INTERPLAY BETWEEN BACKGROUND TURBULENCE AND DARRIEUS-LANDAU INSTABILITY IN PREMIXED FLAMES VIA A MODEL EQUATION

BY

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THESIS

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Abstract

The effect of the Darrieus Landau instability on a premixed flame in a turbulent environment is investigated using a model equation known as the Michelson Sivashinsky equation, in both one and two dimensions. The equation is externally forced with noise representative of weak turbulence in the flow and the response of the flame to both unstructured and structured noise is examined. It is found that for a given noise intensity there exists a threshold domain size, beyond which the instability amplifies the perturbations due to noise and causes the formation of wrinkles on the flame surface, leading to an increase in the propagation velocity. Scaling laws are proposed for the increase in propagation velocity with variations in noise intensity and a bifurcation parameter, which includes the effects of parameters such as the domain size, equivalence ratio and reaction rates. Self fractalization of the flame is observed at high noise intensities. Effects of the scale of noise, analogous to the scale of turbulence, are also examined and a resonant behavior is found to exist at certain scales. In the two dimensional case, cellular structures, similar to the ones observed experimentally on expanding spherical flames, are observed on the flame surface. Scaling laws similar to the one dimensional case are proposed.
To my family...
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Chapter 1

Introduction

Premixed flames in most combustion devices such as IC engines, gas turbines etc. occur in a turbulent environment, which makes the study of interaction between flames and turbulence of practical importance. One of the seminal works in this field was done by Damköhler [1] who proposed two distinct limiting regimes: (a) large scale turbulence regime with purely kinematic interaction between turbulence and the flame, and (b) small scale regime where the smallest turbulent eddies penetrate the flame thickness and affect the transport mechanisms within the flame. Since then a lot of work has been done in developing phase diagrams identifying the different mechanisms of turbulent combustion [2, 3, 4]. The phase diagram developed by Peters [3] is shown in Fig. 1.1. The large scale turbulence regime of Damköhler corresponds to the wrinkled flamelet regime in Fig. 1.1. In this regime turbulence intensity ($v'$) is lesser than the laminar flame speed ($S_L$) and it distorts the flame at scales much larger compared to the flame thickness. In the corrugated flamelets regime, turbulence intensity is higher than the laminar flame speed, but the interaction of the flame with turbulence is still kinematic as the flame thickness ($l_f$) is lesser compared to the smallest turbulent scales, i.e. the Kolmogorov scales ($l_K$). The thin reaction zone is characterized by Kolmogorov eddies of size lesser than the flame thickness ($l_K < l_f$) which penetrate into the preheat zone\(^1\) of the flame and increase scalar mixing. These eddies are still larger compared to the reaction zone, an inner layer of the flame, which is of thickness $l_\delta$, and therefore cannot affect the chemical reactions. In the broken reaction zones regime Kolmogorov eddies penetrate into the reaction zone of the flame ($l_K < l_\delta$) causing the chemistry to break down locally, followed by a decrease in temperature, which ultimately causes the flame to extinguish. This zone corresponds to the small scale regime of Damköhler.

In addition to the effect of flow interactions, premixed flames are also subject to various instabilities, which can be thermal diffusive or hydrodynamic in nature. The thermal diffusive instability [5] is caused due to non-unity Lewis number, implying nonequidiffusion and leads to the formation of cellular structures on the flame surface and/or oscillations of the flame surface. The hydrodynamic instability, also known as

\(^1\)The structure of a premixed flame can be divided into the preheat zone where appreciable temperature rise occurs, and an inner layer known as the reaction zone, where the chemical reactions take place. The inner layer is responsible for keeping the reaction processes alive. Further details can be had from [4].
the Darrieus-Landau instability [6, 7], is caused by the gas expansion produced by heat release in a flame, causing the flow lines across the front to deviate towards the normal to the flame which leads to wrinkling of the flame. Large scale flames, often found in practical applications, are susceptible to this instability leading to the formation of sharp crests on the flame pointing towards the burned side [8] or the roughened surface observed on expanding flames in turbulent flow fields [9]. This instability, acting together with turbulence in the flow, is known to play an important role in turbulent flame propagation. The aim of the current work is to systematically examine and characterize the synergistic role played by the Darrieus-Landau instability and turbulence in the wrinkled flamelet regime of Fig. 1.1.

The multitude of processes that occur in combustion systems viz. multi-component flows, heat and mass transfer and chemical reactions, occur at a wide range of temporal and spatial scales. The flame is very thin, with a thickness of the order of a millimeter representing the diffusion length scale, compared to the flow scales which are of the order of the vessel dimension. The chemical reactions of the combustible mixtures are generally confined to a thin reaction layer inside the flame. The time scales over which the chemical reactions evolve are relatively short compared to the flow or diffusion time scales. Resolving such a multi-scale problem and faithfully representing all the physico-chemical phenomena on all scales demands huge amounts of computing resources, making direct numerical simulations of such reactive systems very difficult. On the other hand the multiscale nature of a premixed flame makes it an ideally suited problem for perturbation or asymptotic analysis. Such methods take advantage of the disparate scales in a problem and...
lead to the formulation of simplified models which often help in understanding the key physics underlying the problem. The asymptotic treatment of premixed flames reduces the flame to a thin sheet separating the burned gases from the fresh unburned gases. Resolving the flame structure on the diffusion scale provides, by asymptotic matching, expressions that relate pressure and velocities across the entire flame and an expression for flame speed. One of the very first applications of this theory, in a primitive form, was by Darrieus [6] and Landau [7] in analyzing the stability properties of planar flames under the assumption of a constant flame speed. They concluded that planar flames were unconditionally unstable and thus the name Darrieus-Landau instability. Their prediction was not totally satisfactory since stable planar flames had been observed experimentally. An improvement was made to the model by Markstein [10] by the inclusion of flame curvature in the flame speed formula through a phenomenological constant now known as the Markstein length. This model predicted that curvature may provide stabilizing effects depending on the sign of the Markstein length, but its relation to the physical parameters was still unknown. Matalon & Matkowsky [11] through a rigorous asymptotic analysis developed a fully non linear generalized model for flame propagation of arbitrary flame shapes in general fluid flows and obtained an expression for flame speed which depended on the local curvature of the flame sheet and the hydrodynamic strain experienced by the flame. They also systematically re-examined the stability problem of a planar flame front and obtained corrections to the Darrieus-Landau formula providing more comprehensive results, incorporating earlier ideas and replacing phenomenology with explicit dependence on physico-chemical parameters (see also [12]). Similar asymptotic analysis were performed by Pelce & Clavin [13] and Frankel & Sivashinsky [14] yielding identical stability results. But these studies were performed under the assumption that the perturbations on the flame surface and in the velocity field were of the order of flame thickness, making them the linearizations of the work in [11], but appropriate for linear stability consideration.

Interaction of premixed flames with turbulent flows can also be analyzed based on the perturbation theory described in the earlier paragraph. The advantage of using this technique is that the analysis is free of closure/modeling assumptions which are generally used to make calculations on turbulent flames tractable. The seminal works of Clavin & Williams [15], Searby & Clavin [16] and Aldredge & Williams [17] examined the effects of low intensity, large scale turbulence on a premixed flame through perturbation analysis. In all these studies the focus was on the interaction of a premixed flame with fluctuations in the upstream velocity and hence the effects of intrinsic flame instabilities were suppressed either by assuming unity Lewis number and neglecting density changes due to combustion [15] or by the stabilizing effects of diffusion and gravity/buoyancy [16, 17]. Clavin & Williams [15] showed that the increase in turbulent flame speed was proportional to the square of the intensity of turbulence. Searby & Clavin [16] used the
same formulation as in Pelce & Clavin [13] and allowed for density changes due to gas expansion. It was shown that apart from gas expansion, the response of a flame to an incoming flow is governed by the flame velocity, the local behavior of the flame front and the characteristics of the incoming flow. Aldredge & Williams [17] adopted the formulations by Clavin & Williams [15] and Searby & Clavin [16], but with the relative turbulence intensity, ratio of a root-mean-square velocity fluctuation to the laminar flame speed, included as an additional independent parameter. They analyzed the modifications introduced by a flame in the turbulence properties of the flow and one of their key findings was that the flame induced anisotropy in an initially isotropic turbulent flow field.

As seen from the earlier paragraph, earlier studies focused on the influence of turbulence on hydrodynamically stable flames. The reason for doing so was that due to the intrinsic instability of the laminar flame front, the turbulent flow field would cause unlimited amplification of long wave disturbances of the front prohibiting the development of a general theory of flame propagation by asymptotic expansion in powers of the turbulence intensity. The only way to inhibit such an unlimited amplification would be to consider the nonlinearity in evolution of a hydrodynamically unstable flame, which is beyond the realm of linear theory. Theoretical progress on the nonlinear development of the hydrodynamic instability has primarily relied on simplified models. One such simplified model describing the evolution of a weakly perturbed flame front is the Michelson-Sivashinsky (MS) equation [18, 19], named after the authors who systematically derived it. It is a nonlinear integrodifferential equation obtained via asymptotic analysis and is valid in the limit of weak thermal expansion or small density changes across the flame. The equation provides valuable physical insight in the development of the hydrodynamical instability by adequately describing the corrugated shapes that develop beyond the onset of instability and the increase in propagation velocity of the flame, though its application is restricted by the fact that density changes in combustion are rarely small. The fully nonlinear evolution of a hydrodynamically unstable flame for arbitrary density changes has been examined numerically within the context of the hydrodynamic theory [11, 12] in the recent works of Rastigejev & Matalon [20, 21]. The current work will utilize the Michelson-Sivashinsky equation to study the evolution of the instability in the weakly nonlinear limit.

Characterization of the interplay between instability and turbulence has been done both via numerical simulations and experiments. One of the approaches in numerical simulations of turbulent flames is based on the so called $G$ equation [22, 23] and the level set approach advocated by Peters [3]. Numerical simulations of premixed turbulent flames in the corrugated flamelet regime were carried out using this approach in [24] and the same methodology was extended in [25] to include the effects of gas-expansion responsible for the development of the hydrodynamic instability. It was shown that the influence of the instability on the
turbulent burning velocity was important only at low turbulence intensities and its influence decreased as the turbulence intensity increased. The work by Treurniet et al. [26] utilized a methodology similar to the one in [25] and illustrated the influence of the instability on the turbulent flame speed. A numerical study carried out by Law et. al [27] also showed dominating influence of the hydrodynamic instability at low turbulent intensities. Numerous experimental configurations such as rod-stabilized v-flames, tube-stabilized conical flames, stagnation-flow stabilized flames [28], weak-swirl stabilized flames [29] and fan-stirred chambers [30] have been employed over the years to study premixed flames in a turbulent environment. The major handicap of these methods is that the turbulence produced is either inhomogeneous or unsteady with an appreciable mean velocity and the flame is subject to straining effects due to the incoming flow. Most of the experimental studies available in the literature fall under the corrugated flamelets and the thin reaction zones regime with very few results in the wrinkled flamelets regime. Kobayashi et al. [31] conducted experiments for very low intensities of turbulence and suggested that the Darrieus Landau hydrodynamic instability played an important role in increasing flame speed at low turbulence intensities. Aldredge et al. [32] provided data in the low intensity turbulence regime employing turbulence Taylor-Couette flow and measuring turbulent flame speeds in a methane-air mixture. They were able to overcome the drawbacks of earlier setups by achieving zero mean velocities and strain rates and suggested similar observations about the role of the hydrodynamic instability in the enhancement of turbulent flame speed.

One of the biggest handicaps of large scale numerical simulations and experiments is that they cannot be used to perform exhaustive parametric studies based on the different parameters in the problem. Such is not the case with a simplified model like the MS equation, which will be used in the current work to study the evolution of the hydrodynamic instability in the weakly nonlinear limit. It can be shown that forcing the MS equation with external noise effectively mimics flame propagation in a weakly turbulent flow. This has resulted in a simplified yet powerful model which can be used to examine the effects of the Darrieus Landau instability on a moderately stretched premixed flame in a weakly turbulent flow. Cambray and Joulin [33] performed numerical integrations of the forced Michelson Sivashinsky equation in order to understand the reaction of premixed flames excited by incoming velocity fluctuations. It was concluded that when the forcing intensity is of the order of the laminar flame speed, omitting the Darrieus Landau instability can lead to under-estimation of the mean propagation velocity of the flame. Procaccia et. al [34] examined the role of random fluctuations on the dynamics of a flame via the forced Michelson Sivashinsky equation. They emphasized on the importance of the interaction between random noise and the hydrodynamic instability and came up with scaling laws for the propagation velocity based the amplitude of noise and the domain size. Rastigejev and Matalon [21] investigated the dynamics of a premixed flame within the context of the
Michelson Sivashinsky equation and within the context of a fully nonlinear hydrodynamic model. They showed that when the domain size is large the flame dynamics are highly sensitive to external noise. Small disturbances get rapidly amplified by the hydrodynamic instability and lead to the formation of small scale wrinkles on the flame profile which causes a significant increase in the flame speed. It was suggested that the formation of wrinkles could be the response of the flame to external noise present in the flow in the form of turbulence. The current work is based on the suggestions put forth in [21] and will carry out a systematic study of the response of the MS equation to external forcing, which will be representative of the effect of Darrieus Landau instability on a turbulent premixed flame. The analysis will be restricted to the wrinkled flamelet regime which is characterized by moderately stretched, unfolded, simply connected flames and can therefore be investigated within the realm of the MS equation.
Chapter 2

Michelson Sivashinsky Equation

2.1 Derivation of the Equation

In the hydrodynamic theory of flame propagation, the diffusion length which characterizes the flame thickness $l_f$ is considered very small when compared to the characteristic size of the domain $L$, due to which a flame can be approximated to a sheet of infinitesimal thickness separating the fresh unburnt mixture from the burnt combustion products. Flow on either side of the flame is assumed to be inviscid and incompressible, a complete description of which can be provided by the Euler equations

\begin{align*}
\nabla \cdot \mathbf{v} &= 0 \\
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} &= -\nabla p, \quad (2.1)
\end{align*}

where $\mathbf{v}$ is the velocity vector, $p$ is the pressure and $\rho$ is the density which is piecewise constant, taking values $\rho_u$ for the unburnt gas and $\rho_b$ for the burnt gas. A flame sheet is mathematically represented by the function $\psi(x,t)$ given by

\begin{align*}
\psi(x,t) = y - f(x,t) = 0, \quad (2.2)
\end{align*}

where $\psi < 0$ refers to the fresh unburnt mixture and $\psi > 0$ is the burnt mixture (Fig. 2.1). The unit normal $\mathbf{n}$ to the flame sheet given by $\mathbf{n} = \nabla \psi / |\nabla \psi|$ is directed towards the burnt side. The propagation velocity of the flame sheet $V_f$, normal to itself, is given by $V_f = -\psi_t / |\nabla \psi|$. Using (2.2) the expressions for $\mathbf{n}$ and $V_f$ can be written as
The flow quantities suffer jump discontinuities across the flame sheet and are subject to the Rankine Hugoniot jump relations given as follows

\[
\begin{align*}
\left[ \rho \left( \mathbf{v} \cdot \mathbf{n} - V_f \right) \right] &= 0, \\
\left[ \mathbf{n} \times \left( \mathbf{v} \times \mathbf{n} \right) \right] &= 0, \\
\left[ p + \rho (\mathbf{v} \cdot \mathbf{n}) (\mathbf{v} \cdot \mathbf{n} - V_f) \right] &= 0,
\end{align*}
\]  

(2.5)

where \( \mathbf{v} = (u, v) \) and \([\ ]\) denotes the jump in the quantity across the flame sheet. The flame speed \( S_f \) is defined as the propagation speed of the sheet relative to the incoming unburnt gas, viz. \( S_f = \mathbf{v}^* \cdot \mathbf{n} - V_f \) where \( \mathbf{v}^* = \mathbf{v} \mid \psi = 0^- \). Using (2.2), the expression for flame speed \( S_f \) can be written as

\[
S_f = \mathbf{v}^* \cdot \mathbf{n} - V_f = \frac{-u^* f_x + v^* - f_i}{\sqrt{1 + f_x^2}}. 
\]  

(2.6)

The flame speed is given by the relation

\[
S_f = S_L (1 - \mathscr{L} \kappa), 
\]  

(2.7)

where \( S_L \) is the laminar flame speed, \( \mathscr{L} \) is the Markstein length and \( \kappa = -\nabla \cdot \mathbf{n} \) is the mean curvature. Laminar flame speed \( S_L \), the propagation speed of a premixed flame, is a unique property of a mixture, indicating its reactivity and exothermicity in a given diffusive medium. The Markstein length \( \mathscr{L} \) is a coefficient of the order of the flame thickness \( l_f \) and is dependent on physico-chemical parameters such as the thermal expansion coefficient, Lewis number, equivalence ratio and global activation energy of the chemical reaction. The dependence of \( \mathscr{L} \) on Lewis number is governed by the deficient reactant, i.e. it depends on Lewis number of the fuel in lean mixtures and Lewis number of the oxidant in rich mixtures.

Equations (2.1), (2.5) and (2.7) admit a simple solution in the form of a planar flame located at \( y = -S_L t \). The velocity and pressure across the flame front are piecewise constants given by
Figure 2.1: Schematic of a flame as an interface separating the burnt gases from the fresh unburnt gases.

\[ v = \begin{cases} 
0 & y < -S_L t \\
(\sigma - 1)S_L & y > -S_L t, 
\end{cases} \]  
(2.8)

\[ p = \begin{cases} 
0 & y < -S_L t \\
-(\sigma - 1)\rho_u S_L^2 & y > -S_L t, 
\end{cases} \]  
(2.9)

where \( \sigma = \rho_u/\rho_b > 1 \) is the thermal expansion coefficient. An asymptotic solution that describes a weakly corrugated flame, can be obtained as a perturbation of the planar flame in the limit of weak thermal expansion, i.e. \( (\sigma - 1) \ll 1 \). The perturbed flame front can be expressed in the form

\[ y = -S_L t + (\sigma - 1)\phi, \]  
(2.10)

where \( \phi = \phi(x, \tau) \) describes the flame front and \( \tau = (\sigma - 1)t \) is the scaled time. The velocity is scaled as \( v = \bar{v} + (\sigma - 1)^2 \tilde{v} \) and pressure as \( p = \bar{p} + (\sigma - 1)^2 \tilde{p} \) where \( \bar{v}, \bar{p} \) correspond to the basic state of the planar flame. The flame speed relations (2.7) and (2.6) give

\[ \frac{-(\sigma - 1)^3\phi_x \ddot{\phi} + (\sigma - 1)^2 \dot{\phi}^2 + S_L - (\sigma - 1)^2 \phi_t}{\sqrt{1 + (\sigma - 1)^2 \phi_x^2}} = S_L - \frac{S_L \mathcal{L}(\sigma - 1)\phi_{xx}}{\sqrt{1 + (\sigma - 1)^2 \phi_x^2}}, \]

where the curvature \( \nabla \cdot \mathbf{n} \approx -\phi_{xx}/\sqrt{1 + (\sigma - 1)^2 \phi_x^2} \). Using the approximation

where
\[\sqrt{1 + x^2} \simeq 1 + \frac{1}{2} x + \ldots\]
\[\sqrt{1 + (\sigma - 1)^2 \phi_x^2} \simeq 1 + \frac{1}{2} (\sigma - 1)^2 \phi_x^2 + \cdots ,\]

and keeping terms of \(O(\sigma - 1)^2\) we get

\[
S_L + (\sigma - 1)^2 \tilde{v}^* - (\sigma - 1)^2 \phi_x = S_L + \frac{1}{2} S_L (\sigma - 1)^2 \phi_x^2 - \mathcal{L} S_L (\sigma - 1) \phi_{xx},
\]
\[
\phi_x + \frac{1}{2} S_L \phi_x^2 - \frac{S_L \mathcal{L}}{(\sigma - 1)} \phi_{xx} - \tilde{v}^* = 0. \tag{2.11}
\]

To evaluate \(v^*\) we need to solve for the flow field. Before substituting the scaled expressions in the Euler equations, it is convenient to move to a coordinate system attached to the planar front. Let \(\hat{y} = y + S_L t\), then

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + S_L \frac{\partial}{\partial \hat{y}}, \quad \frac{\partial}{\partial \hat{y}} = \frac{\partial}{\partial y}.
\]

The Euler equations can be written as

\[
\begin{align*}
    u_x + v_y &= 0, \\
    \rho u_t + \rho uu_x + \rho (u + S_L) u_y &= -p_x, \\
    \rho v_t + \rho vv_x + \rho (v + S_L) u_y &= -p_y.
\end{align*}
\]

Substituting the scaled expressions for pressure and velocity in the Euler equations and retaining terms of \(O(\sigma - 1)^2\) we get

\[
\begin{align*}
    \tilde{u}_x + \tilde{v}_y &= 0, \\
    \rho_u S_L \tilde{u}_y &= -\tilde{p}_x, \\
    \rho_u S_L \tilde{v}_y &= -\tilde{p}_y.
\end{align*}
\]

These equations are valid for \(\hat{y} > 0\) and \(\hat{y} < 0\) since \(\rho_b = \rho_u/\sigma \approx \rho_u/1 + (\sigma - 1) \sim \rho_u[1 - (\sigma - 1)] \sim\)
\[ \rho_u + O(\sigma - 1). \]

The jump conditions across the perturbed flame sheet are given by (2.5). Since the perturbation of the flame sheet is small, the jump conditions are transferred across \( \dot{y} = 0 \) by doing a Taylor expansion about \( \dot{y} = 0 \). Retaining terms of \( O(\sigma - 1)^2 \) we get

\[
\begin{align*}
[\tilde{v}] &= 0, \\
[\tilde{u}] &= -\phi_x, \\
[\tilde{p}] &= 0.
\end{align*}
\] (2.12)

Non-dimensionalizing the system with the transverse domain of integration \( L \) as the unit for length, \( S_L \) as the unit for velocity, \( L/S_L \) as the unit for time and \( \rho_u S_L^2 \) as the unit for pressure we obtain

\[
\begin{align*}
\tilde{u}_x + \tilde{v}_y &= 0, \quad (2.13) \\
\tilde{u}_y &= -\tilde{p}_x, \quad (2.14) \\
\tilde{v}_y &= -\tilde{p}_y, \quad (2.15)
\end{align*}
\]

as the non-dimensionalized Euler equations. The non-dimensional evolution equation for the flame profile can be written as

\[ \phi_\tau + \frac{1}{2} \phi_x^2 - \alpha \phi_{xx} - \tilde{v}^* = 0. \] (2.16)

The parameter \( \alpha = \mathcal{L}/L(\sigma - 1) \) is the scaled Markstein number and as seen from the expression, is inversely proportional to the transverse domain of integration. Equations (2.14) and (2.15) together give the following

\[ \nabla^2 \tilde{p} = 0. \] (2.17)

The Fourier transform of a function, say \( h(x, y, \tau) \), is given by

\[
\mathcal{F}(h(x, y, \tau)) \equiv h_k(y, \tau) = \int_{-\infty}^{\infty} e^{-ikx} h(x, y, \tau) dx,
\] (2.18)
and its inverse is given by
\[ h(x, y, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} h_k(y, \tau) dk. \] (2.19)

Taking Fourier transform of (2.17) we obtain
\[ \tilde{p}_{k_y} - k^2 \tilde{p}_k = 0, \] (2.20)

which on solving on either of the sheet results in
\[ \tilde{p}_k = \begin{cases} 
C_1 e^{k|y|} & y < 0 \\
C_2 e^{-k|y|} & y > 0.
\end{cases} \] (2.21)

From (2.15),
\[ \tilde{v}_{k_y} = \begin{cases} 
-k|C_1 e^{k|y|} & y < 0 \\
|k|C_2 e^{-k|y|} & y > 0.
\end{cases} \] (2.22)

Integrating with respect to y,
\[ \tilde{v}_k = \begin{cases} 
-C_1 e^{k|y|} + C_3 & y < 0 \\
-C_2 e^{-k|y|} + C_4 & y > 0.
\end{cases} \] (2.23)

Since the flame is propagating in a quiescent flow, the velocity field in the unburnt gases far away from
the flame front tends to 0, i.e. \( \lim_{y \to -\infty} \tilde{v} = 0 \) which sets \( C_3 = 0 \), resulting in
\[ \tilde{v}_k = \begin{cases} 
-C_1 e^{k|y|} & y < 0 \\
-C_2 e^{-k|y|} + C_4 & y > 0.
\end{cases} \] (2.24)
From (2.13),

\[
\tilde{u}_x = -\tilde{v}_y,
\]
\[
\tilde{u}_k = \frac{i}{k} \tilde{v}_y.
\]

From (2.22),

\[
\tilde{u}_k = \begin{cases} 
-\frac{|k|}{k} i C_1 e^{|k|y} & y < 0 \\
\frac{|k|}{k} C_2 e^{-|k|y} & y > 0.
\end{cases}
\] (2.25)

Applying the jump conditions given by (2.12) to (2.21), (2.24) and (2.25) we get

\[
C_1 = C_2 = -\frac{|k|}{2} \phi_k,
\]
\[
C_4 = 0.
\]

Transforming into physical space via inverse Fourier transform, as shown in (2.19) we get

\[
\tilde{v}^* = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k| e^{ik(x-\xi)} \phi(\xi, \tau) dk d\xi \equiv \frac{1}{2} I\{\phi\}.
\] (2.26)

The operator \(I\{\phi\}\) is a linear operator which in Fourier constitutes a multiplication by \(|k|\) i.e. \(I\{\cos(kx)\} = |k| \cos(kx)\). Equation (2.16) can now be written as

\[
\phi_{\tau} + \frac{1}{2} \phi_x^2 - \alpha \phi_{xx} - \frac{1}{2} I\{\phi\} = 0.
\] (2.27)

This equation is known as the Michelson Sivashinsky (MS) equation. Its dimensional form is the following

\[
\phi_{\tau} + \frac{1}{2} S_L \phi_x^2 - \frac{S_L \mathcal{Z}}{(\sigma - 1)} \phi_{xx} - \frac{1}{2} I\{\phi\} = 0.
\] (2.28)
2.2 Analytical solution

The Michelson Sivashinsky equation (2.27) admits exact solutions of the form, \( \phi = -U \tau + \Phi(x) \) on a finite domain \( 0 \leq x \leq 1 \) with periodic boundary conditions. The solution corresponds to steadily propagating patterns in the \( y < 0 \) direction at a constant speed \( U \) without change in shape. \( U \) represents the increment in the flame speed \( S_L \) due to the formation of a corrugated structure and shall hence forth be referred to as the incremental propagation velocity. It can be obtained by substituting \( \phi = -U \tau + \Phi(x) \) into (2.27) and integrating throughout the domain. One finds that,

\[
U = \frac{1}{2} \langle \Phi_x^2 \rangle \equiv \frac{1}{2} \int_0^1 \Phi_x^2 \, dx. \tag{2.29}
\]

Solutions of the form mentioned above were derived in [35] by a pole decomposition technique (see also [36, 37]). The poles, denoted by \( z_n \), come in conjugate pairs in the complex plane and can be represented as \( z_n = x_n + iy_n \) and its conjugate as \( z^*_n = x_n - iy_n \). The solution of (2.27) is given by a finite sum of functions of these poles. The motion of the poles in the complex plane is governed by the laws of pole dynamics derived from the MS equation. It was noticed by Thual, Frisch and Hénon [35] that the poles tend to attract each other horizontally (along the real axis), repel each other vertically (along the imaginary axis) and are also subjected to a drift towards the real axis. This results in the vertical alignment of the poles and they ultimately coalesce into a single vertical line with a common real part. The corresponding solutions are referred to as the coalescent pole solutions and are characterized by the number of pairs of complex conjugate poles \( N \) that contribute to the solution. Of particular interest are the coalescent solutions for which all the poles are time independent, referred to as the steady coalescent pole solutions. A \( N \) pole solution of this type takes the form

\[
\Phi_N(x) = -2\alpha \sum_{a=1}^{N} \ln \frac{1}{2} \left[ \cosh(y_n) - \cos(2\pi x - x_c) \right], \tag{2.30}
\]

where \( x_c \) is the common real part and \( y_n \) is the imaginary part of the poles. The imaginary parts of the \( N \) poles are evaluated by solving \( N \) nonlinear equations, which result from the equation governing the pole dynamics, given by

\[
\sum_{l=1, l \neq n}^{N} \left[ \coth \left( \frac{y_n - y_l}{2} \right) + \coth \left( \frac{y_n + y_l}{2} \right) \right] + \coth(y_n) - \frac{1}{4\pi\alpha} = 0. \tag{2.31}
\]

In the physical plane, these pole solutions correspond to cusp like patterns with the real part corresponding to the location of the cusp and the imaginary part to its height. The analytical expression for the
incremental propagation velocity of a $N$ pole solution was obtained as

$$U_P = 2\pi \alpha N (1 - 4\pi N\alpha).$$

At this point it is appropriate to represent the solution in terms of the parameter $\gamma = 1/\alpha$, directly proportional to the transverse size of the domain of integration, to discuss the properties of these pole solutions. The equation for $U_P$ in terms of $\gamma$ is given by

$$U_P = 2\pi N (1 - \frac{4\pi N}{\gamma} \frac{1}{\gamma}).$$

For every $\gamma$ there is a maximum limit $N_0(\gamma)$ on number of poles possessed by the coalescent pole solution and is given by the following relation,

$$N_0 = \begin{cases} \text{Int} \left[ \frac{\gamma}{8\pi} + \frac{1}{2} \right] & \text{if } \frac{\gamma}{8\pi} + \frac{1}{2} \text{ is not an integer} \\ \text{Int} \left[ \frac{\gamma}{8\pi} - \frac{1}{2} \right] & \text{otherwise.} \end{cases}$$

The 0 pole solution corresponds to a planar flame and exists for all $\gamma > 0$. At $\gamma = 4\pi$ a 1 pole solution bifurcates from the 0 pole solution. At $\gamma = 12\pi$ a 2 pole solution emerges from the 1 pole solution. In general a $N$ pole solution emerges from a $N - 1$ pole solution at $\gamma = 4\pi (2N - 1)$. Thus for a given value of $\gamma$ there exist multiple pole solutions. Vaynblat and Matalon [36, 37] showed that out of all the existing pole solutions, the one which possessed the maximum number of poles $N_0(\gamma)$ for a given value of $\gamma$ is the one and only asymptotically stable solution. Therefore the 0 pole solution is stable for $0 < \gamma < 4\pi$, the 1 pole solution is stable for $4\pi < \gamma < 12\pi$ and so on. As $\gamma$ increases, solutions with increasing number of poles emerge leading to the formation of sharper and sharper cusps. A few pole solutions with $N_0 = 1 - 4$ are shown in Fig. 2.2(a) and it can be seen that as the number of poles increases the profiles get sharper. The incremental propagation velocity $U_P$ increases with increasing $\gamma$ and asymptotes to a value of $U_\infty = 0.125$ when $\gamma \to \infty$ as shown in Fig. 2.2(b).
Figure 2.2: (a) Pole solutions for $N_0 = 1, 2, 3, 4$. (b) Incremental propagation velocity $U_P$ as a function of $\gamma$. The bifurcations of the 1 pole, 2 pole, 3 pole and 4 pole solutions are seen at $\gamma = 4\pi, 12\pi, 20\pi, 28\pi$ respectively.
Chapter 3

Michelson Sivashinsky equation with external noise

3.1 Motivation for the study

The MS equation can be integrated numerically at a chosen value of $\gamma$ starting from random initial conditions and the cusp-like profile corresponding to the $N$ pole solution (where $N = N_0(\gamma)$), propagating at $U$ given by (2.32) can be recovered. At very high values of $\gamma$, say $\gamma = 200$, the integration resulted in the appearance of sporadic wrinkles on the profile, superimposed on the pole solution, which traveled along the flame surface and merged at the tip. The incremental propagation velocity $U$ was consistently higher than the analytical value and had a sudden increase in its value every time a wrinkle was formed. This sudden increase can be explained due to the fact that $U$ is directly related to the surface area of the flame and the formation of wrinkles increases the surface area causing an increase in $U$. Formation of such wrinkles and consequent increase in the propagation velocity has been reported in a number of studies in literature [21, 38, 39, 40]. Based on the results of Vaynblat & Matalon [36, 37], it was evident that the analytical steady state solution did not admit the formation of such wrinkles. This led to the following questions,

- Is the formation of wrinkles and associated increase in speed an inherent response of the equation to numerical noise?

- If so, can it be controlled and characterized in some way?

In order to answer the questions posed above it was decided to study the effects of external noise on the Michelson Sivashinsky equation. This was based on the presumption that in experiments the external noise is a result of background turbulence in the upstream flow. To facilitate such a study, the derivation of the MS equation was modified to include the effect of a turbulent flow field, as opposed to a quiescent flow field in the original derivation. The modification is discussed in the next section.
3.2 Derivation of the Michelson Sivashinsky equation with external noise

If the flame is assumed to propagate in a turbulent flow field, characterized with zero mean velocity and turbulent fluctuations $\eta(x,\tau)$, the far field conditions will change to

$$\lim_{y \to -\infty} \tilde{v} = \eta(x,\tau). \quad (3.1)$$

The derivation procedure is exactly the same as described in the earlier section until (2.23). From (3.1) it can be obtained that

$$\tilde{v}_k = \begin{cases} 
-C_1 e^{|k|y} + \eta_k & y < 0 \\
-C_2 e^{-|k|y} + C_4 & y > 0.
\end{cases} \quad (3.2)$$

Following the remaining steps of the derivation as outlined earlier and using the jump conditions, it can be shown that

$$\tilde{v}_k = \frac{|k|}{2} \phi_k(\tau) + \eta_k(\tau), \quad (3.3)$$

$$\tilde{v}^* = \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{ikx} |k| \phi_k(\tau) dk + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \eta_k(\tau) dk. \quad (3.4)$$

Using (2.19) we get

$$\tilde{v}^* = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k| e^{ik(x-\zeta)} \phi(\zeta, \tau) d\zeta dk + \eta(x,\tau). \quad (3.5)$$

The Michelson Sivashinsky equation changes to

$$\phi_\tau = \alpha \phi_{xx} - \frac{1}{2} \phi_x^2 + \frac{1}{2} I\{\phi\} + \eta(x,\tau). \quad (3.6)$$
Thus it is shown that forcing the MS equation with a forcing term $\eta(x, \tau)$ mimics the propagation of a premixed flame in a flow field having turbulent fluctuations characterized by $\eta(x, \tau)$. Numerical simulations of (3.6) will be carried out in the forthcoming sections.

### 3.3 Generation of noise representative of turbulence

To perform a comprehensive study of the response of the MS equation to external forcing, it is necessary that we force it with a signal which can be completely characterized in space and time. A signal with a prescribed two point statistic, such as the autocorrelation function in space and/or time would be an ideal choice. The definition of spatial autocorrelation function for a signal $\eta(x, y, \tau)$ is given by

$$R_{\eta\eta}(r) = \frac{\eta(x, \tau)\eta(x + r, \tau)}{\eta(x, \tau)^2}, \quad (3.7)$$

where $r$ is the three dimensional separation vector. Batchelor [41] showed that in the final period of decay of homogeneous turbulence, the autocorrelation function at a time $t$ takes the form

$$R_{\eta\eta}(r) = \exp \left( -\frac{\pi r^2}{4\ell^2} \right), \quad (3.8)$$

where $\ell = \ell(t) = \sqrt{2\pi \nu (t - t_0)}$ ($\nu$ is the kinematic viscosity and $t_0$ is an initial instant in time) is the length scale of turbulence and can be prescribed explicitly in this case. A method for generating signals with the above mentioned autocorrelation function has been developed by Klein et. al [42] and its description for the one dimensional case is as follows. A set of uniformly distributed random numbers $\lambda_m$ with $\lambda_m \lambda_n = \delta_{mn}$, is first transformed into a Gaussian distributed set with the mean $\xi_m = 0$ and variance $\varsigma = 1.0$. This is done using a transformation function given by

$$\xi_m = Y(\lambda_m) = \text{erf}^{-1}(\lambda_m)\sqrt{2\varsigma} + \overline{\lambda_m}. \quad (3.9)$$

The uniform deviates $\lambda_m$, the function $Y$ and the Gaussian deviates $\xi_m$ are shown in Fig. 3.1 (a)-(c). The transformed series is then convoluted with a digital linear non recursive filter with filter coefficients $b_n$,

$$\eta_m = \sum_{n=-N}^{n=N} b_n \xi_{m+n}. \quad (3.10)$$

The filter coefficients $b_n$ are such that the convolution converts an uncorrelated random data series, $\xi_m$
Figure 3.1: (a) Distribution of uniform deviates $\lambda_m \in [0, 1]$ (b) Transformation Function $Y(\lambda_m)$ (c) Distribution of gaussian deviates $\xi_m$, with $\xi_m = 0.0$ and $\varsigma = 1.0$
into a correlated signal, \( \eta_m \). Using the statistical properties of \( \xi_m \), the autocorrelation function of \( \eta_m \) can be related to the filter coefficients as

\[
\frac{\eta_m \eta_{m+k}}{\eta_m \eta_m} = \frac{\sum_{j=-N+k}^k b_j b_{j-k}}{\sum_{j=-N}^N b_j^2},
\]

(3.11)

where the functional form of the autocorrelation is given by (3.8). Expressing \( \ell \) and \( r \) as multiples of the spatial grid spacing, \( n \Delta x \) and \( k \Delta x \) respectively, the values of the filter coefficients can be obtained by solving (3.11) via Newton Rhapsont method. A much simpler, closed form expression to obtain filter coefficients was suggested in [42] and is given as

\[
b_p \approx \frac{\tilde{b}_p}{\left( \sum_{j=-N}^N \tilde{b}_j^2 \right)^{1/2}},
\]

(3.12)

\[
\tilde{b}_p := \exp\left( -\frac{\pi p^2}{4n^2} \right).
\]

(3.13)

\( N \) is a quantity related to the extent of the filter. For the above equation to be considerably accurate, \( N \) should be large enough to capture at least twice the length scale, which implies that \( N \geq 2n \). Note that based on Taylor’s hypothesis one can interpret \( \ell \) as representative of a spatial as well as a temporal scale and in the latter case, \( R_{\eta \eta} \) will be a time correlation function. Extension of this method to higher dimensions (2D or 3D) is fairly simple. Digital filters for higher dimensions are generated by the convolution of multiple one dimensional filters. A snapshot of the 3D noise at a particular instant of time is shown in Fig 3.2(a) and the autocorrelations in space are shown in 3.2(b). In the most general case, this method will produce a noise signal \( \eta(x, \tau) \), representative of homogeneous turbulence, whose spatial and temporal autocorrelation functions \( R_{\eta \eta}(r) \) are known. Signals representative of homogeneous isotropic turbulence can be produced by setting the spatial scales to be equal in all spatial dimensions. The intensity of the noise is taken to be the r.m.s value of the signal denoted as \( \eta' \).
Figure 3.2: (a) Noise field $\eta(x, y)$ with spatial scale $\ell_x = \ell_y = 0.08$. Contours are equally spaced with $\eta \in [-1, 1]$ (dashed contours for negative values). (b) Autocorrelation functions $R_{\eta\eta}(r)$ for $\ell_x = \ell_y = 0.08$ (filled symbols) and $\ell_x = \ell_y = 0.2$ (empty symbols). Dashed lines: analytical autocorrelation functions.
3.4 Numerical integration of the forced Michelson Sivashinsky equation

The Michelson Sivshinsky is solved numerically using a pseudospectral method. The function $\phi(x, \tau)$ on $0 \leq x \leq 1$ is represented at uniformly distributed collocation points $x_n = n/N$, where $n = 1, \ldots, N$, by the truncated Fourier series given by

$$
\phi_n \equiv \phi(x_n, \tau) = \sum_{j=-N/2+1}^{N/2} \phi_j(\tau) e^{2\pi i j n/N},
$$

(3.14)

where $\phi_j$ are the discrete Fourier coefficients given by

$$
\phi_j(\tau) = \frac{1}{N} \sum_{j=-N/2+1}^{N/2} \phi_n e^{-2\pi i j n/N}.
$$

(3.15)

Substituting in (2.27) we get

$$
\phi_j(\tau) = \pi |j| \phi_j - 4\pi^2 j^2 \phi_j - P(j) + \eta_j,
$$

(3.16)

$$
P(j) = (2\pi i \frac{1}{N} \sum_{j=-N/2+1}^{N/2} \phi_j(\tau) e^{2\pi i j n/N})^2,
$$

(3.17)

where $\eta_j$ are the discrete Fourier coefficients of the external noise. The non-linear term $P(j)$ is evaluated by going back and forth from Fourier to physical space. Integration in time is performed by explicitly treating the nonlinear term and implicitly treating the curvature and non-local terms, with the noise being added at every time step. Such an implicit-explicit scheme relaxes the severe stability restriction on the time step, which would arise in the case of a completely explicit scheme. One of the popular choices for such a scheme is a combination of the Crank Nicholson and Adams Bashforth methods [43]. This combination would result in a second order accuracy in time in the absence of the noise term.
Chapter 4

Results and Discussion

The analytical behavior of the MS equation has completely been understood via the pole solutions and their stability analysis based on the eigen values and eigen functions of the PDE. On the other hand, the equation’s behavior in the presence of noise is not as clearly understood. The effects of external forcing on the MS equation will be discussed here and an attempt will be made to characterize the equation’s response to the forcing. The characterization will primarily be based on the variations of the incremental propagation velocity $U$ with the various system parameters ($\gamma, \eta', \ell$). The following section discusses the effects of forcing the equation with unstructured or white noise of varying intensities.

4.1 Effect of white noise

The influence of white noise on the MS equation will primarily be explained by examining the variation of incremental propagation velocity $U$ with $\gamma$ for different values of $\eta'$ (Fig. 4.1) and variation with $\eta'$ for different values of $\gamma$ (Fig. 4.2). Fig. 4.1 is divided into two regions viz., planar corresponding to planar flames for $\gamma < 4\pi$ and corrugated corresponding to corrugated “cusp-like” flames for $\gamma > 4\pi$. To facilitate a better understanding of the plot, the quantity plotted in Fig. 4.2 is $U/U_p$ instead of $U$, where $U_p$ is the velocity of the $N = N_0(\gamma)$ pole solution given by (2.32). For a given value of $\eta'$ (say 0.2) it can be seen in Fig. 4.1 that $U$ scales with $\gamma$. Similarly for a fixed value of $\gamma$ (say 170) Fig. 4.2 shows that $U$ scales with $\eta'$. It has been found that the variation of $U$ with $\gamma$ and noise intensity $\eta'$ can be summarized by the following scaling law,

$$U \sim \psi(\eta', \gamma) \times (\eta')^{\beta_1(\eta', \gamma)} \times (\gamma)^{\beta_2(\eta')},$$

(4.1)

where the scaling with $\gamma$ is a measure of the influence of hydrodynamic instability on the flame, the scaling with $\eta'$ is the influence of noise and $\psi$ is a generic function. For a planar flame subjected to large scale low intensity turbulence, it has been established by Clavin & Williams [15] that the leading order increase in the propagation velocity is due to an increase in the surface area of the flame caused by wrinkling and is
Figure 4.1: Incremental propagation velocity as a function of $\gamma$ for variable noise intensity $\eta'$. 

Figure 4.2: Ratio of incremental propagation velocity to pole velocity $U/U_P$ as a function of noise intensity $\eta'$ in the stable regime $\gamma < 4\pi$. Instability dominated regimes (2) – (5) are defined as $U \sim (\eta')^{0-2}$. 
Table 4.1: Values of the scaling exponents $\beta_1$ and $\beta_2$ for the regions (1) - (5) shown in Fig. 4.1 and 4.2

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>2</td>
<td>1.5-2</td>
<td>1-1.5</td>
<td>0.1-1</td>
<td>0-0.1</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>1.043</td>
<td>Refer Fig. 4.3 (b)</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

proportional to the square of the turbulence intensity. The effect of the flame hydrodynamic instability was neglected in their analysis. This result is recovered via the forced MS equation for $\gamma < 4\pi$ and can be seen as straight lines with slope 2 in region (1) of Fig. 4.2. But it turns out that the hydrodynamic instability has an effect as well. There is an additional scaling with the bifurcation parameter $\gamma$. This is observed in Fig. 4.2 in the form of a shift in the straight lines of region (1) as $\gamma$ is varied. The exponents $\beta_1$ and $\beta_2$ in (4.1) are constants with values 2 and 1.043 respectively. The function $\psi(\eta', \gamma)$ also takes a constant value of 0.0059. The scaling law can therefore be represented as

$$U = 5.9 \times 10^{-3} (\eta')^2 (\gamma)^{1.043}.$$  \tag{4.2}

The presence of noise in dynamical systems is known to disrupt sharp bifurcations and cause smooth transitions, such as the smooth transition from conduction to convection in the presence of thermal noise [44]. A similar phenomenon is observed here. The presence of noise causes a smooth transition from a planar to corrugated flame. This transition, which is characterized by an abrupt increase in $U$ in the absence of noise, occurs gradually in the presence of noise as seen in Fig. 4.1.

For $\gamma > 4\pi$, when the flame is corrugated or has a “cusp-like” configuration, the dependence of $U$ on $\eta'$ is no longer quadratic. The scaling law takes the form,

$$U \sim (\eta')^{\beta_1(\eta')} \times (\gamma)^{\beta_2(\eta')}.$$  \tag{4.3}

The scaling exponents $\beta_1$ and $\beta_2$ are functions of the noise intensity $\eta'$ as shown in Fig. 4.3(a) and (b). It is observed that for every value of noise intensity $\eta'$, there is a threshold value of $\gamma$ below which the noise does not influence the flame and this threshold value shifts towards higher values of $\gamma$ as the level of noise decreases. In Fig. 4.1 for every constant $\eta'$ curve, the value of $U$ corresponds to its analytical value ($U_P$) until this threshold value, beyond which $U$ starts following the scaling law given by equation (4.3). An equivalent representation of this is the line corresponding to $U/U_P = 1$ in Fig. 4.2. The curve marked as $\eta' = 0.0$ in Fig. 4.1 corresponds to the solution of MS equation under the influence of numerical noise only. It can be seen that even for a level of noise as low as numerical noise, there exists a similar threshold value of
Figure 4.3: Exponents $\beta_1(\eta', \gamma)$, $\beta_2(\eta')$, $\psi(\eta', \gamma)$.

$\gamma$ beyond which the scaling law is followed. Increasing the accuracy of numerical integration by either using a higher resolution or a different method, such as a spectral Galerkin method instead of a pseudospectral method, can only increase the threshold value of $\gamma$ but cannot get rid of it.

At high values of $\gamma$ (which corresponds to a large domain size $L$) the flame dynamics is highly sensitive to external noise and disturbances of very small magnitude get rapidly amplified by the instability causing the formation of wrinkles. Such is not the case at low values of $\gamma$ and therefore the magnitude of noise needed to cause the formation of wrinkles is pretty high. This phenomenon is clearly illustrated in Fig. 4.4.

Considering $\eta' = 0.001$ (Fig. 4.4(b)), the noise has no influence on the flame for $\gamma = 20$. At $\gamma = 50$ it causes rigid motion of the flame without altering its shape significantly. At $\gamma = 200$, the amplification of noise by the instability is enough to cause the formation of small scale wrinkles on the profile. These wrinkles are very similar to the ones obtained by Rastigejev & Matalon [21] while integrating the MS equation from arbitrary initial conditions. This confirms their conjecture that the multiscale nature of the flame is a result of a permanent source of external noise, viz. turbulent fluctuations in the flow. To cause similar wrinkling at $\gamma = 50$ the magnitude of noise is required is 2 orders higher in magnitude (Fig. 4.4(c)).

Based on the values of the scaling exponent $\beta_1$, the combined response of the flame to noise and instability can either be “turbulence dominated” or “instability dominated”. When the wrinkling of the flame and the increase in the propagation velocity is primarily due to noise (or turbulence) and the influence of the instability is very weak, the flame’s behavior is said to be “turbulence dominated”. This behavior is observed in the planar flame regime ($\gamma < 4\pi$) marked as (1) and the value of $\beta_1$ in this region is 2. When $\gamma > 4\pi$ and the flame is corrugated or has a “cusp-like” configuration, the value of $\beta_1$ is lesser than 2 (Fig. 4.2) implying the flame’s greater resilience to noise. The regions marked as (2) – (5) constitute the “instability dominated” regime where $\beta_1 < 2$. Range of values of $\beta_1$ in each region can be had from Table 4.1. Very
Figure 4.4: (a) Flame profiles corresponding to the pole solution at different values of $\gamma$. Effect of different magnitudes of noise intensity ($\eta'$) on the flame profiles (b) $\eta' = 0.001$ (c) $\eta' = 0.2$ (d) $\eta' = 0.8$

Figure 4.5: Probability density function of the flame height $\phi(x,t)$ for (a) $\gamma = 30$ (b) $\gamma = 40$ (c) $\gamma = 100$ (white noise intensity $\eta' = 0.1$).

High values of $\eta'$ would be required to overcome the effect of the instability and cause the flame’s behavior to be “turbulence dominated” again. Fig. 4.3 (a) shows that for $\gamma > 4\pi$, the variation of $U$ with $\eta'$ will tend towards a quadratic behavior only at very high values of $\eta'$, of the order $\eta' \sim 10$.

For values of $\eta' < 0.3$, the scaling law given by (4.3) is followed quite well. For larger values of $\eta'$, the scaling law breaks down as the function $\psi$ and the exponent $\beta_1$ start varying $\gamma$ along with the variation with $\eta'$ as seen in Fig. 4.3 (a) and (c). The scaling law now changes to the most general one as given by (4.1). At very large values of noise intensity, it is seen that the exponents $\beta_1$ and $\beta_2$ asymptote to the values 2 and 1.08 respectively. At such high noise intensities the hydrodynamic instability no longer has any influence on the flame, implying that the “cusp-like” structure no longer exists and the flame starts responding to noise like a planar flame.
The influence of noise on the flame profiles can be well understood from the probability density function of the flame height. Fig. 4.5 shows the probability density function (pdf) of the flame height $\phi(x, \tau)$ for different values of $\gamma$ at a constant noise intensity $\eta' = 0.1$. The dashed lines represent the pdf of the analytical pole solution which has a bimodal structure (a pdf with two distinct maxima). The bimodal structure is due to the “cusp-like” profile of the flame. The first peak corresponds to the smooth trough of the profile towards the unburnt side and the second peak is the sharper crest towards the burnt side. The shape of the pdf is reflective of the sensitivity of the flame profile to noise. As discussed earlier, at low values of $\gamma$ (Fig. 4.5(a)), the flame is least sensitive to noise and thus the pdf still maintains its bimodal structure. As $\gamma$ increases, the sensitivity of the flame to noise increases. The bimodal structure starts to disappear and the pdf becomes wider (Fig. 4.5(b)). At values of $\gamma$ beyond the threshold value, the structure of the flame profile is altered by the formation of small scale wrinkles. This decreases the probability of finding the flame at a particular location and widens the pdf. The bimodal structure completely disappears and it is skewed towards the unburnt side of the flame (Fig.4.5(c)). Pdfs similar to Fig.4.5(c) have been obtained experimentally by Plessing et. al [45] and Wirth et. al [46]. The skewed pdfs were attributed to the asymmetric influence of burning velocity on the flame profile. Based on the current results one may conjecture that the hydrodynamic instability, which causes the flame to shift from a planar shape to a “cusp-like” shape, is the underlying reason for a skewed pdf.

### 4.2 Effect of scale

Noise used in the previous section was unstructured with the spatial length scale $\ell$ in (3.8) set to 0. To study the effects of turbulence scale on the flame, a finite values of $\ell$ are used to generate noise which is structured. The autocorrelation function $R_{\eta\eta}(r)$ is related to $\ell$ by the expression $\ell = \int_0^\infty R_{\eta\eta}(r)dr$. Aldredge and Williams [17] studied the behavior of a planar flame perturbed with low intensity turbulence and showed the existence of a resonating length scale $\ell_{\text{res}}$ at which the flame propagation velocity is the maximum. Propagation velocities at all other scales greater and smaller than $\ell_{\text{res}}$ were found to be lesser. A constant noise intensity of $\eta' = 0.2$ is chosen and noise of varying spatial length scales is used to force the MS equation. Three values of the parameter $\gamma$ are chosen corresponding to a 0 pole solution ($\gamma = 10$), 1 pole solution ($\gamma = 20$) and 3 pole solution ($\gamma = 80$). Results similar to the ones obtained by Aldredge and Williams are obtained for the planar flame (Fig. 4.6(a)) showing the existence of a resonating length scale. Results similar to the planar flame are obtained at higher values of $\gamma$ for the “cusp-like” flame as well, but with the magnitude of $l_{\text{res}}$ decreasing with an increase in $\gamma$. 

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To explain the dependence of $\ell_{res}$ on $\gamma$, the eigen value problem of the MS equation, linearized about a N pole solution $\Phi_N(x)$ and subjected to a perturbation $\psi(x,t)$ is examined. Substituting $\phi(x,\tau) = \Phi_N(x) + \psi(x,\tau)$ in the MS equation results in the following evolution equation for the perturbation,

$$
\psi_\tau = \alpha \psi_{xx} + 1/2 \{ \psi \} - (\Phi_N)_x \psi_x.
$$

(4.4)

Representing $\psi(x,t) = \sum_{k=-\infty}^{+\infty} a_k(t)e^{2\pi ikx}$ and $\Phi_N(x) = \sum_{m=-\infty}^{+\infty} A_k e^{2\pi ikx}$, the above equation can be written in Fourier space as,

$$
\dot{a}_k = 1/2 \pi |k| - \alpha 4\pi^2 k^2 - 2\pi i \sum_{n=-\infty}^{+\infty} (k-n) A_{k-n} a_n.
$$

(4.5)

Vaynblat & Matalon [36][37] analyzed the above equation by assuming a solution of the form $e^{\omega t}$ and determined the family of eigen values which controlled the stability of the pole solutions. This family is given by

$$
\omega_k^N = \pi k - 4\pi^2 k^2 \alpha - 8\pi^2 k N \alpha.
$$

(4.6)

For every $\gamma = 1/\alpha$, there is a maximum number of steady coalescent poles possessed by the solution denoted by $N_0(\gamma)$ (given by (2.33)) and $N_0 + 1$ families of eigen values $\{\omega_0^N, \cdots, \omega_{N_0}^N\}$ each corresponding
to a pole solution. As mentioned earlier, the solution with the maximum number of poles \( N_0(\gamma) \) is the one and only stable solution, since it is the only solution for which \( \omega_k^N \) is negative at all integral values of \( k \). At any given point in time, the unstable eigen values \( \omega_k^N \) (where \( N < N_0 \)) continue to persist along with the stable eigen value \( \omega_k^{N_0} \). In the noiseless case, at a specified value of \( \gamma \), an initial perturbation evolves and ultimately results in the pole solution with \( N_0(\gamma) \) poles. In the presence of noise, it is the unstable eigen values that determine the mechanism of amplification of the constant perturbation viz. the noise. The scale of noise at which the amplification is a maximum is determined by the mode \( k \) possessing the highest growth rate \( \omega_k^N \).

In Fig. 4.7 the \( N_0 + 1 \) dispersion relations are plotted for \( \gamma/2\pi = 7 \) \((N_0 = 2)\) and \( \gamma/2\pi = 15 \) \((N_0 = 4)\). Out of all the unstable eigen values, the most critical set of eigen values are the ones corresponding to the 0 pole solution since at every \( k \) the eigen values of the 0 pole solution are the highest, implying maximum amplification of disturbances. The wavenumber at which the maximum growth rate \( (\omega_k^0) \) occurs is given by \( k_{res} = \gamma/8\pi \). This shows that the resonating length scale \( (\ell_{res}) \) which is inversely proportional to \( k_{res} \) is also inversely proportional to \( \gamma \) i.e. \( \ell_{res} \sim 1/\gamma \). To verify this dependence, the values of \( \ell_{res} \) which were obtained from the peaks of the least square fits of the data in Fig. 4.6(a), are plotted against \( \gamma \). A relation of the form \( \ell_{res} \sim 1/\gamma^{0.8} \) (Fig. 4.8) is obtained, which conforms to the prediction of \( \ell_{res} \) being inversely proportional to \( \gamma \).

The scaling laws in (4.2) and (4.3) are derived for the forcing of the MS equation with unstructured noise. The effect of scale is included in the form of the function \( \chi(\ell, \gamma) \). This function is representative of
the incremental propagation velocity $U$ at a particular value of $\ell$, $\eta'$ and $\gamma$ normalized by the value of $U$ at the same $\eta'$ and $\gamma$ when the equation is forced with unstructured noise. A plot of the function $\chi(\ell, \gamma)$ at $\eta' = 0.2$ and $\gamma = 10, 20, 80$ is shown in Fig. 4.6(b). The scaling law for $U$ now becomes

$$U \sim \chi(\ell, \gamma) \times (\eta')^{\beta_1(\eta')} \times (\gamma)^{\beta_2(\eta')}.$$ \hspace{1cm} (4.7)

The function $\chi(\ell, \gamma)$ can be interpreted as an amplification factor coming into play when the noise has a finite length scale. We argue that at negligibly small scales, there is no amplification caused due to resonating scales and $U$ has no functional dependence on $\ell$. As seen from Fig. 4.6(b), the magnitude of amplification goes to 1 as $\ell \to 0$, thus eliminating any effects due to the spatial length scale when the equation is forced with white noise and recovering the scaling law given in (4.3).

### 4.3 Fractalization of the flame

A fractal can be defined as a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole [47]. The fractal dimension $d_F$ is a quantity which gives an indication of how completely a fractal appears to fill space as one zooms down to finer and finer scales. The classical example of fractal dimension lies in the answer to the question - “How long is the coastline of Britain ?”. If we measure the length of the coastline using rulers of size $\epsilon$, the total length would turn out to be $N\epsilon$ where $N$ is the total number of rulers used. In the case of regular geometry, this quantity would be constant irrespective of the value of $\epsilon$ used. But in the case of a fractal geometry like the coast of Britain, it increases with decrease in the value of $\epsilon$, i.e. with the use of smaller rulers. The reason for this
behavior is that the coastline is rough for any value of $\epsilon$ considered and can never be measured accurately. If the quantity $1/\epsilon$ is plotted against $N$ in a log-log plot, it would result in a straight line, the slope of which is the fractal dimension, $d_F$ and every time $\epsilon$ is decreased, the quantity $N\epsilon$ increases by $\epsilon^{1-d_F}$. Fractals are observed in many important processes in engineering namely surface growth [48], soot agglomeration [49] and in phenomena such as turbulence [50]. Sreenivasan and Meneveau [50] measured the fractal dimension of different turbulent flows such as a boundary layer ($d_F = 2.4$), an axisymmetric jet ($d_F = 2.32$), a plane wake ($d_F = 2.37$) and a mixing layer ($d_F = 2.4$). The surface of a turbulent flame can also be considered fractal since it is wrinkled over a wide range of scales and the scales are self similar. Denet [51] used the Frankel equation, which is the generalization of the MS equation for a closed front, to describe turbulent flame subjected to a hydrodynamic instability. A significant excess fractal dimension is observed with increasing thermal expansion, equivalent to an increase in $\gamma$. Cambray and Joulin [52] proposed an asymptotic law for the ensemble averaged power density spectrum of wrinkling of a thin unstable premixed flame and an attempt was made to measure the fractal dimension of the flame. They concluded, based on Denet’s work, that in order for a significant excess fractal dimension to exist, the MS equation has to be subjected to external forcing.

In the present study, as the value of $\eta'$ is increased, the flame starts exhibiting a multiscale behavior with structures of three distinct scales present on the flame profile. There is the large scale cusp (for $\gamma > 4\pi$) characteristic of the hydrodynamic instability, smaller cusps or wrinkles formed as a result of the flame’s sensitivity to noise and then there are a number of small scale structures in addition to the wrinkles. The wrinkles along with the small scale structures on it are similar in appearance to the large scale cusp along with the wrinkles. Such a self similar structure is typical of a fractal curve. The fractal dimension of the flame profiles is calculated using an algorithm which measures the length of the flame profile with a ruler of a particular size. The measurements are then repeated by halving the ruler size at every iteration. The log-log plot of the measured length vs the size of the ruler used results in a straight line, the slope of which gives the fractal dimension $d_F$. The fractal dimension is found to be a function of both $\gamma$ and $\eta'$. Fig. 4.10 shows the excess fractal dimension, given by $d_F - 1$, as a function of $\gamma$, while Fig. 4.11 shows the excess fractal dimension as a function of $\eta'$. The dashed line in Fig. 4.10 is the excess fractal dimension of the analytical pole solution for different values of $\gamma$. This is the output of the algorithm when it is used on the pole solutions and can be considered as the reference level for all further calculations of $d_F$. The empty circles in Fig. 4.10 is the case where the MS equation is under the influence of numerical noise only. The increased sensitivity to noise at large values of $\gamma$ ($\gamma \geq 200$) causes the curve to depart from the analytical $d_F$ curve and start following a power law scaling with $\gamma$.  

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As $\eta'$ increases, the flame starts having an appreciable value of $d_F$ starting with the fractalization of the profiles at high values of $\gamma$, and at large $\eta'$ the profiles at lower values of $\gamma$ also fractalize. The fractal dimension $d_F$ follows the scaling law given by

$$d_F = 1 + E(\eta') \times \gamma^{\beta_F(\eta')}.$$  \hspace{1cm} (4.8)

The power law scaling with $\gamma$ can be seen as straight lines in a log-log plot of the excess fractal dimension $(d_F - 1)$ vs $\gamma$ (Fig. 4.10). The exponent $\beta_F$ is a function of $\eta'$ as shown in Fig. 4.9(a). As $\eta'$ increases, the value of $\beta_F$ decreases and it appears that $\beta_F \to 0$ as $\eta' \to \infty$ since the influence of instability is completely overcome at high values of $\eta'$ as discussed earlier. The dependence of $d_F$ on $\eta'$ cannot be expressed as power law but the function $E(\eta')$ (Fig. 4.9(b)), which is obtained as a least squares fit of the data, is the same for all values of $\gamma$. The increase in the fractal dimension from one value of $\eta'$ to a higher value starts decreasing at values of $\eta'$ larger than 0.2 and $d_F$ seems to approach an asymptotic value. Extrapolation of the least squares fit of the data in Fig. 4.11 gives $d_F = 1.23$ as $\eta \to \infty$, implying a 23% increase in dimension.

Fig. 4.12(a) and (b) are space-time plots of the height of the flame front. The bright continuous curve represents the tip of the flame profile. The smaller branch like curves represent the wrinkles which are formed on the profile and ultimately travel upwards to merge at the tip. At $\gamma = 200$ and $\eta' = 0.1$ (Fig. 4.12(a)), when the fractalization has not kicked in, we see the existence of the large cusp and the wrinkles as shown by the snapshot of the profile at $\tau = 0.5$. At $\gamma = 400$ and $\eta' = 1.0$ (Fig. 4.12(b)), the flame front is completely fractal and structures of different scales are visible. There is still the existence of two cusp like large scale structures along with intermediate scale wrinkles. But now we see the creation of a large number of small scale structures which merge into the wrinkles which ultimately merge at the tip of the large cusps. A snapshot of the flame profile at $\tau = 0.5$ shown in Fig. 4.12(b) clearly illustrates its fractal nature.

### 4.4 Two dimensional Michelson Sivashinsky equation

The two dimensional MS equation, governing the evolution of a weakly curved, hydrodynamically unstable flame surface represented as $z - f(x, y, t) = 0$, is given by,

$$\phi_{\tau} + \frac{1}{2} |\nabla \phi|^2 - \alpha \nabla^2 \phi - I\{\phi\} = 0.$$  \hspace{1cm} (4.9)

The operator $I\{\phi\}$ now corresponds to multiplication by $\sqrt{k_x^2 + k_y^2}$ in Fourier space where $k_x$ and $k_y$ are the $x$ and $y$ components respectively of the wave number vector $k$. Joulin [53] showed that in a square
Figure 4.9: (a) Scaling exponent $\beta_F$ (b) The function $E(\eta')$

Figure 4.10: Fractal dimension of flame profile as a function of $\gamma$ for variable white noise rms $\eta'$. Empty dots refer to the 'noisless' case ($\eta' = 0$) for which only numerical noise is present. Labels (a) and (b) are representative conditions depicted in Fig. 4.12(a)-(b). Considering a scaling $d_F - 1 \sim \gamma^\delta$ the exponent $\delta \in (2, 0.58)$ for $\eta' \in (0, 1)$
Figure 4.11: Fractal dimension of flame profile as a function of noise intensity $\eta'$ with $\gamma$ as a parameter. Extrapolation of the fitted $\gamma = 300, 400$ data yields $d_F \rightarrow 1.23$ for $\eta' \rightarrow \infty$.

Figure 4.12: Space-time plots of flame height for conditions (a) and (b) depicted in Fig.4.10. Also shown is the flame profile at $t = 0.5$. 
domain, if $\phi_1(x, \tau)$ and $\phi_2(y, \tau)$ are the solutions of the one dimensional MS equation in the $x$ and $y$ direction respectively, $\phi_1(x, \tau) + \phi_2(y, \tau)$ is the solution of (4.9) in two dimensions. Numerical integration of (4.9) with periodic boundary conditions [54] resulted in a stationary solution which was the sum of two one dimensional solutions.

Considering a general two dimensional rectangular domain with sides of lengths $L_1$ and $L_2$, the one dimensional MS equation for each dimension would result in two parameters $\alpha_1 = \mathcal{L}/(\sigma - 1)L_1$ and $\alpha_2 = \mathcal{L}/(\sigma - 1)L_2$. If $\lambda = L_1/L_2$ is the domain aspect ratio, then $\gamma_2 = \gamma_1/\lambda$. The analytical expression for the incremental propagation velocity in two dimensions turns out to be the sum of the incremental propagation velocity along each dimension given by (2.32) and can be written as

$$U_P(\gamma_1, \gamma_2) = U_P(\gamma_1) + U_P(\gamma_1/\lambda). \quad (4.10)$$

A plot of $U_P(\gamma_1, \gamma_2)$ for an aspect ratio of 4 is shown by the thick solid line in Fig. 4.14. The dashed lines in Fig. 4.14 represent the analytical values of $U_P(\gamma_1)$ and $U_P(\gamma_1/\lambda)$ in each of the two dimensions respectively. Numerical integration of (4.9) exhibits similar characteristics as the one dimensional equation. Wrinkles are formed on the flame surface at high values of $\gamma$, which travel along the surface and coalescence at the tip. Fig. 4.13(a) shows the two dimensional pole solution, which is the sum of two one dimensional solutions and Fig. 4.13(b) - (f) show the evolution of the wrinkles starting with their formation, travel along the flame surface and merging at the tip.

The forced two dimensional MS equation is given by

$$\phi_\tau = \alpha \nabla^2 \phi - \frac{1}{2} |\nabla \phi|^2 + I\{\phi\} + \eta(x,y,\tau) \quad (4.11)$$

In the analytical case, the solution of the two dimensional MS equation is decoupled in the $x$ and $y$ directions as mentioned earlier. Forcing the two dimensional equation with noise causes a coupling in the $x$ and $y$ direction, resulting in the formation of cellular structures on the flame surface as shown in Fig. 4.15(a) and (b). Such cellular structures have commonly been associated with the thermal diffusive instability in spherically expanding flames [55, 56]. Formation of such cellular structures which are solely due to the hydrodynamic instability have also been observed experimentally [57, 58]. Groff [57] observed the transition of smooth spherical laminar flames to polyhedral-cellular flames in lean propane-air mixtures. In order to ensure that the cellular structures were only due to the hydrodynamic instability, the role of the thermal diffusive instability was eliminated by suitably varying the Lewis number. Similar structures were observed by Law et. al [58] on the surface of an acetylene-air flame. The near-equidiffusive nature of acetylene-air
Figure 4.13: (a)-(f) Snapshots of the corrugated flame surface at different times showing the evolution of small scale wrinkles.
Figure 4.14: Incremental propagation speed (filled symbols) for a two-dimensional flame front in a rectangular domain of aspect ratio $\lambda = 4$ and variable turbulence intensity $\eta'$. Bold line is the incremental speed for the two-dimensional flame in laminar conditions.

mixture eliminated any influence of the thermal diffusive instability on cell formation. Fig 4.15(a) shows the flame profile obtained by forcing the MS equation with unstructured noise of intensity $\eta' = 0.05$ at $\gamma = 200$. As in the one dimensional case, the flame exhibits a multi scale structure. There are the remains of two large scale cusps which form a tent like structure, wrinkles on the flame surface which form the cells and a large number of small scale structures within each cell. Imparting a finite length scale to the noise completely eliminates the tent like structure and the flame is completely cellular as seen in Fig 4.15(b). Groff [57] mentioned that no particular relation could be established between the experimental parameters and the cell sizes which were observed. In a similar way the number of cells formed on the flame do not show any predictable variations with the working parameters, as seen in Fig. 4.15 (a) and (b). An increase in the value of $\gamma$ creates sharper wrinkles resulting in sharper cell boundaries without affecting the number of cells. Changes in the lengthscale and intensity of the noise affect the structure of the flame but do not have any significant effect on the number of cells. Such an independence from the working parameters is also noticed in the one dimensional case for the number of wrinkles on the flame profile (which are responsible for cell formation on a flame surface). The influence of noise on the incremental propagation velocity for the two dimensional equation is similar to the one dimensional case as shown in Fig 4.14. A gradual instead of an abrupt bifurcation and the existence of a threshold value of $\gamma$ below which the hydrodynamic instability
Figure 4.15: 2-dimensional flame profiles (viewed from the top) color coded according to $|\nabla \phi|$ (a) $\gamma = 200 \eta' = 0.05$ forcing: white noise (b) $\gamma = 180 \eta' = 0.22$ forcing: correlated noise $\ell_x = \ell_y = 0.05$.

does not amplify continuous perturbations caused by noise are observed here. The incremental propagation velocity scales with $\gamma$ as $U \sim \gamma^{\beta_2}$ just as in the one dimensional case.
Chapter 5

Conclusion

The nonlinear evolution of a hydrodynamically unstable premixed flame propagating in a weakly turbulent flow is discussed within the context of the Michelson Sivashinsky equation for weak thermal expansion. The effect of weak turbulence was simulated by forcing the equation with external noise characterized by a fixed amplitude and spatial and/or temporal scales. The influence of both unstructured and structured noise on the flame was examined. In the presence of turbulence, planar flames followed the well established quadratic scaling with turbulence intensity, but with an additional influence of the hydrodynamic instability. It was shown that background noise/turbulence affected the dynamics of the system, causing the system to feel the effects of the instability at much lower values of the bifurcation parameter $\gamma$ as compared to the noiseless case. This resulted in a gradual transition from a planar to a corrugated or “cusp-like” flame, as opposed to an abrupt one in the noiseless case.

Corrugated flames demonstrated greater resilience to turbulence as compared to planar flames. The sensitivity of the dynamics of corrugated flames to background turbulence was shown to be a function of $\gamma$, which was directly proportional to the domain size, and turbulence intensity. For large flames, disturbances caused by low intensity background turbulence were rapidly magnified by the hydrodynamic mechanism resulting in the formation of small scale wrinkles on the flame profile, which led to an increase in the propagation velocity. Small flames on the other hand, required higher turbulence intensities to cause the formation of such wrinkles. It was shown that for a given turbulence intensity, there exists a threshold value of domain size beyond which the hydrodynamic or Darrieus-Landau mechanism is triggered causing the amplification of small disturbances. Scaling laws were established for the incremental propagation velocity based on the turbulence intensity and $\gamma$. The behavior of the flame in a turbulent environment was classified as “instability dominated” or “turbulence dominated” based on the scaling with turbulence intensity.

When forced with structured noise characterized by a spatial scale, the flame exhibited a resonant behavior causing the greatest increase in propagation velocity at a particular length scale. Such a behavior was observed for both planar and corrugated flames and it was shown that the value of the resonating length scale scaled as $\sim 1/\gamma$. It was shown that maximum amplification of an initial perturbation was caused by
the eigen values of the planar flame solution, which existed at all values of $\gamma$. The wavenumber at which maximum growth rate occurred scaled as $\sim \gamma$ which explained the scaling of the resonating length scale as $\sim 1/\gamma$. The effect of finite spatial scales was included in the scaling laws via an amplification factor which was active at finite scales. At large turbulence intensities (within the realm of the wrinkled flamelet regime) the flame profile started exhibiting fractal-like behavior. The fractal dimension of the flame was calculated and an asymptotic value of 1.23 was estimated for very large turbulence intensity. Scaling laws, similar to the ones for the incremental propagation velocity, were established for the excess fractal dimension.

The two dimensional Michelson Sivashinsky equation was also examined in this study. Other than the parameter $\gamma$, the aspect ratio of the domain was shown to be another parameter that governed the behavior of the flame in the two dimensional case. Forcing the equation with external noise resulted in the formation of cellular structures on the flame surface, which closely resembled the structure on the surface of expanding spherical flames observed experimentally. The incremental propagation velocity followed scaling laws similar to the one dimensional case. At large turbulence intensities, the flame exhibited a fractal-like multiscale structure just as the one dimensional case.

The present study of the forced Michelson Sivashinsky equation has clearly illustrated that though formulated for small density changes which do not characterize combustion, this equation effectively captures the physics of hydrodynamically unstable flames in the presence of background turbulence. The results obtained in this study can serve as a good starting point for future works that try to examine the effects of the Darrieus-Landau instability on a turbulent premixed flame for arbitrary density changes and in the different regimes of turbulent combustion.
References


