ON P-OPERATOR SPACES AND THEIR APPLICATIONS

BY

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Abstract

There have been a lot of research done on the relationship between locally compact groups and algebras associated with them. For example, Johnson proved that a locally compact group $G$ is amenable if and only if the convolution algebra $L_1(G)$ is amenable as a Banach algebra, and Ruan showed that $G$ is amenable if and only if the Fourier algebra $A(G)$ of $G$ is operator amenable. Motivated by Ruan’s work, we want to study $G$ through tools from $p$-operator spaces. We first introduce the $p$-operator space and various $p$-operator space tensor products. We then study $p$-operator space approximation property and $p$-operator space completely bounded approximation property which are related to $p$-operator space injective tensor product. We then apply these properties to the study of the pseudofunction algebra $PF_p(G)$, the pseudomeasure algebra $PM_p(G)$, and the Figà-Talamanca-Herz Algebra $A_p(G)$. Especially we show that if $G$ is discrete, the most of approximation properties for the reduced group C*-algebra $C^*_r(G)$, the group von Neumann algebra $VN(G)$, and the Fourier algebra $A(G)$ (related to amenability, weak amenability, and approximation property of $G$) have natural $p$-analogues for $PF_p(G)$, $PM_p(G)$, and $A_p(G)$. With help of Herz’s work, we also study the stability of these properties. Finally we discuss the properties $C_p$, $C'_p$, and $C''_p$ which are natural $p$-analogues of properties $C$, $C'$, and $C''$. 
To my family
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**List of Abbreviations**

- **p-cb**  
  $p$-completely bounded

- **p-AP**  
  $p$-approximation property

- **p-OAP**  
  $p$-operator space approximation property

- **p-CBAP**  
  $p$-operator space completely bounded approximation property
List of Symbols

\( \mathbb{C} \) the field of complex numbers
\( \mathbb{N} \) the set of natural numbers, \( \mathbb{N} = \{1, 2, \ldots\} \).
\( X \equiv Y \) \( X \) is isometrically isomorphic to \( Y \)
\( X \overset{\text{pcb}}{\cong} Y \) \( X \) is \( p \)-completely isometrically isomorphic to \( Y \)
\( \mathcal{CB}_p(X, Y) \) the space of \( p \)-completely bounded operators between \( X \) and \( Y \)
\( \mathcal{CB}^*(X', Y') \) the space of weak*-continuous \( p \)-completely bounded operators between \( X' \) and \( Y' \)
\( \mathbb{M}_{n,r}(X) \) the linear space of all \( n \times r \) matrices with entries in \( X \)
\( \mathbb{M}_n(X) \) the linear space of all \( n \times n \) matrices with entries in \( X \)
\( \mathbb{M}_{n,r,\mathbb{C}} = \mathbb{M}_{n,r}(\mathbb{C}) \)
\( \mathbb{M}_n(\mathbb{C}) = \mathbb{M}_n \)
\( \mathbb{M}_\infty(X) \) the linear space of all infinite matrices with entries in \( X \)
\( \mathbb{M}_\infty = \mathbb{M}_\infty(\mathbb{C}) \)
\( \mathbb{M}_{\text{fin}}^\infty(X) \) the linear space of all infinite matrices with only finitely many nonzero entries from \( X \)
\( \mathbb{M}_{n,r}(X) \) the normed linear space obtained by specifying a norm on \( \mathbb{M}_{n,r}(X) \)
\( \mathbb{M}_n(X) = \mathbb{M}_{n,n}(X) \)
\( \mathbb{M}_{n,r}(\mathbb{C}) \)
\( \mathbb{M}_n(\mathbb{C}) \)
\( \mathbb{M}_\infty(X) \) the space of all \( x \in \mathbb{M}_\infty(X) \) with \( \|x\| < \infty \)
\( \mathbb{M}_\infty = \mathbb{M}_\infty(\mathbb{C}) \)
\( \mathcal{K}_\infty(X) \) the closure of \( \mathbb{M}_{\text{fin}}^\infty(X) \) in \( \mathbb{M}_\infty(X) \)
\( \otimes_p \) \( p \)-tensor product
\( \oplus_p \) \( p \)-direct sum
\( E^n \) \( n \)-fold \( p \)-direct sum \( E \oplus_p \cdots \oplus_p E \) of an \( SQ_p \) space \( E \).
\( \ell_p^n(X) = \ell_p^n \otimes_p X \)
\( \| \cdot \|_{\text{pcb}} \) \( p \)-completely bounded norm
\[
X \otimes Y \quad \text{the algebraic tensor product of } X \text{ and } Y
\]
\[
X \hat{\otimes} Y \quad p\text{-operator space projective tensor product of } X \text{ and } Y
\]
\[
X \vee p Y \quad p\text{-operator space injective tensor product of } X \text{ and } Y
\]
\[
X \hat{\vee} Y \quad p\text{-operator space Haagerup tensor product of } X \text{ and } Y
\]
\[
X \otimes Y \quad \text{Banach space projective tensor product of } X \text{ and } Y
\]
\[
X \hat{\otimes} Y \quad \text{Banach space injective tensor product of } X \text{ and } Y
\]
\[
\langle \cdot, \cdot \rangle \quad \text{scalar pairing, } \langle [v_{ij}], [w_{ij}] \rangle = \sum_{i,j} \langle v_{ij}, w_{ij} \rangle
\]
\[
\langle \langle \cdot \rangle \rangle \quad \text{matrix pairing, } \langle \langle [v_{ij}], [w_{kl}] \rangle \rangle = \sum_{i,k} \sum_{j,l} \langle v_{ij}, w_{kl} \rangle_{(i,k),(j,l)}
\]
\[
K(L_p(\mu)) \quad \text{the space of compact operators on } L_p(\mu)
\]
\[
\mathcal{N}(L_p(\mu)) \quad \text{the space of nuclear operators on } L_p(\mu)
\]
\[
\mathcal{N}_n = \mathcal{N}(\ell_p^n)
\]
\[
\mathcal{N}_\infty = \mathcal{N}(\ell_\infty)
\]
\[
\| \cdot \|_p \quad \ell_p\text{-norm of a vector (or a matrix): } \|v_i\| = (\sum_i |v_i|^p)^{1/p}, \|a_{ij}\| = \left(\sum_{i,j} |a_{ij}|^p\right)^{1/p}
\]
\[
\| \cdot \|_{1,n} \quad \text{a norm on } \mathbb{M}_n(V) \text{ given by } \|v\|_{1,n} = \inf\{\|a\|_p \|w\|_p : r \in \mathbb{N}, v = \alpha w\beta, \alpha \in \mathbb{M}_{n,r}, \beta \in \mathbb{M}_{r,n}, w \in M_r(V)\}
\]
\[
\mathcal{N}_n(V) \quad \text{the normed space } (\mathbb{M}_n(V), \| \cdot \|_{1,n})
\]
\[
\epsilon_{ij} \quad \text{standard basis for } \mathbb{M}_n
\]
\[
\text{Ball}(X) \quad \text{the closed unit ball of a Banach space } X. X_1 \text{ is also used.}
\]
\[
X_1 \quad \text{the closed unit ball of a Banach space } X. \text{ Ball}(X) \text{ is also used.}
\]
\[
\lambda_p(s) \quad \text{the left regular representation, } (\lambda_p(s)f)(t) = f(s^{-1}t) \text{ for } f \in L_p(G) \text{ and } s \in G.
\]
\[
\rho_p(s) \quad \text{the right regular representation, } (\rho_p(s)f)(t) = f(ts)\Delta(s)^{1/p} \text{ for } f \in L_p(G) \text{ and } s \in G.
\]
\[
C(G) \quad \text{the space of all continuous functions on } G
\]
\[
C_b(G) \quad \text{the space of all bounded continuous functions on } G
\]
\[
C_{00}(G) \quad \text{the space of compactly supported functions on } G
\]
\[
C_0(G) \quad \text{the norm closure of } C_{00}(G) \text{ in } C(G), \text{ the space of continuous functions on } G \text{ which vanish at infinity}
\]
\[
A_p(G) \quad \text{the Figà-Talamanca-Herz Algebra of } G
\]
\[
A_{p,c}(G) \quad \text{functions in } A_p(G) \text{ with compact support}
\]
\[
A_p(K) \quad \text{functions that are restrictions to } K \text{ of functions in } A_p(G).
\]
\[
PF_p(G) \quad \text{the } p\text{-pseudofunction algebra of } G, \text{ that is, the norm closure of } \lambda_p(L_1(G)) \text{ in } B(L_p(G))
\]
\[
PM_p(G) \quad \text{the } p\text{-pseudomeasure algebra of } G, \text{ that is, the weak*-closure of } \lambda_p(L_1(G)) \text{ in } B(L_p(G)) = \text{the weak*-closure of span}\{\lambda_p(s) : s \in G\} \text{ in } B(L_p(G))
\]
\( \xi \star \eta \) the convolution of \( \xi \) and \( \eta \): 
\[ \xi \star \eta(x) = \int_G \xi(y)\eta(y^{-1}x)dy \]

\( \mathbb{1}_S \) the characteristic function of \( S \)

\( \mathbb{1}_K \) the constant function on \( K \), that is, \( \mathbb{1}_K(x) = 1 \) for all \( x \in K \)

\( |S| \) the (Haar) measure of \( S \).

\( x\varphi \) the left translation of a function \( \varphi \) by \( x \); 
\[ x\varphi(t) = \varphi(xt). \]
Chapter 1

Introduction to $p$-Operator Spaces

We introduce and study basics on $p$-operator spaces, which can be regarded as $p$-generalization of operator spaces. We define $p$-operator spaces and $p$-completely bounded maps, and give some examples.

1.1 Preliminaries

Throughout this writing, we always assume $1 < p < \infty$ unless stated otherwise. Given $p$, its conjugate exponent is denoted by $p'$ so that $1/p + 1/p' = 1$. Every Banach space is over $\mathbb{C}$, the field of complex numbers. If $X$ and $Y$ are normed linear spaces, $\mathcal{B}(X,Y)$ will denote the normed linear space of the bounded linear operators from $X$ into $Y$ and we will use $\mathcal{B}(X)$ for $\mathcal{B}(X,X)$. For a normed linear space $X$, the dual space $X'$ of $X$ is the Banach space defined by $X' = \mathcal{B}(X,\mathbb{C})$, that is, the space of all continuous linear functionals on $X$. For a measure space $(\Omega, \Sigma, \mu)$, if there is no risk of confusion, we will simply write $L^p(\mu)$ for $L^p(\Omega, \Sigma, \mu)$.

1.2 $SQ_p$ Spaces

Definition 1.2.1. A Banach space $E$ is called an $L_p$ space if it is isometrically isomorphic to some $L_p(\mu)$. A Banach space $E$ is called an $SQ_p$ (denoted $E \in SQ_p$) space\footnote{In the literature, e.g. [Run05], $QSL_p$ space is also used.} if it is isometrically isomorphic to a quotient of a subspace of an $L_p$ space.

Remark 1.2.2. $E \in SQ_p$ if and only if $E$ is isometrically isomorphic to a subspace of quotient of an $L_p$ space. Indeed, if $X \subseteq Y \subseteq L_p(\mu)$ for some measure $\mu$, then $Y/X \subseteq L_p(\mu)/X$. Conversely, if $\pi_X : L_p(\mu) \to L_p(\mu)/X$ denotes the canonical quotient map and if $W \subseteq L_p(\mu)/X$, then one can show that $\pi_X^{-1}(W)/X$ is isometrically isomorphic to $W$ via the map $\Phi : \pi_X^{-1}(W)/X \to W$ defined by $\Phi(f + X) = \pi_X(f)$, $f \in \pi_X^{-1}(W)$.

To discuss properties of $SQ_p$ spaces, we need some facts on Banach spaces. We include a lemma for convenience.
Lemma 1.2.3. Let $X \subseteq Y \subseteq Z$ be Banach spaces and let
\[ X^\perp = \{ \varphi \in Z' : \varphi|_X \equiv 0 \} \quad \text{and} \quad Y^\perp = \{ \varphi \in Z' : \varphi|_Y \equiv 0 \}. \]

Then there is an isometric isomorphism
\[ (Y/X)' \cong X^\perp/Y^\perp. \]

Proof. First note that the function $\Phi : (Y/X)' \to \{ \varphi \in Y' : \varphi|_X \equiv 0 \}$ defined by
\[ \Phi(f)(y) = f(\hat{y}), \quad f \in Y/X, \quad y \in Y, \]
where $\hat{y} := y + X \in Y/X$, is an isometric isomorphism onto its image. Note also that the function $\Psi : Y' \to Z'/Y^\perp$ defined by
\[ \Psi(\varphi) = \tilde{\varphi} + Y^\perp, \quad \varphi \in Y' \]
is also a well-defined isometric isomorphism onto its image, where $\tilde{\varphi}$ is a Hahn-Banach extension of $\varphi$ to $Z'$. The result follows since the image of $(Y/X)'$ under $\Psi \circ \Phi$ is $X^\perp/Y^\perp$. □

Remark 1.2.4. Now we can state some properties of $SQ_p$ spaces.

1. Every $SQ_p$ space is reflexive by [Meg98, Theorem 1.11.16, Corollary 1.11.18].

2. By 1 above and Lemma 1.2.3, $E$ is an $SQ_p$ space if and only if $E'$ is an $SQ_{p'}$ space.

3. A calculation similar to that in Remark 1.2.2 shows that $SQ_p$ space is closed under taking subspaces and quotients.

4. Since every subspace or quotient of a Hilbert space is again a Hilbert space, it follows that $SQ_2 = \{ \text{Hilbert spaces} \}$.

5. If $E$ and $F$ are $SQ_p$ spaces, then so is $E \oplus_p F$, where $\oplus_p$ is the $p$-direct sum defined on $E \oplus F$ by
\[ \|e \oplus f\| = (\|e\|^p + \|f\|^p)^{1/p} \]
for $e \in E$ and $f \in F$. Similarly, we can define the $n$-fold $p$-direct sum $E_1 \oplus_p \cdots \oplus_p E_n$ of $SQ_p$ spaces $E_1, \ldots, E_n$. In particular, if $E_1 = \cdots = E_n = E$, and there is no risk of confusion, then we will use the notation $E^n$ for $E \oplus_p \cdots \oplus_p E$.

6. If $p = 1$ or $p = \infty$, then $SQ_p = \{ \text{Banach spaces} \}$. To see this, it suffices to check that $\Phi : \ell_1(X_1) \to X$ (resp. $\Psi : X \to \ell_\infty(X_1')$) given by $\{a_x\} \mapsto \sum_x a_x x$ (respectively, $x \mapsto \Psi(x)(f) = f(x), f \in \ell_\infty(X_1')$) is
a quotient map (respectively, an isometry). If $X$ is a separable Banach space, then $X$ can be regarded as a quotient of $\ell_1$ [Mor01, Theorem 2.19] or a subspace of $\ell_\infty$. To see this, first note that $X'_\ell$ is weak*-metrizable [Meg98, Theorem 2.6.23] and therefore weak*-separable by Alaoglu’s theorem. Let $\{f_n\}_{n=1}^\infty$ be a weak*-dense subset of $X'_\ell$ and define a map from $X$ to $\ell_\infty$ by $x \mapsto \{f_n(x)\}_{n=1}^\infty$: this gives an isometry.

7. By [Her71, Corollary 2], if $p \leq q \leq 2$ or $2 \leq q \leq p$, then an $L_q$ space is an $SQ_p$ space. Therefore, if $p \leq q \leq 2$ or $2 \leq q \leq p$, then every $SQ_q$ space is an $SQ_p$ space.

8. For another characterization of $SQ_p$ spaces, see [Kwa72].

1.3 $p$-Operator Spaces

Let $(\Omega, \Sigma, \mu)$ be a measure space and let $X$ be a Banach space. Let $L_p(\mu, X)$ be the space of Bochner $p$-integrable functions from $\Omega$ to $X$.\footnote{See [DF93, Appendix B12] for details.} We define a norm on the algebraic tensor product $L_p(\mu) \otimes X$ by embedding $L_p(\mu) \otimes X$ into $L_p(\mu, X)$ in the natural way, that is, $f \otimes x \mapsto f(\cdot)x$ for $f \in L_p(\mu)$ and $x \in X$. Let $L_p(\mu) \otimes_p X$ denote the completion of $L_p(\mu) \otimes X$ in $L_p(\mu, X)$ with respect to the norm in $L_p(\mu, X)$. It follows easily that $L_p(\mu) \otimes_p X$ is isometrically isomorphic to $L_p(\mu, X)$.\footnote{See [DF93, §7.2] for details.}

If $X = L_p(\Omega', \Sigma', \mu')$, then we have

$$L_p(\mu) \otimes_p L_p(\mu') \cong L_p(\mu \times \mu'),$$

where $\mu \times \mu$ denotes the product measure on $\Sigma \times \Sigma'$ [DF93, §7.2]. In particular, if $I$ and $J$ are index sets, then we have

$$\ell_p(I) \otimes_p \ell_p(J) = \ell_p(I \times J).$$

If $L_p(\mu) = \ell_p^n, \mathbb{C}^n$ equipped with $\ell_p$-norm, then we will also use notation $\ell_p^n(X)$ for $\ell_p^n \otimes_p X = X^n$.

Now we are ready to define the main subject of the thesis.

Definition 1.3.1. A Banach space $X$ is called a concrete $p$-operator space if $X$ is a closed subspace of $\mathcal{B}(E)$ for some $E \in SQ_p$.

Let $M_n(X)$ denote the linear space of all $n \times n$ matrices with entries in $X$. For a concrete $p$-operator space $X \subseteq \mathcal{B}(E)$ and for each $n \in \mathbb{N}$, define a norm $\| \cdot \|_n$ on $M_n(X)$ by identifying $M_n(X)$ as a subspace of $B(\ell_p^n \otimes_p E) = B(\ell_p^n(E))$, and let $M(X)$ denote the corresponding normed space. The norms $\| \cdot \|_n$ then satisfy

2See [DF93, Appendix B12] for details.
3See [DF93, §7.2] for details.
for \( u \in M_n(X) \) and \( v \in M_m(X) \), we have \( \|u \oplus v\|_{M_{n+m}(X)} = \max\{\|u\|_n, \|v\|_m\} \).

\( M_p \) for \( u \in M_m(X), \alpha \in M_{n,m}, \) and \( \beta \in M_{m,n} \), we have \( \|\alpha u \beta\|_n \leq \|\alpha\|\|u\|_m\|\beta\| \), where \( \|\alpha\| \) is the norm of \( \alpha \) as a member of \( B(\ell^m_{p}, \ell^n_{p}) \), and similarly for \( \beta \).

\textbf{Remark 1.3.2.} When \( p = 2 \), these are Ruan’s axioms and 2-operator spaces are simply operator spaces because the \( SQ_2 \) spaces are exactly the same as Hilbert spaces.

As in operator spaces, we can also define abstract \( p \)-operator spaces.

\textbf{Definition 1.3.3.} An \textit{abstract \( p \)-operator space} is a Banach space \( X \) together with a sequence of norms \( \| \cdot \|_n \) defined on \( M_n(X) \) satisfying the conditions \( D_\infty \) and \( M_p \) above.

Thanks to the following theorem by Le Merdy, we do not distinguish between concrete \( p \)-operator spaces and abstract \( p \)-operator spaces and we will merely speak of \( p \)-operator spaces.

\textbf{Theorem 1.3.4.} [LeM96, Theorem 4.1] An abstract \( p \)-operator space \( X \) can be isometrically embedded in \( B(E) \) for some \( E \in SQ_p \) in such a way that the canonical norms on \( M_n(X) \) arising from this embedding agree with the given norms.

\textbf{Example 1.3.5.}

1. Suppose \( E \) and \( F \) are \( SQ_p \) spaces and let \( L = E \oplus_p F \), then the mapping

\[ x \mapsto \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix} \]

is an isometric embedding of \( B(E, F) \) into \( B(L) \) and using this we can view \( B(E, F) \) as a \( p \)-operator space. Note that \( M_n(B(E, F)) \) is isometrically isomorphic to \( B(\ell^n_{p}(E), \ell^n_{p}(F)) \).

2. The identification \( L_p(\mu) = B(\mathbb{C}, L_p(\mu)) \subseteq B(\mathbb{C} \oplus_p L_p(\mu)) \) gives a \( p \)-operator space structure on \( L_p(\mu) \) called the \textit{column \( p \)-operator space structure} of \( L_p(\mu) \), which we denote by \( L_p^c(\mu) \). Similarly, we denote by \( L_p^r(\mu) \) the \( p \)-operator space structure on \( L_p^r(\mu) \) which is called the \textit{row \( p \)-operator space structure} of \( L_p^r(\mu) \) and defined by the identification \( L_p^r(\mu) = B(L_p(\mu), \mathbb{C}) \). In general, we can define \( E^c \) and \( (E')^r \) for any \( E \in SQ_p \).

3. Unless stated otherwise, we give \( \mathbb{C} \) the obvious \( p \)-operator space structure, that is, \( M_n(\mathbb{C}) = B(\ell^n_{p}) \).\footnote{We will also use \( M_n \) to denote \( B(\ell^n_{p}) \).}
1.4 $p$-Completely Bounded Maps

Note that a linear map $u: X \to Y$ between $p$-operator spaces $X$ and $Y$ induces a map $u_n : M_n(X) \to M_n(Y)$ by applying $u$ entrywise.

**Definition 1.4.1.** We say that $u$ is $p$-completely bounded ($p$-cb) if $\|u\|_{\text{cb}} := \sup_n \|u_n\| < \infty$. Similarly, we define the notions of $p$-completely contractive, $p$-completely isometric, and $p$-completely quotient. We write $\mathcal{CB}_p(X,Y)$ for the space of all $p$-completely bounded maps from $X$ into $Y$ and $\mathcal{CB}_p(X)$ for the space $\mathcal{CB}_p(X,X)$.

Before proceeding, we discuss subspaces and quotients of a $p$-operator spaces as in [Daw10]. If $Y$ is a subspace of a $p$-operator space $X$, then inclusions $M_n(Y) \subseteq M_n(X)$ and the corresponding norms determine a $p$-operator space matrix norm on $Y$. This determines the $p$-operator subspace structure.

**Example 1.4.2.** Let $\mathcal{K}(L_p(\mu)) \subseteq \mathcal{B}(L_p(\mu))$ denote the space of compact operators on $L_p(\mu)$, then $\mathcal{K}(L_p(\mu))$ has a $p$-operator space structure inherited from that of $\mathcal{B}(L_p(\mu))$. Using the fact that a compact operator on $L_p(\mu)$ can be approximated by finite rank operators [Rya02, Example 4.5 and Corollary 4.13]$^5$, it is easily shown that $M_n(\mathcal{K}(L_p(\mu)))$ can be identified with $\mathcal{K}(L_p(\mu)^n) = \mathcal{K}(L_p(\mu) \oplus_p \cdots \oplus_p L_p(\mu))$.

Given a closed subspace $Y$ of a $p$-operator space $X$, we use the identification $M_n(X/Y) = M_n(X)/M_n(Y)$ to define a norm on $M_n(X/Y)$, and it is easy to check that $X/Y$ becomes a $p$-operator space and that the quotient map $\pi : X \to X/Y$ is a $p$-completely quotient map. This determines the $p$-operator quotient structure.

For a $p$-operator space $X$ and for each $n \in \mathbb{N}$, we can give $M_n(X)$ a natural $p$-operator space structure using the identification $M_r(M_n(X)) = M_{rn}(X)$. We also want to turn the mapping space $\mathcal{CB}_p(X,Y)$ between two $p$-operator spaces $X$ and $Y$ into a $p$-operator space: let us define a norm on $M_n(\mathcal{CB}_p(X,Y))$ by identifying this space with $\mathcal{CB}_p(X,M_n(Y))$. Using Le Merdy’s theorem, one can show that $\mathcal{CB}_p(X,Y)$ itself is a $p$-operator space. In particular, the $p$-operator dual space of $X$ is defined to be $\mathcal{CB}_p(X,\mathbb{C})$. The next lemma by Daws shows that we may identify the Banach dual space $X'$ of $X$ with the $p$-operator dual space $\mathcal{CB}_p(X,\mathbb{C})$ of $X$.

**Lemma 1.4.3.** [Daw10, Lemma 4.2] Let $X$ be a $p$-operator space, and let $\varphi \in X'$, the Banach dual of $X$. Then $\varphi$ is $p$-completely bounded as a map to $\mathbb{C}$. Moreover, $\|\varphi\|_{\text{cb}} = \|\varphi\|$. 

**Remark 1.4.4.** If $\varphi = [\varphi_{ij}] \in M_n(X') = M_n(\mathcal{CB}_p(X,\mathbb{C})) = \mathcal{CB}_p(X,M_n)$ for some $p$-operator space $X$, then

$$\|\varphi\| = \sup\{\|\langle \varphi, x \rangle\| : m \in \mathbb{N}, x \in M_m(X), \|x\| \leq 1\},$$

That is, $L_p(\mu)$ has the Approximation Property.
Proposition 1.4.5. Let $X$ be a $p$-operator space and $\varphi : X \to M_n = \mathcal{B}(\ell^m_p)$ be bounded and linear. Then $\varphi$ is $p$-completely bounded and $\|\varphi\|_{pcb} \leq n\|\varphi_n\|$.\footnote{When $p = 2$, we have $\|\varphi\|_{cb} = \|\varphi_n\|$. See [ER00, Proposition 2.2.2].}

Proof. Fix $m \geq n$. We need to show that $\|\varphi_m\| \leq n\|\varphi_n\|$. Let $x = [x_{ij}] \in M_m(X)$. Choose $\tilde{\xi} \in \ell^m_p$ and $\tilde{\eta} \in \ell^m_p$ such that $\|\tilde{\xi}\| = \|\tilde{\eta}\| = 1$ and consider $\langle \tilde{\eta}, \varphi_m(x)\tilde{\xi} \rangle$. One can find $mn \times mn$ permutation matrices $P$ and $Q$ such that

$$
\Phi := Q\varphi_m(x)P = \begin{bmatrix}
[\varphi_{1,1}(x_{ij})] & \cdots & [\varphi_{1,n}(x_{ij})] \\
\vdots & \ddots & \vdots \\
[\varphi_{n,1}(x_{ij})] & \cdots & [\varphi_{n,n}(x_{ij})]
\end{bmatrix}.
$$

Letting $\xi = P^{-1}\tilde{\xi}$ and $\eta = (Q^T)^{-1}\tilde{\eta}$, we have $\|\xi\| = \|\eta\| = 1$ and $\langle \eta, \varphi_m(x)\tilde{\xi} \rangle = \langle \eta, \Phi \xi \rangle$. Write

$$
\xi = \begin{bmatrix}
\xi_1 \\
\vdots \\
\xi_n
\end{bmatrix}, \quad
\eta = \begin{bmatrix}
\eta_1 \\
\vdots \\
\eta_n
\end{bmatrix}, \quad
\xi_k = \begin{bmatrix}
\xi_{k,1} \\
\vdots \\
\xi_{k,m}
\end{bmatrix} \in \ell^m_p, \quad
\eta_k = \begin{bmatrix}
\eta_{k,1} \\
\vdots \\
\eta_{k,m}
\end{bmatrix} \in \ell^m_p, \quad 1 \leq k, l \leq n,
$$

and for each $1 \leq k, l \leq n$, put $u_k = \frac{1}{\|\xi_k\|} \xi_k$ and $v_l = \frac{1}{\|\eta_l\|} \eta_l$. Finally letting

$$
\alpha = \begin{bmatrix}
\|\xi_1\| \\
\vdots \\
\|\xi_n\|
\end{bmatrix}, \quad
\beta = \begin{bmatrix}
\|\eta_1\| \\
\vdots \\
\|\eta_n\|
\end{bmatrix}, \quad
U = \begin{bmatrix}
u_1 \\
\vdots \\
u_n \
\end{bmatrix}_{mn \times n}, \quad
V = \begin{bmatrix}
v_1 \\
\vdots \\
v_n \
\end{bmatrix}_{mn \times n}, \quad
$$

we obtain $\langle \eta, \Phi \xi \rangle = \langle V\beta, \Phi U\alpha \rangle = \langle \beta, V^T\Phi U\alpha \rangle$ with $U, V$ contractive and $\|\alpha\|_{\ell^m_p} = \|\beta\|_{\ell^m_p} = 1$. Thus the result will follow once we show that $\|V^T\Phi U\|_{\mathcal{B}(\ell^2_p)} \leq n\|\varphi_n\|$. Note that

$$
V^T\Phi U = \begin{bmatrix}
\begin{bmatrix}v_1^T \\
\vdots \\
v_n^T
\end{bmatrix}_{n \times mn} \begin{bmatrix}u_1 & \cdots & u_n\end{bmatrix}_{mn \times mn} \\
\begin{bmatrix}v_1^T & \cdots & v_n^T\end{bmatrix}_{n \times mn} \begin{bmatrix}u_1 & \cdots & u_n\end{bmatrix}_{mn \times mn}
\end{bmatrix}
$$

where $\langle \cdot, \cdot \rangle$ is the matrix paring as in [ER00, §1.1].

We can slightly generalize Lemma 1.4.3 as in the following proposition.
\[
\begin{bmatrix}
\varphi(v_k^T x u_l)_{k,l}
\end{bmatrix}_{n \times n}.
\]

Denote \(\varphi(v_k^T x u_l) \in M_n\) by \(y^{kl}\), then the last matrix above is nothing but
\[
\begin{bmatrix}
(y^{11})_{1,1} & \cdots & (y^{1n})_{1,n} \\
\vdots & \ddots & \vdots \\
(y^{n1})_{n,1} & \cdots & (y^{nn})_{n,n}
\end{bmatrix},
\]
which can be computed as
\[
R[y^{kl}]_{n^2 \times n^2} C = R\varphi_n \begin{pmatrix} V^T & U \end{pmatrix} C,
\]
where \(R\) is an \(n \times n^2\) matrix whose only nonzero entry is 1 at positions \((1,1), (2, n+2), (3, 2n+3), \cdots\), and at \((n, n^2)\) and \(C = R^T\). Since
\[
\begin{bmatrix} x & x & \cdots & x \\
x & x & \cdots & x \\
\vdots & \vdots & \ddots & \vdots \\
x & x & \cdots & x
\end{bmatrix} = \begin{bmatrix} x \\
x \\
\vdots \\
x
\end{bmatrix} + \begin{bmatrix} x \\
x \\
\vdots \\
x
\end{bmatrix} + \cdots + \begin{bmatrix} x \\
x \\
\vdots \\
x
\end{bmatrix}, \tag{1.1}
\]
we get
\[
\|V^T \Phi U\|_{\mathcal{B}(l^p_n)} \leq \|R\| \cdot \|\varphi_n\| \cdot \|V^T\| \cdot \|U\| \cdot \|C\| 
\]
(by (1.1)) \leq n\|\varphi_n\|\|x\|,
and this completes the proof.
1.5\quad \textit{p-Operator Spaces on }L_p\text{ Space}

When \( E = L_p(\mu) \) and if \( X \subseteq \mathcal{B}(E) = \mathcal{B}(L_p(\mu)) \), then we say that \( X \) is a \textit{p-operator space on }L_p\text{ space}. These \( p \)-operator spaces are often easier to work with as we will see soon. Let \( \kappa_X : X \to X'' \) denote the canonical inclusion from a Banach space \( X \) into its second dual. Contrary to operator spaces, \( \kappa_X \) is \textit{not} always \( p \)-completely isometric. Thanks to the following theorem by Daws, however, we can easily characterize those \( p \)-operator spaces with the property that the canonical inclusion is \( p \)-completely isometric.

**Proposition 1.5.1.** \cite[Proposition 4.4]{Daw10} Let \( X \) be a \textit{p-operator space}. Then \( \kappa_X \) is a \textit{p-complete contraction}. Moreover, \( \kappa_X \) is a \textit{p-complete isometry} if and only if \( X \subseteq \mathcal{B}(L_p(\mu)) \) \textit{p-completely isometrically} for some measure \( \mu \).

We say that a \( p \)-operator space \( X \) is \textit{reflexive} if the canonical isometric inclusion \( \kappa_X : X \to X'' \) is a \( p \)-completely isometric isomorphism from \( X \) onto \( X'' \).

**Lemma 1.5.2.** A \textit{p-operator space} \( X \) is reflexive if and only if \( X \) is reflexive as a Banach space and there is a measure \( \mu \) such that \( X \subseteq \mathcal{B}(L_p(\mu)) \). In particular, for any measure \( \mu \), \( L_p(\mu) \) and \( L_{p'}(\mu) \) are reflexive.

**Proof.** \( \leftarrow \Rightarrow \) If \( X \) is reflexive as a \( p \)-operator space, then clearly \( X \) is reflexive as a Banach space. By \cite[Proposition 4.4]{Daw10}, if \( \kappa_X \) is \( p \)-completely isometric, then there exists a measure \( \mu \) such that \( X \subseteq \mathcal{B}(L_p(\mu)) \).

\( \Rightarrow \iff \) Since \( X \subseteq \mathcal{B}(L_p(\mu)) \), by \cite[Proposition 4.4]{Daw10}, for each \( n \in \mathbb{N} \), \((\kappa_X)_n : M_n(L_p(\mu)) \to M_n(L_{p'}(\mu))^{''} \) is surjective follows from the fact that \( \kappa_X : X \to X'' \) is surjective. \( \square \)

As pointed out in \cite[§4.1]{Daw10}, Le Merdy gives an example of (even finite dimensional) \( p \)-operator spaces \( X \) such that \( \kappa_X \) is not a \( p \)-complete isometry, and this is a first significant problem we confront with when extending results from operator spaces.\(^8\) At the same time, Proposition 1.5.1 shows that \( p \)-operator spaces on \( L_p \) space are easier to work with.

Lemma 1.5.3 below exhibits another problem we face when extending results from operator spaces to \( p \)-operator spaces.\(^9\)

**Lemma 1.5.3.** \cite[Lemmas 4.5 and 4.6]{Daw10} Let \( X \) and \( Y \) be \textit{p-operator spaces}. If \( u \in \mathcal{CB}_p(X,Y) \), then the adjoint mapping \( u' \) belongs to \( \mathcal{CB}_p(Y',X') \) with \( \|u'\|_{pcb} \leq \|u\|_{pcb} \). If \( u \) is a \textit{p-complete quotient map}, then \( u' \) is a \textit{p-complete isometry}.

**Remark 1.5.4.** We do not know whether we always have \( \|u'\|_{pcb} = \|u\|_{pcb} \). We cannot simply apply the same argument as in operator spaces theory because we lack \( p \)-analogue of Roger Smith’s lemma \cite[Proposition 3.2.1]{ER00}.

\(^7\)That is, \( \kappa_X \) is a \( p \)-complete isometry if and only if \( X \) is a \( p \)-operator space on \( L_p \) space.

\(^8\)For any operator space \( X \), the canonical inclusion is always completely isometric \cite[Proposition 3.2.1]{ER00}.

\(^9\)For operator spaces \( X,Y \) and for \( u \in \mathcal{CB}(X,Y) \), we always have \( \|u\|_{cb} = \|u'\|_{cb} \) \cite[Proposition 3.2.2]{ER00}.
Lemma 2.2.1. Similarly, when $u$ is a $p$-complete isometry, we do not know whether $u'$ is a $p$-complete quotient because a $p$-analogue of Arveson-Wittstock-Hahn-Banach theorem is not available yet at this moment.\footnote{Having a positive answer to this question is equivalent to $p$-analogue of Arveson-Wittstock-Hahn-Banach Theorem because \[ (u')_n : M_n(Y') = CB_p(Y, M_n) \to CB_p(X, M_n) = M_n(X') \] is given by the restriction.}

However, with an additional condition that $Y$ is a $p$-operator space on $L_p$ space, we do get the equality in Lemma 1.5.3 as explained in the following.

**Proposition 1.5.5.** Let $X$ and $Y$ be $p$-operator spaces with $Y \subseteq B(L_p(\mu))$ for some measure $\mu$. If $u \in CB_p(X,Y)$, then the adjoint $\|u'\|_{pcb} = \|u\|_{pcb}$.

**Proof.**

\[
\|u\|_{pcb} = \sup \{ m_n(\langle x_{ij}, \varphi_{kl} \rangle) : n \in \mathbb{N}, [x_{ij}] \in M_n(X), \|x_{ij}\| \leq 1, \|\varphi_{kl}\| \leq 1 \}
= \sup \{ m_n(\langle x_{ij}, \varphi_{kl} \rangle) : n \in \mathbb{N}, [x_{ij}] \in M_n(Y), \|x_{ij}\| \leq 1, \|\varphi_{kl}\| \leq 1 \}
= \sup \{ m_n(\langle x_{ij}, u'(\varphi_{kl}) \rangle) : n \in \mathbb{N}, [x_{ij}] \in M_n(Y'), \|x_{ij}\| \leq 1, \|\varphi_{kl}\| \leq 1 \}
= \|u'\|_{pcb},
\]

where the second equality comes from Proposition 1.5.1.

We close this section with the following proposition.

**Proposition 1.5.6.** Let $X \subseteq B(L_p(\mu))$ and $Y \subseteq B(L_p(\nu))$ be $p$-operator spaces. Then the adjoint mapping

\[ \Phi : CB_p(X,Y) \to CB_p(Y',X'), \quad T \mapsto T' \]

is a $p$-completely isometric isomorphism from $CB_p(X,Y)$ onto $CB_p(Y',X')$ if and only if either $X = \{0\}$ or $Y$ is reflexive.

**Proof.** ($\Rightarrow$) Suppose $X \neq \{0\}$. By Lemma 1.5.2, to show that $Y$ is reflexive, it suffices to show that $Y$ is reflexive as a Banach space. Let $\varphi \in Y''$. Let $x \neq 0$ be a vector in $X$, then there exists $x' \in X'$ such that $x'(x) = 1$. Define $S : Y' \to X'$ by $S(f) = \varphi(f)x'$, then $S \in CB_p(Y',X')$ and hence $S = T'$ for some $T \in CB_p(X,Y)$. Since $\varphi(f) = \langle S(f), x \rangle = \langle f, Tx \rangle$ for all $f \in Y'$, we see that the canonical inclusion $\kappa_Y$ is
onto, that is, \( Y \) is reflexive as a Banach space.

\((\iff)\) If \( X = \{0\} \), then the result is trivial. Suppose that \( Y \) is reflexive and fix \( n \in \mathbb{N} \). Let \( S = [S_{ij}] \in M_n(\mathcal{CB}_p(Y',X')) = \mathcal{CB}_p(Y',M_n(X')) = \mathcal{CB}_p(Y',\mathcal{CB}_p(X,M_n)) \). Since \( X \subseteq \mathcal{B}(L_p(\mu)) \), for each \( i, j \), we can define \( T_{ij} = S_{ij}|_X \in \mathcal{CB}_p(X,Y) \). Now for any \( f \in Y' \) and for any \( x \in X \),

\[
S(f)(x) = [(S_{ij}(f), \kappa_X(x))] = [(f, T_{ij}(x))] = [(T'_{ij}(f), x)].
\]

This shows that \( S = T' \), where \( T = [T_{ij}] \in M_n(\mathcal{CB}_p(X,Y)) \) and hence the adjoint mapping is onto. \( \square \)

### 1.6 Examples of \( p \)-Completely Isometric Isomorphisms

In this section, we will give specific examples of \( p \)-operator spaces and identify some of them via \( p \)-completely isometric isomorphisms. Let \( \mathcal{N}(L_p(\mu)) \) denote the Banach space of all nuclear operators on \( L_p(\mu) \). Since \( L_p(\mu) \) has the Approximation Property [Rya02, Example 4.5], we have

\[
\mathcal{N}(L_p(\mu)) = L_p'(\mu) \tilde{\otimes} L_p(\mu), \tag{1.2}
\]

where \( \tilde{\otimes} \) denotes the Banach space projective tensor product [Rya02, Chapter 2]. Since \( L_p(\mu) \) is reflexive (hence has the Radon-Nikodým property, [Rya02, Corollary 5.45]), we have the isometric isomorphisms

\[
\mathcal{K}(L_p(\mu))' = \mathcal{N}(L_p(\mu)) \quad \text{and} \quad \mathcal{N}(L_p(\mu))' = \mathcal{B}(L_p(\mu)). \tag{1.3}
\]

Giving \( \mathcal{K}(L_p(\mu)) \) a \( p \)-operator subspace structure in \( \mathcal{B}(L_p(\mu)) \) and giving \( \mathcal{N}(L_p(\mu)) \) a \( p \)-operator space structure by duality, that is, by regarding \( \mathcal{N}(L_p(\mu)) \) as a \( p \)-operator subspace of \( \mathcal{B}(L_p(\mu))' \)[12], we can regard \( \mathcal{K}(L_p(\mu)) \) and \( \mathcal{N}(L_p(\mu)) \) as \( p \)-operator spaces. With this structure, we can say more about the second isometry in (1.3) as in the following lemma.

**Proposition 1.6.1.** [Daw10, Lemma 5.1] With the dual \( p \)-operator space structure on \( \mathcal{N}(L_p(\mu)) \), we have

\[
\mathcal{N}(L_p(\mu))' = \mathcal{B}(L_p(\mu)) \text{ \( p \)-completely isometrically.}
\]

As an application of Proposition 1.6.1, let us take a closer look at the representation of a dual \( p \)-operator

[11]See Corollary 4.8, Corollary 4.13, and Theorem 5.33 in [Rya02]. See also [Rya02, §2.2].

[12]This \( p \)-operator space structure on \( \mathcal{N}(L_p(\mu)) \) will be called the dual \( p \)-operator space structure on \( \mathcal{N}(L_p(\mu)) \).
space. Since $X'$ is again a $p$-operator space for a $p$-operator space $X$, by Theorem 1.3.4, there exists $E \in SQ_p$ such that $X' \subseteq B(E)$ $p$-completely isometrically. Daws showed that in fact we can choose $E$ to be $\ell_p(I)$ for some index set $I$ [Daw10, Theorem 4.3]. We can still improve Daws’ result as in the following.

**Proposition 1.6.2.** Let $V$ be a $p$-operator space. Then there exists an index set $I$ such that $V'$ can be identified with a weak*-closed subspace of $B(\ell_p(I))$ and the restriction map

$$
\pi : \omega \in \mathcal{N}(\ell_p(I)) \to \omega|_{V'} \in V
$$

is $p$-completely quotient. Therefore, we have the $p$-complete isometry $V = \mathcal{N}(\ell_p(I))/V'_\perp$, where $V'_\perp$ is the pre-annihilator of $V'$ in $\mathcal{N}(\ell_p(I))$.

**Proof.** We can apply a construction similar to that given in the proof of [ER00, Proposition 3.2.4] by constructing $s_n = M_n(V)_1$ and $s = \bigcup_{n=1}^\infty s_n$. We can obtain an index set $I$ such that $\ell_p(I) = \bigoplus_{x \in s} \ell_p^{n(x)}$. The map

$$
\Phi^\sigma : f \in V' \to \text{diag} \{ f_n(x) : x \in s, n \in \mathbb{N} \} \in \prod_{x \in s} B(\ell_p^{n(x)}) \subseteq B(\ell_p(I))
$$

is a weak*-continuous $p$-completely isometric inclusion. In this case, we can identify $V'$ with the weak*-closed subspace $\Phi^\sigma(V')$ in $B(\ell_p(I))$ and $V$ is equal to the quotient $\mathcal{N}(\ell_p(I))/V'_\perp$ via the restriction map

$$
\pi : \omega \in \mathcal{N}(\ell_p(I)) \to \omega|_{V'} \in V.
$$

Now for each $x = [x_{ij}] \in M_n(V)_1 = s_n$, we let $t_{n(x)}$ be the canonical inclusion of $\ell_p^{n(x)}$ into $\ell_p(I)$ and $P_{n(x)}$ be the contractive projection from $\ell_p(I)$ onto $\ell_p^{n(x)}$. Then the truncation map $\omega(y) = P_{n(x)}y_{n(x)}$ on $B(\ell_p(I))$ defines a contractive element $\omega \in M_n(\mathcal{N}(\ell_p(I)))$, which satisfies $\pi_n(\omega) = x$. This shows that $\pi$ is actually a $p$-complete quotient map from $\mathcal{N}(\ell_p(I))$ onto $V$. The last part follows from Lemma 1.5.3. 

The first isometric isomorphism in (1.3) also turns out to be $p$-completely isometric with the dual $p$-operator space structure on $\mathcal{N}(L_p(\mu))$. To prove this, we first need to study a norm structure on $M_n(V)$ for a general $p$-operator space $V$.

**Lemma 1.6.3.** Let $1 \leq p, p' \leq \infty$ with $1/p' + 1/p = 1$. Let $\lambda = \{ \lambda_j \}_{1 \leq j \leq n}$ be a finite sequence in $\mathbb{C}$. Then

$$
\| \lambda \|_{\ell_p^n} \leq n^{1/p-1/p'} \cdot \| \lambda \|_{\ell_p^{n'}}.
$$

**Proof.** There is nothing to prove if $p = p' = 2$. It is trivial if $p = 1$. If $p > p'$, then $\| \lambda \|_2 \leq \| \lambda \|_{\ell_p^n} \leq n^{1/p-1/p'} \cdot \| \lambda \|_{\ell_p^{n'}}$ since $n^{1/p-1/p'} \geq 1$. Finally, assume $1 < p < p'$ and let $q = \frac{p'}{p} > 1$ and let $q'$ be the
conjugate exponent to $q$. By Hölder's inequality,
\[
\|\lambda\|_{p}^{p} \leq \left( \sum_{j=1}^{n} |\lambda_{j}|^{pq} \right)^{1/q} \cdot n^{1/q} = \left( \sum_{j=1}^{n} |\lambda_{j}|^{p'} \right)^{p/p'} \cdot n^{1-p/p'}
\]
and hence $\|\lambda\|_{e} \leq n^{1/p-1/p'} \cdot \|\lambda\|_{p'}$. 

**Lemma 1.6.4.** Let $\alpha = [\alpha_{ij}] \in M_{n,r}$ and $\beta = [\beta_{kl}] \in M_{r,n}$. Let $1 < p, p' < \infty$ with $1/p' + 1/p = 1$. Then we have\[ \|\alpha\|_{B(\ell_{p}^{r},\ell_{p'}^{n})} \leq \|\alpha\|_{p'} \cdot n^{1/p-1/p'} \quad \text{and} \quad \|\beta\|_{B(\ell_{p}^{r},\ell_{p'}^{n})} \leq \|\beta\|_{p} \cdot n^{1/p-1/p'}, \]
where
\[ \|\alpha\|_{p'} = \left( \sum_{i=1}^{n} \sum_{j=1}^{r} |\alpha_{ij}|^{p'} \right)^{1/p'} \quad \text{and} \quad \|\beta\|_{p} = \left( \sum_{k=1}^{r} \sum_{i=1}^{n} |\beta_{ki}|^{p} \right)^{1/p}. \]

**Proof.** Suppose $\xi = \{\xi_{j}\}_{j=1}^{n}$ is a unit vector in $\ell_{p}^{r}$. For each $i$, $1 \leq i \leq n$, let $\eta_{i} = \sum_{j=1}^{r} \alpha_{ij} \xi_{j}$, then by Hölder's inequality, $\eta_{i} \leq \left( \sum_{j=1}^{r} |\alpha_{ij}|^{p'} \right)^{1/p'}$ and by Lemma 1.6.3,
\[
\left( \sum_{i=1}^{n} \eta_{i}^{p} \right)^{1/p} \leq n^{1/p-1/p'} \cdot \left( \sum_{i=1}^{n} \eta_{i}^{p'} \right)^{1/p'} \leq n^{1/p-1/p'} \cdot \|\alpha\|_{p'}
\]
and hence we get $\|\alpha\|_{B(\ell_{p}^{r},\ell_{p'}^{n})} \leq n^{1/p-1/p'} \cdot \|\alpha\|_{p'}$. To prove the second inequality, let $\gamma : \ell_{p}^{r} \to \ell_{p'}^{n}$ be the adjoint operator of $\beta$. Then by the argument above we have
\[
\|\gamma\|_{B(\ell_{p}^{r},\ell_{p'}^{n})} \leq \|\gamma\|_{p} \cdot n^{1/p-1/p'},
\]
Since $\|\gamma\|_{B(\ell_{p}^{r},\ell_{p'}^{n})} = \|\beta\|_{B(\ell_{p}^{r},\ell_{p'}^{n})}$ and $\|\gamma\|_{p} = \|\beta\|_{p}$, we get the desired inequality. 

Let $V$ be a $p$-operator space. Fix $n \in \mathbb{N}$ and define $\|\cdot\|_{1,n} : M_{n}(V) \to [0, \infty)$ by
\[
\|v\|_{1,n} = \inf \{ \|\alpha\|_{p'} \|w\| \|\beta\|_{p} : r \in \mathbb{N}, \ v = \alpha w \beta, \ \alpha \in M_{n,r}, \ \beta \in M_{r,n}, \ w \in M_{r}(V) \}, \quad (1.4)
\]
where $\|\cdot\|_{p'}$ and $\|\cdot\|_{p}$ as in Lemma 1.6.4.

**Proposition 1.6.5.** Suppose that $V$ is a $p$-operator space and $n \in \mathbb{N}$. Then $\|\cdot\|_{1,n}$ defines a norm on $M_{n}(V)$.

**Proof.** Suppose $v_{1}, v_{2} \in M_{n}(V)$. Let $\epsilon > 0$. For $i = 1, 2$, we can find $\alpha_{i}, \beta_{i},$ and $w_{i}$ such that $v_{i} = \alpha_{i} w_{i} \beta_{i}$
with \( \| w_i \| \leq 1 \) and
\[
\| \alpha_i \|_{p'} < (\| v_i \|_{1,n} + \epsilon)^{1/p'}, \quad \| \beta_i \|_p < (\| v_i \|_{1,n} + \epsilon)^{1/p}.
\] (1.5)

Let
\[
\alpha = [\alpha_1 \alpha_2], \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \text{and} \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},
\]
then \( \| \alpha \|_{p'} = \| \alpha_1 \|_{p'} + \| \alpha_2 \|_{p'}, \quad \| \beta \|_p = \| \beta_1 \|_p + \| \beta_2 \|_p, \) and \( \| w \| \leq 1. \) Since \( v_1 + v_2 = \alpha w \beta, \) it follows that
\[
\| v_1 + v_2 \|_{1,n} \leq \| \alpha \|_{p'} \| \beta \|_p
\] (Young’s inequality)
\[
= \frac{\| \alpha \|_{p'}^p + \| \beta \|_p^p}{p'}
\]
\[
= \frac{\| \alpha_1 \|_{p'}^p + \| \alpha_2 \|_{p'}^p + \| \beta_1 \|_p^p + \| \beta_2 \|_p^p}{p'}
\]
(by (1.5))
\[
< \frac{\| v_1 \|_{1,n} + \| v_2 \|_{1,n} + 2\epsilon}{p'} + \frac{\| v_1 \|_{1,n} + \| v_2 \|_{1,n} + 2\epsilon}{p}
\]
\[
= \| v_1 \|_{1,n} + \| v_2 \|_{1,n} + 2\epsilon.
\]

Since \( \epsilon \) is arbitrary, we get \( \| v_1 + v_2 \|_{1,n} \leq \| v_1 \|_{1,n} + \| v_2 \|_{1,n}. \)

For any \( c \in \mathbb{C}, \) if \( v = \alpha w \beta, \) then we have \( cv = \alpha (cw) \beta \) and hence \( \| cv \|_{1,n} \leq \| \alpha \|_{p'} |c| \| w \| \| \beta \|_p. \) Taking the infimum, we get
\[
\| cv \|_{1,n} \leq |c| \| v \|_{1,n}.
\] (1.6)

Replacing \( c \) by \( 1/c \) and \( v \) by \( cv \) in (1.6) gives
\[
|c| \| v \|_{1,n} \leq \| cv \|_{1,n},
\] (1.7)
so (1.6) together with (1.7) gives \( \| cv \|_{1,n} = |c| \| v \|_{1,n}. \)

Finally, suppose \( \| v \|_{1,n} = 0. \) To show that \( v = 0, \) it suffices to show that
\[
\| v \| \leq n^{2 \frac{1}{1/p-1/p'}} \cdot \| v \|_{1,n}.
\] (1.8)

Indeed, if \( v = \alpha w \beta \) with \( \alpha \in M_{n,r}, \beta \in M_{r,n}, \) and \( w \in M_r(v), \) then
\[
\| v \| \leq \| \alpha \| \| w \| \| \beta \|
\]
(by Lemma 1.6.4)
\[
\leq \| \alpha \|_{p'} \cdot n^{1/1/p-1/p'} \cdot \| w \| \cdot \| \beta \|_p \cdot n^{1/1/p-1/p'}
\]
Definition 1.6.6. For a $p$-operator space $V$, let $\mathcal{N}_n(V)$ denote the normed space $(\mathcal{M}_n(V), \| \cdot \|_{1,n})$.

We want to study the Banach dual of $\mathcal{N}_n(V)$. Let $f = [f_{ij}] \in M_n(V') = CB_p(V, M_n)$. By Remark 1.4.4, we have

$$\|f\| = \sup \{ \|\langle f, \tilde{v} \rangle\| : r \in \mathbb{N}, \tilde{v} = [\tilde{v}_{kl}] \in M_r(V), \|\tilde{v}\| \leq 1 \}.$$ 

Let $D^p_{n \times r}$ denote the closed unit ball of $\ell^p_{n \times r}$, then

$$\|f\| = \sup \{ \|\langle f, \tilde{v} \rangle\eta, \xi \| : r \in \mathbb{N}, \tilde{v} = [\tilde{v}_{kl}] \in M_r(V), \|\tilde{v}\| \leq 1, \eta \in D^p_{n \times r}, \xi \in D^p_{n \times r}' \}$$

$$= \sup \left\{ \left\| \sum_{i,j,k,l} f_{ij}(\tilde{v}_{kl})\eta_{(j,l)}\xi_{(i,k)} \right\| : r \in \mathbb{N}, \tilde{v} = [\tilde{v}_{kl}] \in M_r(V), \|\tilde{v}\| \leq 1, \eta \in D^p_{n \times r}, \xi \in D^p_{n \times r}' \right\}$$

$$= \sup \left\{ \left\| \sum_{i,j=1}^{n} \sum_{k,l=1}^{r} f_{ij} \xi_{(i,k)}\tilde{v}_{kl}\eta_{(j,l)} \right\| : r \in \mathbb{N}, \tilde{v} = [\tilde{v}_{kl}] \in M_r(V), \|\tilde{v}\| \leq 1, \eta \in D^p_{n \times r}, \xi \in D^p_{n \times r}' \right\}.$$ 

Note that $\sum_{k,l=1}^{r} \xi_{(i,k)}\tilde{v}_{kl}\eta_{(j,l)}$ is the $(i, j)$-entry of the matrix product $\alpha \tilde{v} \beta$, where

$$\alpha = \begin{bmatrix} \xi_{(1,1)} & \cdots & \xi_{(1,r)} \\ \vdots & \ddots & \vdots \\ \xi_{(n,1)} & \cdots & \xi_{(n,r)} \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} \eta_{(1,1)} & \cdots & \eta_{(n,1)} \\ \vdots & \ddots & \vdots \\ \beta_{(1,r)} & \cdots & \eta_{(n,r)} \end{bmatrix},$$

so

$$\|f\| = \sup \left\{ \left\| \sum_{i,j=1}^{n} \langle f_{ij}, (\alpha \tilde{v} \beta)_{ij} \rangle \right\| : \|\tilde{v}\| \leq 1, \|\alpha\|_{p'} \leq 1, \|\beta\|_p \leq 1 \right\}$$

$$= \sup \left\{ \|\langle f, v \rangle\| : v = \alpha \tilde{v} \beta, \|\tilde{v}\| \leq 1, \|\alpha\|_{p'} \leq 1, \|\beta\|_p \leq 1 \right\}$$

$$= \sup \left\{ \|\langle f, v \rangle\| : \|v\|_{1,n} \leq 1 \right\}. \quad (1.9)$$

Lemma 1.6.7. For a $p$-operator space $V$, the scalar pairing determines the isometric identification $\mathcal{N}_n(V)' = M_n(V').$

Proof. Define $\Phi : M_n(V') \to \mathcal{N}_n(V)'$ by $f \mapsto \langle f, \cdot \rangle$.

$\Phi$ is one-to-one: suppose $\Phi(f) = 0$, and let $v \in V$. For each $i, j$, consider an element $E_{ij}(v) \in \mathcal{N}_n(V)$ whose only nonzero element is $v$ at $(i, j)$-position, then $\langle f, E_{ij}(v) \rangle = f_{ij}(v) = 0$ and hence $f_{ij} = 0$. 

Taking the infimum, (1.8) follows.
Φ is onto: Let $G \in \mathcal{N}_n(V)'$. For each $i, j$, define $E_{ij} : V \to \mathcal{N}_n(V)$ by $v \mapsto E_{ij}(v)$, where $E_{ij}(v)$ as as above. Let $f_{ij} = G \circ E_{ij} : V \to \mathbb{C}$ and let $f = [f_{ij}] \in M_n(V')$, then it follows that $\Phi(f) = G$.

Φ is isometric: this is immediate from (1.9).

Now we are ready to show that the first isometric isomorphism in (1.3) is also $p$-completely isometric with the dual $p$-operator space structure on $\mathcal{N}(L_p(\mu))$.

**Proposition 1.6.8.** With the dual $p$-operator space structure on $\mathcal{N}(L_p(\mu))$, we have $\mathcal{K}(L_p(\mu))' = \mathcal{N}(L_p(\mu))$ $p$-completely isometrically.

**Proof.** Fix $n \in \mathbb{N}$. We need to show that

$$M_n(\mathcal{N}(L_p(\mu))) = CB_p(\mathcal{K}(L_p(\mu)), M_n)$$

isometrically. By the argument between Propositions 5.3 and 5.4 in [Daw10], we have an isometry

$$M_n(\mathcal{N}(L_p(\mu))) = \mathcal{N}_n(\mathcal{K}(L_p(\mu)))'. \quad (13)$$

Now the result follows from Lemma 1.6.7. \qed

To explore more examples of $p$-completely isometric isomorphisms, we will make use of the following lemmas.

**Lemma 1.6.9.** Let $\eta = [\eta_{ij}] \in M_{m,n}(L_{p_C}(\mu))$ and $\|\eta_{ij}\| < \epsilon$ for all $i, j$. Then $\|\eta\|_{M_{m,n}(L_{p}(\mu))} < m^{1/p} n^{1/p'} \epsilon$.

**Proof.** Since $\|\eta\|_{M_{m,n}(L_{p}(\mu))} = \|\eta\|_{\mathcal{B}(L_{p_C}(\mu), L_{p}(\mu))}$,

$$\|\eta\|_{M_{m,n}(L_{p}(\mu))} = \sup_{\|\lambda\|_1 \leq 1} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} \eta_{ij} \right)^{1/p}.$$  

Here for $\|\lambda\| \leq 1$, by Hölder’s inequality,

$$\left\| \sum_{j=1}^{n} \lambda_j \eta_{ij} \right\| \leq \sum_{j=1}^{n} |\lambda_j| \|\eta_{ij}\| \leq \left( \sum_{j=1}^{n} \|\eta_{ij}\|^{p'} \right)^{1/p'} \leq n^{1/p'} \epsilon$$

and the result follows. \qed

\[13\] In [Daw10], Daws used the notation $T_n(\mathcal{K}(L_p(\mu)))$ for $\mathcal{N}_n(\mathcal{K}(L_p(\mu)))$. 

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Lemma 1.6.10. Let \( \mu \) be a measure. Then for any \( \xi = [\xi_{ij}] \in M_m(L_p^\mu(\mu)) \) and for any \( \epsilon > 0 \), there exist a \( k(m) \in \mathbb{N} \), a subspace \( F \) of \( L_p(\mu) \) which is isometrically isomorphic to \( \ell_p^{k(m)} \), \( \{f_1, \cdots, f_{k(m)}\} \subseteq F \), and matrices \( \{\alpha^1, \cdots, \alpha^{k(m)}\} \) such that

\[
\|\xi - \tilde{\xi}\|_{M_m(L_p^\mu(\mu))} < \epsilon, 
\]

where

\[
\tilde{\xi} = \sum_{t=1}^{k(m)} f_t \otimes \alpha^t.
\]

Moreover, we can have

\[
\|\tilde{\xi}\| = \left\| \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^{k(m)} \end{bmatrix} \right\|_{M_m(k(m),m)}. 
\]

Proof. By standard properties of \( L_p(\mu) \), there exist a \( k(m) \in \mathbb{N} \), a subspace \( F \) of \( L_p(\mu) \) which is isometrically isomorphic to \( \ell_p^{k(m)} \), and \( \{\tilde{\xi}_{ij}\}_{i,j=1}^{m} \subseteq F \) such that \( \|\xi_{ij} - \tilde{\xi}_{ij}\| < \frac{\epsilon}{m} \) for each \( i, j \).\(^{14}\) Let \( \{f_1, \cdots, f_{k(m)}\} \subseteq F \) correspond to the canonical basis of \( \ell_p^{k(m)} \) and write \( \tilde{\xi}_{ij} = \sum_{t=1}^{k(m)} \alpha^t_{ij} f_t \), then we have

\[
\tilde{\xi} := [\tilde{\xi}_{ij}] = \sum_{t=1}^{k(m)} f_t \otimes \alpha^t,
\]

where \( \alpha^t = [\alpha^t_{ij}] \in M_m \). By Lemma 1.6.9, \( \|\xi - \tilde{\xi}\|_{M_m(L_p^\mu(\mu))} < \epsilon \). Finally,

\[
\|\tilde{\xi}\| = \left\| \sum_{t=1}^{k(m)} \alpha^t_{ij} f_t \right\|_{B(\ell_p^{m}, \ell_p^{m}(L_p(\mu)))}
= \sup_{\lambda \in \ell_p^{m}} \left( \sum_{i=1}^{m} \left( \sum_{j=1}^{m} \left( \sum_{t=1}^{k(m)} \alpha^t_{ij} \lambda_j f_t \right)^p \right)^{1/p} \right)^{1/p}
= \sup_{\lambda \in \ell_p^{m}} \left( \sum_{i=1}^{m} \left( \sum_{j=1}^{m} \left( \sum_{t=1}^{k(m)} \alpha^t_{ij} \lambda_j \right) f_t \right)^p \right)^{1/p}
= \sup_{\lambda \in \ell_p^{m}} \left( \sum_{i=1}^{m} \sum_{j=1}^{m} \left| \sum_{t=1}^{k(m)} \alpha^t_{ij} \lambda_j \right|^p \right)^{1/p}
= \sup_{\lambda \in \ell_p^{m}} \left( \sum_{i=1}^{m} \sum_{j=1}^{m} \left| \sum_{t=1}^{k(m)} \alpha^t_{ij} \lambda_j \right|^p \right)^{1/p}
\]

\(^{14}\)See [LP68]. We will keep refer this property to the rigid \( L_p \)-structure of \( L_p(\mu) \).
Proposition 1.6.11. For any $L_p(\mu)$ and $L_p(\nu)$, there exists a natural $p$-completely isometric isomorphism

$$CB_p(L_c^\mu(\mu), L_c^\nu(\nu)) = B(L_p(\mu), L_p(\nu)).$$

Proof. Fix $n \in \mathbb{N}$. For any matrix $T = [T_{kl}] \in M_n(B(L_p(\mu), L_p(\nu))) = B(\ell_p^\mu(L_p(\mu)), \ell_p^\nu(L_p(\nu)))$, the corresponding mapping $\tilde{T} \in M_n(CB_p(L_c^\mu(\mu), L_c^\nu(\nu))) = CB_p(L_c^\mu(\mu), M_n(L_c^\nu(\nu)))$ is defined by $\tilde{T}(\xi) = [T_{kl}(\xi)]$ for all $\xi \in L_c^\mu(\mu)$. We wish to show that $\|\tilde{T}\|_{cb} = \|T\|$. Fix $\xi = [\xi_{ij}] \in M_n(L_c^\mu(\mu))$ and let $\epsilon > 0$. As in the proof of Lemma 1.6.10, there exist a $k(m) \in \mathbb{N}$, a subspace $F$ of $L_p(\mu)$ which is isometrically isomorphic to $\ell_p^{k(m)}$, and $\{\tilde{\xi}_{ij}\}_{i,j=1}^m \subseteq F$ such that

$$\|\hat{\xi}_{ij} - \tilde{\xi}_{ij}\| < \min\left\{\frac{\epsilon}{nm\|T\|}, \frac{\epsilon}{m}\right\}$$

for each $i, j$. In particular,

$$\|\hat{\xi} - \tilde{\xi}\| < \epsilon$$

by Lemma 1.6.9, where $\hat{\xi} = [\tilde{\xi}_{ij}]$. Let $\{f_1, \ldots, f_{k(m)}\} \subseteq F$ correspond to the canonical basis of $\ell_p^{k(m)}$ and write $\hat{\xi}_{ij} = \sum_{t=1}^{k(m)} \alpha_t^i f_t$, then $\tilde{T}_m(\tilde{\xi}) = \sum_{t=1}^{k(m)} [T_{kl}(\xi_t)] \otimes \alpha^t$. Since $\|T_{kl}\| \leq \|T\|$ for each $k$ and $l$, from (1.10) above and Lemma 1.6.9, we get

$$\|\tilde{T}_m(\tilde{\xi}) - \tilde{T}_m(\hat{\xi})\| < \epsilon.$$  (1.12)

Again as in the proof of Lemma 1.6.10, there exist an $l(m) \in \mathbb{N}$, a subspace $G$ of $L_p(\nu)$ which is isometrically isomorphic to $\ell_p^{l(m)}$, and $\{\eta_{kl}^t\}_{k,l,t} \subseteq G$ such that

$$\|T_{kl}(f_t) - \eta_{kt}^l\| < \min\left\{\frac{\epsilon}{nk(m)^{1/p}}, \frac{\epsilon}{n\sum_{t=1}^{k(m)} \|\alpha^t\|}\right\}$$

for all $k, l$, and $t$. In particular, by Lemma 1.6.9,

$$\|[T_{kl}(f_t)] - [\eta_{kt}^l]\| < \frac{\epsilon}{\sum_{t=1}^{k(m)} \|\alpha^t\|}$$

for each $t$. (1.14)
Let \( \{g_1, \ldots, g_{l(m)}\} \subseteq G \) correspond to the canonical basis of \( \ell_p^{l(m)} \) and write \( \eta_{kl}^t = \sum_{s=1}^{l(m)} \beta_{kl}^{st} g_s \) so that for each \( t \)

\[
[\eta_{kl}^t] = \sum_{s=1}^{l(m)} g_s \otimes \beta_{kl}^{st},
\]

where \( \beta^{st} = [\beta_{kl}^{st}] \in M_n \). Define \( T_0 : \ell_p^n(F) \to \ell_p^n(G) \) by

\[
\varphi := \begin{bmatrix}
\phi_1 \\
\vdots \\
\phi_n
\end{bmatrix} \quad \xrightarrow{T_0} \quad \begin{bmatrix}
\sum_{t,s} \gamma_{t}^l \eta_{kl}^t \\
\vdots \\
\sum_{t,s} \gamma_{n}^l \eta_{nl}^t
\end{bmatrix},
\]

where \( \varphi_t = \sum_{t=1}^{k(m)} \gamma_{l}^t f_t \). Note that

\[
\|T_0(\varphi)\|^p = \sum_{k=1}^{n} \left\| \sum_{t,s} \gamma_{t}^l \eta_{kl}^t \right\|^p
\]

\[
= \sum_{k=1}^{n} \left\| \sum_{s,t} \gamma_{t}^l \beta_{kl}^{st} g_s \right\|^p
\]

\[
= \sum_{k=1}^{n} \sum_{s,t} \left\| \sum_{t,s} \beta_{kl}^{st} \gamma_{t}^l \right\|^p
\]

\[
= \|\beta\gamma\|^p,
\]

where

\[
\beta = \begin{bmatrix}
\beta_{11} & \cdots & \beta_{1k(m)} \\
\vdots & \ddots & \vdots \\
\beta_{l(m)1} & \cdots & \beta_{l(m)k(m)}
\end{bmatrix}, \quad \gamma_t = \begin{bmatrix}
\gamma_{1}^t \\
\vdots \\
\gamma_{n}^t
\end{bmatrix}, \quad \text{and} \quad \gamma = \begin{bmatrix}
\gamma_{1} \\
\vdots \\
\gamma_{k(m)}
\end{bmatrix}.
\]

Since \( \|\varphi\| = \left( \sum_{t,s} |\gamma_{l}^t|^p \right)^{1/p} = \|\gamma\| \), it follows that

\[
\|T_0\| = \|\beta\|. \tag{1.16}
\]
Now for all $\varphi \in \ell_p^n(F)$, we obtain

$$
||T(\varphi) - T_0(\varphi)||^p = \left\| \left[ \sum_{t,l} \gamma_l^t(T_{2t}(f_t) - \eta^t_{kl}) \right] \right\|^p
= \left\| \sum_{k=1}^n \left\| \sum_{t,l} \gamma_l^t(T_{kl}(f_t) - \eta^t_{kl}) \right\|^p
\leq \sum_{k=1}^n \left( \sum_{t,l} \gamma_l^t \left\| T_{kl}(f_t) - \eta^t_{kl} \right\| \right)^p
$$

(Hölder’s inequality)

(by (1.13)) \leq \left\| \varphi \right\|^p \sum_{k=1}^n \left( n k(m) \frac{\epsilon}{n^p k(m)} \right)^{p/p'}
= \epsilon^p ||\varphi||^p

and hence

$$
||T_0|| \leq ||T||_{\ell_p^n(F)} + ||T||_{\ell_p^n(F)} - T_0 || \leq ||T|| + \epsilon.
$$

(1.17)

Since

$$
||\tilde{T}_m(\tilde{\xi}) - \sum_{t=1}^{k(m)} [\eta^t_{kl}] \otimes \alpha^t || = \left\| \sum_{t=1}^{k(m)} [T_{kl}(f_t)] \otimes \alpha^t - \sum_{t=1}^{k(m)} [\eta^t_{kl}] \otimes \alpha^t \right\|
\leq \sum_{t=1}^{k(m)} ||[T_{kl}(f_t)] - [\eta^t_{kl}] || \left\| \alpha^t \right\|
$$

(by (1.14)) \leq \epsilon,

we finally have

$$
||\tilde{T}_m(\tilde{\xi})|| \leq ||\tilde{T}_m(\tilde{\xi})|| + ||\tilde{T}_m(\tilde{\xi}) - \tilde{T}_m(\tilde{\xi})||
\leq \sum_{t=1}^{k(m)} [\eta^t_{kl}] \otimes \alpha^t + \left\| \tilde{T}_m(\tilde{\xi}) - \sum_{t=1}^{k(m)} [\eta^t_{kl}] \otimes \alpha^t \right\| + \epsilon
\leq \sum_{t=1}^{k(m)} [\eta^t_{kl}] \otimes \alpha^t + 2\epsilon
\leq \sum_{t=1}^{k(m)} \sum_{s=1}^{l(m)} g_s \otimes \beta^s_t \otimes \alpha^t || + 2\epsilon
\leq \left\| \sum_{t=1}^{k(m)} \beta^t_l \otimes \alpha^t \right\| + 2\epsilon
\leq \left\| \beta^{11} \ldots \beta^{1k(m)} \right\| \left\| \alpha^1 \right\| + 2\epsilon
\leq \left\| \beta^{l(m)1} \ldots \beta^{l(m)k(m)} \right\| \left\| \alpha^{k(m)} \right\| + 2\epsilon
\leq \left\| \beta^{l(m)1} \ldots \beta^{l(m)k(m)} \right\| \left\| \alpha^{k(m)} \right\| + 2\epsilon
$$

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(by (1.16)) \[ = \|T_0\| \|\bar{\xi}\| + 2\varepsilon. \]

(by (1.17) and (1.11)) \[ \leq (\|T\| + \varepsilon)(\|\xi\| + \varepsilon) + 2\varepsilon. \]

This shows that \( \|\bar{T}\|_{pcb} \leq \|T\| \). For the reverse inequality, fix

\[ \xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \in \ell_p^n(L_p(\mu)), \]

then there exist a \( q(n) \in \mathbb{N} \), a subspace \( F \) of \( L_p(\mu) \) which is isometrically isomorphic to \( \ell_p^{q(n)} \), \( \{f_1, \ldots, f_{q(n)}\} \subseteq F \) corresponding to the canonical basis in \( \ell_p^{q(n)} \), and a matrix \( \delta = [\delta_{lt}] \in M_{q(n),n} \) such that \( \|\xi_l - \bar{\xi}_l\| < \min \left\{ \frac{\varepsilon}{n^{1/p}}, \frac{\varepsilon}{n^{1/p}\|T\|} \right\} \) for all \( l \), where \( \bar{\xi}_l = \sum_{t=1}^{q(n)} \delta_{lt} f_t \). Note that

\[ \|\xi - \bar{\xi}\| \leq \min \left\{ \varepsilon, \frac{\varepsilon}{\|T\|} \right\}. \]  

(1.19)

Put

\[ \check{\xi} = \begin{bmatrix} \check{\xi}_1 \\ \vdots \\ \check{\xi}_n \end{bmatrix} \quad \text{and} \quad \delta_t = \begin{bmatrix} \delta_{t1} \\ \vdots \\ \delta_{tn} \end{bmatrix}, \]

then by (1.19)

\[ \left\| T(\xi) - T(\check{\xi}) \right\| \leq \|T\| \|\xi - \check{\xi}\| < \varepsilon \]  

(1.20)

and

\[ T(\check{\xi}) = \begin{bmatrix} \sum_{t=1}^{n} T_{1t}(\check{\xi}_t) \\ \vdots \\ \sum_{t=1}^{n} T_{nt}(\check{\xi}_t) \end{bmatrix} \in M_{n, q(n)} \]

\[ = \begin{bmatrix} T(f_1) \cdots T(f_{q(n)}) \end{bmatrix} \in M_{n, q(n)} \cdot \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_{q(n)} \end{bmatrix} \]
Therefore,

\[ \|T(\tilde{\xi})\| \leq \left\| \tilde{T}_{1,q(n)}( [ f_1 \cdots f_{q(n)} ] ) \right\| \left\| \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_{q(n)} \end{bmatrix} \right\| \]

(1.21)

which completes the proof. \qed

**Corollary 1.6.12.** For any \( L_p(\mu) \) and \( L_p(\nu) \), we have the \( p \)-completely isometric isomorphisms

\[ (L_p^c(\mu))' = L_p^c(\mu), \quad (L_p^c(\nu))' = L_p^c(\mu), \quad \text{and} \quad \text{CB}_p(L_p^r(\mu), L_p^r(\nu)) = B(L_p(\nu), L_p(\mu)). \]

**Proof.** By Proposition 1.6.11,

\[ (L_p^c(\mu))' = \text{CB}_p(L_p^c(\mu), \mathbb{C}) = B(L_p(\mu), \mathbb{C}) = B(L_p^r(\mu), \mathbb{C}) = L_p^c(\mu). \]

By Lemma 1.5.2 and above calculation,

\[ (L_p^c(\mu))' = (L_p^c(\mu))^\prime = L_p^c(\mu). \]
Finally, by Proposition 1.5.6,

\[ CB_p(L_{p'}^r(\mu), L_{p'}^r(\nu)) = CB_p((L_{p'}^c(\mu))', (L_{p'}^c(\nu))') = CB_p(L_p^c(\nu), L_p^c(\mu)) = B(L_p(\nu), L_p(\mu)). \]

We close this section with one more example. Let \( G \) be an index set. Then \( \ell_\infty(G) \) has a natural \( p \)-operator space structure inherited from the embedding \( \ell_\infty(G) \subseteq B(\ell_p(G)) \). We will always assume that \( \ell_\infty(G) \) is given this \( p \)-operator space structure unless stated otherwise. Now give \( \ell_1(G) \) the dual \( p \)-operator space structure, that is, \( \ell_1(G) \) is given the \( p \)-operator space structure regarded as a subspace of \( \ell_\infty(G)' \).

**Proposition 1.6.13.** \( \ell_1(G)' = \ell_\infty(G) \) \( p \)-completely isometrically.

**Proof.** We claim that \( \ell_\infty(G) \) is weak*-closed in \( B(\ell_p(G)) \); once this is done, the result follows from [Daw10, Proposition 5.5]. To this end, suppose that \( \{ f_i \} \subseteq \ell_\infty(G) \) and that \( f_i \to T \) in the weak*-topology. If \( s, t \in G \) with \( s \neq t \), then

\[ T_{st} = \langle \delta_p^s, T\delta_p^t \rangle = \lim_i f_i(s)\delta_p^t(s) = 0. \]

This shows that \( T \subseteq \ell_\infty(G) \). \( \square \)

Just like \( p = 2 \) case, we have the following characterization of the norm in \( M_n(\ell_\infty(G)) \).

**Proposition 1.6.14.** If \( [u_{ij}] \in M_n(\ell_\infty(G)) \), then \( \|[u_{ij}]\|_{M_n(\ell_\infty(G))} = \sup_{s \in G} \|[u_{ij}(s)]\|_{B(\ell_p^n)}. \)

**Proof.** For all \( s \in G \),

\[ \|[u_{ij}(s)]\|_{B(\ell_p^n)} = \sup \left\{ \sum_{i,j=1}^{n} a_{i}u_{ij}(s)b_{j} : \sum_{i=1}^{n} |a_{i}|^{p'} \leq 1, \sum_{j=1}^{n} |b_{j}|^{p} \leq 1 \right\}. \]

Since

\[ \begin{align*}
\sum_{i,j=1}^{n} a_{i}u_{ij}(s)b_{j} &= \sum_{i,j=1}^{n} \sum_{x \in G} a_{i}\delta_{p}^{x}(x)u_{ij}(x)b_{j}\delta_{p}^{x}(x) \\
&= \left\langle [a_{i}\delta_{p}^{x}], [u_{ij}][b_{j}\delta_{p}^{x}] \right\rangle,
\end{align*} \]

we get

\[ \left| \sum_{i,j=1}^{n} a_{i}u_{ij}(s)b_{j} \right| \leq \|[u_{ij}]\|_{M_n(\ell_\infty(G))}. \]
For the converse, suppose that \( g = (g_i)_{i=1}^n \in \ell_p^\otimes \ell_p'(G) \) with \( \sum_{i,x} |g_i(x)|^{p'} \leq 1 \). Similarly, suppose that \( f = (f_j)_{j=1}^n \in \ell_p^\otimes \ell_p(G) \) with \( \sum_{j,x} |f_j(x)|^p \leq 1 \). Then

\[
\left| \sum_{i,j} \sum_x g_i(x) u_{ij}(x) f_j(x) \right|
\leq \sum_x \left| \sum_{i,j} g_i(x) u_{ij}(x) f_j(x) \right|
\leq \sum_x \| [u_{ij}(x)] \|_{g(\ell_p^n)} \left( \sum_{i=1}^n |g_i(x)|^{p'} \right)^{1/p'} \left( \sum_{j=1}^n |f_j(x)|^p \right)^{1/p}
\leq \sup_{s \in G} \| [u_{ij}(s)] \|_{g'(\ell_p^n)}
\]

and this completes the proof. \( \Box \)
Chapter 2

Tensor Products of \( p \)-Operator Spaces

In this chapter, we study various tensor products on \( p \)-operator spaces. We mainly focus on the following three tensor products: \( p \)-projective tensor product, \( p \)-injective tensor product, and \( p \)-Haagerup tensor product.

2.1 \( p \)-Projective Tensor Product

The main source for this section is [Daw10].

**Definition 2.1.1.** Let \( X, Y \) be \( p \)-operator spaces. For \( u \in M_n(X \otimes Y) \), let

\[
\|u\|_{\wedge p} = \inf \{ \|\alpha\| \|v\| \|w\| \|\beta\| : u = \alpha (v \otimes w) \beta \},
\]

where the infimum is taken over \( r, s \in \mathbb{N}, \alpha \in M_{n,r \times s}, v \in M_r(X), w \in M_s(Y), \) and \( \beta \in M_{r \times s,n} \).

It was Daws who first defined and studied the \( p \)-projective tensor product. Note that \( \|\cdot\|_{\wedge p} \) gives the algebraic tensor product \( X \otimes Y \) a \( p \)-operator space structure [Daw10, Proposition 4.8]. Furthermore, \( \|\cdot\|_{\wedge p} \) is the largest subcross \( p \)-operator space norm on \( X \otimes Y \) in the sense that \( \|x \otimes y\| \leq \|x\|_p \|y\|_p \) for all \( x \in M_r(X) \) and all \( y \in M_s(Y) \) [Daw10, Proposition 4.8]. The \( p \)-operator space projective tensor product is defined to be the completion of \( X \otimes Y \) with respect to this norm and is denoted by \( X ^{\wedge p} \otimes Y \).

**Remark 2.1.2.**

1. One can show that \( p \)-operator space projective tensor product is commutative, i.e., \( X ^{\wedge p} \otimes Y = Y ^{\wedge p} \otimes X \) \( p \)-completely isometrically.

2. By universality of the Banach space projective tensor product \( \bigotimes \) [BLM04, A.3.3], we have

\[
\|u\|_{\wedge p} \leq \|u\|_{\pi}
\]

for all \( u \in X \otimes Y \).
Let \( V, W \) and \( Z \) be \( p \)-operator spaces, and let \( \psi : V \times W \to Z \) be a bilinear map. Define bilinear maps 
\[ \psi_{r,s,t,u} : M_{r,s}(V) \times M_{t,u}(W) \to M_{r \times t, s \times u}(Z), \quad (v, w) \mapsto (\psi(v_{i,j}, w_{k,l})), \]
and let \( \psi_{r,s} = \psi_{r,r,s,s} \). Finally define 
\[ \| \psi \|_{\text{Jpcb}} = \sup\{ \| \psi_{r,s} \| : r, s \in \mathbb{N} \}. \]

We say that \( \psi \) is jointly \( p \)-completely bounded (respectively, jointly \( p \)-completely contractive) if \( \| \psi \|_{\text{Jpcb}} < \infty \) (respectively, \( \| \psi \|_{\text{Jpcb}} \leq 1 \)). The space of all jointly \( p \)-completely bounded maps from \( V \times W \) to \( Z \) will be denoted by \( \mathcal{CB}_p(V \times W, Z) \) and this space can be turned into a \( p \)-operator space in the same way as for \( \mathcal{CB}_p(V, W) \). Here we collect some results on the \( p \)-projective tensor product for convenience.

**Proposition 2.1.3.** [Daw10, Proposition 4.9] Let \( X, Y, \) and \( Z \) be \( p \)-operator spaces. Then we have natural \( p \)-completely isometric identifications 
\[ \mathcal{CB}_p(X \hat{\otimes} Y, Z) = \mathcal{CB}_p(X \times Y, Z) = \mathcal{CB}_p(X, \mathcal{CB}_p(Y, Z)). \]

In particular, 
\[ (X \hat{\otimes} Y)' = \mathcal{CB}_p(X, Y'). \]

For \( p \)-operator spaces \( X \) and \( Y \), let \( \mathcal{CB}_p^*(X', Y') \) denote the space of all weak*-weak*-continuous \( p \)-completely bounded maps between \( X' \) and \( Y' \).

**Corollary 2.1.4.** Let \( X \) be a \( p \)-operator space on \( L_p \) space and \( Y \) be a \( p \)-operator space. Then we have 
\[ \mathcal{CB}_p(X, Y') = \mathcal{CB}_p^*(X'', Y') \quad (2.1) \]
\( p \)-completely isometrically.

**Proof.** Let \( u \in \mathcal{CB}_p(X, Y') \). It is easy to verify that \( \tilde{u} = \kappa_{Y'} \circ u'' : X'' \to Y' \) defines a weak*-weak*-continuous \( p \)-completely bounded extension of \( u \) with \( \| \tilde{u} \|_{\text{pceb}} \leq \| u \|_{\text{pceb}} \). In fact, by Goldstine’s Theorem, \( \tilde{u} \) is a unique weak*-weak*-continuous extension of \( u \). Since \( X \subseteq X'' \) \( p \)-completely isometrically, we also have \( \| \tilde{u} \|_{\text{pceb}} \geq \| u \|_{\text{pceb}} \). If \( T \in \mathcal{CB}_p^*(X'', Y') \), then we must have \( T = T|_X \) and this shows that (2.1) holds.
isometrically. To show that (2.1) is a $p$-complete isometry, let us fix $n \in \mathbb{N}$. Then we have isometries

$$M_n(CB_p(X, Y')) = CB_p(X, M_n(Y')) = CB_p(X, CB_p(Y, M_n)) = CB_p(X, (N_n ^\wedge p Y)').$$

Since we already showed that (2.1) holds isometrically, we have

$$CB_p(X, (N_n ^\wedge p Y')) = CB_p(X'', (N_n ^\wedge p Y'))$$

isometrically and it follows that

$$M_n(CB_p(X, Y')) = CB_p(X'', (N_n ^\wedge p Y')) = CB_p(X'', CB_p(Y, M_n)) = CB_p(X'', M_n(Y')) = M_n(CB_p(X'', Y'))$$

isometrically. This completes the proof.

As in operator spaces, the $p$-operator space projective tensor product is projective in the following sense:

***Proposition 2.1.5.*** [Daw10, Proposition 4.10] Let $X, \tilde{X}, Y$, and $\tilde{Y}$ be $p$-operator spaces. If $u : X \to \tilde{X}$ and $v : Y \to \tilde{Y}$ are $p$-complete quotient maps, then $u \otimes v$ extends to a $p$-complete quotient map $u \otimes v : X ^\wedge p Y \to \tilde{X} ^\wedge p \tilde{Y}$.

In operator spaces, the trace class operators on a Hilbert space $H$ can be expressed in terms of operator space projective tensor product. To be more precise, we have the following natural isometries

$$(H^c)^{\otimes} H^c \cong H^\pi \otimes H \cong T(H),$$

where $T(H)$ denotes the space of all trace class operators on $H$ [ER00, Proposition 8.2.1]. We have the following $p$-analogue of this result.

***Proposition 2.1.6.*** For a measure $\mu$, we have an isometric isomorphism $N(L_p(\mu)) \cong L_p'(\mu) ^\wedge p L_p^c(\mu)$.

**Proof.** Note that the adjoint of the canonical contraction

$$\varphi : N(L_p(\mu)) \overset{(1.2)}{=} L_p'(\mu) ^\pi L_p(\mu) \to L_p'(\mu) ^\wedge p L_p^c(\mu)$$

is the natural mapping $CB_p(L_p^c(\mu)) \to B(L_p(\mu))$, which is an isometric surjection by Proposition 1.6.11. The

---

1 Note that $M_n(Y') = CB_p(Y, M_n)$ $p$-completely isometrically.
2 See Remark 2.1.2.
result follows using the arguments in [ER00, §A.2].

Since \( p \)-operator space projective tensor product resembles the Banach space projective tensor product in many ways, we can expect that the \( p \)-operator space \( \mathcal{N}(L_p(\mu)) \) to behave well with respect to the \( p \)-operator space projective tensor product. Our next goal is to make this clear. Let \( \mathcal{N}_n = \mathcal{N}(\ell^n_p) \) so that \( B(\ell^n_p)' = K(\ell^n_p)' = \mathcal{N}_n \) and \( \mathcal{N}_n' = B(\ell^n_p) \) \( p \)-completely isometrically. For a \( p \)-operator space \( V \), we wish to study the Banach space structure of \( \mathcal{N}_n \wedge_p \otimes V \).

**Proposition 2.1.7.** For any \( p \)-operator space \( V \), we have a natural isometry

\[
\mathcal{N}_n(V) \cong \mathcal{N}_n \wedge_p \otimes V,
\]

where \( \mathcal{N}_n(V) \) denotes the normed space in Definition 1.6.6.

**Proof.** By Proposition 2.1.3, we know that \( (\mathcal{N}_n \wedge_p \otimes V)' \) is isometrically isomorphic to \( CB_p(V, M_n) = M_n(V') \).

Let’s examine the duality between \( \mathcal{N}_n \wedge_p \otimes V \) and \( M_n(V') \) more closely. Note that every element in \( \mathcal{N}_n \), being a linear map from \( \ell^n_p \) to \( \ell^n_p \), can be written as a linear combination of \( \epsilon_{ij} \), where \( \{\epsilon_{ij}\}_{i,j=1}^n \) denotes a standard basis for \( M_n \). If \( v = \sum_{i,j} \epsilon_{ij} \otimes v_{ij} \in \mathcal{N}_n \wedge_p \otimes V \) and \( f = [f_{kl}] \in M_n(V') \), then

\[
\left\langle f; \sum_{i,j} \epsilon_{ij} \otimes v_{ij} \right\rangle_{M_n(V'), M_n \wedge_p \otimes V} = \sum_{i,j} \langle \epsilon_{ij}, [f_{kl}(v_{ij})]_{k,l=1}^n \rangle_{\mathcal{N}_n, M_n}
\]

\[
= \sum_{i,j} f_{ij}(v_{ij})
\]

\[
= \langle f, v \rangle_{M_n(V'), \mathcal{N}_n(V)}.
\]

Therefore,

\[
\|v\|_{\mathcal{N}_n \wedge_p \otimes V} \leq \sup \left\{ \left\| \left\langle f; \sum_{i,j} \epsilon_{ij} \otimes v_{ij} \right\rangle_{M_n(V'), \mathcal{N}_n \wedge_p \otimes V} \right\| : \|f\| \leq 1, f \in M_n(V') \right\}
\]

\[
= \sup \left\{ \|f\|_{M_n(V')} : \|f\| \leq 1, f \in M_n(V') \right\}
\]

\[
= \|v\|_{\mathcal{N}_n(V)}.
\]

\[ ^3 \text{That is, if } \psi : V \to W \text{ is a bounded linear map between two Banach spaces } V \text{ and } W, \text{ then } \psi \text{ is an isometry if and only if } \psi' \text{ is a quotient mapping. Under the same assumption, } \psi \text{ is a quotient mapping if and only if } \psi' \text{ is an isometry.} \]
2.2 $p$-Haagerup Tensor Product

In [LeM96], Le Merdy considered the $p$-operator space Haagerup tensor product. Let $X, Y$ be $p$-operator spaces. Given $v = [v_{ir}] \in M_{n,k}(X)$ and $w = [w_{rj}] \in M_{k,m}(Y)$, define $v \odot w = [\sum_{r=1}^{k} v_{ir} \otimes w_{rj}] \in M_{n,m}(X \otimes Y)$.

**Definition 2.2.1.** Let $X, Y$ be $p$-operator spaces. For $u \in M_{n,m}(X \otimes Y)$, we define a norm

$$
\|u\|_{h_p} = \inf \{\|v\|\|w\| : v \in M_{n,k}(X), w \in M_{k,m}(Y), u = v \odot w\}.
$$

The $p$-operator space Haagerup tensor product is defined to be the completion of $(X \otimes Y, \| \cdot \|_{h_p})$.

**Remark 2.2.2.**

1. The $p$-Haagerup tensor product $^h \otimes$ is associative [LeM96, Remark 2.5].

2. $\| \cdot \|_{h_p}$ is a subcross norm, because for $v \in M_r(V)$ and $w \in M_s(W)$, $v \otimes w = (v \otimes I_s) \odot (I_r \otimes w)$. Hence it follows from [Daw10, Proposition 4.8] that $\| \cdot \|_{h_p} \leq \| \cdot \|_{\wedge_p}$ on the algebraic tensor product $V \otimes W$.

Just in operator spaces, sometimes $^h \otimes$ and $^\wedge \otimes$ produce the same norm. To give an example, we need a lemma.

**Lemma 2.2.3.** Let $V$ and $W$ are $p$-operator spaces. Let $v = [v_{ij}] \in M_{n,m}(V)$ and $w = [w_{kl}] \in M_{m,n}(W)$ with $\|w_{kl}\| \leq \epsilon$ for all $k,l$, then $\|v \otimes w\|_{\wedge_p} \leq c n^2 m \|v\|$.

**Proof.** Observe that

$$
v \otimes w = \sum_{i,l=1}^{n} \sum_{k=1}^{m} 1_{i,k} (v_{ik} \otimes w_{kl}) \begin{bmatrix} 0 & \cdots & 0 & 1_{i,k} & 0 & \cdots & 0 \end{bmatrix}_{1 \times n}.
$$

Since $\|v_{ik}\| \leq \|v\|$ for all $i,k$, we get

$$
\|v \otimes w\|_{\wedge_p} \leq \sum_{i,l=1}^{n} \sum_{k=1}^{m} \|v\| \epsilon = n^2 m \|v\| \epsilon.
$$
Proposition 2.2.4. Let $V$ be a $p$-operator space and $\mu$ a measure. Then we have the $p$-complete isometries

$$V \overset{\overset{h_p}{\otimes}}{\overset{h_p}{\otimes}} L_p^c(\mu) = V \overset{\overset{\Lambda_p}{\otimes}}{\overset{\Lambda_p}{\otimes}} L_p^c(\mu), \quad L_p'(\mu) \overset{\overset{h_p}{\otimes}}{\overset{h_p}{\otimes}} V = L_p'(\mu) \overset{\overset{\Lambda_p}{\otimes}}{\overset{\Lambda_p}{\otimes}} V.$$

Proof. We prove only the first identification; the second one is similar. Suppose $u \in M_n(V \otimes L_p^c(\mu))$ with $\|u\|_{h_p} < 1$. By Remark 2.2.2, it suffices to show that $\|u\|_{\Lambda_p} < 1$. There exist an $m \in \mathbb{N}$, $v = [v_{ij}] \in M_{n,m}(V)$, and $\xi = [\xi_{kl}] \in M_{m,n}(L_p^c(\mu))$ such that $u = v \otimes \xi$ with $\|v\|, \|\xi\| < 1$. Let $\epsilon > 0$. As in the proof of Lemma 1.6.10, there exist a $k(m) \in \mathbb{N}$, a subspace $F$ of $L_p(\mu)$ which is isometrically isomorphic to $\ell_p^k(m)$, and $\{\tilde{\xi}_{kl}\} \subseteq F$ such that

$$\|\xi_{kl} - \tilde{\xi}_{kl}\| < \min\left\{\frac{\epsilon}{2m^{1/p}n^{1/p'}}, \frac{\epsilon}{2n^2m}\right\} \quad \text{for each } k,l. \quad (2.2)$$

Let $\{f_1, \cdots, f_{k(m)}\} \subseteq F$ correspond to the canonical basis of $\ell_p^k(m)$ and write $\tilde{\xi}_{kl} = \sum_{t=1}^{k(m)} \alpha^t_{kl} f_t$. If we let $\tilde{\xi} = [\tilde{\xi}_{kl}] \in M_{m,n}(L_p^c(\mu))$, then

$$\tilde{\xi} = \sum_{t=1}^{k(m)} f_t \otimes \alpha^t = [f_1 \cdots f_{k(m)}] \otimes \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^{k(m)} \end{bmatrix}$$

and by Lemma 1.6.9 and (2.2), we obtain $\|\xi - \tilde{\xi}\| < \frac{\epsilon}{2}$. In particular,

$$\left\| \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^{k(m)} \end{bmatrix} \right\| = \|\tilde{\xi}\| < \|\xi\| + \frac{\epsilon}{2} < 1 + \frac{\epsilon}{2}.$$

If we let $\tilde{u} = [\tilde{u}_{kl}] = v \otimes \tilde{\xi} \in M_n(V \otimes L_p^c(\mu))$, then

$$\tilde{u} = \left[ \sum_{j,t} \alpha^t_j v_{ij} \otimes f_t \right]_{(i,l)} = (v \otimes [f_1 \cdots f_{k(m)}]) \begin{bmatrix} \tilde{\alpha}^1 \\ \vdots \\ \tilde{\alpha}^m \end{bmatrix},$$

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where
\[ \tilde{\alpha}^k = \begin{bmatrix} \alpha_{1k} & \cdots & \alpha_{kn} \\ \vdots & \ddots & \vdots \\ \alpha_{k1} & \cdots & \alpha_{kn} \end{bmatrix} \in M_{k(m),n} \]
for each \( k = 1, \cdots, m \). Since
\[ \|f_1 \cdots f_{k(m)}\| = 1 \]
and
\[ \left\| \begin{bmatrix} \tilde{\alpha}^1 \\ \vdots \\ \tilde{\alpha}^m \end{bmatrix} \right\| = \left\| \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^{k(m)} \end{bmatrix} \right\|, \]
we have
\[ \|\tilde{u}\|_p \leq \|v\|_p \|\tilde{\xi}\| < 1 + \frac{\epsilon}{2}. \]
Since \( u - \tilde{u} = v \odot (\xi - \tilde{\xi}) \), by Lemma 2.2.3 and (2.2), we have
\[ \|u - \tilde{u}\|_p < \frac{\epsilon}{2}, \]
so \( \|u\|_p \leq \|\tilde{u}\|_p + \|u - \tilde{u}\|_p < 1 + \epsilon \). Since \( \epsilon \) is arbitrary, we are done.

**Corollary 2.2.5.** For a measure \( \mu \), we have an isometric isomorphism \( \mathcal{N}(L_p(\mu)) \cong L_p^c(\mu) \otimes h_p L^c_p(\mu) \).

**Proof.** This follows immediately from Propositions 2.1.6 and 2.2.4.

**Corollary 2.2.6.** Given a \( p \)-operator space \( V \) and measures \( \mu \) and \( \nu \), we have the \( p \)-completely isometric isometry
\[ CB_p(V, B(L_p(\mu), L_p(\nu))) = (L_p^c(\nu) \otimes V \otimes L_p^c(\mu))^\prime. \]

**Proof.** Since \( p \)-Haagerup tensor product is associative and \( p \)-projective tensor product is commutative, by Proposition 2.2.4,
\[ L_p^c(\nu) \otimes V \otimes L_p^c(\mu) = L_p^c(\nu) \otimes L_p^c(\mu) \otimes V = V \otimes L_p^c(\mu) \otimes L_p^c(\nu). \]
Therefore, by Proposition 2.1.3, Corollary 1.6.12, and Proposition 1.6.11,
\[ (L_p^c(\nu) \otimes V \otimes L_p^c(\mu))^\prime = (V \otimes L_p^c(\mu) \otimes L_p^c(\nu))^\prime = CB_p(V, (L_p^c(\mu) \otimes L_p^c(\nu))^\prime) = CB_p(V, CB_p(L_p^c(\mu), (L_p^c(\nu))^\prime)) = CB_p(V, CB_p(L_p^c(\mu), L_p^c(\nu))) \]
\[ \mathcal{B}(X, Y) \cong ((Y')^r \otimes X^c)' \].

If we replace \( L_p \) spaces by \( SQ_p \) spaces in Corollary 2.2.6, then we have a slightly weaker result.

**Lemma 2.2.7.** If \( X \) and \( Y \) are \( SQ_p \) spaces, then we have the isometric isomorphism

\[ \mathcal{B}(X, Y) \cong ((Y')^r \otimes X^c)' \].

**Proof.** Define

\[ \Phi : \mathcal{B}(X, Y) \to ((Y')^r \otimes X^c)' \] by \( \Phi(u)(y' \otimes x) = \langle u(x), y' \rangle, \quad u \in \mathcal{B}(X, Y), \ x \in X, \ y' \in Y' \)

and

\[ \Psi : ((Y')^r \otimes X^c)' \to \mathcal{B}(X, Y) \] by \( \langle \Psi(f)(x), y' \rangle = f(y' \otimes x), \quad f \in ((Y')^r \otimes X^c)', \ x \in X, \ y' \in Y' \).

Then it is easy to show that \( \Psi \) and \( \Phi \) are inverses of each other. For all \( u \in \mathcal{B}(X, Y) \),

\[
\| u \| = \sup \{ |\langle u(x), y' \rangle| : \| x \| \leq 1, \| y' \| \leq 1, x \in X, y' \in Y' \} \\
= \sup \{ \| \Phi(u)(y' \otimes x) \| : \| x \| \leq 1, \| y' \| \leq 1, x \in X, y' \in Y' \} \\
\leq \| \Phi(u) \|,
\]

where the last inequality comes from the fact that \( p \)-Haagerup tensor product is a subcross norm. Now suppose \( z \in (Y')^r \otimes X^c \) with \( \| z \|_{hp} \leq 1 \). Let \( \epsilon > 0 \), then by definition of the \( p \)-Haagerup tensor product, there exist \( y'_1, \ldots, y'_n \in Y' \) and \( x_1, \ldots, x_n \in X \) such that \( z = \sum_{i=1}^n y'_i \otimes x_i \) and

\[
\| [y'_1 \cdots y'_n] \| \leq \left( \sum_{i=1}^n \| y'_i \|^{p'} \right)^{1/p'} \left( \sum_{i=1}^n \| x_i \|^{p} \right)^{1/p} < 1 + \epsilon.
\]

\[ \text{Recall that every } SQ_p \text{ space is reflexive. See Remark 1.2.4.} \]
By Hölder’s inequality,
\[
|\Phi(u)(z)| = \left| \sum_{i=1}^{n} (u(x_i), y'_i) \right| \leq \|u\| \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} \left( \sum_{i=1}^{n} \|y'_i\|^{p'} \right)^{1/p'} < \|u\|(1 + \epsilon)
\]
and this shows that \(\|\Phi(u)\| \leq \|u\|\). Therefore \(\Phi\) is an isometric isomorphism.

\[\square\]

**Remark 2.2.8.** Lemma 2.2.7 is enough to show that the \(p\)-Haagerup tensor product is *not* injective for \(p \neq 2\), in the sense of Remark 2.3.5.\(^5\) Motivated by [LeM96, Remark 6.2], let \(E\) be a subspace of \(L_p(\mu)\) which is not an \(L_p\) space.\(^6\) Consider the inclusion mapping \(J : (E')^r \otimes L_c^c \to (E')^r \otimes L_c^c\). We claim that \(J\) is not even isometric. Indeed, if \(J\) were isometric, the adjoint \(J' : B(L_p(\mu), E) \to B(E, E)\)\(^7\) would be a quotient map given by the restriction. In particular, \(id_E\) must extend to a map from \(L_p(\mu)\) onto \(E\) and this would imply that \(E\) is a 1-complemented subspace of \(L_p(\mu)\). This is equivalent to saying that \(E\) is an \(L_p\) space, which is a contradiction.

The fact that the \(p\)-Haagerup tensor product is not injective can be used to show that there is no \(p\)-analogue of polar decomposition. To make it precise, suppose \(\beta \in \mathbb{M}_{r,n}\) with \(r > n\). Regard \(\beta\) as an operator from \(\ell_2^n\) to \(\ell_2^r\). If \(\beta\) has full rank, then we can always write \(\beta = \tau \beta_0\), where \(\tau \in B(\ell_2^n, \ell_2^r)\) is an isometry and \(\beta_0 \in B(\ell_2^n, \ell_2^n)\) satisfies \(\|\beta\| = \|\beta_0\|\).\(^8\) It leads us to the following question.

**Question 2.2.9.** Do we have a similar decomposition if \(p \neq 2\)? That is to say, if \(p \neq 2\), can we always write a full rank matrix \(\beta \in \mathbb{M}_{r,n}\) as \(\beta = \tau \beta_0\) where \(\tau \in B(\ell_p^n, \ell_p^r)\) is an isometry and \(\beta_0 \in B(\ell_p^n, \ell_p^n)\) satisfies \(\|\beta\|_{B(\ell_p^n, \ell_p^r)} = \|\beta_0\|_{B(\ell_p^n, \ell_p^n)}\)?

If the answer to Question 2.2.9 were yes, then the same argument as in [ER00, Lemma 9.2.3 and Proposition 9.2.5] could be used to show that \(p\)-Haagerup tensor product is injective. Since \(p\)-Haagerup tensor product is not injective, we conclude that the answer to Question 2.2.9 is no.

### 2.3 \(p\)-Injective Tensor Product

**Definition 2.3.1.** Let \(X, Y\) be \(p\)-operator spaces. Regarding the algebraic tensor product \(X \otimes Y\) as a subspace of \(CB_p(X', Y)\), we define the \(p\)-operator space *injective tensor product* \(X \overset{\text{\(p\)}}{\otimes} Y\) to be the completion of \(X \otimes Y\) in \(CB_p(X', Y)\).

---

\(^5\)The Haagerup tensor product for operator spaces is injective [ER00].

\(^6\)\(\ell_2^n\) can be regarded as a subspace of \(L_p(\mathbb{C}^n, dx)\). See [DF93, Proposition 8.7]. Note that a Hilbert space cannot be isometric to any \(L_p\) space, \(p \neq 2\), since \(L_p\) spaces do not have the parallelogram law unless \(p = 2\).

\(^7\)Note that \(E\) is reflexive, see Remark 1.2.4.

\(^8\)One can take \(\beta_0 = |\beta|\) using polar decomposition. See the proof of [ER00, Lemma 9.2.3].
Remark 2.3.2.

1. For each \( u \in M_n(X \otimes Y) \), we have \( \|u\|_{\varphi_p} \leq \|u\|_{h_p} \). To see this, suppose \( u = v \circ w \) with \( v \in M_{n,r}(X) \) and \( w \in M_{r,n}(Y) \). It follows that

\[
\|u\|_{\varphi_p} = \sup \left\{ \left\| \sum_{k=1}^{r} \varphi_{st}(v_{ik})w_{kj} \right\| : m \in \mathbb{N}, \varphi = [\varphi_{st}] \in M_m(X')_1 \right\}
\]

\[
\leq \sup \left\{ \|\langle \varphi, v \rangle\| \|w\| : m \in \mathbb{N}, \varphi = [\varphi_{st}] \in M_m(X')_1 \right\}
\]

\[
\leq \|v\| \|w\|.
\]

Taking infimum over \( v \) and \( w \), we get the desired inequality.

2. By definition of the Banach space injective tensor product \( \otimes \), we have

\[
\|u\|_{\varphi_p} = \|u\|_{CB_p(X',Y)} \leq \|u\|_{h_p} \leq \|u\|_{\Lambda_p} \leq \|u\|_{\pi}.
\]

for every \( u \in X \otimes Y \).

3. If \( Y \subseteq B(L_p(\nu)) \), then the p-operator space injective matrix norm \( \|\cdot\|_{\varphi_p} \) on \( X \otimes Y \) satisfies

\[
\|u\|_{\varphi_p} = \sup \{ \|((\varphi \circ \psi)_n(u)) : m, k \in \mathbb{N}, \varphi \in M_m(X')_1, \psi \in M_k(Y')_1 \}
\]

for each matrix \( u \in M_n(X \otimes Y) \).

4. If \( X \subseteq B(L_p(\mu)) \) as well, then \( X \otimes Y = Y \otimes X \) \( p \)-completely isometrically.

Now we are ready to compare various tensor norms on \( X \otimes Y \).

**Proposition 2.3.3.** If \( X \) and \( Y \) are \( p \)-operator spaces, then the various tensor norms \( X \otimes Y \) are ordered as follows:

\[
\|\cdot\|_e \leq \|\cdot\|_{\varphi_p} \leq \|\cdot\|_{h_p} \leq \|\cdot\|_{\Lambda_p} \leq \|\cdot\|_{\pi}.
\]

**Proof.** Combine Remarks 2.1.2, 2.2.2, and 2.3.2. \( \square \)

**Proposition 2.3.4.** If \( X \subseteq B(L_p(\mu)) \), then we have \( p \)-complete isometric isomorphisms

\[
M_n(X) = M_n \otimes_p X = CB_p^*(X', M_n).
\]

---

9 In particular, \( \|u\|_{\varphi_p} \) is a subcross norm.

10 For Banach spaces \( E \) and \( F \), the norm of \( x \in E \otimes F \) is given by regarding \( x \) as a member of \( B(E', F) \) [Rya02, §3.1].
Proof. For each \( m \in \mathbb{N} \), we have
\[
M_m(M_n(X)) \subseteq M_m(M_n(X'')) = CB_p(X', M_{mn}).
\]
By definition, \( M_m(M_n \odot_p X) \subseteq M_m(CB_p(X', M_n)) = CB_p(X', M_{mn}) \) and the first identification follows. For
the second identification, simple calculation shows that \( M_n(X) \subseteq CB_p(X', M_n) \). The other inclusion comes
from applying [Con90, Theorem V.1.2] to each component of \( T = [T_{ij}] \in CB_p(X', M_n) \).

Remark 2.3.5. At this moment, we do not know whether the \( p \)-operator space injective tensor product
is injective that is, if \( u : X \to \tilde{X} \) and \( v : Y \to \tilde{Y} \) are \( p \)-completely isometric injections, then we do not
know whether \( u \otimes v \) always extend to a \( p \)-completely isometric injection \( u \otimes v : X \otimes_p Y \to \tilde{X} \otimes_p \tilde{Y} \). But
if we assume that all the \( p \)-operator spaces under consideration are on \( L_p \) space, then we can show that
\( u \otimes v : X \otimes_p Y \to \tilde{X} \otimes_p \tilde{Y} \) is a \( p \)-complete isometry as in the following proposition. This fact supports that
the terminology \( p \)-injective tensor product is still reasonable.

Proposition 2.3.6. For \( i = 1, 2 \), suppose \( X_i \subseteq Y_i \subseteq B(L_p(\mu_i)) \). Then
\[
X_1 \otimes_p X_2 \subseteq Y_1 \otimes_p Y_2
\]
\( p \)-completely isometrically.

Proof. For \( i = 1, 2 \), let \( \varphi_i : X_i \hookrightarrow Y_i \) denote the \( (p \text{-completely isometric}) \) inclusion. Since \( \varphi_1 \otimes \varphi_2 = (\varphi_1 \otimes id_{Y_2}) \circ (id_{X_1} \otimes \varphi_2) \), by Remark 2.3.2 above, it suffices to show that
\[
id_{X_1} \otimes \varphi_2 : X_1 \otimes_p X_2 \to X_1 \otimes_p Y_2
\]
is \( p \)-completely isometric. Note that the following diagram commutes:

Since \( X_1 \otimes_p X_2 \subseteq CB_p(X_1', X_2) \), \( X_1 \otimes_p Y_2 \subseteq CB_p(X_1', Y_2) \), and \( CB_p(X_1', X_2) \subseteq CB_p(X_1', Y_2) \) \( p \)-completely
isometrically, we conclude that \( id_{X_1} \otimes \varphi_2 \) is \( p \)-completely isometric. \( \square \)
Theorem 2.3.7. If $X \subseteq B(L_p(\nu))$, then there are $p$-complete isometries

\[ K(L_p(\mu)) \overset{\gamma^p}{\otimes} X \hookrightarrow B(L_p(\mu) \otimes_p L_p(\nu)), \]

\[ B(L_p(\mu)) \overset{\gamma^p}{\otimes} X \hookrightarrow B(L_p(\mu) \otimes_p L_p(\nu)). \]

In particular, if $V \subseteq B(L_p(\mu))$ and $W \subseteq B(L_p(\nu))$, then $V \otimes_p W \hookrightarrow B(L_p(\mu) \otimes_p L_p(\nu))$ $p$-completely isometrically.

Proof. By Proposition 1.6.8, we have a $p$-complete isometry $K(L_p(\mu)) \overset{\gamma^p}{\otimes} X \hookrightarrow CB_p(N(L_p(\mu)), X)$. By [Daw10, Proposition 5.3], we have the isometry $N(L_p(\mu)) \overset{\gamma^p}{\otimes} N(L_p(\nu)) = N(L_p(\mu) \otimes_p L_p(\nu))$ and hence we get

\[ CB_p(N(L_p(\mu)), B(L_p(\nu))) = (N'(L_p(\mu)) \overset{\gamma^p}{\otimes} N'(L_p(\nu)))' = (N'(L_p(\mu) \otimes_p L_p(\nu)))' = B(L_p(\mu) \otimes_p L_p(\nu)) \] (2.3)
isometrically. Therefore,

\[
M_n((N'(L_p(\mu)) \overset{\gamma^p}{\otimes} N'(L_p(\nu)))') = CB_p(N'(L_p(\mu)) \overset{\gamma^p}{\otimes} N'(L_p(\nu)), M_n) \\
= CB_p(N'(L_p(\mu)), CB_p(N'(L_p(\nu)), M_n)) \\
= CB_p(N'(L_p(\mu)), CB_p(B(L_p(\nu)))) \\
= CB_p(N'(L_p(\mu)), CB_p(\ell_p^{n} \otimes_p L_p(\nu))) \\
(\text{by (2.3)}) = B(L_p(\mu) \otimes_p \ell_p^{n} \otimes_p L_p(\nu)) \\
= M_n(B(L_p(\mu) \otimes_p L_p(\nu)))
\]

and this means that we have the $p$-complete isometry

\[ (N'(L_p(\mu)) \overset{\gamma^p}{\otimes} N'(L_p(\nu)))' \cong B(L_p(\mu) \otimes_p L_p(\nu)). \] (2.4)

Therefore, (2.3) is in fact a $p$-complete isometry and this in particular shows that

\[ K(L_p(\mu)) \overset{\gamma^p}{\otimes} X \subseteq CB_p(N'(L_p(\mu)), X) \subseteq CB_p(N'(L_p(\mu)), B(L_p(\nu))) = B(L_p(\mu) \otimes_p L_p(\nu)) \]

$p$-completely isometrically.
For the second identification, note that we have the following $p$-completely isometric inclusions

\[ B(L_p(\mu)) \overset{\psi_p}{\otimes} X \subseteq CB_p(B(L_p(\mu))^{'}, X) \subseteq CB_p(B(L_p(\mu))^{'}, B(L_p(\nu))). \tag{2.5} \]

By Proposition 2.1.7 and [Daw10, Proposition 5.4], we get isometric isomorphisms

\[ M_n(B(L_p(\mu))^{'}) = CB_p(B(L_p(\mu)), M_n) = (B(L_p(\mu)) \overset{\psi_p}{\otimes} N_n) = (N_n(B(L_p(\mu))))' = (M_n(N(L_p(\mu))))'' \tag{2.6} \]

and hence the closed unit ball of $M_n(N(L_p(\mu)))$ is weak*-dense in the closed unit ball of $M_n(B(L_p(\mu))^{'})$. To be more precise, by Lemma 1.6.7, (2.6) means that for all $\psi = [\psi_{ij}] \in M_n(B(L_p(\mu))^{'}, 1)$, there is a net $\psi_{ij} = \sum_{i,j}\langle \psi_{ij}, T_{ij} \rangle \rightarrow \sum_{i,j}\langle \psi_{ij}, T_{ij} \rangle.$ (2.7)

Let $T_0 \in B(L_p(\mu))$ and let $\delta > 0$. By considering $T = \epsilon_{ij} \otimes T_0 \in M_n(B(L_p(\mu)))$, (2.7) in particular yields that for each $i, j$, there is $\gamma_{ij}$ such that

\[ |\langle \psi_{ij}^{\gamma} - \psi_{ij}, T_0 \rangle| < \delta \quad \text{for all } \gamma \geq \gamma_{ij}. \tag{2.8} \]

Consider the following diagram

\[ \begin{array}{ccc}
M_n(N(L_p(\mu))) & \overset{\psi_p}{\longrightarrow} & M_n(B(L_p(\mu))^{'}) \\
\| & \| & \| \\
CB_p(B(L_p(\mu)), M_n) & \longrightarrow & CB_p(B(L_p(\mu)), M_n),
\end{array} \]

where the first column comes from Proposition 2.3.4. (2.7) and (2.8) mean that $\psi^{\gamma} \in CB_p(B(L_p(\mu)), M_n)_1$ converges to $\psi$ in the point-norm topology because if $T_0 \in B(L_p(\mu))$, then

\[ \|\psi^{\gamma}(T_0) - \psi(T_0)\| \leq \sum_{i,j} |\langle \psi_{ij}^{\gamma} - \psi_{ij}, T_0 \rangle| \leq n^2 \delta \]

for $\gamma$ large enough. Therefore, using the same argument as in [ER00, Proposition 8.1.2], we can replace $CB_p(B(L_p(\mu))^{'}, B(L_p(\nu)))$ in (2.5) by $CB_p(N(L_p(\mu)), B(L_p(\nu)))$ and the result follows from the previous case. The remaining part follows from Proposition 2.3.6.

\[ ^{11}\text{See Definition 1.6.6 for the definition of } N_n(B(L_p(\mu))). \]
Remark 2.3.8. In fact, in Theorem 2.3.7, one can show that $K(L_p(\mu)) \otimes X \hookrightarrow B(L_p(\mu)) \otimes X$ using Proposition 2.3.6. We gave a different proof to exhibit that (2.4) is a $p$-complete isometry.

The following result provides another reason why the terminology $p$-injective tensor product is reasonable.

**Theorem 2.3.9.** Let $V$ and $W$ be $p$-operator spaces on $L_p$ spaces. Then there exist two index sets $I$ and $J$ such that we can identify $V$ and $W$ with $p$-operator subspaces of $B(\ell_p(I))$ and $B(\ell_p(J))$, respectively, and the canonical inclusion

$$V \otimes W \hookrightarrow B(\ell_p(I) \otimes_p \ell_p(J))$$

is a $p$-completely isometric injection.

**Proof.** By assumption, $V \subseteq V''$ (respectively, $W \subseteq W''$) $p$-completely isometrically. By Proposition 1.6.2, there is an index set $I$ (respectively, $J$) such that $V'' \subseteq B(\ell_p(I))$ (respectively, $W'' \subseteq B(\ell_p(J))$). We can conclude from Proposition 2.3.6 and Theorem 2.3.7 that the canonical inclusions

$$V \otimes W \hookrightarrow V'' \otimes W'' \hookrightarrow B(\ell_p(I)) \otimes_p B(\ell_p(I)) \hookrightarrow B(\ell_p(I) \otimes_p \ell_p(J))$$

are $p$-completely isometric injections. 

Our next result is a $p$-operator space injective tensor product counterpart of Proposition 2.2.4.

**Proposition 2.3.10.** Let $\mu$ and $\nu$ be measures and let $V \subseteq B(L_p(\nu))$. Then we have the $p$-complete isometries

$$L_p^c(\mu) \otimes V \cong L_p^c(\mu) \otimes V, \quad V \otimes L_p^r(\mu) = V \otimes L_p^r(\mu).$$

**Proof.** Let us only give the sketch of the proof for the column space $L_p^c(\mu)$. The proof for the row space $L_p^r(\mu)$ is similar. Let us first assume that $L_p(\mu) = \ell_p^n$. Then we have the $p$-completely isometric inclusion

$$(\ell_p^n) \otimes_p V = V \otimes (\ell_p^n) \otimes C B_p(V', (\ell_p^n)c = M_{n,1}(V'')$$

It follows that every $v \in M_m((\ell_p^n) \otimes_p V) = M_{m,n}(V)$ can be expressed by $v = I_{mn} \odot v$. Since we can regard $I_{mn}$ as a contractive element in $M_{m,mn}((\ell_p^n)c) = M_{mn}$, we get $\|v\|_{h_p} \leq \|v\|_{v_p}$. The general case follows by applying the rigid $L_p$-structure of $L_p(\mu)$ [LP68]. 

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Theorem 2.3.11. Let $V$ be a $p$-operator space on $L_p$ space. For any measure $\mu$, we have the $p$-complete isometries

$$\mathcal{K}(L_p(\mu)) \overset{\vee_p}{\otimes} V = L_p^c(\mu) \overset{h_p}{\otimes} h_p L_p^c(\mu) = L_p^c(\mu) \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} L_p^c(\mu).$$

In particular, we have

$$\mathcal{K}(L_p(\mu)) = L_p^c(\mu) \overset{h_p}{\otimes} L_p^c(\mu) = L_p^c(\mu) \overset{\vee_p}{\otimes} L_p^c(\mu).$$

Proof. If $L_p(\mu) = \ell^n_p$, we have $\mathcal{K}(\ell^n_p) = \mathcal{B}(\ell^n_p) = M_n$. In this case, we obtain

$$\mathcal{B}(\ell^n_p) \overset{\vee_p}{\otimes} V = M_n(V) = (\ell^n_p)^c \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} (\ell^n_p)^c = (\ell^n_p) \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} (\ell^n_p)^c$$

by [LeM96, Proposition 6.3] and Proposition 2.3.10. For general case, we need to apply the rigid $L_p$-structure of $L_p(\mu)$, i.e. we need to consider an increasing net $\{F_\alpha\}$ of finite dimensional subspaces in $L_p(\mu)$ such that each $F_\alpha$ is isometric to some $\ell_p^{d(\alpha)}$ and the union $\bigcup_\alpha F_\alpha$ is norm-dense in $L_p(\mu)$. For each $\alpha$, we may identify the dual space $F_\alpha'$ with a subspace of $L_p'(\mu)$ and $F_\alpha'$ is isometric to $\ell_p^{d(\alpha)}$. In this case, the $p$-operator spaces $F_\alpha^c$, $(F_\alpha')^r$, and $\mathcal{B}(F_\alpha)$ are 1-complemented subspaces of $L_p^c(\mu)$, $L_p^r(\mu)$, and $\mathcal{K}(L_p(\mu))$, respectively. Then we obtain an increasing net of $p$-operator spaces

$$\mathcal{B}(F_\alpha) \overset{\vee_p}{\otimes} V = F_\alpha^c \overset{h_p}{\otimes} V \overset{h_p}{\otimes} (F_\alpha')^r = F_\alpha^c \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} (F_\alpha')^c.$$

Since the unions $\bigcup_\alpha \mathcal{B}(F_\alpha)$, $\bigcup_\alpha F_\alpha^c \overset{h_p}{\otimes} V \overset{h_p}{\otimes} (F_\alpha')^r$, and $\bigcup_\alpha F_\alpha^c \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} (F_\alpha')^r$ are norm-dense in $\mathcal{K}(L_p(\mu))$, $L_p^c(\mu) \overset{h_p}{\otimes} V \overset{h_p}{\otimes} L_p^c(\mu)$, and $L_p^c(\mu) \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} L_p^c(\mu)$, respectively, thanks to Proposition 2.3.6, we obtain the desired $p$-complete isometries

$$\mathcal{K}(L_p(\mu)) \overset{\vee_p}{\otimes} V = L_p^c(\mu) \overset{h_p}{\otimes} V \overset{h_p}{\otimes} L_p^c(\mu) = L_p^c(\mu) \overset{\vee_p}{\otimes} V \overset{\vee_p}{\otimes} L_p^c(\mu).$$

The second part is immediate by taking $V = \mathbb{C}$. \hfill $\square$

Now let us discuss the duality property between the $p$-operator space injective tensor product and $p$-operator space projective tensor product.\textsuperscript{12}

Theorem 2.3.12. Let $V$ be a $p$-operator space on $L_p$ space. For each $n \in \mathbb{N}$, we have the isometric isomorphism

$$(M_n \overset{\vee_p}{\otimes} V)' = N_n \overset{\wedge_p}{\otimes} V'.$$

\textsuperscript{12}See also Lemma 5.1.1.
Proof. Let us first assume that $u$ is a contractive linear functional contained in $(M_n \otimes \mathcal{B}(E))' = \mathcal{B}(E^n)'$ with $E = L_p(\mu)$ and $\|u\| = 1$. Since $\|u\|_{pch} = \|u\| = 1$ (Lemma 1.4.3), $u$ is actually a $p$-completely contractive linear functional from $\mathcal{B}(E^n)$ into $\mathbb{C}$. We can apply Pisier’s representation theorem [Pis90, Theorem 2.1(c)] to obtain an $L_p$ space, a unital $p$-completely contractive homomorphism $\hat{\pi}\mathcal{B}(E^n) \to \mathcal{B}(L_p)$ and contractive vectors $\eta \in L_p$, $\xi \in L_{p'}$ such that

$$u([a_{ij}]) = \langle \xi, \hat{\pi}([a_{ij}])\eta \rangle.$$ 

Now using the submatrix system $\{e_{ij} \otimes 1\}$ in $M_n \otimes \mathcal{B}(E) = \mathcal{B}(E^n)$, we can split the range space $L_p$ into the following $n$-copies of $\ell_p$-direct sum

$$L_p = \hat{\pi}(e_{11} \otimes 1)L_p \oplus \cdots \oplus \hat{\pi}(e_{nn} \otimes 1)L_p.$$ 

It is easy to see that $\hat{\pi}(e_{11} \otimes 1)L_p$ is a closed and contractively complemented subspace of $L_p$ and thus is an $L_p$ space, which we denote by $L_p(\nu)$. All other spaces $\hat{\pi}(e_{ij} \otimes 1)L_p(1 \leq i \leq n)$ are isometrically isomorphic to $L_p(\nu)$ via $\{\hat{\pi}(e_{ij} \otimes 1)\}$. Therefore (up to an isometric isomorphism), we can obtain a unital $p$-completely contractive homomorphism $\pi : \mathcal{B}(E) \to \mathcal{B}(L_p(\nu))$ and contractive vectors $[\xi_i] \in L_{p'}(\nu)^n$ and $[\eta_j] \in L_p(\nu)^n$ such that

$$u([a_{ij}]) = \langle [\xi_i], (id_{M_n} \otimes \pi)([a_{ij}])[\eta_j] \rangle = \langle [\xi_i], [\pi(a_{ij})][\eta_j] \rangle = \sum_{i,j=1}^n \langle \xi_i, \pi(a_{ij})\eta_j \rangle.$$ 

Now let us assume that the vectors $\eta_1, \ldots, \eta_n$ are contained in a finite dimensional subspace $F \subseteq L_p(\nu)$ such that $F$ is isometric to $\ell^k_p$. If we let $f_1, \ldots, f_k$ correspond to the canonical basis of $\ell^k_p$, we can express each $\eta_j$ as $\eta_j = \sum_{v=1}^k \beta_{j,v}f_v$ with $\sum_{j,v} |\beta_{j,v}|^p \leq 1$. Similarly, we may assume that the vectors $\xi_1, \ldots, \xi_n$ are contained in a finite dimensional subspace $G \subseteq L_{p'}(\nu)$ such that $G$ is isometric to $\ell^l_{p'}$. In this case, we can express $\xi_i$ as $\xi_i = \sum_{w=1}^l \alpha_{i,w}g_w$ with $\sum_{i,w} |\alpha_{i,w}|^{p'} \leq 1$. Let $\iota_F$ and $\iota_G$ be the embedding of $F$ and $G$ into $L_p(\nu)$ and $L_{p'}(\nu)$, respectively. Then the map

$$\Phi = [\Phi_{w,v}] : a \in \mathcal{B}(E) \to (\iota_G)'\pi(a)\iota_F \in (\iota_G)'\mathcal{B}(L_p(\nu))\iota_F \cong M_{l,k}$$ 

defines a contractive element in $M_{l,k}(\mathcal{B}(E)') = \mathcal{B}p(\mathcal{B}(E), M_{l,k})$ such that

$$u = \langle [\xi_i], (id_{M_n} \otimes \pi)[\eta_j] \rangle = [\alpha_{i,w}][id_{M_n} \otimes \Phi][\beta_{j,v}].$$ 

This shows that $u$ corresponds to a contractive element in $N_n \otimes \mathcal{B}(E)'$.

In general, the vectors $\{\eta_j\}_{j=1}^n$ (respectively, $\{\xi_i\}_{i=1}^n$) can be approximated by vectors in some sufficiently
large finite dimensional subspaces $F \subseteq L_p(\nu)$ such that $F \cong \ell^p_P$ (respectively, $G \subseteq L_p(\nu)$ such that $G \cong \ell^p_P$).

It follows that $u$ is a limit of contractive elements in $\mathcal{N}_n \hat{\otimes} \mathcal{B}(E)'$ and thus $u$ itself is contractive in $\mathcal{N}_n \hat{\otimes} \mathcal{B}(E)'$. Therefore we have the isometric isomorphism

$$(M_n \hat{\otimes} \mathcal{B}(E))' = \mathcal{N}_n \hat{\otimes} \mathcal{B}(E)'$$

Now for general $p$-operator space $V \subseteq \mathcal{B}(L_p(\mu))$, if $u$ is a contractive linear functional in $(M_n \hat{\otimes} V)'$, we may extend $u$ to a contractive linear functional $\tilde{u} \in (M_n \hat{\otimes} \mathcal{B}(L_p(\mu)))'$. From the above discussion, we can find a $p$-complete contraction $\Phi : \mathcal{B}(L_p(\mu)) \to M_{l,k}$ and contractive $[\alpha_{i,w}]$ and $[\beta_{j,v}]$ such that $\tilde{u} = [\alpha_{i,w}](id_{M_n} \otimes \Phi)[\beta_{j,v}]$ is contractive in $\mathcal{N}_n \hat{\otimes} \mathcal{B}(E)'$. Then $\Phi|V$ is a $p$-complete contraction from $V$ into $M_{l,k}$ and thus $u = [\alpha_{i,w}](id_{M_n} \otimes \Phi|V)[\beta_{j,v}]$ is contractive in $\mathcal{N}_n \hat{\otimes} V'$. This completes the proof.

2.3.13. Let $V$ and $W$ be $p$-operator spaces on $L_p$ space. Then the canonical inclusion

$$V' \hat{\otimes} W \hookrightarrow \mathcal{CB}_p(V,W)$$

is a $p$-completely isometric injection.

Proof. It is known from Proposition 2.3.4 and Theorem 2.3.12 that for each $n \in \mathbb{N}$, we have the isometric isomorphisms

$$M_n(V'') = \mathcal{CB}_p(V', M_n) = (\mathcal{N}_n \hat{\otimes} V')' = M_n(V)''.$$ 

Therefore we can replace $V''$ by $V$ in the definition $V' \hat{\otimes} W \hookrightarrow \mathcal{CB}_p(V'', W)$ of the $p$-operator space injective tensor product as in the proof of the second half of Theorem 2.3.7.

2.4 Infinite Matrices

As in operator spaces, we can develop the theory of infinite matrices for $p$-operator spaces. Most of the ideas in this section come from those of [ER00, Chapter 11]. For the convenience of the reader, we briefly introduce some $p$-operator space analogues. Suppose that $V$ is a $p$-operator space. We denote by $\mathbb{M}_\infty(V)$ the linear space of all infinite matrices $[v_{ij}]$ with $v_{ij} \in V$. For $1 \leq r, s < \infty$, we identify matrix spaces $\mathbb{M}_{r,s}(V)$ and $\mathbb{M}_r(V)$ in the obvious manner, and we let $v^r$ denote the truncation of $v \in \mathbb{M}_\infty(V)$ to $\mathbb{M}_r(V)$. If $r \leq s$, then

$$\|v^r\| = \|(I_r \oplus 0_{s-r})v^s(I_r \oplus 0_{s-r})\| \leq \|v^s\|.$$
If \( v \in M_\infty(V) \), then we define
\[
\|v\| = \sup\{\|v^r\| : r \in \mathbb{N}\} = \lim_{r \to \infty} \|v^r\|,
\]
and we define \( M_\infty(V) \) to be the space of all \( v \in M_\infty(V) \) for which \( \|v\| < \infty \).

For any \( m \in \mathbb{N} \), we identify \( M_m(M_\infty(V)) \) with \( M_m \otimes V \). Let \( M_m(M_\infty(V)) \) denote \( M_m(M_\infty(V)) \) with the corresponding norm.

**Proposition 2.4.1.** If \( V \) is a \( p \)-operator space, then \( M_\infty(V) \) is a \( p \)-operator space.

**Proof.** This follows exactly as for operator spaces, as in [ER00, Proposition 10.1.1].

We define \( K_\infty(V) \) to be the closure of \( M^\text{lin}_\infty(V) \) in \( M_\infty(V) \), where \( M^\text{lin}_\infty(V) \) denotes the linear space of matrices with only finitely many nonzero entries. Suppose that \( V \subseteq B(L_p(\mu)) \), then the column mappings in the diagram
\[
\begin{array}{c c}
M_r(V) & \cong & M_r \otimes V \\
\downarrow & & \downarrow \\
K_\infty(V) & \cong & K_\infty \otimes V
\end{array}
\]
are isometric, and in each case the union of their ranges is dense. It follows that we have the isometry
\[
K_\infty(V) \cong K_\infty \otimes V.
\]

One can also consider the spaces \( M_{n,\infty}(V) \) and \( M_\infty,n(V) \). If \( v \in M_{n,\infty}(V) \), then we interpret the truncation \( v^r \) as an element of \( M_r(V) \) for \( r \leq n \), as an element of \( M_{n,r}(V) \) for \( r \geq n \), and we use a similar convention for \( M_\infty,n(V) \). We define \( M_{n,\infty}(V) \) and \( M_\infty,n(V) \) just as for \( M_\infty(V) \). In particular, we wish to study \( M_{n,\infty} \) and \( M_\infty,n \) more closely.

**Lemma 2.4.2.** Let \( \alpha = [\alpha_{ij}] \in M_{n,\infty} \) and \( \beta = [\beta_{kl}] \in M_\infty,n \). Let \( 1 < p, p' < \infty \) with \( 1/p + 1/p' = 1 \). Then we have
\[
\|\alpha\|_{\mathcal{B}(\ell_p, \ell_{p'})} \leq \|\alpha\|_{p'} \cdot n^{1/p-1/p'} \quad \text{and} \quad \|\beta\|_{\mathcal{B}(\ell_p, \ell_{p'})} \leq \|\beta\|_p \cdot n^{1/p-1/p'},
\]
where
\[
\|\alpha\|_{p'} = \left( \sum_{i=1}^n \sum_{j=1}^\infty |\alpha_{ij}|^{p'} \right)^{1/p'} \quad \text{and} \quad \|\beta\|_p = \left( \sum_{k=1}^\infty \sum_{l=1}^n |\beta_{kl}|^p \right)^{1/p}.
\]

**Proof.** The proof is almost identical to that of Lemma 1.6.4. We insert the proof for convenience. Suppose \( \xi = (\xi_j) \) is a unit vector in \( \ell_p \). For each \( i \), \( 1 \leq i \leq n \), let \( \eta_i = \left| \sum_{j=1}^\infty \alpha_{ij} \xi_j \right| \), then by Hölder’s inequality,
\( \eta_i \leq \left( \sum_{j=1}^{\infty} |\alpha_{ij}|^{p'} \right)^{1/p'} \) and by Lemma 1.6.3,
\[
\left( \sum_{i=1}^{n} \eta_i^p \right)^{1/p} \leq n^{1/p-1/p'} \cdot \left( \sum_{i=1}^{n} \eta_i^{p'} \right)^{1/p'} \leq n^{1/p-1/p'} \cdot \| \alpha \|_{p'}
\]
and hence we get \( \| \alpha \|_{B(\ell_p', \ell_{p'})} \leq n^{1/p-1/p'} \cdot \| \alpha \|_{p'} \). To prove the second inequality, let \( \gamma : \ell_p' \to \ell_{p'}^n \) be the adjoint operator of \( \beta \). Then by the argument above we have
\[
\| \gamma \|_{B(\ell_p', \ell_{p'})} \leq \| \beta \|_{p} \cdot n^{1/p-1/p'}.
\]
Since \( \| \gamma \|_{B(\ell_p', \ell_{p'})} = \| \beta \|_{B(\ell_{p'}', \ell_{p'})} \) and \( \| \gamma \|_{p} = \| \beta \|_{p} \), we get the desired inequality. \( \square \)

For any matrices \( \alpha \in M_{n, \infty} \) and \( \beta \in M_{\infty, n} \), we have
\[
\lim_{r \to \infty} \| \alpha - \alpha^r \| = \lim_{r \to \infty} \| \beta - \beta^r \| = 0. \quad (2.9)
\]
To see the first equality of (2.9), by Lemma 2.4.2, it suffices to show that \( \lim_{r \to \infty} \| \alpha - \alpha^r \|_{p'} = 0 \), but this is a simple consequence of \( \ell_{p'} \)-convergence. Similar argument works for the second equality in (2.9), and we see that \( M_{n, \infty} = K_{n, \infty} \) and \( M_{\infty, n} = K_{\infty, n} \).

If \( v \in M_{\infty}(V) \) and \( \alpha \in M_{\infty}^{\text{fin}} \), then we define \( \alpha v \in M_{\infty}(V) \) by
\[
(\alpha v)_{i,j} = \sum_k \alpha_{i,k} v_{k,j}.
\]
As in the 2-operator spaces, it turns out that indeed \( \alpha v \in M_{\infty}(V) \). Given \( \alpha \in K_{\infty} \) and \( r \leq s \),
\[
\| \alpha^s v - \alpha^r v \| \leq \| \alpha^s - \alpha^r \| \| v \|,
\]
and thus \( \alpha^r v \) is Cauchy. We let
\[
\alpha v = \lim_{r \to \infty} \alpha^r v. \quad (2.10)
\]
Similarly, for \( \beta \in K_{\infty} \), let
\[
v\beta = \lim_{r \to \infty} v/\beta^r. \quad (2.11)
\]

**Lemma 2.4.3.** With operations in (2.10) and (2.11), \( M_{\infty}(V) \) is \( K_{\infty} \)-bimodule.
**Proof.** We follow the argument in [ER00]. Let $\alpha, \beta \in K_\infty$, and $v \in M_\infty(V)$. Firstly, we show that $(\alpha \beta) v = \alpha (\beta v)$. Let $E^r = I_r \oplus 0 \in K_\infty$, then $\|\alpha (I - E^r)\| \to 0$ as $r \to \infty$ since this is evident for $\alpha \in M_r$, and the general case follows from the density argument. Therefore,

$$
\|(\alpha \beta)^r - \alpha^r \beta^r\| = \|E^r \alpha (I - E^r) \beta E^r\| \to 0
$$

and

$$(\alpha \beta) v = \lim_{r \to \infty} (\alpha^r \beta^r) v = \lim_{r \to \infty} \alpha^r (\beta^r v).$$

But we have

$$
\|\alpha (\beta v) - \alpha^r (\beta^r v)\| \leq \|\alpha (\beta - \beta^r) v\| + \|\alpha - \alpha^r\| \|\beta^r v\| \to 0,
$$

and associativity follows. The same argument applies on the right side. Lastly, noting that $(\alpha^r v) \beta^r = \alpha^r (v \beta^r) = \alpha^r v \beta^r$, we show that

$$(\alpha v) \beta = \lim_{r \to \infty} \alpha^r v \beta^r = \alpha (v \beta). \tag{2.12}$$

Indeed,

$$
\|((\alpha v) \beta - (\alpha^r v) \beta^r)\| \leq \|((\alpha v) \beta - (\alpha^r v) \beta)\| + \|((\alpha^r v) \beta - (\alpha^r v) \beta^r)\|
= \|((\alpha v - \alpha^r v) \beta)\| + \|\alpha^r v (\beta - \beta^r)\|
\leq \|\alpha v - \alpha^r v\| \|\beta\| + \|\alpha^r v\| \|\beta - \beta^r\| \to 0
$$

as $r \to \infty$ and the first equality of (2.12) follows. Likewise, we can prove the second equality of (2.12) and this completes the proof.

The above argument applies, as well, to the space $M_{n,\infty}(V)$ and $M_{\infty,n}(V)$. If $\alpha \in M_{n,\infty} = K_{n,\infty}$, $v \in M_\infty(V)$, and $\beta \in M_{\infty,n} = K_{\infty,n}$, then we have a corresponding element

$$\alpha v \beta = \lim_{r \to \infty} \alpha^r v \beta^r \in M_n(V). \tag{2.13}$$

Following [ER00, Theorem 10.1.4], for any $p$-operator space $V$, we get $p$-completely isometric isomorphism

$$(N_\infty \hat{\otimes} V)' \cong M_\infty(V'), \tag{2.14}$$

where $N_\infty$ denotes the space of all nuclear operators on $\ell_p$.

Now let us suppose that $v \in M_\infty(V)$, $w \in M_\infty(W)$, $\alpha \in M_{n,\infty_2}$, and $\beta \in M_{\infty_2,n}$. We have $v \otimes w \in \ell_p \otimes \ell_q$.
$M_\infty^2(V \hat{\otimes} W)$ since
\[ \|(v \otimes w)^{r \times s}\|_{\Lambda_p} = \|v^r \otimes w^s\|_{\Lambda_p} \leq \|v\|^r \|w\|^s \leq \|v\| \|w\|, \]
and thus from (2.13) we have a well-defined element $u = \alpha(v \otimes w)\beta \in M_n(V \hat{\otimes} W)$. Moreover, proceeding as for 2-operator spaces, we get
\[ \|u\|_{\Lambda_p} \leq \|\alpha\| \|v\| \|w\| \|\beta\|. \quad (2.15) \]

**Proposition 2.4.4.** Given $p$-operator spaces $V$ and $W$, and $u \in M_n(V \hat{\otimes} W)$,
\[ \|u\|_{\Lambda_p} = \inf \{\|\alpha\| \|v\| \|w\| \|\beta\| : u = \alpha(v \otimes w)\beta\}, \]
where the infimum is taken over all such representations with $v \in M_\infty(V)$, $w \in M_\infty(W)$, $\alpha \in M_{n,\infty^2}$, and $\beta \in M_{\infty^2,n}$. Furthermore, we may assume that $v \in K_\infty(V)$ and $w \in K_\infty(W)$.

**Proof.** By (2.15), it suffices to show that $\|u\|_{\Lambda_p} \geq \inf \{\|\alpha\| \|v\| \|w\| \|\beta\| : u = \alpha(v \otimes w)\beta\}$. If $u \in M_n(V \otimes W)$ and $\|u\|_{\Lambda_p} < 1$, then there exists a sequence $\{u_k\}$ of elements in $M_n(V \otimes W)$ such that
\[ u = \sum_{k=1}^{\infty} u_k \quad \text{and} \quad \sum_{k=1}^{\infty} \|u_k\|_{\Lambda_p} < 1. \]
Let $0 < \epsilon < 1 - \sum_{k=1}^{\infty} \|u_k\|_{\Lambda_p}$. For each $u_k \in M_n(V \otimes W)$ we can choose $v_k \in M_{p_k}(V)$, $w_k \in M_{q_k}(W)$, $\alpha_k \in M_{n,p_k \times q_k}$, and $\beta_k \in M_{p_k \times q_k, n}$ such that
\[ u_k = \alpha_k(v_k \otimes w_k)\beta_k \]
and
\[ \|\alpha_k\| \|v_k\| \|w_k\| \|\beta_k\| < \|u_k\|_{\Lambda_p} + \frac{\epsilon}{2^k}. \]
Without loss of generality, we can suppose that $\|v_k\| = \|w_k\| = 1$ and
\[ \|\alpha_k\|^{p'} = \|\beta_k\|^p < \|u_k\|_{\Lambda_p} + \frac{\epsilon}{2^k}. \quad (2.16) \]
Let $q = \max\{p', p\}$. Since $\sum_k (\|u_k\|_{\Lambda_p} + \frac{\epsilon}{2^k}) < 1$, we can find a sequence $\{c_k\}$ such that $c_k \geq 1$, $c_k \to \infty$, and yet we still have
\[ \sum_{k=1}^{\infty} c_k^q (\|u_k\|_{\Lambda_p} + \frac{\epsilon}{2^k}) < 1. \quad (2.17) \]
Let
\[ v = \bigoplus_{k=1}^{\infty} c_k^{-1} v_k \quad \text{and} \quad w = \bigoplus_{k=1}^{\infty} c_k^{-1} w_k, \]
then \( v \in \mathcal{K}_\infty(V) \) and \( w \in \mathcal{K}_\infty(W) \) with \( \|v\|, \|w\| \leq 1 \). Let
\[ \alpha = [c_1 \alpha_1 \ 0 \ 12 \ \cdots \ 0_{21} \ c_2 \alpha_2 \ \cdots \ 0_{23} \ \cdots], \]
where \( 0_{rs} \) denotes the \( n \) by \( p_r \times q_s \) zero matrix. We claim that \( \alpha \in M_{n,\infty^2} \) with \( \|\alpha\| < 1 \). Indeed, let
\[ \xi = [\xi_{11} \ \xi_{12} \ \cdots \ \xi_{21} \ \xi_{22} \ \cdots] \]
be a unit column vector in \( \ell_p \), where \( \xi_{rs} \) is a \( p_r \times q_s \)-dimensional row vector. Then
\[ \|\alpha\xi\| = \left\| \sum_k c_k \alpha_k \xi_{kk} \right\| \leq \sum_k \|c_k \alpha_k \xi_{kk}\| \leq \left( \sum_k \|c_k \alpha_k\|^{p'} \right)^{1/p'} \left( \sum_k \|\xi_{kk}\|^p \right)^{1/p} < 1 \]
by (2.16) and (2.17). This shows that \( \|\alpha\| < 1 \). Let \( \beta \) be the transpose of
\[ [c_1 \beta_1^T \ 0 \ 12 \ \cdots \ 0_{21} \ c_2 \beta_2^T \ \cdots \ 0_{23} \ \cdots], \]
Again, by (2.16) and (2.17), it follows easily that \( \beta \in M_{\infty^2,n} \) with \( \|\beta\| < 1 \). Now the result follows since \( u = \alpha(v \otimes w)\beta \).

We can also represent elements in the \( p \)-operator spaces Haagerup tensor product in terms of infinite matrices. First of all, using an argument similar to that used right before Proposition 2.4.4, we get that if \( v \in M_{n,\infty}(V) \), \( x \in M_\infty(X) \), and \( w \in M_{\infty,n}(W) \), then we have a well-defined element \( u = v \otimes x \otimes w \) in \( M_n(V \otimes_{h_p} X \otimes_{h_p} W) \) and moreover \( \|u\|_{h_p} \leq \|v\|\|x\|\|w\| \).

**Proposition 2.4.5.** *Given \( p \)-operator spaces \( V, W, X \) and \( u \in M_n(V \otimes_{h_p} X \otimes_{h_p} W) \),
\[ \|u\|_{h_p} = \inf\{\|v\|\|x\|\|w\| : u = v \otimes x \otimes w\}, \]
where the infimum is taken over all such representations with \( v \in M_{n,\infty}(V) \), \( x \in M_\infty(X) \), and \( w \in M_{\infty,n}(W) \). Furthermore, we may assume that \( x \in \mathcal{K}_\infty(X) \).*

**Proof.** If \( u \in M_n(V \otimes_{h_p} X \otimes_{h_p} W) \) and \( \|u\|_{h_p} < 1 \), then there exists a sequence \( \{u_k\} \) of elements in \( M_n(V \otimes_{h_p} X \otimes_{h_p} W) \) such that
\[ u = \sum_{k=1}^{\infty} u_k \quad \text{and} \quad \sum_{k=1}^{\infty} \|u_k\|_{h_p} < 1. \]
Let \( 0 < \epsilon < 1 - \sum_{k=1}^{\infty} \|u_k\|_{h_p} \). For each \( u_k \in M_n(V \otimes_{h_p} X \otimes_{h_p} W) \) we can choose \( v_k \in M_{n,p_k}(V), x_k \in \mathcal{K}_{\infty}(X) \), and
$M_{p_k \times q_k}(X)$, and $w_k \in M_{q_k,n}$ such that
\[ u_k = v_k \odot x_k \odot w_k \]

and
\[ \|v_k\|\|x_k\|\|w_k\| < \|u_k\| + \frac{\varepsilon}{2^k}. \]

Without loss of generality, we can suppose that $\|x_k\| = 1$ and
\[ \|v_k\|^p = \|w_k\|^p < \|u_k\| + \frac{\varepsilon}{2^k}. \quad (2.18) \]

Let $q = \max\{p', p\}$. Since $\sum_k (\|u_k\| + \frac{\varepsilon}{2^k}) < 1$, we can find a sequence $\{c_k\}$ such that $c_k \geq 1$, $c_k \to \infty$, and yet we still have
\[ \sum_{k=1}^{\infty} c_k^q (\|u_k\| + \frac{\varepsilon}{2^k}) < 1. \quad (2.19) \]

Let
\[ x = \bigoplus_{k=1}^{\infty} c_k^{-2} x_k, \]

then $x \in \mathcal{K}_\infty(X)$ with $\|x\| \leq 1$. Let
\[ v = [c_1 v_1 \ v_2 \ \cdots] \quad \text{and} \quad w = \begin{bmatrix} c_1 w_1 \\ c_2 w_2 \\ \vdots \end{bmatrix}, \]

then as in the proof of Proposition 2.4.4, we get $v \in M_{n,\infty}(V)$ and $w \in M_{\infty,n}(W)$ with $\|v\|, \|w\| \leq 1$. Finally, $u = v \odot x \odot w$ and this finishes the proof. \qed
Chapter 3

Figà-Talamanca-Herz Algebras

3.1 Basics on Locally Compact Groups

Throughout this section $G$ will denote a locally compact group. Whenever it makes sense, the convolution $\xi \ast \eta$ of two functions $\xi$ and $\eta$ on $G$ is defined to be

$$\xi \ast \eta(x) = \int_G \xi(y)\eta(y^{-1}x)dy,$$

where $dy$ means the left Haar measure.

Lemma 3.1.1. Let $K$ be a compact subset of $G$. Then there is a function $f \in C_{00}(G)$ such that $f|_K \equiv 1$.

Proof. Let $L$ be a compact neighborhood of the neutral element $e \in G$, then $KL$ is also compact being the image of a compact set under a continuous function: $K \times L \ni (k,l) \mapsto kl$. Consider the function

$$f = \frac{1}{|L|}\mathbb{1}_{KL} \ast \mathbb{1}_{L^{-1}}.$$

Since

$$f(x) = \frac{1}{|L|} \int_{KL} \mathbb{1}_{L^{-1}}(y^{-1}x)dy = \frac{1}{|L|} \int_{KL} \mathbb{1}_L(x^{-1}y)dy = \frac{|KL \cap xL|}{|L|},$$

for every $k \in K$, we get

$$f(k) = \frac{|KL \cap kL|}{|L|} = \frac{|xL|}{|L|} = 1.$$

$f \in C_0(G)$ because $f$ is given by the convolution of functions with compact supports [HR79, (20.16)] and $f \in C_{00}(G)$ because $f$ vanishes off the compact set $KLL^{-1}$. \qed

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3.2 $p$-Pseudofunction Algebras and $p$-Pseudomeasure Algebras

Let $G$ be a locally compact group and let $1 < p < \infty$. For each $s \in G$, there exists an isometric isomorphism $\lambda_p(s)$ on $L_p(G)$, called the left regular representation, given by

$$\lambda_p(s)(\xi)(t) = \xi(s^{-1}t), \quad \xi \in L_p(G), \quad s, t \in G.$$  

It is well-known (for instance, [Run02]) that $\lambda_p : G \to \mathcal{B}(L_p(G))$ is strong operator continuous (equivalently, weak operator continuous).\(^1\) Note that $\lambda_p$ induces a representation $\lambda_p : L_1(G) \to \mathcal{B}(L_p(G))$ via integration.

We let $PF_p(G)$ denote the $p$-pseudofunction algebra, which is defined to be the norm closure of $\lambda_p(L_1(G))$ in $\mathcal{B}(L_p(G))$ so that for $f \in L_1(G)$, the norm of $\lambda_p(f)$ is defined by

$$\|\lambda_p(f)\| = \sup\{\|f \ast \xi\|_p : \xi \in L_p(G), \|\xi\|_p \leq 1\}. \quad (3.1)$$

**Remark 3.2.1.**

1. $PF_p(G)$ is indeed a Banach algebra, because $\lambda_p(f \ast g) = \lambda_p(f)\lambda_p(g)$ for all $f, g \in L_1(G)$.

2. By Young’s inequality for convolution

$$\|\lambda_p(f)\| \leq \|f\|_1 \quad (3.2)$$

for any $f \in L_1(G)$. In particular, when $p = 1$, $\|\lambda_1(f)\| = \|f\|_1$ for all $f \in L_1(G)$ since $L_1(G)$ has a contractive approximate identity.

3. $G$ is amenable if and only if for each positive $f \in L_1(G)$ one has $\|\lambda_p(f)\| = \|f\|_1$ [Lep68].

4. $PF_p(G)$ has a contractive approximate identity arising from contractive approximate identity of $L_1(G)$.

   If $G$ is discrete, then $PF_p(G)$ has a unit.

5. If $p = 2$, $PF_2(G)$ is nothing but the reduced group C*-algebra of $G$.

We let $PM_p(G)$ denote the $p$-pseudomeasure algebra, which is defined to be the weak operator topology closure of $\lambda_p(L_1(G))$ in $\mathcal{B}(L_p(G))$. By Krein-Šmulian Theorem, $PM_p(G)$ is the same as the weak* closure of $\lambda_p(L_1(G))$ in $\mathcal{B}(L_p(G))$.\(^2\)

---

\(^1\)Similarly one can define the right regular representation $\rho_p$, given by $\rho_p(s)(\xi)(t) = \xi(ts)\Delta(s)^{1/p}, \xi \in L_p(G)$, and $s, t \in G$, where $\Delta$ denotes the modular function.

\(^2\)Equivalently, the weak* closure of span$\{\lambda_p(s) : s \in G\}$ in $\mathcal{B}(L_p(G))$. So $PM_p(G)$ is always unital.
3.3 Introduction to Figà-Talamanca-Herz Algebras

In this section we introduce a \( p \)-generalization of the Fourier algebra.

**Definition 3.3.1.** Let \( G \) be a locally compact group. Let \( \Lambda_p : L_p'(G) \otimes L_p(G) \to C_0(G) \) be defined by

\[
\Lambda_p(g \otimes f)(s) = \langle g, \lambda_p(s)(f) \rangle, \quad s \in G, \ f \in L_p(G), \ g \in L_p'(G).
\]

The **Figà-Talamanca-Herz Algebra** \( A_p(G) \) is defined to be the coimage of \( \Lambda_p \).

**Remark 3.3.2.** We collect some facts about \( A_p(G) \).

1. \( A_p(G) \) consists of those \( f \in C_0(G) \) such that there are sequences \( (\xi_n) \subseteq L_p'(G) \) and \( (\eta_n) \subseteq L_p(G) \) with \( \sum_{n=1}^{\infty} \|\xi_n\|\|\eta_n\| < \infty \) and \( f = \sum_{n=1}^{\infty} \xi_n \ast \eta_n \). Note that \( \|f\|_{A_p(G)} = \inf \left\{ \sum_{n=1}^{\infty} \|\xi_n\|\|\eta_n\| : f = \sum_{n=1}^{\infty} \xi_n \ast \eta_n \right\} \).

\( \Lambda_p \) maps into \( C_0(G) \) by Theorem [HR79, (20.16)]. Moreover, \( \| \cdot \|_{A_p(G)} \geq \| \cdot \|_{\infty} \) by Young’s inequality for convolution [HR79, (20.18)]. In particular, the convergence in \( \| \cdot \|_{A_p(G)} \) implies uniform (hence pointwise) convergence.

2. \( A_p(G) \) is a commutative Banach algebra under pointwise operations [Her71].

3. \( G \) is amenable if and only if \( A_p(G) \) has bounded (by 1) approximate identities for any \( p, 1 < p < \infty \) [Her73, Theorem 6].

4. By Lemma 3.1.1, for any compact subset \( K \) of \( G \), there exists \( \varphi \in A_p(G) \) with \( \varphi \equiv 1 \) on \( K \). See also 9 below.

5. If \( G \) is amenable, given a compact subset \( K \) and \( \epsilon > 0 \), there exists \( \varphi \in A_p(G) \) with \( \varphi \equiv 1 \) on \( K \) and \( \|\varphi\|_{A_p(G)} \leq 1 + \epsilon \). See Theorem 4 and Section 0 in [Her73].

6. \( A_p(G) \) is isometrically isomorphic to \( A_{p'}(G) \) via \( f \mapsto \tilde{f} \). If \( G \) is abelian, then \( A_p(G) = A_{p'}(G) \).

7. The Banach space dual \( A_p(G)' \) can be identified with \( PM_p(G) \) [Run02].

8. Let \( f \in L_1(G) \), then \( f \) defines an element \( \Phi_f \in A_p(G)' \) by \( \int f(x)\omega(x)dx, \ \omega \in A_p(G) \). The norm \( \|\Phi_f\|_{A_p(G)'} \) equals \( \|\lambda_p(f)\| \) introduced in (3.1) above [Cow79, §4].

9. \( A_p(G) \) is a regular tauberian algebra in the sense of [Her73, Proposition 3].

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\(^3\)Some authors swap \( p \) and \( p' \). For example, Runde in [Run02] uses \( A_{p'}(G) \) instead of \( A_p(G) \) in our definition.

\(^4\)A Banach algebra \( A \) is said to be a regular tauberian algebra of functions on \( G \) if the following three conditions hold:
10. \( A_p(G) \) is dense in \( C_0(G) \) in \( \| \cdot \|_\infty \). To see this, it suffices to show that for any \( f \in C_0(G) \) and for any \( \epsilon > 0 \), there exists \( u \in A_p(G) \) such that \( \| u - f \|_\infty < \epsilon \). Since \( f \in C_0(G) \), \( f \) is (right) uniformly continuous [Fol95, Proposition (2.6)], so there is an open neighborhood \( U \) of the neutral element of \( G \) such that \( \| f(y) - f(\cdot) \|_\infty < \epsilon \) for all \( y \in U \). By [Fol95, Proposition (2.19)] and (inner) regularity of the Haar measure, we can replace \( U \) with a compact set \( K \) with \( 0 < |K| < \infty \). Define \( u := \frac{1}{|K|} f \ast \mathbb{1}_K^{-1} \), then \( u \in A_p(G) \) and for all \( t \in G \),

\[
|u(t) - f(t)| = \left| \frac{1}{|K|} \int_{s \in G} f(s) \mathbb{1}_{K^{-1}}(s^{-1}t) ds - f(t) \right| \\
= \left| \frac{1}{|K|} \int_{s \in K} (f(s) - f(t)) ds \right| \\
(y = t^{-1}s) \leq \frac{1}{|K|} \int_{y \in K} |f(ty) - f(t)| dy < \epsilon.
\]

11. If \( x_i \to e \) in \( G \), then \( \| x_i u - u \|_{A_p(G)} \to 0 \). This follows from the fact that \( A_{p,e}(G) = A_p(G) \cap C_0(G) \) is dense in \( A_p(G) \).\(^5\)

12. \( A_p(G) \) is closed under the left and right translations since for \( \varphi = \xi \ast \check{\eta}, \xi \in L_{p'}(G), \eta \in L_p(G) \), we get

\[
\varphi(s^{-1}x) = (\lambda_{p'}(s)\xi \ast \check{\eta})(x) \quad \text{and} \quad \varphi(x) = (\xi \ast \rho_p(s)\check{\eta})(x).
\]

Since \( PM_p(G) \subseteq B(L_p(G)) \) is a \( p \)-operator space, \( A_p(G) \) has a dual \( p \)-operator space structure and \( A_p(G)' = PM_p(G) \) \( p \)-completely isometrically [Daw10, Proposition 5.5]. Unless stated otherwise, we assume that \( A_p(G) \) carries this \( p \)-operator space structure.

**Definition 3.3.3.** A linear map \( T : A_p(G) \to A_p(G) \) is said to be a **multiplier** if \( T(uv) = T(u)v \) for all \( u, v \in A_p(G) \). The set of multipliers of \( A_p(G) \) is denoted by \( MA_p(G) \).

**Remark 3.3.4.** If \( T \in MA_p(G) \), then \( T \) is necessarily bounded. To see this, suppose \( u_n \to u \) and \( Tu_n \to v \) in \( A_p(G) \). Fix \( x \in G \) and take \( h \in A_p(G) \) such that \( h(x) = 1 \). Then

\[
v(x) = v(hx) = \lim_n Tu_n(x)h(x) = \lim_n T(u_n,h)(x) = \lim_n u_n(x)Th(x) = u(x)Th(x) = Tu(x)h(x) = Tu(x)
\]

(a) Given a compact set \( K \subseteq G \) and a closed subset \( F \) disjoint from \( K \), there exists \( u \in A \) such that \( u \equiv 1 \) on \( K \) and \( u \equiv 0 \) on \( F \).

(b) The elements of compact support are dense in \( A \).

(c) If \( M \) is a continuous multiplicative linear functional on \( A \) whose support (in the sense of [Her73, §3]) is a single point \( \{ x \} \subseteq G \), then \( M = \delta_x \), i.e., \( \langle u, M \rangle = u(x) \) for all \( u \in A \).

\(^5\)See Remark 3.3.2.9.
and the result follows by the Closed Graph Theorem.

To each \( T \in MA_p(G) \), we associate a function \( h_T : G \to \mathbb{C} \) in the following way: for \( x \in G \), take \( u \in A_p(G) \) with \( u(x) = 1 \) and define \( h_T(x) = (Tu)(x) \). Note that \( h_T \) is well-defined since if \( v \in A_p(G) \) is another function satisfying \( v(x) = 1 \), then

\[
(Tu)(x) = (Tu)(x)v(x) = T(uv)(x) = T(vu)(x) = (Tv)(x)u(x) = (Tv)(x).
\]

It is not difficult to show that the mapping \( T \mapsto h_T \) is injective and this gives an alternative definition of multipliers.

**Definition 3.3.5.** A complex-valued function \( u \) on \( G \) is said to be a multiplier for \( A_p(G) \) if the linear map \( m_u(v) = uv \) maps \( A_p(G) \) to \( A_p(G) \).

**Definition 3.3.6.** For \( u \in MA_p(G) \), let \( M_u : PM_p(G) \to PM_p(G) \) denote the weak*-continuous linear map defined by \( M_u = m'_u \) and \( \bar{M}_u \) denote the restriction of \( M_u \) to \( PF_p(G) \). \( M_{cb}A_p(G) \) is defined to be the space of \( p \)-completely bounded multipliers, endowed with the norm \( \|u\|_{M_{cb}A_p(G)} = \|M_u\|_{pcb} \).

**Remark 3.3.7.**

1. If \( u \in MA_p(G) \), then \( m_u \) is necessarily in \( C_b(G) \). To see this, suppose \( g_i \to g \) in \( G \). Let \( K \) be a compact neighborhood of the neutral element \( e \) of \( G \) and define \( \psi \in A_p(G) \) by \( \psi(s) = \langle 1_K, \lambda_p(s)\mathbb{1}_{g^{-1}K} \rangle \), then \( \psi(g) = |K| > 0 \). Since \( u\psi \in A_p(G) \subseteq C_0(G) \) (Remark 3.3.2), \( u(g_i)\psi(g_i) \to u(g)\psi(g) = u(g)|K| \). Since \( \psi(g_i) \to \psi(g) = |K| \) (See Remark 3.3.2), we conclude that \( u(g_i) \to u(g) \). For boundedness, one can rely on Theorem 3.3.8 below.

2. If \( u \in MA_p(G) \), then \( m_u \) is a bounded linear map on \( A_p(G) \). This follows from the closed graph theorem and the fact that \( \psi_n \to \psi \) in \( A_p(G) \) implies \( \psi_n \to \psi \) pointwise.

We have some useful characterizations of these spaces.

**Theorem 3.3.8.** [Daw10, Lemma 8.2, Theorem 8.3]

1. \( u \in MA_p(G) \) if and only if there exists a bounded, weak*-continuous operator \( M : PM_p(G) \to PM_p(G) \) such that \( M(\lambda_p(s)) = u(s)\lambda_p(s) \).

2. \( u \in M_{cb}A_p(G) \) if and only if there exist \( E \in SQ_p \) and bounded continuous maps \( \alpha : G \to E \) and \( \beta : G \to E' \) such that \( u(ts^{-1}) = (\beta(t), \alpha(s)) \) for \( s, t \in G \).

**Remark 3.3.9.**
1. Theorem 3.3.8.1 shows that the range of $M_u$ is contained in $PF_p(G)$.

2. Theorem 3.3.8.2 shows that the left and right translations are isometries in $M_{cb}A_p(G)$.

3. Since $M_{cb}A_p(G) \subseteq CB_p(A_p(G))$, $M_{cb}A_p(G)$ has a natural induced $p$-operator space structure. In fact, if $u = [u_{ij}] \in M_n(M_{cb}A_p(G))$, then

$$
\|u\|_{M_n(M_{cb}A_p(G))} = \|u\|_{M_n(CB_p(A_p(G)))} = \|u\|_{CB_p(A_p(G)), M_n(A_p(G))} = \sup \{ \|u_{ij}\|_{M_m(A_p(G))} : m \in \mathbb{N}, \ \varphi = [\varphi_{kl}] \in M_m(A_p(G)) \} = \sup \{ \|u_{ij}\|_{M_m((PM_p(G))^t)} : m \in \mathbb{N}, \ \varphi = [\varphi_{kl}] \in M_m(A_p(G)) \} = \sup \{ \|u_{ij}\|_{CB_p(PM_p(G), M_m(A_p(G)))} : m \in \mathbb{N}, \ \varphi = [\varphi_{kl}] \in M_m(A_p(G)) \} = \sup \{ \|u_{ij}\|_{(PM_p(G)), M_n(PM_p(G))} : m \in \mathbb{N}, \ \varphi = [\varphi_{kl}] \in M_m(A_p(G)) \} = \|M_u\|_{CB_p(PM_p(G))}.$$

4. If $G$ is a discrete group, the inclusion mapping $\iota : M_{cb}A_p(G) \to \ell_\infty(G)$ is $p$-completely contractive.\footnote{See the paragraph right before Proposition 1.6.13 for the discussion of the $p$-operator space structure of $\ell_\infty(G)$.}

Indeed, it is easy to show that $\iota$ is contractive (See Remark 3.3.7). To prove that $\iota$ is $p$-completely contractive, fix $n \in \mathbb{N}$ and let $[u_{ij}] \in M_n(M_{cb}A_p(G))$. Then for each $s \in G$, we have

$$
\|u_{ij}(s)\|_{\mathcal{B}(\ell^p_n)} = \sup \left\{ \left\| \sum_{i,j=1}^n a_i u_{ij}(s) b_j \right\| : \sum_{i=1}^n |a_i|^p' \leq 1, \ \sum_{j=1}^n |b_j|^p \leq 1 \right\} = \sup \left\{ |v(s)| : v = [a_i]^T [u_{ij}] [b_j], \sum_{i=1}^n |a_i|^p' \leq 1, \ \sum_{j=1}^n |b_j|^p \leq 1 \right\} \leq \sup \left\{ \|v\|_{\ell_\infty(G)} : v = [a_i]^T [u_{ij}] [b_j], \sum_{i=1}^n |a_i|^p' \leq 1, \ \sum_{j=1}^n |b_j|^p \leq 1 \right\} \leq \sup \left\{ \|v\|_{M_{cb}A_p(G)} : v = [a_i]^T [u_{ij}] [b_j], \sum_{i=1}^n |a_i|^p' \leq 1, \ \sum_{j=1}^n |b_j|^p \leq 1 \right\}.
$$
By Proposition 1.6.14, taking supremum over \( s \in G \), we get the desired result.

We close this section with an important observation on \( A_p(G) \) given by Miao. We record it for convenience.

**Lemma 3.3.10.** Let \( \varphi \in MA_p(G) \) and \( a \in A_{p,c}(G) \). Then the map \( x \mapsto (x\varphi)a \) from \( G \) to \( A_p(G) \) is continuous.\(^8\)

**Proof.** If \( x \) is in a neighborhood \( V \) of the neutral element \( e \) of \( G \) such that \( \bar{V} \) is compact, then by Remark 3.3.2.9, there is \( u \in A_p(G) \) such that \( u(t) = 1 \) for all \( t \in \bar{V} \cdot \text{supp}(a) \). Hence for all \( x \in V \), \( (x\varphi)a = (x(\varphi u))a \) and \( \varphi a = (\varphi u)a \). Thus for all \( x \in V \), we have by Remark 3.3.2.11

\[
\|(x\varphi)a - \varphi a\|_{A_p(G)} = \|(x(\varphi u))a - (\varphi u)a\|_{A_p(G)} \leq \|a\|_{A_p(G)} \|x(\varphi u) - \varphi u\|_{A_p(G)} \to 0
\]
as \( x \to e \). \(\Box\)

**Corollary 3.3.11.** Let \( \varphi \in MA_p(G) \) and \( a \in A_{p,c}(G) \). Let \( V \) be a neighborhood of \( e \). If \( f_V \in L_1(G) \) with \( \|f_V\|_1 = 1 \) and \( f_V(t) = 0 \) for \( t \not\in V \), then

\[
\lim_{V \to e} \int_G \langle (x\varphi)a, T \rangle f_V(x) \, dx = \langle \varphi a, T \rangle
\]

for all \( T \in PM_p(G) \).

**Proof.** It is immediate since

\[
\left| \int_G \langle (x\varphi)a - \varphi a, T \rangle f_V(x) \, dx \right| = \left| \int_G \langle (x\varphi)a - \varphi a, T \rangle f_V(x) \, dx \right| \leq \|(x\varphi)a - \varphi a\|_{A_p(G)} \|T\|.
\]

\(\Box\)

**Corollary 3.3.12.** Let \( \varphi \in MA_p(G) \) and \( a \in A_{p,c}(G) \). For any \( f \in L_1(G) \), there is an element \( \eta \in A_p(G) \) such that \( \eta = \int_G (x^{-1}\varphi)a f(x) \, dx \) and

\[
\langle T, \eta \rangle = \int_G \langle (x^{-1}\varphi)a, T \rangle f(x) \, dx
\]

\(^7\)\( A_{p,c}(G) = A_p(G) \cap C_{00}(G) \) is dense in \( A_p(G) \). See Remark 3.3.2.9.

\(^8\)Note that \( x\varphi \in MA_p(G) \). Indeed, if \( u \in A_p(G) \), then \( x\varphi \cdot u = x \langle \varphi \cdot x^{-1} \rangle u \in A_p(G) \) by Remark 3.3.2.12.
for all $T \in PM_p(G)$. Moreover, $\eta = (f \star \varphi)a$ and hence

$$
\langle T, (f \star \varphi)a \rangle = \int_G \langle (x^{-1}\varphi)a, T \rangle f(x)dx
$$

for all $T \in PM_p(G)$.

**Proof.** This follows from [Ped89, page.76], because $x \mapsto (x^{-1}\varphi)a$ is a bounded continuous function and $f(x)dx$ is a bounded Radon measure. Second part follows from the calculation

$$(f \star \varphi)a(t) = \int_G f(x)\varphi(x^{-1}t)a(t)dx = \int_G (x^{-1}\varphi(t))a(t)f(x)dx.$$  

\[\square\]

**Corollary 3.3.13.** Let $\varphi \in M_{cb}A_p(G)$ and $v \in L_{1}(G)$ with $\|v\|_{1} = 1$. Then $v \star \varphi \in M_{cb}A_p(G)$ and $\|v \star \varphi\|_{M_{cb}A_p(G)} \leq \|\varphi\|_{M_{cb}A_p(G)}$.

**Proof.** Let $n \in \mathbb{N}$ and let $T = [T_{ij}] \in M_{n}(PM_p(G))$. Suppose that $f = \{f_{j}\}_{j=1}^{n} \in L_{p}(G) \otimes_{p'} \ell_{p}^{n}$ and $g = \{g_{i}\}_{i=1}^{n} \in L_{p'}(G) \otimes_{p'} \ell_{p'}^{n}$ have compact supports. Then

$$
|\langle g, (M_{v \star \varphi})_{n}(T)(f) \rangle| = \left| \sum_{i,j=1}^{n} \langle g_{i}, M_{v \star \varphi}(T_{ij})(f_{j}) \rangle \right|
$$

$$
= \left| \sum_{i,j=1}^{n} \langle g_{i} \star \hat{f}_{j}, M_{v \star \varphi}(T_{ij}) \rangle \right|
$$

$$
= \left| \sum_{i,j=1}^{n} \langle (g_{i} \star \hat{f}_{j})(v \star \varphi), T_{ij} \rangle \right|
$$

(Corollary 3.3.12)$$
\leq \left| \sum_{i,j=1}^{n} \int_{G} \langle (g_{i} \star \hat{f}_{j})(x^{-1}\varphi), T_{ij} \rangle v(x)dx \right|
$$

$$
= \left| \sum_{i,j=1}^{n} \int_{G} \langle g_{i}, M_{x^{-1}\varphi}(T_{ij})(f_{j}) \rangle v(x)dx \right|
$$

$$
= \left| \int_{G} \langle g, (M_{x^{-1}\varphi})_{n}(T)(f) \rangle v(x)dx \right|
$$

$$
\leq \|g\|_{p'} \|x^{-1}\varphi\|_{M_{cb}A_p(G)} \|T\|\|f\|_{p}
$$

(Remark 3.3.9.2)$$
\leq \|g\|_{p'} \|\varphi\|_{M_{cb}A_p(G)} \|T\|\|f\|_{p}
$$

This completes the proof.  \[\square\]
3.4 Other Related Spaces and Some Open Questions

In this section, we collect some results from [Run05].

**Definition 3.4.1.** Let $G$ be a locally compact group. A *representation* of $G$ is a pair $(\pi, E)$ where $E$ is a Banach space and $\pi$ is a group homomorphism from $G$ into the invertible isometries on $E$ which is continuous with respect to the given topology on $G$ and the strong operator topology on $\mathcal{B}(E)$. We denote by $\text{Rep}_p(G)$ the collection of all representations of $G$ on a $SQ_p$ space. A *coefficient function* of $(\pi, E)$ is a function $f : G \to \mathbb{C}$ of the form

$$f(x) = \langle \pi(x)\xi, \varphi \rangle \quad (x \in G),$$

where $\xi \in E$ and $\varphi \in E'$. Finally, we let

$$B_p(G) = \{ f : G \to \mathbb{C} : f \text{ is a coefficient of some } (\pi, E) \in \text{Rep}_p(G) \}.$$

**Remark 3.4.2.** Let $G$ be a locally compact group.

1. Any representation $(\pi, E)$ of $G$ induces a representation of the group algebra $L_1(G)$ on $E$ via integration, that is,

$$\pi(f) := \int_G f(x)\pi(x)dx \quad (f \in L_1(G)).$$

2. A representation $(\pi, E)$ is called *cyclic* if there is $\xi \in E$ such that $\pi(L_1(G))\xi$ is dense in $E$. We let

$$\text{Cyc}_p(G) := \{ (\pi, E) : (\pi, E) \text{ is cyclic} \}$$

and for $f \in B_p(G)$, we define $\|f\|_{B_p(G)}$ as the infimum over all expressions $\sum_{n=1}^{\infty} \|\xi_n\|\|\varphi_n\|$, where, for each $n \in \mathbb{N}$, there is $(\pi_n, E_n) \in \text{Cyc}_p(G)$ with $\xi_n \in E_n$ and $\varphi_n \in E'_n$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\|\|\varphi_n\| < \infty \quad \text{and} \quad f(x) = \sum_{n=1}^{\infty} \langle \pi_n(x)\xi_n, \varphi_n \rangle \quad (x \in G).$$

Then $\| \cdot \|_{B_p(G)}$ defines a norm on $B_p(G)$.

Runde proved the following result.

**Theorem 3.4.3.** Let $G$ be a locally compact group and let $B_p(G)$ be equipped with $\| \cdot \|_{B_p(G)}$. Then

1. $B_p(G)$ is a commutative Banach algebra.

2. There are contractive inclusions $A_p(G) \subseteq B_p(G) \subseteq MA_p(G)$.
3. For \(2 \leq q \leq p\) or \(p \leq q \leq 2\), the inclusion \(B_q(G) \subseteq B_p(G)\) is a contraction.

Proof. Theorem 4.7 and Corollary 5.3 in [Run05].

Theorem 3.4.4. Let \(G\) be a locally compact group, then \(PF_p(G)'\) embeds contractively into \(B_p(G)\).

Proof. Theorem 6.6 in [Run05].

Let \(f \in L_1(G)\), then \(f\) defines a bounded linear functional \(\Phi_f\) on \(A_p(G) \subseteq C_0(G)\) via integration: for \(u \in A_p(G)\), \(\Phi_f(u) = \int_G f(s)u(s)ds\).

Lemma 3.4.5. With notations as above, \(\|\Phi_f\| = \|\lambda_p(f)\|\) for all \(f \in L_1(G)\).

Proof. This is proved in [Cow79]. We include the proof for convenience. By (3.1),

\[
\|\lambda_p(f)\| = \sup\{\|\langle \xi, f \ast \eta \rangle\| : \xi \in L_{p'}(G), \eta \in L_p(G), \|\xi\|, \|\eta\| \leq 1\}.
\]

Since

\[
\langle \xi, f \ast \eta \rangle = \int_{s \in G} \xi(s) \int_{t \in G} f(t)\eta(t^{-1}s)dt ds
\]

\[
= \int_{t \in G} f(t) \int_{s \in G} \xi(s)\eta(t^{-1}s)ds dt
\]

\[
= \Phi_f(\xi \ast \eta),
\]

it follows that \(\|\lambda_p(f)\| \leq \|\Phi_f\|\). For the other direction, let \(\epsilon > 0\) and let \(u \in A_p(G)\) with \(\|u\|_{A_p(G)} = 1\) such that \(|\Phi_f(u)| \geq \|\Phi_f\| - \epsilon\). Let \(u = \sum_{n=1}^{\infty} \xi_n \ast \tilde{\eta}_n\) with \(\sum_{n=1}^{\infty} \|\xi_n\|_{p'} \|\eta_n\|_p < 1 + \epsilon\), then

\[
|\Phi_f(u)| = \left| \sum_{n=1}^{\infty} \langle \xi_n, f \ast \eta_n \rangle \right|
\]

\[
\leq \sum_{n=1}^{\infty} \|\lambda_p(f)\| \|\xi_n\|_{p'} \|\eta_n\|_p
\]

\[
< (1 + \epsilon) \|\lambda_p(f)\|
\]

and hence \(\|\lambda_p(f)\| > \frac{\|\Phi_f\| - \epsilon}{1 + \epsilon}\). Letting \(\epsilon \to 0\), we get the desired inequality.

Remark 3.4.6. By Remark 3.3.7, Theorems 3.4.3, 3.4.4, and Lemma 3.4.5, we have contractive inclusions

\[A_p(G) \subseteq PF_p(G)' \subseteq B_p(G) \subseteq M_{cb}A_p(G) \subseteq MA_p(G) \subseteq C_b(G).\]

Runde showed that some of these sets agree when \(G\) is amenable.
Theorem 3.4.7. [Run05, Theorem 6.7] Let $G$ be an amenable locally compact group, then $PF_p(G)'$, $B_p(G)$, and $MA_p(G)$ are equal with identical norms.

By Remark 3.4.6, if $G$ is amenable then $PF_p(G)' = B_p(G) = M_{cb}A_p(G) = MA_p(G)$. In fact, a certain equality among these sets implies the amenability of $G$ as in the following remark.

Remark 3.4.8. For a locally compact group $G$,

1. $PF_p(G)' = MA_p(G)$ if and only if $G$ is amenable [Cow79].
2. $B_2(G) = MA_2(G)$ if and only if $G$ is amenable. [Neb82] for discrete case, [Los84] for general case.
3. If $G$ is discrete, $B_2(G) = M_{cb}A_2(G)$ if and only if $G$ is amenable [Boz85].

Remark 3.4.8 leads us to the following questions.\(^9\)

Question 3.4.9. For a locally compact group $G$ (see Theorem 3.4.7 and the explanation right after that),

1. Do we have $G$ is amenable if (and only if) $PF_p(G)' = M_{cb}A_p(G)$?
2. Do we have $G$ is amenable if (and only if) $PF_p(G)' = B_p(G)$?
3. Do we have $G$ is amenable if (and only if) $B_p(G) = MA_p(G)$?
4. Do we have $G$ is amenable if (and only if) $B_p(G) = M_{cb}A_p(G)$?

3.5 Amenability and Multiplier Algebras

In this section, $G$ will always denote a locally compact group. Let $K$ be a compact subset of $G$. Let $A_p(K)$ denote the space of restrictions to $K$ of functions in $A_p(G)$ equipped with the norm defined by

$$\|w\|_{A_p(K)} = \inf\{|\|u\|_{A_p(G)} : w = u|_K, u \in A_p(G)\}.$$  

In other words, we identify $A_p(K)$ with the quotient space $A_p(G)/M_K$, where $M_K = \{u \in A_p(G) : u|_K \equiv 0\}$.

In [Cow79], Cowling gives a useful characterization of $PF_p(G)'$.

Theorem 3.5.1. [Cow79, Theorem 4] $w \in L_\infty(G)$ belongs to $PF_p(G)'$ and has norm at most $C$ if and only if, for any compact subset $K$ of $G$,

$$w|_K \in A_p(K) \quad and \quad \|w|_K\|_{A_p(K)} \leq C.$$  

\(^9\)We have positive answers for the first two in Question 3.4.9. See Theorem 3.5.3 below.
We recall the proof of the following theorem originally proved by Cowling [Cow79].

**Theorem 3.5.2.** For a locally compact group $G$, $PF_p(G)' = MA_p(G)$ isometrically if and only if $G$ is amenable.

*Proof. (⇒) First of all, by Remark 3.3.2, for each compact $K$, the constant function $1_K \in A_p(K)$. If we assume that $\sup\{\|1_K\|_{A_p(K)} : K \text{ compact}\} < \infty$, then we would be able to find a net $\{u_K \in A_p(G) : K \text{ compact}\}$ such that $u_K \equiv 1$ on $K$ for each $K$ and $C := \sup\{\|u_K\|_{A_p(G)}\} < \infty$. Now let $\varphi \in A_p(G)$ and $\epsilon > 0$. Consider $\varphi_0 \in A_{p,\epsilon}(G)$ such that $\|\varphi - \varphi_0\| < \epsilon$. For every $K$ containing the support of $\varphi_0$, we get $\varphi_0 u_K = \varphi_0$ and it follows that

$$
\|\varphi u_K - \varphi\| = \|(\varphi - \varphi_0)u_K + \varphi_0 u_K - u\| \leq (C + 1)\epsilon.
$$

This shows that $\{u_K \in A_p(G) : K \text{ compact}\}$ is a bounded approximate identity for $A_p(G)$ and hence $G$ must be amenable. Therefore, if $G$ were not amenable, then we must have $\sup\{\|1_K\|_{A_p(K)} : K \text{ compact}\} = \infty$. By Theorem 3.5.1, the constant function $1_G$ is not in $PF_p(G)'$ but $1_G$ is always in $MA_p(G)$.

(⇐) From Theorems 3.4.3 and 3.4.4, we see that $PF_p(G)' \subseteq MA_p(G)$ contractively. Suppose $w \in MA_p(G)$ and let $\epsilon > 0$. If $G$ is amenable, then by Remark 3.3.2, for every compact $K$ there exists $u_K$ such that $u_K \equiv 1$ on $K$ with $\|u_K\|_{A_p(G)} \leq 1 + \epsilon$. Now $w|_K = (wu_K)|_K \in A_p(K)$ and hence $\|w|_K\|_{A_p(K)} \leq \|w\|_{MA_p(G)}(1 + \epsilon)$. Since $\epsilon$ is arbitrary, it follows that $\|w|_K\|_{A_p(K)} \leq \|w\|_{MA_p(G)}$ and by Theorem 3.5.1, $w \in PF_p(G)'$. Finally, to prove that $\|w\|_{MA_p(G)} \leq \|w\|_{PF_p(G)'},$ it suffices to show that $\|w\varphi_0\|_{A_p(G)} \leq \|w\|_{PF_p(G)'}\|\varphi_0\|_{A_p(G)}$ for any compactly supported $\varphi_0$. Let $\epsilon > 0$ and $K$ be the support of $\varphi_0$. By definition of $A_p(K)$, one can find $u \in A_p(G)$ such that $u|_K = w|_K$ and $\|u\|_{A_p(G)} \leq \|w\|_{A_p(G)} + \epsilon$. Now by Theorem 3.5.1,

$$
\|w\varphi_0\|_{A_p(G)} = \|u\varphi_0\|_{A_p(G)} \leq (\|w|_K\|_{A_p(K)} + \epsilon)\|\varphi_0\|_{A_p(G)} \leq (\|w\|_{PF_p(G)'} + \epsilon)\|\varphi_0\|_{A_p(G)}
$$

and the result follows.

We can now answer the first and the second questions in Question 3.4.9 positively.

**Theorem 3.5.3.** For a locally compact group $G$, $PF_p(G)' = M_{cb}A_p(G)$ (respectively, $PF_p(G)' = B_p(G)$) isometrically if and only if $G$ is amenable.

*Proof. (⇐) This direction comes from Theorem 3.5.2.

(⇒) The proof is almost identical to that of Theorem 3.5.2. The only point to note is that $1_G$ is always in $M_{cb}A_p(G)$ (respectively, $B_p(G)$) as well.
### 3.6 $Q_{pcb}(G)$ as the Predual of $M_{cb}A_p(G)$

In this section, we will show that $M_{cb}A_p(G)$ is a dual space, which is essential to study $p$-operator space approximation property. To begin with, let $f \in L_1(G)$. For each $\varphi \in M_{cb}A_p(G) \subseteq C_b(G)$, the integration $\int_G f(t)\varphi(t)dt$ defines a bounded linear functional on $M_{cb}A_p(G)$ and hence we can embed $L_1(G)$ into $(M_{cb}A_p(G))'$. Let $\| \cdot \|_{Q,p}$ denote the norm on $L_1(G)$ inherited from this structure and let $Q_{pcb}(G)$ denote the norm closure of $(L_1(G), \| \cdot \|_{Q,p})$ in $(M_{cb}A_p(G))'$.

**Remark 3.6.1.** Let $f \in L_1(G)$. For each $\psi \in MA_p(G) \subseteq C_b(G)$, the integration $\int_G f(t)\psi(t)dt$ defines a bounded linear functional on $MA_p(G)$ and hence we can embed $L_1(G)$ into $(MA_p(G))'$. Let $Q_p(G)$ denote the norm closure of $L_1(G)$ with respect to the norm of $(MA_p(G))'$, then we have $Q_p(G)' = MA_p(G)$ [Mia04, Theorem 3.2].

**Proposition 3.6.2 (Miao).** Let $G$ be a locally compact group, then we have an isometric isomorphism $Q_{pcb}(G)' = M_{cb}A_p(G)$.

**Proof.** We give a proof for convenience. First of all, it is clear that each $\varphi \in M_{cb}A_p(G)$ is in $Q_{pcb}(G)'$ with $\| \varphi \|_{Q_{pcb}(G)'} \leq \| \varphi \|_{M_{cb}A_p(G)}$. Conversely, let $m \in Q_{pcb}(G)'$ and $\| m \|_{Q_{pcb}(G)'} = 1$. Since $Q_{pcb}(G)$ is a closed subspace of $(M_{cb}A_p(G))'$, by the Hahn-Banach Theorem, we can extend $m$ to a linear functional $\tilde{m}$ on $(M_{cb}A_p(G))'$ with $\| \tilde{m} \|_{(M_{cb}A_p(G))''} = 1$. By Goldstine’s theorem, there is a net $\{m_\alpha\}$ in $M_{cb}A_p(G)$ such that $\| m_\alpha \|_{M_{cb}A_p(G)} \leq 1$ and $m_\alpha \to \tilde{m}$ in the $\sigma((M_{cb}A_p(G))'', (M_{cb}A_p(G))')$-topology. In particular, $\langle m_\alpha, f \rangle \to \langle m, f \rangle$ for any $f \in Q_{pcb}(G)$.

Let $f \in L_1(G)$. Since $M_{cb}A_p(G) \subseteq MA_p(G)$ with $\| \cdot \|_{MA_p(G)} \leq \| \cdot \|_{M_{cb}A_p(G)}$ on $M_{cb}A_p(G)$, we have $\| f \|_{Q_{pcb}(G)} \leq \| f \|_{Q_p(G)}$ and hence $m \in Q_p(G)' = MA_p(G)$ by Remark 3.6.1. We need to show that $\mathfrak{M} = m' : PM_p(G) \to PM_p(G)$ is $p$-completely contractive. So let $n \in \mathbb{N}$ and let $T = [T_{ij}] \in M_n(PM_p(G))$.

Suppose that $f = \{f_i\}_{i=1}^n \in L_p(G) \otimes_p \ell^p_n$ and $g = \{g_j\}_{j=1}^n \in L_{p'}(G) \otimes_{p'} \ell^p_n$ have compact supports. For each neighborhood $V$ of the neutral element $e$ of $G$, choose an $f_V$ in $A_p(G)$ such that $f_V(t) = 0$ for $t \notin V$ and $\| f_V \|_1 = 1$. Then we have

\[
\langle g, \mathfrak{M}_n(T)(f) \rangle = \sum_{i,j=1}^n \langle g_i, \mathfrak{M}(T_{ij})(f_j) \rangle
\]

\[
= \sum_{i,j=1}^n \langle g_i * \hat{f}_j, \mathfrak{M}(T_{ij}) \rangle
\]

\[
= \sum_{i,j=1}^n \langle (g_i * \hat{f}_j) m, T_{ij} \rangle
\]
\[(\text{Corollary 3.3.11}) = \left| \lim_{V} \sum_{i,j=1}^{n} \int_{G} \langle (g_{i} \ast f_{j})(x^{-1}m), T_{ij} \rangle f_{V}(x) dx \right| \]

\[(\text{Corollary 3.3.12}) = \left| \lim_{V} \sum_{i,j=1}^{n} \langle (g_{i} \ast f_{j}), M_{f_{V} \ast m}(T_{ij}) \rangle \right| \]

Here note that $\langle (g_{i} \ast f_{j}), M_{f_{V} \ast m}(T_{ij}) \rangle = \langle \omega_{T_{ij}, g_{i} \ast f_{j}, f_{V}}, m \rangle$, where $\omega_{T_{ij}, g_{i} \ast f_{j}, f_{V}} \in L_{1}(G)$ [Mia09, Lemma 3.1]. Therefore

\[
\left| \lim_{V} \sum_{i,j=1}^{n} \langle (g_{i} \ast f_{j}), M_{f_{V} \ast m}(T_{ij}) \rangle \right| = \left| \lim_{V} \sum_{i,j=1}^{n} \langle \omega_{T_{ij}, g_{i} \ast f_{j}, f_{V}}, m \rangle \right|
\]

\[
= \left| \lim_{\alpha} \lim_{V} \sum_{i,j=1}^{n} \langle \omega_{T_{ij}, g_{i} \ast f_{j}, f_{V}}, m_{\alpha} \rangle \right|
\]

\[
= \left| \lim_{\alpha} \lim_{V} \sum_{i,j=1}^{n} \langle g_{i}, M_{f_{V} \ast m_{\alpha}}(T_{ij})(f_{j}) \rangle \right|
\]

\[
= \left| \lim_{\alpha} \langle g, (M_{f_{V} \ast m_{\alpha}})(T)(f) \rangle \right|
\]

\[
= \lim_{\alpha} \| g \|_{p}^{\prime} \| f_{V} \ast m_{\alpha} \|_{M_{cb} A_{p}(G)} \| T \| \| f \|_{p}
\]

\[(\text{Corollary 3.3.13}) \leq \| g \|_{p}^{\prime} \| m_{\alpha} \|_{M_{cb} A_{p}(G)} \| T \| \| f \|_{p}
\]

\[
\leq \| g \|_{p}^{\prime} \| T \| \| f \|_{p}
\]

and it follows that $\mathfrak{M}$ is $p$-completely contractive.

\[\square\]

### 3.7 Description of $Q_{pcb}(G)$ for a Discrete Group $G$

For future applications, we want to characterize $Q_{pcb}(G)$ for a discrete group $G$. We begin with a lemma.

**Lemma 3.7.1.** Let $\mu$ and $\nu$ be measures.

1. If $A \subseteq B(L_{p}(\mu))$ is a $p$-operator space and $T : A \to A$ is a $p$-completely bounded map, then there exists a unique bounded map $\hat{T} : B(L_{p}(\nu)) \hat{\otimes} A \to B(L_{p}(\nu)) \hat{\otimes} A$ such that

\[
\hat{T}(b \otimes a) = b \otimes T(a), \quad b \in B(L_{p}(\nu)), \quad a \in A,
\]

with $\| \hat{T} \| \leq \| T \|_{pcb}$.

2. If $M$ is a weak*-$\ast$-closed subalgebra of $B(L_{p}(\mu))$, and $T : M \to M$ is a weak* continuous $p$-completely
bounded map, then there exists a unique weak*-continuous bounded map $\tilde{T} : \mathcal{B}(L_p(\nu)) \otimes M \to \mathcal{B}(L_p(\nu)) \otimes M$ such that

$$\tilde{T}(d \otimes c) = d \otimes T(c), \quad d \in \mathcal{B}(L_p(\nu)), \quad c \in M,$$

where $\mathcal{B}(L_p(\nu)) \otimes M$ denotes the weak*-closure of the algebraic tensor product $\mathcal{B}(L_p(\nu)) \otimes M$ in $\mathcal{B}(L_p(\nu \times \mu))$. Moreover, we have $\|\tilde{T}\| \leq \|T\|_{\text{pcb}}$.

**Remark 3.7.2.**

1. Once Lemma 3.7.1 is established, we can justify (and will use) the notation $\tilde{T} = id_{L_p(\nu)} \otimes T$.

2. In fact, $\tilde{T}$ is $p$-completely bounded with $\|\tilde{T}\|_{\text{pcb}} \leq \|T\|_{\text{pcb}}$. See Lemma 3.7.3 below.

**Proof of Lemma 3.7.1.** 1. We use the rigid $L_p$-structure of $L_p(\nu)$. Let $\{E_\gamma\}$ be a net of finite-dimensional subspaces of $L_p(\nu)$, directed by inclusion, such that $\bigcup_\gamma E_\gamma$ is dense in $L_p(\nu)$ and each $E_\gamma$ is isometric to $\ell_p^m(\gamma)$ with $m(\gamma) = \dim E_\gamma$ [LP68]. Let $P_\gamma$ denote the norm 1 projection from $L_p(\nu)$ onto $E_\gamma$, so that we identify $P_\gamma \mathcal{B}(L_p(\nu)) P_\gamma$ with $M_{m(\gamma)}$. Define an operator $T_0$ on $\mathcal{B}(L_p(\nu)) \otimes_{\text{alg}} A$ by $T_0(\sum b_i \otimes a_i) = \sum b_i \otimes T(a_i)$, then for every $\sum b_i \otimes a_i \in \mathcal{B}(L_p(\nu)) \otimes_{\text{alg}} A$, we have

$$\left\| (P_\gamma \otimes id_{L_p(\nu)}) T_0 \left( \sum b_i \otimes a_i \right) (P_\gamma \otimes id_{L_p(\nu)}) \right\| \leq \|T_0\|_{\text{pcb}} \left\| (P_\gamma \otimes id_{L_p(\nu)}) \left( \sum b_i \otimes a_i \right) (P_\gamma \otimes id_{L_p(\nu)}) \right\|$$

Since $(P_\gamma)$ converges strongly to $id_{L_p(\nu)}$, it follows that

$$\left\| T_0 \left( \sum b_i \otimes a_i \right) \right\| \leq \|T\|_{\text{pcb}} \left\| \sum b_i \otimes a_i \right\|$$

and $T_0$ has a bounded extension $\tilde{T}$ to $\mathcal{B}(L_p(\nu)) \otimes_{\text{wk}} A$.

2. This is [Daw10, Theorem 6.4]. \qed

**Lemma 3.7.3.** Let $T$ be as in Lemma 3.7.1. Then $id_{L_p(\nu)} \otimes T$ (See Remark 3.7.2 for definition) is $p$-completely bounded with $\|id_{L_p(\nu)} \otimes T\|_{\text{pcb}} \leq \|T\|_{\text{pcb}}$.

**Proof.** Fix $n \in \mathbb{N}$ and consider $(id_{L_p(\nu)} \otimes T)_n : M_n(\mathcal{B}(L_p(\nu)) \otimes_{\text{wk}} A) \to M_n(\mathcal{B}(L_p(\nu)) \otimes_{\text{wk}} A)$. Since $M_n(\mathcal{B}(L_p(\nu)) \otimes_{\text{wk}} A) \cong \mathcal{B}(L_p(\nu)^n) \otimes_{\text{wk}} A$, we can identify $(id_{L_p(\nu)} \otimes T)_n$ with $id_{L_p(\nu)^n} \otimes T$. The result
Proposition 3.7.4. Let $G$ be a discrete group. Then

$$Q_{pcb}(G) = \{ \omega_{a,\varphi} : a \in \mathcal{B}(\ell_p) \quad \varphi \in (\mathcal{B}(\ell_p) \otimes PF_p(G))' \}.$$ 

Proof. \(\supseteq\): Since \(\|\omega_{a,\varphi}\| \leq \|a\|\|\varphi\|\) and \(\text{span}\{\lambda_p(s) : s \in G\}\) is norm dense in \(PF_p(G)\), it suffices to show that \(\omega_{a,\varphi} \in Q_{pcb}(G)\) for \(a = b \otimes \lambda_s\) with \(b \in \mathcal{B}(\ell_p)\). Indeed, for \(u \in M_{cb}A_p(G)\), we get

$$\omega_{b \otimes \lambda_s,\varphi}(u) = (id_{\mathcal{B}(\ell_p)} \otimes M_u)(b \otimes \lambda_s, \varphi) = \langle b \otimes u(s)\lambda_s, \varphi \rangle = u(s)\langle b \otimes \lambda_s, \varphi \rangle = \sum_{t \in G} u(t)g(t),$$

where \(g(\cdot) = \langle b \otimes \lambda_s, \varphi \rangle \delta_s(\cdot)\) is a function in \(\ell_1(G)\). By definition of \(Q_{pcb}(G)\), this shows that \(\omega_{b \otimes \lambda_s,\varphi} \in Q_{pcb}(G)\).

\(\subseteq\): We follow an idea similar to that given in [HK94]. Let

$$S = \{ \omega_{a,\varphi} : a \in \mathcal{B}(\ell_p) \quad \varphi \in (\mathcal{B}(\ell_p) \otimes PM_p(G))'_1 \}.$$ 

Then by the argument above, \(S\) is contained in the closed unit ball of \(Q_{pcb}(G)\). It is also easy to check that \(S\) is balanced. We claim that \(S\) is convex. To this end, first note that we can identify \(\mathcal{B}(\ell_p \oplus \ell_p) \otimes PF_p(G)\) with \(M_2(\mathcal{B}(\ell_p)) \otimes PF_p(G) = M_2(\mathcal{B}(\ell_p) \otimes PF_p(G))\). By Theorem 2.3.12,

$$(M_2(\mathcal{B}(\ell_p) \otimes PF_p(G))')' = \mathcal{N}_2 \otimes (\mathcal{B}(\ell_p) \otimes PF_p(G))'$$

isometrically, where \(\mathcal{N}_2\) denotes the space of nuclear operators on \(\ell_p^2\), and hence we can also identify \((\mathcal{B}(\ell_p \oplus \ell_p) \otimes PF_p(G))'\) with \(\mathcal{N}_2 \otimes (\mathcal{B}(\ell_p) \otimes PF_p(G))'\). In particular, the duality is given in such a way that if \(b = [b_{ij}] \in \mathcal{B}(\ell_p \oplus \ell_p \otimes PF_p(G)\) and \(\varphi = [\varphi_{ij}] \in (\mathcal{B}(\ell_p \oplus \ell_p) \otimes PF_p(G))'\), then

$$\langle b, \varphi \rangle = \sum_{i,j=1}^2 \langle b_{ij}, \varphi_{ij} \rangle.$$

If \(T\) is a \(p\)-completely bounded operator on \(PF_p(G)\), then for any \(a = [a_{ij}] \in \mathcal{B}(\ell_p \oplus \ell_p) \otimes PF_p(G)\), we have

$$id_{\mathcal{B}(\ell_p \oplus \ell_p)} \otimes T(a) = [id_{\mathcal{B}(\ell_p)} \otimes T(a_{ij})]$$

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(See Lemma 3.7.1 for the definition of $id_{B(\ell_p \oplus \ell_p)} \otimes T$ and $id_{B(\ell_p)} \otimes T$). Now let $\omega_{a_1, \varphi_1}$ and $\omega_{a_2, \varphi_2}$ be elements of $S$, and suppose $0 \leq \lambda \leq 1$. Let

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in B(\ell_p \oplus \ell_p) \otimes PF_p(G), \quad \varphi = \begin{bmatrix} \lambda \varphi_1 & 0 \\ 0 & (1-\lambda)\varphi_2 \end{bmatrix} \in (B(\ell_p \oplus \ell_p) \otimes PF_p(G))'. $$

Then $\|a\| = \max\{\|a_1\|, \|a_2\|\} \leq 1$ and $\|\varphi\| \leq \lambda\|\varphi_1\| + (1-\lambda)\|\varphi_2\| \leq 1$. Now for any $p$-completely bounded operator $T$ on $PF_p(G)$,

$$\omega_{a, \varphi}(T) = \langle id_{B(\ell_p \oplus \ell_p)} \otimes T(a), \varphi \rangle = \langle id_{B(\ell_p)} \otimes T(a_1), \lambda \varphi_1 \rangle + \langle id_{B(\ell_p)} \otimes T(a_2), (1-\lambda)\varphi_2 \rangle = (\lambda \omega_{a_1, \varphi_1} + (1-\lambda)\omega_{a_2, \varphi_2})(T).$$

Moreover, since $\ell_p$ and $\ell_p \oplus \ell_p$ are isometrically isomorphic, $\omega_{a, \varphi} = \omega_{b, \psi}$ for some $b \in (B(\ell_p) \otimes PF_p(G))_1$ and some $\psi \in ((B(\ell_p) \otimes PF_p(G))')_1$. This shows that $S$ is convex.

Now we claim that $S$ is norm dense in the closed unit ball of $Q_{pcb}(G)$. Suppose this is not the case and $\omega \in Q_{pcb}(G)_1$ is not in the closure of $S$. Using the geometric Hahn-Banach theorem and the fact that $S$ is balanced, we can find $u \in M_{cb}A_p(G)$ such that

$$|\langle \omega_{a, \varphi}, u \rangle| \leq 1 < \langle \omega, u \rangle, \quad \forall \omega_{a, \varphi} \in S.$$

However, this implies

$$\|u\|_{pcb} \leq \sup\{\|id_{B(\ell_p)} \otimes M_u(a)\| : a \in (B(\ell_p) \otimes PF_p(G))_1\} = \sup\{\|id_{B(\ell_p)} \otimes M_u(a)\|, \varphi \} : a \in (B(\ell_p) \otimes PF_p(G))_1, \varphi \in ((B(\ell_p) \otimes PF_p(G))')_1\} = \sup\{\|\omega_{a, \varphi}(u)\| : a \in (B(\ell_p) \otimes PF_p(G))_1, \varphi \in ((B(\ell_p) \otimes PF_p(G))')_1\} \leq 1.$$

But then this would imply $|\langle \omega, u \rangle| \leq 1$, a contradiction. This shows that $S$ is norm dense in the closed unit ball of $Q_{pcb}(G)$.

Let $\ell_p^{(\infty)}$ denote the $p$-direct sum of a countably infinite number of copies of $\ell_p$. Since $\ell_p$ is isometrically isomorphic to $\ell_p^{(\infty)}$, to complete the proof it suffices to show that every $\omega \in Q_{pcb}(G)$ is of the form $\omega = \omega_{a, \varphi}$ for some $a \in B(\ell_p^{(\infty)} \otimes PF_p(G))$ and some $\varphi \in (B(\ell_p^{(\infty)}) \otimes PF_p(G))'$. Without loss of generality, we can assume that $\omega \in Q_{pcb}(G)_1$. Then there is an $\omega_1 \in S$ such that $\|\omega - \omega_1\| < \frac{1}{2}$. Since $2(\omega - \omega_1) \in Q_{pcb}(G)_1,$
there is an \( \omega_2 \in S \) such that \( \| \omega - \omega_1 - \frac{1}{2} \omega_2 \| < \frac{1}{2^2} \). Continuing in this fashion, we find a sequence \( \{\omega_n\} \) in \( S \) such that
\[
\left\| \omega - \sum_{i=1}^{n} \frac{1}{2^{i-1}} \omega_i \right\| < \frac{1}{2^n}.
\]
Thus \( \omega = \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} \omega_i \). Since \( \omega_i \in S \), there are sequences \( b_i \in (B(\ell_p) \otimes PF_p(G))_1 \) and \( \psi_i \in ((B(\ell_p) \otimes PF_p(G))'_1 \) such that \( \omega_i = \omega_{b_i,\psi_i} \). Let \( \alpha_i = (2^{-i+1})^{1/2} \), let \( a_i = \alpha_i b_i \), and let \( \varphi_i = \alpha_i \psi_i \). Let \( a \in M_\infty(B(\ell_p) \otimes PF_p(G)) = B(\ell_p^{\infty}) \otimes PF_p(G) \) be the diagonal matrix with diagonal entries \( a_1, a_2, \ldots \).

Since \( \|a_i\| \to 0 \), \( a \) in fact lies in \( K(\ell_p^{\infty}) \otimes PF_p(G) \). Moreover, since \( \sum_{i=1}^{\infty} \|\varphi_i\| < \infty \), we can define \( \varphi_0 \in (K(\ell_p^{\infty}) \otimes PF_p(G))' = N_\infty \otimes (B(\ell_p) \otimes PF_p(G))' \) by
\[
\varphi_0([a_{ij}]) = \sum_{i=1}^{\infty} \langle a_{ii}, \varphi_i \rangle, \quad [a_{ij}] \in K(\ell_p^{\infty}) \otimes PF_p(G).
\]
Extend \( \varphi_0 \) to \( \varphi \in (B(\ell_p^{\infty}) \otimes PF_p(G))' \) using Hahn-Banach theorem, then it follows that \( \omega = \omega_{a,\varphi} \) and this completes the proof. \( \square \)

**Remark 3.7.5.** The same argument as above actually works to show that
\[
\mathcal{Q}_{pcb}(G) = \{ \omega_{a,\varphi} : a \in K(\ell_p) \otimes PF_p(G), \quad \varphi \in (K(\ell_p) \otimes PF_p(G))' \}.
\]

**Proposition 3.7.6.** Let \( G \) be a discrete group. Then
\[
\mathcal{Q}_{pcb}(G) = \{ \omega_{a,\varphi} : a \in K(\ell_p) \otimes PF_p(G), \quad \varphi \in \mathcal{N}(\ell_p \otimes \ell_p(G)) \},
\]
where \( \mathcal{N}(\ell_p \otimes \ell_p(G)) \) denotes the space of nuclear operators on \( \ell_p \otimes \ell_p(G) \).

**Proof.** \( \supseteq \): This direction is obtained by the same argument as in Proposition 3.7.4.

\( \subseteq \): Let
\[
S = \{ \omega_{a,\varphi} : a \in (K(\ell_p) \otimes PF_p(G))_1, \quad \varphi \in (\mathcal{N}(\ell_p \otimes \ell_p(G))_1) \}.
\]
Then by the argument above, \( S \) is contained in the closed unit ball of \( \mathcal{Q}_{pcb}(G) \). It is easy to check that \( S \) is bounded. We claim that \( S \) is convex. Let \( \omega_{a_1,\varphi_1} \) and \( \omega_{a_2,\varphi_2} \) be elements of \( S \), and suppose \( 0 \leq \lambda \leq 1 \). Let
\[
a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \in M_2(K(\ell_p) \otimes PF_p(G))
\]
and
\[
\varphi = \begin{bmatrix}
\lambda \varphi_1 & 0 \\
0 & (1 - \lambda) \varphi_2
\end{bmatrix} \in N_2 \hat{\otimes} T(\ell_p \otimes_p \ell_p(G)),
\]
then \( \varphi \in (M_2(K(\ell_p) \hat{\otimes} PF_p(G)))' = N_2 \hat{\otimes} (K(\ell_p) \hat{\otimes} PF_p(G))' \) with \( \|a\| \leq 1 \) and \( \|\varphi\| \leq 1 \). Now, after isometric identification
\[
M_2(K(\ell_p) \hat{\otimes} PF_p(G)) = K(\ell_p \oplus_p \ell_p) \hat{\otimes} PF_p(G)
\]
and
\[
N_2 \hat{\otimes} N(\ell_p \otimes_p \ell_p(G)) = N(\ell_p \otimes \ell_p(G)) = N(\ell_p \oplus \ell_p) \hat{\otimes} N(\ell_p(G)),
\]
we have for any \( p \)-completely bounded operator \( T \) on \( PF_p(G) \),
\[
\omega_{a,\varphi}(T) = \langle id_{K(\ell_p \oplus \ell_p)} \otimes T(a), \varphi \rangle = \langle id_{K(\ell_p)} \otimes T(a_1), \lambda \varphi_1 \rangle + \langle id_{K(\ell_p)} \otimes T(a_2), (1 - \lambda) \varphi_2 \rangle
\]
\[
= (\lambda \omega_{a_1,\varphi_1} + (1 - \lambda) \omega_{a_2,\varphi_2})(T).
\]
Moreover, since \( \ell_p \) and \( \ell_p \oplus \ell_p \) are isometrically isomorphic, \( \omega_{a,\varphi} = \omega_{b,\psi} \) for some \( b \in (K(\ell_p) \hat{\otimes} PF_p(G))_1 \) and some \( \psi \in (N(\ell_p \otimes \ell_p(G)))_1 \). This shows that \( S \) is convex.

Now we claim that \( S \) is norm dense in the closed unit ball of \( Q_{pcb}(G) \). Suppose this is not the case and \( \omega \in Q_{pcb}(G)_1 \) is not in the closure of \( S \). Using the geometric Hahn-Banach theorem and the fact that \( S \) is balanced, we can find \( u \in M_{cb}A_p(G) \) such that
\[
|\langle \omega_{a,\varphi}, u \rangle| \leq 1 < \langle \omega, u \rangle, \quad \forall \omega_{a,\varphi} \in S.
\]
However, this implies
\[
\|u\|_{pcb} \leq \sup\{|id_{K(\ell_p)} \otimes \bar{M}_u(a) : a \in (K(\ell_p) \hat{\otimes} PF_p(G))_1\}
\]
\[
= \sup\{|id_{K(\ell_p)} \otimes \bar{M}_u(a), \varphi) : a \in (K(\ell_p) \hat{\otimes} PF_p(G))_1, \varphi \in (N(\ell_p \otimes \ell_p(G)))_1\}
\]
\[
= \sup\{|\omega_{a,\varphi}(u) : a \in (K(\ell_p) \hat{\otimes} PF_p(G))_1, \varphi \in (N(\ell_p \otimes \ell_p(G)))_1\}
\]
\[
\leq 1.
\]
But then this would imply \( |\langle \omega, u \rangle| \leq 1 \), a contradiction. This shows that \( S \) is norm dense in the closed unit ball of \( Q_{pcb}(G) \).

Let \( \ell_p^{(\infty)} \) denote the \( p \)-direct sum of a countably infinite number of copies of \( \ell_p \). Since \( \ell_p \) is isometrically
Remark 3.7.7. The same argument as above actually works to show that every \( \omega \in Q_{pcb}(G) \) is of the form \( \omega = \omega_{a,\varphi} \) for some \( a \in K(\ell_p^{(\infty)}) \cong P\mathcal{F}_p(G) \) and some \( \varphi \in \mathcal{N}(\ell_p^{(\infty)}) \ominus \mathcal{N}(\ell_p(G)) \). Without loss of generality, we can assume that \( \omega \in Q_{pcb}(G)_1 \). Then there is an \( \omega_1 \in S \) such that \( \|\omega - \omega_1\| < \frac{1}{2} \). Since \( 2(\omega - \omega_1) \in Q_{pcb}(G)_1 \), there is an \( \omega_2 \in S \) such that \( \|\omega - \omega_1 - \frac{1}{2}\omega_2\| < \frac{1}{2^2} \). Continuing in this fashion, we find a sequence \( \{\omega_n\} \) in \( S \) such that
\[
\|\omega - \sum_{i=1}^n \frac{1}{2^{i-1}}\omega_i\| < \frac{1}{2^n}.
\]
Thus \( \omega = \sum_{i=1}^{\infty} \frac{1}{2^{i-1}}\omega_i \). Since \( \omega_i \in S \), there are sequences \( b_i \in (\mathcal{K}(\ell_p) \cong P\mathcal{F}_p(G))_1 \) and \( \psi_i \in (\mathcal{N}(\ell_p \otimes_p \ell_p(G)))_1 \) such that \( \omega_i = \omega_{b_i,\psi_i} \). Let \( \alpha_i = (2^{-i+1})^{1/2} \), let \( a_i = \alpha_i b_i \), and let \( \varphi_i = \alpha_i \psi_i \). Let \( a \in M_\infty(\mathcal{K}(\ell_p) \cong P\mathcal{F}_p(G)) = B(\ell_p^{(\infty)}) \otimes P\mathcal{F}_p(G) \) be the diagonal matrix with diagonal entries \( a_1, a_2, \cdots \). Since \( \|a_i\| \to 0 \), \( a \) in fact lies in \( \mathcal{K}(\ell_p^{(\infty)}) \otimes P\mathcal{F}_p(G) \). Moreover, since \( \sum_{i=1}^\infty \|\varphi_i\| < \infty \), we can define \( \varphi \in \mathcal{N}(\ell_p^{(\infty)}) \ominus \mathcal{N}(\ell_p \otimes_p \ell_p(G)) = \mathcal{N}(\ell_p^{(\infty)}) \otimes \mathcal{N}(\ell_p(G)) \) by
\[
\varphi([a_{ij}]) = \sum_{i=1}^\infty \langle a_{ii}, \varphi_i \rangle, \quad [a_{ij}] \in K(\ell_p^{(\infty)}) \otimes P\mathcal{F}_p(G).
\]
Now it follows that \( \omega = \omega_{a,\varphi} \) and this completes the proof. 

Remark 3.7.7. The same argument as above actually works to show that
\[
Q_{pcb}(G) = \{\omega_{b,\varphi} : b \in \mathcal{B}(\ell_p) \cong P\mathcal{F}_p(G), \varphi \in \mathcal{N}(\ell_p \otimes_p \ell_p(G))\}
\]
for a discrete group \( G \).

### 3.8 More on \( Q_{pcb}(G) \): for General Locally Compact Group \( G \)

**Theorem 3.8.1.** \( Q_{pcb}(G) \supseteq \{\omega_{a,\varphi, f} : a \in PM_p(G) \otimes K(\ell_p), \varphi \in A_p(G) \otimes \mathcal{N}(\ell_p), f \in A_{p,c}(G)\} \).

**Proof.** It suffices to show that if \( a \in PM_p(G), \varphi \in A_{p,c}(G) \), and \( f \in A_{p,c}(G) \), then \( \omega_{a,\varphi, f} \in Q_{pcb}(G) \), where \( \omega_{a,\varphi, f} = (M_{f*a}(a), \varphi) \). We will find a \( g \in L_1(G) \) such that \( \omega_{a,\varphi, f} = \int u(x)g(x)dx \) for all \( u \in M_{c,b}A_p(G) \). Let \( K = (\text{supp} f)^{-1} \text{supp} \varphi \). It is easy to show that
\[
([f * u] \varphi)(x) = (f * 1_K u)(x) = \int_G f_y(x)\varphi(x)(1_K u)(y)^{-1} \quad \text{for all } x \in G,
\]
where \( f_y(\cdot) = f(y) \). Define a map \( \Phi : G \to A_p(G) \) by \( [\Phi(y)](x) = f_y(x)\varphi(x) \), and define a measure \( \mu \) on \( G \) by \( d\mu(y) = (1_K u)(y^{-1})dy \), then \( d\mu \) is a bounded Radon measure. Take \( v = \int_G \Phi(y)d\mu(y) \) (Analysis Now
p.70) in $A_p(G)$ so that

$$\langle b, v \rangle = \int_G \langle b, \Phi(y) \rangle d\mu(y) \quad \text{for all } b \in PM_p(G).$$

In particular, if $b = \lambda_p(x)$, then we get $v(x) = (f \ast 1_K u)(x)$ and thus

$$\omega_{a,v,f}(u) = \langle a, v \rangle = \int_G \langle a, \Phi(y) \rangle d\mu(y) = \int_G u(y^{-1}) \langle a, \Phi(y) \rangle 1_K(y^{-1}) dy = \int_G u(y) g(y) dy,$$

where $g(y) = \langle a, \Phi(y^{-1}) \rangle 1_K(y) \Delta(y^{-1})$. Now it is easy to verify that $g \in L_1(G)$. \hfill \square
Chapter 4

Approximation Properties

4.1 \( p \)-Conditional Expectation

In this chapter, \( G \) always denotes a discrete group, unless stated otherwise. Let \( A_{p,c}(G) = C_{00}(G) \cap A_p(G) \), where \( C_{00}(G) \) denotes the set of compactly supported functions on \( G \). Define

\[
\Delta : A_{p,c}(G) \to A_{p,c}(G \times G), \quad \Delta \varphi = \sum_{g \in G} \varphi(g) \delta_{(g,g)}.
\]

Note that \( \Delta \varphi(s,t) = \delta_s(t) \varphi(s) \).

Lemma 4.1.1. The map \( \Delta \) extends to a linear contraction from \( A_p(G) \) into \( A_p(G \times G) \).

Proof. Let \( \epsilon > 0 \). Suppose \( \varphi \in A_{p,c}(G) \), then we can express \( \varphi \) as \( \varphi = \sum_n \xi_n * \tilde{\eta}_n \) with \( \xi_n \in \ell_{p'}(G) \), \( \eta_n \in \ell_p(G) \), and \( \sum_n \|\xi_n\| \|\eta_n\| < \|\varphi\|_{A_p(G)} + \epsilon \). Define \( \tilde{\xi}_n \in \ell_{p'}'(G \times G) \) by \( \tilde{\xi}_n(s,t) = \delta_s(t) \xi_n(s) \) and \( \tilde{\eta}_n \in \ell_p(G \times G) \) by \( \tilde{\eta}_n(s,t) = \delta_s(t) \eta_n(s) \). It is easy to show that \( \|\tilde{\xi}_n\|_{p'} = \|\xi_n\|_{p'} \) and \( \|\tilde{\eta}_n\|_p = \|\eta_n\|_p \).

Moreover,

\[
\sum_n \tilde{\xi}_n * \tilde{\eta}_n(s,t) \\
= \sum_n \left( \sum_{(g,h)} \tilde{\xi}_n(g,h) \tilde{\eta}_n(s^{-1}g,t^{-1}h) \right) \\
= \sum_n \sum_{(g,h)} \delta_g(h) \xi_n(g) \delta_{s^{-1}}(t^{-1}h) \eta_n(s^{-1}g) \\
= \sum_n \sum_g \xi_n(g) \delta_s(t) \eta_n(s^{-1}g) \\
= \delta_s(t) \varphi(s) \\
= \Delta \varphi(s,t).
\]

Therefore, we get

\[
\|\Delta \varphi\|_{A_p(G \times G)} \leq \|\varphi\|_{A_p(G)} + \epsilon
\]

and hence \( \|\Delta\| \leq 1 \). Since \( A_{p,c}(G) \) is norm dense in \( A_p(G) \) [Run02], we can extend \( \Delta \) to a contraction from \( A_p(G) \) into \( A_p(G \times G) \).
Remark 4.1.2. In particular, the proof above shows that if $\xi \ast \eta \in A_p,c(G)$ with $\xi \in \ell_p'(G)$ and $\eta \in \ell_p(G)$, then $\Delta(\xi \ast \eta) = \hat{\xi} \ast \hat{\eta}$. In fact, more is true as we have the following

Lemma 4.1.3. With the same notations as above, if $\xi \in \ell_p'(G)$ and $\eta \in \ell_p(G)$, then $\Delta(\xi \ast \eta) = \hat{\xi} \ast \hat{\eta}$.

Proof. Since $C_{00}(G)$ is dense in $\ell_p'(G)$, we can find a sequence $(f_n) \in C_{00}(G)$ such that $f_n \to \xi$ and $\|f_n\| \leq \|\xi\|_{p'}$ for all $n$. Similarly, there is a sequence $(g_n) \in C_{00}(G)$ such that $g_n \to \eta$ and $\|g_n\| \leq \|\eta\|_p$. It is easy to check that $f_n \ast \tilde{g}_n \to \xi \ast \tilde{\eta}$ in $A_p(G)$ and hence $\Delta(f_n \ast \tilde{g}_n) \to \Delta(\xi \ast \tilde{\eta})$ in $A_p(G \times G)$. However, since $f_n \ast \tilde{g}_n \in C_{00}(G)$, from Remark 4.1.2, we get $\Delta(f_n \ast \tilde{g}_n) = \hat{f}_n \ast \tilde{\eta}_n$, which converges to $\hat{\xi} \ast \hat{\eta}$ in $A_p(G \times G)$. \qed

Define an isometry $\Gamma_p : \ell_p(G) \to \ell_p(G \times G)$ by $\delta_g \mapsto \delta_{(g,g)}$. Note that $\ell_p'(G) \otimes \ell_p(G) = \mathcal{N}(\ell_p(G))$, the space of nuclear operators on $\ell_p(G)$, then $\Gamma_{p',p} \cong \Gamma_p$ induces a map from $\ell_p'(G) \otimes \ell_p(G)$ to $\mathcal{N}(\ell_p(G) \times G) = \mathcal{N}(\ell_p(G)) \otimes \mathcal{N}(\ell_p(G))$ is a contraction. As in [Daw10], define a map $\Lambda_p : \mathcal{N}(\ell_p(G)) \to C_0(G)$ by

$$
\Lambda_p(g \otimes f)(s) = \langle g, \lambda_p(s)(f) \rangle \quad (s \in G, f \in \ell_p(G), g \in \ell_p'(G)),
$$

then it is easy to show that $\Lambda_p$ induces a map from $A_p(G)$ to $A_p(G \times G)$, which coincides with $\Delta$ above.

Proposition 4.1.4. Let $\rho : PM_{p}(G \times G) \to PM_{p}(G)$ denote the adjoint of $\Delta$. Then $\rho$ is $p$-completely contractive.

Proof. Fix $n \in \mathbb{N}$. Let $[T_{ij}] \in M_n(PM_p(G \times G))$. To estimate the norm of $[\rho(T_{ij})] \in M_n(PM_p(G))$, choose $f_i \in \ell_{p'}(G), g_j \in \ell_p(G), 1 \leq i, j \leq n$. We have

$$
\begin{align*}
\langle \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \begin{bmatrix} \rho(T_{ij}) \\ \vdots \\ g_n \end{bmatrix} \rangle &\in \ell_{p'}(G) \otimes \ell_p(G)
= \sum_{i,j=1}^n \langle f_i, \rho(T_{ij})g_j \rangle_{\ell_{p'}(G) \otimes \ell_p(G)} \\
\end{align*}
$$

$$(\Delta' = \rho) \quad \sum_{i,j=1}^n \langle f_i \ast \tilde{g}_j, \rho(T_{ij}) \rangle_{A_p(G), PM_p(G)}
= \sum_{i,j=1}^n \langle f_i \ast \tilde{g}_j, T_{ij} \rangle_{A_p(G \times G), PM_p(G \times G)}
$$

(Lemma 4.1.3)

$$
\begin{align*}
\langle \begin{bmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_n \end{bmatrix}, \begin{bmatrix} T_{ij} \end{bmatrix}, \begin{bmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_n \end{bmatrix} \rangle &\in \ell_{p'}(G \times G) \otimes \ell_p(G \times G)
= \sum_{i,j=1}^n \langle f_i \ast \tilde{g}_j, T_{ij} \rangle_{\ell_{p'}(G \times G), \ell_p(G \times G)}
\end{align*}
$$

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Since \( \| f_i \|_{p'} = \| \tilde{f}_i \|_{p'} \) and \( \| g_j \|_p = \| \tilde{g}_j \|_p \), 1 \( \leq i, j \leq n \), we are done.

**Lemma 4.1.5.** For all \( s, t \in G \), \( \rho(\lambda_{s,t}) = \delta_s(t)\lambda_s \).

**Proof.** For all \( g, s, t \in G \), by the duality between \( A_p(G) \) (respectively, \( A_p(G \times G) \)) and \( PM_p(G) \) (respectively, \( PM_p(G \times G) \)), we get

\[
\langle \delta_g, \rho(\lambda_{s,t}) \rangle = \langle \Delta \delta_g, \lambda_{s,t} \rangle = \langle \delta_g(s), \lambda_s(t) \rangle = \delta_g(s)\delta_s(t) = \delta_s(t)\langle \delta_g, \lambda_s \rangle = \langle \delta_g, \delta_s(t)\lambda_s \rangle
\]

and the result follows because \( A_{p,c}(G) \) is dense in \( A_p(G) \).

As in [Daw10], define \( W_p : \ell_p(G \times G) \to \ell_p(G \times G) \) by \( (W_p\xi)(s,t) = \xi(s, st) \), then \( W_p \) is an isometric isomorphism on \( \ell_p(G \times G) \) with the inverse \( (W_p^{-1}\eta)(s,t) = \eta(s, s^{-1}t) \). It is straightforward to check that \( (W_p^{-1})' = W_p \). Define

\[
\gamma : PM_p(G) \to \mathcal{B}(\ell_p(G \times G)) = \mathcal{B}(\ell_p(G) \otimes_p \ell_p(G)), \quad T \mapsto W_p^{-1}(T \otimes I)W_p \quad (T \in PM_p(G)).
\]

Before we proceed, we need a lemma about the Banach space projective tensor product.

**Lemma 4.1.6.** Let \( X \overset{\pi}{\otimes} Y \) denote the Banach space projective tensor product of Banach spaces \( X \) and \( Y \). Let \( 1 < p', p < \infty \) with \( 1/p' + 1/p = 1 \). Then for every \( u \in X \overset{\pi}{\otimes} Y \), the norm \( \pi(u) \) of \( u \) is given by

\[
\pi(u) = \inf \left\{ \left( \sum_{n=1}^{\infty} \| x_n \|_{p'} \right)^{1/p'} \left( \sum_{n=1}^{\infty} \| y_n \|_p \right)^{1/p} \right\}, \quad (4.1)
\]

where the infimum is taken over all expressions \( u = \sum_{n=1}^{\infty} x_n \otimes y_n \) with

\[
\left( \sum_{n=1}^{\infty} \| x_n \|_{p'} \right)^{1/p'} \left( \sum_{n=1}^{\infty} \| y_n \|_p \right)^{1/p} < \infty.
\]

**Proof.** Let \( u \in X \overset{\pi}{\otimes} Y \) and \( \epsilon > 0 \). It is well known that

\[
\pi(u) = \inf \left\{ \sum_{n=1}^{\infty} \| x_n \|_p \| y_n \| : \sum_{n=1}^{\infty} \| x_n \|_p \| y_n \| < \infty, \quad u = \sum_{n=1}^{\infty} x_n \otimes y_n \right\}. \quad (4.2)
\]

Therefore \( u \) can be written as \( u = \sum_{n=1}^{\infty} \xi_n \otimes \eta_n \), \( \xi_n(\neq 0) \in X \), \( \eta_n(\neq 0) \in Y \) with \( \sum_{n=1}^{\infty} \| \xi_n \| \| \eta_n \| < \pi(u) + \epsilon \).

For each \( n \), let \( \lambda_n = \| \eta_n \|^{p'/pp'}/\| \xi_n \|^{pp'}/p' \) so that

\[
\| \lambda_n \xi_n \|_{p'} = \left( \frac{1}{\lambda^n} \right)^{p'} = \| \xi_n \| \| \eta_n \|. \quad (70)
\]
Now let \( x_n = \lambda_n \xi_n \) and \( y_n = \frac{1}{\lambda_n} \eta_n \) then \( u = \sum_{n=1}^{\infty} x_n \otimes y_n \) and

\[
\left( \sum_{n=1}^{\infty} \|x_n\|^{p'} \right)^{1/p'} \left( \sum_{n=1}^{\infty} \|y_n\|^p \right)^{1/p} = \sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| < \pi(u) + \epsilon.
\]

Since \( \epsilon \) is arbitrary, this shows that the right hand side of (4.1) is less than or equal to \( \pi(u) \). To show the reverse inequality, let’s assume, for contradiction, that the right hand side of (4.1) is strictly less than \( \pi(u) \).

This means \( u \) can be written as \( u = \sum_{n=1}^{\infty} x_n \otimes y_n \) with \( (\sum_{n=1}^{\infty} \|x_n\|^{p'})^{1/p'} (\sum_{n=1}^{\infty} \|y_n\|^p)^{1/p} < \pi(u) \). By Hölder’s inequality, we would get \( \sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \pi(u) \) but this is impossible because of (4.2). \( \square \)

**Lemma 4.1.7.** \( \gamma \) is weak*-weak* continuous.

**Proof.** Suppose \( T_\alpha, T \in PM_p(G) \) and \( T_\alpha \to T \) in the weak* topology. For every \( \sum_n f_n \otimes g_n \in \ell_{p'}(G) \otimes \ell_p(G) \) with \( f_n \in \ell_{p'}(G), g_n \in \ell_p(G) \), and \( \sum_n \|f_n\| \|g_n\| < \infty \), we get

\[
\sum_n \langle f_n, T_\alpha g_n \rangle \to \sum_n \langle f_n, T g_n \rangle.
\]

We claim that \( T_\alpha \otimes I \to T \otimes I \) in the weak* topology in \( B(\ell_p(G \times G)) \). By Lemma 4.1.6, every \( u \in \ell_{p'}(G \times G) \otimes \ell_p(G \times G) \) can be expressed as \( u = \sum_n \xi_n \otimes \eta_n \) with \( \xi_n \in \ell_{p'}(G \times G), \eta_n \in \ell_p(G \times G) \), and \( (\sum_n \|\xi_n\|^{p'})^{1/p'} (\sum_n \|\eta_n\|^p)^{1/p} < \infty \). Since \( \ell_{p'}(G \times G) = \ell_{p'}(G) \otimes \ell_{p'}(G) \), each \( \xi_n \) can be written as \( \xi_n = \sum_{s \in G} f_s^n \otimes \delta_s \), where \( f_s^n \in \ell_{p'}(G) \) with \( \sum_{s \in G} \|f_s^n\|^{p'} = \|\xi_n\|^{p'} \). In particular, \( f_s^n \) is nonzero only for at most countably many \( s \in G \). Similarly each \( \eta_n \) can be expressed as \( \eta_n = \sum_{t \in G} g_t^n \otimes \delta_t \), where \( g_t^n \in \ell_p(G) \) with \( \sum_{t \in G} \|g_t^n\|^p = \|\eta_n\|^p \). Now

\[
\sum_n \langle \xi_n, (T_\alpha \otimes I)\eta_n \rangle = \sum_n \sum_{s,t \in G} \langle f_s^n \otimes \delta_s, T_\alpha g_t^n \otimes \delta_t \rangle = \sum_n \sum_{s \in G} \langle f_s^n, T_\alpha g_s^n \rangle.
\]

Here the last term is a countable sum and

\[
\sum_n \sum_{s \in G} \|f_s^n\| \|g_s^n\| \leq \left( \sum_n \sum_{s \in G} \|f_s^n\|^{p'} \right)^{1/p'} \left( \sum_n \sum_{s \in G} \|g_s^n\|^p \right)^{1/p} \leq \left( \sum_n \|\xi_n\|^{p'} \right)^{1/p'} \left( \sum_n \|\eta_n\|^p \right)^{1/p} < \infty
\]

and hence \( \sum_n \sum_{s \in G} \langle f_s^n, T_\alpha g_s^n \rangle \to \sum_n \sum_{s \in G} \langle f_s^n, T g_s^n \rangle \). This shows that \( T_\alpha \otimes I \to T \otimes I \) in the weak* topology in \( B(\ell_p(G \times G)) \). Finally, for every \( u = \sum_n \xi_n \otimes \eta_n \) with \( \xi_n \in \ell_{p'}(G \times G), \eta_n \in \ell_p(G \times G) \), and
\[
(\sum_{n} \|\xi_n\|^{p'})^{1/p'} (\sum_{n} \|\eta_n\|^{p})^{1/p} < \infty,
\]
we obtain
\[
\langle \gamma(T_\alpha), u \rangle = \sum_{n} \langle \xi_n, W_p^{-1}(T_\alpha \otimes I) W_p \eta_n \rangle
\]
\[
= \sum_{n} \langle W_p \xi_n, (T_\alpha \otimes I) W_p \eta_n \rangle
\]
\[
\rightarrow \sum_{n} \langle W_p \xi_n, (T \otimes I) W_p \eta_n \rangle
\]
\[
= \langle \gamma(T), u \rangle
\]
since \((W_p^{-1})' = W_{p'}\) and \(W_{p'}, W_p\) are isometries and \(T_\alpha \otimes I \rightarrow T \otimes I\) in the weak* topology.

**Remark 4.1.8.** It is easy to show that \(\gamma(\lambda_s) = \lambda_{(s,s)}\) and Lemma 4.1.7 shows that the range of \(\gamma\) is contained in \(PM_p(G) \bar{\otimes} PM_p(G) = PM_p(G \times G)\).

Let us define the \(p\)-trace \(tr_p : PM_p(G \times G) \rightarrow \mathbb{C}\) by \(tr_p(T) = \langle T \delta^p_{(e,e)}, \delta^p_{(e,e)} \rangle\).

**Proposition 4.1.9.** Define \(E : PM_p(G \times G) \rightarrow PM_p(G \times G)\) by \(E = \gamma \circ \rho\). Then

1. The range of \(E\) is the weak* closure of \(\text{span}\{\lambda_{(s,s)}^p : s \in G\}\)

2. \(E\) is weak*-weak* continuous;

3. \(E^2 = E\);

4. \(tr_p \circ E = tr_p\);

5. \(E\) is unital and \(p\)-completely contractive.

**Proof.** (1) is easy to verify. (2) follows from definition of \(\rho\) and Lemma 4.1.7. (3) and (4) are immediate from Lemma 4.1.5 and Remark 4.1.8. For (5), it is obvious that \(E\) is unital. By Proposition 4.1.4, it suffices to show that \(\gamma\) is \(p\)-completely contractive. Fix \(n \in \mathbb{N}\) and let \([T_{ij}] \in M_n(PM_p(G))\). To compute the norm of \([\gamma(T_{ij})]\), let \(\xi_i \in \ell_{p'}(G \times G), \eta_j \in \ell_{p}(G \times G), 1 \leq i, j \leq n\). Since \(\ell_{p'}(G \times G) = \ell_{p'}(G) \otimes \ell_{p'}(G)\), each \(\xi_i\) can be written as \(\xi_i = \sum_{s \in G} f^i_s \otimes \delta^p_s\), where \(f^i_s \in \ell_{p'}(G)\) with \(\sum_{s \in G} \|f^i_s\|^{p'} = \|\xi_i\|^{p'}\). Likewise, each \(\eta_j\) can
be written as $\eta_j = \sum_{t \in G} g^j_t \otimes \delta^p_t$, where $g^j_t \in \ell_p(G)$ with $\sum_{t \in G} \|g^j_t\|^p = \|\eta_j\|^p$. Since

$$\left\langle \begin{array}{c}
\xi_1 \\
\vdots \\
\xi_n
\end{array} , \begin{array}{c}
\gamma(T_{ij}) \\
\vdots \\
\eta_n
\end{array} \right\rangle_{\ell_p^\prime \otimes \ell_p(G \times G), \ell_p^\prime \otimes \ell_p(G \times G)} = \left\langle \begin{array}{c}
\xi_1 \\
\vdots \\
\xi_n
\end{array} , \begin{array}{c}
W^{-1}_p(T_{ij} \otimes I)W_p \\
\vdots \\
\eta_n
\end{array} \right\rangle_{\ell_p^\prime \otimes \ell_p(G \times G), \ell_p^\prime \otimes \ell_p(G \times G)}$$

$$((W^{-1}_p)^\prime = W_p^\prime) = \left\langle \begin{array}{c}
W_p\xi_1 \\
\vdots \\
W_p\xi_n
\end{array} , \begin{array}{c}
T_{ij} \otimes I \\
\vdots \\
W_p\eta_n
\end{array} \right\rangle_{\ell_p^\prime \otimes \ell_p(G \times G), \ell_p^\prime \otimes \ell_p(G \times G)}$$

and $W_p^\prime$ (respectively, $W_p$) is an isometric isomorphism on $\ell_p^\prime(G \times G)$ (respectively, $\ell_p(G \times G)$), for norm calculation of $[\gamma(T_{ij})]$, we can replace the last term above by

$$\left\langle \begin{array}{c}
\xi_1 \\
\vdots \\
\xi_n
\end{array} , \begin{array}{c}
T_{ij} \otimes I \\
\vdots \\
\eta_n
\end{array} \right\rangle_{\ell_p^\prime \otimes \ell_p(G \times G), \ell_p^\prime \otimes \ell_p(G \times G)} = \sum_{i,j=1}^n \langle \xi_i, (T_{ij} \otimes I)\eta_j \rangle_{\ell_p^\prime(G \times G), \ell_p(G \times G)}$$

$$= \sum_{i,j=1}^n \langle \sum_{s \in G} f^i_s \otimes \delta^p_s, \sum_{t \in G} T_{ij}g^j_t \otimes \delta^p_t \rangle_{\ell_p^\prime(G \times G), \ell_p(G \times G)}$$

$$= \sum_{i,j=1}^n \sum_{s \in G} \langle f^i_s, T_{ij}g^j_s \rangle_{\ell_p^\prime(G), \ell_p(G)}$$

$$= \sum_{s \in G} \sum_{i,j=1}^n \langle f^i_s, T_{ij}g^j_s \rangle_{\ell_p^\prime(G), \ell_p(G)}$$

$$= \sum_{s \in G} \left\langle \begin{array}{c}
f^i_s \\
\vdots \\
f^n_s
\end{array} , \begin{array}{c}
T_{ij} \\
\vdots \\
g^n_s\end{array} \right\rangle_{\ell_p^\prime \otimes \ell_p(G), \ell_p^\prime \otimes \ell_p(G)}.$$
Here

\[
\sum_{s \in G} \left( \begin{bmatrix} f_1^s & \cdots & f_n^s \\ \vdots & \ddots & \vdots \\ g_1^s & \cdots & g_n^s \end{bmatrix} T_{ij} \right) \leq \|T_{ij}\| \cdot \sum_{s \in G} \left( \sum_{i=1}^n \|f_i^s\|^p \right)^{1/p'} \left( \sum_{j=1}^n \|g_j^s\|^p \right)^{1/p}
\]

(Hölder’s inequality)

\[
\leq \|T_{ij}\| \cdot \left( \sum_{s \in G} \sum_{i=1}^n \|f_i^s\|^p \right)^{1/p'} \left( \sum_{s \in G} \sum_{j=1}^n \|g_j^s\|^p \right)^{1/p}
\]

\[
= \|T_{ij}\| \cdot \left( \sum_{s=1}^n \sum_{s \in G} \|f_i^s\|^p \right)^{1/p'} \left( \sum_{j=1}^n \sum_{s \in G} \|g_j^s\|^p \right)^{1/p}
\]

This completes the proof. \qed

As an application, we get a $p$-analogue of Lemma 2.5 in [Haa86].

**Corollary 4.1.10.** Let $G$ be a discrete group and let $T$ be a weak*-continuous $p$-completely bounded map on $PM_p(G)$ or $p$-completely bounded map on $PF_p(G)$. Define $\varphi : G \to \mathbb{C}$ by

\[
\varphi(s) = \langle T(\lambda_s^p)\delta_c^p, \delta_c^p \rangle.
\]

Then

1. $\varphi \in M_{cb}A_p(G)$ with $\|\varphi\|_{M_{cb}A_p(G)} \leq \|T\|_{pcb}$.

2. If $T$ is of finite rank, then $\varphi \in \ell_p(G) \subseteq A_p(G)$.

**Proof.** (1) First of all, suppose that $T$ is a weak*-continuous $p$-completely bounded map on $PM_p(G)$. For simplicity of notation, let $M = PM_p(G)$. Define

\[
S = \rho \circ (T \otimes id_{PM_p(G)}) \circ \gamma,
\]

where $T \otimes id_{PM_p(G)}$ as in Lemma 3.7.1, then $S$ is a weak*-continuous $p$-completely bounded map on $PM_p(G)$ with $\|S\|_{pcb} \leq \|T\|_{pcb}$. Suppose that $T(\lambda_s^p)\delta_c^p = \sum_{t \in G} c_{s,t}^p \delta_t$ with $c_{s,t} \in \mathbb{C}$ and $\sum_{t \in G} |c_{s,t}|^p < \infty$. We claim that $S(\lambda_s) = c_{s,s} \lambda_s$. To show this, let $\psi = \sum_{g \in G} a_g^p \delta_g \in A_{p,c}(G)$ with $a_g \in \mathbb{C}$. Since

\[
\langle T(\lambda_s) \otimes \lambda_s, \delta_g \circ \delta_g \rangle_{PM_p(G \times G), A_p(G \times G)} = \langle \delta_{c_{s,t}^p} \otimes \delta_{c_{s,t}^p}, \sum_{t \in G} c_{s,t} \delta_t^p \otimes \delta_t^p \rangle_{\ell_p(G \times G), \ell_p(G \times G)},
\]

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we obtain
\[
\langle S(\lambda_s), \psi \rangle = \langle (\rho \circ (T \otimes id_{PM_p(G)}) \circ \gamma)(\lambda_s), \psi \rangle
\]
\[(\rho = \Delta') = \langle T(\lambda_s) \otimes \lambda_s, \Delta \psi \rangle = \sum_{g \in G} a_g \langle T(\lambda_s) \otimes \lambda_s, \delta_g \otimes \delta_g \rangle = a_s c_{s,s} = \langle c_{s,s} \lambda_s, \psi \rangle
\]
and this proves the claim. On the other hand, by definition, \( \varphi(s) = c_{s,s} \) and this shows that \( S = M_\varphi \) and therefore \( \|\varphi\|_{M,A_p(G)} = \|S\|_{pcb} \leq \|T\|_{pcb} \).

Now assume that \( T \) is a \( p \)-completely bounded map on \( PF_p(G) \). Define
\[
S = \rho \circ (T \otimes id_{PF_p(G)}) \circ \gamma|_{PF_p(G)},
\]
where \( T \otimes id_{PF_p(G)} \) as in Lemma 3.7.1, then \( S \) is a well-defined \( p \)-completely bounded map on \( PF_p(G) \) the result follows using the same argument as above.

(2) We adapt the idea used in [Haa86]. Without loss of generality, we can assume that \( T = f \otimes b \), where \( f \in PM_p(G)' \) and \( b \in PM_p(G) \). Then \( T(\lambda_s) = f(\lambda_s)b \) and
\[
\varphi(s) = \langle T(\lambda_s)\delta_p^e, \delta_p^s \rangle = f(\lambda_s)\langle b\delta_p^e, \delta_p^s \rangle.
\]
This shows that \( \varphi(s) \) is the \( s \)-component of \( b\delta_p^e \in \ell_p(G) \).

\[\Box\]

4.2 \( p \)-AP and \( p \)-OAP

Let \( G \) be a locally compact group. We say that \( G \) has the \( p \)-approximation property (\( p \)-AP) if there is a net \( \{u_\alpha\} \) in \( A_p(G) \) such that \( u_\alpha \to 1 \) in the \( \sigma(M_{cb}A_p(G), Q_{pcb}(G)) \)-topology. Let \( V \) be a \( p \)-operator space.

We say that \( V \) has the \( p \)-operator space approximation property (\( p \)-OAP) if there is a net \( \{T_\alpha\} \) of bounded finite rank maps on \( V \) such that for every \( a \in K(\ell_p) \otimes V \), \( (id_{K(\ell_p)} \otimes T_\alpha)(a) \to a \) in the norm topology (See Lemma 3.7.1 for the definition of \( id_{K(\ell_p)} \otimes T_\alpha \)). \( V \) is said to have the strong \( p \)-operator space approximation property (strong \( p \)-OAP) if there is a net \( \{T_\alpha\} \) of finite rank maps on \( V \) such that for every \( a \in B(\ell_p) \otimes V \), \( (id_{B(\ell_p)} \otimes T_\alpha)(a) \to a \) in the norm topology.

Remark 4.2.1. 2-AP is the same as the approximation property studied in [HK94]. Similarly, 2-OAP is the same as the operator space approximation property in [ER00].
Lemma 4.2.2. Suppose that \( T, T' \in \mathcal{CB}_p(\mathcal{P} F_p(G), \mathcal{B}(\ell_p(G))) \) and that \( T \to T' \) in the stable point-norm topology. Then for any \( L_p(\mu) \) and for any \( c \in K(L_p(\mu)) \),

\[
\text{id}_{K(L_p(\mu))} \otimes T_{\alpha}(c) \to \text{id}_{K(L_p(\mu))} \otimes T(c)
\]

in norm.

Proof. By Theorem 2.3.11, we have a \( p \)-complete isometry

\[
K(L_p(\mu))^\tau_p \otimes \mathcal{P} F_p(G) = L_p^c(\mu)^{\tau_p} \otimes \mathcal{P} F_p(G) \otimes L_p^c(\mu).
\]

By Proposition 2.4.5, \( c \in K(L_p(\mu)) \) can be written as \( v \otimes x \otimes w \), where \( v \in M_{1,\infty}(L_p^c(\mu)) \), \( x \in K(\mathcal{P} F_p(G)) \), and \( w \in M_{1,\infty}(L_p^c(\mu)) \) and hence

\[
\| \text{id}_{K(L_p(\mu))} \otimes (T_{\alpha} - T)(c) \| \\
= \| v \otimes (\text{id}_{K(L_p(\mu))} \otimes (T_{\alpha} - T))(x) \otimes w \| \\
\leq \| v \| \| (\text{id}_{K(L_p(\mu))} \otimes (T_{\alpha} - T))(x) \| \| w \| \\
\to 0.
\]

\[\square\]

Lemma 4.2.3. Suppose that \( T, T' \in \mathcal{CB}_p(\mathcal{P} F_p(G), \mathcal{B}(\ell_p(G))) \) and that \( T \to T' \) in the stable point-norm topology. Then \( T_{\alpha} \to T' \) in the \( \sigma(\mathcal{CB}_p(\mathcal{P} F_p(G), \mathcal{B}(\ell_p(G))), \mathcal{P} F_p(G) \otimes \mathcal{N}(\ell_p(G))) \) topology.

Proof. By assumption, for any \( a \in K(\ell_p)^{\tau_p} \otimes \mathcal{P} F_p(G) \), \( \text{id}_{K(\ell_p)} \otimes T_{\alpha}(a) \to \text{id}_{K(\ell_p)} \otimes T(a) \) in norm. Let \( u \in \mathcal{P} F_p(G) \otimes \mathcal{N}(\ell_p(G)) \), then by Proposition 2.4.4, we may assume that \( u = \gamma(a \otimes f)\delta = \sum_{i,j,k,l} \gamma_{ik} a_{ij} \otimes f_{kl} \delta_{jl} \),

where \( \gamma \in M_{1,\infty} \), \( \delta \in M_{1,\infty} \), \( a \in K(\ell_p)^{\tau_p} \otimes \mathcal{P} F_p(G) \), and \( f \in K(\ell_p)^{\tau_p} \otimes \mathcal{N}(\ell_p(G)) \). The operators \( \bar{\gamma} \in M_{1,\infty} \) and \( \bar{\delta} \in M_{1,\infty} \) defined by

\[
\bar{\gamma} = [\gamma^1 \gamma^2 \cdots ] , \quad \text{where} \quad \gamma^i = [\gamma_{i1} \gamma_{i2} \cdots ]
\]

and

\[
\bar{\delta} = \begin{bmatrix}
\delta^1 \\
\delta^2 \\
\vdots
\end{bmatrix} , \quad \text{where} \quad \delta^i = \begin{bmatrix}
\delta_{1i} \\
\delta_{2i} \\
\vdots
\end{bmatrix}
\]

have the same norm as \( \gamma \) and \( \delta \). Let \( \gamma^n, \bar{\gamma}^n, a^n, f^n, \delta^n \), and \( \bar{\delta}^n \) be the truncations of \( \gamma, \bar{\gamma}, a, f, \delta \), and \( \bar{\delta} \),
respectively, then it follows easily that $u$ is the norm limit of $\gamma^n (a^n \otimes f^n)\delta^n$ in $PF_p(G) \overset{\wedge}{\otimes} N(\ell_p(G))$ and that $\bar{\gamma} \circ f \circ \bar{\delta}$ is the norm limit of $\bar{\gamma}^n \circ f^n \circ \bar{\delta}^n$ in $\ell_p^h \otimes N(\ell_p(G)) \otimes \ell_p^c = N_{\infty} \overset{\wedge}{\otimes} N(\ell_p(G))$. Since
\begin{align*}
\langle T_\alpha, u \rangle &= \lim_{n \to \infty} \sum_{i,j,k,l \leq n} \gamma_{ik} \langle T_\alpha(a_{ij}), f_{kl} \rangle \delta_{jl} \\
&= \lim_{n \to \infty} \langle (T_\alpha)_n(a^n), \bar{\gamma}^n \circ f^n \circ \bar{\delta}^n \rangle \\
&= \lim_{n \to \infty} \langle id_{K(\ell_p)} \otimes T_\alpha(a), \bar{\gamma}^n \circ f^n \circ \bar{\delta}^n \rangle \\
&= \langle id_{K(\ell_p)} \otimes T_\alpha(a), \bar{\gamma} \circ f \circ \bar{\delta}, \rangle,
\end{align*}
the result follows. \(\square\)

**Lemma 4.2.4.** Suppose that $T_\alpha, T \in CB_p(PF_p(G), B(\ell_p(G)))$. Then

$$T_\alpha \to T$$

in the $\sigma(CB_p(PF_p(G), B(\ell_p(G))), PF_p(G) \overset{\wedge}{\otimes} N(\ell_p(G)))$ topology

if and only if for every $b \in B(\ell_p) \overset{\vee}{\otimes} PF_p(G)$ and $\varphi \in N(\ell_p \otimes_p \ell_p(G))$,

$$\langle id_{B(\ell_p)} \otimes T_\alpha(b), \varphi \rangle \to \langle id_{B(\ell_p)} \otimes T(b), \varphi \rangle.$$

**Proof.** Let $u \in PF_p(G) \overset{\wedge}{\otimes} N(\ell_p(G))$, then by Proposition 2.4.4, we may assume that $u = \gamma(b \otimes f)\delta = \sum_{i,j,k,l} \gamma_{ik} b_{ij} \otimes f_{kl} \delta_{jl}$, where $\gamma \in M_{1,\infty^2}$, $\delta \in M_{\infty^2,1}$, $b \in B(\ell_p) \overset{\vee}{\otimes} PF_p(G)$, and $f \in B(\ell_p) \overset{\vee}{\otimes} N(\ell_p(G))$. The operators $\bar{\gamma} \in M_{1,\infty}$ and $\bar{\delta} \in M_{\infty,1}$ defined by

$$\bar{\gamma} = [\gamma^1 \gamma^2 \cdots] \quad \text{and} \quad \gamma^i = [\gamma_{i1} \gamma_{i2} \cdots]$$

and

$$\bar{\delta} = \begin{bmatrix}
\delta^1 \\
\delta^2 \\
\vdots
\end{bmatrix} \quad \text{and} \quad \delta^i = \begin{bmatrix}
\delta_{1i} \\
\delta_{2i} \\
\vdots
\end{bmatrix}$$

have the same norm as $\gamma$ and $\delta$. Let $\gamma^n, \bar{\gamma}^n, b^n, f^n, \delta^n$, and $\bar{\delta}^n$ be the truncations of $\gamma, \bar{\gamma}, b, f, \delta, \text{and} \bar{\delta}$, respectively, then it follows easily that $u$ is the norm limit of $\gamma^n (b^n \otimes f^n)\delta^n$ in $PF_p(G) \overset{\wedge}{\otimes} N(\ell_p(G))$ and that $\bar{\gamma} \circ f \circ \bar{\delta}$ is the norm limit of $\bar{\gamma}^n \circ f^n \circ \bar{\delta}^n$ in $\ell_p^h \otimes N(\ell_p(G)) \otimes \ell_p^c = N_{\infty} \overset{\wedge}{\otimes} N(\ell_p(G)) = N(\ell_p \otimes_p \ell_p(G))$.\[77\]
Since

\[
(T_n, u) = \lim_{n \to \infty} \sum_{i,j,k,l \leq n} \gamma_{ij} (T_n(b_{ij}), f_{kl}) \delta_{jl}
\]

\[
= \lim_{n \to \infty} ((T_n)_n(b^n), \gamma^n \circ f^n \circ \delta^n)
\]

\[
= \lim_{n \to \infty} (id_{\mathcal{B}(\ell_p)} \otimes T_n(b), \gamma^n \circ f^n \circ \delta^n)
\]

\[
= \langle id_{\mathcal{B}(\ell_p)} \otimes T_n(b), \gamma \circ f \circ \delta \rangle,
\]

we get the result.

\[\square\]

**Proposition 4.2.5.** Suppose that \(T_n, T \in C\mathcal{B}_p(PF_p(G), \mathcal{B}(\ell_p(G)))\) and that \(T_n \to T\) in the stable point-norm topology. Then for any \(c \in \mathcal{B}(L_p(\mu))^\vee_p \otimes PF_p(G)\) and \(\varphi \in \mathcal{N}(L_p(\mu) \otimes_p \ell_p(G))\), we have

\[
\langle id_{\mathcal{B}(L_p(\mu))} \otimes T_n(c), \varphi \rangle \to \langle id_{\mathcal{B}(L_p(\mu))} \otimes T(c), \varphi \rangle.
\]

**Proof.** Suppose \(c \in \mathcal{B}(L_p(\mu))^\vee_p \otimes PF_p(G)\) and \(\varphi \in \mathcal{N}(L_p(\mu) \otimes_p \ell_p(G))\). By Proposition 2.4.4, we can write

\[
\varphi = \gamma([f_{ij}] \otimes [g_{kl}]) = \sum_{i,j,k,l \leq n} \gamma_{ij} f_{ij} \otimes g_{kl} \delta_{jl},
\]

where \(\gamma \in M_{1,\infty^2}, \delta \in M_{\infty^2,1}, f = [f_{ij}] \in \mathcal{K}(\ell_p)^{\vee_p} \otimes \mathcal{N}(L_p(\mu))\), and \(g = [g_{kl}] \in \mathcal{K}(\ell_p)^{\vee_p} \otimes \mathcal{N}(\ell_p(G))\). Note that the map \(\tilde{f} : \mathcal{B}(L_p(\mu)) \otimes PF_p(G) \to \mathcal{K}(\ell_p)^{\vee_p} \otimes PF_p(G)\) defined by

\[
\tilde{f}(x \otimes y) = [f_{ij}(x) y]
\]

extends to a map from \(\mathcal{B}(L_p(\mu))^{\vee_p} \otimes PF_p(G)\) to \(\mathcal{K}(\ell_p)^{\vee_p} \otimes PF_p(G)\). Now, using the same notation as in the proof of Lemma 4.2.3, we claim that

\[
\langle id_{\mathcal{B}(L_p(\mu))} \otimes T_n(c), \varphi \rangle = \langle id_{\mathcal{K}(\ell_p)} \otimes T_n(\tilde{f}(c)), \gamma \circ g \circ \delta \rangle.
\]

(4.3)

It suffices to consider \(c = a \otimes b\), where \(a \in \mathcal{B}(L_p(\mu))\) and \(b \in PF_p(G)\). Then

\[
\langle id_{\mathcal{B}(L_p(\mu))} \otimes T_n(c), \varphi \rangle = \lim_{n \to \infty} \sum_{i,j,k,l \leq n} \langle a \otimes T_n(b), \gamma_{ij} f_{ij} \otimes g_{kl} \delta_{jl} \rangle
\]

\[
= \lim_{n \to \infty} \sum_{i,j,k,l \leq n} \gamma_{ij} \langle f_{ij}, a \rangle g_{kl} \delta_{jl}
\]

\[
= \lim_{n \to \infty} \langle [f_{ij}(T_n(a))] \otimes b, \gamma^n \circ f^n \circ \delta^n \rangle
\]

\[
= \langle id_{\mathcal{K}(\ell_p)} \otimes T_n(\tilde{f}(a \otimes b)), \gamma \circ g \circ \delta \rangle,
\]

and the claim is proved. Finally, the result follows since \(id_{\mathcal{K}(\ell_p)} \otimes T_n(\tilde{f}(c)) \to id_{\mathcal{K}(\ell_p)} \otimes T(\tilde{f}(c))\) in norm. \[\square\]
Lemma 4.2.6. For any $x \in \mathcal{B}(\ell_p) \mathbin{\hat\otimes} PF_p(G) \mathbin{\hat\otimes} PF_p(G)$ and any $\varphi \in \mathcal{N}(\ell_p \otimes_p \ell_p(G))$, we have

$$(\text{id}_{\mathcal{B}(\ell_p)} \otimes \rho(x), \varphi) = \langle x, \text{id}_{\mathcal{N}(\ell_p)} \otimes \Gamma_{p',p}(\varphi) \rangle,$$

where $\text{id}_{\mathcal{N}(\ell_p)} \otimes \Gamma_{p',p} : \mathcal{N}(\ell_p) \mathbin{\hat\otimes} \mathcal{N}(\ell_p(G)) \to \mathcal{N}(\ell_p) \mathbin{\hat\otimes} \mathcal{N}(\ell_p(G)) \otimes \mathcal{N}(\ell_p(G))$.

Proof. It is enough to verify this for $x = T \otimes \lambda_s \otimes \lambda_t$ and $\varphi = \psi \otimes \delta_w \otimes \delta_w$, where $T \in \mathcal{B}(\ell_p)$, $\lambda_s, \lambda_t \in PF_p(G)$, $\psi \in \mathcal{N}(\ell_p(G))$, and $\delta_w \in \ell_p(G)$. Indeed, by Lemma 4.1.5,

$$\langle \text{id}_{\mathcal{B}(\ell_p)} \otimes \rho(x), \varphi \rangle = \delta_w(T \otimes \lambda_s, \varphi) = \delta_s(t)\delta_w(sz)(T, \psi).$$

On the other hand

$$\langle x, \text{id}_{\mathcal{N}(\ell_p)} \otimes \Gamma_{p',p}(\varphi) \rangle = \langle x, \psi \otimes \delta_w \otimes \delta_w \rangle = \langle T, \psi \rangle \delta_w(sz) \delta_w(sz),$$

and the result follows.

Theorem 4.2.7. If $PF_p(G)$ has the $p$-OAP, then $G$ has the $p$-AP.

Proof. Let $(T_\alpha)$ be a net of bounded finite rank maps on $PF_p(G)$ such that $(\text{id}_{K(\ell_p)} \otimes T_\alpha)(a) \to a$ in the norm topology for every $a \in K(\ell_p) \otimes PF_p(G) = K(\ell_p) \mathbin{\hat\otimes} PF_p(G)$. Define $u_\alpha : G \to \mathbb{C}$ by $u_\alpha(s) = \langle T_\alpha(\lambda_s)\delta_w, \delta_w \rangle$, then by Corollary 4.1.10, $u_\alpha \in \ell_p(G) \subseteq A_p(G)$ and moreover, it is easy to check that

$$\tilde{M}_{u_\alpha} = \rho \circ (\text{id}_{PF_p(G)} \otimes T_\alpha) \circ \gamma|_{PF_p(G)}.$$

We claim that $u_\alpha \to 1$ in the $\sigma(M_{cb}A_p(G), Q_{pcb}(G))$ topology. Let $a \in \mathcal{B}(\ell_p) \mathbin{\hat\otimes} PF_p(G)$ and $\varphi \in \mathcal{N}(\ell_p \otimes_p \ell_p(G))$ (See Remark 3.7.7). Since

$$\omega_{a,\varphi}(u_\alpha) = \langle \text{id}_{\mathcal{B}(\ell_p)} \otimes \tilde{M}_{u_\alpha}(a), \varphi \rangle,$$

we need to show that

$$\langle \text{id}_{\mathcal{B}(\ell_p)} \otimes \tilde{M}_{u_\alpha}(a), \varphi \rangle \to \langle a, \varphi \rangle.$$  \hspace{1cm} \text{(4.4)}

Since

$$\text{id}_{\mathcal{B}(\ell_p)} \otimes \tilde{M}_{u_\alpha} = (\text{id}_{\mathcal{B}(\ell_p)} \otimes \rho) \circ (\text{id}_{\mathcal{B}(\ell_p)} \otimes (\text{id}_{PF_p(G)} \otimes T_\alpha)) \circ (\text{id}_{\mathcal{B}(\ell_p)} \otimes \gamma|_{PF_p(G)}),$$

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from Lemma 4.2.6, (4.4) becomes

\[(\text{id}_{\mathcal{B}(\ell_p)} \otimes (\text{id}_{PF_p(G)} \otimes T_{\alpha})) \circ (\text{id}_{\mathcal{B}(\ell_p)} \otimes \gamma|_{PF_p(G)})(a), \text{id}_{N(\ell_p)} \otimes \Gamma'_{p',p}(\varphi))\]

and the result follows from Proposition 4.2.5.

We can summarize this section in the following

**Corollary 4.2.8.** Let $G$ be a discrete group. Then the following are equivalent

1. $G$ has the $p$-AP;
2. there exists a net $\{\varphi_\alpha\} \subseteq A_{p,c}(G)$ such that $\text{id}_{\mathcal{K}(\ell_p)} \otimes \bar{M}_{\varphi_\alpha}(a) \to a$ in norm for every $a \in \mathcal{K}(\ell_p) \otimes PF_p(G)$;
3. $PF_p(G)$ has the $p$-OAP;
4. there exists a net $\{\varphi_\alpha\} \subseteq A_{p,c}(G)$ such that $\text{id}_{\mathcal{B}(\ell_p)} \otimes \bar{M}_{\varphi_\alpha}(a) \to a$ in norm for every $a \in \mathcal{B}(\ell_p) \otimes PF_p(G)$;
5. $PF_p(G)$ has the strong $p$-OAP;

Proof. 3$\implies$ 1 was proved in Theorem 4.2.7. Since 4$\implies$ 5, 4$\implies$ 2, 2$\implies$ 3, and 5$\implies$ 3 are clear, to complete the proof, we only need to show that 1$\implies$ 4. Suppose that $G$ has the $p$-AP, then there exists a net $\{\psi_\beta\} \subseteq A_p(G)$ such that $\psi_\beta \to 1$ with respect to $\sigma(M_{cb}A_p(G), Q_{pcb}(G))-\text{topology}$. Moreover we can assume that $\{\psi_\beta\} \subseteq A_{p,c}(G)$. Then $\bar{M}_{\psi_\beta}$ is a net of finite rank maps on $PF_p(G)$ such that for every $a \in \mathcal{B}(\ell_p) \otimes PF_p(G)$ and $\varphi \in (\mathcal{B}(\ell_p) \otimes PF_p(G))'$,

\[(\text{id}_{\mathcal{B}(\ell_p)} \otimes \bar{M}_{\psi_\beta}(a), \varphi) = \langle \omega_{a,\varphi}, \psi_\beta \rangle \to \langle \omega_{a,\varphi}, 1 \rangle = \langle a, \varphi \rangle.\]

Therefore $\text{id}_{\mathcal{B}(\ell_p)} \otimes \bar{M}_{\psi_\beta}(a) \to a$ in the weak topology. By a standard convexity argument, we can choose a net $\varphi_\alpha$ such that $\text{id}_{\mathcal{B}(\ell_p)} \otimes \bar{M}_{\varphi_\alpha}(a) \to a$ in norm for every $a \in \mathcal{B}(\ell_p) \otimes PF_p(G)$.

This completes the proof.

### 4.3 $p$-Weak Amenability and $p$-CBAP

In this section we study properties which are stronger than those in the previous section. A locally compact group $G$ is said to be $p$-weakly amenable if $A_p(G)$ has an approximate identity that is bounded in the
Theorem 4.3.2. Let $G$ be a discrete group. Then the following are equivalent:

1. $A_p(G)$ has an approximate identity $\{\varphi_\alpha\}$ such that $\|\varphi_\alpha\|_{M_{cb}A_p(G)} < K$;

2. there exists a net $\{T_\alpha\}$ of finite rank maps on $PF_p(G)$ such that $\|T_\alpha\| < K$ and $\|T_\alpha(x) - x\| \to 0$ for all $x \in PF_p(G)$;

3. there exists a net $\{T_\alpha\}$ of weak*-continuous finite rank maps on $PM_p(G)$ such that $\|T_\alpha\| < K$ and $(T_\alpha(x) - x, \varphi) \to 0$ for all $x \in PM_p(G)$.

Proof. $1 \implies 3$: We may assume that $\varphi_\alpha \in A_{p,c}(G)$. Then $\{m_{\varphi_\alpha}\}$ are finite rank $p$-cb maps on $A_p(G)$ such that $\|m_{\varphi_\alpha}\|_{p,cb} < K$ and $\|m_{\varphi_\alpha}(\omega) - \omega\|_{A_p(G)} \to 0$ for all $\omega \in A_p(G)$. Then $T_\alpha := m_{\varphi_\alpha}^*$ satisfy 3.

$1 \implies 2$: First note that for each $s \in G$, $\|\delta_s\|_{A_p(G)} = 1$, see Remark 3.3.2. From 1, we can conclude that $|\varphi_\alpha(s) - 1| = \|\varphi_\alpha(s) - 1\|_{A_p(G)} = \|\varphi_\alpha(\delta_s - \delta_s)\|_{A_p(G)} \to 0$. Since $M_{\varphi_\alpha}(\lambda_s) = \varphi_\alpha(\lambda_s)\lambda_s$, $\|M_{\varphi_\alpha}\|_{p,cb} < K$ and

$$\|M_{\varphi_\alpha}(\lambda_s) - \lambda_s\| = \|\varphi_\alpha(s)\lambda_s - \lambda_s\| = |\varphi_\alpha(s) - 1|\|\lambda_s\| \to 0.$$

Then we can obtain 2.

$2 \implies 1$: Given $\{T_\alpha\}$, define $\varphi_\alpha : G \to C$ by $\varphi(s) = \langle T(\lambda_s^\varepsilon)\delta_s^\varepsilon, \delta_s^s \rangle$. By Corollary 4.1.10, $\varphi_\alpha \in A_p(G) \subseteq M_{cb}A_p(G)$ with $\|\varphi_\alpha\|_{M_{cb}A_p(G)} < K$. Since $\|T_\alpha(\lambda_s^\varepsilon) - \lambda_s^\varepsilon\| \to 0$, we can conclude that $|\varphi_\alpha - 1| \to 0$. Then for any $\psi \in A_{p,c}(G)$, we get $\|\varphi_\alpha \psi - \psi\|_{A_p(G)} \to 0$. This implies that we have this true for all $\psi \in A_p(G)$.

$3 \implies 1$: For each $s \in G$, it is easy to see that the linear functional

$$a \in PM_p(G) \mapsto \langle a\delta_s^\varepsilon, \delta_s^s \rangle$$

Remark 4.3.1. 1. 2-weak amenability is the same as weak amenability studied in Haagerup’s unpublished paper [Haa86]. Similarly, 2-CBAP is the same as CBAP.

2. Following Theorem 11.3.3 in [ER00], one can show that if $V$ has the p-CBAP, then $V$ has the p-OAP.
is contained in $A_p(G)$. Then

$$\varphi_\alpha(s) = \langle T_\alpha(\lambda_\alpha^p)\delta^p_\alpha, \delta^p_\alpha \rangle \to \langle \delta^p_\alpha, \delta^p_\alpha \rangle = 1.$$  

Now we can get the proof as in $2 \implies 1$.

\[ \square \]

### 4.4 Comparison with the Classical Case

We take a closer look at Theorem 3.3.8.

**Theorem 4.4.1.** Let $G$ be a discrete group and $\mu$ a measure. Let $\mathcal{B}(L_p(\mu)) \hat{\otimes} P M_p(G)$ denote the weak* closure of $\mathcal{B}(L_p(\mu)) \otimes_{\text{alg}} P M_p(G)$ in $\mathcal{B}(L_p(\mu) \otimes \ell_p(G))$. Then the following properties of a function $\varphi : G \to \mathcal{B}(L_p(\mu))$ are equivalent:

1. there exists a weak*-continuous $p$-complete contraction $M_\varphi : P M_p(G) \to \mathcal{B}(L_p(\mu)) \hat{\otimes} P M_p(G)$ such that

   $$M_u(\lambda_p(s)) = \varphi(s) \otimes \lambda_p(s);$$

2. there exists a $p$-complete contraction $\tilde{M}_\varphi : P F_p(G) \to \mathcal{B}(L_p(\mu)) \hat{\otimes} P F_p(G)$ such that

   $$\tilde{M}_u(\lambda_p(s)) = \varphi(s) \otimes \lambda_p(s);$$

3. There exist an $SQ_p$ space $E$ and bounded maps $\alpha : G \to \mathcal{B}(L_p(\mu), E)$ and $\beta : G \to \mathcal{B}(E, L_p(\mu))$ such that $\sup_{t \in G} \|x(t)\| \leq 1$, $\sup_{s \in G} \|y(s)\| \leq 1$ and

   $$\forall s, t \in G, \quad \varphi(st^{-1}) = y(s)x(t).$$

**Proof.** $1 \implies 2$ is obvious.

$2 \implies 3$ First of all, note that $\lambda(\theta) \otimes \varphi(\theta)$ makes sense by Theorem 7.9 in [DF93]. By Theorem 4.1 in [Daw10], there are an $SQ_p$ space $K$, a $p$-representation $\pi : P F_p(G) \to \mathcal{B}(K)$, and operators $U : \ell_p(G) \otimes_p L_p(\mu) \to K$, $V : \ell_p(G) \otimes_p L_p(\mu) \to \mathcal{B}(E, L_p(\mu))$ such that

$$\forall \theta \in G, \quad \lambda(\theta) \otimes \varphi(\theta) = M_\varphi(\lambda(\theta)) = V \pi(\lambda(\theta)) U.$$  

For $t \in G$, define $x(t) \in \mathcal{B}(L_p(\mu), K)$ by

$$x(t)f = \pi(\lambda(t^{-1}))U(\delta^p_\alpha \otimes f), \quad f \in L_p(\mu).$$
For $s \in G$, define $y(s) \in \mathcal{B}(K, L_p(\mu))$ by

$$y(s)k = R_s V \pi(\lambda(s))k, \quad k \in K,$$

where $R_s \in \mathcal{B}(\ell_p(G) \otimes_p L_p(\mu), L_p(\mu))$ defined by $R_s \left( \sum_{t \in G} \delta_t \otimes g_t \right) = g_s$. Note that $\sup_{t \in G} \|x(t)\| \leq 1$ and $\sup_{s \in G} \|y(s)\| \leq 1$. Now for all $f \in L_p(\mu)$ and for all $f' \in L_{p'}(\mu)$,

$$\langle y(s)x(t)f, f' \rangle$$

$$= \langle R_s V \pi(\lambda(s))\pi(\lambda(t^{-1}))U(\delta^p_t \otimes f), f' \rangle$$

$$= \langle R_s V \pi(\lambda(st^{-1}))U(\delta^p_t \otimes f), f' \rangle$$

$$= \langle R_s(\delta^p_t \otimes \varphi(st^{-1}))f, f' \rangle$$

$$= \langle \varphi(st^{-1})f, f' \rangle$$

and the result follows.

(b)$\implies$(a) Suppose $K \subseteq L_p(\nu)/E$ for some measure $\nu$ and a subspace $E$ of $L_p(\nu)$. Let $Q : L_p(\nu) \to L_p(\nu)/E$ denote the quotient mapping. By the argument in Section 7.3 in [DF93], for any $T \in \mathcal{B}(\ell_p(G))$, $T \otimes id_{L_p(\nu)} \in \mathcal{B}(\ell_p(G) \otimes_p L_p(\nu))$. By Proposition 7.4 in [DF93], $T \otimes id_{L_p(\nu)}$ induces a continuous mapping $T \otimes id_{L_p(\nu)/E} : \ell_p(G) \otimes_p L_p(\nu)/E \to \ell_p(G) \otimes_p L_p(\nu)/E$. Finally let $\pi : PF_p(G) \to \mathcal{B}(\ell_p(G) \otimes_p K)$ denote the operator defined by $\pi(T) = T \otimes id_K$. Define operators $U$ and $V$ by

$$U : \ell_p(G) \otimes_p L_p(\mu) \to \ell_p(G) \otimes_p K, \quad \delta^p_t \otimes f \mapsto \delta^p_t \otimes x(t)f$$

and

$$V : \ell_p(G) \otimes_p K \to \ell_p(G) \otimes_p L_p(\mu), \quad \delta^p_s \otimes k \mapsto \delta^p_s \otimes y(s)k,$$

then $\|U\|, \|V\| \leq 1$ and for all $f \in L_p(\mu)$ and for all $f' \in L_{p'}(\mu)$, we get

$$\langle V\pi(\lambda(\theta))U(\delta^p_t \otimes f), \delta^p_{s'} \otimes f' \rangle$$

$$= \langle V\pi(\lambda(\theta))(\delta^p_t \otimes x(t)f), \delta^p_{s'} \otimes f' \rangle$$

$$= \langle V(\delta^p_{st} \otimes x(t)f), \delta^p_{s'} \otimes f' \rangle$$

$$= \langle \delta^p_{st} \otimes y(s)x(t)f, \delta^p_{s'} \otimes f' \rangle$$
\[
\langle \varphi(st^{-1})f, f' \rangle, \quad \text{if } \theta = st^{-1} \\
0, \quad \text{otherwise}
\]

\[
= \langle (\lambda(\theta) \otimes \varphi(\theta)) (\delta^\theta \otimes f), \delta^\theta \otimes f' \rangle
\]

hence \(M_p(\lambda(\theta)) = V\pi(\lambda(\theta))U\) and the result follows by the converse of Theorem 4.1 in [Daw10] once we can prove that the mapping \(T \mapsto \pi(T)\) is \(p\)-completely contractive. To this end, let \([T_{ij}] \in M_n(\mathbb{K}_p(G))\) with \(\|T_{ij}\| \leq 1\). Write \(T_{ij} \delta_t = \sum_{s \in G} a_{ij}^{s,t} \delta_s\) and for each \(1 \leq j \leq n\), let \(\xi_j = \sum_{t \in G} \xi_t^j \delta_t \in \ell_p(G)\) with \(\left(\sum_{j=1}^n \|\xi_j\|^p\right)^{1/p} = \left(\sum_{j,t} |\xi_t^j|^p\right)^{1/p} \leq 1\). Then we get

\[
1 \geq \left\| \begin{bmatrix} T_{ij} \\ \vdots \\ T_{in} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \right\|^p = \sum_{i} \left\| \sum_{j} T_{ij} \xi_j \right\|^p \geq \sum_{i,s} \left\| \sum_{j,t} a_{ij}^{s,t} \xi_t^j \right\|^p. \tag{4.5}
\]

For each \(1 \leq j \leq n\), let \(k_j = \sum_{t \in G} \delta_t \otimes k_t^j \in \ell_p(G) \otimes_p K\) with \(\left(\sum_{j=1}^n \|k_j\|^p\right)^{1/p} = \left(\sum_{j,t} \|k_t^j\|^p\right)^{1/p} \leq 1\). Then we get

\[
\left\| \begin{bmatrix} \pi(T_{ij}) \\ \vdots \\ \pi(T_{in}) \end{bmatrix} \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} \right\|^p = \sum_{i} \left\| \sum_{j} \pi(T_{ij}) k_j \right\|^p = \sum_{i,s} \left\| \sum_{j,t} a_{ij}^{s,t} k_t^j \right\|^p.
\]

By Theorem 3.2 in [LeM96] (originally due to [Kwa72]), the last term \(\sum_{i,s} \left\| \sum_{j,t} a_{ij}^{s,t} k_t^j \right\|^p\) is dominated by \(\sum_{i,s} \left\| \sum_{j,t} a_{ij}^{s,t} \xi_t^j \right\|^p\), which is in turn dominated by 1 by (4.5). This completes the proof.

\(\square\)

**Remark 4.4.2.** Suppose \(2 \leq q \leq p\) or \(p \leq q \leq 2\). By [Her71, Corollary 2] and Theorem 4.4.1, it follows that the identity mapping \(J_{q,p} : M_{cb}A_q(G) \to M_{cb}A_p(G)\) is contractive. Taking the adjoint, we have that
$J'_{q,p} : (M_{cb}A_p(G))' \to (M_{cb}A_q(G))'$ is also contractive and that $\|f\|_{Q,p} \geq \|f\|_{Q,q}$ for all $f \in L_1(G)$ (See the discussion in the beginning of Section 3.6). Therefore $Q_{pcb}(G) \subseteq Q_{qcb}(G)$ contractively.

**Proposition 4.4.3.** Suppose $2 \leq q \leq p$ or $p \leq q \leq 2$. If a discrete group $G$ is $q$-weakly amenable, then it is also $p$-weakly amenable.

**Proof.** Since $G$ is $q$-weakly amenable, there exist an approximate identity $\{\omega_\alpha \} \in A_{q,c}(G)$ and $C > 0$ such that $\|\omega_\alpha\|_{M_{cb}A_q(G)} \leq C$ for all $\alpha$. By Remark 4.4.2, we see that $\omega_\alpha \in M_{cb}A_p(G)$ with $\|\omega_\alpha\|_{M_{cb}A_p(G)} \leq C$ for each $\alpha$. Define $M_{\omega_\alpha}$ on $PF_p(G)$ by $M_{\omega_\alpha}(\lambda_p(s)) = \omega_\alpha(s)\lambda_p(s)$, then by Theorem 3.3.8, it follows that $\{M_{\omega_\alpha}\}$ are finite rank bounded maps on $PF_p(G)$ with $\|M_{\omega_\alpha}\| \leq C$ for all $\alpha$. Since $|\omega_\alpha(s) - 1| = \|\omega_\alpha\delta_s - \delta_s\|_{A_q(G)} \to 0$ for each $s \in G$, we conclude that $\|M_{\omega_\alpha}(x) - x\| \to 0$ for all $x \in PF_p(G)$. Now the result follows from Theorem 4.3.2.

**Proposition 4.4.4.** Suppose $2 \leq q \leq p$ or $p \leq q \leq 2$. If a discrete group $G$ has the $q$-AP, then it also has the $p$-AP.

**Proof.** If $G$ has the $q$-AP, then there exists a net $\{u_\alpha\} \subseteq A_{q,c}(G)$ such that $\langle u_\alpha, \omega \rangle \to \langle 1, \omega \rangle$ for all $\omega \in Q_{qcb}(G)$. Since $G$ is discrete, $\{u_\alpha\} \subseteq A_{p,c}(G)$ and by Remark 4.4.2, we have $\langle u_\alpha, \omega \rangle \to \langle 1, \omega \rangle$ for all $\omega \in Q_{pcb}(G)$.

### 4.5 1-Nuclearity

**Lemma 4.5.1.** Let $x = [x_{ij}] \in B(\ell_1^m)$. Then

$$\|x\|_{B(\ell_1^m)} = \max_{1 \leq j \leq m} \left\{ \sum_{i=1}^m |x_{ij}| \right\}.$$ 

**Proof.** Easy.

Suppose $G$ is a countable discrete group. Write $G = \bigcup_{n=1}^\infty F_n$ with $\{e\} \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$, each $F_n$ finite, and $|F_n| = N_n$. For each $n \in \mathbb{N}$, define $P_n \in B(\ell_1(G))$ by

$$P_n \left( \sum_{s \in G} a_s \delta_s \right) = \sum_{s \in F_n} a_s \delta_s,$$
then we can identify $P_n B(\ell_1(G)) P_n$ with $M_{N_n} = B(\ell_1^{N_n})$.

\[
\begin{array}{ccc}
P_n B(\ell_1(G)) P_n & \xrightarrow{\varphi_n} & P_n B(\ell_1(G)) P_n \\
\ell_1(G) & \xrightarrow{\psi_n} & \ell_1(G)
\end{array}
\]

Define

\[
\varphi_n : \ell_1(G) \to P_n B(\ell_1(G)) P_n, \quad f \mapsto P_n \lambda(f) P_n
\]

and

\[
\psi_n : P_n B(\ell_1(G)) P_n \to \ell_1(G), \quad [a_{u,v}]_{u,v \in F_n} \mapsto \sum_{u \in F_n} a_{u,e} \delta_u.
\]

**Claim 4.5.2.** For all $f \in \ell_1(G)$, $\|\psi_n \circ \varphi_n(f) - f\| \to 0$.

**Proof.** For $u \in F_n$, we have $\langle \delta_u, \varphi_n(f) \delta_e \rangle = \langle \delta_u, P_n f \rangle = f(u)$ and this calculation shows that $\psi_n \circ \varphi_n(f) = \chi_{F_n} f$. \hfill $\square$

Note that $\ell_1(G)$ can be identified with $\lambda(\ell_1(G)) \subseteq B(\ell_1(G))$ as convolution operators and naturally equipped with the 1-operator space structure inherited from this inclusion. For notational convenience, we fix $n$ and drop the subscript $n$.

**Claim 4.5.3.** $\varphi$ is 1-completely contractive.

**Proof.** Clear. \hfill $\square$

**Claim 4.5.4.** $\psi$ is 1-completely contractive.

**Proof.** Fix $k \in \mathbb{N}$. Let $[T_{ij}]_{1 \leq i,j \leq k} \in M_k(P_n B(\ell_1(G)) P_n)$ with $\|[T_{ij}]\| \leq 1$. We need to show that $\|\psi([T_{ij}])\| \leq 1$. To this end, suppose $g_j \in \ell_1(G)$ with $\sum_{j=1}^k \|g_j\|_1 \leq 1$, then

\[
\begin{align*}
\left\| \psi([T_{ij}]) \left[ \begin{array}{c}
g_1 \\
\vdots \\
g_k
\end{array} \right] \right\| & \leq \sum_{i=1}^k \left\| \sum_{j=1}^k \psi(T_{ij}) * g_j \right\| \\
& \leq \sum_{i=1}^k \sum_{j=1}^k \|\psi(T_{ij})\|_1 \|g_j\|_1
\end{align*}
\]
\[
\sum_{j=1}^{k} \|g_j\|_1 \sum_{i=1}^{k} \|\psi(T_{ij})\|_1
\]
\[
(\text{By Lemma 4.5.1}) \leq \sum_{j=1}^{k} \|g_j\|_1 \|T_{ij}\|
\]
\[
\leq 1.
\]

This completes the proof. \(\square\)

4.6 Stability of Approximation Properties

**Lemma 4.6.1.** Let \(H\) be a closed subgroup of a locally compact group \(G\). For any function \(u\) on \(G\), let \(u|_H\) denote the restriction of \(u\) to \(H\).

1. If \(u \in MA_p(G)\), then \(u|_H \in MA_p(H)\) and

\[
\|u|_H\|_{MA_p(H)} \leq \|u\|_{MA_p(G)}.
\]

2. If \(u \in M_{cb}A_p(G)\), then \(u|_H \in M_{cb}A_p(H)\) and

\[
\|u|_H\|_{M_{cb}A_p(H)} \leq \|u\|_{M_{cb}A_p(G)}.
\]

**Proof.** (a) By Theorem 1 in [Her71], the restriction of functions gives a quotient mapping from \(A_p(G)\) to \(A_p(H)\). Therefore, for \(\varphi \in A_p(H)\), we get

\[
\|\varphi\|_{A_p(H)} = \inf\{\|\tilde{\varphi}\|_{A_p(G)} : \tilde{\varphi} \in A_p(G), \tilde{\varphi}|_H = \varphi\}.
\]

Suppose \(u \in MA_p(G)\). For \(\varphi \in A_p(H)\), there exists \(\tilde{\varphi}\) with \(\tilde{\varphi}|_H = \varphi\). Since \(u|_H \varphi = (u\tilde{\varphi})|_H\) and \(u\tilde{\varphi} \in A_p(G)\), again by Theorem 1 in [Her71] we get \(u|_H \varphi \in A_p(H)\). This shows that \(u|_H \in MA_p(H)\) and moreover,

\[
\|u|_H\|_{MA_p(H)} = \inf\{\|u|_H \varphi\|_{A_p(H)} : \varphi \in A_p(H), \|\varphi\|_{A_p(H)} \leq 1\}
\]
\[
\leq \inf\{\|u\tilde{\varphi}\|_{A_p(G)} : \tilde{\varphi} \in A_p(G), \tilde{\varphi}|_H = \varphi, \|\varphi\|_{A_p(H)} \leq 1\}
\]
\[
\leq \|u\|_{MA_p(G)}.
\]

(b) By Theorem 8.3 in [Daw10], there exists \(E \in SQ_p\) and bounded continuous maps \(\alpha : G \to E\) and \(\beta : G \to E'\) such that \(u(ts^{-1}) = \langle \beta(t), \alpha(s) \rangle\) for all \(t, s \in G\). It follows that \(u|_H(\tau \sigma^{-1}) = \langle \beta|_H(\tau), \alpha|_H(\sigma) \rangle\)
for all $\tau, \sigma \in H$. This shows that $u|_H \in M_{cb}A_p(H)$ and that \(\|u|_H\|_{M_{cb}A_p(H)} \leq \|u\|_{M_{cb}A_p(G)}\).

\[\textbf{Theorem 4.6.2.}\] Let $H$ be a closed subgroup of a locally compact group $G$. If $G$ is $p$-weakly amenable, then $H$ is also $p$-weakly amenable.

\textit{Proof.} Let \(\{u_i\}\) be an approximate identity in $A_p(G)$ such that \(\sup_i \|u_i\|_{M_{cb}A_p(G)} \leq k\) for some $k$. Put $v_i = u_i|_H$. We claim that \(\{v_i\}\) is an approximate identity in $A_p(H)$ such that \(\sup_i \|u_i\|_{M_{cb}A_p(G)} \leq k\).

Indeed, for any $\varphi \in A_p(H)$, we can always find $\tilde{\varphi} \in A_p(G)$ such that $\tilde{\varphi}|_H = \varphi$ and

\[\|v_i \varphi - \varphi\|_{A_p(H)} = \|(u_i \tilde{\varphi} - \tilde{\varphi})|_H\|_{A_p(H)} \leq \|u_i \tilde{\varphi} - \tilde{\varphi}\|_{A_p(G)} \to 0.\]

Since $\sup_i \|u_i\|_{M_{cb}A_p(G)} \leq k$ by Lemma 4.6.1, this completes the proof. \(\Box\)

\[\textbf{Theorem 4.6.3.}\] Let $H$ be a subgroup of a discrete group $G$. If $G$ has $p$-AP, then $H$ also has $p$-AP.

\textit{Proof.} Suppose $\{u_i\}$ is a net in $A_p(G)$ such that $u_i \to 1_G$ in $\sigma(M_{cb}A_p(G), Q_{pcb}(G))$-topology. Let $v_i = u_i|_H$.

We claim that $v_i \to 1_H$ in $\sigma(M_{cb}A_p(H), Q_{pcb}(H))$-topology. Suppose $\xi \in Q_{pcb}(H)$, then $\xi = \lim_n f_n$ for some $f_n \in L_1(H)$ in $M_{cb}A_p(H)'$. Define $g_n \in L_1(G)$ by

\[g_n(x) = \begin{cases} f_n(x), & x \in H, \\ 0, & x \in G \setminus H. \end{cases}\]

Since

\[\|g_n - g_m\|_{M_{cb}A_p(G)'} = \sup \left\{ \sum_{x \in G} (g_n(x) - g_m(x))\varphi(x) : \varphi \in M_{cb}A_p(G), \|\varphi\|_{M_{cb}A_p(G)} \leq 1 \right\} \]

\[= \sup \left\{ \sum_{x \in H} (f_n(x) - f_m(x))\varphi|_H(x) : \varphi \in M_{cb}A_p(G), \|\varphi\|_{M_{cb}A_p(G)} \leq 1 \right\} \]

\[= \|f_n - f_m\|_{M_{cb}A_p(H)'} \]

\(\{g_n\}\) is Cauchy in $M_{cb}A_p(G)'$. Let $\eta = \lim_n g_n$ in $M_{cb}A_p(H)'$, then

\[\langle v_i, \xi \rangle = \lim_n \sum_{x \in H} v_i(x)f_n(x) = \lim_n \sum_{x \in G} u_i(x)g_n(x) \]

\[= \langle u_i, \eta \rangle \]

\[= \langle 1_G, \eta \rangle \]

\[= \lim_n \sum_{x \in G} g_n(x) = \lim_n \sum_{x \in H} f_n(x) \]

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This completes the proof. \(\square\)

**Remark 4.6.4.** Even when \(p = 2\), it is unlikely that \(G\) has the \(p\)-AP implies \(G/H\) has the \(p\)-AP. See [HK94].

**Lemma 4.6.5.** Let \(G\) be a locally compact group, and suppose that \(H\) is a closed subgroup of \(G\) such that \(\Delta_G(h) = \Delta_H(h)\) for all \(h \in H\). For each \(f, g \in C_00(G)\), let \(\Phi_{f,g}\) denote the map defined on \(M_{cb}A_p(H)\) by

\[
\Phi_{f,g}(u) = f \ast udh \star \check{g}, \quad u \in M_{cb}A_p(H),
\]

where \(dh\) is a fixed left Haar measure on \(H\). Then \(\Phi_{f,g}\) is a bounded linear map from \(M_{cb}A_p(H)\) into \(M_{cb}A_p(G)\) that is \(\sigma(M_{cb}A_p(H), Q_{pcb}(H)) - \sigma(M_{cb}A_p(G), Q_{pcb}(G))\) continuous.

**Proof.** Let \(\pi_H : G \to G/H\) denote the canonical mapping onto the left cosets of \(H\) and write \(\dot{x} = \pi_H(x)\). Let \(T_H : C_00(G) \to C_00(G/H)\) be as in [RS00, Definition 3.3.9], that is,

\[
T_H f(\dot{x}) = \int_H f(xh)dh, \quad f \in C_00(G), \quad \dot{x} \in G/H.
\]

By [RS00, Propositions 8.1.1, 8.1.3, 8.1.4], there is a measure \(\mu\) on \(G/H\) that is invariant under the natural action of \(G\) on \(G/H\) and satisfies the relation

\[
\int_{G/H} T_H f(\dot{x})d\mu(\dot{x}) = \int_G f(x)dx, \quad f \in C_00(G), \quad (4.6)
\]

where \(dx\) is a fixed Haar measure on \(G\). Let \(g \in C_00(G)\) and let \(u \in M_{cb}A_p(H)\). Then

\[
(udh \star \check{g})(y) = \int_H u(h)g(y^{-1}h)dh, \quad \forall y \in G.
\]

Using the fact that \(u\) is bounded and \(g\) is uniformly continuous, one can easily show that \(udh \star \check{g}\) is continuous, and hence for each \(x \in G\), the function \(y \mapsto f(y)(udh \star \check{g})(y^{-1}x)\) is in \(C_00(G)\). Now, by (4.6),

\[
\Phi_{f,g}(u)(x) = \int_G f(y)(udh \star \check{g})(y^{-1}x)dy
\]

\[
\quad = \int_{G/H} \int_H f(yh)(udh \star \check{g})(h^{-1}y^{-1}x)dhd\mu(\check{y})
\]

\[
\quad = \int_{G/H} \int_H f(yh) \int_H u(k)g(x^{-1}yhk)dkdhd\mu(\check{y})
\]

\[
\quad = \int_{G/H} \int_H f(yh)u(k)g(x^{-1}yhk)dkdhd\mu(\check{y}). \quad (4.7)
\]

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Let $\epsilon > 0$. Since $u \in M_{cb}A_p(H)$, by [Daw10, Theorem 8.3], there exist $E \in SQ_p$ and bounded maps $\alpha : H \to E$ and $\beta : H \to E'$ such that $u(s^{-1}t) = \langle \beta(s), \alpha(t) \rangle$ for all $t, s \in H$ and $\|\alpha\|\|\beta\| < \|u\|_{M_{cb}A_p(H)} + \epsilon$. Define $P : G \to E'$ (respectively, $Q : G \to E$) by

$$P(x) = \int_H f(xh)\beta(h)dh \quad \text{respectively, } Q(x) = \int_H g(xh)\alpha(h), \quad x \in G.$$  

For any $x \in G$ and for any $\varphi \in E'' = E$, we have

$$|\langle P(x), \varphi \rangle| \leq \int_H |f(xh)| |\langle \beta(h), \varphi \rangle| dh \leq \|\beta\| \|\varphi\| \int_H |f(xh)| dh,$$

and hence

$$\|P(x)\| \leq \|\beta\| \int_H |f(xh)| dh, \quad \forall x \in G. \quad (4.8)$$

Since

$$\|P(x) - P(y)\| \leq \|\beta\| \int_H |f(xh) - f(yh)| dh$$

and $f \in C_{00}(G)$, it follows that $P$ is continuous. Similarly,

$$\|Q(x)\| \leq \|\alpha\| \int_H |g(xh)| dh, \quad \forall x \in G, \quad (4.9)$$

and in particular, $Q$ is also continuous. Now by (4.7),

$$\Phi_{f,g}(u)(x^{-1}y) = \int_{G/H} \int_H \int_H f(zh)u(k)g(y^{-1}xzkh)dkdhd\mu(z)$$

$k \mapsto h^{-1}k$ = $\int_{G/H} \int_H \int_H f(zh)\alpha(h^{-1}k)g(y^{-1}xz)dkdhd\mu(z)$

$z \mapsto x^{-1}z$ = $\int_{G/H} \int_H \int_H f(x^{-1}zh)u(h^{-1}k)g(y^{-1}zk)dkdhd\mu(z)$

= $\int_{G/H} \int_H \int_H f(x^{-1}zh)g(y^{-1}zk)\langle \beta(h), \alpha(k) \rangle dkdhd\mu(z)$

= $\int_{G/H} \langle P(x^{-1}z), Q(y^{-1}z) \rangle d\mu(z)$,

where the $G$-invariance of $\mu$ is used in the third equality. Let $\rho, \omega$ be as in the proof of [HK94, Lemma 1.16].
Then
\[
\int_G \|Q(y^{-1}x)\|^p \, d\omega(x) = \int_{G/H} \|Q(y^{-1}\rho(\hat{x}))\|^p \, d\mu(\hat{x})
\]
(by (4.9)) \leq \int_{G/H} \left( \|\alpha\|_\infty \int_H \|g(y^{-1}\rho(\hat{x})h)\| \, dh \right)^p \, d\mu(\hat{x})
\[
(\pi_H(\rho(\hat{x})) = \hat{x}) = \int_{G/H} \left( \|\alpha\|_\infty \int_H \|g(y^{-1}xh)\| \, dh \right)^p \, d\mu(\hat{x})
\]
\[
(G\text{-invariance of } \mu) = \int_{G/H} \left( \|\alpha\|_\infty \int_H \|g(xh)\| \, dh \right)^p \, d\mu(\hat{x})
\]

Put \( L(g) = \int_{G/H} \left( \int_H |g(xh)| \, dh \right)^p \, d\mu(\hat{x}) \) and \( F = L_p(G,E,\omega) = L_p(G,\omega) \otimes_p E \), then \( F \) is an \( SQ_p \) space
(See Theorem 3.1 and the remarks after it in [Run05]) and the function \( B : G \to F \) defined by
\[
B(x)(y) = Q(x^{-1}y), \quad x, y \in G,
\]
satisfies \( \|B(x)\|^p \leq \|\alpha\|^p_\infty L(g) \) for all \( x \in G \). Moreover, for any \( y, z \in G \),
\[
\|B(y) - B(z)\|^p = \int_G \|Q(y^{-1}x) - Q(z^{-1}x)\|^p \, d\omega(x)
\]
\[
= \int_{G/H} \|Q(y^{-1}\rho(\hat{x})) - Q(z^{-1}\rho(\hat{x}))\|^p \, d\mu(\hat{x})
\]
(by (4.10)) \leq \int_{G/H} \|\alpha\|^p_\infty \left( \int_H \|g(y^{-1}\rho(\hat{x})h) - g(z^{-1}\rho(\hat{x})h)\| \, dh \right)^p \, d\mu(\hat{x})
\[
\leq \int_{G/H} \|\alpha\|^p_\infty \left( \int_H \|g(y^{-1}xh) - g(z^{-1}xh)\| \, dh \right)^p \, d\mu(\hat{x})
\]
\[
\leq \int_{G/H} \|\alpha\|^p_\infty 2^{p-1} |T_H|\|g\|^p_{\infty} \int_H \|g(y^{-1}xh) - g(z^{-1}xh)\| \, dh \, d\mu(\hat{x})
\]
\[
= \|\alpha\|^p_\infty 2^{p-1} |T_H|\|g\|^p_{\infty} \int_G \|g(y^{-1}x) - g(z^{-1}x)\| \, dx
\]
and it follows that \( B \) is continuous. Similarly,
\[
\int_G \|P(y^{-1}x)\|^{p'} \, d\omega(x) \leq \int_{G/H} \left( \|\beta\|_\infty \int_H |f(xh)| \, dh \right)^{p'} \, d\mu(\hat{x}).
\]

Put \( M(f) = \int_{G/H} \left( \int_H |g(xh)| \, dh \right)^p \, d\mu(\hat{x}) \) and define \( A : G \to L_{p'}(G,\omega) \otimes_{p'} E' \subseteq F' \) [DF93, Proposition 15.10] by \( A(x)(y) = P(x^{-1}y) \), then \( \|A(x)\|^{p'} \leq \|\beta\|^{p'}_\infty M(f) \) for all \( x \in G \). Moreover, the same argument as above gives
\[
\|A(y) - A(z)\|^{p'} \leq \|\beta\|^{p'}_\infty 2^{p'-1} |T_H|\|g\|^{p'-1}_{\infty} \int_G \|f(y^{-1}x) - f(z^{-1}x)\| \, dx
\]
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and hence \( A \) is also continuous. Now it follows that

\[
(A(x), B(y)) = \int_G (P(x^{-1}z), Q(y^{-1}z))d\omega(z)
\]

\[
= \int_G \int_H \int_H f(x^{-1}zh)g(y^{-1}zk)(\beta(h), \alpha(k))dhdkd\omega(z)
\]

\[
= \int_{G/H} \int_H \int_H f(x^{-1}zh)g(y^{-1}zk)(\beta(h), \alpha(k))dhdkd\mu(\hat{z})
\]

\[
= \Phi_{f,g}(u)(x^{-1}y)
\]

and this shows that \( \Phi_{f,g}(u) \in M_{cb}A_p(G) \). Moreover, by taking the infimum over \( \alpha \) and \( \beta \), we get

\[
\|\Phi_{f,g}\| \leq L(g)^{1/p} M(f)^{1/p'}.
\]

It remains to show that \( \Phi_{f,g} \) is \( \sigma(M_{cb}A_p(H), Q_{pcb}(H)) \) \( \sigma(M_{cb}A_p(G), Q_{pcb}(G)) \) continuous. To this end, it suffices to show that \( \Phi_{f,g}' \) maps \( Q_{pcb}(G) \) into \( Q_{pcb}(H) \). Since \( L_1(G) \) is dense in \( Q_{pcb}(G) \), it suffices to show that \( \Phi_{f,g}'(\xi) \in Q_{pcb}(H) \) for every \( \xi \in L_1(G) \). So let \( \xi \in L_1(G) \) and define a function \( \xi_{f,g} \) on \( H \) by

\[
\xi_{f,g}(h) = \int_G f(x)(\xi \ast g)(xh)dx, \quad h \in H.
\]

It is easy to check that \( \xi_{f,g} \in L_1(H) \) and that

\[
\langle u, \Phi_{f,g}'(\xi) \rangle = \langle \Phi_{f,g}(u), \xi \rangle = \int_H u(k)\xi_{f,g}(k)dk = \langle u, \xi_{f,g} \rangle,
\]

from which it follows that \( \Phi_{f,g}'(\xi) = \xi_{f,g} \in L_1(H) \subseteq Q_{pcb}(H) \).

**Theorem 4.6.6.** Let \( G \) be a locally compact group and \( H \) a closed normal subgroup of \( G \). If \( H \) and \( G/H \) have the \( p \)-AP, then \( G \) has the \( p \)-AP.

**Proof.** Let \( \{u_i\} \subseteq A_{p,c}(H) \) be such that \( u_i \to 1_H \) in the \( \sigma(M_{cb}A_p(H), Q_{pcb}(H)) \) topology. By Lemma 4.6.5 and [RS00, Proposition 3.3.17], for any \( f, g, \in C_{00}(G), \Phi_{f,g}(u_i) \to \Phi_{f,g}(1_H) \) in the \( \sigma(M_{cb}A_p(G), Q_{pcb}(G)) \) topology. Since \( \Phi_{f,g}(u_i) \in A_{p,c}(G) \) for all \( i \), it suffices to show that \( 1_G \) is in the \( \sigma(M_{cb}A_p(G), Q_{pcb}(G)) \) closure of \( \{\Phi_{f,g}(1_H) : f, g \in C_{00}(G)\} \). Let \( f, g \in C_{00}(G) \) and put \( \xi = T_H f, \eta = T_H g \). It follows from (4.7) that

\[
\Phi_{f,g}(1_H)(x) = \int_{G/H} \int_H \int_H f(yh)g(x^{-1}yk)dhdkd\mu(y)
\]

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\[ \int_{G/H} \xi(\hat{y})\eta(\hat{x}^{-1}\hat{y})d\mu(\hat{y}) = \xi \ast \hat{\eta}(\hat{x}), \quad \forall x \in G. \]

Since $T_H$ is a map from $C_{00}(G)$ onto $C_{00}(G/H)$ [RS00, Proposition 3.4.2], we obtain

\[ \{ \Phi_{f,g}(1_H) : f, g \in C_{00}(G) \} = \{ (\xi \ast \hat{\eta}) \circ \pi_H : \xi, \eta \in C_{00}(G/H) \}. \quad (4.11) \]

Moreover, $T_H$ extends to a map from $L_1(G)$ onto $L_1(G/H)$ [RS00, Proposition 3.4.5]. Define a map $\Psi$ on $M_{cb}A_p(G/H)$ by $\Psi(u) = u \circ \pi_H$, $u \in M_{cb}A_p(G/H)$. Since $\Psi(u)(st^{-1}) = u(st^{-1})$, we see that $\Psi$ maps $M_{cb}A_p(G/H)$ contractively into $M_{cb}A_p(G)$. We claim that $\Psi$ is $\sigma(M_{cb}A_p(G/H), Q_{pcb}(G/H)) - \sigma(M_{cb}A_p(G), Q_{pcb}(G))$ continuous. Let $\zeta \in L_1(G)$, then for any $u \in M_{cb}A_p(G/H)$, we get

\[ \langle u, T_H \zeta \rangle = \int_{G/H} u(\hat{x}) \int_H \zeta(xh)dhd\mu(\hat{x}) \]
\[ = \int_{G/H} \int_H u(\hat{x})\zeta(xh)dhd\mu(\hat{x}) \]
\[ = \int_{G/H} \int_H \Psi(u)(xh)\zeta(xh)dhd\mu(\hat{x}) \]
\[ = \langle \Psi(u), \zeta \rangle \]

and using the same argument as in the proof of Lemma 4.6.5, we get the desired continuity of $\Psi$. Let $R$ denote the linear span of $\{ \xi \ast \hat{\eta} : \xi, \eta \in C_{00}(G/H) \}$, then $R$ is dense in $A_p(G/H)$ in the $A_p(G/H)$ norm and hence $1_{G/H}$ is in the $\sigma(M_{cb}A_p(G/H), Q_{pcb}(G/H))$ closure of $R$. Now from the continuity of $\Psi$ and (4.11), we conclude that $1_G$ is in the $\sigma(M_{cb}A_p(G), Q_{pcb}(G))$ closure of $\{ \Phi_{f,g}(1_H) : f, g \in C_{00}(G) \}$ as desired. \qed
Chapter 5

Conditions $C_p$, $C'_p$, and $C''_p$ for $p$-Operator Spaces

5.1 Introduction

Conditions $C$, $C'$, and $C''$ for operator spaces are studied in [ER00, Chapter 14]. To be precise, it is known that an operator space $W$ is locally reflexive if and only if $W$ satisfies condition $C''$ [ER00, Theorem 14.3.1]. It is also known that an operator space $V$ is exact if and only if $V$ satisfies condition $C'$ [ER00, Theorem 14.4.1]. In this chapter, we define $p$-analogues of these definitions, which will be called conditions $C_p$, $C'_p$, and $C''_p$, and show that a $p$-operator space satisfies condition $C_p$ if and only if it satisfies both conditions $C'_p$ and $C''_p$.

Lemma 5.1.1. Let $V$ and $W$ be $p$-operator spaces. Then the bilinear mapping

$$\tilde{\Psi} : V' \times W' \to (V \overset{\vee}{\otimes} W)', \quad (f, g) \mapsto f \otimes g$$

is jointly $p$-completely contractive and hence the canonical mapping $\Psi : V' \overset{\wedge}{\otimes} W' \to (V \overset{\vee}{\otimes} W)'$ is $p$-completely contractive.

Proof. The second half follows from Proposition 2.1.3, so it suffices to show that the bilinear map $\tilde{\Psi}$ is jointly $p$-completely contractive. Recall the definition

$$\tilde{\Psi}_{r,s} : M_r(V') \times M_s(W') \to M_{rs}((V \overset{\vee}{\otimes} W)'), \quad ([f_{ij}],[g_{kl}]) \mapsto [f_{ij} \otimes g_{kl}], \quad r, s \in \mathbb{N}.$$ 

Here we can identify $[f_{ij} \otimes g_{kl}]$ with $[f_{ij}] \otimes [g_{kl}] : V \overset{\vee}{\otimes} W \to M_{rs}$ and hence we get $\|[f_{ij} \otimes g_{kl}]\| \leq \|[f_{ij}]\| \cdot \|[g_{kl}]\|$.

Lemma 5.1.2. Let $V \subseteq B(L_p(\mu))$ and $W \subseteq B(L_p(\nu))$. Then $\|\cdot\|_{\overset{\vee}{\otimes}}$ is a subcross matrix norm. In particular, for every $u \in M_n(V \otimes W)$, we have $\|u\|_{\overset{\vee}{\otimes}} \leq \|u\|_{\overset{\wedge}{\otimes}}$.
Proof. Let $v = [v_{ij}] \in M_r(V)$ and $w = [w_{kl}] \in M_q(W)$, then by Remark 2.3.2,

$$\|v \otimes w\|_{\nu_p} = \sup \{\|(f \otimes g)_{\nu_p}(v \otimes w)\| : f \in M_s(V'), g \in M_t(W')\}.$$ 

Note that $(f \otimes g)_{\nu_p}(v \otimes w) = f_r(v) \otimes g_q(w)$ and hence

$$\|(f \otimes g)_{\nu_p}(v \otimes w)\| = \|f_r(v)\| \cdot \|g_q(w)\| \leq \|v\| \cdot \|w\|.$$ 

The second half of the lemma follows immediately because $\cdot \|_{\nu_p}$ is the largest subcross matrix norm. \hfill \Box

### 5.2 Conditions $C'_p$, $C''_p$, and $C'_p$ for $p$-Operator Spaces

Let $V \subseteq \mathcal{B}(L_p(\mu))$ and $W \subseteq \mathcal{B}(L_p(\nu))$. Fix $\varphi \in (V \overline{\otimes} W)'$. For $v_0 \in V$, we define a bounded linear functional $v_0 \varphi$ on $W$ by

$$v_0 \varphi(w) = \varphi(v_0 \otimes w), \quad w \in W.$$ 

In general, when $v_0 = [v_{ij}] \in M_r(V)$ and $\varphi = [\varphi_{kl}] \in M_n((V \overline{\otimes} W)')$, we define $v_0 \varphi = [v_{ij} \varphi_{kl}] \in M_{rn}(W')$. Similarly, for $w_0 \in W$, we define $\varphi_{w_0} \in V'$ by

$$\varphi_{w_0}(v) = \varphi(v \otimes w_0), \quad v \in V.$$ 

As in $v_0 \varphi$ above, we can extend the definition of $\varphi_{w_0}$ for $w_0 \in M_r(W)$ and $\varphi \in M_n((V \overline{\otimes} W)')$.

Define a linear map $\Phi_{V,W}^r : V \otimes W'' \rightarrow (V \overline{\otimes} W)''$ by

$$\Phi_{V,W}^r(v \otimes w'')(\varphi) = \langle v \varphi, w'' \rangle_{V',W''}, \quad v \in V, \quad w'' \in W'', \quad \varphi \in (V \overline{\otimes} W)'.$$ 

Similarly, define a linear map $\Phi_{V,W}^l : V'' \otimes W \rightarrow (V \overline{\otimes} W)''$ by

$$\Phi_{V,W}^l(v'' \otimes w)(\varphi) = \langle \varphi_{w''}, v'' \rangle_{V'',W''}, \quad v'' \in V'', \quad w \in W, \quad \varphi \in (V \overline{\otimes} W)' .$$

**Lemma 5.2.1.** The map $\Phi_{V,W}^r$ (respectively, $\Phi_{V,W}^l$) defined above extends to a $p$-completely contractive map $\Phi_{V,W}^r : V \overline{\otimes} W'' \rightarrow (V \overline{\otimes} W)''$ (respectively, $\Phi_{V,W}^l : V'' \otimes W \rightarrow (V \overline{\otimes} W)'$).

**Proof.** Consider the bilinear map $\Phi : V \times W'' \rightarrow (V \overline{\otimes} W)''$ given by

$$(v, w'') \mapsto (\varphi \mapsto \langle v \varphi, w'' \rangle_{V',W''}).$$
We get
\[ \Phi_{r,s} : M_r(V) \times M_s(W') \to M_{rs}((V \otimes^p W)''), \quad ([v_{ij}], [w_{kl}'']) \mapsto [\Phi(v_{ij}, w_{kl}'')]. \]

Following the notation in [ER00], we obtain
\[ \|\Phi(v_{ij}, w_{kl}'')\| = \sup_n \left\{ \|\langle \Phi_{r,s}(v, w''), \varphi \rangle\| : \varphi \in M_n((V \otimes^p W)'), \|\varphi\| \leq 1 \right\}. \]

Since \( \langle \Phi_{r,s}(v, w''), \varphi \rangle = \langle v^\varphi, w'' \rangle \), we have
\[ \|\langle \Phi_{r,s}(v, w''), \varphi \rangle\| = \|v^\varphi\| \leq \|v\| \|

and the result follows because
\[ \|v^\varphi\| = \sup_m \{ \|\langle v^\varphi, w \rangle\|_{M_{rm}(W')} : w \in M_m(W'), \|w\| \leq 1 \} \]
\[ = \sup_m \{ \|\langle \varphi, v \otimes w \rangle\|_{M_{rm}(W')} : w \in M_m(W'), \|w\| \leq 1 \} \]
\[ \leq \|\varphi\| \|v\| \]
\[ \leq \|v\|. \]

Let \( \Psi : V' \otimes^p W' \to (V \otimes W)' \) denote the canonical map, and consider the following commutative diagram
\[
\begin{align*}
V \otimes W'' &\xrightarrow{\Phi_{V,W}^p} (V \otimes^p W)''' \xrightarrow{\Psi'} (V' \otimes^p W)' \\
\mathcal{CB}_{p,F}^p(V', W'') &\xrightarrow{\iota} \mathcal{CB}_p(V', W'')
\end{align*}
\]

where \( \mathcal{CB}_{p,F}^p(V', W'') \) denotes the space of all weak*-continuous \( p \)-completely bounded finite rank maps from \( V' \) to \( W'' \) and \( \iota \) is the inclusion map. This commutative diagram shows that \( \Phi_{V,W}^p \) is one-to-one, so one can equip \( V \otimes W'' \) with the \( p \)-operator space norm inherited from \( (V \otimes W)''' \), which will be denoted by, following the notation in [ER00], \( V \otimes_{V,p} : W'' \). We say that \( V \) satisfies condition \( C'_p \) (or \( V \) has property \( C'_p \)) if this induced norm coincides with the injective tensor product norm with every \( W \subseteq \mathcal{B}(L_p(\nu)) \).

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Similarly, the following diagram

\[ V'' \otimes W \xrightarrow{\Phi_{V,W}} (V \wedge_p W)'' \xrightarrow{\Psi'} (V' \wedge_p W')' \]
\[ \xrightarrow{\text{CB}_{p,F}(W',V'')} \]  

is also commutative, \( \Phi_{V,W} \) is one-to-one, and one can hence equip \( V'' \otimes W \) with the \( p \)-operator space norm inherited from \( (V \wedge_p W)'' \), which will be denoted by \( V'' : \wedge_p W \). We say that \( V \) satisfies condition \( C''_p \) (or \( V \) has property \( C''_p \)) if this induced norm coincides with the injective tensor product norm with every \( W \subseteq B(L_p(\nu)) \).

In order to define condition \( C_p \) for \( p \)-operator spaces, we consider the following diagram

\[ (V \wedge_p W)'' \xrightarrow{(\Phi_{V,W})''} (V \wedge_p W)''' \xrightarrow{P} (V \wedge_p W)''', \]

where \( P \) is the restriction mapping. Note that Lemma 5.2.1 was used here to consider \( (\Phi_{V,W})'' \) and \( (\Phi_{V,W})''' \).

For \( p \)-operator spaces \( V \subseteq B(L_p(\mu)) \), \( W \subseteq B(L_p(\nu)) \), we consider the following \( p \)-complete contraction:

\[ (V \wedge_p W)'' \xrightarrow{\text{PCB}(V,W')} \xrightarrow{\text{adj}} \text{CB}_p(W'',V') \equiv (V \wedge_p W)''. \]

For \( \varphi \in (V \wedge_p W)' \), let \( \varphi^\wedge \in (V \wedge_p W'')' \) denote the image of \( \varphi \) under this map. Then we have

\[ \varphi^\wedge (v \otimes w'') = \langle v \varphi, w'' \rangle_{W'',W'''} = \Phi_{V,W}'(v \otimes w'')(\varphi), \quad v \in V, \quad w'' \in W''. \]

Moreover, \( \varphi^\wedge \) is weak*-continuous in the second variable. Similarly, we also consider the \( p \)-complete contraction

\[ (V \wedge_p W)' \xrightarrow{\text{PCB}(V,W')} \xrightarrow{\text{adj}} \text{CB}_p(W'',V') \equiv (V'' \wedge_p W)''. \]
and define $\wedge \varphi$, and then we get that

$$\wedge \varphi(v'' \otimes w) = \langle \varphi_w, v'' \rangle_{V'', W''} = \Phi_{V'' W''}^R(v'' \otimes w)(\varphi), \quad v'' \in V'', \ w \in W,$$

and that $\wedge \varphi$ is weak*-continuous in the first variable.

Following the idea in [Han07], we have the next result.

**Theorem 5.2.2.** Let $V \subseteq B(L_p(\mu))$ and $W \subseteq B(L_p(\nu))$. Let $\alpha$ be a subcross matrix norm on $V \otimes W$ and denote by $V \otimes_\alpha W$ the resulting normed space. Then the following are equivalent.

1. There exists a separately weak*-continuous extension $\Phi : V'' \otimes W'' \to (V \otimes_\alpha W)''$ of the natural inclusion $\iota : V \otimes W \to (V \otimes_\alpha W)''$.

2. The following diagram commutes

3. For every $\varphi \in (V \otimes_\alpha W)'$, two functionals $(\wedge \varphi)^\wedge$ and $\wedge (\varphi^\wedge)$ coincide on $V'' \otimes W''$.

4. For every $\varphi \in (V \otimes_\alpha W)'$, $L_\varphi : V \to W'$ is weakly compact, where $\langle L_\varphi(v), w \rangle = \varphi(v \otimes w)$, $v \in V$, $w \in W$.

**Proof.** 2 $\iff$ 3: Every $\varphi \in (V \otimes_\alpha W)'$ can be regarded as a bounded linear functional on $V \wedge^p W$ because $\| \cdot \|_{\wedge^p}$ is the largest subcross matrix norm and both $\wedge \varphi$ and $\varphi^\wedge$ extend $\varphi$. We have

$$\varphi^\wedge(v \otimes w'') = \langle \varphi_w, v'' \rangle_{W', W''} = \langle \Phi_{V'' W''}^R(v \otimes w'', \varphi) \rangle = \langle v \otimes w'', ((\Phi_{V', W'}^R)' \circ \kappa)(\varphi) \rangle,$$

where $\kappa$ is the natural inclusion from $(V \wedge^p W)'$ into $(V \wedge^p W)''$. This shows that $\varphi^\wedge = ((\Phi_{V', W'}^R)' \circ \kappa)(\varphi)$. Similarly, $\wedge \varphi = ((\Phi_{V'' W''}^R)' \circ \kappa)(\varphi)$. Let $\Phi_1$ (respectively, $\Phi_2$) be the composition of the upper chain
(respectively, lower chain) in the diagram in 2, i.e., \( \Phi_1 = P \circ (\Phi_{V,W}^R)'' \circ \Phi_{V,W''}^L \) and \( \Phi_2 = P \circ (\Phi_{V,W}^L)'' \circ \Phi_{V',W}^R \), then we have

\[
\langle \Phi_1(v'' \otimes w''), \varphi \rangle = (P \circ (\Phi_{V,W}^R)'' \circ \Phi_{V,W''}^L)(v'' \otimes w''), \varphi)_{(V \otimes_a W)''} = \langle \Phi_{V,W''}^L(v'' \otimes w''), ((\Phi_{V,W}^L)' \circ \kappa)(\varphi))_{(V \otimes_a W)''} = \langle \Phi_{V,W''}^L(v'' \otimes w''), \varphi^\wedge \rangle_{(V \otimes_a W)''} = ((\varphi^\wedge)_{W''}, v'')_{V'',V''} = ^\wedge (\varphi^\wedge)(v'' \otimes w'').
\]

Similarly,

\[
\Phi_2(v'' \otimes w''), \varphi = (\wedge \varphi)^\wedge (v'' \otimes w''),
\]

and thus \( \Phi_1 = \Phi_2 \Leftrightarrow ^\wedge (\varphi^\wedge) = (\wedge \varphi)^\wedge \) on \( V'' \otimes W'' \).

\( I \Leftrightarrow 2: \) According to calculations above, \( \Phi_1 \) (respectively, \( \Phi_2 \)) is left (respectively, right) weak*-continuous. If \( \Phi_1 = \Phi_2 \), then it \( (\Phi_1 = \Phi_2 \Leftrightarrow \Phi) \) is a separately weak*-continuous map which extends the natural inclusion \( \iota : V \otimes W \to (V \otimes_a W)' \). Conversely, suppose that \( \Phi \) is the separately weak*-continuous extension of \( \iota \).

For \( v'' \in V'' \) (respectively, \( w'' \in W'' \)), let us take a net \( \{v_i\} \) in \( V \) converging to \( v'' \) in the weak* topology (respectively, \( \{w_j\} \) in \( W \) converging to \( w'' \) in the weak* topology). It follows that

\[
\langle \Phi_1(v'' \otimes w''), \varphi \rangle = ^\wedge (\varphi^\wedge)(v'' \otimes w'') = \lim_i ^\wedge (\varphi^\wedge)(v_i \otimes w'') = \lim_i \varphi^\wedge(v_i \otimes w') = \lim_i \lim_j \varphi^\wedge(v_i \otimes w_j) = \lim_i \lim_j \langle \Phi(v_i \otimes w_j), \varphi \rangle = \langle \Phi(v'' \otimes w''), \varphi \rangle = \lim_j \varphi^\wedge(v_j \otimes w') = \lim_j \varphi^\wedge(v_j \otimes w_j) = \lim_j \lim_i \varphi^\wedge(v_i \otimes w_j) = \lim_j (\wedge \varphi)^\wedge(v'' \otimes w_j) = (\wedge \varphi)^\wedge(v'' \otimes w'') = \langle \Phi_2(v'' \otimes w''), \varphi \rangle,
\]

so we conclude that \( \Phi_1 = \Phi = \Phi_2 \).

\( 3 \Leftrightarrow 4: \) Let \( \varphi \in (V \otimes_a W)' \). Since \( \langle L_{\varphi^\wedge}(v), w'' \rangle_{W''} = \varphi^\wedge(v \otimes w'') = \varphi^\wedge(v \otimes w') = \langle v, \varphi, w'' \rangle_{W',W''} = (L_{\varphi}(v), w'')_{W',W''} = \langle \iota_{W'}, L_{\varphi}(v), w'' \rangle_{W''} \), we get

\[
L_{\varphi^\wedge} = \iota_{W'} \circ L_{\varphi}.
\]

On the other hand, observe that \( \langle L^\wedge(\varphi^\wedge), w \rangle_{W',W} = ^\wedge \varphi(v'' \otimes w) = \langle \varphi_w, v'' \rangle_{V',V''} \) and that \( P_{W''} \circ
Since
\[ \langle L' \circ \iota_W(w), v \rangle_{V', V} = \langle \iota_W(w), L'_\phi(v) \rangle_{W', W} = \langle w, L' \phi(v) \rangle_{W; W'} = \varphi(v \otimes w) = \varphi(v), \]
we obtain
\[ L^\wedge \varphi = P_{W'} \circ L'' \varphi. \]

Therefore, we get
\[ L_{(\wedge \varphi)\wedge} = \iota_{W'} \circ L_{\wedge \varphi} = \iota_{W'} \circ P_{W'} \circ L'' \varphi \]
and
\[ L_{\wedge (\varphi\wedge)} = P_{W''} \circ L_{\varphi\wedge} = P_{W''} \circ (\iota_{W'} \circ L' \varphi)'' = P_{W''} \circ \iota_{W'} \circ L'' \varphi = L'' \varphi. \]

This shows that
\[ (\wedge \varphi)\wedge = \wedge (\varphi\wedge) \text{ on } V'' \otimes W'' \]
\[ \iff L_{(\wedge \varphi)\wedge} = L_{(\varphi\wedge)} \]
\[ \iff L_{\varphi\wedge}''(V'') \subseteq W' \]
\[ \iff L_\varphi \text{ is weakly compact.} \]

(See [Meg98, Theorem 3.5.8] for the last equivalence)

\[ \square \]

**Theorem 5.2.3.** Let \( V \subseteq B(L_p(\mu)) \) and \( W \subseteq B(L_p(\nu)) \). For every \( \varphi \in (V \hat{\otimes} W)' \), \( L_\varphi \) is weakly compact.

**Proof.** Without loss of generality, we may assume \( \|\varphi\|_b = 1 \). Using Theorem 2.3.9, we have two index sets \( I \) and \( J \) and \( p \)-complete isometries
\[ \Phi^V : V \hookrightarrow V'' \hookrightarrow B(\ell_p(I)) \quad \text{and} \quad \Phi^W : W \hookrightarrow W'' \hookrightarrow B(\ell_p(J)). \]

Consider the diagram below:

\[
\begin{array}{ccc}
V \hat{\otimes} W & \xrightarrow{\varphi} & C \\
\downarrow & & \\
B(\ell_p(I)) \hat{\otimes} B(\ell_p(J)) & \xrightarrow{\varphi} & B(\ell_p(I \otimes J))
\end{array}
\]

By Hahn-Banach Theorem, \( \varphi \) extends to \( \tilde{\varphi} : B(\ell_p(I \otimes J)) \to C \). Applying the same technique as in the proof
of Theorem 2.3.12, we can find a measure space \((\Omega, \Sigma, \theta)\) together with two vectors \(\xi \in L_p(\theta),\) \(\eta \in L_p'(\theta),\) and a unital \(p\)-completely contractive homomorphism \(\pi : B(\ell_p(I \otimes J)) \to B(L_p(\theta))\) such that \(\tilde{\varphi}(\cdot) = \langle \pi(\cdot)\xi, \eta \rangle.\)
Define \(T : B(\ell_p(I)) \to B(\ell_p(J))'\) by
\[
(T(x), y) = \tilde{\varphi}(x \otimes y), \quad x \in B(\ell_p(I)), \; y \in B(\ell_p(J)).
\]
Then it is easy to check that the following diagram is commutative:

\[
\begin{array}{ccc}
V & \xrightarrow{L_{\varphi}} & W' \\
\phi^I \downarrow & & \downarrow (\Phi^W)' \\
B(\ell_p(I)) & \xrightarrow{T} & B(\ell_p(J))'
\end{array}
\]

Define \(R : B(\ell_p(I)) \to L_p(\theta)\) and \(S : B(\ell_p(J)) \to L_p'(\theta)\) by
\[
R(x) = \pi(x \otimes 1)\xi, \quad x \in B(\ell_p(I)), \quad \text{and} \quad S(y) = (\pi(1 \otimes y)')\eta, \quad y \in B(\ell_p(J)),
\]
then the diagram

\[
\begin{array}{ccc}
B(\ell_p(I)) & \xrightarrow{T} & B(\ell_p(J))' \\
\downarrow R & & \downarrow S' \\
L_p(\theta) & &
\end{array}
\]

is commutative, because
\[
\langle S'R(x), y \rangle = \langle R(x), S(y) \rangle = \langle \pi(x \otimes 1)\xi, (\pi(1 \otimes y)')\eta \rangle = \langle \pi(x \otimes y)\xi, \eta \rangle = \tilde{\varphi}(x \otimes y) = (T(x), y).
\]
Combining these two commutative diagrams, we finally have \(L_{\varphi} = (\Phi^W)'S'R\Phi^V,\) that is, \(L_{\varphi}\) is factorized through a reflexive Banach space \(L_p(\theta),\) so \(L_{\varphi}\) is a weakly compact operator [Meg98, Propositions 3.5.4 and 3.5.11].

**Corollary 5.2.4.** There exists a (necessarily unique) separately weak*-continuous extension
\[
\Phi : V'' \otimes W'' \to (V \otimes W)^{\prime\prime}
\]

of the natural inclusion \(\iota : V \otimes W \to (V \otimes W)^{\prime\prime}.\)

**Proof.** Combine Theorem 5.2.2 and Theorem 5.2.3. Uniqueness follows from separate weak*-continuity.

\[\square\]
Now we are ready to define condition $C_p$ for $p$-operator spaces. Let $\Phi$ be as in Corollary 5.2.4. The following commutative diagram

$$
\begin{array}{ccc}
V'' \otimes W'' & \xrightarrow{\Phi} & (V \otimes W)'', \\
\downarrow & & \downarrow \\
CB_{p,F}(V',W'') & \xrightarrow{\iota} & CB_p(V',W'')
\end{array}
$$

shows that $\Phi$ is injective. Thus we can equip $V'' \otimes W''$ with the $p$-operator space structure induced by $\Phi$, which will be denoted by $V'' : \otimes_p : W''$. We say that $V \subseteq B(L_p(\mu))$ satisfies condition $C_p$ (or has property $C_p$) if the map $\Phi$ is isometric with respect to the injective tensor product norm with every $W \subseteq B(L_p(\nu))$.

**Proposition 5.2.5.** Suppose that $V \subseteq B(L_p(\mu))$. Then $V$ satisfies condition $C_p$ if and only if $V$ satisfies both condition $C'_p$ and $C''_p$.

**Proof.** Suppose that $V$ satisfies condition $C_p$ and $W \subseteq B(L_p(\nu))$. Note that, even though we do not have a $p$-analogue of Arveson-Wittstock-Hahn-Banach theorem, we still have $p$-completely isometric embedding $V \otimes_{\vee_p} W'' \subseteq V'' \otimes_{\vee_p} W''$ and the bottom row in the following commutative diagram

$$
\begin{array}{ccc}
V \otimes_{\vee_p} W'' & \xrightarrow{\Phi} & V \otimes_{\vee_p} W'' \\
\downarrow & & \downarrow \\
V'' \otimes_{\vee_p} W'' & \xrightarrow{\Phi} & V'' \otimes_{\vee_p} W''
\end{array}
$$

is isometric. Therefore the top row is also isometric and hence $V$ satisfies condition $C'_p$. That $V$ satisfies condition $C''_p$ can be proved using a similar argument.

On the other hand, if $V$ satisfies condition $C''_p$, we get

$$
V'' \otimes_{\vee_p} W'' = V'' : \otimes_{\vee_p} : W'' \hookrightarrow (V \otimes_{\vee_p} W'')''.
$$

If $V$ also satisfies condition $C'_p$, then

$$
V \otimes_{\vee_p} W'' = V \otimes_{\vee_p} : W'' \hookrightarrow (V \otimes_{\vee_p} W)'',
$$

and hence we have isometric inclusion

$$
V'' \otimes_{\vee_p} W'' \hookrightarrow (V \otimes_{\vee_p} W)''.
$$
Since $V'' \otimes_{\vee_p} W'' \subset (V \otimes_{\vee_p} W)''$ and $(V \otimes_{\vee_p} W)'' \hookrightarrow (V \otimes_{\vee_p} W)'''$ isometrically, the inclusion $V'' \otimes_{\vee_p} W'' \subseteq (V \otimes_{\vee_p} W)''$ must be isometric.
References


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Ph.D. in Mathematics Urbana, IL
Dissertation Title: On \( p \)-Operator Spaces and Their Applications August 2010
Advisor: Professor Zhong-Jin Ruan

Korea University Seoul, South Korea
M.S. in Applied Mathematics February 2001

Korea University Seoul, South Korea
B.S. in Mathematics February 1999

RESEARCH INTEREST

- \( p \)-operator spaces and their applications
  - Extending results on classical operator spaces to \( p \)-operator spaces
  - Property \( A_p \) and exactness of discrete groups

- harmonic analysis on locally compact group
  - Characterization of amenability of a locally compact group
  - Applications of theory of \( p \)-operator spaces, especially to the study of Figà-Talamanca-Herz algebras and related algebras/multipliers

PUBLICATION

TALKS

• **Annual Wabash Miniconference** October 3, 2009
  
  – Indiana University Purdue University at Indianapolis
  
  – Title: $p$-operator spaces and approximation properties

• **Student seminar** Summer 2007 – on QWEP conjecture

• **Working seminar** Fall 2006 – on von Neumann algebras

TEACHING EXPERIENCE

• **Full Instructor** 9 semesters, University of Illinois at Urbana-Champaign
  
  – Class organization, exam making/grading with full responsibility
  
  – Four appearances on the ‘List of Teachers Ranked as Excellent’, twice ranked within top 10%
  
  – Introductory Matrix Theory (Sp ’06, Fa ’06, Sp ’07, Su ’07, Fa ’08‡, Sp ’09‡, Fa ’09‡, Sp ’10)
  
  – Elementary Linear Algebra (Su ’09‡)

• **Discussion Session Leader** 4 semesters, University of Illinois at Urbana-Champaign
  
  – Two appearances on the ‘List of Teachers Ranked as Excellent’
  
  – Calculus II (Sp ’08‡)
  
  – Calculus I (Fa ’07)
  
  – Calculus of Several Variables (Fa ’05‡)
  
  – Algebra (Fa ’04)

• One of the finalists for the Graduate Teaching Award, Department of Mathematics, University of Illinois at Urbana-Champaign (2010)

† means appearance on the ‘List of Teachers Ranked as Excellent’

‡ means appearance on the ‘List of Teachers Ranked as Excellent’ within top 10%.

HONORS AND AWARDS

• **Graduate College Scholarship** University of Illinois at Urbana-Champaign
  
  – 2003-2004 academic year and 2004-2005 academic year
  
  – Each came with full tuition waiver and $7,500 stipend

• **Research Experiences for Graduate Students** University of Illinois at Urbana-Champaign
  
  – Project on vector-valued integration under the direction of Professor Marius Junge
– Came with summer tuition waiver and $3,000 stipend

• **Korea University Presidential Award** Korea University, Seoul, South Korea
  – For best undergraduate academic achievement in College of Science
  – Came with full tuition waiver for senior semester

• **Freshmen Scholarship** Korea University, Seoul, South Korea
  – For excellence in the national university entrance examination
  – Came with partial tuition waiver

• **Research Assistantship** University of Illinois at Urbana-Champaign
  – Summer ’06, ’07, ’08, ’09, ’10
  – Research Assistant of Professor Zhong-Jin Ruan