A DISCRETE EULERIAN MODEL FOR SPHERICAL WAVE PROPAGATION IN COMPRESSIBLE MEDIA

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E. Faccioli
and
A. H.-S. Ang

Issued as a Technical Report of a Research Program Carried Out Under

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I. INTRODUCTION

1.1. General

A rapidly increasing interest in the effects of high-energy underground explosions has stimulated in recent years a considerable number of investigations devoted to the prediction of ground motions in the neighborhood of a blast source. In many instances this has led to the development of computer codes capable of describing, by means of discrete numerical techniques, the propagation of the pressure waves generated by the blast in the surrounding medium. The problem posed is of considerable mathematical complexity, as it can be easily realized through a brief review of the phenomena observed in connection with an underground atomic burst.

Experimental evidence indicates that a zone of vaporized and molten material is generated by the blast in the immediate vicinity of the source; the extent of this region depends on the properties of the medium, the yield of the explosion and other factors. In this zone the material behaves probably like a fluid and it seems therefore appropriate to term it a "hydrodynamic" zone. When the energy level of the outward propagating blast wave has been sufficiently lowered by heat dissipation and spherical divergence, a subsequent zone is formed which is characterized by permanent distortions and large density changes; the rigidity and other properties of the solid medium then play an increasingly important role.

The final stage of the propagation process occurs with a gradual transition from the zone of permanent distortions into a zone where the
material particles undergo a purely elastic motion. A complete solution of the problem including all the different ranges of material behavior has so far been obtained only for extremely simplified situations.

For this reason, many recent research efforts have been directed toward a partial theoretical solution of the problem, i.e. one which applies for a single range (or possibly two ranges) of material behavior. A good part of these investigations were concerned with the motions and pressures in the hydrodynamic zone because of the enormous amount of energy which is released by the explosion in the early stage of propagation. At the same time, extensive experimental research has been promoted to obtain adequate description of the mechanical and thermodynamical properties of various materials under extremely high pressures.

Comparatively lesser attention has been devoted to the study of wave propagation in the range where the medium experiences large distortions and density changes but where the assumptions of purely hydrodynamic flow would no longer be adequate. This region may be quite extensive and information relative to the distribution of stresses and motions in this region could be of significant importance in the design of protective structures. In fact the design of an underground structure to resist the effects of a nuclear explosion requires knowledge of the pressure and wave motions at different ranges from the burst. However, the scarcity of knowledge about compressible material behavior with permanent deformations under multi-axial states of stress represents a major obstacle to obtaining a realistic model of wave propagation in this range.
1.2. Object and Scope of the Research

The objective of this research is to develop a numerical method for describing wave motions in spherically symmetric compressible solids with nonlinear behavior. One important application of such a method is the study or prediction of the dynamic environment in the permanent-deformation zone generated by an underground contained nuclear blast.

In such an undertaking, the continuum is simulated by a discrete conceptual model of the spherical solid. The discrete model provides a clear and simple physical framework for the formulation of the basic conservation principles governing the problems of compressible flow, both in the hydrodynamic and non-hydrodynamic range. The basic relationships for the model can be developed quite generally to cover both ranges of flow, but as stated above, emphasis in the applications will be directed to the non-hydrodynamic problem.

The assumption of spherically symmetric geometry and the idealization of an infinite medium will be used throughout in order to simulate a fully contained underground nuclear burst. The results obtained are approximate, because the method of solution is essentially a centered finite difference approach and errors may be introduced by the use of artificial viscosity terms which were found necessary for smoothing flow discontinuities. However, some hitherto unsolved problems can be better solved with the proposed technique, including the effects of strongly nonlinear constitutive relationships of compressible media. Because of the current interest in problems of ground motions induced by nuclear bursts, attention will be focused on the dynamic characteristics of a representative class of earth media.
The discrete conservation equations derived for the model are compared with the corresponding continuum conservation principles both in integral and differential form and the conservative form of the model equations is pointed out.

A section is devoted to the review of certain aspects of the theory of discontinuous flow with special reference to the relation between the concept of weak solutions of the conservation equations and the artificial viscosity terms used for propagating shock waves.

The features of the proposed method are illustrated with two problems of dynamic expansion of a spherical cavity in an infinite medium using various assumptions of material behavior. The effect of a nonlinear bulk-type behavior is analyzed, as well as that of a more complete stress-strain description, including deviatoric effects. Also considered is the effect of different unloading hypotheses on the rate of decay of wave amplitudes.

A non-rigorous analysis of the stability properties and of the truncation error of the finite difference equations of the model is discussed. A rigorous analysis of the discretization error introduced by the model is not attempted but the effect of successively finer space and time zoning on the convergence of the solution is shown. This point is of great importance; in the absence of a more rigorous analytic examination, no computer code should be considered to give a satisfactory solution to this problem unless a "demonstrated convergent" solution has been illustrated.

All the computations were made on an IBM 7094 digital computer from programs written in FORTRAN II language.
1.3. Related Previous Work

All of the mathematical models used for solving ground motion problems in numerical form are obtained by representing a solid continuum through a finite number of points and by variously defining the basic field quantities at these points. Thus, by applying the fundamental conservation principles to small elements of the continuum, the conservation equations in finite difference form can be obtained. Ignoring higher than first order terms and taking the limit as the discrete element shrinks to a point in the space-time domain, one obtains the equations of continuum mechanics. Clearly, the absence of the limiting procedure and the possible omission of important second order terms represent the primary difference between the difference equations and the differential (or integral) equations of continuum mechanics and the primary source of errors. It is also evident that by finer and finer discretization of the continuum the difference equations approach the proper differential equations. However, this does not mean that the solution of the difference equations necessarily approaches in the limit the exact solution, since this result has been rigorously proven only in some special cases. This point should be recognized whenever discrete techniques are used and the validity of the results obtained thereby should be evaluated properly.

From the point of view of the description of wave motions, the mathematical models used in existing codes can be classified as being either of Lagrangian or Eulerian type. According to the so-called Lagrangian point of view (actually due to Euler), one describes the trajectories of individual particles as a function of time with reference
to some initial reference position taken at time $t_0$. On the other hand, with the Eulerian description one does not follow individual particles but rather watches what happens at every fixed point in space as a function of time.

In a discrete approach, the Lagrangian viewpoint can be used in various ways. One can initially set a lattice of reference points in the medium and write the finite difference conservation equations for the distorting lattice as direct analogues of the differential equations. One-dimensional spherically symmetric calculations of this type can be found for instance in Ref. (29).

In another Lagrangian approach, which has the features of a physical analogue, the medium is subdivided into finite portions, each conveniently related to some reference point. The description of motions is obtained by writing the conservation principles for each zone or for an appropriate set of zones centered at a reference point and following its trajectory. The relevant field equations at the reference points are obtained through suitable definitions of the inertial and kinematical properties of the neighboring zones. A rather sophisticated model of this nature is described for example in Ref. (36).

All the Lagrangian models suffer from the following drawback. When the medium undergoes large deformations, the lattice of reference points can be so severely distorted that the finite difference equations no longer produce uniform approximation and, as such, are bound to cause rapid deterioration of the calculations unless some elaborate rezoning procedures are introduced. On the other hand, Lagrangian models usually allow for a convenient treatment of physical boundaries in a problem,
and may also produce finite difference schemes with better stability properties than the corresponding Eulerian schemes.

The most convenient way of setting a reference frame for the Eulerian models is to think of a set of fictitious cells centered around the points of a fixed space lattice. These cells are the smallest "subsystem" in which the conservation processes are defined. The laws governing such processes can be stated, in an elementary way, in terms of the field quantities associated with the material that flows into or out of each cell during motion. Such "transport phenomena" involving the transfer of material from one cell to another clearly play a central role in the Eulerian description, and some of the most significant differences among the various discrete Eulerian models are precisely related to different representations of the transport phenomena. One- and two-dimensional hydrodynamic calculations using Eulerian models can be found for example in Refs. (27) and (31).

For non-hydrodynamic motions in one space dimension an Eulerian analogue which does not use the cell representation has been proposed in Ref. (28).

Because of the nondistorting space lattice, Eulerian models seem to be most appropriate for problems involving large material deformations. These also lead to a more convenient analysis when fixed physical boundaries are present in the region of flow. The drawbacks consist chiefly of the diffusion effects caused by the discretization of transport phenomena and in the difficulties arising from the treatment of physical boundaries that are in motion. In these cases, the position of the boundaries will change smoothly and continuously and will not
coincide with the points of the reference mesh, therefore, adding a further unknown to the problem.

Another interesting category of models make use of the so-called "particle-in-cell" (PIC) method, in which the most convenient aspects of the Lagrangian and Eulerian approaches are combined. Here the basic cell of the Eulerian models is retained but the material within each cell is divided into many particles, each carrying a certain fraction of the total mass contained in the cell. The flow of material through the cell boundaries is achieved by the Lagrangian motion of the single particles between cells, whereas the conservation requirements are computed with an Eulerian scheme. This method has proved successful in the solution of a number of one- and two-dimensional problems. However, its applicability seems to be somewhat limited to the range of hydrodynamic behavior. Moreover, the vast amount of computational labor required by this method makes its use inadvisable for problems where other methods of solution are possible. The latest version of the PIC method is described in Ref. (7).

The above review provides some of the reasons for the need of a new Eulerian model for prediction of wave motions in both the hydrodynamic and non-hydrodynamic regimes. It is believed that, for one-dimensional problems, the Eulerian framework described herein gives a discretization procedure with certain desirable properties, including a clear visualization of the physical principles involved.

The Eulerian model proposed in this research can be properly regarded as a physical analogue for problems involving hydrodynamic flow, but as it will be shown, it can as well be employed in continuum problems.
involving a non-isotropic stress tensor, through a convenient spatial arrangement of the cells and a proper definition of the local stress components. In this respect the present model can be viewed as a generalization of the model proposed by A. H.-S. Ang (3) and extensively used for static and dynamic problems of solid media; see for example an important application in Ref. (4). The two models share the common property of being derived on clear physical grounds and of producing centered finite difference analogues of the conservation equations.

1.4. Notation

When not otherwise specified, compressive stresses are considered positive and tensile stresses negative. Both the suffix notation and the common engineering definitions of stress and strain are employed in the formulas of Chapter III.

Many symbols are defined where they appear for the first time in the text. A list of the symbols not defined elsewhere follows.

\[ c_1 = \left[ \frac{1}{\rho_0} \left( K + \frac{4}{3} G \right) \right]^{1/2}, \text{ elastic dilatational velocity of wave propagation.} \]

\[ c_s = \left( \frac{G}{\rho_0} \right)^{1/2}, \text{ elastic shear velocity of wave propagation} \]

\[ \nabla \text{ gradient operator} \]

\[ \nabla \cdot \text{ divergence operator} \]

\[ \Delta t \quad \text{time increment} \]

\[ \Delta r \quad \text{radial mesh length} \]

\[ E \quad \text{total energy (internal + kinetic) per unit mass} \]

\[ e = \rho E, \text{ total energy per unit volume} \]
\( f \)  
generic surface traction per unit area

\( F_r \)  
radial component of the total force \( \int f \mathbf{n} \, dS \)

\( m = \rho v \)  
momentum per unit volume

\( \mathbf{n} \)  
outward unit normal vector

\( p \)  
pressure

\( P = P(t) \)  
applied boundary pressure

\( \rho \)  
density, or mass per unit volume

\( \rho_0 \)  
initial density for undisturbed states

\( r \)  
radial Eulerian coordinate

\( \sigma_r \)  
radial stress

\( \sigma_a, \sigma_{\text{axial}} \)  
alxial stress

\( \sigma_\theta, \sigma_\phi \)  
circumferential or tangential stress

\( S \)  
entropy per unit volume

\( t \)  
time

\( \mathbf{v} \)  
velocity vector

\( v \)  
radial velocity in spherically symmetric flow

\( W_i \)  
initial water content
II. CONSERVATION LAWS

2.1. Integral and Differential Form of the Conservation Equations for Continuous Compressible Flow

The equations governing the wave motions of compressible media express the basic conservation requirements of continuum mechanics, which are as follows:

(a) conservation of mass
(b) conservation of momentum
(c) conservation of total energy (first law of thermodynamics).

With the addition of an equation of state accounting for the specific properties of the medium, the above system allows for a complete description of continuous compressible flow.

An ideal medium is considered here, in which no viscous friction and no heat conduction can occur. The conservation principles will be stated with reference to a "control volume" fixed in space, through which the medium flows without discontinuities. The notion of a "control volume" is clearly implied by the Eulerian viewpoint.

Neglecting the effect of body forces, the conservation equations in Eulerian form read as follows

\[ \int_{t} \frac{\partial \rho}{\partial t} \, dt + \int_{S} \rho \vec{v} \cdot \vec{n} \, dS = 0 \]  
(conservation of mass) \hspace{1cm} (1)

\[ \int_{t} \frac{\partial (\rho \vec{v})}{\partial t} \, dt + \int_{S} \vec{v} (\rho \vec{v} \cdot \vec{n}) \, dS = \int_{S} f \vec{n} \, dS \]  
(conservation of momentum) \hspace{1cm} (2)

\[ \int_{t} \frac{\partial (\rho E)}{\partial t} \, dt + \int_{S} \rho \vec{E} \vec{v} \cdot \vec{n} \, dS = \int_{S} f \vec{v} \cdot \vec{n} \, dS \]  
(conservation of total energy) \hspace{1cm} (3)
where \( \tau \) is the control volume, bounded by the surface \( S \). In each of
the above equations the first integral on the left side can be interpreted as the time rate of accumulation of the total quantity (mass, momentum, etc.) that is being conserved; let this quantity be denoted by \( B \). The second integral on the left side represents the algebraic difference between the outflux of \( B \) through the control surface \( S \) and the influx of \( B \) through the same surface. The integral on the right side of Eq. (2) represents the total force exerted on \( \tau \) through \( S \) by the surrounding medium, whereas the corresponding integral in Eq. (3) is the rate of work performed on \( S \) by all the surface tractions. (18)

Then, each of the above equations essentially states that \( B \) changes at a rate equal to its flux through the boundary \( S \) in a manner compatible with the action exerted on it by the neighboring medium. Any dissipative mechanism within \( \tau \) is thus clearly ignored in Eq. (3).

For purposes of simplicity consider a control volume with spherical geometry as shown in Fig. 1. This means that the flow will occur only through the curved surfaces \( S_1 \) and \( S_2 \) and in directions normal to these surfaces indicated by the arrows in Fig. 1. Moreover if there is complete symmetry, \( f, \rho, v \) and \( E \) are constant on \( S_1 \) and \( S_2 \).

Under these conditions, Eqs. (1), (2), and (3) reduce to

\[
\int_{\tau} \frac{\partial}{\partial t} \ d\tau = (\rho v)_{1} A_{1} - (\rho v)_{2} A_{2} \quad (4)
\]

\[
\int_{\tau} \frac{\partial (\rho v)}{\partial t} \ d\tau = (\rho v^2)_{1} A_{1} - (\rho v^2)_{2} A_{2} + F_r \quad (5)
\]

\[
\int_{\tau} \frac{\partial (\rho E)}{\partial t} \ d\tau = (\rho v E - f v)_{1} A_{1} - (\rho v E - f v)_{2} A_{2} \quad (6)
\]
where $A_1$, $A_2$ are the areas of the surfaces $S_1$, $S_2$ and the subscripts 1 and 2 denote the constant values of the respective quantities over $S_1$ and $S_2$. Applying the mean value integral theorem to the volume integrals appearing in Eqs. (4), (5), and (6), these equations yield,

\[
\left[ \frac{\partial \tau}{\partial t} \right]_Q = (\rho v)_{1}A_1 - (\rho v)_{2}A_2
\]

\[
\left[ \frac{\partial (\rho v)}{\partial t} \right]_Q = (\rho v^2)_{1}A_1 - (\rho v^2)_{2}A_2 + F_r
\]

\[
\left[ \frac{\partial (\rho E)}{\partial t} \right]_Q = (\rho vE - f v)_{1}A_1 - (\rho vE - f v)_{2}A_2
\]

where $\left[ \ldots \right]_Q$ indicates that the derivatives are evaluated at some point $Q$ in $\tau$. It should be observed that no approximation has so far been introduced. It will be seen that the conservation equations of the model are closely related to the exact equations, Eqs. (7) through (9).

Although the physical character of the conservation principles is best described in the form as shown in Eqs. (1) through (3), it is convenient to express them also in differential form. Using the divergence theorem for the surface integral in Eq. (1) yields,

\[
\int_{\tau} \left[ \frac{\partial \tau}{\partial t} + \nabla \cdot (\rho \vec{v}) \right] d\tau = 0
\]

from which, for arbitrary control volume $\tau$,

\[
\frac{\partial \tau}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]

Using again the divergence theorem, Eq. (2) can be rewritten as,
\begin{equation}
\int_\tau \left[ \frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) \right] \, dt = + \int_S \mathbf{f} \cdot \mathbf{n} \, dS \tag{11}
\end{equation}

But \( \frac{\partial (\rho \mathbf{v})}{\partial t} = -\mathbf{v} \cdot \nabla \left( \rho \mathbf{v} \right) + \rho \frac{\partial \mathbf{v}}{\partial t} \) from Eq. (10),

and \( \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = \nabla \cdot (\rho \mathbf{v} \mathbf{v}) + \rho \mathbf{v} \cdot \nabla \mathbf{v} \)

and also \( \int_S \mathbf{f} \mathbf{n} \, dS = \int_\tau \nabla f \, d\tau \)

Therefore Eq. (11) becomes

\begin{equation}
\int_\tau \left[ \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla p \right] \, dt = 0 \tag{12}
\end{equation}

With the further assumption of purely hydrodynamic motion, the only
surface traction is the hydrostatic pressure exerted on \( \tau \) by the sur-
rounding fluid, and then from Eq. (12)

\begin{equation}
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = 0 \tag{13}
\end{equation}

But, in vector notation

\( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{D \mathbf{v}}{Dt} \)

where \( \frac{D}{Dt} \) is the total derivative operator. Therefore Eq. (13) can be
rewritten as

\begin{equation}
\rho \frac{D \mathbf{v}}{Dt} = -\nabla p \tag{14}
\end{equation}

Similarly, Eq. (3) can be shown to be

\begin{equation}
\rho \frac{D \mathbf{v}}{Dt} = -\nabla \cdot (\rho \mathbf{v}) \tag{15}
\end{equation}
Equations (10), (14), and (15) are the well known differential equations of classical hydrodynamics in their standard form. Except for Eq. (10), the specific physical character of the conservation principles is not obvious in this form. However, these can be rewritten in such a way that the analogy with Eqs. (7) through (9) appears more clearly.

For simplicity, consider the system of Eqs. (10), (14), and (15) in the case of spherically symmetric flow, which are,

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial r} + \frac{\partial v^2}{\partial r} = 0 \quad \text{(conservation of mass)} \tag{16}
\]

\[
\rho \left( \frac{\partial v}{\partial t} + \frac{\partial v}{\partial r} + \frac{\partial (\rho v)}{\partial r} \right) = 0 \quad \text{(conservation of momentum)} \tag{17}
\]

\[
\rho \left( \frac{\partial E}{\partial t} + v \frac{\partial E}{\partial r} + \frac{\partial (\rho v)}{\partial r} + \frac{2pv}{r} \right) = 0 \quad \text{(conservation of energy)} \tag{18}
\]

Adding the term \( v \frac{\partial \rho}{\partial t} \) to both sides of Eq. (17) and using Eq. (16) yield,

\[
\frac{\partial (\rho v)}{\partial t} = - \frac{\partial (\rho + \rho v^2)}{\partial r} - 2 \frac{v^2}{r} \tag{19}
\]

With an analogous treatment Eq. (18) becomes:

\[
\frac{\partial (\rho E)}{\partial t} = - \frac{\partial (\rho vE + \rho v)}{\partial r} - 2 \frac{v(p + \rho E)}{r} \tag{20}
\]

For the case of purely hydrodynamic flow and using \( \Delta \theta = \Delta \rho \), Eqs. (7) through (9) read

\[
\left[ \frac{\partial}{\partial t} \right]_Q \tau = (\rho v)_1 A_1 - (\rho v)_2 A_2 \tag{21}
\]

\[
\left[ \frac{\partial (\rho v)}{\partial t} \right]_Q \tau = (p + \rho v^2)_1 A_1 - (p + \rho v^2)_2 A_2 + 4 \int_{S_e}^{} p \sin \frac{\Delta \theta}{2} dS \tag{22}
\]
\[
\frac{\partial (\rho \mathbf{E})}{\partial t} \bigg|_Q = (\rho \mathbf{v} + \rho \mathbf{v}^2) \mathbf{A}_1 - (\rho \mathbf{v} + \rho \mathbf{v}^2) \mathbf{A}_2
\]  

where the forces have been taken as positive in the direction of increasing \( r \) (see Fig. 1). By letting \( \tau \to 0 \) in the above expressions, Eqs. (16), (19), and (20) are obtained (where the terms \( 2 \frac{\rho \mathbf{v}}{r} \), \( 2 \frac{\rho \mathbf{v}^2}{r} \), and \( 2 \frac{(\rho \mathbf{v} + \rho \mathbf{v}^2)}{r} \) clearly account for the spherical geometry of the system).

It may be observed that the differential equations, Eqs. (19) and (20), which were derived from the integral form of the conservation laws as given by Eqs. (2) and (3) through a limiting process, preserve their conservative form, whereas Eqs. (17) and (18), which are special cases of Eqs. (14) and (15) do not. The difference between the two forms is due to the explicit coupling of mass conservation with the conservation of momentum and energy that was used to obtain the standard hydrodynamic equations, Eqs. (14) and (15), from the integral equations. If the coupling is not used, the conservative character remains clear throughout.

It has been previously assumed that the medium behaves like a fluid, which is possible only for solid media in an extremely high pressure range. For lower pressures, the rigidity of the medium may not be negligible even when relatively large distortions and density changes occur. In this latter case, the explicit form of the surface traction changes but otherwise the conservation equations given above remain valid. The hydrostatic pressure is, therefore, replaced with the components of a stress tensor.

Referring again to the case of spherically symmetric motions, and expressing the surface tractions by means of the components of the
local stress tensor \((\sigma_\text{r}, \sigma_\theta = \sigma_\phi)\), Eqs. (8) and (9) yield as \(\tau \to 0\),

\[
\frac{\partial (\rho v)}{\partial t} = - \frac{\partial (\sigma_\text{r} + \rho v^2)}{\partial r} + \frac{2(\rho_\theta - \sigma_\text{r} - \rho v^2)}{r}
\]

\[
\frac{\partial (\rho E)}{\partial t} = - \frac{\partial (\sigma_\text{r} v + \rho vE)}{\partial r} - \frac{2v(\sigma_\text{r} + \rho E)}{r}
\]

which correspond to Eqs. (19) and (20) of the purely hydrodynamic situation.

2.2. **Significance of Assumptions**

The meaning of the assumptions made at the beginning of Sect. 2.1 requires some clarification.

Neglecting the effects of heat conduction and viscosity is equivalent to assuming that the specific entropy of a particle of the medium remains constant or, according to Courant and Friedrichs, (10) that the changes in state at a material point are adiabatic. Since a particle has no meaning in the Eulerian description, one can translate the same concept into Eulerian terms by saying that changes of state of the portion of fluid which happens to be inside the control volume at a certain time, are adiabatic.

The requirement of "adiabaticity" can be satisfied through the first law of thermodynamics by stating that the change of total energy in the control volume is due solely to the work done by the pressure forces acting on its surface, since no heat flow can take place through the boundaries. This is expressed by Eq. (3) or (15) and it can
interval $dt$ is $\omega d\theta$ tr velocity, the mass transport terms become,

$$
\Delta M_i = \int \int \int_{t+\Delta t} \rho(i-1,t)v(i-1,t)dt \, r(i-1)d\phi \, r(i-1)d\theta
- \frac{\Delta \rho}{2} - \frac{\Delta \theta}{2}
$$

$$
= r^2(i-1)\Delta \phi \Delta \theta \int_t^{t+\Delta t} \rho(i-1,t)v(i-1,t)dt
$$

$$
= r^2(i-1)\rho(i-1,t^{*})v(i-1,t^{*})\Delta t \Delta \phi \Delta \theta
$$

(27)

where $t \leq t^{*} \leq t + \Delta t$, according to the mean-value integral theorem.

Analogously,

$$
\Delta M_o = \int \int \int_{t+\Delta t} \rho(i+1,t)v(i+1,t)dt \, r(i+1)d\phi \, r(i+1)d\theta
- \frac{\Delta \rho}{2} - \frac{\Delta \theta}{2}
$$

$$
= r^2(i+1)\rho(i+1,t^{**})v(i+1,t^{**})\Delta t \Delta \phi \Delta \theta
$$

(28)

where $t \leq t^{**} \leq t + \Delta t$.

The total volume of the cell centered at $i$ is

$$
\tau(i) = \int \int \int dr \, r d\phi d\theta = \left[ r^2(i)\Delta r + \frac{\Delta r^3}{12} \right] \Delta \phi \Delta \theta
$$

$$
= r^2(i) \left[ 1 + \frac{1}{12} \frac{\Delta r^2}{r^2(i)} \right] \Delta r \Delta \phi \Delta \theta \simeq r^2(i)\Delta r \Delta \phi \Delta \theta
$$

(29)

where the term $\frac{\Delta r^2}{12r^2(i)}$ has been considered negligible in comparison with unity.
The mass in cell \( i \) is, therefore,

\[
M^{t+\Delta t}(i) = \rho(i,t+\Delta t)t^i(i) \quad \text{and} \quad M^t(i) = \rho(i,t)t^i(i)
\]

Then Eq. (26a) yields

\[
\rho(i,t+\Delta t)r^2(i)\Delta r\Delta \phi = \rho(i,t)r^2(i)\Delta r\Delta \phi
\]

\[
+ r^2(i-1)\rho(i-1,t^*)v(i-1,t^*)\Delta t\Delta \phi
\]

\[
- r^2(i+1)\rho(i+1,t^{**})v(i+1,t^{**})\Delta t\Delta \phi
\]

(30)

It should be observed that

\[
r^2(i+1) = \left[ r(i) + \frac{\Delta r}{2} \right]^2 = r^2(i)\left[ 1 + \frac{\Delta r}{r(i)} + \frac{\Delta r^2}{4r^2(i)} \right]
\]

and also,

\[
r^2(i-1) = r^2(i)\left[ 1 - \frac{\Delta r}{r(i)} + \frac{\Delta r^2}{4r^2(i)} \right]
\]

Then,

\[
\frac{r^2(i+1)}{\Delta r^2(i)} \approx \frac{1}{\Delta r} + \frac{1}{r(i)}
\]

(31)

and

\[
\frac{r^2(i-1)}{\Delta r^2(i)} \approx \frac{1}{\Delta r} - \frac{1}{r(i)}
\]

(31a)

in which the term \( \frac{\Delta r^2}{4r^2(i)} \) has been considered negligible in comparison with unity.

With these approximations, and also taking \( t^* \approx t^{**} \), Eq. (30) becomes
The mass in cell \( i \) is, therefore,

\[
M^{i+\Delta t}(i) = \rho(i,t+\Delta t)\tau(i) \quad \text{and} \quad M^{i}(i) = \rho(i,t)\tau(i)
\]

Then Eq. (26a) yields

\[
\rho(i,t+\Delta t)r^2(i)\Delta r\Delta \phi \Delta \theta = \rho(i,t)r^2(i)\Delta r\Delta \phi \Delta \theta + r^2(i-1)\rho(i-1,t^*)v(i-1,t^*)\Delta t\Delta \phi \Delta \theta - r^2(i+1)\rho(i+1,t^{**})v(i+1,t^{**})\Delta t\Delta \phi \Delta \theta
\]

(30)

It should be observed that

\[
r^2(i+1) = \left[ r(i) + \frac{\Delta r}{2} \right]^2 = r^2(i)\left[ 1 + \frac{\Delta r}{r(i)} + \frac{\Delta r^2}{4r^2(i)} \right]
\]

and also,

\[
r^2(i-1) = r^2(i)\left[ 1 - \frac{\Delta r}{r(i)} + \frac{\Delta r^2}{4r^2(i)} \right]
\]

Then,

\[
\frac{r^2(i+1)}{\Delta r^2(i)} \approx \frac{1}{\Delta r} + \frac{1}{r(i)}
\]

(31)

and

\[
\frac{r^2(i-1)}{\Delta r^2(i)} \approx \frac{1}{\Delta r} - \frac{1}{r(i)}
\]

(31a)

in which the term \( \frac{\Delta r^2}{4r^2(i)} \) has been considered negligible in comparison with unity.

With these approximations, and also taking \( t^* = t^{**} \), Eq. (30) becomes
\[ \varrho(i, t+\Delta t) = \varrho(i, t) + (\Delta t/\Delta x)[\varrho(i-1, t^*)v(i-1, t^*) - \varrho(i+1, t^*)v(i+1, t^*)] \]

- \( (\Delta t/r(i))[\varrho(i-1, t^*)v(i-1, t^*) + \varrho(i+1, t^*)v(i+1, t^*)] \) \hspace{1cm} (32)

By letting \( \Delta t \to 0 \) and \( \Delta x \to 0 \), the continuity equation of Eq. (16) is obtained. Therefore Eq. (32) can be considered as a centered finite difference analogue of Eq. (16). It should be observed that this is a direct consequence of Eq. (26) or, more explicitly Eq. (30), which represents an approximate statement of the exact conservation principle of Eq. (7).

In a similar manner, a relationship expressing momentum conservation can be formulated. The following scalar form of the law of impulse and momentum for cell \( i \) over a time interval \( \Delta t \) is considered;

\[ T_{t+\Delta t}(i) = T^t(i) + \Delta T^I(i-1) - \Delta T^O(i+1) + \Delta I(i) \] \hspace{1cm} (33)

where \( T \) is the momentum of the total mass contained in the cell at a specified time, \( \Delta T^I \) and \( \Delta T^O \) are the momenta of the inflowing and outflowing masses during \( \Delta t \), and \( \Delta I(i) \) is the change of impulse defined as

\[ \Delta I(i) = \int_0^{t+\Delta t} F(i, t)dt - \int_0^t F(i, t)dt = \int_t^{t+\Delta t} F(i, t)dt = F(i, t^*)\Delta t \] \hspace{1cm} (34)

Here \( F(i, t) \) represents the resultant force in the direction of motion as a consequence of the forces acting on the cell boundaries. When pure hydrodynamic flow is considered,
\Delta I(i) = \left[ p(i-1,t*)r^2(i-1) - p(i+1,t*)r^2(i+1) \right] \Delta \varphi \Delta \theta \Delta t \\
+ 2p(i,t*) \sin \frac{\Delta \theta}{2} r(i) \Delta r \Delta \varphi \Delta t + 2p(i,t*) \sin \frac{\Delta \varphi}{2} r(i) \Delta r \Delta \theta \Delta t \\
- \left[ p(i-1,t*)r^2(i-1) - p(i+1)r^2(i+1) \right] \Delta t \Delta r \Delta \varphi \Delta \theta \\
+ 2p(i,t*)r(i) \Delta r \Delta \varphi \Delta \theta \Delta t \\
(35)

where the forces in the direction of increasing \( r \) are taken as positive (Fig. 1).

If a complete stress description is desired, involving a radial stress \( \sigma_r \) and two tangential stresses \( \sigma_\theta \), \( \sigma_\phi \), the resultant of the forces acting in the direction of motion yields (see Fig. 1):

\Delta I(i) = \left[ \sigma_r(i-1,t*)r^2(i-1) - \sigma_r(i+1,t*)r^2(i+1) \right] \Delta \varphi \Delta \theta \Delta t \\
+ 2\sigma_\theta(i,t*) \sin \frac{\Delta \theta}{2} r(i) \Delta r \Delta \varphi \Delta t \\
+ 2\sigma_\phi(i,t*) \sin \frac{\Delta \varphi}{2} r(i) \Delta r \Delta \theta \Delta t \\
(36)

Assuming that \( \Delta \varphi \) and \( \Delta \theta \) are small and recalling that \( \sigma_\theta = \sigma_\phi \) for spherically symmetric flow, Eq. (36) becomes

\Delta I(i) = \left[ \sigma_r(i-1,t*)r^2(i-1) - \sigma_r(i+1,t*)r^2(i+1) \right] \Delta \varphi \Delta \theta \Delta t \\
+ 2\sigma_\theta(i,t*)r(i) \Delta r \Delta \varphi \Delta \theta \Delta t \\
(37)

The momentum transport terms \( \Delta T_i(i-1) \) and \( \Delta T_o(i+1) \) in Eq. (33) can be defined in the same way as the mass transport terms of Eq. (26). The momentum associated with the mass flowing through a fixed surface element \( \varphi \Delta \theta \Delta \varphi \Delta \theta \Delta \varphi \Delta \theta \) during \( \Delta t \), is \( \rho v^2 \varphi \Delta \varphi \Delta \theta \Delta \varphi \Delta \theta \). With a procedure similar to that leading to Eqs. (27) and (28), these momentum transport terms can be shown to be,
Clearly,

\[ T^+(i) = \rho(i,t)\nu(i,t)\tau(i) \]

With Eq. (38) and the approximations of Eqs. (29) and (31), and taking \( t^{**} = t^* \), Eq. (33) yields:

\[
\rho(i,t+\Delta t)\nu(i,t+\Delta t) = \rho(i,t)\nu(i,t)
+ (\Delta t/\sigma)[\rho(i-1,t^*)\nu^2(i-1,t^*)-\rho(i+1,t^*)\nu^2(i+1,t^*)]
- (\Delta t/\kappa(i))[\rho(i-1,t^*)\nu^2(i-1,t^*)+\rho(i+1,t^*)\nu^2(i+1,t^*)]
+ \frac{\Delta T(i)}{\tau(i)}
\]

For hydrodynamic flow Eq. (35) applies and Eq. (39) becomes

\[
\rho(i,t+\Delta t)\nu(i,t+\Delta t) = \rho(i,t)\nu(i,t) + (\Delta t/\sigma)[\rho(i-1,t^*)\nu^2(i-1,t^*)+\rho(i-1,t^*)\nu^2(i-1,t^*)]
- \rho(i+1,t^*)\nu(i+1,t^*)\nu^2(i+1,t^*)]
- (\Delta t/\kappa(i))[\rho(i-1,t^*)\nu^2(i-1,t^*)+\rho(i+1,t^*)\nu^2(i+1,t^*)]
+ (\Delta t/\kappa(i))[2\rho(i,t^*)-\rho(i-1,t^*)-\rho(i+1,t^*)]
\]

It should be observed that the last term on the right side of Eq. (40) does not have an analogue in the corresponding differential equation of Eq. (19). However, it is clear that this term vanishes if \( \rho(i,t^*) \) is assumed equal to \( 1/2[\rho(i-1,t^*)+\rho(i+1,t^*)] \) which would indeed be the case
if $\Delta r \to 0$. If Eq. (37) applies, then Eq. (39) yields the following:

$$
\rho(i,t+\Delta t)v(i,t+\Delta t) = \rho(i,t)v(i,t) + (\Delta t/\Delta r)[\sigma_r(i-1,t^*) + \rho(i-1,t^*)v^2(i-1,t^*)]
$$

$$
- \sigma_r(i+1,t^*) - \rho(i+1,t^*)v^2(i+1,t^*)]
$$

$$
- (\Delta t/r(i))[\sigma_r(i-1,t^*) + \rho(i-1,t^*)v^2(i-1,t^*)]
$$

$$
+ \sigma_r(i+1,t^*) + \rho(i+1,t^*)v^2(i+1,t^*) + 2\sigma_d(i,t^*)] \quad (41)
$$

It is easily recognized that Eqs. (40) and (41) are centered finite difference analogues of Eqs. (19) and (24), respectively.

The analogy of Eq. (33) with Eq. (8) is slightly less obvious than for the case of mass conservation; however, it can be observed that Eq. (33), relating impulse and momentum, is simply another way of expressing Newton's Law. Once the impulse term has been explicitly written, the analogy becomes clearer.

For the conservation of energy, each cell element must express the first law of thermodynamics, taking into account the mass flux through its boundaries.

Neglecting heat transfer, the total energy balance for cell $i$ over a time interval $\Delta t$, can be expressed as

$$
H^t+\Delta t(i) = H^t(i) + \Delta H_I(i-1) - \Delta H_O(i+1) + \Delta W(i) \quad (42)
$$

where $H$ is the total energy of the mass contained in a cell element at a specified time, $\Delta H_I$ and $\Delta H_O$ are the energy transport terms and $\Delta W$ is the work performed by the surface forces acting on the cell boundaries. Obviously, $\Delta W$ is found by multiplying the components of the surface tractions acting on each face of a cell by the appropriate displacement
components. Assuming that the forces acting in the direction of motion are positive and observing that only the radial forces perform work

$$\Delta W = [-p(i+1,t^*)r^2(i+1)v(i+1,t^*)]$$

$$+ p(i-1,t^*)r^2(i-1)v(i-1,t^*) \Delta t \Delta \phi \Delta \theta$$

(43)

where the direction of flow is as shown in Fig. 1, and purely hydrodynamic motion is assumed.

For a non-hydrodynamic stress state, the pressure is simply replaced by the radial stress; thus,

$$\Delta W = \left[ \sigma_r(i-1,t^*)r^2(i-1)v(i-1,t^*) \right.$$ 

$$- \sigma_r(i+1,t^*)r^2(i+1)v(i+1,t^*) \Delta t \Delta \phi \Delta \theta$$

(44)

The energy transport terms can be expressed by observing that the total energy associated with the mass flowing radially through a fixed surface element of area \(r^2 \Delta \phi \Delta \theta\) during \(dt\) is \(\rho E v d\theta d\sigma d\theta\), where \(E\) is the total specific energy. Following the same procedure that leads to Eqs. (27) and (28) yields,

$$\Delta H_1 = r^2(i-1)\rho(i+1,t^*)v(i-1,t^*)E(i-1,t)\Delta t \Delta \phi \Delta \theta$$

(45a)

$$\Delta H_0 = r^2(i+1)\rho(i+1,t^*)v(i+1,t^*)E(i+1,t^*)\Delta t \Delta \phi \Delta \theta$$

(45b)

In the same terms,

$$H^{t+\Delta t}(i) = \rho(i,t+\Delta t)E(i,t+\Delta t)\tau(i)$$

$$H^t(i) = \rho(i,t)E(i,t)\tau(i),$$

and, using the above relations for \(H\), \(\Delta H\), and \(\Delta W\) and neglecting the 2nd
order terms, and taking $t^{**} \equiv t^*$, Eq. (42) yields,

$$\rho(i, t + \Delta t)E(i, t + \Delta t) = \rho(i, t)E(i, t) + (\Delta t/\Delta r) [p(i-1, t^*)v(i-1, t^*)$$

$$+ \rho(i-1, t^*)v(i-1, t^*)E(i-1, t^*) - p(i+1, t^*)v(i+1, t^*)$$

$$- \rho(i+1, t^*)v(i+1, t^*)E(i+1, t^*)]$$

$$- (\Delta t/r(i))[p(i-1, t^*)v(i-1, t^*)$$

$$+ \rho(i-1, t^*)v(i-1, t^*)E(i-1, t^*) + p(i+1, t^*)v(i+1, t^*)$$

$$+ \rho(i+1, t^*)v(i+1, t^*)E(i+1, t^*)]$$

(46)

for the hydrodynamic case. The same equation, with $p$ replaced by $\sigma$, applies also for the non-hydrodynamic case. Equation (46) is clearly a centered finite difference analogue of Eq. (20). The set of difference equations, Eq. (32), (40), and (46) or, alternatively the corresponding equations written for a non-hydrodynamic situation, constitutes the system of conservation equations for the proposed Eulerian model.

Since no coupling with the conservation of mass was required, the equation derived through the proposed model, Eqs. (32), (40) and (46), remain in the conservative form; that is, the resulting equations are directly related to Eqs. (7), (8), and (9), or to Eqs. (16), (19), and (20), respectively.

The order of approximation and the stability properties of the finite difference equations are discussed in a subsequent chapter. However, two of the approximations involved in the above schemes can be briefly discussed here. The first is related to the omission of the second order $O(\Delta r^2/r^2)$ terms. Clearly, in calculations involving the singular point $r=0$, and where a fixed $\Delta r$ is used, the $O(\Delta r^2/r^2)$ terms
may not be negligible in the vicinity of small values of \( r \). The second point arises in connection with the time instants \( t^* \), etc., where \( t \leq t^* \leq t + \Delta t \), at which the transport phenomena are supposed to take place. In the absence of a limiting process, \( t^* \) cannot be determined precisely as required in the mean value integral theorem. Therefore, any choice of \( t^* \), that satisfies the restriction \( t \leq t^* \leq t + \Delta t \), will generally suffice but will depend on the integration procedure. Consequently, the choice of \( t^* \) may affect the stability requirements of the computational schemes. For instance, choosing \( t^* = t \) or \( t^* = t + \Delta t \) will result, respectively, in an explicit or implicit integration scheme, whose stability requirements are different.

It should be pointed out that, because of the arrangement of the interior cells in the present spherically symmetric model (Fig. 2), the field variables at the boundaries of a cell are well defined. This automatically produces central differencing schemes, which means that no uncontrollable diffusion effect is caused by the transport terms. In many of the existing Eulerian models the field variables at a cell boundary must be defined by means of interpolation procedures which invariably affects the stability of the calculations. Since this is a crucial point, the advantages and disadvantages of the proposed scheme will be discussed separately later.

**B. Boundary Cells**

As pointed out earlier, a difficulty inherent in any Eulerian model is the treatment of moving boundaries. In problems of contained nuclear explosions, the close-in effects may be determined by assuming an expanding pressure pulse applied on a spherical cavity. The boundary
of the cavity, therefore, will be in motion. This motion will generally be smooth and its position will generally not coincide with the original Eulerian grid.

In order to describe the motion of the boundary, the model must permit the definition of the boundary at all times. This means that at the cavity, some sort of Lagrangian description has to be used. However, it will be shown that, by introducing a cell with variable geometry, the present model can allow boundary discontinuities, leaving the Eulerian treatment of the neighboring interior cells completely unchanged.

At a generic time after the onset of motion, the boundary will be at some position between two fixed reference grids as shown in Fig. 3. In this figure, N is the first of the fixed reference grids inside the region filled with material, whereas B and BM are reference points moving with the material inside the partially empty cell. In the following, the quantities labeled with B refer to the boundary while the quantities labeled by BM refer to the center of the portion of the boundary cell which is filled with material and are defined as the average quantities inside this cell of variable volume.

Since no mass flows into the boundary cell from the left side, the conservation of mass, momentum, and energy of the boundary cell would be,

\[
M^{t+\Delta t}_{BM} = M^t_{BM} - \Delta M^t_{BM}(N) \tag{48}
\]
\[
T^{t+\Delta t}_{BM} = T^t_{BM} - \Delta T^t_{BM}(N) + \Delta T^t_{BM}(BM) \tag{49}
\]
\[
H^{t+\Delta t}_{BM} = H^t_{BM} - \Delta H^t_{BM}(N) + \Delta H^t_{BM}(BM) \tag{50}
\]
where the symbols have the same meaning as in Eqs. (26), (33), and (42).

The basic difference with respect to the interior equations is that
the volume of the material considered here is no longer constant but
changes as a result of the motion of the boundary.

If it is assumed that in a time interval \( \Delta t \) the boundary
advances from the position \( r(N) - \Delta \xi(t) \) to the new position \( r(N) - \Delta \xi(t) + v(B, t^*) \Delta t \), then the volume of the boundary cell is,

\[
\tau(BM, t) \cong r(BM, t)^2 \Delta \xi(t) \Delta \rho \Delta \theta = \left[ r(N) - \Delta \xi(t)/2 \right] \Delta \xi(t) \Delta \rho \Delta \theta
\]

\[
= r(N)^2 \Delta \xi(t) \left[ 1 - \frac{\Delta \xi(t)}{r(N)} + \frac{1}{4} \frac{\Delta \xi(t)^2}{r(N)^2} \right] \Delta \rho \Delta \theta
\]

\[
\cong r(N)^2 \Delta \xi(t) \left[ 1 - \frac{\Delta \xi(t)}{r(N)} \right] \Delta \rho \Delta \theta
\]  

(51)

and

\[
\tau(BM, t+\Delta t) \cong r(BM, t+\Delta t)^2 \left[ \Delta \xi(t) - v(B, t^*) \Delta t \right] \Delta \rho \Delta \theta
\]

\[
= \left[ r(N) - (\Delta \xi(t) - v(B, t^*) \Delta t)/2 \right] \Delta \xi(t) - v(B, t^*) \Delta t \Delta \rho \Delta \theta
\]

\[
= r(N)^2 \Delta \xi(t) \left[ 1 - \frac{v(B, t^*) \Delta t}{\Delta \xi(t)} - \frac{\Delta \xi(t)}{r(N)} + \frac{3}{4} \frac{\Delta \xi(t) v(B, t^*) \Delta t}{r(N)^2} \right]
\]

\[
+ \frac{3}{4} \frac{v(B, t^*) \Delta t^2}{r(N)^2} - \frac{1}{4} \frac{v(B, t^*)^3 \Delta t^3}{r(N)^2 \Delta \xi(t)} \Delta \rho \Delta \theta
\]

\[
\cong r(N)^2 \Delta \xi(t) \left[ 1 - \frac{v(B, t^*) \Delta t}{\Delta \xi(t)} - \frac{\Delta \xi(t)}{r(N)} \right]
\]

\[
+ 2 \frac{v(B, t^*) \Delta t}{r(N)} - \frac{v(B, t^*) \Delta t}{r(N) \Delta \xi(t)} \Delta \rho \Delta \theta
\]  

(52)
where terms of orders $O(\Delta t^2/r^2)$ and $O(\Delta t^3)$ have been neglected.

The total mass, momentum, and energy of a boundary cell then can be given as,

\[
M^{t+\Delta t}(BM) = \rho(BM,t+\Delta t)\tau(BM,t+\Delta t); \quad M^{t}(BM) = \rho(BM,t)\tau(BM,t)
\]

\[
T^{t+\Delta t}(BM) = \rho(BM,t+\Delta t)v(BM,t+\Delta t)\tau(BM,t+\Delta t); \quad T^{t}(BM) = \rho(BM,t)v(BM,t)\tau(BM,t)
\]

\[
H^{t+\Delta t}(BM) = \rho(BM,t+\Delta t)E(BM,t+\Delta t)\tau(BM,t+\Delta t); \quad H^{t}(BM) = \rho(BM,t)E(BM,t)\tau(BM,t)
\]

Then Eq. (48) yields the conservation of mass for the boundary cell as follows:

\[
\rho(BM,t+\Delta t) = \frac{\rho(BM,t)\Delta \xi(t)[1-\Delta \xi(t)/r(N)]-\rho(N,t*)v(N,t*)\Delta t}{\Delta \xi(t)[1-\Delta \xi(t)/r(N)]-v(B,t*)\Delta t[1-2\Delta \xi(t)/r(N)+v(B,t*)\Delta t/r(N)]}
\]

(53)

where the outflux term at the point N is computed exactly as for the interior cells.

Similarly, Eq. (49) yields,

\[
\Delta I = [P(t*)r^2(B,t*)-p(N,t*)r^2(N)]\Delta t\Delta \phi\Delta \theta
\]

\[
+ 2\rho(BM,t*)\sin \frac{\Delta \theta}{2} r(BM,t*)\Delta \phi \Delta t
\]

\[
+ 2\rho(BM,t*)\sin \frac{\Delta \phi}{2} r(BM,t*)\Delta \theta \Delta t
\]

(54)

\[
\Delta I \neq [P(t*)r^2(B,t*)-p(N,t*)r^2(N)]\Delta t\Delta \phi\Delta \theta
\]

\[
+ 2\rho(BM,t*)r(BM,t*)\Delta \phi \Delta \theta \Delta t
\]

for the hydrodynamic case, and

\[
\Delta I = [\sigma_\phi(B,t*)r^2(B,t*)-\sigma_\phi(N,t*)r^2(N)]\Delta \phi \Delta \theta \Delta t
\]

\[
+ 2\sigma_\phi(BM,t*)\sin \frac{\Delta \theta}{2} r(BM,t*)\Delta \phi \Delta t
\]

\[
+ 2\sigma_\phi(BM,t*)\sin \frac{\Delta \phi}{2} r(BM,t*)\Delta \theta \Delta t
\]
for the non-hydrodynamic case. These last two equations then yield,

\[
\rho(BM,t+\Delta t)v(BM,t+\Delta t) = \left\{ \rho(BM,t)v(BM,t)\Delta \xi(t)\left[1-\Delta \xi(t)/r(N)\right] \right.
\]

\[
-\rho(N,t*)v^2(N,t*)\Delta t + [P(t*) - p(N,t*)]
\]

\[
-2\Delta \xi(t)(P(t*) - p(BM,t*))/r(N)\Delta t \right\}
\]

\[
\left\{ \Delta \xi(t)[1-\Delta \xi(t)/r(N)] \right.
\]

\[
-v(B,t*)\Delta t[1-2\Delta \xi(t)/r(N)] + v(B,t*)\Delta t/r(N) \right\}
\]

and

\[
\rho(BM,t+\Delta t)v(BM,t+\Delta t) = \left\{ \rho(BM,t)v(BM,t)\Delta \xi(t)\left[1-\Delta \xi(t)/r(N)\right] \right.
\]

\[
-\rho(N,t*)v^2(N,t*)\Delta t + [\sigma_\tau(B,t*) - \sigma_\tau(N,t*)]
\]

\[
-2\Delta \xi(t)(\sigma_\tau(B,t*) - \sigma_\theta(BM,t*))/r(N)\Delta t \right\}
\]

\[
\left\{ \Delta \xi(t)[1-\Delta \xi(t)/r(N)] \right.
\]

\[
-v(B,t*)\Delta t[1-2\Delta \xi(t)/r(N)] + v(B,t*)\Delta t/r(N) \right\}
\]

(54)

(55)

for the conservation of momentum in the hydrodynamic and non-hydrodynamic situations, respectively. The expression for the conservation of energy can be similarly obtained.

It should now be observed that, using the above method, the boundary velocity shows up as an additional unknown in the conservation equations; no additional equation is immediately available for its determination. This means that an extrapolation procedure has to be introduced to compute this quantity; such a procedure will depend on the integration procedure used in the overall solution process. One
such extrapolation procedure will be discussed in detail in a later chapter.

It is interesting to note that with a slight rearrangement of terms, and then taking the limit as $\Delta t \to 0$ and $\Delta x \to 0$ in Eqs. (53) and (54),

\[ v_b = v_r \]  
\[ (56) \]

and from

\[ P_b - P_r = \rho_r v_r (v_r - v_b) \]

\[ P_b = P_r \]  
\[ (57) \]

where "b" denotes quantities at the boundary and "r" the quantities immediately at the right of it.

These two apparently trivial relationships can be interpreted by considering the moving boundary as a contact discontinuity. In fact no mass flow occurs across the boundary and, therefore, Eqs. (56) and (57) represent a degenerate solution of the Rankine-Hugoniot equations.

2.4. Discontinuities Arising In Compressible Flow

**General Remarks.** It is definitely beyond the scope of this work to present a complete account of the basic theory concerning flow discontinuities in compressible media. The following discussion will be limited to certain physical considerations relative to flow discontinuities, and to certain aspects of the generalized or "weak" solutions of the conservation equations and their effects on the numerical calculations.
It is well known in the study of compressible media that some initially continuous motions cannot be maintained indefinitely. This has been well illustrated by the classical example of a piston moving inside a tube filled with gas.\(^{10}\) The propagation of disturbances in the gas is governed by the sound speed \(c = \sqrt{\frac{dp}{dP}}\); for a gas \(c\) can be taken as a monotonically increasing function of \(p\). If the piston is pushed inside the tube with a velocity that increases slowly from rest, the motion will give rise to waves travelling at sound speed \(c\). As the velocity of the piston increases, the density of the gas and consequently also the sound speed \(c\) increase. Therefore, the waves that are generated at the later times, when the piston is moving at a faster velocity, will overtake the earlier waves that are traveling at a lower speed. As a consequence the profile of the compressive velocity wave, considered as a function of distance, becomes steeper and steeper and will eventually be vertical at some point. At such time, a shock wave is propagated into the gas as a jump in velocity, density and pressure, and thus causes the breakdown of the continuity of motion. As suggested by Courant and Friedrichs: "The character of the resulting discontinuity is similar to the breaking of water waves which become steeper and steeper as more slowly progressing parts are overtaken by faster ones."

Although the occurrence of a discontinuity can be predicted and analyzed on purely mathematical grounds, it may be physically realized that one of the factors influencing the formation of such discontinuities is the form of the equation of state \(p = f(p,S)\) that characterizes the medium in the tube. For most actual media, the
Density increases with pressure; also \( \frac{\partial^2 p}{\partial \rho^2} \geq 0 \) and \( \frac{\partial p}{\partial \rho} \geq 0 \), which corresponds to the situation assumed in the piston problem. However, if a medium with \( \frac{\partial p}{\partial \rho} > 0 \), \( \frac{\partial^2 p}{\partial \rho^2} \leq 0 \), \( \rho_0 \leq \rho < \infty \) is considered, the later influences of the piston would give rise to lower sound speeds and, therefore the subsequent piston would give rise to lower sound speeds and, therefore the subsequent waves would travel slower than the earlier ones, thus tending to flatten out the compressive wave. In this case, therefore, no shock waves would arise and the motion would remain continuous throughout.

From a mathematical point of view, the occurrence of discontinuities raises questions of uniqueness of the solution of the conservation equations. This question will be examined more closely below.

The system of conservation equations governing compressible flow in one space variable \( x \) may be written in matrix notation as

\[
U_t + F_x + B = 0 .
\]  

(55)

Here the subscripts stand for partial derivatives, \( U \) is an unknown vector of 3 components, \( F = F(U) \) is a nonlinear vector function of \( x, t, U \), and \( B \) is a vector of 3 components that are functions of \( x, t, \) and \( U \). In the case of spherical flow as considered in Section 2.2,

\[
U = \begin{bmatrix} \rho \\ m \\ e \end{bmatrix} \quad B = \begin{bmatrix} \frac{2m}{x} \\ \frac{2m^2}{\rho x} \\ \frac{2(e+p)m}{\rho x} \end{bmatrix} \quad F = \begin{bmatrix} m \\ p + \frac{m^2}{\rho} \\ \frac{(e+p)m}{\rho} \end{bmatrix}
\]  

(58a)
in which m and e are momentum and total energy per unit volume. The same system in the nonconservative form is

\[ Y_t + A(Y)Y + C = 0 \]  

(59)

where,

\[
Y = \begin{bmatrix} \rho \\ v \\ E \end{bmatrix}, \quad C = \begin{bmatrix} \frac{2 \rho v}{x} \\ 0 \\ \frac{2 \rho v}{p} \end{bmatrix}, \quad A = \begin{bmatrix} v & \rho & 0 \\ c_0^2 \rho & v & 0 \\ \frac{vc_0^2}{\rho} & \rho & v \end{bmatrix}
\]

p = f(\rho, s), and \( c^2 = \frac{\partial p}{\partial \rho} \).

The quasilinear system of Eq. (59) is hyperbolic, since the eigenvalues of the matrix A are all real and distinct for all values of Y, as may be easily verified. Most problems of compressible flow in one space variable give rise to mixed initial and boundary value problems associated with Eq. (58) or (59). However, since the theory of initial value problems is better developed than the corresponding theory for mixed problems, some results of initial value problems will be discussed here in order to provide some insight into the behavior of the solution of the mixed problem. The initial value problem for a system like Eq. (59) consists of determining solutions Y from their initial state \( Y(x, 0) = \varphi \) for all future time. But it has been stated\(^{25}\) that "solutions in the classical sense of quasilinear hyperbolic systems [like Eq. (59)] develop discontinuities after a finite time, regardless of the smoothness of the initial data, and therefore, cannot be continued
indefinitely as regular solutions." For the example of the piston problem with isentropic gas flow, an exhaustive discussion is given in Ref. (10). In this case, the time at which the solution loses its uniqueness and continuity can be determined exactly. In fact, after a finite time, the characteristics start forming an envelope in the x-t plane, on which the contributions to the solution of the single characteristics conflict with each other. For this case the actual analytical expression of the envelope can be found and the breakdown of the compression wave analyzed exactly. A general method for determining the critical time in a system such as Eq. (59) for a given initial condition is given in Ref. (22) and its application to the case of spherical isentropic flow is also shown.

In order to circumvent the difficulty arising from the loss of continuity and uniqueness of the classical solutions, an extended and weaker form of solution of the conservation equations will be sought. In this weaker sense, a solution will permit discontinuity and will not require differentiability.

However, before defining the weak form of solution and its implications with respect to the numerical calculations of discontinuous compressible flow, it is necessary to examine how the conservation equations are going to be affected locally by the presence of a discontinuity, and how some of the hypotheses concerning the thermodynamics of continuous flow must now be changed.

**Physical Characteristics of Flow Discontinuities.** The **Generalized Rankine-Hugoniot Equations.** The previous assumptions that viscous friction is negligible and that the entropy of a particle is
constant cannot be justified when the pressure and temperature gradients are not small. In this case, where the physical phenomenon includes a shock wave, a large amount of energy is dissipated through viscous friction and heat conduction in a very narrow zone in which the gradients of pressure and velocity are high. Outside this zone the flow remains essentially adiabatic and is governed by the conservation equations. The mathematical idealization replaces these narrow dissipation zones with sharp surfaces across which some of the field quantities have infinite gradients and naturally requires that conservation of mass, momentum, and energy is satisfied across the discontinuity. In addition, the dissipative character at a discontinuity must be included; this is done by adding to the conservation equations the condition that the entropy does not decrease across the discontinuity. It must be recalled that in a continuous flow, the conservation of energy reduces to the constancy of the specific entropy. For a discontinuous flow this is no longer true; the energy is conserved across the jump, but in addition the condition of non-decreasing entropy must be required as an independent relationship. Therefore "a discontinuous process is completely determined by the three conservation laws and the entropy condition. The original differential equations, valid in the region of continuous flow, together with the conditions expressing the conservation laws and the entropy condition across a discontinuity surface, suffice to determine the flow without describing in detail the irreversible process across a discontinuity surface."

Under these assumptions, it can be described how the conservation laws are modified locally by the presence of a discontinuity.
Any conservation law such as Eqs. (1), (2), and (3), can be written in the more general form

\[
\frac{D}{Dt} \int_{\tau} U \, dt = \int_{S} (U \nabla - F) \cdot \vec{n} \, ds
\]  

(60)

This can be shown by recalling that the "material rate of change" of a quantity \( B = \int_{\tau} U \, dt \) of the material that is instantaneously occupying the volume \( \tau \), but is also passing through it, is given by

\[
\frac{DB}{Dt} = \frac{D}{Dt} \int_{\tau} U \, dt = \int_{\tau} \frac{\partial U}{\partial t} \, dt + \int_{S} U \nabla \cdot \vec{n} \, ds
\]  

(61)

in which \( \vec{v} \) is the velocity field of the fluid flowing through \( \tau \), and \( \vec{n} \) is the outward normal of the surface \( S \) that bounds \( \tau \). The same statement holds true if one assumes that the surface \( S \) that bounds \( \tau \) is moving in space and that an element \( dS \) of the surface moves with velocity \( \vec{v} \). \( U \) can be any scalar field defined in the same space.

Using Eq. (61) in Eq. (60) yields,

\[
\int_{\tau} \frac{\partial U}{\partial t} \, dt = - \int_{S} \vec{F} \cdot \vec{n} \, ds
\]  

(62)

which is precisely the general form of the integral principles given in Eqs. (1), (2), and (3). The terms \( U \) and \( F \) in Eq. (62) are:

\[
U = \begin{bmatrix} \rho \\ \rho \vec{v} \\ \rho E \end{bmatrix} \quad F = \begin{bmatrix} \rho \vec{v} \\ (\vec{n} \vec{v} - \nabla \rho \vec{v}) \\ (\vec{v} - \rho \vec{E}) \end{bmatrix}
\]  

(62a)
Using the Gaussian divergence theorem in Eq. (62), the equation becomes

$$\frac{\partial \bar{U}}{\partial t} + \nabla \cdot \bar{F} = 0$$  \hspace{1cm} (63)

which expresses the divergence-free character of the conservation equations.

If it is assumed that \( U \) and \( \bar{F} \) contain jump discontinuities across a surface \( \sigma(\bar{r},t) = \text{constant} \) in a 4-space, then within the narrow region of the surface,

$$\frac{\partial \sigma}{\partial t} \, dt + \sum \frac{\partial \sigma}{\partial x^i} \, dx^i \, dx^2 + \sum \frac{\partial \sigma}{\partial x^j} \, dx^j \, dx^3 = \frac{\partial \sigma}{\partial t} \, dt + \nabla_\sigma \cdot d\bar{r} = 0$$

where \( \bar{r}(x^1, x^2, x^3) \) is a position vector, \( \nabla_\sigma \) denotes the gradient operator acting on the space variables only and \( d\bar{r} \) is a space increment with components \( dx^1, dx^2, dx^3 \). Dividing this equation by \( dt \) and taking the limit gives

$$\frac{\partial \sigma}{\partial t} + \nabla_\sigma \cdot \bar{q} = 0$$  \hspace{1cm} (64)

where \( \bar{q} \) is the local velocity of propagation of the discontinuity surface \( \sigma \). For simplicity of reference, let the volume in Fig. 4 be the same as in Fig. 1 and let \( \sigma(\bar{r},t) = \text{constant} \) be a spherical discontinuity surface concentric with the spherical surfaces \( S_1 \) and \( S_2 \) as shown in Fig. 4. Moreover \( S_e = S - S_1 - S_2 \), where \( S_e \) consists of the plane portion of the total bounding surface \( S \). For this configuration Eq. (60) becomes
\[
\frac{D}{Dt} \int \tau \, U = \int_{S_1} (\overline{U} - UF)_1 \cdot \overline{n}_1 dS + \int_{S_2} (\overline{U} - F) \cdot \overline{n}_2 dS + \int_{S_e} (\overline{U} - F) \cdot \overline{n} dS
\]  

(65)

where \((\quad)\) means that the quantity in the parentheses is evaluated for the appropriate part of \(S\). As \(\tau \to 0\), the surfaces \(S_1\) and \(S_2\) coincide, in the limit, with \(\sigma(r, t) = \text{constant}\) and so \(\overline{n}_2 = -\overline{n}_1\). The integral over the volume and the integral over \(S_e\) vanish because the integrands do not contain singularities in the domain \(\tau\), and Eq. (65) reduces to

\[
(Uq - F)_1 \cdot \overline{n}_1 + (Uq - F)_2 \cdot \overline{n}_2 = 0
\]

since the motion of \(S_1\) and \(S_2\) will in the limit coincide with the motion of \(\sigma\). If \([X]\) denotes the jump \(X_1 - X_2\) in the quantity \(X\) across the surface \(\sigma\), this equation then reads

\[
[Uq \cdot \overline{n}_1 - F \cdot \overline{n}_1] = 0
\]  

(66)

By definition the unit vector normal to \(\sigma\) is given by:

\[
\overline{n}_1 = \frac{\nabla \sigma}{|\nabla \sigma|}
\]

and by Eq. (64):

\[
\overline{q} \cdot \overline{n}_1 = -\frac{\partial \sigma}{\partial t} = \overline{\mu}
\]  

(67)

where \(\mu\) is the local velocity of propagation of the discontinuity along \(\overline{n}_1\). Combining Eqs. (66) and (67) yields,

\[
[U\mu - F \cdot \overline{n}_1] = 0 \quad \text{or} \quad \mu[U] = [F \cdot \overline{n}_1]
\]  

(68)
which is known as the generalized Rankine-Hugoniot relation. Equation (68) is quite general since it does not depend on the choice of the volume \( \mathbb{T} \) and can be shown to hold for any number of independent variables.

For the case of purely hydrodynamic flow in one space variable, Eq. (68) takes the simple form:

\[
\mathbf{u}[U] = [F]
\]

which expresses the continuity of the field \((U,F)\) across a discontinuity surface. From Eq. (62a) or (58a), Eq. (69) implies the following:

\[
\begin{align*}
\mathbf{u}[p] &= [m] \\
\mathbf{u}[m] &= [p + \frac{m^2}{\rho}] \\
\mathbf{u}[e] &= [(e+p) \frac{m}{\rho}]
\end{align*}
\]

which are the well known Rankine-Hugoniot equations of gas dynamics.

A number of equivalent ways of writing Eq. (70) are given in Ref. (10). In a true shock there is always mass flow across the discontinuity surface. If no mass flow occurs across the surface, the relative velocities of the particles on either side of the discontinuity are 0 as can be seen from the first of Eq. (70); in this case the surface is called a "contact discontinuity." Such a surface moves with the material and separates two zones of different density but continuity of pressure and velocity is preserved.

The various important properties of a shock transition have been extensively analyzed by a number of authors; e.g., in Refs. (10)
and (22). Among the most important properties in a hydrodynamic shock are the following:

1. Shocks are compressive, i.e., density and pressure rise across the shock front. This is true if and only if an increase in entropy occurs across the discontinuity.

2. The speed of the shock is always supersonic with respect to the material ahead of the discontinuity and subsonic with respect to the material behind the shock, if \( \frac{\partial^2 p}{\partial \rho^2} \geq 0 \) and \( \frac{\partial p}{\partial \rho} \geq 0 \).

Taking the value \([p]\) across the discontinuity as a measure of the shock strength, it can also be easily shown that as \([p] \to 0\) the speed of propagation of the discontinuity \(\mu\) tends to the sound speed \(c = (dp/d\rho)^{1/2}\). Since the shocks for which \(\mu\) does not significantly exceed \(c\) are classified as "weak shocks," it can be said that a sound wave is an infinitely weak shock.

For a medium in which the pressure depends on the density alone and not on the entropy, the first two conditions of Eq. (70) suffice to describe the shock transition. Here the energy jump relation can be used as a means of determining the energy balance after the problem has been solved. This may be related to the considerations regarding isentropic continuous flow. The same simplification also applies to "weak shocks." In fact it can be proved that for weak shocks the jump in entropy is of third order in the shock strength and, therefore, can be safely neglected.
2.5. **Weak Solutions and Viscosity Methods**

Consider now the hyperbolic system Eq. (58), in conservative form and in one space variable,

\[ U_t + F_x + B = 0 \]  \hspace{2cm} (58)

with the initial conditions \( U(x,0) = \phi(x) \).

Assume a class of test vectors \( W(x,t) \) that are differentiable with respect to \( x \) and \( t \), and vanish identically outside a bounded domain \( D \) in the \((x,t)\)-space. Multiplying Eq. (58) by a test vector \( W \), and then integrating the resulting equation over all values of \( x \), it can be shown by integration by parts that, \(22\)

\[- \int_{-\infty}^{\infty} W(x,0)\phi(x)dx + \int_{0}^{\infty} \int_{-\infty}^{\infty} [-W_tU+W_tF+WB]dxdt = 0 \]  \hspace{2cm} (71)

Any vector \( U(x,t) \) satisfying Eq. (71) for all test vectors \( W \) in \( D \) is defined as a "weak solution" of Eq. (58) with the given initial conditions. This definition makes it clear that weak solutions need not be differentiable and may also be regarded as expressing the divergence-free character of the vector field \((U,F)\) in a generalized or weak sense. \(25\)

A first and important property of a weak solution is the following. If \( U_1 \) and \( U_2 \) are two "genuine" solutions of Eq. (61) defined on adjacent sides of a discontinuity curve \( \sigma(x,t) = \text{constant} \), the two solutions taken together may be considered to constitute a weak solution of Eq. (58) if the jumps in \( U \) and \( F \) satisfy the Rankine-Hugoniot
conditions of Eq. (69). This is shown for example in Ref. (11). Therefore, Eq. (71) makes it possible to define in some sense a continuation of the solutions of Eq. (58) beyond a shock discontinuity. It should be clear that if a true solution of Eq. (58) exists, it also satisfies Eq. (71) and, therefore, is also a weak solution.

A solution in this generalized or weak sense, therefore, cannot be expected to possess all the properties of a true solution. In fact the initial values alone do not, in general determine a unique weak solution. Examples proving this statement are given in Ref. (22). This means, therefore that the problem as originally stated is not a meaningful one for weak solution, unless an additional condition can be imposed to assure the uniqueness of a weak solution. In the words of Lax, (34) "if we believe that our mathematical model does describe an aspect of the physical world, then there is indeed assigned to each initial function a unique weak solution, namely the one that occurs in nature. The problem is to characterize mathematically this physically relevant solution." The underlying hypothesis here is the well known postulate that in a physical problem the solution depends continuously on the initial data. Lax (25) has shown that for a hyperbolic system of the type of Eq. (58) the basic requirement of non-decreasing entropy across a shock must characterize a physically relevant weak solution and was able to show also that, for a restricted class of initial data (corresponding to Riemann's problem) the entropy condition insures uniqueness, and on this basis the weak solution can be constructed. However, no proof of uniqueness has yet been given in general.
An additional principle, therefore, must be selected on physical grounds, which at the same time should indicate a way of actually constructing the desired weak solution. It was observed earlier that in actual physical processes, dissipative effects due to viscosity and heat conduction tend to smooth out flow discontinuities such that, in reality, a shock surface is replaced by a thin transition zone in which the gradients of the field quantities are very high but not infinite, as implied in the mathematical idealization. This suggests the important mathematical criterion that weak solutions occurring in nature are limits of viscous flows.

Mathematically, these observations can be stated by first introducing an artificial viscous term in Eq. (58); thus,

$$U_t + F_x + B = \lambda U_{xx}$$  \hspace{1cm} (72)

which is a nonlinear parabolic system, where $\lambda$ is an artificial viscosity constant. It may be conjectured* that if the initial data are fixed and $\lambda$ taken smaller and smaller, the corresponding solutions $U_\lambda(x,t)$ converge to a limit $U(x,t)$ in the interval $0 \leq t \leq T$. If this conjecture is valid, it can be rigorously shown that the limit $U(x,t)$ of the sequence $U_\lambda(x,t)$ is a weak solution of Eq. (58).

Although the procedure suggested by Eq. (72) is one of the simplest, there are obviously other possibilities for introducing a viscosity term. If there is convergence in the sense described above

* Rigorous proofs have been given only for single conservation equations of special types, but there is considerable numerical evidence supporting this conjecture for hydrodynamic problems.
all these different methods may be expected to yield the same weak solution in the limit.

The use of viscosity as described above is of great importance. Aside from its physical plausibility, it also suggests a convenient and powerful approximate method of handling flow discontinuities in practical applications.

Von Neumann and Richtmyer\(^{(38)}\) have shown that if an appropriate dissipative term of the type just discussed is introduced in the one-dimensional flow equations, a narrow zone of smooth but very steep transition replaces each sharp shock jump in the solution. There is no need then to introduce into the calculations the internal boundary conditions represented by the Rankine-Hugoniot equations because the dissipation effects will be relevant only at the locations where shock transitions occur and will be of negligible effect elsewhere. This means that the jumps in the solution are determined automatically, in whatever form they arise. At the same time, the basic conservation laws on which the Hugoniot conditions were based are retained, and the jump conditions still hold across the transition layer. This layer is regarded as thin in comparison with other dimensions occurring in the problem. It can also be proved\(^{(32)}\) that the "smeared" shock travels with the same speed and produces the same entropy change as a true shock under the same conditions.

Although, in general, the convergence of the sequence of solutions corresponding to decreasing values of the artificial viscosity term has not been proved, the Von Neumann-Richtmyer method described above shows that the viscosity approach is sound and is a promising
Another way of formulating a limiting process, that could lead to weak solutions, is obtained by replacing Eq. (61) with a special type of finite difference equations. The idea is due to Lax and Wendroff, in which a system of the following form is considered,

$$U_t = F_x$$

A special difference scheme is devised that has the following property. If \( v(x,t) \) denote a solution of the difference equations, such that as \( \Delta x \to 0 \) and \( \Delta t \to 0 \), \( v(x,t) \) converges to some function \( u(x,t) \), then \( u(x,t) \) is a weak solution of Eq. (73) with given initial data.

The Lax-Wendroff method lends itself conveniently to the treatment of discontinuities because, although no explicit dissipative term is introduced in it, the difference scheme itself has a dissipative character that tends to damp out the high-frequency components of a solution as the flow progresses.
III. MATERIAL BEHAVIOR

3.1. Introductory Remarks

It was pointed out earlier that for a complete description of flow, an equation of state must be added to the conservation laws. Generally speaking, this additional relationship accounts for the thermodynamical properties of the specific medium in terms of pressure, density and entropy (or internal energy). However, it was also observed that if conditions of constant entropy can be assumed throughout, the description of flow is considerably simplified and only a "mechanical" equation of state is needed relating pressure to density. For non-isotropic stress states, this "mechanical" relationship is more generally represented by a law relating stresses to strains (or their respective rates) and density. Actual observations on underground explosions and a number of calculations have shown that at the termination of the hydrodynamic zone only a very small fraction of the initial energy input is left in the pressure wave and therefore the assumption of isentropic flow for subsequent ground motion seems adequate. This assumption will underlie all the rest of this work. Unfortunately though, only few experimental data are available on the stress behavior of earth materials (particularly granular media) beyond the elastic limit and up to pressures of 5 - 10 kilobars. Nevertheless, important attempts have been made towards the development of mathematical models capable of handling the strongly nonlinear physical properties of a large class of earth media in terms of appropriate constitutive relationships.
For problems of soil dynamics within the scope of interest of this work, the most interesting model has perhaps been proposed, together with some applications, by S. S. Grigorian. (14,15,16) The basic characteristics of this model will be outlined in this section with reference to spherically symmetric problems. Some remarks are added concerning its application to dynamic problems and the determination of some experimental functions involved in the description.

3.2. Grigorian's Model

A soil can be viewed as an aggregate of mineral particles, water, and air. In an undisturbed state, the mineral particles are often cemented together and form a porous skeleton which, under sufficiently small loads, will resist deformation as a linearly-elastic Hookean medium. As the load increases, an increasing fraction of it will be carried by the water and the air filling the pores, whereas the skeleton will gradually break up; a further increase in load will not only bring to a total collapse of the cemented skeleton but will also start fracturing the mineral particles themselves with a clearly irreversible process of compaction and comminution. It is evident that a Hookean model is no longer applicable to describe this process. If the load is reduced at this stage, the resulting decrease in density will be very small in comparison with the relatively large density change taking place during loading. This is obviously a result of the irreversibility of recompaction and fracture of the mineral particles, because the complete breakdown of the skeleton reduces the soil to a material without memory of the initial conditions. In this range, it
is then clear that the loading and unloading processes should be
described by different laws. With a significant further increase of the
load, the porosity of the medium will eventually be eliminated and the
soil will compress and expand reversibly in the same way as metals and
non-porous rocks.

It may be expected that the range of irreversible behavior is
greatly affected by the initial water content of the soil. The
importance of this factor will be illustrated below with some experi-
mental results for a typical class of soils under dynamic loading.

As a basis for a stress-strain description, two kinds of
forces must be taken into consideration:

1) Dry friction forces between the particles in contact

2) Elastic forces inside the particles themselves.

The stresses arising from the friction forces should depend only on the
instantaneous state of deformation and not on the final total displace-
ments because of the irreversible character of compaction and fracturing.
This means the state of stress should depend on the deformation rates
and not on the final deformations as is the case for materials with
memory. The presence of elastic forces inside the particles means
there must be elastic components in the stress tensor.

The total state of deformation itself is supposed to be
derived from the superposition of a volumetric and a shear deformation,
as it is commonly done in mathematical models for solid media. In an
isotropic medium it can reasonably be assumed that the magnitude of the
volumetric deformation is determined by the mean stress (hydrostatic
pressure) $p = -1/3 \sigma_{ii}$, and the density changes in the medium. This
assumption leads then to a relationship between pressure and density, which must account for the possibility of both reversible and irreversible volume changes for increasing p. This is in contrast with the models of the various plastic theories for dense non-porous materials where the volumetric deformation is assumed as elastic or completely negligible.

On the basis of these considerations and of experiments the qualitative aspect of the pressure density relation $p = f(\rho)$, is shown in Fig. 5. Below the point $(p_1, \rho_1)$ is the range of irreversible behavior; if the loading process is arrested at any point $(p^*, \rho^*)$ below $(p_1, \rho_1)$ and the pressure is decreased, unloading will occur along a different path and, if the medium has some cohesion, it will terminate at a point on the curve $p_0 = \Phi(\rho_0)$ which represents the state of complete unloading. If the medium has no cohesion, it can withstand no tensile stresses and $p_0 = 0$.

If the pressure is carried beyond $p_1$, reversible volumetric deformation takes place as for an elastic material. If unloading starts from a point $(p_2, \rho_2)$ and the pressure decreases below $p_1$, the unloading path is qualitatively illustrated by the dashed line shown in Fig. 5. An analytical representation for these rather complicated properties of volume deformations can be found in Ref. (15).

The analytical relationship for the shear deformation is based on the Prandtl-Reuss mathematical model of elastic-plastic solids. It is assumed that a part of the infinitesimally small rate of shear
deformation of an element will become plastic when the deviator of the stress tensor satisfies a certain plasticity condition.

This condition makes the plastic limit depend on the pressure and is written

\[ J_2 = \frac{1}{2} s_{ij} s_{ij} = F(p) \]  \hspace{1cm} (77)

where \( s_{ij} = a_{ij} + b_{ij} p \) is the stress deviator and \( F \) is a non-decreasing function of its argument. The explicit form of \( F \) has to be determined by experiments. For a rather large class of granular media it can be assumed that

\[ F(p) = (\alpha p + b)^2 \hspace{1cm} \alpha, b \text{ constants} \]  \hspace{1cm} (78)

Equation (77) is of the same form as the well known Coulomb-Mohr criterion of failure and it can be considered as an approximation or generalization of it. It should be observed that Eq. (77) is a condition of ideal plasticity. For the case of a spherically symmetric state of stress one can set \( \sigma_r = -p + s_r \) and \( \sigma_\theta = -p - s_r/2 \), \( s_r = 2(\sigma_r - \sigma_\theta)/3 \) being the stress deviator. The yield condition, Eq. (77), then simply reads

\[ J_2 = \frac{1}{2} (\sigma_r - \sigma_\theta)^2 = \frac{3}{4} s_r^2 = F(p) \]

or

\[ s_r^2 = \frac{4}{3} F(p) \]  \hspace{1cm} (79)

According to the Prandtl-Reuss theory, when plastic flow occurs, there will be a plastic component of the strain-rate deviator directly proportional to the stress deviator, or
\[ \dot{\varepsilon}_{ij} = \lambda s_{ij} \]  

(80)  

where \( \dot{\varepsilon}_{ij} \) is the strain-rate tensor.

A correct description of the elastic strain rate poses the question of time differentiation of a stress tensor in a case of large deformations. The problem is discussed by Grigorian and solved by using Jaumann's definition of a stress rate tensor. For a problem in principal stress and strain, however, Jaumann's definition coincides with the usual time derivative. For a spherically symmetric stress problem, the complete elastic-plastic law reads:

\[ \frac{ds_r}{dt} + \lambda s_r = \frac{4G}{3} \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right) \]  

(81)  

The plastic multiplier \( \lambda \) can be determined through the plasticity condition, Eq. (77) or Eq. (79). Multiplying Eq. (81) by \( s_r \) one has

\[ s_r \frac{ds_r}{dt} + \lambda s_r s_r = \frac{4}{3} s_r G \left[ \frac{\partial v}{\partial r} - \frac{v}{r} \right] \]  

or

\[ \frac{1}{2} \frac{d(s_r^2)}{dt} + \frac{4}{3} \lambda F(p) = \frac{4}{3} s_r G \left[ \frac{\partial v}{\partial r} - \frac{v}{r} \right] \]  

But

\[ \frac{d(s_r^2)}{dt} = \frac{4}{3} F'(p) \frac{dp}{dt} \]  

yielding,

\[ \lambda = \frac{2GW - F'(p)dp/dt}{2F(p)} \]  

(82)
where \( \dot{W} = s \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right) \) is the plastic rate of work (distortion).

It follows that plastic shear may occur only when both the plasticity criterion and the inequality \( \lambda > 0 \) are satisfied. This inequality implies

\[
2GW > F'(p) \frac{dp}{dt}
\]

or

\[
2Gs \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right) > F'(p) \frac{dp}{dt} \tag{83}
\]

For a granular medium it is reasonable to assume that the shear modulus is a function of the density and furthermore, if the yield condition is satisfied in compression, Eq. (79) gives

\[
s_r = -\frac{2}{\sqrt{3}} \sqrt{F(p)} = -\frac{2}{3} \sqrt{2F(p)} \tag{84}
\]

and Eq. (83) is then equivalent to:

\[
-\frac{4}{3} G(\rho) \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right) > \frac{F'(p)}{\sqrt{2F(p)}} \frac{dp}{dt} \tag{85}
\]

This shows that, since \( G(\rho) \) is naturally assumed to be a bounded quantity whereas \( F(p) \) may increase without limits as \( p \to \infty \), the equality of Eq. (85) may not be satisfied. It is also possible that, after plastic flow is started, a reverse transition from plastic to elastic shear under monotonically increasing compression might occur, depending on the form of the function \( p = p(\rho) \) and \( F = F(p) \). It should be observed that, under increasing load, such a transition cannot take place if plastic shear is governed by the von Mises yield criterion. A detailed
discussion on this point for the case of one-dimensional dynamic flow is given in Ref. (16). If only elastic shear occurs, the stress-strain law is obviously expressed by Eq. (81) with $\lambda = 0$.

In its original statement, Mohr's limit criterion postulates that, at the onset of slippage in a granular medium, the shear stress in any plane cannot exceed the friction force caused by the normal pressure acting on the section. If the medium has negligible cohesion, the limiting condition can be written in terms of the major and minor compressive stresses $\sigma_1$ and $\sigma_3$ as

$$\sigma_1 - \sigma_3 = (\sigma_1 + \sigma_3) \sin \varphi$$

where $\varphi$ is the angle of internal friction. This is the same as

$$\frac{\sigma_1}{\sigma_3} = \beta = \frac{1 + \sin \varphi}{1 - \sin \varphi}$$

stating the constancy of the ratio of the major to the minor principal stress. The ratio $\beta$ can then be obtained through an experimental determination of the Mohr envelope for the material under consideration.

On the other hand, if Eq. (78) holds with $b = 0$, Eq. (84) can be rewritten as

$$\sigma_r - \sigma_\theta = \frac{\alpha}{\sqrt{3}} (\sigma_r + 2\sigma_\theta)$$

and introducing the ratio $\beta = \sigma_r / \sigma_\theta$ gives

$$\alpha = \sqrt{3} \frac{(\beta - 1)}{(\beta + 2)} = 2 \sqrt{3} \frac{\sin \varphi}{3 - \sin \varphi}$$
If the cohesion $c$ is not negligible it can be shown that

$$c \cot \varphi = \frac{b}{\alpha} \quad \text{and} \quad b = \frac{2\sqrt{3}c \cos \varphi}{3 - \sin \varphi}$$

Experimental values for the constants $\alpha$, $b$ obtained from direct field data on sandy soils, are presented in Ref. (1). Other data from direct measurements on a small scale laboratory explosion in a sandy soil, supporting the assumption of constant $\beta$ during plastic flow, are reported in Ref. (33). A different technique, based on uniaxial compression tests, for determining the function $F(p)$ is described by Grigorian. (15)

3.3. Flow Discontinuities and Dissipation Phenomena Associated with Grigorian's Equations

An exhaustive treatment of dynamic motion governed by the Grigorian's equations (one-dimensional piston problem) is contained in Ref. (16). Some aspects of this analysis, without repeating any detailed analytical treatment, are restated here. First of all it should be observed that a transition from elastic to plastic shear, governed by Eqs. (84) and (85), occurs with a sudden jump in the dilatational sound velocity. This can be easily seen assuming for a moment that in the initial state of loading the medium behaves like a linearly elastic solid. Then $G = \text{constant}$ and the initial slope of the pressure-density curve is given by the bulk-modulus $K$. In this case, if the changes in density and accelerations are small, the dilatational sound speed is approximately,

$$c_l = \left[\frac{1}{\rho_0} \left( K + \frac{4}{3} \frac{G}{c^2} \right) \right]^{1/2}$$
When plastic flow commences, the radial stress $\sigma_r$ may in general be expressed as a function of density only, or

$$
\sigma_r = -p + s_r = -f(p) - \frac{2}{3} [3F(p)]^{1/2} = -f(p) - \frac{2}{3} [3F(f(p))]^{1/2}
$$

if compression is considered.

Then

$$
- \frac{d\sigma_r}{dp} = f'(p)[1 + \frac{F'(p)}{3F(p)}]
$$

and the plastic sound velocity is,

$$
c_{pl} = \left(\frac{1}{\rho_o} \frac{d\sigma_r}{dp}\right)^{1/2} = \left[\frac{f'(\rho_o)}{\rho_o} \left[1 + \frac{F'(p)}{\sqrt{3} F(p)}\right]\right]^{1/2}
$$

(87)

Therefore, even if the slope of the pressure-density curve remains constant [$f'(\rho)/\rho_o = (K/\rho_o)$], a sudden discontinuity occurs in $c$ because of the change in the analytical dependence of the deviator on the density. If the inequality

$$
\frac{d(cp)}{dp} > 0
$$

(88)

holds in the region where the elastic-plastic transition occurs (in compression), then a stable (compressive) shock will arise as a result of the velocity discontinuity. If the volumetric deformation in this range is linearly elastic

$$
p(\rho/\rho_o) = K(\rho/\rho_o - 1)
$$

and if $F(p)$ is given by Eq. (78), then
\[ c_{pl} = \left( \frac{K}{\rho_0} \left( 1 + \frac{\alpha_0}{\sqrt{3}} \right) \right)^{1/2} = \text{constant} \] (89)

which shows that the inequality of Eq. (88) is satisfied by Eq. (89) as well as by Eq. (86). If the medium has an S-shaped volumetric curve, as shown in Fig. 5, it may happen that \( d(c_p)/dp < 0 \) for values of \( p \) close to \( \rho_o \), whereas \( d(c_p)/dp > 0 \) for larger \( \rho \), because of the presence of an inflection point. The uniaxial wave-propagation in a nonlinear elastic material with such a characteristic has been previously analyzed by Barenblatt.(5) For small initial compression, the motion starts out as a continuous compressive wave. As compression increases the latter parts of the input signal start travelling faster and faster, tending to overcome the elastic wave front and forming a shock front, analogous to the classical hydrodynamic problem discussed in Sect. 2.3. In this case however, the shock front, is preceded by the continuous elastic precursor wave. If compression increases without limits, the precursor will eventually be absorbed by the jump. The basic features of this propagation pattern have actually been recorded in field measurements on conventional centrally symmetric explosions and also in the elastic-plastic zone created by underground atomic bursts.(1)

Another discontinuity in the sound velocity arises if unloading is started at a point \((p^*, \rho^*)\), in the range of irreversible behavior of the pressure-density curve. As shown in Fig. 5, a sudden jump in the slope \( dp/d\rho \) occurs at such a point. Theoretical considerations show that both a compressive shock and a continuous rarefaction wave could be caused by this type of discontinuity if Eq. (88) holds.
However, consideration of the stability at the wave front leads one to
discard the first possibility. The continuous rarefaction wave may
travel at a speed greater than the existing compressive loading wave
even when the latter contains a shock front. Therefore, the rarefaction
(unloading) front may eventually overtake the whole region of compressive
motion initially ahead of it, causing thereby a very rapid wave attenua-
tion. Actually, the irreversible region at relatively low stresses is
an important feature of Grigorian's model, since field measurements do
not indicate any stress level below which dissipation is unimportant. (17)
Another significant characteristic of the model is the choice of a yield
criterion of the form of Eq. (77). If von Mises' criterion were chosen
instead, no plastic yielding can occur whenever the peak radial stress
is less than a certain level, and the subsequent attenuation of the
wave is completely elastic, i.e., nondissipative. But if the yield
stress depends on the pressure, as stated by Eq. (77), plastic dissipa-
tion occurs at lower stresses resulting in a faster attenuation of
peak stresses, which has a closer correlation with field data, as shown
by some numerical results to be discussed later.

3.4. Experimental Pressure-Density Curves for a Typical Granular Medium

Some interesting experimental results in the pressure range
for which Grigorian's model may be used, were obtained by Hendron and
Davisson (20) from static and dynamic confined uniaxial tests on samples
of playa-silt from the Nevada Test site. In these tests, the radial
stresses were measured in addition to the axial stress and the axial
strain, thus allowing for the calculation of the mean pressure.
In Fig. 6 the loading portions for two tests on samples with different initial degree of saturation are compared. The dashed parts of the curves are extrapolated. Above the points indicating the strains at saturation, the slope represents the compressibility of water (300 ksi), since it is assumed that the voids are completely filled by water, after the porous skeleton is broken. It can be clearly seen that the initial water content greatly affects the slope of the upper portion of the curves and is therefore bound to play a major role in the propagation and attenuation of a shock wave. A plot of radial stress (loading) vs. axial stress for the same samples is shown in Fig. 7. When the point of saturation is reached, the curves should plot as straight lines with a slope of 45° ($\sigma_a = \sigma_r$). Assuming the density of the soil is constant in the volume instantaneously occupied by the specimen, one has, for the confined uniaxial tests under consideration

$$\rho = \rho_0/(1 - \Delta V/V_0) = \rho_0/(1 - \epsilon_{\text{axial}})$$

and

$$p = (\sigma_{\text{axial}} + 2\sigma_{\text{radial}})/3$$

The pressure-density curves obtained by means of the above relationships and using the data of Figs. 6 and 7 are shown in Fig. 8. These curves clearly exhibit the general characteristics illustrated earlier for the Grigorian model.
IV. NUMERICAL METHODS

4.1. Integration Technique for the Interior Cells

It is easily realized that the solution of the equations of conservation given in Chapter II requires numerical methods. Indeed, the development of the discrete Eulerian model that leads to centered difference equations of the conservation laws in the conservative form is precisely for this purpose, with specific reference to problems of spherical geometry. The solution of these equations requires a stable marching process in the space-time domain. For simplicity, only the hydrodynamic form of the equations will be considered; the same basic conclusions hold also for the non-hydrodynamic stress description.

The marching process involves updating the field quantities from a generic time \( t \) (at which all the quantities are supposed known) to time \( t+\Delta t \). This is governed by a two-step procedure that is partially iterative. Introduce the field variable for momentum

\[
m(I,t) = \rho(I,t)v(I,t),
\]

and let \( t^* = t \) in Eq. (40) and \( t^* = t+\Delta t \) in Eq. (39). Referring to Fig. 2, Eq. (40) then becomes

\[
m(i,t+\Delta t) = m(i,t) - (\Delta t/\Delta r)\left\{\frac{m^2(i+1,t)}{\rho(i+1,t)} - \frac{m^2(i-1,t)}{\rho(i-1,t)} + \frac{[p(i+1,t)+q(i+1,t)] - (\Delta t/r(i))[m^2(i+1,t)/\rho(i+1,t)]}{2} + \frac{m^2(i-1,t)/\rho(i-1,t)}{(\Delta t/r(i))[p(i+1,t)-2p(i,t)+p(i-1,t)]}\right\}
\]

and Eq. (32) becomes
\[ p(i,t+\Delta t) = p(i,t) - (\Delta t/\Delta r)[m(i+1,t+\Delta t) - m(i-1,t+\Delta t)] \]
\[ - (\Delta t/r(i))[m(i+1,t+\Delta t) + m(i-1,t+\Delta t)] \]  

(92)

In the momentum equation, Eq. (91), the additional terms \( q \) are artificial viscosity terms introduced to enhance the stability properties of the difference schemes in the presence of flow discontinuities. The explicit form of \( q \) will be given below.

Equations (91) and (92) are used in the first sweep of the integration procedure from \( t \) to \( t+\Delta t \). It should be observed that the newly computed values of the momentum are introduced directly in the continuity equation. It should also be observed that at the end of the first sweep all the field variables are updated at all the mesh points. The second sweep, therefore, uses the newly computed values at all the mesh points in the same way as a purely implicit scheme.

In fact, setting \( t^* = t + \Delta t \) in Eqs. (40) and (32) yields, respectively,

\[ m_2(i,t+\Delta t) = m(i,t) - (\Delta t/\Delta r)[m_2(i+1,t+\Delta t)/\rho_2(i+1,t+\Delta t) \]
\[ + [p_1(i+1,t+\Delta t) + q_1(i+1,t+\Delta t)] - m_2(i-1,t+\Delta t)/\rho_1(i-1,t+\Delta t) \]
\[ - [p_1(i-1,t+\Delta t) + q_1(i-1,t+\Delta t)] \]
\[ - (\Delta t/r(i))[m_2(i+1,t+\Delta t)/\rho_1(i+1,t+\Delta t) \]
\[ + m_2(i-1,t+\Delta t)/\rho_1(i-1,t+\Delta t) \]
\[ - (\Delta t/r(i))[p_1(i+1,t+\Delta t) - p_1(i,t+\Delta t) + p_1(i-1,t+\Delta t)] \]  

(93)

and

\[ \rho_2(i,t+\Delta t) = \rho(i,t) - (\Delta t/\Delta r)[m_2(i+1,t+\Delta t) - m_2(i-1,t+\Delta t)] \]
\[ - (\Delta t/r(i))[m_2(i+1,t+\Delta t) + m_2(i-1,t+\Delta t)] \]  

(94)
where the subscript "1" is used to label the quantities obtained at the end of the first sweep, and the subscript "2" labels the final quantities for time \( t+\Delta t \). Equations (93) and (94) are used in the second sweep, which completes the integration procedure for the interior cells over a single time step \( \Delta t \).

The artificial viscosity terms actually used in the computations are of two forms, as follows:

1. Linear form,
   \[
   q(i,t) = B\rho(i,t)c(i,t)[m(i-1,t)/\rho(i-1,t) - m(i+1,t)/\rho(i+1,t)]
   \]

2. Quadratic form (original definition by von Neumann and Richtmyer)
   \[
   q(i,t) = \frac{1}{2} \alpha^2 [\rho(i,t)+\rho(i,t-\Delta t)](m^2(i+1,t)/\rho(i+1,t) - m^2(i-1,t)/\rho(i-1,t))
   \]

The physical and mathematical implications of these artificial terms have been illustrated earlier. However, additional information including specific analytical forms for \( q \) can be found in Ref. (32).

Actually there are no rigid requirements in the correct choice of the viscosity terms. It is often chosen through computational experiments and depends very much on the problem under consideration.

The constants \( B \) and \( \alpha \) are also usually chosen by computational experiments such that a shock discontinuity is spread over 2 or 3 space mesh lengths.

4.2. Computational Algorithm and Treatment of Boundary Motions

With reference to Fig. 3 which shows the expansion of a spherical cavity, and setting \( t^* = t \), the momentum equation, Eq. (54), for the boundary cell can be written as
where \( v(B,t) \) is the particle velocity at the cavity. The computational steps used for updating the boundary values and for combining these with the interior field values are as follows (described for the first sweep):

1. With Eq. (95), \( m(BM,t+\Delta t) \) is calculated for the boundary cell. Similarly, the momentum values of all interior cells, \( i \geq N + 1 \), are updated using Eq. (91).

2. On the basis of the interpolation formula

\[
\frac{r(BM,t)}{r(N+1)} = \left[ \frac{m(BM,t+\Delta t)}{m(N+1,t+\Delta t)} \right]^{\alpha_1}
\]

the decay parameter \( \alpha_1 \) is obtained. Then the momentum at \( N \) is computed as,

\[
m(N,t+\Delta t) = m(N+1,t+\Delta t) \left[ \frac{r(N)}{r(N+1)} \right]^{\alpha_1}
\]

3. With the knowledge of \( m(N,t+\Delta t) \), the second part, Eq. (92), of the first sweep can be performed; thus updating the values of \( p \) at all the interior cells starting from \( N + 1 \). The corresponding values of \( p \) and of the local sound velocity are then obtained from the prescribed pressure-density relationship.

4. The boundary pressure is a prescribed input quantity and, once loading or unloading are specified, the pressure \( p \) is assumed to
be uniquely determined. Hence, from the pressure-density function \( p = f(\rho) \), the density at the cavity is determined as,

\[
\rho(B, t+\Delta t) = f^{-1}[P(B, t+\Delta t)]
\]

(5) A value of the decay parameter \( \alpha_2 \) is computed from

\[
\frac{r(B, t)}{r(N+1)} = \left[ \frac{\rho(B, t+\Delta t)}{\rho(N+1, t+\Delta t)} \right]^{\alpha_2}
\]

Then, using \( \alpha_2 \) both \( \rho(BM, t+\Delta t) \) and \( \rho(N, t+\Delta t) \) can be calculated by means of an interpolation equation similar to Eq. (97). Alternatively, the density at the boundary may be determined through the continuity condition of Eq. (53); however, computationally this has proved to be rather inconvenient. The direct interpolation scheme used is obviously possible only for the simplified conditions of isentropic flow assumed herein.

(6) The updated velocities for all points, except the boundary point B, are then obtained as

\[
\begin{align*}
v(i, t+\Delta t) &= \frac{m(i, t+\Delta t)}{\rho(i, t+\Delta t)} \quad i \geq N + 1 \\
v(BM, t+\Delta t) &= \frac{m(BM, t+\Delta t)}{\rho(BM, t+\Delta t)}
\end{align*}
\]

also,

\[
v(BM, t+\Delta t) = \frac{m(BM, t+\Delta t)}{\rho(BM, t+\Delta t)}
\]

(7) The boundary velocity \( v(B, t+\Delta t) \) is computed by a least-square linear extrapolation, using the values \( v(BM, t+\Delta t), v(N+1, t+\Delta t) \) and \( v(N+3, t+\Delta t) \), namely the values at the center of the first three cells on the same row, including the boundary cell. The weights used for \( v(BM, t+\Delta t) \) and \( v(N+1, t+\Delta t) \) are the masses \( M(BM, t+\Delta t) \) and \( M(N+1, t+\Delta t) \), whereas the weight for \( v(N+3, t+\Delta t) \) is the mass \( M(N+3, t+\Delta t) \) multiplied by a factor which is inversely proportional to \( M(BM, t+\Delta t) \), precisely,

\[
1 - \frac{M(BM, t)}{M_0(BM, t)}
\]

Here \( M_0 \) is the mass of the current boundary cell at the
start of the emptying process. This procedure is adopted for the following reasons:

(a) The boundary cell becomes increasingly empty as the cavity expands; consequently the accuracy of $v(B, t+\Delta t)$ may decrease as $M(B) \rightarrow 0$. Its weight, therefore, ought to decrease accordingly.

(b) The interpolation and weighting procedure should have a smooth transition from an empty cell to a full cell. The above choice of the weights insures the required smoothness.

The seven steps described above complete the calculations for the first sweep. For the second sweep Eqs. (93) and (94) are used in place of Eqs. (91) and (92) for the interior cells, while for the boundary cell the boundary momentum equation is written, consistent with Eq. (93), as follows:

$$
m_2(B, t+\Delta t) = [m(B, t)\Delta^2(t)[1-\Delta^2(t)/r(N)]

- m_1(N, t+\Delta t)\Delta t/\rho \sqrt{N, t+\Delta t} + [t(t+\Delta t)-t_1(N, t+\Delta t)]

- 2\Delta^2(t)(t(t+\Delta t)-t_1(B, t)/r(N))]\Delta t /

\left(\Delta^2(t)[1-\Delta^2(t)/r(N)] - v_1(B, t+\Delta t)\Delta t[1-2\Delta^2(t)/r(N)]

+ v_1(B, t+\Delta t)\Delta t/r(N)) \right). \tag{99}
$$

All the other steps are repeated exactly as in the first sweep. Once the final value $v_2(B, t+\Delta t)$ is obtained, the position of the boundary with reference to point $N$ can be updated using

$$
\Delta^2(t+\Delta t) = \Delta^2(t) - v_2(B, t+\Delta t)\Delta t
$$

If $\Delta^2(t+\Delta t) \leq 0$, it means that the boundary cell has been completely
emptied. In this event, \( \Delta \xi = \Delta r \) is set for the next cycle of computation and the boundary cell is advanced to the next cell of the same row.

It should be observed that no artificial viscosity terms are used in Eq. (95) and (99). This involves better stability conditions in the case where the cavity pressures are applied with a finite rise time.

Certain modifications are required in the treatment of the boundary when deviatoric stresses are also considered. First of all, the computation of the quantities at the boundary must permit the calculation of both the mean pressure and the stress deviator. These two stresses must necessarily be distinguished in order to allow the determination of the boundary density from the given p-p relationship. For this purpose it is simply assumed that the ratio of the hydrostatic pressure to the (known) total radial stress at the cavity is the same as that of the first interior cell obtained from the stresses of the last cycle of computation. For the first time step (\( t = \Delta t \)), it is assumed that \( p(B,\Delta t) = \sigma_r(B,\Delta t)/3 \).

Once the boundary values of \( p \) and the deviator \( \sigma_r \) are computed at the beginning of each time step, the above procedure is repeated with essentially no change, except for the appropriate momentum equations. The deviatoric stress at the center of the boundary cell is interpolated from the boundary value and the deviator of the first interior cell through an equation similar to Eq. (97). The deviatoric stresses in the interior cells are obtained from the velocities through an appropriate constitutive relationship.
4.3. **Truncation Error and Stability Properties of the Difference Equations**

A Taylor series expansion of the relevant field quantities about the point \((i,t)\) will show that the truncation errors of Eqs. (91) and (92) are, respectively, as follows:

\[
\frac{\partial q}{\partial r} + \frac{1}{2} \frac{\partial^2 m}{\partial t^2} \Delta t + O(\Delta r^2)
\]

and

\[
\left(\frac{1}{2} \frac{\partial^2 \bar{p}}{\partial t^2} + \frac{\partial^2 \bar{m}}{\partial r \partial t} + \frac{2}{r} \frac{\partial \bar{m}}{\partial t}\right) \Delta t + O(\Delta r^2)
\]

where \(\bar{p}\) and \(\bar{m}\) represent an exact solution of the differential equations of conservation, Eqs. (16) and (19), in a region of continuous flow; i.e., where all the derivatives of \(\bar{m}\) and \(\bar{p}\) are bounded.

In order to insure stability of the solution of the difference equations, the time mesh is usually coarser than the space mesh. It can, therefore, be assumed that besides \(\partial q/\partial r\), the dominant error terms are of \(O(\Delta r^2)\). On the other hand \(q\) can be defined in such a way that \(\partial q/\partial r\) becomes very small in the regions of continuous flow. Obviously, this does not hold in the vicinity of a flow discontinuity; however, in this latter region the error term is precisely what makes the propagation of the discontinuity possible by smearing it over a small finite length, and thus preserving the continuity of the derivatives in the transition region.

Usually the truncation error shows how closely a solution of the difference equations agrees with that of the corresponding differential equations as the mesh is made finer and finer. If the
difference scheme is stable, a smaller truncation error usually implies a faster convergence of the numerical solution to the solution of the differential equations. However, if stability is violated this is no longer true, and it would be misleading to judge the quality of a difference scheme solely on the smallness of its truncation error.

In the present case the error terms of $O(\Delta r^2)$ arise from the centered difference formulation of the transport terms in the model equations. This is not necessarily an optimum treatment, particularly if there is formation of shock waves in regions where the particle velocities are much lower than the local sound velocity. Computational results show that such regions are usually critical in the assessment of the stability of explicit Eulerian difference schemes. For this reason, certain Eulerian codes based on the so-called FLIC method, in which the difference schemes are totally explicit, have been used to calculate the transport terms with a mass directly proportional to the density of the cell from which the fluid is flowing. This is known also as the "donor cell" differencing method. It leads to backward space differences and, therefore, the truncation error terms are of $O(\Delta r)$. Some of these terms are similar to viscosity terms in the sense that they cause an artificial diffusion of mass and momentum in regions where rapid changes in these quantities take place. This procedure, therefore, has a stabilizing influence in the presence of shocks, but it also has the disadvantage that the size of the diffusion terms cannot be as well controlled as with the use of artificial terms. An interesting comparison of results obtained with various differencing
methods and various types of viscosity for one-dimensional Eulerian equations is given in Ref. (27).

Because of the rather complicated form of the equations for the boundary cell, no rigorous analysis of the stability question in which the boundary conditions are included was attempted. Only a linear stability analysis for the interior difference equations is outlined; this is intended to provide some insight into the properties of the mixed explicit-implicit system represented by Eqs. (91) and (92) and the stabilizing influence of the viscosity terms. Such an analysis is made following the well known von Neumann's method of testing the growth of small perturbations (in the form of Fourier series components) superimposed on the solution of the differential problem. The von Neumann's method yields the necessary and sufficient conditions for stability only in the case of linear equations with constant coefficients. However, the method is extremely useful also for truly nonlinear difference problems when a suitable linearization can be applied. The procedure can be outlined as follows\(^{(32)}\):

(1) Suppose that \( \bar{u}(I\Delta r,n\Delta t) \) is a solution of the differential problem and \( u^n_I \) is the solution of the corresponding difference equations. Let \( u^n_I = \bar{u}(I\Delta r,n\Delta t) + \delta u^n_I \), where \( \delta u^n_I \) is a small perturbation of first order. Substituting this in the difference equations and dropping terms of second and higher orders in \( \delta u \), the equations of first variation in \( \delta u \) are obtained. These equations are linear in \( \delta u \) with the coefficients depending on \( \bar{u} \).

(2) At each point \( (I\Delta r,n\Delta t) \) consider the equations of the first variation with the (locally constant) values of the coefficients
at that point, and test for stability on the basis of a Fourier analysis. This is done by taking \( \delta u_I^n = \delta u_0 \xi_1^n \), where \( \xi_1^I = e^{ik\Delta r} \) and \( \xi^n = e^{\alpha \Delta t} \), in the equations of the first variation. Since the system is linear with constant coefficients, the vanishing of the determinant of the coefficients is required for the existence of a non-trivial solution. This gives one determinantal equation for \( \xi \).

In order to insure stability, the roots of the determinantal equation must be such that \( |\xi| \leq 1 \). This guarantees that a given Fourier component does not grow with time at the specified point. If \( |\xi| \leq 1 \) is satisfied at all the points of the discretized domain, then there is reasonable assurance for an overall stability of the differencing scheme.

Under certain conditions and for certain classes of equations, local stability can be shown to be a necessary and sufficient requirement for overall stability. However this is not true in general. An extensive discussion on this point is given in Chapter 5 of Ref. (32).

Now consider Eqs. (91) and (92), and let

\[
\rho(I,t) = \rho_I^n = \overline{\rho} + \delta\rho_I^n ;
\]

and

\[
m(I,t) = m_I^n = \overline{m} + \delta m_I^n
\]

Using the linear viscosity in Eq. (91),

\[
q(I,t) = B\rho(I,t)c(I,t)[m(I-1,t)/\rho(I-1,t) - m(I+1,t)/\rho(I+1,t)]
\]

and recalling that \( c^2 = dp/d\rho \), the equations of the first variation resulting from Eqs. (91) and (92) are, respectively,
and $o (102)$

where $\lambda = \Delta t / \Delta r$, and $\nu = n / \rho$. The spherical divergence terms, being of higher order, disappear in the above equations. The coefficients $c$ and $v$ are zero order quantities and can be regarded as constants in any small region. Now let $\delta o = \delta o e^{ikl n}$ and $\delta m = \delta m e^{ikl n}$ in Eqs. (101) and (102). After some simplifications, these yield

\[
[(s-1) + 4ilsnk + 4\lambda Bcsin^2 k] \delta m
\]

\[
+ [2i\lambda (c^2 \nu^2) sink - 4\lambda Bvsin^2 k] \delta o = 0
\]

and

\[
2ilsnk \delta m + (s-1) \delta o = 0
\]

These two equations, therefore, constitute a linear system with constant coefficients. The two unknowns are $\delta o$ and $\delta m$. The requirement for a non-trivial solution yields the following determinantal equation

\[
\delta^2 + [i\lambda \nu (2 + \lambda B^2 c) + \lambda B^2 c + \lambda^2 A^2 (c^2 \nu^2) - 2] \delta
\]

\[
- (2i\lambda \nu + \lambda BcA^2) + 1 = 0
\]

(103)

where $A = 2sink$.

According to von Neumann's method, local stability is insured if the roots of Eq. (103) satisfy $|\delta| \leq 1$. The solution of Eq. (103) for $\delta$, and the evaluation of $|\delta|$, lead to rather complicated expressions.
where complex terms appear under the radical sign. It seems preferable, therefore, to use an algebraic criterion for determining whether unity is an upper bound of $|\xi|$. First let,

$$\lambda \Delta v = M \quad ; \quad \lambda \Delta A^2 c = Q \quad ; \quad (c^2 - v^2)\lambda A^2 - 2 = K$$

and also,

$$\beta_1 = [M^2(2+Q)^2 + (K+Q)^2]^{1/2} \quad ; \quad \beta_2 = [\lambda M^2 + (1-Q)^2]^{1/2} ;$$

it can then be shown (37) that $|\xi| \leq 1$ if the inequalities

$$1 - \beta_1 - \beta_2 > 0 \quad (104)$$

and

$$1 - \beta_1 > 0 \quad (105)$$

are both satisfied. It is readily observed that if the artificial viscosity terms are absent, Eqs. (104) and (105) become

$$1 - (\lambda M^2 + K^2)^{1/2} - (1+\lambda M^2)^{1/2} > 0 \quad (106)$$

$$1 - (\lambda M^2 + K^2) > 0 \quad (107)$$

Clearly, no value of $M$ and $K$ can satisfy Eq. (106); hence, the stability of Eqs. (91) and (92) cannot be assured if no viscosity terms are used.

Next consider the case where $v^2/c^2 << 1$. It will be seen in the next chapter that this requirement is often fulfilled for wave propagations in earth media under low pressures. If this condition is satisfied, the inequalities of Eqs. (104) and (105) yield

$$2 - \lambda A^2 c^2 > 0$$
and \[ 3 - \lambda c B A^2 - (\lambda c)^2 A^2 > 0 \]

Since the largest possible value of \( A^2 \) is 4, and \( c, \sigma, \) and \( B \) are all assumed to be positive quantities, the above inequalities also imply

\[ 1 - 2\lambda^2 c^2 > 0 \]

and

\[ 3 - 4\lambda c - 4\lambda^2 c^2 > 0. \]

These lead, respectively, to the following requirements

\[ \frac{c\Delta t}{\Delta r} < \frac{\sqrt{2}}{2} \quad (108a) \]

and

\[ \frac{c\Delta t}{\Delta r} < \frac{1}{2} (B^2 + 3)^{1/2} - \frac{B}{2} \quad (108b) \]

If the more severe of these two inequalities is satisfied, stability is indicated. For instance, if \( B = 0.5 \), Eq. (108b) yields

\[ \frac{c\Delta t}{\Delta r} < 0.65 < \sqrt{2}/2 \]

These restrictions therefore, are more stringent than the well known Courant-Friedrichs condition, which says

\[ \frac{c\Delta t}{\Delta r} \leq 1. \quad (109) \]

In the general case when \( v \) is of the same order of magnitude as \( c \), the fulfillment of Eqs. (104) and (105) leads to

\[ (K+Q)^2 < 1 - \lambda^2 (2+Q)^2 + 4\lambda^2 (1-Q)^2 + 2(4\lambda^2 + (1-Q)^2)^{1/2} \quad (110) \]

and

\[ (K+Q)^2 < 1 - \lambda^2 (2+Q)^2 \quad (111) \]

From here, it is readily seen that Eq. (111) prevails if
Values of the ratio \( \lambda \) which satisfy the inequalities of Eqs. (110) and (111) must be determined numerically once \( B, c, \) and \( v \) are given or known. It should be emphasized that conditional stability, therefore, can be established locally for Eqs. (91) and (92).

Considering now the second sweep of the integration procedure, let the system of Eqs. (93) and (94) be regarded as a purely implicit scheme. This is not a bad approximation, provided the calculations of the first sweep are stable. Neglecting the viscosity terms, it can be shown that the determinantal equation corresponding to the first variation of the above equations is

\[
\xi^2 [2i\lambda \nu + \lambda^2 (c^2 - v^2)A^2 + 2i\lambda A] - 2\xi (1+i\lambda v) + 1 = 0
\]

whose roots are:

\[
\xi_{1,2} = \frac{[1+\lambda A(v+c)]}{[1+(c^2-v^2)\lambda^2 A^2 + 2i\lambda A]} \]

Hence,

\[
|\xi_{1,2}| = \frac{[1+\lambda^2 (v+c)^2]^{1/2}}{[1-(c^2-v^2)\lambda^2 A^2 + 4\lambda^2 A^2 v^2]^{1/2}}
\]

It can then be shown that \( |\xi_{1,2}| \leq 1 \) for any value of \( c \) and \( v \). This, therefore, insures stability in the small.

Finally, it should be pointed out that the local stability criterion discussed above provides only a rough guide if a shock discontinuity is propagated through the region under consideration. In fact, the whole stability concept is plausible only when \( \Delta t \) and \( \Delta r \) are
much smaller than the total space and time involved in the physical problem, which in turn also implies that $\Delta r$ must be much smaller than the shock thickness. This latter requirement, however, cannot be satisfied by any discrete approach if the definition of the shock has to be sharp. In such cases, where shocks can develop, an approximate stability criterion of the following form

$$(c + |v|)\Delta t/\Delta r < 1$$

can be used (32) for purely hydrodynamic situations. Specific values of $(c + |v|)\Delta t/\Delta r$ used in the present study are illustrated in Chapter V.
5.1. Introduction

No exact solution is known for the propagation of spherical waves in solid media in which the material behavior is described by an arbitrary nonlinear pressure-density relationship and an elastic-plastic stress-strain equation. Approximate analytical solutions determined under greatly simplified assumptions, can be found in Refs. (8), (23) and (33). In fact, even for the simplest case of one-dimensional plane geometry, the analytical solution of the boundary and initial value problem, in which shock and unloading waves can occur in a general material, remains a formidable problem. Consequently, the comparison of numerical calculations with exact solutions is as yet not possible. Therefore, from a purely theoretical standpoint the prediction of wave motions using computer methods must be considered with some care. However, because of the current interest in the prediction of ground motions resulting from explosive forces, particularly those induced by nuclear bursts, some field data are available for contained bursts that may be used to verify the reliability of the numerical calculation method. The data for regions that are relatively far from the center of burst appear to be reasonably good.

In this study, two groups of numerical computations were performed to test the validity of the proposed numerical model, as well as to illustrate its application to the prediction of spherical ground motions in earth media.
(1) Elastic-plastic wave propagation in granite; for this problem it is possible to compare the numerical results in terms of the decay of peak radial stress and peak radial velocity with certain experimental data recorded at the Nevada Test Site (Hardhat Event). A linear pressure-density relationship and two failure criteria of the material were used in the calculations.

(2) Shock-wave formation and propagation in playa silt; the problem is first solved by assuming that no deviatoric stress component is acting and that the pressure-density relationship is as illustrated in Fig. 8. The shock is propagated by means of artificial viscosity terms. Because of the steep slope of the p-p curve at the higher pressure regions, the early stage of this problem, during which strong shocks are formed, constitutes a severe test of the stability and convergence properties of the difference equations. For this reason, the effects of refining the mesh size and of varying the viscosity terms are illustrated.

Calculations were performed also for the same problem in the presence of deviatoric effects as described by the Grigorian theory. No significant differences were found in the propagation process except for the very early stage of loading.

No comparison with field data can be shown for this second group of problems since no test data for deep underground explosions seem to be available.

5.2. Elastic-Plastic Propagation in Granite

The experimental data recorded in the low pressure region (below 4 kbars) for the Hardhat event \(^{(8,34)}\) (Nevada Test Site, 1962),
were used to test the spherical calculations of the proposed Eulerian model. Hardhat involved a 5-kiloton nuclear explosion in granite at a depth of 950 feet, for which a considerable number of particle acceleration and velocity measurements were obtained over a wide range of distances, extending from the hydrodynamic region to the elastic region. Detailed descriptions of the data and their sources can be found in Refs. (8) and (34).

The calculations were carried out for the regions in which the material can be effectively regarded as isotropic and homogeneous. Also, a linear pressure-density relationship of the form $p = K(\rho / \rho_o - 1)$ was assumed. The same material constants as those described in Ref. (9) were used; namely,

$K = 0.361$ megabar $\approx 5235$ ksi
$G = 0.315$ megabar $\approx 4570$ ksi.

These values are in agreement with the in-situ seismic measurements of 17,850 ft/sec and 10,000 ft/sec for the dilatational and shear velocities, respectively, which can be verified on the basis of the following elastic relationships

$c_f^2 \rho_o = K + \frac{1}{2} G$  
$c_s^2 \rho_o = G$

with $\rho_o = 2.67$ gm/cm$^3$ as the initial density of the material.

Two different sets of calculations were performed: the first set assumes a linearly elastic behavior with the plastic equations of Grigorian, while the second set also uses a linearly elastic behavior but with the yield condition of von Mises. The latter assumption implies that when the condition of positive plastic work
\[ \dot{W} = s_r \left( \frac{\partial V}{\partial r} - \frac{V}{r} \right) > 0 \]

is satisfied, the stress deviator \( s_r \) assumes a constant value

\[ s_{r,\text{plastic}} = -\frac{2k_o}{3} \quad (110) \]

since in this case plastic flow is supposed to be independent of the mean pressure. According to the original definition which is based on static loading of metals, the value \( k_o \) is the yield limit in uniaxial tension; but for applications to dynamic problems this definition must be modified and it is preferable to regard \( k_o \) as a plastic modulus determined directly from dynamic test data. Certain experimental evidence indicates that the dynamic strength should be approximately twice the static yield strength. Therefore, based on these assumptions, and on the data and the numerical results discussed in Ref. (9), the value of \( k_o = 3 \) kilobars was selected for the Von Mises criterion.

For both sets of calculations, the elastic and the unloading behavior of the stress deviator is as described by Eq. (81) with \( \lambda = \infty \) and \( G = \text{constant} \). When Eq. (78) is used as the yield condition, the values of \( b = 0.5 \text{ ksi} \cong 35 \text{ bars} \) and \( \alpha = 0.835 \) were assumed. The small value of \( b \) suggests a rather low degree of cohesion whereas the value of \( \alpha \) corresponds approximately to a ratio of \( \beta = 6 \) and, therefore, to an angle of internal friction \( \varphi \) of \( 45^0 \).

It is further assumed that the medium cannot withstand tensile stresses higher than \( T_o = 1 \text{ ksi} \cong 70 \text{ bars} \) and that when this limit is exceeded either by \( \sigma_r \) or \( \sigma_\theta \), \( G = 0 \). If only radial cracks develop, it
would be sounder to assume a one-dimensional state of stress in the

cracked region, with \( \sigma_0 = 0 \); see Ref. (39).

The calculations were started at a distance of 200 ft from the
center of the explosion; this distance was taken as the initial radius
of the cavity for the spherical model. At the station \( r = 203 \) ft, a
smooth pressure-history measurement made by Heusinkfeld, et al. is
available. (21) This profile shows a peak pressure of approximately
4 kilobars, a rise time of approximately 1.6 msec., and a fall-off time
\( t_d \) of approximately 2 msec. This pressure pulse was approximated with
the following function

\[
P(t) = \frac{P_{\text{max}}}{t_r} \left( 1 - \left( \frac{t}{t_r} \right)^2 \right)
\]

(111)

with \( P_{\text{max}} = 4 \) kilobars and \( t_r = 2 \) msec. This is then applied at the
cavity to simulate the starting conditions of the explosion.

The logarithmic decay of the peak radial stress with distance,
scaled to 5 kilotons (according to the usual cube root scaling law) is
illustrated in Fig. 9. The curve calculated with the Coulomb-Mohr
(simplified Grigorian) behavior plots essentially as a straight line,
on which the peak pressure falls off approximately as \( r^{-2.03} \). It does
not deviate much from the curve obtained with the approximate analytical
method of Bishop. (8) The equation given by Bishop for the attenuation
of the radial stress is,

\[
\frac{\sigma_r}{\sigma_{r_1}} = \left( \frac{r}{r_{1_r}} \right)^{-2}
\]

where \( \sigma_{r_1} = 45,400 \) kilobars and \( r_{1_r} = 4 \) ft are the initial cavity pressure
and radius assumed for the explosion. Under the Coulomb-Mohr assumption, it was found that the material yields at very low stress levels (\(\sigma_r\) is approximately 0.55 kilobars) and that plastic dissipation is present at all stress levels over the entire region of the calculations; that is, up to a distance of about 760 ft. from the center of the explosion. The extensive plastic dissipation indicated by the Coulomb-Mohr hypothesis seems to be in rather good agreement with field measurements. It can be observed in Figs. 11 and 12 that this type of plastic behavior does not produce radial or tangential cracks in the unloading portion of a pulse if a nonzero tensile strength is assumed for the material.

A somewhat different behavior is exhibited by the results obtained with the von Mises yield condition as shown in Fig. 9. In this case, plastic dissipation vanishes below the assumed yield limit of 3 kilobars; plastic flow stops at a range of approximately 230 feet; beyond this distance the propagation is entirely elastic. In this curve, the decay of the peak stress is influenced also by the formation of radial cracks immediately following the main pulse which are caused by the tangential stresses exceeding the tensile strength of the granite. Although the stress distribution in the cracked region may not be completely realistic, the propagation of the cracking front as a rarefaction shock, see Figs. 11 and 12, is theoretically correct.\(^{(39)}\)

It can also be observed from Fig. 9 that the curve calculated by Butkovich with the SOC code, taken from Ref. (9), appears to lie definitely above the experimental data. SOC is a Lagrangian code, developed by Seidl,\(^{(35)}\) which supposedly can describe a variety of
plastic crushed and cracked materials by means of rather sophisticated routines. For the curve shown, Butkovich used a von Mises condition with a compressive yield strength of 2.5 kilobars.

The graph for the decay of the peak radial velocity is given in Fig. 10; this shows essentially the same features as those shown in Fig. 9. A more pronounced difference in the velocities between the Coulomb-Mohr and the von Mises assumptions is indicated in the early stage of propagation. The experimental points appear to indicate a slower attenuation than the corresponding computational results in the near region. However, the results obtained with von Mises hypothesis seem to indicate that if a lower compressive strength were assumed, resulting in more extensive plastic flow, a better agreement with the data might follow.

The effect of cracking in the unloading portion of a pulse is clearly illustrated by the stress profiles shown in Figs. 11 and 12. In both cases the peak radial stress is lower than the assumed yield limit of 3 kilobars and therefore propagation is entirely governed by the elastic equations. It is perhaps interesting to observe that cracking of the material leads to the formation of a second peak travelling after the main signal.

A comparison of the radial stress histories calculated with the two different hypotheses at various locations is shown in Fig. 13. The histories for the von Mises case exhibit a sudden drop when the cracks start opening, as would be expected.

It can be observed that the difference in the time of arrival of pulses obtained with the two yield conditions gets more significant
at the larger distances (almost 2 msec at 445 ft). There is also a
discrepancy in the arrival time between the calculated and observed
pulse at 360 feet; this may partly be attributed to an error in the
measuring instrument.\(^{(9)}\) However, the agreement in rise time and
peak stress between the calculated Coulomb-Mohr history and the field
data is otherwise good.

For the calculations described above, a mesh length of
20 inches and linear artificial viscosity terms with \( B = 0.2 \) were used.
A uniform time increment \( \Delta t \) was chosen such that

\[
\frac{c \Delta t}{\Delta r} = \frac{1}{3}.
\]

Additional calculations were performed without the use of viscosity
terms, and no appreciable difference was noted in the results; thus
indicating that the algebraic criterion of Eqs. (104) and (105) might
be more severe than necessary.

5.3. Shock Wave Propagation in Playa-Silt

Numerical calculations were performed also with the model for
problems where shock-waves can develop as a result of the equation of
state of the material. In particular, the material is playa silt with
a pressure-density relationship as shown in Fig. 8. No experimental
data from contained underground bursts in this medium appear to be
available. Nevertheless it is deemed important to ascertain the capa-
ability of the model to predict the wave propagation in a granular
material, such as playa silt, in the range of behavior for which the
Grigorian theory is applicable. As indicated earlier, playa silt is
one of the very few earth materials for which sufficient laboratory data are available to permit a suitable description of the pressure-density relation for pressures up to about 1.5 kilobars (22 ksi).

Numerical calculations of spherical shock propagation in playa silt at very high pressure levels (hydrodynamic zone) were performed by Erkman using a Lagrangian code. Since the pressure levels considered by Erkman were much higher than those considered herein, it is not possible to use any of Erkman's results for the present problem.

In the present calculations, it was first assumed that the state of stress is hydrodynamic and, therefore, that it is completely described by the pressure-density relation \( p = f(\rho) \). The experimental curve of the medium with the higher \( W_i \) in Fig. 8 was chosen for this purpose. For computational convenience, the lower portion of the curve is approximated by means of 5th degree polynomials such as to produce a smooth and continuous fit of the function as well as of its first derivative. The straight part of the curve was extrapolated with a constant slope up to a pressure of 30 ksi. Assuming a density at rest \( \rho_0 = 1.69 \text{ gm/cm}^3 \), the sound speed corresponding to the straight portion of the \( p-\rho \) curve is approximately 3300 ft/sec. In general, the experimental curves obtained by Hendron and Davison\(^{(20)}\) show that unloading from a peak pressure occurs initially with a steeper slope than the loading path, and that in some cases the unloading path is practically vertical. However, for specimens with a relatively high initial water content, the loading-unloading loop tends to be closed, giving some credence to the assumption of Grigorian's theory.
The present calculations, which make use of the Grigorian theory, are based on the assumption that the $p-\rho$ relation is reversible beyond a certain pressure level and consequently unloading follows a straight line as shown in Fig. 5. Unloading paths that are initiated from a point $(p^*,\rho^*)$ lying in the range below $(p_1,\rho_1)$, as shown in Fig. 5, follow a straight line having the same slope as the reversible range. For this group of calculations the following parameters were used:

(a) Initial cavity radius $r_o = 5000 \text{ in} \approx 416 \text{ ft}$

(b) Pressure pulse applied at cavity is given by Eq. (111), with $t_r = 10 \text{ msec}$ and $p_{max} = 30 \text{ ksi} = 2.07 \text{ kilobars}$

(c) Stability requirement: $0.3 \geq (c+v) \Delta t/\Delta x \geq 0.2$.

The space mesh $\Delta x$ was chosen by computational experimentation such that it would be compatible with the limitations of the computer storage and also define a shock over a reasonably small zone. Using linear viscosity terms with $B = 0.5$ it was found that the effective shock thickness is spread over 3 mesh lengths, and that a value of $\Delta x = 10 \text{ inches}$ provides a sufficiently fine description of the jumps. However, rather large differences in the time of arrival of the disturbance were found by varying $\Delta x$ from 5 to 20 inches. This effect is clearly shown by the velocity histories of Fig. 14, and also by the shock and precursor wave paths shown in Fig. 15. In Fig. 15, it can be observed that the difference in the arrival times of the precursor wave obtained for $\Delta x = 5''$ and $\Delta x = 20''$ is about 2 milliseconds. This gives an indication of the error that might be expected from an increase in $\Delta x$. 
A special series of calculations was performed for the purpose of comparing the relative effectiveness of the linear and quadratic viscosity terms, as given in Sec. 4.1, for the propagation of discontinuities. The pressure profiles obtained with the two types of artificial viscosity are compared in Fig. 16. The viscosity terms were employed both during loading and unloading, with $B = 0.5$ for the linear case, and $a = 1.6$ for the quadratic case. The latter value was chosen after some tests with $a$ varying between 1 and 2. A mesh length of 20 inches was used in both cases. It is evident that the quadratic terms do not provide sufficient damping for the oscillations immediately behind the shock, whereas the linear terms give remarkably smoother profiles without causing any significant erosion of the jump amplitude. For this reason it seemed preferable to use linear viscosity terms with $B = 0.5$ and the results that follow herewith were determined on this basis.

The results of the final calculations are plotted in Figs. 17 through 20. The time histories of velocities and pressures at different radial locations are shown in Fig. 17 and 18, respectively. It can be observed that the results for $r = 5015$ in. are not given beyond the time of approximately 18 msec; this means that the cavity has been expanded to a radius larger than 5015 inches. The space profiles for velocity and pressure at different times are given in Figs. 19 and 20, respectively. It should be observed in Fig. 19 that at later time instants, the peak particle velocities do not occur at the jump but occur instead at the cavity. The peak pressures, however, remain invariably at the jump as shown in Fig. 20.
The history of the expansion of the cavity is plotted in Fig. 21. The results shown in Figs. 17 through 20 were obtained using the viscosity terms in loading only, namely

\[ q(I,t) \neq 0 \quad \text{when: } v(I-1,t) - v(I+1,t) > 0 \]
\[ q(I,t) = 0 \quad \text{otherwise.} \]

No noticeable differences were detected by using the viscosity terms also in unloading.

The problems described above were solved also using a stress-strain equation that includes the deviatoric effects following the Grigorian theory. Based on in-situ seismic measurements, a value of 20 ksi was selected for the shear modulus \( G \). The playa-silt was regarded as a cohesionless soil and a value \( \beta = 3 \) corresponding to an angle of friction \( \varphi = 30^\circ \) was chosen. This leads to a value of \( \alpha = 0.695 \).

These calculations indicate that the differences in the results obtained with and without the deviatoric effects are rather small. Yielding apparently occurs at extremely low stress levels (about 100 psi) and the deviator remains plastic up to the point where the pressure-density curve starts turning upward very rapidly. At this point a reverse transition from plastic to elastic flow occurs, as predicted by Grigorian's equations, and from then on the deviator remains practically constant whereas the bulk of the stress under increasing load is carried by the hydrostatic pressure. Changing \( G \) to 40 ksi and \( \beta \) to 4 did not introduce appreciable variations in the above pattern.

Typical space profiles of the radial stress corresponding to the two stress descriptions are compared in Fig. 22. As would be
expected, the curves show noticeable difference during the initial stage of loading; the presence of the deviator appears to delay the formation of the shock wave. It can also be observed that, in accordance with theory, the precursor wave travels slower under hydrodynamic assumption. This is clearly shown by the initial stress-histories plotted in Fig. 23. The transition from plastic to elastic flow occurs when \( \sigma_r \approx 6 \text{ ksi} \) as shown in Fig. 22. On the basis of these results and in the framework of Grigorian's theory it can be concluded that a stress description using only the hydrostatic component is sufficiently accurate for the prediction of ground shock motions in granular media whose behavioral characteristics are similar to playa silt. This conclusion also follows from the fact that in Grigorian's equations the shear modulus determines the magnitude of the elastic deviator and, therefore, also the level of yielding. Since for granular cohesionless materials the shear rigidity is very small, the overall deviatoric effects can be expected to be negligible.

5.4. Effect of Sound Speed Discontinuities of Unloading Waves on Stability

As mentioned in Sect. 3.2, not only loading and compaction but also the unloading characteristics of playa silt are strongly influenced by the initial moisture content. For the pressure levels under consideration, which do not exceed 20 - 30 ksi, the experiments by Hendron and Davisson show that for low moisture contents the material unloads with a steeper slope than that of the corresponding loading path. This means that the assumption of reversibility of the \( p\sigma \) curve in the higher pressure ranges, which is made in the Grigorian theory, may not be
appropriate in the case of relatively dry granular media. In order to
determine the effects of a steeper unloading path, certain calculations
were performed using the volumetric curve of the dry medium indicated
in Fig. 8. Unloading from the peak pressure was assumed to follow a
straight line having a slope twice as steep as the slope of the loading
path. This produces an abrupt jump in the sound speed when the unloading
process starts. The following parameters were used:

(a) Initial cavity radius $r_0 = 1000$ in.
(b) Pressure input at the cavity is given by Eq. (111)
    
    with $t_r = 8$ msec and $p_{max} = 12$ ksi.
(c) Linear artificial viscosity with $B = 0.5$ both in loading
    and unloading
(d) $\Delta r = 10$ inches, and $(c + v) \Delta t/\Delta r = 0.2$ was required.

A number of pressure-distance profiles obtained from these calculations
are shown in Fig. 24. The abrupt change in $c$ clearly produces undesir-
able oscillations in the rarefaction portion of these profiles that grow
slowly with time. The regular pattern of the oscillations plus the fact
that at any given time these oscillations start exactly at the space
location where the discontinuity in $c$ occurs, indicate that some
numerical instability is present. Elimination of the viscosity terms
during unloading did not improve the situation much. Also, imposing a
more severe stability condition in terms of a smaller time step $\Delta t$
(up to $(c + v) \Delta t/\Delta r = 0.1$) did not yield much improvement. Other
authors seem to have observed the same difficulty arising from unloading
waves that travel with a sudden increase in the sound speed from that
of the corresponding loading wave. However, no analysis of such instability is available.
VI. SUMMARY AND CONCLUSIONS

In the present study, a mathematical model is presented for the numerical solution of a broad class of spherically symmetric wave propagation problems in continuous media. The model is essentially a discrete cell analogue through which a discrete Eulerian formulation of the conservation equations can be derived on clear physical grounds. The similarity between the integral form of the conservation principles and the "conservative" form of the difference equations derived through the model is preserved and emphasized. The elementary statements of conservation on which the model is based also allow for a consistent treatment of the difficult problem of a moving boundary, such as that represented by a spherical cavity expanding in an infinite medium.

Certain physical and mathematical aspects of the theory of discontinuous flow are analyzed and some attention is devoted to the concept of a weak solution for a system of hyperbolic equations. This examination shows the plausibility of the numerical treatment of shock waves by means of the well known artificial viscosity methods. A linear stability analysis for the general interior difference equations indicated the positive effect of the artificial viscosity terms on the stability of the approximate numerical solution.

The solutions obtained for a number of problems, in which the material behavior is described by the equations of Grigorian, show a good agreement with existing experimental data. Calculations involving situations where shocks can develop in a granular medium were also
performed; these indicated that shocks are well defined and no significant oscillations occur if the unloading follows a reversible path.

A purely hydrodynamic formulation for non-isentropic flow problems can also be obtained with the model, provided the numerical technique described herein for the treatment of the boundary can be suitably modified and an appropriate equation of state is available.
REFERENCES


FIG. 1 SPHERICALLY SYMMETRIC EULERIAN CELL
FIG. 2 CROSS SECTION OF SPHERICAL MODEL
FIG. 3 BOUNDARY CELL

The Shaded Area Represents the Region Filled With Material.
FIG. 4  CELL WITH DISCONTINUITY SURFACE
\[ \sigma(\tau, t) = \text{constant} \]
FIG. 5 PRESSURE-DENSITY RELATIONSHIP FOR SOILS WITH INITIAL BONDS
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FIG. 23 DEVIATORIC EFFECTS ON RADIAL STRESS HISTORY AT \( r = 5055 \) in.
FIG. 24 EFFECT OF SOUNDSPEED DISCONTINUITY ON UNLOADING WAVES
A Discrete-Eulerian Model for Spherical Wave Propagation in Compressible Media

An Eulerian model based on cell-analogues of the integral conservation laws of continuous media is proposed, which leads to finite-difference equations explicitly conserving mass, momentum and energy. The model is primarily intended for describing ground motions in the relatively low pressure regions of the dynamic environment of underground nuclear bursts. A class of boundary value problems that can be handled with the model is represented by the expansion of a spherical cavity under a prescribed pressure pulse simulating the effects of the close-in region. Illustrative problems include the elastic-plastic wave propagation in granite, and shock formation and attenuation in playa silt.
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