A chromatic art gallery problem

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Abstract

The art gallery problem asks for the smallest number of guards required to see every point of the interior of a polygon $P$. We introduce and study a similar problem called the chromatic art gallery problem. Suppose that two members of a finite point guard set $S \subset P$ must be given different colors if their visible regions overlap. What is the minimum number of colors required to color any guard set (not necessarily a minimal guard set) of a polygon $P$? We call this number, $\chi_G(P)$, the chromatic guard number of $P$. We believe this problem has never been examined before, and it has potential applications to robotics, surveillance, sensor networks, and other areas. We show that for any spiral polygon $P_{spi}$, $\chi_G(P_{spi}) \leq 2$, and for any staircase polygon (strictly monotone orthogonal polygon) $P_{sta}$, $\chi_G(P_{sta}) \leq 3$. We also show that for any positive integer $k$, there exists a polygon $P_k$ with $3k^2 + 2$ vertices such that $\chi_G(P_k) \geq k$. 
1 Introduction

Suppose a robot is navigating a region populated with radio beacons. The robot is equipped with the following primitives: drive toward the beacon, drive away from the beacon, and drive along the level sets of the beacon’s intensity (similar to the model in [24]). If this robot were to be in an area where two different beacons were broadcasting on the same frequency, it may become confused and the action that it takes when ordered to perform a certain primitive could become unpredictable. The same phenomenon could happen with other sensing and actuation models. A robot navigating visually and being told “drive toward the red landmark” may get confused if there are two red landmarks in its visibility region. This raises a natural question: How many classes of partially distinguishable guards are required to guard a given area? In this paper, we try to answer this question for bounded simply-connected polygonal areas. We assume that a robot cannot see a given landmark if the polygon boundary is in the way.

Spiral polygons are a heavily studied area in visibility. Special results for this class of polygons are available for the watchman route problem [18], the weakly cooperative guard problem [12], the visibility graph recognition problem [4], point visibility isomorphisms [14], and triangulation [25]. An algorithm for decomposing general polygons into a minimum number of spiral polygons was described in [11]. We choose to focus on spiral polygons because we think that they could be a useful building block in solving the chromatic guard number problem for general polygons.

There are two commonly used definitions for staircase polygons (see Figure 1). The one we use is that a staircase polygon is a strictly monotone orthogonal polygon. This definition was also used in [8], which found an asymptotically tight bound on the number of guards required to solve the prison yard problem for these polygons. The problem of placing a maximum area staircase of this kind in a planar point set was studied in [16]. They have also been examined in the context of self-avoiding walks for physics modelling in [22] and [23]. These polygons always take the form of two convex right angle vertices joined by two subchains of alternating convex and reflex vertices (described in greater detail in Section 4.2). The other definition, which is a special case of the one we use, is that a staircase polygon is an orthogonal convex fan. This definition is used in [1]. These polygons trivially have a chromatic guard number of one (they are star-shaped). From this point forward, “staircase polygons” will refer exclusively to strictly monotone orthogonal polygons. We choose to focus on these staircase polygons because of their potential as building blocks for a bound on the chromatic guard number of general orthogonal polygons.

Section 2 contains the formal definition of the problem. Section 3 contains a proof of a lower bound on the chromatic guard number for general polygons. Section 4 contains upper bounds on the chromatic guard number for spiral polygons and staircase polygons. Section 5 discusses directions of future research.
2 Problem definition

Let a polygon $P$ be a closed, simply connected, polygonal subset of $\mathbb{R}^2$ with boundary $\partial P$. A point $p \in P$ is visible from point $q \in P$ if the closed segment $\overline{pq}$ is a subset of $P$. The visibility polygon $V(p)$ of a point $p \in P$ is defined as $V(p) = \{ q \in P \mid q \text{ is visible from } p \}$. Let a guard set $S$ be a finite set of points in $P$ such that $\bigcup_{s \in S} V(s) = P$. The members of a guard set are referred to as guards. A pair of guards $s, t \in S$ is called conflicting if $V(s) \cap V(t) \neq \emptyset$. Let $C(S)$ be the minimum number of colors required to color a guard set $S$ such that no two conflicting guards are assigned the same color. Let $T(P)$ be the set of all guard sets of $P$. Let $\chi_G(P) = \min_{S \in T(P)} C(S)$. We will call this value $\chi_G(P)$ the chromatic guard number of the polygon $P$. Note that the number of guards used can be as high or low as is convenient. We want to minimize the number of colors used, not the number of guards.

The notion of conflict can be phrased in terms of link distance. The link distance between two points $p, q \in P$ (denoted $LD(p, q)$) is the minimum number of line segments required to connect $p$ and $q$ via a polygonal path. Each line segment must be a subset of $P$.

**Theorem 1.** Two guards $s_1, s_2 \in P$ conflict if and only if $LD(s_1, s_2) \leq 2$.

**Proof.** If $LD(s_1, s_2) = 1$, then they are mutually visible, and obviously conflict.

If $LD(s_1, s_2) = 2$, then there exists a point $r \in P$, such that $\overline{s_1r}, \overline{rs_2} \subseteq P$. Since $\overline{s_1r} \subseteq P$, $r \in V(s_1)$. Since $\overline{rs_2} \subseteq P$, $r \in V(s_2)$. Because $r$ is in $V(s_1)$ and $V(s_2)$, the intersection of $V(s_1)$ and $V(s_2)$ is non-empty; therefore $s_1$ and $s_2$ conflict.

If $s_1$ and $s_2$ conflict, then let $r$ be a point in the intersection of $V(s_1)$ and $V(s_2)$. Since $r \in V(s_1)$, $\overline{s_1r} \subseteq P$. Since $r \in V(s_2)$, $\overline{rs_2} \subseteq P$. Because $\overline{s_1r}, \overline{rs_2} \subseteq P$, $LD(s_1, s_2) \leq 2$. \qed

3 Lower bounds on the chromatic guard number

**Theorem 2.** For every positive integer $k$, there exists a polygon $P$ with $3k^2 + 2$ vertices such that $\chi_G(P) \geq k$.

**Proof.** The polygon $P$ is a version of the standard “comb” used to show the occasional necessity of $\lceil n/3 \rceil$ guards in the standard art gallery problem [3]. The vertex list of $P$ is $[(0, 1), (1, 0), (2, 1), (4, 1), (5, 0), (6, 1) \ldots (4k^2-4, 1), (4k^2-3, 0), (4k^2-2, 1), (4k^2-2, 2k-2), (0, 2k-2)]$. This polygon has $3k^2 + 2$ vertices, and it consists of a closed rectangular region (the body region) with corners $(0, 1), (4k^2-2, 1), (4k^2-2, 2k-2), (0, 2k-2)$ that has $k^2$ notches attached to the bottom edge. Call the vertices with a $y$ coordinate of zero apex points. Note that each notch has a unique apex point. A guard with coordinates $(x, y)$ will be referred to as an apex guard if $y < 1$ and will be referred to as a body guard if $y \geq 1$ (See Figure 2).

Each body guard can guard up to $k$ distinct notches. However, since the visibility polygon of a body guard includes the entire body region, and every guard’s visibility polygon intersects the body region, a body guard will conflict with every other guard in the polygon. Let $m_{body}$ be the number of body guards used in a guard set of $P$.

Each apex guard can guard only one notch. However, two apex guards will not conflict if they are placed far enough away from each other. Since the top edge of $P$ has a $y$ coordinate of $2k-2$, two apex guards are only forced to conflict if the distance between the apex points of their corresponding notches is $4k$ or less. Let a set of $k$ notches be consecutive if the maximum distance between the apex points of any two notches in the set is $4k$. Let $m_{apex}$ be the maximum number of apex guards in any consecutive set of $k$ notches in $P$. 

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Suppose the polygon \( P \) has a guard set \( S \) assigned to it that requires only \( \chi_G(P) \) colors. Consider a set of \( k \) consecutive notches in \( P \) that contains \( m_{\text{apex}} \) apex guards. All of these apex guards will conflict with each other, and all of these apex guards will conflict with all of the body guards. Therefore, \( \chi_G(P) \geq m_{\text{apex}} + m_{\text{body}} \). Now, note that each body guard can guard at most \( k \) notches. Since there are \( k^2 \) notches, by the pigeonhole principle, apex guards can guard at most \( km_{\text{apex}} \) notches (see Figure 2). Since each notch must be guarded, \( km_{\text{apex}} + km_{\text{body}} \geq k^2 \), so \( m_{\text{apex}} + m_{\text{body}} \geq k \). Therefore \( \chi_G(P) \geq m_{\text{apex}} + m_{\text{body}} \geq k \).

\[ \square \]

4 Upper bounds on the chromatic guard number

One could just give every guard its own color. Any polygon \( P \) with \( n \) vertices can be guarded by \( \lfloor n/3 \rfloor \) guards (the art gallery theorem [3]), so \( \chi_G(P) \leq \lfloor n/3 \rfloor \). However, this bound is unsatisfying, because colors can often be reused. There exist polygons with an arbitrarily high number of vertices that require only two colors. We prove bounds better than \( \lfloor n/3 \rfloor \) for two categories of polygons.

4.1 Spiral polygons

A chain is a series of points \([p_1,p_2,\ldots,p_n]\) along with line segments connecting consecutive points. A subchain is a chain that forms part of the boundary of a polygon. The points \( p_1 \) and \( p_n \) are called endpoints, and all other points are internal vertices. A convex subchain is a subchain where all the internal vertices have an internal angle of less than \( \pi \) radians. A reflex subchain is a subchain where all the internal vertices have an internal angle of greater than \( \pi \) radians. Note that convex and reflex subchains can trivially consist of a single line segment (if there are no internal vertices). A spiral polygon is a polygon with exactly one maximal reflex subchain (all reflex subchains of the spiral polygon must be contained within the maximal reflex subchain).

**Theorem 3.** For any spiral polygon \( P \), \( \chi_G(P) \leq 2 \).

**Proof.** The spiral polygon consists of two subchains, a reflex subchain, and a convex subchain. Let \( v_s \) and \( v_t \) be the endpoints of the reflex subchain. Without loss of generality, assume that the path along the convex subchain from \( v_s \) to \( v_t \) runs clockwise. The guards will all be placed along the edges of the convex subchain.

Call the \( n \)th guard placed \( s_n \). Place \( s_1 \) at \( v_s \). Let \( p_n \) be the point most clockwise along the convex subchain that is visible from \( s_n \). Let \( b_n \) be the most counterclockwise vertex along the reflex subchain visible from \( s_n \). Let \( g_n \) be the vertex immediately clockwise from \( b_n \). Let
Figure 3: [top left] A spiral polygon $P$. The convex subchain is highlighted in red, and the reflex subchain is highlighted in blue. [top right] The first guard $s_1$ is placed on vertex $v_s$. The points $p_1, b_1, g_1,$ and $r_1$ are marked and the interval that $s_2$ can be placed in is highlighted in green. [bottom left] Recursively showing that placed guards form a guard set. The subpolygon $P_1$ is assumed to be guarded by $s_1$. The region that $s_2$ is responsible for is $P_2$, bounded by the reflex subchain between $b_1$ and $b_2$, the edge between $p_2$ and $b_2$, the convex subchain between $p_2$ and $p_1$, and the edge between $b_1$ and $p_1$. The subpolygon $P_2$ has been triangulated, indicating that $s_2$ can guard the whole subpolygon. The triangle with endpoints $p_2$, $b_2$, and $s_2$ is degenerate, as those three points are collinear. [bottom right] A guard placement and 2-coloring.

$r_n$ be the point on the convex subchain collinear with $g_n$ and $b_n$ and visible from both. Note that $p_n$ and $r_n$ define the endpoints of an interval along the convex subchain. Place $s_{n+1}$ at a point on this interval that is not one of the endpoints. Note that this means that $s_{n+1} \notin V(s_n)$. Terminate when a guard can see $v_t$ (see Figure 3).

We can show that this is a guard set for the polygon by triangulating the polygon using the polygon vertices, the members of $S$, and the points $p_i$ and showing that each triangle has a member of $S$ as one of its vertices. Suppose that the polygon bounded by the edges starting from $p_n$ counterclockwise along the boundary of $P$ until $b_n$ and the edge between $p_n$ and $b_n$ has already been triangulated such that each triangle contains a vertex in the set $\{s_i | i \leq n\}$. We must show that $s_{n+1}$ can guard the subpolygon bordered by the edges counterclockwise from $p_{n+1}$ to $p_n$, the edge between $p_n$ and $b_n$, the vertices counterclockwise from $b_n$ to $b_{n+1}$, and the edge between $b_{n+1}$ and $p_{n+1}$ (call this subpolygon $P_{n+1}$). If each of these vertices in the subpolygon is visible from $s_{n+1}$, then the subpolygon can be triangulated by connecting each vertex to $s_{n+1}$, meaning that $s_{n+1}$ guards the entire subpolygon (see Figure 3).
Since $s_{n+1}$ is placed on the interval in between $p_n$ and $r_n$, it must be able to see the entire edge between $g_n$ and $b_n$, meaning that $b_n$ is visible from $s_{n+1}$. The vertex $b_{n+1}$ is visible from $s_{n+1}$ by definition. Examine the polygon consisting of the edges along the reflex subchain between $b_n$ and $b_{n+1}$, $s_{n+1}b_n$, and $s_{n+1}b_{n+1}$. Since all the vertices along the reflex subchain are reflex, they cannot have edges between each other in a triangulation, so in any triangulation, they must all be connected to $s_{n+1}$ (see Figure 4). The point $p_{n+1}$ is visible to $s_{n+1}$ by definition. The point $p_n$ is visible to $s_{n+1}$ because $s_{n+1}$ is on the interval between $p_n$ and $r_n$, and the only reflex vertex which could obstruct part of that interval’s view of another part of that interval would have to lie in between $b_n$ and $g_n$ on the reflex subchain (by definition, there are no such vertices). Because the vertices in between $p_n$ and $p_{n+1}$ lie on a convex subchain, if $s_{n+1}$ can see both $p_n$ and $p_{n+1}$, then $s_{n+1}$ can see all the vertices in between. This means that $P_{n+1}$ can be triangulated with every triangle having $s_{n+1}$ as an endpoint, so $s_{n+1}$ guards $P_{n+1}$ (the triangle with endpoints $p_{n+1}$, $b_{n+1}$, and $s_{n+1}$ is degenerate, as those three points are colinear, but this is not a problem). This technique still works if $s_{n+1}$ can see $v_t$ (in this case, $p_{n+1} = b_{n+1} = v_t$). This implies inductively that $S$ is a guard set for $P$.

Because all the guards are along the convex subchain, if two guards conflict, their visibility polygons must intersect somewhere along the convex subchain. Also, since $s_n \notin V(s_{n+1})$ and $s_n \notin V(s_{n-1})$, $s_{n+1}$ cannot conflict with $s_{n-1}$, or there would be no room along the convex subchain to place $s_n$. Therefore, all evenly indexed guards can be colored red, and all odd indexed guards can be colored blue, so $\chi_G(P) \leq 2$.

4.2 Staircase polygons

An orthogonal polygon is a polygon in which all angles are right angles. An alternating subchain is a subchain with at least one internal vertex, with the first and last internal vertices being convex, and with consecutive internal vertices alternating between convex and reflex. A staircase polygon is an orthogonal polygon consisting of two convex vertices, $v_w$ and $v_z$, connected by two alternating subchains. For simplicity, we will assume without loss of generality that orthogonal
Figure 5: [left] A staircase polygon $P$ with vertices $v_w$ and $v_z$ identified. The lower subchain is highlighted in red, and the upper subchain is highlighted in blue. [middle] The guard $s_1$ is placed on the neighbor of $v_w$ on the lower subchain. The guard $s_2$ is placed on the rightmost convex vertex in $V(s_1)$. [right] A guard placement and coloring for $P$ that uses only three colors.

polygons are always oriented such that each edge is either vertical or horizontal, and that $v_w$ is the top left vertex, and that $v_z$ is the bottom right vertex. Put the polygon on a coordinate plane with $v_w$ at the $(0,0)$ coordinate, let right be the positive $x$ direction, and let up be the positive $y$ direction. As mentioned earlier, “staircase polygon” is a synonym for strictly monotone orthogonal polygon.

**Theorem 4.** For any staircase polygon $P$, $\chi_G(P) \leq 3$.

*Proof.* Due to our assumptions about the orientation of the polygon $P$, one of the alternating subchains is going to be above the other one. Call the higher subchain the *upper subchain* and call the other subchain the *lower subchain*. Place a guard $s_1$ on the neighbor of $v_w$ along the lower subchain. If guard $s_i$ has been placed on the lower subchain, then place guard $s_{i+1}$ on the right-most convex vertex on the upper subchain that is contained in $V(s_i)$. If guard $s_i$ has been placed on the upper subchain, then place guard $s_{i+1}$ on the right-most convex vertex on the lower subchain that is contained in $V(s_i)$. Stop placing guards when a guard can see $v_z$, and let $m$ be the number of guards placed (See Figure 5).

First, it must be shown that $s_i$ and $s_{i+2}$ are not placed on the same vertex. Suppose without loss of generality that $s_i$ is on the lower subchain. Note that the rightmost convex vertex on the lower subchain in $V(s_{i+1})$ must also be the lowest convex vertex on the lower subchain in $V(s_{i+1})$. Note also that a ray extended downward from $s_{i+1}$ must intersect the horizontal edge incident to $s_{i+2}$ (otherwise $s_{i+2}$ would not be the rightmost convex vertex on the lower subchain). If this is the same horizontal edge that is incident to $s_i$, then the point where the ray intersects the horizontal edge incident to $s_i$ must be a convex vertex (call it $v_f$). Since the convex vertex $v_f$ neighbors the convex vertex $v_i$ along a horizontal edge, and since $v_f$ is to the right of $v_i$, $v_f$ must be $v_z$. Therefore, $s_{i+2}$ would only be placed on the same vertex as $s_i$ if $v_z$ is visible from $s_{i+1}$. Since we stop placing guards once a guard can see $v_z$, two guards will never be placed on the same vertex.
Next, it must be shown that this is a guard set for the staircase polygon. Suppose without loss of generality that guard \( s_i \) is placed on the lower subchain. Assume that the set \([s_1, s_2 \ldots s_i]\) forms a guard set for the subpolygon that lies above the guard \( s_i \) (call this subpolygon \( P_i \)). We must show that the set \([s_1, s_2 \ldots s_i+1]\) forms a guard set for the subpolygon that lies to the left of guard \( s_{i+1} \) (call this subpolygon \( P_{i+1} \)). Let \( p_{i+1} \) be the point where a ray extended downward from \( s_{i+1} \) intersects the lower subchain. Note that each vertex on the lower subchain between \( s_i \) and \( p_{i+1} \) is visible from \( s_{i+1} \). We have to show that \( s_{i+1} \) guards \( P_{i+1}\setminus P_i \). Let \( v_i^r \) be the reflex vertex to the right of \( s_i \) on the lower subchain. Let \( Q_{i+1} \) be the subpolygon below \( s_{i+1} \) and to the left of \( s_{i+1} \) (See Figure 6). Clearly, \( Q_{i+1} \supseteq P_{i+1}\setminus P_i \) (as \( s_{i+1} \) cannot be lower than \( s_i \)). Note that every vertex of \( Q_{i+1} \) that is not connected to \( s_{i+1} \) by an edge of \( Q_{i+1} \) is on the lower subchain. For any given vertex \( v \) in \( Q_{i+1} \) that is not connected to \( s_{i+1} \) by an edge of \( Q_{i+1} \), all edges of \( Q_{i+1} \) not incident to \( s_{i+1} \) that lie above \( v \) must also lie to the left of \( v \), and all edges of \( Q_{i+1} \) not incident to \( s_{i+1} \) that lie to the right of \( v \) must also lie below \( v \). Since \( s_{i+1} \) is never lower than \( v \), and never to the right of \( v \), every vertex \( v \) of \( Q_{i+1} \) must be visible from \( s_{i+1} \). This means that one could triangulate \( Q_{i+1} \) such that each triangle has \( s_{i+1} \) as one of its corners. Therefore, the guard \( s_{i+1} \) can guard \( Q_{i+1} \) by itself. Therefore, the set \([s_1, s_2 \ldots s_m]\) forms a guard set for \( P \).

Finally, it must be shown that the guard set \([s_1, s_2 \ldots s_m]\) can be colored with three colors. Suppose guard \( s_i \) is placed on the lower chain. Let \( y_i \) be the \( y \)-coordinate of the lowest point visible from \( s_i \). Note that, because \( s_i \) is on a convex right-angle vertex on the lower subchain, \( V(s_i) \) is bordered on the bottom by a horizontal line at the same height as the horizontal edge incident to \( s_i \); therefore \( y_i \) is just the \( y \)-coordinate of \( s_i \). Let \( y_{i+3} \) be the \( y \)-coordinate of the highest point in \( V(s_{i+3}) \). Because \( s_{i+3} \) is on a convex right-angle vertex on the upper subchain, \( V(s_{i+3}) \) is bordered on top by a horizontal line at the same height as the horizontal edge incident to \( s_{i+3} \); therefore \( y_{i+3} \) is just the \( y \)-coordinate of \( s_{i+3} \). Now, we must show that \( y_i > y_{i+3} \). In the portion of the proof that showed that each guard is placed on a unique vertex, we demonstrated that the \( y \)-coordinate of \( s_{i+1} \) (call it \( y_{i+1} \)) has to be higher than the \( y \)-coordinate of \( s_{i+3} \). If \( y_i \leq y_{i+3} \), then \( y_i \leq y_{i+3} < y_{i+1} \). However, this is impossible, because \( s_{i+1} \) was placed on the rightmost (and thus, lowest) vertex on the upper chain that was in \( V(s_i) \). Therefore, \( y_i > y_{i+3} \). Since the highest point in \( V(s_{i+3}) \) is lower than the lowest point in \( V(s_i) \), \( s_i \) and \( s_{i+3} \) cannot conflict (see Figure 7).

Since \( s_i \) and \( s_{i+3} \) do not conflict, we can color all guards with an index of 0 mod 3 with green, all guards with an index of 1 mod 3 with red, and all guards with an index of 2 mod 3 with blue. Therefore \( \chi_G(P) \leq 3 \).

We have assumed throughout this proof that guard \( s_i \) was placed on the lower subchain. However, the arguments made above still apply if \( s_i \) was placed on the upper subchain (reflect the polygon over the \( y = -x \) line).

\[\square\]

5 Conclusion

One direction of future research would be to find bounds for other categories of polygons. Finding a bound better than \( \chi_G(P) \leq \lfloor n/3 \rfloor \) for general polygons is the most obvious target (sources that examine visibility problems in general polygons include [3], [5], [6], [20]), but orthogonal polygons [8], [9], [13], [19], and monotone polygons [2], [17], are also heavily studied in visibility. Visibility in curvilinear bounded regions has also been researched [10]. Allowing polygons with holes is another possibility, as is placing further restrictions on the placement of
Figure 6: [top left] A polygon $P$ with a guard placement. [top middle] The region $P_1$ that $s_1$ is responsible for guarding. [top right] The region $P_2$ that $s_1$ and $s_2$ are responsible for guarding. [bottom left] The region $P_2 \setminus P_1$ that $s_2$ is responsible for guarding. [bottom middle] The region $Q_2$, which consists of the portion of $P$ below and to the left of $s_2$. This region is a superset of $P_2 \setminus P_1$. [bottom right] A triangulation of $Q_2$ where all triangles have a vertex at the location of $s_2$, showing that $s_2$ guards $Q_2$.

Figure 7: [left] A staircase polygon $P$ with a guard placement. [right] The regions $V(s_1)$ and $V(s_4)$ are shown. Note that the lowest point in $V(s_1)$ is higher than the highest point in $V(s_4)$, as the horizontal line incident to $s_1$’s vertex is higher than the horizontal line incident to $s_4$’s vertex.
guards, perhaps forcing the guards to be strongly cooperative [21] or weakly cooperative [15].

The problem could also be attacked from a visibility graph context. The structure of standard visibility graphs for general polygons is still not completely understood, but [7] gives four necessary conditions for visibility graphs. It is likely that analogues of these four conditions could be made for “2-link” visibility.

Finally, for practical robotics purposes, it would be useful to make a more realistic model of when guards conflict. For example, using a model where a robot has limited vision, so two guards sufficiently far from each other will not conflict even if there is no obstacle between them. Alternately, it may be useful to make a model where the “signal” from a guard degrades as the robot gets further away, perhaps degrading faster if it must go through an obstacle.

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