresponding hysteresis parameters.

The stiffnesses were determined using the mode-matching technique (Sect. 3.1.1), based on the first translational mode shape and frequency given in the U.S. Dep't. of Commerce (1973) report. Figure 6.10 shows a comparison of the first four shear-beam mode shapes and those of the corresponding "exact" mode shapes; the latter were obtained by the direct stiffness method (U.S. Dep't. of Commerce, 1973). Table 6.5 shows a comparison of the seven translational periods. The equivalent story strengths were estimated from the design maximum story shears.

The story masses were estimated by summing the actual tributary dead weights at each floor, and estimating the weights of furnishings, mechanical and electrical equipment, exterior walls and windows, and partitions, as reported in U.S. Dep't. of Commerce (1973).

The hysteresis parameters summarized in Table 6.4b were determined using the recommendations for concrete structures presented in
Fig. 6.9 Orion Avenue Holiday Inn, Van Nuys, California (after U.S. Dep't. of Commerce, 1973)
Table 6.4a Parameters for Equivalent Shear-Beam System

<table>
<thead>
<tr>
<th>Story</th>
<th>Lateral Stiffness (kips/in)</th>
<th>Strength (kips)</th>
<th>Mass (kips-sec²/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7409</td>
<td>591</td>
<td>4.74</td>
</tr>
<tr>
<td>2</td>
<td>8047</td>
<td>561</td>
<td>3.78</td>
</tr>
<tr>
<td>3</td>
<td>6589</td>
<td>561</td>
<td>3.78</td>
</tr>
<tr>
<td>4</td>
<td>6254</td>
<td>561</td>
<td>3.78</td>
</tr>
<tr>
<td>5</td>
<td>6135</td>
<td>561</td>
<td>3.78</td>
</tr>
<tr>
<td>6</td>
<td>5844</td>
<td>561</td>
<td>3.78</td>
</tr>
<tr>
<td>7</td>
<td>4748</td>
<td>496</td>
<td>3.65</td>
</tr>
</tbody>
</table>

Table 6.4b Hysteresis Parameters

<table>
<thead>
<tr>
<th>Story</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td>225</td>
<td>297</td>
<td>199</td>
<td>179</td>
<td>172</td>
<td>156</td>
<td>132</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>-75</td>
<td>-99</td>
<td>-67</td>
<td>-60</td>
<td>-57</td>
<td>-52</td>
<td>-44</td>
</tr>
</tbody>
</table>

\( A = 1.0, \alpha = 0.02, n = 2 \) (all stories)
Fig. 6.10 Comparison of Exact and Equivalent Shear-Beam Mode Shapes
Table 6.5 Comparison of "Exact" and Shear-Beam System Modal Periods (secs.)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>0.700</td>
<td>0.230</td>
<td>0.130</td>
<td>0.084</td>
<td>0.058</td>
<td>0.044</td>
<td>0.037</td>
</tr>
<tr>
<td>Equivalent Shear Beam</td>
<td>0.700</td>
<td>0.250</td>
<td>0.160</td>
<td>0.120</td>
<td>0.098</td>
<td>0.085</td>
<td>0.078</td>
</tr>
</tbody>
</table>

Sect. 3.2.2. The specific values of $\beta$ and $\gamma$ were obtained using $\beta/\gamma=3$ with $\gamma<0$, in conjunction with Eqs. 3.12 and 3.13.

Finally, viscous damping was taken to be approximately 1%, 2%, 4% and 5% of critical for the first four modes, respectively, and the restoring force behavior was assumed to follow the displacement-dependent stiffness degrading rule (Sect. 3.2.1).

6.3.2 Load Description

The ground motion was modeled as discussed in Chapter 2, assuming an intermediate soil condition. The power spectral density function is, therefore, described by the Kanai-Tajimi spectrum with mean parameters $\bar{\omega} = 16.5$ and $\bar{\beta} = 0.8$. The variation of the ground motion intensity with time follows the envelope of Amin and Ang (1968), as shown in Fig. 2.3, with deterministic parameters $t_1 = 1.5$ sec. and $c = 0.18$ sec. and a mean strong-phase duration $(t_2 - t_1)$ of 7.0 sec. As the soil condition is intermediate, the relationship between the power spectral intensity, $s$, and the expected maximum acceleration given by Eq. 6.1 applies for this case as well.

The seismic hazard assumed for the reliability analysis is that of the Los Angeles area; the building was located in the San Fernando Valley. The hazard curves were evaluated using the method of Der Kiureghian and Ang (see Chapter 2), based on the geologic source data given by Kiremidjian and Shah (1975). The value $\beta = 1.6$ (slope of the magnitude-recurrence curve) was used, which is a representative
value for the faults in the area. Also, the parameters $a=1.596$ and $b=7.560$ of the magnitude-slip length equation were assumed (Ambraseys and Tchalenko, 1968). Finally, the attenuation equation is that proposed by Donovan (1973),

$$a = 1.10e^{0.5m(R+25)^{-1.32}}$$

(6.3)

where $a$ is the maximum ground acceleration expressed as a fraction of gravity, $m$ is the earthquake magnitude in Richter scale, and $R$ is the shortest distance between the site and the slipped area in km. The annual and 50-year hazard curves obtained from the analysis are shown in Fig. 6.11.

6.3.3 Discussion of Results

The structure was subjected to five levels of excitation with maximum accelerations ranging from 0.05g to 0.45g in 0.1g intervals. As the degrading restoring force model and nonstationary load intensity envelope were used, a nonstationary response analysis was necessary. The method of Yang and Liu (1981), described in Sect. 4.2.1, was used to evaluate the maximum response statistics from the nonstationary random vibration results. The actual structure was subjected to a maximum acceleration of 0.25g during the 1971 San Fernando earthquake. Therefore, it will be particularly interesting to first examine the response and damage predicted by the present analysis procedure for this load level. Also, the intensity variation of the actual ground motion is similar to that modeled in the analysis, thereby allowing the predicted and actual damage to be compared (although the random process excitation has a frequency content representative of an average of many earthquakes rather than a single specific event).

Expected Drift -- At the load intensity of 0.25g acceleration, the expected maximum drifts of the first four stories were calculated to be 1.59%, 1.63%, 1.27%, and 0.68% of story height, respectively.
Fig. 6.11 Seismic Hazard Curves for Los Angeles Area
The corresponding drift values obtained using the actual recorded ground motion and a modal analysis procedure (U.S. Dep’t. of Commerce, 1973) were 0.9%, 1.14%, 1.48%, and 1.50%, respectively. Although this analysis used an adjusted frequency to account for the loss of structural stiffness due to yielding and material deterioration, it is based on the linear elastic mode shapes. The interstory drifts, therefore, are essentially constrained to conform to the dominant (the first) mode shape. This is not the case for the nonlinear analysis method used herein, which indicates that the maximum interstory drifts occur in the lower stories due to yielding.

Based on the damage functions presented in Fig. 5.1, and the qualitative interpretations of the damage intensities, the expected maximum story drifts calculated herein indicate substantial structural damage (using the linear damage function of Fig. 5.1) and greater than major nonstructural damage (using the elliptical damage function). This description fits into the upper end of Whitman’s damage state 5 (see Table 5.1). Thus, the predicted repair costs, in terms of replacement cost of the building, are in the range of 10-15%. As reported (U.S. Dep’t. of Commerce, 1973), the actual structural damage was minor, and the nonstructural damage was extensive, requiring major repairs and replacements. The total repair cost was approximately 11% of the initial construction cost (the repair cost in terms of the replacement cost would probably be lower).

The agreement between the analytically predicted repair cost and those actually observed appears to be reasonable, keeping in mind that the predicted 10-15% cost ratio is based on the response of the building in the transverse direction, which experienced the most severe shaking and that the actual building repair cost reflects the damage to the entire structure.

Although the cost of damage compared well, there is some discrepancy in the actual amount of structural damage predicted. Because of the flat plate construction of the building, the structural system is relatively flexible; therefore the linear damage function (Fig. 5.1), that was derived for more typical designs may be conservative in this case. Reinforced concrete structural elements of
typical proportions with moderate axial loads may be capable of sustaining drifts of as much as 2% without serious spalling of the concrete (Algan, 1982). For these reasons, the linear function for structural damage of Fig. 5.1 is modified in the reliability analysis as follows: the damage intensity is taken as zero for a damage index (drift, in this case) of 0.7% instead of 0.5%, and the damage intensity is taken as one for a damage index of 2.2%, instead of 2.0%. It is realized that this type of calibration procedure would not be possible if actual data were not available; however, this allows for a more meaningful illustration.

Response Variance -- A reliability analysis was performed for the first two stories, as it was determined from the evaluation of the expected maximum response for all seven stories that these would be the critical stories.

Based on the results of the four-story steel frame example, it was assumed that the stiffness and strength uncertainty of the first three stories, and the load duration uncertainty, would dominate the response uncertainty of the first two stories. Therefore, the derivatives of the first and second story maximum response, with respect to these seven parameters only, were calculated. The standard deviations of the stiffness and strength were evaluated using the coefficients of variation given in Table 3.4. The c.o.v. of the load duration was obtained from Table 2.5. As in the steel frame example, stiffness and strength were assumed to be perfectly correlated, whereas the load duration was assumed to be independent of these parameters. The total variances of the maximum drifts of the first two stories arising from the model parameter uncertainties were evaluated using Eq. 3.32.

The total response uncertainty must also include the effects of the randomness in the earthquake ground motions and the uncertainty in the mathematical idealization of the structure. The dispersive error in the mathematical idealization of the structure contributes a coefficient of variation of 0.22 to the estimated drifts of reinforced concrete structures (Sect. 3.3.1), whereas the response variance due to the randomness in the earthquake ground motion is evaluated from the random vibration analysis.
Table 6.6 shows the percentage contribution to the total variance of the first-story maximum drift, from each of the three sources of uncertainty. Observe that the contribution from the parameter uncertainty decreases with increasing load intensity; this contrasts the behavior found in the steel frame example. Actually, the response coefficient of variation due only to parameter uncertainty decreases as the load increases, causing a net overall decrease in total response coefficient of variation. This behavior is probably a result of the nature of the material deterioration. Since the concrete was modeled with the displacement-depending stiffness degrading rule, the stiffness at high response levels is primarily governed by displacements. Therefore, as the load increases, the response becomes less sensitive to the initial stiffness. Also, the restoring force model assumes that the strength does not deteriorate; thus, at the high response levels, the large reductions in stiffness prevent the force levels from approaching the material strength, rendering the response less sensitive to the actual material strength. The total response coefficient of variation, however, is still larger than that obtained for the steel structure due to the greater material uncertainties associated with reinforced concrete.

**Lifetime Damage and Safety Evaluation** -- Figure 6.12 shows the mean and total standard deviation of the maximum interstory drift

<table>
<thead>
<tr>
<th>$E[a_{\text{max}}]$ (g)</th>
<th>Structural Modeling</th>
<th>Randomness of Loading History</th>
<th>Parameter Uncertainties</th>
<th>Total Response Coefficient of Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>11</td>
<td>11</td>
<td>78</td>
<td>0.77</td>
</tr>
<tr>
<td>0.15</td>
<td>10</td>
<td>16</td>
<td>74</td>
<td>0.79</td>
</tr>
<tr>
<td>0.25</td>
<td>13</td>
<td>23</td>
<td>64</td>
<td>0.69</td>
</tr>
<tr>
<td>0.35</td>
<td>14</td>
<td>26</td>
<td>60</td>
<td>0.66</td>
</tr>
<tr>
<td>0.45</td>
<td>16</td>
<td>30</td>
<td>54</td>
<td>0.62</td>
</tr>
</tbody>
</table>
Fig. 6.12 Maximum Drift Statistics
(considering all three sources of uncertainty), plotted as a function of the expected maximum acceleration for the first two stories. The drift is expressed in percent of story height, as this quantity will be used to estimate damage.

Based on the results of Baber and Wen (1980) a Type I extreme value distribution is assumed for the maximum drift (fit using the statistics evaluated above). With the seismic hazard curves obtained for the Los Angeles area, the lifetime exceedance probabilities for particular values of drift were calculated. The results are plotted in Fig. 6.13, showing both the annual and 50-year exceedance probabilities. The dashed lines show the probabilities corresponding to the mean maximum response; i.e., assuming no uncertainty in the calculated response. The effects are not quite as pronounced here as for the steel structure due to the relatively higher exceedance probability levels being considered. However, significant error may still arise if the response uncertainties are neglected. For example, the annual probability of the first-story drift exceeding 2.5\% (a level at which structural failures may occur), changes from approximately $3 \times 10^{-3}$ (333 year return period) to $5 \times 10^{-3}$ (200 year return period) when the response uncertainties are added.

Based on these drift exceedance probabilities and damage functions of the form shown in Fig. 5.1, both nonstructural and structural damage probabilities are evaluated. The structural damage function used is a modified version of the linear form of Algan (1982) given by

$$D_S = \begin{cases} 
0 & ; \quad \Delta < 0.7 \\
\frac{2}{3} \Delta - \frac{7}{15} & ; \quad 0.7 \leq \Delta \leq 2.2 \\
1 & ; \quad 2.2 < \Delta 
\end{cases}$$

(6.4)

where $\Delta$ is interstory drift (since the lateral force-resisting system is a frame) expressed as a percent of story height. The elliptical function of Fig. 5.1, appropriate for nonstructural damage, is given by
Fig. 6.13 Drift Exceedance Probability Curves
In order to account for the prediction uncertainty in the drift-damage relation, the procedure outlined in Eqs. 5.1-5.2 was used. The mean and coefficient of variation of the Bayesian correction variable \( N \) were taken as 1.0 and 0.35, respectively, based on the results of Scawthorn, et al. (1981), discussed in Chapter 5. The value used for the coefficient of variation, is actually somewhat smaller than that reported, reflecting the fact that the current analysis considers the type of lateral force resisting system (i.e., frame, wall, or combination) and distinguishes between structural and nonstructural damage, whereas the c.o.v. obtained by Scawthorn, et al. (1981) was based on a drift-damage regression of a very general analysis and class of structures (mid-rise reinforced concrete). The distribution of \( N \) is assumed to be triangular.

Figure 6.14 shows the annual and 50-year exceedance probabilities for the first and second stories. The 50-year probabilities of exceeding the damage levels actually experienced during the 1971 San Fernando earthquake (i.e., nonstructural damage intensity \( \geq 0.95 \) and structural damage intensity around 0.6) are of particular interest. From these figures, the pertinent probabilities are approximately 35%, for both stories. These imply that the structure had a 35% chance of experiencing damage equal to or greater than that sustained in the 1971 San Fernando earthquake during its lifetime (assuming a 50-year life).

Finally, the safety of the structure may be evaluated, as discussed in Chapter 5, in terms of the probability of the structural damage intensity exceeding 0.9. This probability is 0.27 for the first story and 0.25 for the second story over the 50-year lifetime. This implies that a full-time occupant of the structure has approximately a 25% chance of being exposed to a potentially hazardous (seismically induced)
Fig. 6.14 Damage Intensity Probability Curves
event, at least once in 50 years. The corresponding return period for this event, is approximately 150 years.
CHAPTER 7

SUMMARY AND CONCLUSIONS

7.1 Summary

A method has been presented for determining the probabilities that a structure will sustain various levels of damage or become unsafe under earthquake loading during its lifetime. Uncertainties in the dynamic analysis associated with both the loading and the prediction of the structural response are considered.

The method is based on a nonlinear random vibration analysis of a structure and an analytical technique for evaluating the sensitivity of the response to various structural and load parameters. Recently obtained data were used to update the parameter values commonly used in the random process representation of earthquakes. The structure was modeled using a shear beam idealization, with equivalent story parameters systematically evaluated in order to reasonably represent actual structural behavior, and an analytical hysteretic degrading model was used to represent the structural restoring force characteristics. A system identification technique was developed to obtain the values of the parameters of the hysteresis model from experimental results, and was applied to several sets of test data. Based on these results, general rules for determining the proper values of the parameters for structural modeling purposes were recommended.

A seismic hazard model was used to evaluate the probabilities (or return periods) associated with all significant ground motion
intensities, and various damage and safety criteria were explored for the purpose of evaluating the lifetime damage probabilities.

7.2 Conclusions

On the basis of this study, the following conclusions may be drawn:

1. The proposed method is a viable and tractable approach for performing lifetime damage and safety analysis of structures located in seismically active regions.

2. The technique for evaluating the story stiffnesses of the shear-beam model by matching one dominant "exact" mode shape (obtained by a direct stiffness analysis) results in a structural model whose overall modal properties reasonably match those obtained by the detailed analysis.

3. The analytical hysteresis model can represent the behavior of both steel and reinforced concrete structures under cyclic loadings. Therefore, it is well suited for predicting damage in terms of maximum displacements and dissipated energy.

4. The method for estimating the hysteresis model parameter values based on response data accurately identifies the parameters.

5. The random vibration analysis predicts expected maximum response with good accuracy (as compared with simulation results), and predicts reasonable interstory drifts as compared with those observed in actual buildings during past earthquakes. Estimation of structural damage (in terms of repair cost) as a function of the maximum interstory drifts, gives results that are within the range of actual repair costs of past earthquake damage.

6. In evaluating the adequacy of a structure, it is insufficient to consider only the damage probabilities associated with an earthquake of a given intensity (e.g., a "design" earthquake). Consideration must be given to all earthquakes of significant intensity and their occurrence probabilities; the lifetime probability of exceeding damaging response levels may be more significantly influenced by earthquakes of moderate intensity with high occurrence probabilities than earthquakes
of large intensity and low occurrence probabilities. This was the case in the four-story steel frame example.

7. Without proper consideration of the uncertainties underlying the analysis, the evaluation of the response exceedance probabilities could be underestimated by as much as an order of magnitude, if the probabilities are in the range of \(10^{-3}\) or smaller (as in the case of design response levels for critical facilities such as hospitals or nuclear power plants).

8. Uncertainties in the structural and ground motion model parameters contribute significantly to the total response uncertainty of a structure. However, in evaluating the response uncertainty for a particular story of a building, it is sufficient to consider the effects of the uncertainty in the story stiffness and strength of that particular story and those of the immediately adjacent stories. The uncertainty in the earthquake strong motion duration also contributes significantly to the total response uncertainty.

7.3 Final Remarks

It is interesting to contrast the proposed approach with a conventional structural safety evaluation. In a conventional approach, uncertainties are not systematically considered; safety factors are used instead to insure a degree of conservatism. The actual risk implicit in the design, therefore, is unknown. In the proposed approach, the degree of variability in the loads and resistances, and the potential inaccuracies of the underlying assumptions are quantified. The safety of a structure and the amount of damage that could occur during the life of a structure may then be expressed in probabilistic terms. Decisions concerning the necessity for structural improvements in order to increase safety may, therefore, be based on these probabilities rather than on an appraisal of the adequacy of a particular safety factor. Also, the lifetime expected damage costs can be evaluated by considering the costs associated with each damage state and the lifetime probability of sustaining each damage state. This would allow a cost-benefit
analysis of structural improvements versus increases in initial construction costs.

It is also important to emphasize, that as the uncertainties due to prediction error decrease (through improvements in modeling, etc.), the probability of damage will decrease (assuming the probability levels being considered are small), allowing a more cost effective design. This, of course, is the motivation behind engineering research—learning as much as possible about the physics of a problem to minimize uncertainty and allow the most economical solution. In another light, by examining the change in the final probabilities the value (or utility) of more complex and sophisticated analysis may be assessed. The proposed methodology, therefore, provides an impetus for conducting research and applying the knowledge obtained, along with a means for assessing the value of the incentive, which is not provided in a conventional approach.

A passage from Broca's Brain (Sagan, 1979), concerning a belief held by Albert Einstein, is referenced in closing.

In the 1920s and 1930s he [Einstein] expressed grave doubts about a basic precept of quantum mechanics: that at the most fundamental level of matter, particles behave in an unpredictable way, as expressed by the Heisenberg uncertainty principle. Einstein said, "God does not play dice with the cosmos."

This evokes appropriate thought in regard to the present work. Perhaps some day enough knowledge and understanding will be amassed so that the exact occurrence time and ground motion of an earthquake as well as the exact structural response can be predicted. Unfortunately, for engineering endeavors such predictive abilities may never be attainable. It is apparent that for now and for a long time to come, we must rely on probabilities.
APENDIX A

EQUIVALENT LINEAR COEFFICIENTS

Applying Eq. 4.2 to Eq. 3.16 for \( \dot{z} \), with \( A, \eta \) and \( \nu \) given by Eq. 3.18 yields the general expression for \( C \) and \( K \). The expected values of \( A, \eta \) and \( \nu \) are slowly varying monotonic functions; hence considerable simplification can be obtained by assuming that the partial derivatives with respect to \( \dot{u} \) and \( z \) are zero. Then, by first order approximation, replacing \( A, \eta \) and \( \nu \) by the respective mean values \( \bar{A}, \bar{\eta} \) and \( \bar{\nu} \), the expressions for \( C \) and \( K \) are (Baber and Wen, 1980)

\[
C_e = E\left\{ \bar{A} - \bar{\nu}(\beta |\dot{u}| \frac{3}{n-1} |z|^{n-1} z + \gamma |z|^n) / \bar{\eta} \right\}
\]

\[
K_e = E\left[ -\bar{\nu}(\beta |\dot{u}| \frac{3}{n-1} |z|^{n-1} z + \gamma |z|^n) / \bar{\eta} \right]
\]

where:

\[
\bar{A} = A_0 - \delta_{\bar{A}}
\]

\[
\bar{\eta} = 1.0 + \delta_{\bar{\eta}}
\]

\[
\bar{\nu} = 1.0 + \delta_{\bar{\nu}}
\]
In which \( \bar{e} \) is the mean dissipated hysteretic energy obtained by taking the expected value of Eq. 3.17. Interchanging expectation and integration, \( \bar{e} \) is given by:

\[
\bar{e} = (1 - \alpha) k \int_{t_0}^{t} E[\dot{u}z] dt
\]  

(A.3)

The term \( E[\dot{u}z] \) is proportional to the expected rate of energy dissipation and is an element of the covariance matrix, obtained in the general random vibration solution (Sect. 4.1.2).

Taking the expectation of the individual terms of Eq. A.1 results in the final expressions for \( C \) and \( K \),

\[
C_e = \frac{1}{\bar{e}} - \frac{\nu(\beta F_1 + \gamma F_2)}{\bar{e}} 
\]

\[
K_e = - \frac{\nu(\beta F_3 + \gamma F_4)}{\bar{e}} 
\]  

(A.4)

where,

\[
F_1 = E[\frac{\partial |\dot{u}|}{\partial u} |z|^{n-1} z] 
\]

\[
F_2 = E[|z|^n] 
\]

\[
F_3 = E[|\dot{u}| \frac{\partial |z|^n}{\partial z}] 
\]

(A.5)

\[
F_4 = E[\dot{u} \frac{\partial |z|^n}{\partial z}] 
\]

Assuming that \( \dot{u} \) and \( z \) are jointly Gaussian variables, the expectations in Eq. A.5 may be evaluated. Performing the necessary integrals (noting that \( \dot{u} \) and \( z \) have zero means) results in
\[ F_1 = \frac{\sigma_z^n}{\pi} \Gamma\left(\frac{n+2}{2}\right) 2^{n/2} \frac{I_s}{2^{n/2}} \]

\[ F_2 = \frac{\sigma_z^n}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) 2^{n/2} \]

\[ F_3 = \frac{n \sigma_z \sigma_u^{n-1}}{\pi} \Gamma\left(\frac{n+2}{2}\right) 2^{n/2} \left\{ \frac{2(1 - \rho_{uz}^2)^{(n+1)/2}}{n} + \rho_{uz} I_s \right\} \quad \text{(A.6)} \]

\[ F_4 = \frac{n}{\sqrt{\pi}} \rho_{uz} \sigma_z^{n-1} \sigma_u \Gamma\left(\frac{n+1}{2}\right) 2^{n/2} \]

where  \( \Gamma(\cdot) \) is the gamma function;

\[ I_s = 2 \int_{-L}^{L} \sin^n \theta d\theta \quad \text{(A.7)} \]

and

\[ L = \tan^{-1} \left( \frac{\sqrt{1 - \rho_{uz}^2}}{\rho_{uz}} \right) \quad \text{(A.8)} \]

From Eq. A.5, it is seen that the equivalent linear coefficients depend on the response statistics,  \( \sigma_z \),  \( \sigma_u \), and  \( \rho_{uz} \). As  \( \bar{u} \) and  \( \bar{z} \) have zero mean, these statistics may be rewritten as

\[ \sigma_z = \sqrt{\mathbb{E}[z^2]} \]

\[ \sigma_u = \sqrt{\mathbb{E}[\bar{u}^2]} \]

and

\[ \rho_{uz} = \frac{\mathbb{E}[\bar{u}z]}{\sqrt{\mathbb{E}[\bar{u}^2]\mathbb{E}[z^2]}} \]
which are terms of the response covariance matrix obtained in the general random vibration solution. Extension of the formulation to multi-degree of freedom systems is straightforward. The basic procedure is identical; the coefficients of the $i^{th}$ degree-of-freedom are now calculated using the $i^{th}$ degree-of-freedom response statistics.

Further details of the procedure and a complete literature review of equivalent linearization techniques may be found in Baber and Wen (1980).
APPENDIX B

REQUIRED DERIVATIVE EXPRESSIONS

B.1 Evaluation of Matrix ∂B/∂p

When the derivatives with respect to the parameters $t_1$, $t_2$, or $c$ must be evaluated, the matrix $\partial B/\partial p$ has one nonzero term given by

$$\frac{\partial b}{\partial t_1} = \begin{cases} 
\frac{4\pi s_0 t^2}{t_1^3} & 0 \leq t \leq t_1 \\
0 & t_1 < t
\end{cases} \quad (B.1)$$

or

$$\frac{\partial b}{\partial t_2} = \begin{cases} 
0 & 0 \leq t < t_2 \\
2\pi s_0 e^{-c(t-t_2)} & t_2 \leq t 
\end{cases} \quad (B.2)$$
B.2 Evaluation of $\frac{\partial z}{\partial p}$

Assuming $\dot{u}>0$ (this will not affect evaluation of $\frac{\partial z}{\partial p}$), separating variables and including the stiffness degradation variable $\eta$, Eq. 3.11 becomes

$$\frac{dz}{(A - \beta|z|^{n-1}z - \gamma|z|^n)/\eta} = du$$

(B.4)

Integrating both sides of Eq. B.4 now gives

$$\int_{z_0}^{z_f} \frac{dz}{A - \beta|z|^{n-1}z - \gamma|z|^n} = \frac{1}{\eta_0} (u_f - u_o)$$

(B.5)

where $\eta$ in Eq. B.4 is properly the value calculated at the previous iteration, thus designated above as $\eta_0$. Using Leibnitz's rule to differentiate Eq. B.5 results in

$$\int_{z_0}^{z_f} \frac{\partial}{\partial p} \left( \frac{1}{A - \beta|z|^{n-1}z - \gamma|z|^n} \right) dz + \frac{\partial z_f}{\partial p} \left( \frac{1}{A - \beta|z_f|^{n-1}z_f - \gamma|z_f|^n} \right)$$

$$- \frac{\partial z_o}{\partial p} \left( \frac{1}{A - \beta|z_0|^{n-1}z_0 - \gamma|z_0|^n} \right) = \frac{1}{\eta_0} \left( \frac{\partial u_f}{\partial p} - \frac{\partial u_o}{\partial p} \right) - \frac{1}{\eta_0} \left( \frac{3\eta_0}{2} (u_f - u_o) \right)$$

(B.6)
This equation may now be solved for $\frac{\partial z}{\partial p}$ assuming $\frac{\partial z}{\partial p}$ is known from the previous iteration (i.e., the calculation of $\frac{\partial z}{\partial p}$) and noting that $\frac{\partial u}{\partial p}$ and $\frac{\partial u}{\partial p}$ have already been evaluated by Eq. 4.44. Observe that when $p$ does not represent one of the hysteresis parameters $A, B, \gamma, or n$ (i.e., when it is a system parameter such as story stiffness or mass), the integral in Eq. B.6 is zero, since its integrand is zero.

### B.3 Evaluation of $\frac{\partial F}{\partial p}$, $\frac{\partial F}{\partial p}$, $\frac{\partial F}{\partial p}$, $\frac{\partial F}{\partial p}$

When the parameter $p$ is not the hysteresis parameter $n$, the relevant expressions (obtained by differentiating Eqs. A.6-A.8) are

\[
\frac{\partial F_1}{\partial p} = \frac{2^{n/2}}{\pi} \Gamma\left(\frac{n+2}{2}\right) \left\{ \frac{\partial I}{\partial p} + \frac{n-1}{2} \frac{\partial \sigma_z}{\partial p} \right\}
\]

(B.7a)

\[
\frac{\partial F_2}{\partial p} = \frac{2^{n/2}}{\pi} \Gamma\left(\frac{n+1}{2}\right) \frac{\partial \sigma_z}{\partial p}
\]

(B.7b)

\[
\frac{\partial F_3}{\partial p} = \frac{n^2}{\pi} \Gamma\left(\frac{n+2}{2}\right) \left\{ \frac{\sigma_z}{z} \frac{\partial z}{\partial p} + \frac{\partial \sigma_z}{\partial p} \right\} \left( 2 \frac{(1-p_{uz})(n+1)/2}{n} \right)
\]

(B.7c)

\[
\frac{\partial F_4}{\partial p} = \frac{n^2}{\pi} \Gamma\left(\frac{n+1}{2}\right) \left\{ \frac{\partial \sigma_z}{\partial p} \right\}
\]

(B.7d)
where

\[
\frac{\partial \sigma_z}{\partial p} = \frac{1}{2 \sigma_z} \frac{\partial E[z^2]}{\partial p} \tag{B.8a}
\]

\[
\frac{\partial \sigma_u}{\partial p} = \frac{1}{2 \sigma_u} \frac{\partial E[u^2]}{\partial p} \tag{B.8b}
\]

\[
\frac{\partial \sigma_{uz}}{\partial p} = \frac{1}{\sqrt{E[u^2]E[z^2]}} \frac{\partial E[uz]}{\partial p} - \frac{1}{2 \sigma_{uz}} \left( E[u^2] \frac{\partial E[z^2]}{\partial p} + E[z^2] \frac{\partial E[u^2]}{\partial p} \right) \tag{B.8c}
\]

\[
\frac{\partial I_s}{\partial p} = -2 \sin^2 L \frac{\partial L}{\partial p} \tag{B.9}
\]

and

\[
\frac{\partial L}{\partial p} = \frac{-1}{\sqrt{1 - \rho_{uz}^2}} \frac{\partial \rho_{uz}}{\partial p} \tag{B.10}
\]

However, when the derivative with respect to the parameter \( n \) is required the expressions are

\[
\frac{\partial F_1}{\partial n} = \frac{1}{\pi} \left[ \sigma_n \Gamma \left( \frac{n+2}{2} \right) z^{n/2} \frac{\partial I_s}{\partial n} + \sigma_n \Gamma \left( \frac{n+2}{2} \right) \frac{\partial (\sigma_n z^{n/2})}{\partial n} \right] I_s
\]

\[
+ \sigma_n \frac{\partial \Gamma \left( \frac{n+2}{2} \right)}{\partial n} z^{n/2} I_s + \frac{\partial \sigma_n}{\partial n} \Gamma \left( \frac{n+2}{2} \right) z^{n/2} I_s \tag{B.11a}
\]

\[
\frac{\partial F_2}{\partial n} = \frac{1}{\sqrt{\pi}} \left[ \sigma_n \Gamma \left( \frac{n+1}{2} \right) z^{(n/2)} + \sigma_n \frac{\partial \Gamma \left( \frac{n+1}{2} \right)}{\partial n} z^{(n/2)} + \frac{\partial \sigma_n}{\partial n} \Gamma \left( \frac{n+1}{2} \right) z^{(n/2)} \right] \tag{B.11b}
\]
\[
\frac{\partial F_3}{\partial n} = \frac{1}{n} \left\{ \begin{array}{l}
(n \sigma_{u z} \sigma_a^{n-1} \Gamma \left( \frac{n+2}{2} \right) \frac{\partial \left( \frac{2^{n/2}}{\partial n} \right)}{\partial n} + n \sigma_{u z} \sigma_a^{n-1} \frac{\partial \Gamma \left( \frac{n+2}{2} \right)}{\partial n} 2^{n/2} \\
+ n \sigma_{u z} \frac{\partial (\sigma_z^{n-1})}{\partial n} \Gamma \left( \frac{n+2}{2} \right) 2^{n/2} + n \sigma_{u z} \sigma_a^{n-1} \Gamma \left( \frac{n+2}{2} \right) 2^{n/2} \\
+ \sigma_{u z} \sigma_z \sigma_a^{n-1} \Gamma \left( \frac{n+2}{2} \right) \frac{2 \left( 1 - \rho_{u z} \right)^{n+1}/2}{n} + \rho_{u z} \left( n+1 \right) / s \right\} \\
+ \left( n \sigma_{u z} \sigma_a^{n-1} \Gamma \left( \frac{n+2}{2} \right) 2^{n/2} \right) \cdot \left\{ 2 \frac{\partial (1 - \rho_{u z} \sigma_z^{n-1})}{\partial n} - \rho_{u z} \frac{\partial I}{\partial n} + \rho_{u z} \frac{\partial I}{\partial n} s \right\}
\right\}
\]

(B.11c)

\[
\frac{\partial F_4}{\partial n} = \frac{1}{\sqrt{n}} \left\{ \begin{array}{l}
n \sigma_{u z} \sigma_a^{n-1} \Gamma \left( \frac{n+1}{2} \right) \frac{\partial \left( \frac{2^{n/2}}{\partial n} \right)}{\partial n} + n \sigma_{u z} \sigma_a^{n-1} \frac{\partial \Gamma \left( \frac{n+1}{2} \right)}{\partial n} 2^{n/2} \\
+ n \sigma_{u z} \sigma_z \frac{\partial \sigma_z^{n-1}}{\partial n} \Gamma \left( \frac{n+1}{2} \right) 2^{n/2} + n \sigma_{u z} \sigma_z \sigma_a^{n-1} \Gamma \left( \frac{n+1}{2} \right) 2^{n/2} \\
+ n \frac{\partial \rho_{u z}}{\partial n} \sigma_z \sigma_a^{n-1} \Gamma \left( \frac{n+1}{2} \right) 2^{n/2} + \rho_{u z} \sigma_z \sigma_a^{n-1} \Gamma \left( \frac{n+1}{2} \right) 2^{n/2} \right\}
\right\}
\]

(B.11d)

where

\[
\frac{\partial \sigma_z}{\partial n} = \sigma_z \left[ \frac{\partial \sigma_z}{\partial n} + \xi \sigma_z \right]
\]

(B.12a)

\[
\frac{\partial \sigma_z^{n-1}}{\partial n} = \sigma_z^{n-1} \left[ \frac{\partial \sigma_z}{\partial n} + \xi \sigma_z \right]
\]

(B.12b)
\[
\frac{\partial}{\partial n} \left[ (1-\rho_{\text{uz}}^2)^{(n+1)/2} \right] = (1-\rho_{\text{uz}}^2)^{(n+1)/2} \left[ \frac{1}{2} \ln(1-\rho_{\text{uz}}^2) \right]
\]

\[
- \frac{n+1}{(1-\rho_{\text{uz}}^2)^{(n+1)/2}} \rho_{\text{uz}} \frac{\partial \rho_{\text{uz}}}{\partial n}
\]

(B.12c)

\[
\frac{\partial (2^{n/2})}{\partial n} = 2^{(n-2)/2} \ln 2
\]

(B.13)

and \( \frac{\partial \sigma}{\partial n}, \frac{\partial \sigma_u}{\partial n}, \) and \( \frac{\partial \rho_{\text{uz}}}{\partial n} \) are as defined in Eq. B.8 with \( p = n \).

The derivative of the Gamma function with respect to its argument is obtained as

\[
\frac{\partial \Gamma(X)}{\partial X} = \Gamma(X) \psi(X-1)
\]

where \( \psi(*) \) is the Digamma function (see, e.g., Hildebrand, 1976). Thus, by letting \( X_1 = (n+2)/2 \), \( X_2 = (n+1)/2 \) and using the chain rule, the derivative of the Gamma function in Eq. B.11 may be evaluated as

\[
\frac{\partial \Gamma\left(\frac{n+2}{2}\right)}{\partial n} = \frac{\partial \Gamma(X_1)}{\partial n} = \frac{\partial \Gamma(X_1)}{\partial X_1} \frac{\partial X_1}{\partial n} = \frac{1}{2} \frac{\partial \Gamma(X_1)}{\partial X_1}
\]

(B.15)

and

\[
\frac{\partial \Gamma\left(\frac{n+1}{2}\right)}{\partial n} = \frac{1}{2} \frac{\partial \Gamma(X_2)}{\partial X_2}
\]

(B.16)
Finally,

\[
\frac{\partial I}{\partial n} = 2\int_{L}^{\pi/2} \frac{\partial (\sin^n \theta)}{\partial n} \ d\theta - 2\sin^n L \frac{\partial L}{\partial n}
\]

\[
= 2\int_{L}^{\pi/2} \sin^n \theta \ell n(\sin \theta) d\theta - 2\sin^n L \frac{\partial L}{\partial n}
\]

where \( \partial L/\partial n \) is defined by Eq. B.10 with \( p = n \).

**B.4 Derivatives with Respect to Story Strength**

Including the parameter \( \nu \) in the model, Eq. 3.12 becomes

\[
\zeta_{\text{max}} = \left( \frac{A}{\nu(\beta + \gamma)} \right)^{1/n}
\]

(B.18)

and the initial yield strength is given as

\[
f_{y_0} = (1 - \alpha)k \left[ \frac{A}{\nu_0(\beta + \gamma)} \right]^{1/n}
\]

(B.19)

where \( \nu \) is the initial value of \( \nu \). Solving Eq. B.19 for \( \nu \) and differentiating the result with respect to \( f_{y_0} \), the proportionality constant relating the variation in the initial yield strength to variation in \( \nu \) is seen to be

\[
\frac{\partial \nu_0}{\partial f_{y_0}} = \frac{-n}{(1 - \alpha)k} \nu_0 \left[ \frac{A}{\nu_0(\beta + \gamma)} \right]^{-1/n}
\]

(B.20)
In general, \( v = 1.0 \); thus, the expression may be simplified to

\[
\frac{\partial v}{\partial f} f = \frac{-n}{(1 - \alpha)k} \left( \frac{A}{\beta + \gamma} \right)^{-1/n} \quad (B.21)
\]

The desired response statistic derivatives are thus obtained via the chain rule as

\[
\frac{\partial S}{\partial f} f = \frac{\partial S}{\partial v} \frac{\partial v}{\partial f} \quad (B.22)
\]

The derivatives with respect to \( v \) are obtained, as are the derivatives with respect to any other parameter after a generalization of Eqs. 4.41 and B.6 is made. That is \( \frac{\partial v}{\partial p} \) (Eq. 4.41) is revised to

\[
\frac{\partial v}{\partial p} = \frac{\partial v}{\partial \zeta} + \frac{\partial v}{\partial \epsilon} \frac{\partial \epsilon}{\partial p} + \frac{\partial \epsilon}{\partial p} \quad (B.23)
\]

and the integral term in Eq. B.6 is rewritten as

\[
\int_{z_0}^{z_f} \frac{1}{A - \nu (\beta |z|^{n-1} z + \gamma |z|^n)} \, dz \quad (B.24)
\]

It should be emphasized that the value of \( v = 1.0 \) is assumed, but \( \frac{\partial v}{\partial p} = 0 \) when \( \nu = 0 \) and \( \frac{\partial v}{\partial p} = 1 \) when \( \nu = 1 \). Thus, \( v \) serves as a convenient parameter through which derivatives with respect to story strength may be obtained.
LIST OF REFERENCES


