

© 2010 by Paul Shannahan Wenger. All rights reserved.

THREE EXISTENCE PROBLEMS IN EXTREMAL GRAPH THEORY

BY

PAUL SHANNAHAN WENGER

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2010

Urbana, Illinois

Doctoral Committee:

Professor Alexandr V. Kostochka, Chair
Professor Douglas B. West, Director of Research
Professor Zoltán Füredi
Research Assistant Professor Sujith Vijay

Abstract

Proving the existence or nonexistence of structures with specified properties is the impetus for many classical results in discrete mathematics. In this thesis we take this approach to three different structural questions rooted in extremal graph theory.

When studying graph representations, we seek efficient ways to encode the structure of a graph. For example, an *interval representation* of a graph G is an assignment of intervals on the real line to the vertices of G such that two vertices are adjacent if and only if their intervals intersect. We consider graphs that have *bar k -visibility representations*, a generalization of both interval representations and another well-studied class of representations known as visibility representations. We obtain results on \mathcal{F}_k , the family of graphs having bar k -visibility representations. We also study $\bigcup_{k=0}^{\infty} \mathcal{F}_k$. In particular, we determine the largest complete graph having a bar k -visibility representation, and we show that there are graphs that do not have bar k -visibility representations for any k .

Graphs arise naturally as models of networks, and there has been much study of the movement of information or resources in graphs. Lampert and Slater [18] introduced *acquisition* in weighted graphs, whereby weight moves around G provided that each move transfers weight from a vertex to a heavier neighbor. Our goal in making acquisition moves is to consolidate all of the weight in G on the minimum number of vertices; this minimum number is the *acquisition number* of G . We study three variations of acquisition in graphs: when a move must transfer all the weight from a vertex to its neighbor, when each move transfers a single unit of weight, and when a move can transfer any positive amount of weight. We consider acquisition numbers in various families of graphs, including paths, cycles, trees, and graphs with diameter 2. We also study, under the various

acquisition models, those graphs in which all the weight can be moved to a single vertex.

Restrictive local conditions often have far-reaching impacts on the global structure of mathematical objects. Some local conditions are so limiting that very few objects satisfy the requirements. For example, suppose that we seek a graph in which every two vertices have exactly one common neighbor. Such graphs are called *friendship graphs*, and Wilf [30] proved that the only such graphs consist of edge-disjoint triangles sharing a common vertex. We study a related structural restriction where similar phenomena occur. For a fixed graph H , we consider those graphs that do not contain H and such that the addition of any edge completes exactly one copy of H . Such a graph is called *uniquely H -saturated*. We study the existence of uniquely H -saturated graphs when H is a path or a cycle. In particular, we determine all of the uniquely C_4 -saturated graphs; there are exactly ten. Interestingly, the uniquely C_5 -saturated graphs are precisely the friendship graphs characterized by Wilf.

To my wife Liz.

Acknowledgments

This project would not have been possible without the support and assistance of many people. I would first like to thank Professors Douglas B. West, Alexandr V. Kostochka, Zoltán Füredi, and Sujith Vijay for serving on my committee and helping me discover my love for combinatorics and graph theory. In particular, I wish to thank advisor, Professor West, for the many hours that he has spent helping me develop my style as both a mathematician and a writer.

I am also indebted to my coauthors on the various projects contained in this dissertation. I thank Jenn Vandebussche, Stephen Hartke, Noah Prince, Pratik Worah, Joshua Cooper, John Lenz, Professor West and Timothy D. LeSaulnier for their contributions. A special thanks goes to Tim for being my initial sounding board for many of the arguments that have ended up in this dissertation.

I would like to thank my family for all of their love and support. I am especially grateful to my parents for raising my sisters and me in an environment that prized education. I am where I am today because of the motivation that they instilled in me to continue learning. After their continual reminders to get my work done, I am sure they are happy to see this document.

Finally, I would like to thank my wife Liz for being with me throughout this process. Thank you for your continual support and for putting up with me through the various emotional stages of grad school. I also appreciate your sense of adventure and the fact that you are on board for wherever math may take us.

Table of Contents

Chapter 1 Introduction	1
1.1 Bar k -Visibility Graphs	2
1.2 Acquisition Parameters in Graphs	4
1.3 Uniquely H -Saturated Graphs	6
1.4 Terminology and Notation	7
Chapter 2 Bar k-Visibility Graphs	12
2.1 Introduction	12
2.2 An Upper Bound on the Number of Edges	14
2.3 Comparing the Families \mathcal{F}_{k-1} and \mathcal{F}_k	18
2.4 Induced Subgraphs	22
2.5 Regular Bar k -Visibility Graphs	24
Chapter 3 Total Acquisition in Graphs	27
3.1 Introduction to Acquisition Parameters	27
3.2 Trees	30
3.3 Total Acquisition Number 1	38
3.4 Operations on Graphs	41
3.5 Diameter 2	46
Chapter 4 Unit Acquisition in Graphs	52
4.1 Introduction	52
4.2 Initial Results	54
4.3 Diameter 2	61
Chapter 5 Fractional Acquisition in Graphs	66
5.1 Introduction	66
5.2 Paths and Cycles	67
5.3 General Graphs	76
Chapter 6 Uniquely H-Saturated Graphs	83
6.1 Introduction	83
6.2 Cycles	86
6.3 Uniquely C_4 -Saturated Graphs	90
6.4 Uniquely C_5 -Saturated Graphs	99

6.5	Uniquely C_6 -Saturated Graphs	100
6.6	Uniquely C_7 -Saturated Graphs	104
References	113

Chapter 1

Introduction

Proving the existence or nonexistence of structures with specified properties is the impetus for many classical results in discrete mathematics. In this thesis we take this approach to three different structural questions rooted in extremal graph theory. Here we present an overview of the thesis, including a description of the results in each chapter. In Section 1.4, we provide the reader with the necessary background terminology and notation.

When studying graph representations, we seek efficient ways to encode the structure of a graph. In Chapter 2, we consider graphs that have a *bar k -visibility representation*, a generalization of both interval representations and another well-studied class of representations known as visibility representations. We obtain results on \mathcal{F}_k , the family of graphs with bar k -visibility representations. We also study $\bigcup_{k=0}^{\infty} \mathcal{F}_k$.

Graphs arise naturally as models of networks, and there has been much study of the movement of information or resources in graphs. Lampert and Slater [18] introduced *acquisition* in weighted graphs, whereby weight moves around G provided that each move transfers weight from a vertex to a heavier neighbor. Our goal in making acquisition moves is to consolidate all of the weight in G on the minimum number of vertices, which is the *acquisition number* of G . In Chapter 3, we study three variations of acquisition in graphs: when a move must transfer all the weight from a vertex to its neighbor, when each move transfers a single unit of weight, and when a move can transfer any positive amount of weight.

Restrictive local conditions often have far-reaching impacts on the global structure of mathematical objects. Some local conditions are so limiting that very few objects satisfy the requirements. For a fixed graph H , we consider those graphs that do not contain H in which the addition

of any edge completes exactly one copy of H . Such a graph is called *uniquely H -saturated*. In Chapter 4, we study the existence of uniquely H -saturated graphs, focusing on the families of paths and cycles. Interestingly, the uniquely C_5 -saturated graphs are precisely the friendship graphs characterized by Wilf [30].

1.1 Bar k -Visibility Graphs

Our first results deal with a problem in graph representation. The idea of representing a graph using a visibility relation has received much attention due to its applications to circuit layout (see [25], and additional references in [1]). We consider a family of closed horizontal bars in the plane and define a *visibility relation* on the bars. An unordered pair of bars is in the visibility relation if there is a vertical line segment connecting the two bars that does not intersect any other bar; such a pair of bars is called a *visibility*, and we say that the bars “see” each other. This yields a graph whose vertices correspond to the bars and whose adjacency relation is the visibility relation on the family of bars. We say that the family of bars is a *bar visibility representation* of this graph. A graph is a *bar visibility graph* if it has a bar visibility representation. (In the literature, these graphs have also been referred to as “visibility graphs” or “strong visibility graphs.”) This particular model was first introduced by Luccio, Mazzone, and Wong [20]. They observed that bar visibility graphs must be planar, and they later provided a characterization of bar visibility graphs under the restriction that bars have distinct x coordinates as endpoints [21].

Since bar visibility graphs must be planar, very few graphs have such representations. By relaxing the requirement that each bar corresponds to a distinct vertex, it is possible to represent every graph. If we partition the family of bars into subsets and consider the visibility relation between these subsets, then there is a graph whose vertices correspond to the sets and edges correspond to the visibility relation. We call this family a *multibar visibility representation* of the graph. For a graph G , the *bar visibility number* of G , introduced by Chang, Hutchinson, Jacobson, Lehel and West in [7], is the minimum over all multibar visibility representations of G of

the maximum number of bars assigned to a vertex. For example, a bar visibility graph has bar visibility number 1. Another possible relaxation of the bar visibility model, introduced by Dean, Evans, Gethner, Laison, Safari, and Trotter in [10], is to allow interrupted lines of sight between bars in the representation. Given a family of bars, we define a new visibility relation, called the *bar k -visibility relation*. An unordered pair of bars is in the bar k -visibility relation if there is a vertical line segment connecting the two bars that intersects at most k other bars. This yields a graph whose vertices correspond to the bars and whose adjacency relation is the visibility relation on the family of bars. We say that the family of bars is a *bar k -visibility representation* of this graph. A graph is a *bar k -visibility graph* if it has a bar k -visibility representation, and \mathcal{F}_k denotes the family of such graphs. In Chapter 2, we prove that there are graphs that do not have a bar k -visibility representation for any k in contrast to the representability of all graphs by giving multiple bars to the vertices. In particular, nonplanar triangle-free graphs are forbidden as induced subgraphs of bar k -visibility graphs for all k .

Dean et al. [10] obtained a bound on the number of edges for graphs in \mathcal{F}_k with n vertices. If G is such a graph and $n \geq 2k + 2$, then G has at most $(k + 1)(3n - \frac{7}{2}k - 5) - 1$ edges. They gave a construction showing that the complete graph K_{4k+4} on $4k + 4$ vertices is in \mathcal{F}_k and conjectured an improved edge bound of $(k + 1)(3n - 4k - 6)$, which is attained by K_{4k+4} . They proved this conjecture for $k \in \{0, 1\}$ and used their edge bound to prove that $K_{5k+5} \notin \mathcal{F}_k$. We prove that the upper bound on the number of edges conjectured in [10] is correct, yielding K_{4k+4} as the largest complete graph that is a bar k -visibility graph.

We also prove for each k that \mathcal{F}_{k-1} and \mathcal{F}_k are incomparable under set inclusion. Finally, inspired by the result that the only regular interval graphs are complete graphs, we prove that if G is regular of degree d with $d < 2k + 2$ and $G \in \mathcal{F}_k$, then G is a complete graph. This bound is sharp; we construct $(2k + 2)$ -regular non-complete graphs with bar k -visibility representations for $k \in \{0, 1, 2, 3, 4\}$. This is joint work with Stephen Hartke and Jennifer Vandenbussche and appears in [15].

1.2 Acquisition Parameters in Graphs

Suppose there is an army of troops deployed at a network of bases, some of which are joined by secured roads, such that there is one unit at each base. The government wishes to withdraw the troops from the bases, but they also want to minimize the number of bases where there will be an airlift. Troops evacuate from one base to another along a secured road, but it is necessary that the target base has at least as many troops to protect the incoming units. What is the minimum number of bases where there have to be airlifts? We model this situation with a graph, where each base is represented by a vertex and the secured roads are represented by edges. The troops are represented by units of weight that move around the graph.

On a weighted graph, an *acquisition move* consists of moving weight from a vertex u to a neighbor v , provided that before the move the weight on v is at least the weight on u . Lampert and Slater [18] introduced acquisition in graphs, and they studied acquisition when all the vertices in G begin with weight 1 and each move transfers all of the weight from a vertex to its neighbor. We refer to such a move as a *total acquisition move*. The minimum number of vertices with positive weight in a graph G after a sequence of total acquisition moves is the *total acquisition number* of G , denoted $a_t(G)$. (In the literature this is called the “acquisition number”.)

Lampert and Slater [18] proved that, for any connected n -vertex graph (with $n \geq 2$), the acquisition number is at most $\lfloor (n+1)/3 \rfloor$, which is sharp. They also proved that the weight on a vertex v never exceeds $2^{d(v)}$, and they used this to establish a lower bound on the total acquisition number of a graph. Furthermore, they showed that the problem of deciding “Is $a_t(G) = 1$?” is NP-complete, though it may be determined in linear time when G is a tree.

Our first results in Chapter 3 concern total acquisition in graphs. Since the set of edges used in any sequence of total acquisition moves is acyclic, our research naturally focuses on trees. We establish several bounds on the total acquisition number of trees with bounded diameter, and we obtain an efficient algorithm to test if the total acquisition number of a tree T is bounded by a fixed value k . We also explore structural aspects of graphs with total acquisition number 1, establishing

several sufficient conditions. In addition, we prove that the total acquisition number of an n -vertex graph with diameter 2 is bounded. Finally, we study the effects of edge deletion and graph products on total acquisition number.

In Chapters 4 and 5 we consider two additional models of acquisition. A *unit acquisition move* moves a single unit of weight from u to v , and a *fractional acquisition move* moves any positive amount of weight from u to v . For a graph G , the *unit acquisition number*, denoted $a_u(G)$, and *fractional acquisition number*, denoted $a_f(G)$, are defined to be the minimum number of vertices with positive weight after a sequence of unit or fractional acquisition moves, respectively.

In Chapter 4 we establish several tight upper bounds on the unit acquisition number of a graph in terms of other parameters. We also prove that in the family of graphs with maximum degree at least 5 the amount of weight that a single vertex can acquire via unit acquisition moves is unbounded, in particular providing an infinite family of trees with maximum degree 5 and unit acquisition number 1. Therefore, there is no lower bound on unit acquisition number in terms of maximum degree analogous to the bound established by Lampert and Slater for total acquisition. We also prove that if G has diameter 2, then $a_u(G) \leq 2$ with only C_5 and the Petersen graph satisfying $a_u(G) = 2$.

In Chapter 5 we determine the fractional acquisition number of every graph. If G is a path or cycle on n vertices, we prove that $a_t(G) = a_u(G) = a_f(G) = \lceil n/4 \rceil$. On the other hand, we show that in a sufficiently long path, the amount of weight that a single vertex can acquire via fractional acquisition moves is arbitrarily large. Finally, our most surprising result is that if G is a connected graph with maximum degree at least 3, then $a_f(G) = 1$. Furthermore, all of the weight in G can be acquired by any vertex with degree at least 2. Our proof provides a procedure to generate fractional acquisition moves achieving this result. This is joint work with Timothy D. LeSaulnier, Noah Prince, Douglas B. West, and Pratik Worah.

1.3 Uniquely H -Saturated Graphs

Perhaps the most famous result in extremal graph theory is Turán's Theorem [27], which states that among the n -vertex graphs having no complete subgraph with $r + 1$ vertices, the one that has the most edges is the n -vertex complete r -partite graph whose part sizes differ by at most 1. In honor of this result, this graph is called *Turán graph* and is denoted $T_{n,r}$. Erdős, Hajnal, and Moon [11] observed an equivalent statement of Turán's theorem: The Turán graph has the maximum number of edges among all n -vertex simple graphs that do not contain K_{r+1} and have the property that the addition of any edge completes a copy of K_{r+1} . This motivates the following definition. For a fixed graph H , a graph G is *H -saturated* if G does not contain H but adding the edge between any two nonadjacent vertices in G produces a graph that does contain H . Erdős, Hajnal, and Moon [11] then determined the n -vertex K_{r+1} -saturated graphs with the fewest edges.

A graph G is *uniquely H -saturated* if G is H -saturated and the addition of any edge joining nonadjacent vertices completes exactly one copy of H . The smallest K_{r+1} -saturated n -vertex graphs, found in [11], are uniquely K_{r+1} -saturated, but the Turán graphs are not. In Chapter 6, we study uniquely H -saturated graphs when H is a path or short cycle. As an example, consider $H = C_3$. Every C_3 -saturated graph has diameter at most 2. All trees with diameter 2 are stars and are uniquely C_3 -saturated. A uniquely C_3 -saturated graph G cannot contain a 3-cycle or a 4-cycle, so such a graph that is not a tree has girth 5, where the girth is the length of the shortest cycle in a graph. Every graph with girth 5 and diameter 2 is uniquely C_3 -saturated. The graphs with diameter d and girth $2d + 1$ are called *Moore graphs*. Hoffman and Singleton [16] proved that besides odd cycles there are only finitely many Moore graphs, all having diameter 2. Thus, except for stars, there are finitely many uniquely C_3 -saturated graphs.

Much study has been devoted to determining the minimum number of edges in an n -vertex C_k -saturated graph. For $k = 3$, the result of Erdős, Hajnal, and Moon [11] suffices. Ollmann [23] determined the extremal graphs for $k = 4$, and Chen [8] resolved the question for $k = 5$. For values of k greater than 5, no exact results are known. On the other hand, we have determined

the uniquely C_k -saturated graphs for $k \leq 7$. When $k = 4$, there are ten such graphs, and our proof is similar to the eigenvalue approach used to prove both the Hoffman-Singleton result on Moore graphs and the “Friendship Theorem”, which states that a graph in which any two distinct vertices have exactly one common neighbor has a vertex adjacent to all others (see Wilf [30]). Structural arguments are used to show that under certain conditions the graphs in question are regular. Counting of walks then yields a polynomial equation involving the adjacency matrix, after which eigenvalue arguments exclude all but a few graphs.

The graphs studied in the Friendship Theorem consist of some number of triangles sharing a single vertex; such graphs are uniquely C_5 -saturated, and we show that except for small complete graphs, the “friendship graphs” are the only uniquely C_5 -saturated graphs. Thus, unlike for C_4 , there are infinitely many uniquely C_5 -saturated graphs. We then show that for $k = 6$ and $k = 7$, there are no uniquely C_k -saturated graphs. This is joint work with Joshua Cooper, Timothy D. LeSaulnier, John Lenz, and Douglas B. West.

1.4 Terminology and Notation

A graph G consists of two sets, $V(G)$ and $E(G)$, called the *vertex set* and *edge set*, respectively. Each member of $V(G)$ is a *vertex*, and each member of $E(G)$ is an *edge*. Each edge is an unordered pair of distinct vertices. We use the notation uv to denote the edge (u, v) , and we say that u and v are *adjacent* if $uv \in E(G)$. If uv is an edge, then u and v are the *endpoints* of uv , and uv is *incident* to u and v . We also frequently refer to an arbitrary edge as e rather than by its endpoints. The *order* of a graph is the size of its vertex set. All graphs are assumed to be of positive, finite order.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case we say that G *contains* H . A subgraph H *spans* G if $V(H) = V(G)$; we also say that H is a *spanning subgraph* of G . A subgraph H of G is an *induced subgraph* if it has the property that two vertices u and v are adjacent in H if and only if they are adjacent in G . Given a vertex set S that is a subset

of $V(G)$, we denote the induced subgraph of G with vertex set S by $G[S]$; we say that S *induces* the graph $G[S]$. Similarly, $G - S = G[V(G) - S]$. To *delete* an edge e from G is to remove it from $E(G)$; we denote the resulting graph by $G - e$.

A graph in which every two vertices are adjacent is called a *complete graph*; we denote the n -vertex complete graph by K_n . A *clique* is a set of vertices in a graph G that induces a complete subgraph. A set of vertices is *independent* if it induces a subgraph with no edges. The maximum size of an independent set in G is denoted $\alpha(G)$. The complement \overline{G} of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in \overline{G} if and only if they are not adjacent in G .

A *path* with n vertices is a graph whose vertices can be named v_1, \dots, v_n so that the edge set is $\{v_i v_{i+1} : 1 \leq i \leq n - 1\}$; with this labeling, we call v_1 and v_n the *endpoints* of the path. The *length* of a path is size of its edge set. We denote the n -vertex path by P_n and call it the *n -path*. We specify the path having vertices v_1, \dots, v_n in order by the notation $\langle v_1, \dots, v_n \rangle$.

A subgraph of a graph G that is a path with endpoints u and v is a *u, v -path* in G . A graph G is *connected* if for every two vertices u and v in G there is a u, v -path. A *component* of a graph is a maximal connected subgraph. If G is a connected graph and S is a set of vertices such that $G - S$ is disconnected, then S is a *cutset*; a cutset of size 1 is called a *cutvertex*. The minimum size of a cutset in a graph G having a cutset is the *connectivity* of G , and G is *k -connected* if G has connectivity at least k . A *block* in a graph G is a maximal 2-connected subgraph. A set S of edges in a connected graph G is an *edgecut* if $G - S$ is disconnected.

A *cycle* with n vertices is a graph whose vertices can be named v_1, \dots, v_n so that the edge set is $\{v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_1 v_n\}$. The *length* of a cycle is the size of its edge set. We denote the n -vertex cycle by C_n and call it the *n -cycle*. We specify the cycle having vertices v_1, \dots, v_n in order by the notation $[v_1, \dots, v_n]$. A set of edges in a graph G that does not contain the edge set of any cycle is called *acyclic*. A connected graph that contains no cycles is called a *tree*; a *forest* is a graph in which every component is a tree. At times we wish to specify a vertex in a tree T that we call the *root*. A *rooted tree* is a pair (T, r) , where $r \in V(G)$. A vertex in a tree with degree at least

3 is a *branch vertex*, and a vertex with degree 1 is a *leaf*. If v is a branch vertex in a tree T , we call the maximal subtrees of T that have v as a leaf the *branches* of v . We will extend the usage of “leaf” to a vertex of degree 1 in any graph; such a vertex is also called a *pendant vertex*, and the edge incident to such a vertex is a *pendant edge*. The *girth* of a graph G is the minimum order of a cycle in G ; if G is a forest, then the girth of G is infinite.

The *distance* between two vertices u and v in a graph G , denoted $d_G(u, v)$ or $d(u, v)$ when the graph is clear, is the length of a shortest u, v -path in G ; if there is no u, v -path, then the distance between u and v is infinite. The *diameter* of a graph G is the maximum of the distances between vertices in G . The Petersen graph is the graph shown in Figure 1.1; it has diameter 2 and girth 5.

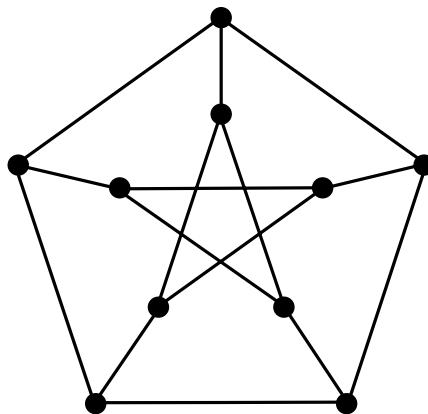


Figure 1.1: The Petersen graph.

The *degree* of a vertex v in a graph G is the number of edges incident to a vertex; it is denoted by $d_G(v)$ or by $d(v)$ when the graph is clear. A graph G is *regular* if every vertex has the same degree; if the degree is k , then G is *k -regular*. The *maximum degree* of a graph G , written $\Delta(G)$, is the maximum over the degrees of the vertices in its vertex set. The *minimum degree* of a graph G , written $\delta(G)$, is the minimum of the degrees of the vertices in the vertex set. The *neighborhood* of a vertex v , written $N(v)$, is the set of all vertices that are adjacent to v ; vertices in the neighborhood are also called the *neighbors* of v . We extend the terminology of neighborhoods to sets of vertices; the terminology we use is nonstandard but will clarify many arguments throughout this dissertation. Given a set S of vertices, the *neighborhood* of S , denoted $N(S)$ is the set of vertices

that are not in S but are adjacent to a vertex in S (in the literature this is called the *boundary* of S and is denoted $\partial(S)$). The *closed neighborhood* $N[v]$ of a vertex v is $N(v) \cup \{v\}$; similarly, the *closed neighborhood* $N[S]$ of a vertex set S is $N(S) \cup S$. The *k th neighborhood of v* , written $N^k(v)$, is the set of vertices that are distance k from v . The *k th neighborhood of S* , written $N^k(S)$, is the set of vertices such that the minimum distance to the vertices in S is k . A vertex set S is a *dominating set* if $S \cup N(S) = V(G)$; the minimum size of a dominating set is denoted by $\gamma(G)$.

Given two graphs G and H , the *Cartesian product* of G and H is the graph $G \square H$ with vertex set $V(G) \times V(H)$ such that (u, v) and (u', v') are adjacent if and only if (i) $u = u'$ and $vv' \in E(H)$, or (ii) $v = v'$ and $uu' \in E(G)$. The *strong product* of G and H is the graph $G \boxtimes H$ with vertex set $V(G) \times V(H)$ such that (u, v) and (u', v') are adjacent if and only if (i) $u = u'$ and $vv' \in E(H)$, (ii) $v = v'$ and $uu' \in E(G)$, or (iii) $uu' \in E(G)$ and $vv' \in E(H)$.

Given a graph G and an edge uv , the process of *subdividing* uv consists of deleting uv from G and adding a vertex x and edges xu and xv to $G - uv$. A graph G' is a *subdivision* of G if G' can be obtained from G using a succession of edge subdivisions.

A set of edges is a *matching* if no two edges are incident to the same vertex. A matching in a graph G is a *perfect matching* if it is the edge set of a spanning subgraph of G .

Let G be an n -vertex graph. The *adjacency matrix* $A(G)$ of G is the $n \times n$ matrix with rows and columns indexed by $V(G)$ such that $A_{u,v}$ is 1 if u and v are adjacent and 0 otherwise. An eigenvalue λ of a square matrix A is a number such that there exists a nonzero vector \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$. The *eigenvalues of G* are the eigenvalues of $A(G)$. The *characteristic polynomial* $p_A(x)$ of a matrix A is $\det(A - xI)$, where I is the n -dimensional identity matrix.

A *weighted graph* is a graph G together with a function $f : V(G) \rightarrow \mathbb{R}$; in this situation we call f a *weight assignment* on G .

We frequently wish to determine how difficult it is to solve a given problem (for instance, does a particular graph G have an independent set of size k ?). Problems are associated by their difficulty, and one such family is that of *NP problems*. Roughly speaking, a problem is in the family NP if it is possible to verify quickly that a particular potential solution is in fact a solution

(for instance, that a set of k vertices is an independent set), but finding a solution is in fact very difficult. A problem is *NP-complete* if it is in the family NP and is also as difficult as any other problem in the family NP. For a formal treatment of NP-completeness, we refer the reader to [13].

For additional background material on graph theory, we refer the reader to [28].

Chapter 2

Bar k -Visibility Graphs

2.1 Introduction

Representations of graphs using visibility in geometric objects have received much attention due to their applications to circuit layout (see [25], additional references in [1]). We consider a family of closed horizontal bars in the plane and define a *visibility relation* on the bars. An unordered pair of bars is in the visibility relation if there is a vertical line segment connecting the two bars that does not intersect any other bar; such a pair of bars is called a *visibility*, and we say that the bars “see” each other. This yields a graph whose vertices correspond to the bars and whose adjacency relation is the visibility relation on the family of bars. We say that the family of bars is a *bar visibility representation* of this graph. A graph is a *bar visibility graph* if it has a bar visibility representation. (In the literature, these graphs have also been called “visibility graphs” or “strong visibility graphs.”)

Bar visibility graphs were first introduced by Luccio, Mazzone, and Wong [20]. They observed that bar visibility graphs must be planar, and they later provided a characterization of bar visibility graphs having bar visibility representations such that bars have distinct x coordinates as endpoints [21]. Later, Tamassia and Tollis [26] proved that a graph is a bar visibility graph if and only if it has an embedding in the plane such that all cut vertices are on the outer face. They also obtained some results concerning connectivity and bar visibility representations. Andreae [1] showed that determining whether a given graph is a bar visibility graph is an NP-complete problem. A well-studied variation of bar visibility graphs is ϵ -visibility graphs, introduced by Melnikov [22]. These graphs are defined just as bar visibility graphs, except that bars are replaced with horizontal segments that

may or may not contain their endpoints. Wismath [31] and, independently, Tamassia and Tollis [26] gave a very simple characterization of ϵ -visibility graphs.

Recently, Dean, Evans, Gethner, Laison, Safari, and Trotter introduced a generalization of bar visibility graphs [10, 9]. A *bar k -visibility representation* of a graph G is a set of bars such that vertices v and w are adjacent in G if and only if a vertical line segment can be drawn joining their associated bars that intersects *at most* k other bars. Note that in contrast to some other visibility models, a sight line of zero width is sufficient in this model.

Let \mathcal{F}_k denote the family of bar k -visibility graphs. Notice that \mathcal{F}_0 is the family of bar visibility graphs defined above. Dean et al. [10] obtained a bound on the number of edges for graphs in \mathcal{F}_k with n vertices. If G is such a graph and $n \geq 2k + 2$, then G has at most $(k + 1)(3n - \frac{7}{2}k - 5) - 1$ edges. They gave a construction showing that the complete graph K_{4k+4} on $4k + 4$ vertices is in \mathcal{F}_k and conjectured an improved edge bound of $(k + 1)(3n - 4k - 6)$, which is attained by K_{4k+4} . They proved this conjecture for $k \in \{0, 1\}$ and used their edge bound to prove $K_{5k+5} \notin \mathcal{F}_k$. We prove that the upper bound on the number of edges conjectured in [10] is correct, yielding K_{4k+4} as the largest complete graph that is a bar k -visibility graph. We also prove for each k that \mathcal{F}_{k-1} and \mathcal{F}_k are incomparable under set inclusion.

Bar k -visibility graphs can be seen as a generalization of interval graphs. An interval graph is based upon an *intersection representation*. Given a family of sets, an unordered pair of sets is in the intersection representation if the sets have a nonempty intersection. An interval graph is a graph that has an intersection representation in which the sets are intervals in the real line. It is easy to see that every interval graph is a bar k -visibility graph when k is at least the number of vertices; hence results on interval graphs motivate many of our investigations into bar k -visibility graphs. Inspired by the result that the only regular interval graphs are complete graphs, we prove that if G is regular of degree d with $d < 2k + 2$ and $G \in \mathcal{F}_k$, then G is a complete graph. This bound is sharp; we construct $(2k + 2)$ -regular non-complete graphs with bar k -visibility representations for $k \in \{0, 1, 2, 3, 4\}$.

Another generalization of interval graphs that has been studied is the idea of t -interval graphs,

where each vertex of G is allotted t distinct intervals in an intersection relation. This idea was extended by Chang, Hutchinson, Jacobson, Lehel, and West to bar visibility graphs in [7], where each vertex of G is permitted t bars in its representation, and vertices are adjacent if there is a direct line of sight between any of their t bars. The *bar visibility number* of a graph G is the minimum t such that there is a visibility representation in which the vertices of G are permitted t bars in the representation. Every graph has a multibar visibility representation; in contrast, we prove that there are graphs that are not bar k -visibility graphs for any value of k .

2.2 An Upper Bound on the Number of Edges

Dean et al. [10] gave an upper bound of $(k + 1)(3n - \frac{7}{2}k - 5) - 1$ on the number of edges in an n -vertex bar k -visibility graph with $n \geq 2k + 2$. They also conjectured an upper bound of $(k + 1)(3n - 4k - 6)$, which holds with equality in their construction for K_{4k+4} . We prove their conjectured upper bound on the number of edges by refining their edge-counting technique. When we are referring to a particular bar k -visibility representation of a graph G , $B(v)$ will denote the bar associated with v and $I(v)$ will denote the projection of $B(v)$ onto the x -axis.

We begin by reproving the bound of Dean et al.

Lemma 2.2.1. *If G is a bar k -visibility graph with more than $2k + 2$ vertices, then G has at most $(k + 1)(3n - \frac{7}{2}k - 5) - 1$ edges.*

Proof. Consider a bar k -visibility representation of G with vertices v_1, \dots, v_n . We may assume that no two bars are at the same height, and hence we index the bars in order such that $B(v_1)$ is the topmost bar and $B(v_n)$ is the bottommost bar. As noted in [10], we may also assume that all $2n$ horizontal coordinates of endpoints of the bars are distinct. If the endpoints are not distinct, then perturbing the endpoints slightly cannot decrease the number of edges in the resulting graph.

We sweep a vertical line from left to right over the representation, counting the visibilities that are created as we move the vertical line. As we sweep the representation, we say that a visibility is new if it is present for the vertical line passing through the horizontal point a but there is a point

b that is less than a such that the visibility is not present for any vertical lines passing through the interval $[b, a)$. When the left endpoint of a bar $B(v)$ is encountered, the only new visibilities involve $B(v)$, and we call the resulting edges the *left edges* of $B(v)$. If the left endpoint of $B(v)$ is the i^{th} left endpoint encountered, then when $i \leq 2k + 2$, $B(v)$ can see at most $i - 1$ other bars. If $i > 2k + 2$, then $B(v)$ may see as many as $k + 1$ bars above and $k + 1$ bars below, for a total of at most $2k + 2$ new edges. Thus the maximum number of edges counted by encountering left endpoints is $(k + 1)(2n - 2k - 3)$, since

$$\begin{aligned} \sum_{i=1}^{2k+2} (i-1) + \sum_{i=2k+3}^n (2k+2) &= \frac{(2k+2)(2k+1)}{2} + (n-2k-2)(2k+2) \\ &= (k+1)(2n-2k-3). \end{aligned}$$

Suppose the right endpoint of a bar $B(w)$ is the j^{th} right endpoint encountered. When $j \leq n - 2k - 2$, up to $k + 1$ bars above $B(w)$ have the potential to each see one new bar below $B(w)$. Hence, at most $k + 1$ new visibilities may be created, and we call the resulting edges the *right edges* of $B(w)$. Once there are only $2k + 2$ bars remaining, each time a bar ends the potential number of new right edges decreases by one. Hence when $n - 2k - 2 < j < n - k - 1$, there are at most $n - k - 1 - j$ new visibilities created. When $j \geq n - k - 1$, no new visibilities are created since every bar already sees every other remaining bar. Thus the maximum number of edges counted by encountering right endpoints is $(k + 1)(n - (3/2)k - 2)$, since

$$\begin{aligned} \sum_{j=1}^{n-2k-2} (k+1) + \sum_{j=n-2k-1}^{n-k-2} (n-k-1-j) + \sum_{j=n-k-1}^n 0 &= (n-2k-2)(k+1) + \frac{k(k+1)}{2} \\ &= (k+1)(n - \frac{3}{2}k - 2). \end{aligned}$$

Summing the number of left edges and right edges yields an upper bound of $(k+1)(3n - (7/2)k - 5)$ edges in G . □

To prove our edge bound, we observe that for a graph G there are two ways that the edge count

in Lemma 2.2.1 can exceed the number of edges in G ; we refer to this gap as overcounting. The first way that there is overcounting in Lemma 2.2.1 is if there are bars that are among both the first $2k + 2$ and last $2k + 2$ bars. The visibilities between a pair of such bars will be counted twice. The second source of overcounting is if a bar among the top $k + 1$ or bottom $k + 1$ bars is not in the first $2k + 2$ bars or last $2k + 2$ bars. The count in Lemma 2.2.1 assumes that each bar that starts after the first $2k + 2$ has $2k + 2$ left edges, and it also assumes that each bar that ends before the last $2k + 2$ has $k + 1$ right edges. However, a bar among the top $k + 1$ or bottom $k + 1$ does not have $k + 1$ bars above and below, and therefore cannot contribute $2k + 2$ left edges or $k + 1$ right edges. By accounting for these two sources of overcounting, we improve the edge bound.

Theorem 2.2.2. *If G is a bar k -visibility graph with more than $2k + 2$ vertices, then G has at most $(k + 1)(3n - 4k - 6)$ edges.*

Proof. Observe that when maximizing the number of visibilities in a representation we may assume that the first $2k + 2$ bars all see each other and the last $2k + 2$ bars all see each other. To achieve this, the topmost and bottom most bars among the first $2k + 2$ should star first. The next bars to start should be the second from the top and the bottom. This pattern continues until the centermost bars begin last. This order is then reversed when considering the right endpoints: the centermost bars end first, and the outermost bars end last.

Let ℓ be the number of bars among the top $k + 1$ whose left endpoints are among the first $2k + 2$ left endpoints and whose right endpoints are among the last $2k + 2$ right endpoints. Similarly, let m be the number of bars among the bottom $k + 1$ having this property. We observe first that each of the ℓm edges joining these two sets of vertices is counted both when these bars begin (as a left edge) and when the bars between them end (as a right edge). The remaining $k + 1 - \ell$ bars among the top $k + 1$ either begin among the last $n - (2k + 2)$ or end among the first $n - (2k + 2)$, or perhaps both. Either the left endpoint of these bars does not contribute $2k + 2$ left edges or the right endpoint of these bars does not contribute $k + 1$ right edges. If $i \leq k + 1$ and $B(v_i)$ begins after the first $2k + 2$ or ends before the last $2k + 2$, then we overcount only by at least $k + 2 - i$;

thus the overcount is smallest when these $k + 1 - \ell$ bars are $B(v_{k+1}), B(v_k), \dots, B(v_{k+1-\ell})$. In this case, our edge bound has overcounted at least

$$1 + 2 + \dots + (k + 1 - \ell) = \binom{k + 2 - \ell}{2}$$

edges. Similarly, we obtained at least an extra $\binom{k+2-m}{2}$ in our edge count by assuming the $k+1-m$ bars from the bottom $k+1$ yielded $2k+2$ edges when they began and $k+1$ edges when they ended. Therefore our graph has at most

$$\left(3n - \frac{7}{2}k - 5\right)(k + 1) - \left[\ell m + \binom{k + 2 - \ell}{2} + \binom{k + 2 - m}{2}\right]$$

edges; we seek to minimize the function

$$\begin{aligned} f(\ell, m) &= \ell m + \binom{k + 2 - \ell}{2} + \binom{k + 2 - m}{2} \\ &= \frac{1}{2}(\ell + m)^2 - \frac{2k + 3}{2}(\ell + m) + (k + 1)(k + 2) \end{aligned}$$

As this is a quadratic in $(\ell + m)$, we find that local extrema occur when $\ell + m = \frac{2k+3}{2}$, yielding a minimum value of $\frac{8k^2+24k+14}{16}$. Hence $f(\ell, m) \geq \frac{1}{2}k^2 + \frac{3}{2}k + \frac{7}{8}$. For the bound on the number of edges, we compute

$$\left(3n - \frac{7}{2}k - 5\right)(k + 1) - \left(\frac{1}{2}k^2 + \frac{3}{2}k + \frac{7}{8}\right) = 3nk + 3n - 4k^2 - 10k - \frac{47}{8}.$$

Since the number of edges is an integer, we obtain

$$|E(G)| \leq 3nk + 3n - 4k^2 - 10k - 6 = (k + 1)(3n - 4k - 6).$$

□

Corollary 2.2.3. *If K_n is a bar k -visibility graph, then $n \leq 4k + 4$.*

Proof. If $n = 4k + 4 + m$, then

$$|E(K_n)| = (4k + 4 + m)(4k + 4 + m - 1)\frac{1}{2} = 8k^2 + 14k + 4mk + 6 + \frac{7}{2}m + \frac{1}{2}m^2.$$

Theorem 2.2.2 requires

$$|E(G)| \leq (k + 1)(3(4k + 4 + m) - 4k - 6) = 8k^2 + 14k + 3mk + 6 + 3m.$$

Hence

$$8k^2 + 14k + 4mk + 6 + \frac{7}{2}m + \frac{1}{2}m^2 \leq 8k^2 + 14k + 3mk + 6 + 3m$$

$$4mk + \frac{7}{2}m + \frac{1}{2}m^2 \leq 3mk + 3m$$

$$mk + \frac{1}{2}(m + m^2) \leq 0,$$

and therefore $m \leq 0$. Hence $n \leq 4k + 4$. □

Dean et al. [10] gave a construction achieving the edge bound in Theorem 2.2.2 for all $n \geq 4k + 4$ (see Figure 2.1). When $n = 4k + 4$, the resulting graph is the complete graph K_{4k+4} . When $n < 4k + 4$, the k -visibility graph with the most edges is a complete graph on n vertices, obtained by omitting any $4k + 4 - n$ bars from the representation of K_{4k+4} .

2.3 Comparing the Families \mathcal{F}_{k-1} and \mathcal{F}_k

Corollary 2.2.3 shows that K_{4k+4} is in \mathcal{F}_k but not \mathcal{F}_{k-1} . A natural question is whether \mathcal{F}_{k-1} is contained in \mathcal{F}_k . In order to answer this question, we will need the following lemma:

Lemma 2.3.1. *Given a bar k -visibility representation of a graph G , let v and w be nonadjacent vertices such that $I(v) \cap I(w) \neq \emptyset$. If a vertical line ℓ intersects $I(v) \cap I(w)$ and crosses $B(x)$, then x is contained in a $(k + 2)$ -clique whose intervals also intersect ℓ .*

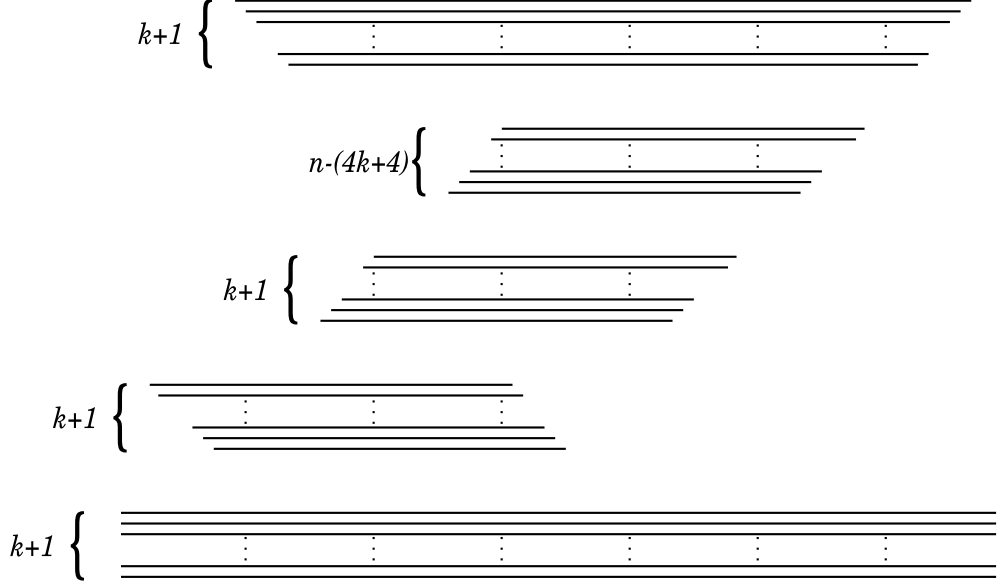


Figure 2.1: A bar k -visibility representation with $(k + 1)(3n - 4k - 6)$ edges.

Proof. Since $vw \notin E(G)$, there must be at least $k + 1$ bars blocking $B(v)$ from $B(w)$. Hence, at least $k + 3$ bars intersect ℓ . Any consecutive $k + 2$ bars among those intersecting ℓ , including $B(x)$, can all see each other, and hence their associated vertices form a $(k + 2)$ -clique. \square

Define the k -wheel W_n^k to be the graph obtained by making every vertex of a k -clique adjacent to every vertex of an n -cycle (see Figure 2.2). We will show that W_n^k is not a bar k -visibility graph. To complete this proof, we use the characterization of interval graphs due to Lekkerkerker and Boland [19]: interval graphs are chordal graphs and do not contain any asteroidal triples. An *asteroidal triple* is a set of three vertices in a graph such that between any two there is a path avoiding the neighborhood of the third. Specifically, we use that fact that induced subdivisions of $K_{1,3}$ are asteroidal triples.

Proposition 2.3.2. *For $n \geq 5$, W_n^k is not a bar k -visibility graph.*

Proof. Suppose that there is a bar k -visibility representation of W_n^k . If there is a vertical line that intersects at least $k + 5$ bars in the representation, then there are at least $k + 1$ bars seeing at least $k + 3$ other bars. However, there are only k vertices in W_n^k with degree at least $k + 3$. Thus a

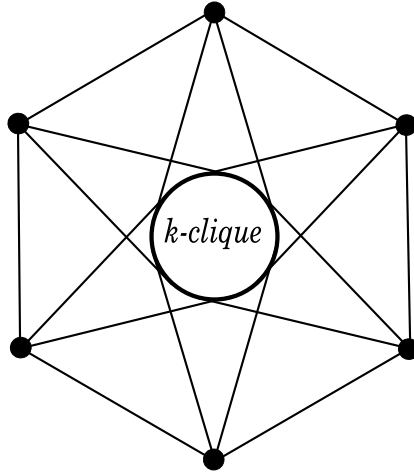


Figure 2.2: W_6^k

vertical line can intersect at most $k + 4$ bars.

Let x be a vertex in the n -cycle whose bar has the right endpoint with the minimum horizontal coordinate, and let a be the value of that coordinate. Let y and w be the neighbors of v on the cycle, and let y' and w' be the other neighbors of y and w on the cycle, respectively. Note that $yw \notin E(G)$, but $I(y)$ and $I(w)$ overlap. Therefore there are at least $k + 1$ bars between $B(y)$ and $B(w)$ in the representation. Because a vertical line intersects at most $k + 4$ bars, there are two cases to consider: when there are $k + 1$ bars between $B(y)$ and $B(w)$ and when there are $k + 2$ bars between $B(y)$ and $B(w)$.

If there are $k + 1$ bars between $B(y)$ and $B(w)$, then y and w are both adjacent to all of the $k + 1$ corresponding vertices. Thus those bars correspond to the k vertices in the central clique and v . If the horizontal coordinates of the right endpoints of $B(y)$ and $B(w)$ are greater than a , then $B(y)$ and $B(w)$ see each other when $B(v)$ ends. If $I(y)$ and $I(w)$ both have a as their right endpoint, then y' and w' , which are distinct, have bars that begin before a . This implies that a vertical line through a intersects at least $k + 5$ bars. Therefore we may assume (by symmetry) that $I(y)$ has a as its right endpoint and that the horizontal coordinate of the right endpoint of $I(w)$ is greater than a .

Because $I(y)$ and $I(y')$ overlap, $I(y')$ includes a . Thus a horizontal line through a intersects

$k + 4$ bars, namely the bars of the central clique as well as the $B(x)$, $B(w)$, $B(y)$ and $B(y')$. As we scan from left to right, there is a first bar that begins after the point a ; let b be the horizontal coordinate of the left endpoint of that bar. Any horizontal line through a point in $[a, b]$ intersects $k + 2$ bars, and the corresponding vertices form a $k + 2$ -clique. Therefore there is an edge joining y' and w , a contradiction.

We now assume that there are $k + 2$ bars between the $B(y)$ and $B(w)$. Because a vertical line can intersect at most $k + 4$ bars, $B(x)$ lies between $B(y)$ and $B(w)$. Therefore, $B(x)$ sees $k + 3$ bars unless it is the bar immediately next to $B(y)$ or $B(w)$. However, if $B(x)$ is immediately next to $B(y)$ (by symmetry), then $B(x)$ sees $k + 2$ bars, not including $B(w)$. Because $xw \in E(G)$, there is a point where $B(x)$ sees $B(w)$. Thus $B(x)$ has at least $k + 3$ visibilities, whereas x has degree $k + 2$. Therefore W_n^k is not a bar k -visibility graph. \square

Note that the construction in Figure 2.3 shows that W_4^k is a bar k -visibility graph, so the result is sharp.

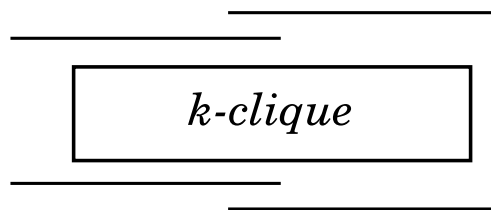


Figure 2.3: A bar k -visibility representation of W_4^k .

The results above combine to give the following:

Theorem 2.3.3. *For all k , the families \mathcal{F}_k and \mathcal{F}_{k-1} are incomparable under inclusion.*

Proof. $\mathcal{F}_k \not\subseteq \mathcal{F}_{k-1}$ follows from Corollary 2.2.3. For the reverse noninclusion, we observe that $W_n^k \in \mathcal{F}_{k-1}$; Figure 2.4 gives a bar $(k - 1)$ -visibility representation. By Proposition 2.3.2, $W_n^k \notin \mathcal{F}_k$; hence $\mathcal{F}_{k-1} \not\subseteq \mathcal{F}_k$. \square

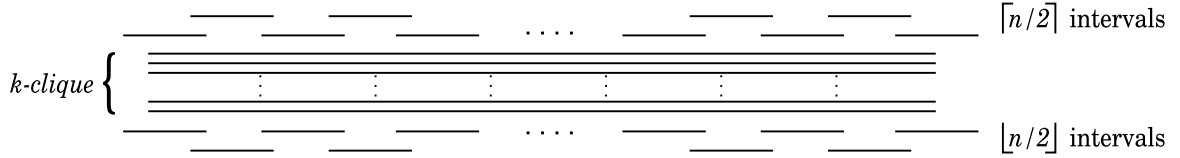


Figure 2.4: A bar $(k - 1)$ -visibility representation of W_n^k .

2.4 Induced Subgraphs

We have already observed that each interval graph lies in some \mathcal{F}_k when k is sufficiently large. It is an easy observation that an interval graph has no induced subgraph consisting of three paths of length at least 2 with a common endpoint. An interval representation of such a graph would contain an interval for the common endpoint, and the three neighbors of that vertex would correspond to three disjoint intervals. It follows that one of those three intervals is contained in the interval of the common endpoint, and thus any neighbor of that vertex is also adjacent to the common endpoint of the paths. We have shown that W_n^k is a bar $(k - 1)$ -visibility graph, and hence bar k -visibility graphs may contain induced long cycles. One may wonder whether an induced subdivision of $K_{1,3}$ prevents a graph from being in \mathcal{F}_k for any k . The following proposition answers this question.

Proposition 2.4.1. *For every tree T and every $k \geq 0$, there exists a graph G such that G contains T as an induced subgraph and G is a bar k -visibility graph.*

Proof. Choose a vertex r of T to be the root. We first place bars for the vertices of T by induction on the distance from r ; later we add bars for vertices in G that are not in T . Assign the root the bar $B(r)$. Having assigned bars to all vertices at distance ℓ from r , we place the bars for the vertices at distance $\ell + 1$ from r as follows: For a vertex v at level ℓ , find its children v_1, \dots, v_m in T . Divide $B(v)$ into $2m - 1$ closed segments, and assign v_i the $(2i - 1)^{st}$ segment. Translate this segment down by some fixed nonzero distance to obtain $B(v_i)$.

Having assigned bars to all vertices of T in this way, if v and w are adjacent in T , then $I(v) \cap I(w) \neq \emptyset$. By placing a k -clique between the bars for consecutive levels, we ensure that only vertices at adjacent levels can see each other. The graph induced by this bar k -visibility

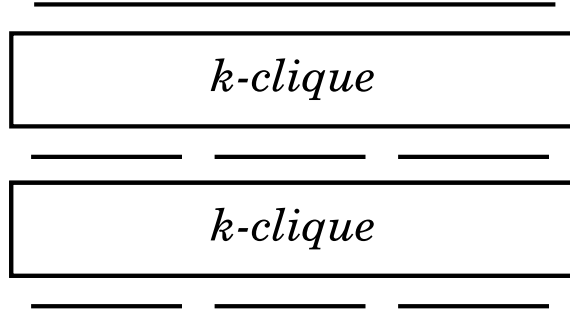


Figure 2.5: A bar k -visibility representation of a graph with an induced subdivided $K_{1,3}$.

representation is the desired graph G . □

Figure 2.5 gives an example when T is obtained from $K_{1,3}$ by subdividing each edge.

On the other hand, certain nonplanar subgraphs *are* forbidden as induced subgraphs.

Proposition 2.4.2. *If a graph G contains a triangle-free nonplanar induced subgraph, then G is not a bar k -visibility graph for any k .*

Proof. Suppose that G is a bar k -visibility graph for some k and has a triangle-free nonplanar induced subgraph H . Fix a bar k -visibility representation of G . Any two adjacent vertices of H are also adjacent in G , and thus the projection of their associated intervals in the bar k -visibility representation of G intersect. A pair of adjacent vertices u and v must exist in H such that any vertical line segment joining $B(u)$ and $B(v)$ intersects the bar of at least one other vertex w in H . Otherwise, if we restrict the bar k -visibility representation of G to the vertices in H , we would obtain a planar representation of H .

Thus, when $B(u)$ sees $B(v)$ in the bar k -visibility representation of G , the line of sight intersects some such $B(w)$. Therefore u , v , and w form a triangle in G that is also in H . This contradicts that H is triangle-free, and hence G is not a bar k -visibility graph. □

2.5 Regular Bar k -Visibility Graphs

It is easy to show that the only connected regular interval graphs are complete graphs. For small degrees in terms of k , this fact remains true for bar k -visibility graphs.

Proposition 2.5.1. *If G is a connected r -regular bar k -visibility graph with $r \leq 2k + 1$, then G is a complete graph. This inequality is sharp when $k \leq 4$*

Proof. Let v be a vertex whose bar begins last; that is, no vertex has a bar whose left endpoint is farther right than that of v . Let $I(v) = [a, b]$, and let v_1, \dots, v_m be the vertices such that $a \in I(v_i)$, ordered from top to bottom by the height of their corresponding bars. Note that $v = v_i$ for some i .

All neighbors of v are among v_1, \dots, v_m , so $r \leq m - 1$. If $m \geq 2k + 3$, then $B(v_{\lfloor m/2 \rfloor})$ can see $2k + 2$ other bars, contradicting regularity. Therefore $m < 2k + 3$, and $B(v_{\lfloor m/2 \rfloor})$ sees at least $m - 1$ bars, yielding $r \geq m - 1$. Hence, $r = d(v) = d(v_{\lfloor m/2 \rfloor}) = m - 1$.

If G is not a complete graph, then there exists at least one vertex whose bar ends before $B(v)$ begins. Let c be the maximum value among the right endpoints of the intervals associated to such vertices. Note that $c < a$, and every bar whose endpoint is to the right of c is $B(v_i)$ for some i . Let z_1, z_2, \dots, z_p be all the vertices whose intervals' right endpoints are c . As G is connected, some v_i must be adjacent to some z_j . Among the bars in $\{B(v_1), \dots, B(v_m)\}$ that see some $B(z_j)$ at the point c , choose i to minimize $|\lfloor m/2 \rfloor - i|$.

We claim that $d(v_i) \geq m$. We know that z_j is adjacent to v_i , so if G is r -regular then there is some v_ℓ that is not adjacent to v_i , $i \neq \ell$. Since any bar that begins after c must see $B(v_i)$ in order to have degree $m - 1$, we conclude $c \in I(v_\ell)$. Consider the point $c + \epsilon$, where ϵ is chosen to be small enough so that no interval begins in $[c, c + \epsilon]$. As v_i was chosen to be the “most central” bar extending left to the point c , there cannot be k intervals containing the point $c + \epsilon$ blocking $B(v_i)$ from $B(v_\ell)$. Therefore $v_i v_\ell \in E(G)$, and hence $d(v_i) \geq m$, contradicting the assumption that G is regular.

When $d = 2k + 2$ and $k \in \{0, 1, 2, 3, 4\}$, there exist d -regular non-complete graphs in \mathcal{F}_k . For $k = 0$, every cycle C_n with $n \geq 4$ is a 2-regular bar 0-visibility graph. Figure 2.6 shows

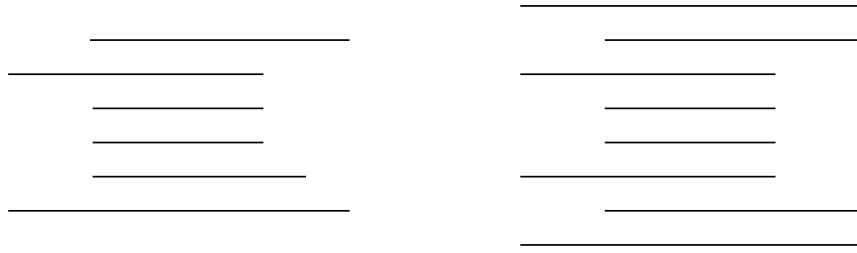


Figure 2.6: Left: A bar 1-visibility representation of a 4-regular graph. Right: A bar 2-visibility representation of a 6-regular graph.

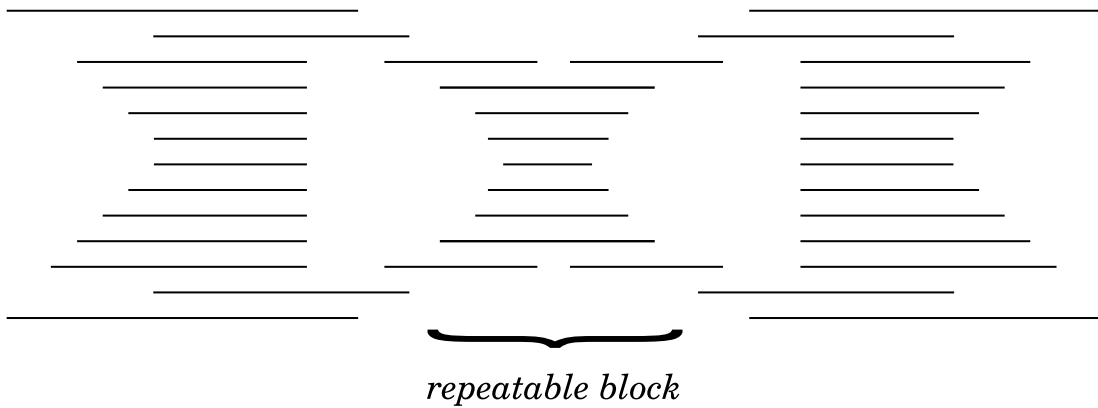


Figure 2.7: A bar 3-visibility representation of an 8-regular graph.

non-complete graphs that are $2k + 2$ regular when $k = 1$ and $k = 2$. Figures 2.7 and 2.8 show constructions for an infinite number of regular graphs of degree $2k + 2$ when $k = 3$ and $k = 4$, respectively. These constructions depend on repeatable blocks of 9 and 11 bars, respectively, that can be repeated horizontally as many times as desired. Consecutive blocks may need to be perturbed vertically a small amount so that the top and bottom bars from a given block can see the top and bottom bars from the next block, but are still disjoint from those bars.

□

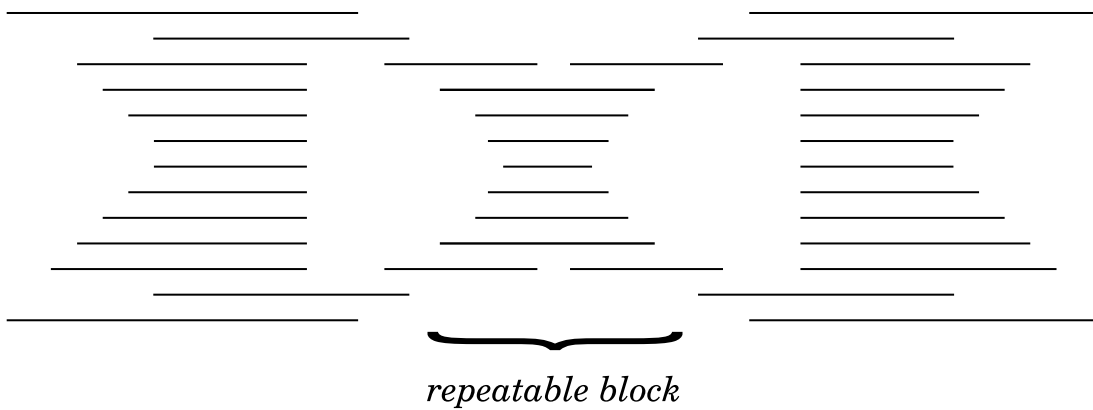


Figure 2.8: A bar 4-visibility representation of a 10-regular graph.

Chapter 3

Total Acquisition in Graphs

3.1 Introduction to Acquisition Parameters

Consider an army dispersed among many cities. There are roads joining some pairs of the cities. We wish to model the process of withdrawing the troops to a small number of cities. It is reasonable to require that troops only move to occupied cities, and that the number of troops moving to a city cannot exceed the number of troops already at that city. The goal of such a procedure is to consolidate all of the troops in the least number of cities. We model this situation with a weighted graph, where each vertex represents a city, the weight at each vertex corresponds to the number of troops present, and edges are roads.

If G is a weighted graph, then an *acquisition move* transfers some amount of weight from a vertex u in G to a neighbor v , provided that the weight on v is at least the weight on u . Our goal in making acquisition moves is to transfer all of the weight in G to the least number of vertices. In this and the following two chapters we study the behavior of three different types of acquisition moves: total acquisition moves, unit acquisition moves, and fractional acquisition moves. We refer to a succession of acquisition moves as an *acquisition protocol*. For simplicity, we use the terms *total acquisition protocol*, *unit acquisition protocol*, and *fractional acquisition protocol* for protocols of total, unit, and fractional acquisition moves respectively.

Lamper and Slater introduced total and unit acquisition in graphs in [18], and their work focused on total acquisition. Let G be a weighted graph in which every vertex has weight 1. If $uv \in E(G)$, and the weight on v is at least the weight on u , then a *total acquisition move* transfers all of the weight from u to v . Thus the weight on v becomes the sum of the previous weights on

u and v , the weight on u becomes 0, and the total weight on G is preserved. The *total acquisition number* of G , denoted $a_t(G)$, is the minimum size of the set of vertices with positive weight after a sequence of total acquisition moves on G . Always total acquisition moves can be made until the set of vertices with positive weight becomes an independent set.

With the same initial setup of G , u , and v , an *unit acquisition move* transfers one unit of weight from u to v . The *unit acquisition number* of G , denoted $a_u(G)$, is the minimum size of the set vertices with positive weight after a unit acquisition protocol on G . In [18], Lampert and Slater referred to certain combinations of unit acquisition moves as a “consolidation move,” but to maintain consistency in our terminology, we do not use this term.

Again with the same initial set up of G , u , and v , a *fractional acquisition move* transfers any positive amount of the weight from u to v . The *fractional acquisition number* of G , denoted $a_f(G)$, is the minimum size of the set vertices with positive weight after a sequence of fractional acquisition moves on G .

Some terminology and basic results that apply to all of these acquisition parameters. Let the notation $a(G)$ denote a generic acquisition number; results that are true about this parameter are true of the total, unit, and fractional acquisition numbers of G . A protocol \mathcal{A} on a graph G is *optimal* if the number of vertices with positive weight after running \mathcal{A} on G is equal to $a(G)$. We study acquisition protocols on graphs in which every vertex starts with weight 1; we denote this weight assignment by the bold 1. A weight assignment w on a graph G is *feasible* if it is possible to achieve w from $\mathbf{1}$ using valid acquisition moves.

Recall that $\alpha(G)$ denotes the maximum size of an independent set in G and that $\gamma(G)$ denotes the minimum size of a dominating set in G .

Observation 3.1.1. *If G is a graph, then $a(G) \leq \alpha(G)$ and $a(G) \leq \gamma(G)$.*

Proof. If an acquisition protocol \mathcal{A} is optimal on a graph G , then adjacent vertices in G cannot have positive weight at the end of \mathcal{A} . Also, given a dominating set S , it is possible to move the weight from each vertex in $V(G) - S$ to a neighbor in S via acquisition moves. \square

We now prove a lemma that provides a lower bound on the acquisition number of a graph G with an arbitrary weight assignment. It is a useful tool to determine lower bounds on the acquisition number of a graph when it can be proved that an optimal protocol contains certain acquisition moves in a given order.

Lemma 3.1.2. *Let G be a weighted graph, and let S be a cut-set in G . If each vertex in S has weight 0 and there are m components in $G - S$ having positive total weight, then $a(G) \geq m$.*

Proof. Because every vertex in S has weight 0, no weight can move to a vertex in S via an acquisition move. Therefore, there is no way to move weight from one component of $G - S$ to another. \square

We now observe a natural relationship among the total, unit, and fractional acquisition numbers of a graph.

Proposition 3.1.3. *For a graph G , $a_t(G) \geq a_u(G) \geq a_f(G)$.*

Proof. Every total acquisition move is a unit acquisition move, and every unit acquisition move is a fractional acquisition move. Thus every optimal total acquisition protocol is a unit protocol, and every optimal unit protocol is a fractional protocol. \square

Proposition 3.1.4. *Any total acquisition protocol or unit acquisition protocol is finite.*

Proof. Let G be an n -vertex graph with weight distribution 1. The sums of the squares of the initial weights in G is n . Every total acquisition move and unit acquisition move increases this sum by at least 2, and the sum cannot exceed n^2 . Therefore every total acquisition protocol and unit acquisition protocol is finite. \square

There are infinite fractional acquisition protocols. As an example, let $G = K_2$ with vertices x and y , and let the i th move in a fractional acquisition protocol transfer weight 2^{-i} from x to y . However, if there is a lower bound on the minimum amount of weight transferred by a move in a fractional acquisition protocol \mathcal{A} on a graph G , then \mathcal{A} is finite.

In Chapter 4, we prove general upper bounds on the unit acquisition number of a graph, and we prove that there is no lower bound based on the maximum degree of a graph. We also study the unit acquisition number of graphs with diameter 2. In Chapter 5 we prove our most surprising result: If a graph G is connected and $\Delta(G) \geq 3$, then the fractional acquisition number of G is 1.

In this chapter, we study total acquisition. In Section 3.2, we study the acquisition number of n -vertex trees with fixed maximum degree or diameter. For every k and D with $k \geq 3$ and $D \geq 6$, there is an n -vertex tree T with maximum degree k , diameter at least D , and acquisition number $(n + 1)/3$, which is in fact the maximum over all graphs with n vertices. Trivially, $a_t(T) = 1$ when T is a tree with diameter at most 3. For n -vertex trees with diameter 4 and diameter 5, we show that the maximum is $\Theta(\sqrt{n})$. We characterize trees T for which $a_t(T) = 1$, which allows us to construct a polynomial-time algorithm to test $a_t(T) \leq k$ for any fixed positive integer k .

In Section 3.3, we give sufficient conditions for a graph to have acquisition number 1. We show that, if $G \neq C_5$, then $a_t(G)$ or $a_t(\overline{G})$ is 1, where \overline{G} denotes the complement of G . Furthermore, if $\delta(G) \geq (|V(G)| - 1)/2$, then $a_t(G) = 1$ (again, unless $G = C_5$), and no lower minimum degree is sufficient.

In Section 3.4, we study the effects of edge deletion and graph products on acquisition numbers. The deletion of a single edge in an n -vertex graph cannot increase the acquisition by more than $6.84\sqrt{n}$, but there is an n -vertex tree having an edge whose deletion increases the acquisition number by more than $\frac{1}{2}\sqrt{n}$.

In Section 3.5, we consider graphs with diameter 2. We conjecture that $a(G)$ is bounded by an absolute constant for such graphs, perhaps by 2. We prove an upper bound of $32 \lg n \lg \lg n$, but if $\text{diam}(G) = 2$ and G contains no 4-cycle and has maximum degree at least 7, then $a(G) = 1$.

3.2 Trees

Study of bounds on acquisition number for trees is motivated by the fact that $a_t(G)$ is determined by the total acquisition numbers of spanning forests of G .

Proposition 3.2.1. *If G is a graph, then $a_t(G) = \min\{a_t(F) : F \in \mathcal{F}\}$, where \mathcal{F} is the set of spanning forests of G with the same number of components as G .*

Proof. Deleting an edge cannot reduce the total acquisition number, so $a_t(F) \geq a_t(G)$ for every spanning forest F . On the other hand, if A is the set of edges along which weight is transferred in an optimal total acquisition protocol in G , then A is acyclic, since weight can never be moved to a vertex once it has weight 0. Hence $a_t(G) \leq a_t(F)$, where F is the spanning forest with edge set A . Adding edges cannot increase the total acquisition number, so the minimum occurs for a spanning forest that contains a spanning tree of each component of G . \square

Proposition 3.2.1 will yield the total acquisition number of the n -vertex cycle from the total acquisition number of the n -vertex path.

Proposition 3.2.2. *If P_n is the n -vertex path, then $a_t(P_n) = \lceil n/4 \rceil$.*

Proof. Lampert and Slater [18] observed that the maximum amount of weight that a vertex of degree d can attain is 2^d . This follows from the fact that each time weight moves onto a vertex, the weight at that vertex at most doubles. Since P_n has maximum degree 2, the maximum amount of weight that a single vertex can attain is 4. Therefore $a_t(P_n) \geq \lceil n/4 \rceil$.

We can partition the vertices of P_n into $\lceil n/4 \rceil$ sets, $\lceil n/4 \rceil - 1$ of which induce 4-vertex paths and one that induces a path on at most four vertices. Each of these subtrees has total acquisition number 1, so $a_t(P_n) \leq \lceil n/4 \rceil$. \square

Corollary 3.2.3. *If C_n is the n -vertex cycle, then $a_t(C_n) = \lceil n/4 \rceil$.*

Proof. All spanning forests with the same number of components as C_n are n -vertex paths. By Proposition 3.2.1 $a_t(C_n) = a_t(P_n)$. \square

Lampert and Slater [18] showed that $\lfloor (n+1)/3 \rfloor$ is the maximum of $a_t(G)$ over n -vertex graphs. For $n \equiv 2 \pmod{3}$, they provided a tree achieving this bound, but its maximum degree is $(n+1)/3$. We generalize the example to show that the bound for n -vertex graphs is sharp among

those with any value of the maximum degree that is at least 3. The total acquisition number of the path is only $\lceil n/4 \rceil$, so the bound is not tight when $\Delta(G) = 2$.

Lemma 3.2.4. *Let x and y be vertices in a tree T . If the unique x, y -path in T contains a vertex having degree 2 in T that is not adjacent to x or y , then the weight from x and y cannot reach a common vertex via total acquisition moves.*

Proof. Let v be a vertex on the unique x, y -path in T that is not adjacent to x or y such that $d(v) = 2$. For weight from x and y to reach the same vertex, vertex v must be used. Consider the first total acquisition move involving v . This move transfers weight 1, and afterwards v or one of its neighbors has weight 0. By Lemma 3.1.2, the weights from x and y cannot then reach the same vertex. □

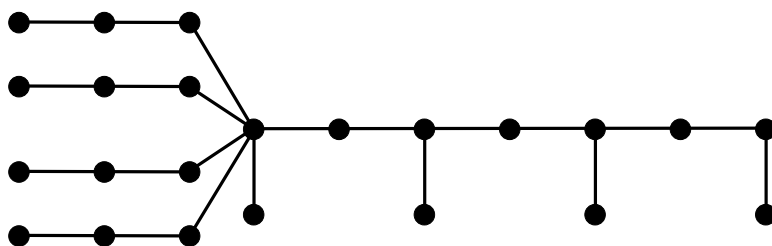


Figure 3.1: The tree $T(4, 3)$

Theorem 3.2.5. *For all $\ell \geq 1$ and $m \geq 1$, there is a tree T with $\Delta(T) = \ell + 2$, $\text{diam}(T) = 2m + 4$, and $a_t(T) = (|V(T)| + 1)/3$.*

Proof. For $\ell \geq 1$ and $m \geq 1$, define a tree $T(\ell, m)$ as follows. Let Q_1, \dots, Q_ℓ be disjoint copies of P_3 . Let R be a path of length $2m$ with vertices v_1, \dots, v_{2m+1} , in order. For $1 \leq i \leq m + 1$, let u_i be a leaf neighbor of v_{2i-1} . Add edges to make v_1 adjacent to one endpoint of each Q_i (the tree $T(4, 3)$ appears in Figure 3.1). Now $T(\ell, m)$ is a tree with $3\ell + 3m + 2$ vertices and $\ell + m + 1$ leaves. It has diameter $2m + 4$ and maximum degree $\ell + 2$.

Because the neighbors of the leaves of $T(\ell, m)$ form a dominating set of size $\ell + m + 1$, it suffices to show that $a_t(T(\ell, m)) \geq (|V(T)| + 1)/3 = \ell + m + 1$. Between each pair of leaves

in T , there is a vertex with degree 2 that is not adjacent to any leaves. By Lemma 3.2.4, it follows that the weight from two leaves cannot reach the same vertex. Since $T(\ell, m)$ has $\ell + m + 1$ leaves, the bound holds. \square

Note that in this construction, reducing ℓ by 1 and increasing m by 1 does not change the number of vertices. Thus, for n -vertex graphs with $n \equiv 2 \pmod{3}$ our construction produces trees with total acquisition number $(n + 1)/3$ for all even values of the diameter from 6 up to $\frac{2}{3}(n + 1)$.

This motivates the study of total acquisition number for n -vertex trees with fixed diameter. Theorem 3.2.5 shows that $a_t(T) = (n + 1)/3$ is achievable when $\text{diam}(T) \geq 6$. When the diameter is 2 or 3, it is easy to see that $a_t(T) = 1$. Theorems 3.2.7 and 3.2.8 settle the remaining cases.

The proof of Theorem 3.2.7 relies on a total acquisition protocol that is also used in Section 3.5. The protocol allows the vertex in the center of a tree of diameter 4 to acquire the maximum amount of weight.

Let T be a tree with diameter 4, and let u be the center of T . Let v_1, \dots, v_k be the neighbors of u labeled in nondecreasing order of degree. Define the u -greedy protocol, denoted $\mathcal{A}(u)$, as follows. Let w_i denote the weight on u at the beginning of step i . Initialize $w_1 = 1$. In step i , move weight $\min\{w_i, d(v_i)\} - 1$ from $N(v_i)$ to v_i ; the weight on v_i is now $\min\{w_i, d(v_i)\}$. Complete step i by transferring the weight on v_i to u . Thus $w_{i+1} = w_i + \min\{w_i, d(v_i)\}$.

We can bound the number of vertices in $N(u)$ that have leaf neighbors with positive weight after running $\mathcal{A}(u)$.

Lemma 3.2.6. *Let T be a tree with diameter 4, let u be the center of T , and run the u -greedy protocol $\mathcal{A}(u)$ on T . If d is the degree of the last neighbor of u having leaf neighbors with positive weight, then at most $\lceil \lg(d) \rceil$ vertices in $N(u)$ have leaf neighbors with positive weight. Consequently, $a_t(T) \leq d \lceil \lg(d) \rceil$.*

Proof. Let S be the set of vertices in $N(u)$ having leaf neighbors with positive weight after $\mathcal{A}(u)$, and let $m = \max\{i: v_i \in S\}$, so $d(v_m) = d$. If $v_i \in S$, then $d(v_i) > w_i$ at step i in $\mathcal{A}(u)$. Therefore, the weight at u doubles during step i for each $i \in S$. Before step m , the weight on u

has doubled at most $\lg(d - 1)$ times, because $w_m < d$. Hence $|S| \leq 1 + \lfloor \lg(d - 1) \rfloor = \lceil \lg d \rceil$. Because each vertex in S has at most $d - 1$ leaf neighbors, it follows that $a_t(T) \leq d \lceil \lg(d) \rceil$. \square

We now prove an upper bound for the total acquisition number of trees with diameter 4 or 5. Our proof uses the u -greedy protocol in trees of diameter 4 when the degree of the central vertex and the maximum degree among the other vertices are suitable. For trees of diameter 5 we delete the central edge and apply our bound for trees of diameter 4.

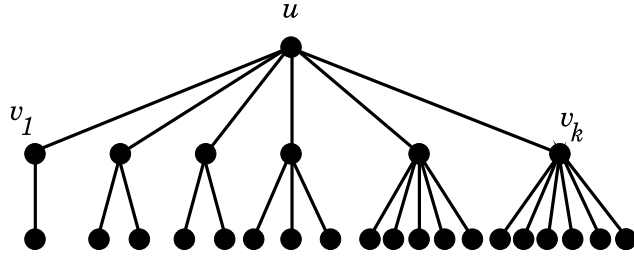


Figure 3.2: Vertex labeling for trees of diameter 4

Theorem 3.2.7. *Let T be an n -vertex tree. If $\text{diam}(T) = 4$, then $a_t(T) \leq \sqrt{n \lg n}$. If $\text{diam}(T) = 5$, then $a_t(T) \leq \sqrt{2} \sqrt{n \lg n}$.*

Proof. We first consider trees with diameter 4, and we prove the bound by induction on n . Let u be the central vertex of T , and label its neighbors v_1, \dots, v_k in nondecreasing order of degree, as shown in Figure 3.2.

If $k \leq \sqrt{n \lg n}$, then it suffices to let $N(u)$ absorb all the weight, which it can do since it is a dominating set. If $k > \sqrt{n \lg n}$, then $k \geq n/2$ if $n \leq 16$. We will show in Lemma 3.3.3 that when G has a vertex of degree at least $|V(G)|/2$ whose neighborhood is a dominating set, $a_t(G) = 1$. Hence we may assume that $k > \sqrt{n \lg n}$ and $n > 16$.

If $d(v_k) \geq \sqrt{n}$, then we let v_k acquire the weight on its leaf neighbors and apply the induction hypothesis to the tree obtained by deleting v_k and its leaf neighbors. Thus $a_t(T) \leq 1 + \sqrt{(n - \sqrt{n}) \lg n}$. Note that $1 + \sqrt{A - B} \leq \sqrt{A}$ if and only if $B \geq 2\sqrt{A} - 1$. Since $\sqrt{n} \lg n \geq 2\sqrt{n \lg n} - 1$ when $n \geq 12$, this case is complete.

Hence we may assume that $k > \sqrt{n \lg n}$, that $d(v_k) < \sqrt{n}$, and that $n > 16$. Now we run the u -greedy protocol $\mathcal{A}(u)$ on T . Let S be the resulting set of vertices in $N(u)$ that have leaf neighbors with positive weight. Let $m = \max\{i: v_i \in S\}$. By Lemma 3.2.6, $a_t(T) \leq d(v_m) \lceil \lg d(v_m) \rceil$.

We will prove that $d(v_m) < 2n/k$. Given this,

$$a_t(T) \leq d(v_m) \lceil \lg d(v_m) \rceil < \frac{2n}{k} \left\lceil \lg \frac{2n}{k} \right\rceil \leq 2\sqrt{\frac{n}{\lg n}} \left\lceil \lg \sqrt{\frac{n}{\lg n}} \right\rceil < \sqrt{n \lg n}.$$

To prove $d(v_m) < 2n/k$, we first argue that $m < k/2$. The computation will be

$$m \leq w_m < d(v_m) \leq d(v_k) < \sqrt{n} \leq k/2.$$

Since u acquires weight with each step, $m \leq w_m$. Since $m \in S$, we have $w_m < d(v_m)$. We are in the case where $d(v_k) < \sqrt{n}$. Finally, $k > \sqrt{n \lg n}$ yields $\sqrt{n} \leq k/2$ provided that $n \geq 16$.

Note that $d(v_m)(k - m) < n$, because each vertex after v_m has at least $d(v_m)$ neighbors. Since $m < k/2$, we have $k - m > k/2$, and therefore $d(v_m) < 2n/k$. This completes the proof for trees with diameter 4.

If T is an n -vertex tree with diameter 5, then the center of T is an edge; call it e . Deleting e yields two trees T' and T'' of diameter at most 4 with m and $n - m$ vertices respectively. We then apply the bound for trees with diameter 4 to T' and T'' to obtain

$$a_t(T) \leq \sqrt{m \lg m} + \sqrt{(n - m) \lg(n - m)}.$$

By concavity of $f(x) = \sqrt{x \lg x}$, this is maximized when $m = n/2$. Therefore

$$a_t(T) \leq 2 \left(\sqrt{(n/2) \lg(n/2)} \right) \leq \sqrt{2} \sqrt{n \lg(n)}.$$

□

We now provide constructions that demonstrate that the bounds of Theorem 3.2.7 has the cor-

rect order of growth.

Theorem 3.2.8. *For sufficiently large n , there is an n -vertex tree T_n with diameter 4 and $a_t(T) \geq (1 - o(1))\sqrt{\frac{1}{2}n \lg n}$, where c is a positive constant.*

Proof. Let T_n have central vertex u , and let the neighbors of u be v_1, \dots, v_k such that the degrees of v_1, \dots, v_k differ by at most 1; let $\lceil m \rceil$ and $\lfloor m \rfloor$ be those degrees. We choose m and k such that $m = \sqrt{\frac{2n}{\lg n}}$; it follows that $k = (1 + o(1))\sqrt{\frac{n \lg n}{2}}$.

Let \mathcal{A} be an optimal total acquisition protocol on T_n , and let q be the number of vertices in $N(u)$ that transfer weight to u . Without loss of generality, we can assume that weight moves from v_1, \dots, v_q to u in order. To minimize the number of vertices with positive weight, \mathcal{A} transfers the weight from all the leaf neighbors of v_i to v_i for each $i > q$. Let T' be the subtree of T_n induced by u, v_1, \dots, v_q , and the children of v_1, \dots, v_q . Since weight moves from v_i to u for $i \in [q]$, weight in T' will end up at u and a set of the leaves in T' . Therefore, to minimize the number of leaves with weight after running \mathcal{A} , the u -greedy protocol on T' should be contained in \mathcal{A} .

If $q < \lg m$, then weight is stranded on at least one leaf neighbor of each v_i with $i \in [q]$, so there is weight on at least k vertices. On the other hand, consolidating the weight on v_1, \dots, v_k leaves weight on at exactly k vertices. Therefore if $q < \lg m$, we may assume that $q = 0$. If all of the weight is consolidated on k vertices, then $a_t(T) \leq k = (1 + o(1))\sqrt{\frac{n \lg n}{2}}$.

If $q > \lg m$, then the u -greedy protocol on T' achieves weight at least m on u . Thus u is able to acquire all of the weight from each v_i when $i > q$. We conclude that if $q > \lg m$, then $q = k$. In this case, the number of vertices with positive weight after \mathcal{A} is given by

$$m \lg m - \sum_{i=1}^{\lg m} (2^i - 1).$$

Simplification yields

$$\begin{aligned}
m \lg m - \sum_{i=1}^{\lg m} (2^i - 1) &= (1 + o(1))(m \lg m) \\
&= (1 + o(1)) \sqrt{\frac{2n}{\lg n}} \lg \sqrt{\frac{2n}{\lg n}} \\
&= (1 + o(1)) \frac{1}{2} \sqrt{\frac{2n}{\lg n}} \lg n \\
&= (1 + o(1)) \sqrt{\frac{n \lg n}{2}}.
\end{aligned}$$

We conclude that $a_t(T) \geq (1 + o(1)) \sqrt{\frac{n \lg n}{2}}$.

□

Lampert and Slater [18] proved that the problem of determining $a_t(G)$ for general graphs is NP-complete. In fact, it is NP-complete even to test whether $a_t(G) = 1$. They asked whether the same statements are true when we require G to be a tree. We answer part of this question by providing for any fixed k a polynomial-time algorithm to determine whether $a(T) \leq k$. We start by characterizing trees with acquisition number 1.

Define a *rooted acquisition tree* to be a rooted tree (T, r) such that there is a total acquisition protocol that transfers all the weight in T to r .

Lemma 3.2.9. *A rooted tree (T, r) is a rooted acquisition tree if and only if*

- 1) $|V(T)| = 1$, or
- 2) T has an edge rr' so that, if T_1 and T_2 are the subtrees of $T - rr'$ containing r and r' , respectively, then (T_1, r) and (T_2, r') are rooted acquisition trees.

Proof. Sufficiency of the condition is clear. For necessity, let rr' be the last edge along which weight is moved in a total acquisition protocol that consolidates all of the weight on r . □

The recursive characterization of rooted acquisition trees in Lemma 3.2.9 is the same as that for *union trees*, a class of trees used as a data structure in computer science. Thus we have:

(T, r) is a rooted acquisition tree if and only if it is a union tree.

Cai [6] characterized union trees and gave a $O(n^2)$ -time algorithm to recognize them. We use it in our polynomial-time algorithm for testing $a_t(T) \leq k$.

Theorem 3.2.10. *For each positive integer k there is an $O(n^{k+2})$ -time algorithm for testing $a_t(T) \leq k$ when T is an n -vertex tree.*

Proof. Since $a_t(T)$ is the least total acquisition number over all spanning forests of T , we have $a_t(T) \leq k$ if and only if there are k disjoint rooted acquisition trees in T that span $V(T)$.

For all $B \subseteq E(T)$ with $|B| = k - 1$, let T_1, \dots, T_k be the components of $T - B$. For each vertex $r \in V(T_i)$, use Cai's algorithm to test whether (T_i, r) is a rooted acquisition tree. Conclude $a_t(T) \leq k$ if and only if, for some B , each T_i contains a vertex r such that (T_i, r) is a rooted acquisition tree for all $1 \leq i \leq k$.

There are $O(n^{k-1})$ choices for B . For a given B , testing vertices as roots takes cubic time since we run a quadratic-time algorithm a linear number of times. Therefore our algorithm takes $O(n^{k+2})$ time. \square

3.3 Total Acquisition Number 1

The goal of this section is to find sufficient conditions for G to have acquisition number 1.

A *dominating clique* in a graph is a dominating set that induces a complete graph.

Proposition 3.3.1. *If G is a graph with a dominating clique, then $a_t(G) = 1$.*

Proof. When K is a dominating clique in G , we can move all weight from $V(G) - K$ onto K and then consolidate all weight onto a single vertex using edges induced by K . \square

Bacsó and Tuza [2] showed that every graph that is P_5 -free and C_5 -free has a dominating clique. This gives our next corollary.

Corollary 3.3.2. *If G is a connected P_5 -free and C_5 -free graph, then $a_t(G) = 1$.*

We now show that if $G \neq C_5$, then $a_t(G) = 1$ or $a_t(\overline{G}) = 1$. We first prove that $a_t(G) = 1$ if the neighborhood of a vertex with sufficiently high degree is a dominating set. We then show that if $G \neq C_5$ and G is $(|V(G)| - 1)/2$ -regular, then $a_t(G) = 1$. Together with Proposition 3.3.1, these results complete the proof.

Lemma 3.3.3. *If G is a graph having a vertex v such that $d(v) \geq |V(G)|/2$ and $N(v)$ dominates G , then $a_t(G) = 1$.*

Proof. Since $N(v)$ dominates G , we can begin by moving all weight from $V(G) - N[v]$ onto $N(v)$. If w is the maximum weight on any vertex of $N(v)$, then there are at least $w - 1$ vertices in $N(v)$ with weight 1, since $|N(v)| > |V(G) - N[v]|$. We can move the weight from each neighbor of v with weight 1 to v so that v has weight at least w , after which we can move all remaining weight to v . \square

Lemma 3.3.4. *If G is an n -vertex, $(n - 1)/2$ -regular graph that is not C_5 , then $a_t(G) = 1$.*

Proof. Since the conclusion holds for K_3 and C_5 is excluded, we may assume $n \geq 7$. Choose $v \in V(G)$. Since any two nonadjacent vertices have a common neighbor, $N(v)$ is a dominating set in G . Let $x \in N(v)$ be a vertex with the most neighbors outside $N[v]$, and let $w = |N(x) - N[v]|$. Let $R = V(G) - N[v]$.

If $w \geq 2$, then move all weight from $N(x) - N[v]$ onto x (giving x weight $w + 1$) and move all the rest of the weight from R onto $N(v) - \{x\}$. Since $d(x) = (n - 1)/2$, some other vertex in $N(v)$ now has weight at least 2. If some such vertex y now has weight at least 3, then at least w vertices in $N(v)$ have weight 1, since weight $2|N(v)| - w - 4$ is distributed among $|N(v)| - 2$ vertices. As in the proof of Lemma 3.3.3, all weight can now be acquired by v . If no such y exists, then some $z \in N(v) - \{x\}$ has weight 2. Now at least $w - 1$ vertices in $N(v)$ have weight 1, so we can move the weight from these vertices onto v , then move the weight from z to v (since $w \geq 2$), and finally move the rest of the weight onto v .

We are left only with the case that $w = 2$. If this happens, each vertex of R has exactly one neighbor in $N[v]$, and $(n - 3)/2$ neighbors in R . Thus R is a clique. Also, x is adjacent to all but

one vertex of $N(v)$. Since $(n - 1)/2 \geq 3$, we can move all weight from $N[v]$ onto x , routing the weight from the nonneighbor of x in $N(v)$ through v . Now move all weight from R onto x by first moving it to the neighbor of x in R . \square

Theorem 3.3.5. *If G is a graph and $G \neq C_5$, then $a_t(G)$ or $a_t(\overline{G})$ equals 1.*

Proof. If $\text{diam}(G) \geq 3$, then any pair of vertices x and y satisfying $d_G(x, y) \geq 3$ form a dominating clique in \overline{G} , so $a_t(\overline{G}) = 1$ by Proposition 3.3.1. We may assume, by symmetry, that G and \overline{G} both have diameter 2.

If G has a vertex v of degree at least $|V(G)|/2$, then $N(v)$ dominates G (since G has diameter 2), so Lemma 3.3.3 implies $a_t(G) = 1$. We may assume, therefore, that $\Delta(G)$ and $\Delta(\overline{G})$ (by symmetry) are at most $(n - 1)/2$. It follows that G is $(n - 1)/2$ -regular, and Lemma 3.3.4 yields $a_t(G) = 1$. \square

Ore's Theorem [24] states that if every pair of nonadjacent vertices in an n -vertex graph have degree sum at least n , then the graph is Hamiltonian. We note that Lemmas 3.3.3 and 3.3.4 hinge on the fact that any two nonadjacent vertices have a common neighbor. In light of this, it is natural to ask if an Ore-type condition guarantees total acquisition number 1. We answer this in the affirmative.

Theorem 3.3.6. *If G is an n -vertex graph with $d(u) + d(v) \geq n - 1$ for all nonadjacent vertices u and v , and $G \neq C_5$, then $a_t(G) = 1$.*

Proof. By the degree requirement, every two nonadjacent vertices have a common neighbor, so $N(v)$ dominates G for every $v \in V(G)$. If there is a vertex v with $d(v) \geq n/2$, then Lemma 3.3.3 applies. If there is no such v , then G is $(n - 1)/2$ -regular and Lemma 3.3.4 applies. \square

As a corollary of Theorem 3.3.6, $\delta(G) \geq (|V(G)| - 1)/2$ implies then $a_t(G) = 1$. This is sharp because $2K_{n/2}$ has minimum degree $((n - 2)/2)$ and acquisition number 2 (when n is even).

3.4 Operations on Graphs

We now turn our attention to the effect of edge deletion and graph products on $a_t(G)$. Deleting a single edge can greatly increase the total acquisition number, as there are graphs G with an edge e for which $a_t(G) = 1$ and $a_t(G - e) = \Theta(\sqrt{|V(G)|})$. However, as we shall see in Theorem 3.4.4, this is the largest that $a_t(G - e) - a_t(G)$ can be.

We begin by proving a lemma that shows there is a great deal of flexibility in the amount of weight that can move to a given vertex.

Lemma 3.4.1. *Let T be a tree, with $v \in V(T)$, and suppose there is a total acquisition protocol on T that yields weight w on v . If $1 \leq k \leq w$, then there is a total acquisition protocol that yields weight k on v .*

Proof. We prove this by induction on k . The base case $k = 1$ is clear. Also, the case $k = w$ is given, so assume $k < w$.

Let \mathcal{A} be a protocol that yields weight w on v . Let $v_i v$ be the edges used to transport weight to v , indexed by their order in \mathcal{A} , and let w_i be the amount of weight sent from v_i to v . Let j be the largest integer for which $1 + \sum_{i=1}^j w_i \leq k$. Let $k' = k - 1 - \sum_{i=1}^j w_i$. Define T' to be the component of $T - vv_{j+1}$ containing v_{j+1} (notice that v_{j+1} exists because $k < w$).

The restriction of \mathcal{A} to T' , yields weight w_{j+1} on v_{j+1} , since weight does not move to v_{j+1} from v . Since $k' < k$, the induction hypothesis implies there is a total acquisition protocol \mathcal{A}' on T' placing weight k' on v_{j+1} .

Consider the following total acquisition protocol on T : first move weight w_i from v_i to v for $1 \leq i \leq j$, then run \mathcal{A}' on T' . Finally move the weight on v_{j+1} to v . This protocol puts weight k on v . □

We now bound the total acquisition number of a rooted acquisition tree when an edge incident to the root is deleted. Recall that $a_t(T) = 1$ for a rooted acquisition tree.

Lemma 3.4.2. *Let (T, r) be an n -vertex rooted acquisition tree and let $rv \in E(T)$. If T' is the component of $T - rv$ containing r , then $a(T') \leq 2\sqrt{n}$.*

Proof. Let the neighbors of r be v_1, \dots, v_k , with $v = v_q$, and let T_i be the subtree of $T - \{r\}$ that contains v_i . Choose the index such that $|V(T_i)|$ is in nondecreasing order. By the recursive definition of rooted acquisition tree, T_i is a rooted acquisition tree, so $a(T_i) = 1$ for all i .

Let t be the smallest index such that $|V(T_t)| > \sqrt{n}$, if such a t exists, and let $t = k + 1$ otherwise. Notice that $t \geq k - \sqrt{n}$.

Define a total acquisition protocol for T' as follows. Transfer all weight from $\bigcup_{i=1}^{q-1} T_i$ onto r . Next, transfer weight $|V(T_{i-1})|$ from T_i to r for $q + 1 \leq i \leq t - 1$; this is possible by Lemma 3.4.1. Finally, transfer all weight from T_i to v_i for $i \geq t$, leaving this weight on v_i . This protocol establishes the following bound on $a_t(T - e)$:

$$a_t(T - e) \leq 1 + \sum_{i=q+1}^t (|V(T_i)| - |V(T_{i-1})|) + \max\{k - t, 0\} \leq 1 + |V(T_{t-1})| + \sqrt{n} \leq 2\sqrt{n}.$$

□

We now extend the result of Lemma 3.4.2 to consider deleting an arbitrary edge in a rooted acquisition tree.

Lemma 3.4.3. *If (T, r) is an n -vertex rooted acquisition tree and $e \in E(T)$, then $a(T - e) \leq 1 + c\sqrt{n}$, where $c = \frac{2\sqrt{2}}{\sqrt{2}-1} < 6.84$.*

Proof. Let x_0 be the endpoint of e whose distance to r is greater. Let the vertices of the x_0, r -path in T be x_0, \dots, x_k , with $r = x_k$. Let $T' = T - \{x_{j-1}x_j : 1 \leq j \leq k\}$ and let T_i be the component of T' containing x_i (See Figure 3.3). Since T' is a spanning subgraph of $T - e$, we have $a_t(T') \geq a_t(T - e)$.

Define S_i to be the component of $T - x_i x_{i+1}$ containing x_i for $i \in \{0, \dots, k\}$ and let $S_k = T$. Following the recursive definition of rooted acquisition tree, we see that (S_i, x_i) is a rooted acquisition tree for all $i \in \{0, \dots, k\}$. Therefore $a_t(T_0) = 1$ and by Lemma 3.4.2, for $i \in [k]$,

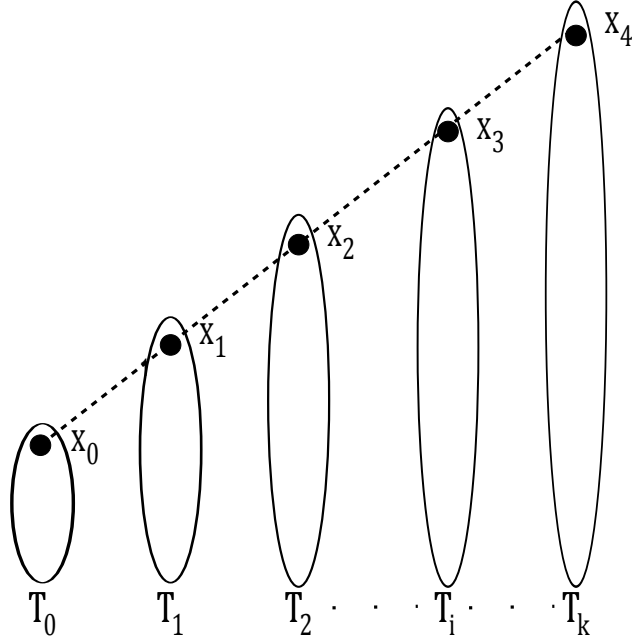


Figure 3.3: The subgraph T' of T .

all the weight from T_i can be moved to at most $2\sqrt{|V(S_i)|}$ vertices (T_i plays the role of T' in the statement of Lemma 3.4.2). Therefore

$$a_t(T') \leq 1 + \sum_{i=1}^k 2\sqrt{|V(S_i)|}. \quad (3.1)$$

Since (S_i, x_i) is a rooted acquisition tree for $i \in [k]$, we know that $|V(S_{i-1})| \leq \frac{1}{2}|V(S_i)|$ (in order for x_i to acquire the weight from S_{i-1}). It follows that $|V(S_i)| \leq n/2^{k-i}$, since $|V(S_k)| = n$. Thus we establish the following bound on $a_t(T)$:

$$a_t(T - e) \leq a_t(T) \leq 1 + \sum_{i=1}^k 2\sqrt{\frac{n}{2^{k-i}}} \leq 1 + 2\sqrt{n} \left(\frac{\sqrt{2}}{\sqrt{2}-1} \right).$$

□

We now prove a similar result for the deletion of an edge in an arbitrary graph.

Theorem 3.4.4. *If G is an n -vertex graph and $e \in E(G)$, then $a_t(G - e) \leq a_t(G) + 6.84\sqrt{n}$.*

Proof. If $a_t(G) = k$, then G contains k acquisition trees T_1, \dots, T_k that together span $V(G)$. If e does not belong to any of these trees, then $a_t(G - e) = a_t(G)$. If $e \in E(T_i)$, then $a_t(T_i - e) \leq 6.84\sqrt{n} + 1$ from Lemma 3.4.3, and $a_t(T_j) = 1$ for $j \neq i$. \square

We now provide a construction of an n -vertex graph in which the deletion of a particular edge increases the total acquisition number by $\sqrt{n}/2$.

Theorem 3.4.5. *For each positive integer n , there exists an n -vertex rooted acquisition tree T and an edge $e \in E(T)$ such that $a_t(T - e) \geq \sqrt{n}/2$.*

Proof. Let $\ell = \lceil \lg \sqrt{n} \rceil$ and let $m = \lceil n/2^\ell \rceil - 1$. We construct a tree T of diameter 4 with central vertex r . Let the neighbors of r be $v_1, \dots, v_{\ell+m}$. Let v_i have degree 2^{i-1} for all $1 \leq i \leq \ell$, and let each v_i with $i > \ell$ have degree 2^ℓ or $2^\ell - 1$ in such a way that T has n vertices. Running the r -greedy protocol on T transfers all weight to r , so $a_t(T) = 1$.

Let $e = rv_1$; we show that $a_t(T - e) \geq \sqrt{n}/2$. Let \mathcal{A} be an optimal total acquisition protocol on $T - e$. If no weight moves from v_i to r for $i > \ell$, then $a_t(T - e) \geq m + 1$, since v_1 is isolated in $T - e$. Since $m = \lceil n/2^\ell \rceil - 1$ and $\ell = \lceil \lg \sqrt{n} \rceil$, we conclude that $a_t(T - e) \geq \sqrt{n}/2$.

If \mathcal{A} transfers weight from v_i to r for some $i > \ell$, then let v_q be the first such vertex. Before r receives weight from v_q , at most $\ell - 1$ vertices have sent weight to r , so the weight on r is at most $2^{\ell-1}$. Therefore v_q can only send weight $2^{\ell-1}$ to r . Since $d(v_q) \geq 2^\ell - 1$, at least $2^{\ell-1} - 1$ leaf neighbors of v_q retain their weight in \mathcal{A} . Because v_1 is isolated in $T - e$, weight remains on at least $2^{\ell-1}$ vertices. Hence $a_t(T - e) \geq \sqrt{n}/2$. \square

We now shift our focus to products of graphs. Let $G \square H$ and $G \boxtimes H$ denote the Cartesian product and strong product of G and H , respectively.

Proposition 3.4.6. *For all graphs G and H , $a_t(G \boxtimes H) \leq a_t(G \square H) \leq a_t(G)a_t(H)$.*

Proof. Note that $a_t(G \boxtimes H) \leq a_t(G \square H)$, since $G \square H \subseteq G \boxtimes H$.

To show that $a_t(G \square H) \leq a_t(G)a_t(H)$, run the same optimal total acquisition protocol in each copy of G , so that all the weight in $G \square H$ lies in $a_t(G)$ copies of H and the weight on each vertex of a copy of H is the same. In the $a_t(G)$ copies of H with positive weight, run an optimal acquisition protocol for H . This leaves positive weight on exactly $a_t(G)a_t(H)$ vertices, yielding the desired bound. \square

The bounds in Proposition 3.4.6 can be arbitrarily loose, even for connected graphs. Let G_m be the graph with $3m$ vertices obtained from a path P_{2m} with vertices v_1, \dots, v_{2m} in order by giving each odd indexed vertex a leaf neighbor.

Proposition 3.4.7. *If k is a positive integer, then*

$$a_t(G_{4k} \square K_2) \leq 3k = \frac{3}{4}a_t(G_{4k})a_t(K_2)$$

and

$$a_t(G_{2k} \boxtimes K_2) \leq k = \frac{1}{2}a_t(G_{2k})a_t(K_2).$$

Proof. First note that G_{4k} contains $4k$ leaves pairwise at distance at least 3, and the weight from no two of these leaves can reach the same vertex (a formal proof of this fact is provided in Lemma 4.2.1). Thus $a_t(G_{4k}) \geq 4k$. The neighbors of the leaves in G_m form a dominating set, so $a_t(G_{4k}) \leq 4k$ by Observation 3.1.1. Therefore $a_t(G_{4k}) = 4k$. Since $a_t(K_2) = 1$, we have $a_t(G_{4k})a_t(K_2) = 4k$.

To see that $a_t(G_{4k} \square K_2) \leq 3k$, delete $k - 1$ edges from each copy of G_{4k} to get k copies of $G_4 \square K_2$. Figure 3.4 shows three groups of vertices in $G_4 \square K_2$, each of which induces a graph with total acquisition number 1 (the copies of K_2 are not pictured). Therefore $a_t(G_{4k} \square K_2) \leq ka_t(G_4 \square K_2) \leq 3k$.

To see that $a_t(G_{2k} \boxtimes K_2) \leq k$, notice that $k(G_2 \boxtimes K_2)$ is a spanning subgraph of $G_{2k} \boxtimes K_2$. It is straightforward to see that $a_t(G_2 \boxtimes K_2) = 1$, and therefore $a_t(G_{2k} \boxtimes K_2) \leq k$. \square

We know of only finitely many G and H such that $a_t(G \boxtimes H) < \frac{1}{2}a_t(G)a_t(H)$. For example,

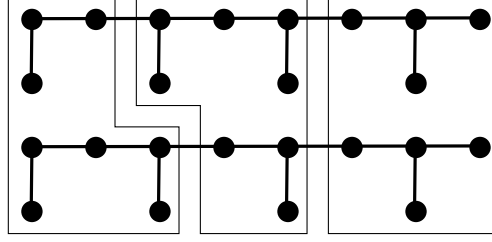


Figure 3.4: $a_t(G_{4k} \square K_2) \leq 3$

$a_t(C_5 \boxtimes C_5) = 1$, while $a_t(C_5)a_t(C_5) = 4$. For both the Cartesian product and the strong product, we do not know how small a_t can be as a function of $a_t(G)$ and $a_t(H)$.

3.5 Diameter 2

Intuitively, graphs with diameter 2 should have small acquisition numbers, since it is easier to move weight smaller distances. Furthermore, the condition that any two nonadjacent vertices have a common neighbor was a recurring theme in the results in Section 3.3.

Because every graph with diameter 2 has a spanning tree with diameter at most 4, it is natural to apply Lemma 3.2.6 to these graphs. Indeed, an immediate corollary of Lemma 3.2.6 is that if G is a graph with diameter 2, $u \in V(G)$, and $d = \max_{v \in N(u)} (N(v) - N[u])$, then $a_t(G) \leq d \lceil \lg(d) \rceil$. However, we conjecture a much stronger upper bound on the total acquisition number of graphs with diameter 2.

Conjecture 3.5.1. *There is an absolute constant c such that, for all graphs G with diameter 2, $a_t(G) \leq c$.*

In fact, we know of no graph with diameter 2 having total acquisition number more than 2. We have computed $a_t(G)$ for various graphs G with diameter 2. The only nontrivial Cartesian products with diameter 2 are Cartesian products of two complete graphs; $a_t(K_r \square K_s) = 1$. Also, $a_t(C_5 \boxtimes C_5) = 1$, and higher powers still have diameter 2 and total acquisition number 1.

Very few graphs have diameter 2 and girth at least 5; these are called *Moore graphs*. They

are regular and exist only for degrees 2, 3, 7, and possibly 57 [16]. With degree less than 7, only C_5 and the Petersen graph arise, and they have total acquisition number 2 (the proof that the total acquisition number of the Petersen graph is 2 is deferred to Theorem 4.3.4).

Another family of graphs with diameter 2 and no 4-cycle are the *polarity graphs*, which arise from projective planes (see Chapter 13 of [29]). In Theorem 3.5.5, we will prove that $a_t(G) = 1$ when $\Delta(G) \geq 7$ and G has diameter 2 and no 4-cycle. There is exactly one polarity graph with each maximum degree 3, 4, 5, and 6; a manual check shows that each has total acquisition number at most 2. Therefore Conjecture 3.5.1 holds with $c = 2$ for Moore graphs and polarity graphs.

In the remainder of this section, we prove stronger upper bounds on the total acquisition number of graphs with diameter 2 than the result of Lemma 3.2.6. These include a logarithmic upper bound in terms of the number of vertices and the full strength of the conjecture for graphs having large maximum degree and no 4-cycles. We begin with the following immediate consequence of Lemma 3.2.6.

Lemma 3.5.2. *Let T be a tree with diameter 4 and let u be the center of T . If d is the maximum degree of a vertex in $N(u)$, then there is a total acquisition protocol on T that moves all but at most $d \lceil \lg(d) \rceil$ weight to u .*

To prove our bound for graphs with diameter 2 it is necessary to present another protocol, which we will use in conjunction with the u -greedy protocol.

Lemma 3.5.3. *Let T be a tree with diameter 4 and let u be the center of T . Let $d_T(u) = d \geq 256$ and let $R = V(T) - N[u]$. If $|R| \leq d \lceil \lg d \rceil$, then $a_t(T) \leq 10 \lg d \lg \lg d$. Also, there is a protocol achieving this bound under which u acquires weight at least $d - 4 \lg d$.*

Proof. Let N_1 be the set of neighbors of u with degree less than $4 \lg d$, let N_2 be the set of neighbors of u with degree at least $4 \lg d$ and less than $d/4$, and let N_3 be the set of neighbors of u with degree at least $d/4$ (See Figure 3.5). For $i \in \{1, 2\}$, let T_i be the subtree of T with vertex set $\bigcup_{x \in N_i} N[x]$. We will apply the u -greedy protocol to T_1 and then show that this gives u enough weight to acquire all of the weight in T_2 . Finally, we transfer all weight from $\bigcup_{x \in N_3} N(x) - \{u\}$ to N_3 .

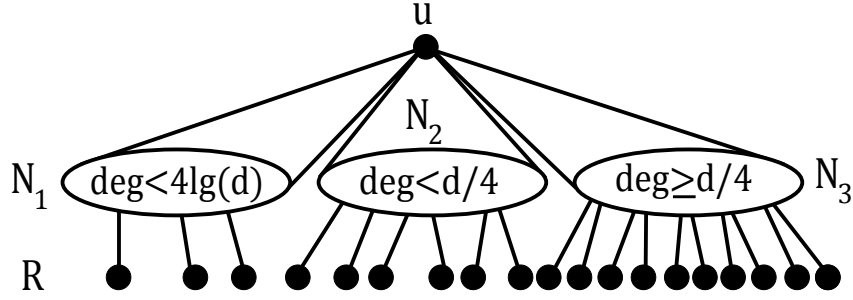


Figure 3.5: The partition of $N(u)$ for Lemma 3.5.3.

We have $|N_1| \geq d/4$, since otherwise $|R| > \frac{3d}{4}4 \lg d - d > d \lceil \lg d \rceil \geq |R|$. By Lemma 3.5.2 there is a total acquisition protocol on T_1 that moves weight from all but $4 \lg d \lceil \lg(4 \lg d) \rceil$ vertices in T_1 to u . Note that $4 \lg d \lceil \lg(4 \lg d) \rceil \leq 8 \lg d + 4 \lg d \lceil \lg \lg d \rceil \leq 8 \lg d \lg \lg d$, since $d \geq 256$. Also, u now has weight at least $|N_1|$. Since $|N_1| \geq d(x)$ for $x \in N_2$, we can transfer all the weight in T_2 to u . Finally, transfer all weight from $\bigcup_{x \in N_3} N(x) - u$ to N_3 .

Since $|R| \leq d \lceil \lg d \rceil$, we have $|N_3| \leq |R|/(d/4) \leq 4 \lceil \lg d \rceil < \frac{3}{2} \lg d \lg \lg d$. Therefore,

$$a_t(T) \leq 1 + 8 \lg d \lg \lg d + |N_3| \leq 10 \lg d \lg \lg d.$$

Also, the weight on u is at least $|N_1| + |N_2|$, which is at least $d - 4 \lg d$. □

We now prove our bound for graphs with diameter 2. The proof consists of applying the u -greedy protocol to a subgraph of G , where u is a vertex of degree $\Delta(G)$. We then apply Lemma 3.5.3 to a subtree where the leaves are the vertices with weight 1 after the application of the u -greedy protocol. The combination of the two protocols yields a much stronger bound.

Theorem 3.5.4. *If G is an n -vertex graph with $\text{diam}(G) = 2$, then $a_t(G) \leq 32 \lg n \lg \lg n$.*

Proof. For the main case, we will need $n > 256^2$. For $n \leq 256^2$, we apply Theorem 3.2.7. We obtain a spanning tree of diameter 4 in G showing that $a_t(G) \leq \sqrt{n \lg n}$, and therefore

$$a_t(G)/\lg n \lg \lg n \leq 16.$$

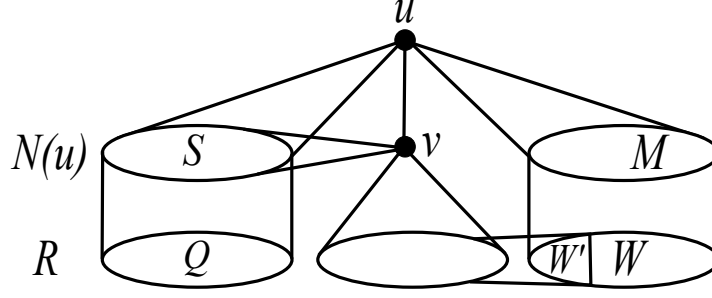


Figure 3.6: The structure of the diameter 2 graph G in Theorem 3.5.4.

Henceforth we assume that $n > 256^2$. Let u be a vertex of maximum degree in G , and let $d = d_G(u)$. Since G has diameter 2, we have $d \geq \sqrt{n-1} = 256$. Let $N(u) = \{v_1, \dots, v_d\}$ and let $R = V(G) - N[u]$.

Among all vertices in $N(u)$, let v be one with the most neighbors in R . Let $r = |N(v) \cap R|$. If $r \leq 256$, then G has a spanning tree T with diameter 4 centered at u such that $\max_{x \in N_T(v)} d_T(x) \leq 256$. Lemma 3.5.2 then implies that $a_t(G) \leq 256 \lceil \lg 256 \rceil$, yielding the desired bound. Thus we may assume that v has more than 256 neighbors in R . Let $S = N(u) \cap N(v)$ and $M = N(u) - S - v$. Let W be the subset of R consisting of vertices with no neighbor in $S \cup \{v\}$ (See Figure 3.6).

Let $k = \max_{x \in M} |N(x) \cap W|$. Lemma 3.5.2 applied to the graph $G[\{u\} \cup M \cup W]$ implies that there is a protocol that leaves weight on at most $k \lceil \lg k \rceil$ vertices in W and transfers the rest of the weight from $M \cup W$ to u . Let W' be the set of vertices in W that retain positive weight.

Since G has diameter 2 and no edges join W' to $S \cup \{v\}$, every vertex in W' has a neighbor in $N(v) \cap R$. Let $H = G[W' \cup N[v] - S - \{u\}]$. Note that $d_H(v) = r$, and thus $d_H(v) > 256$; we will apply Lemma 3.5.3 to a suitable spanning tree of H . Breadth-first search in H starting at v yields a spanning tree T' of diameter 4 with v as its root, such that $d_{T'}(v) = r \geq k$ and $|V(H) - N_H(v)| \leq k \lceil \lg k \rceil$. Because $k \leq r$, Lemma 3.5.3 implies $a_t(H) \leq 10 \lg r \lg \lg r$, and there is a protocol achieving this bound such that the weight at v is m , where $m \geq r - 4 \lg r$.

Let $Q = R - W - N(v)$. The only remaining vertices outside of $V(H) \cup \{u\}$ with positive weight are in $S \cup Q$. Among vertices in S , let y be one with the most neighbors in Q . If $m > |Q \cap N(y)|$, then the weight on v is greater than $|Q \cap N(v_i)|$ for all $v_i \in S$, and we are able to

transfer all weight from $S \cup Q$ to v . By then transferring weight along the edge uv , the weight is consolidated on at most $1 + 10 \lg d \lg \lg d$ vertices.

If $m \leq |Q \cap N(y)|$, then transfer weight from $m - 1$ vertices in Q to y and then move weight m from y to v . After this move, the weight on v is $2m$, and $2m \geq 2r - 8 \lg r$. Since $r \geq 256$, we have $2m > r$, so the weight now on v exceeds $|N(v_i) \cap Q|$ for each $v_i \in S$. Thus there is positive weight on at most $d - m$ vertices in $S \cup Q$, and $r - m \leq 4 \lg r$.

Thus all of the weight in G lies on u, v , at most $10 \lg d \lg \lg d$ vertices in H , and at most $4 \lg r$ vertices in Q . Since $r \leq d$

$$a_t(G) \leq 2 + 10 \lg d \lg \lg d + 4 \lg d \leq 12 \lg n \lg \lg n.$$

In all cases, $a_t(G) \leq 32 \lg(n) \lg \lg(n)$.

□

Finally, we obtain the desired bound when G has no 4-cycles and has a vertex with degree at least 7.

Theorem 3.5.5. *Let G be a graph such that $\text{diam}(G) = 2$ and G contains no 4-cycles. If $\Delta(G) \geq 7$, then $a_t(G) = 1$.*

Proof. If G has a dominating vertex, then the result is trivial. Let $q = \Delta(G)$; we may assume that $q < n - 1$.

As remarked by Bondy, Erdős, and Fajtlowicz in [5], G either has a dominating vertex or $\delta(G) \geq q - 1$. Let v be a vertex of maximum degree in G , and let $N(v) = \{v_1, \dots, v_q\}$. Observe that $N(v_i) \cap N(v_j) = \{v\}$ for all i and j , otherwise G contains C_4 . Also, if v_i is not adjacent to v_j and $x \in N(v_i) - v$, then the common neighbor of x and v_j is not v_i . Hence every $x \in N(v_i) - v$ has exactly one neighbor in $N(v_j) - v$. By symmetry, there is a perfect matching between $N(v_i) - v$ and $N(v_j) - v$.

Let $k = \lceil \lg q \rceil + 1$, and let $S = \{v_1, \dots, v_k\}$. Because $q \geq 7$, we have $2 \lceil \lg q \rceil + 1 \leq q$. Each vertex in S has at most one neighbor in $N(v)$, so each has at least 2 nonneighbors in $N(v) - S$

(there are at least three vertices in $N(v) - S$). Without loss of generality, assume that v_i and v_{i+k} are not adjacent for $i \in [k - 1]$ and that v_1 and v_{k+3} are not adjacent.

Run the v -greedy protocol on the subgraph induced by $\{v\} \cup S \cup (N(S) \cap N^2(v))$. Because $\delta(G) \geq q - 1$, after the protocol is finished, the weight on v is at least $2q - 1$ and there is no weight on $N(v_k) - N[v]$. Let x be a vertex in $N(v_1) - \{v\}$, and transfer the weight from $N(v_1) - \{v, x\}$ to $N(v_{k+1}) - \{v\}$ via the perfect matching. All of this weight can then be consolidated on v_{k+1} . Now, move all weight still on $N(v_i) - N[v]$ to $N(v_{i+k}) - N[v]$, $i \in \{2, \dots, k - 1\}$, using the perfect matchings between the sets. Also, move the weight from x to $N(v_{k+3})$ (this is possible because $v_1 v_{k+3} \notin E(G)$). It is then possible to consolidate all of this weight on T , and no vertex in T has weight greater than $2q - 1$. Now transfer the rest of the weight on $V(G) - \{v\}$ to v . Thus $a_t(G) = 1$. \square

Finally, connected P_4 -free graphs have diameter 2, and by Corollary 3.3.2 they have acquisition number 1.

Chapter 4

Unit Acquisition in Graphs

4.1 Introduction

Lampert and Slater referred to the movement of a positive integer amount of weight from a vertex to a neighbor with at least as much weight as a *consolidation* [18]. It is easy to see that any consolidation transferring an integer amount m of weight from u to v can be achieved by m consecutive moves that each transfer a single unit from u to v ; hence it suffices to use only moves transferring unit weight. A *unit acquisition move* on a weighted graph G transfers a unit of weight from a vertex to a neighbor with at least as much weight. For brevity we refer to a sequence of unit acquisition moves as a *unit protocol*. The *unit acquisition number of G* , denoted $a_u(G)$, is the minimum size of the set of vertices with positive weight at the end of a unit acquisition protocol when each vertex initially has weight 1.

In Section 4.2, we prove several basic results about unit protocols, and we then establish various upper bounds for the unit acquisition number of a graph. We also consider lower bounds that depend on the maximum degree of a graph. Lampert and Slater [18] proved that such a bound exists for total acquisition, using the fact that the weight on a vertex at most doubles each time weight moves to it. We prove that no similar bound exists for the unit acquisition number of a graph by constructing an infinite family of trees having maximum degree 5 and unit acquisition number 1.

In Section 4.3, we consider the unit acquisition number of graphs with diameter 2. We prove that the unit acquisition number satisfies the bound conjectured for total acquisition in Conjecture 3.5.1: If G has diameter 2, then $a_u(G) \leq 2$. Furthermore, the only graphs for which equality

holds are C_5 and the Petersen graph.

Each unit acquisition move is determined by two adjacent vertices and a direction. We denote the allowed movement of one unit from u to v by $u \rightarrow v$, and $(u \rightarrow v)^m$ denotes m such moves in succession. For ease of exposition, we refer to a unit of weight as a “chip.” When a chip is at vertex u , we define the *height* of the chip to be the weight on u when that chip arrived at u . It will be useful to consider the weight at each vertex to be a stack of chips, and we assume that if a chip moves from u to v , then the transferred chip is taken from the top of the stack at u and placed on the top of the stack at v . Therefore, the height of a chip during a unit protocol is nondecreasing.

In this chapter and Chapter 5, we will rely on the unique path that joins a pair of vertices in a tree. Therefore we introduce the following notation: If u and v are vertices in a tree T , let $T(u, v)$ denote the unique path joining u and v in T .

In particular, we are interested in paths with weight assignments that allow all of the weight on the path to be acquired by a single vertex. An *ascending path* is a path $\langle v_1, v_2, \dots, v_n \rangle$ with a weight assignment w such that $w(v_1) \leq w(v_2)$ and $w(v_i) < w(v_{i+1})$ for $i \in \{2, 3, \dots, n-1\}$. When it is convenient, we will say that such a path P *ascends to* v_n , or that P is v_n -*ascending*. Also, P is *strictly ascending* if $w(v_1) < w(v_2)$. A weighted tree T is *ascending* or is an *ascending tree* if there is a vertex $v \in V(T)$ such that for every vertex u in the tree, $T(u, v)$ is v -ascending.

Observation 4.1.1. *Let G be a weighted graph, and let v be a vertex in G . If for every vertex $u \in V(G)$ there is a u, v -path that [strictly] ascends to v , then there is a spanning tree T of G for which the weight assignment is [strictly] ascending.*

We will frequently use a protocol that moves weight along an ascending path.

Definition 4.1.2. Let P be an n -vertex path with a weight assignment w that ascends from v_1 to v_n . Let c be the top chip on the stack at v_1 . Define the *path protocol*, denoted $\mathcal{A}(v_1, v_n)$, as follows. Transfer c from v_1 to v_n while moving no other chips; let step i in the protocol move c from v_i to v_{i+1} . After the i th step, the new weight on v_{i+1} is $w(v_{i+1}) + 1$; since $w(v_{i+2}) \geq w(v_{i+1}) + 1$, the protocol can continue. On step $n - 1$, the chip reaches v_n . We denote k repeated applications of

the path protocol $\mathcal{A}(v_1, v_n)$ by $\mathcal{A}(v_1, v_n)^k$.

Using this terminology, we prove a useful lemma.

Lemma 4.1.3. *If a tree T has a weight assignment w that ascends to a vertex r , then $a_u(T) = 1$ and r can acquire all of the weight in T .*

Proof. We use induction on the total of the weight not on r . If the total is 0, then all of the weight is on r . Otherwise, use the path protocol to move one chip to r from a vertex of the tree having positive weight that is farthest from r , and then apply the induction hypothesis. \square

We will often apply Lemma 4.1.3 when T is a path in a graph to acquire all of the weight on the path.

Recall that a feasible weight assignment is a weight assignment that can be achieved from the assignment $\mathbf{1}$ using permissible moves (in this case unit acquisition moves). Lemma 4.1.3 and Observation 4.1.1 yield a useful corollary for general graphs. We say that a weight assignment on a graph G *ascends* to a vertex v if the vertices of G with positive weight are spanned by a subtree that ascends to v .

Corollary 4.1.4. *If G is a graph and there is a feasible weight assignment on a graph G that ascends to a vertex v , then $a_u(G) = 1$.*

4.2 Initial Results

We have shown that the edges of a total acquisition protocol in a graph G form a forest contained in G . This is not true in general for a unit acquisition protocol, and we provide a graph in which optimal unit acquisition protocols use edge sets that contain cycles.

First we prove a general lemma that provides lower bounds for $a_u(G)$.

Lemma 4.2.1. *Let G be a graph, and let u and v be two vertices in G . If S is a minimal u, v -cut in G such that every vertex in S has degree 2 and u and v have no neighbors in S , then the chips from u and v cannot reach the same vertex via unit acquisition moves.*

Proof. Because S is a minimal u, v -cut in which every vertex has degree 2, no edges are induced by S . Let x be a vertex in S and consider the first unit acquisition move that uses an edge incident x . This move transfers exactly one chip, and subsequently either x or one of its neighbors has weight 0. In both cases, one of the edges incident to x has an endpoint with weight 0, and no subsequent unit acquisition moves can use that edge. Also, the chip that is transferred is the chip from x or the neighbor of x ; hence it is not the chip from u or v . Thus the first move involving an edge incident to a vertex in S specifies an edge that cannot be used again, and the set of all such edges is a u, v -edge-cut. Because the chips from u and v cannot be transported across any of the edges in the u, v -edge-cut, it is not possible for the chips to reach the same vertex. \square

We prove a similar lemma for trees.

Lemma 4.2.2. *If a tree T has a vertex v that is not a leaf and is adjacent to no leaves, then $a_u(T) \neq 1$.*

Proof. The first acquisition move involving v leaves either v or a neighbor of v with weight 0. Since these vertices are all cut vertices, it follows that $a_u(T) \geq 2$. \square

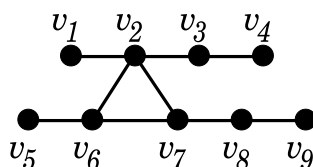


Figure 4.1: A graph in which optimal unit protocols use a cyclic set of edges

Proposition 4.2.3. *If G is the graph shown in Figure 4.1, then $a_u(G) = 1$. If e is an edge in the triangle in G , then $a_u(G - e) = 2$.*

Proof. The following unit protocol proves that $a_u(G) = 1$. First perform $v_1 \rightarrow v_2$, $v_4 \rightarrow v_3$, $(v_3 \rightarrow v_2)^2$, $v_5 \rightarrow v_6$, $v_6 \rightarrow v_2$, and $v_6 \rightarrow v_7$. At this point, the weight assignment ascends to v_2 and Corollary 4.1.4.

We now show that $a_u(G - e) \neq 1$ if $e \in \{v_2v_6, v_2v_7, v_6v_7\}$. If we delete any of those three edges, then $G - e$ is a tree in which v_7 is a non-leaf that is adjacent to no leaves. Therefore $a_u(G - e) \geq 2$ by Lemma 4.2.2. \square

We now establish the unit acquisition numbers of paths and cycles. Interestingly, they are the same as the total acquisition numbers.

Proposition 4.2.4. *If P_n is the path on n vertices, then $a_u(P_n) = \lceil n/4 \rceil$.*

Proof. If $n \leq 4$, then $a_u(P_n) = 1$. Split $V(P_n)$ into $\lceil n/4 \rceil - 1$ paths on four vertices and one path on at most four vertices. Each of these paths satisfy $a_u(G) = 1$, so $a_u(P_n) \leq \lceil n/4 \rceil$.

To prove the lower bound, assume $n \geq 5$, and let v_1, v_2, v_3, v_4 , and v_5 be five (not necessarily consecutive) vertices in P_n , indexed in order. The vertex v_3 has degree 2, is not adjacent to v_1 or v_5 , and is a minimal v_1, v_5 -cut. By Lemma 4.2.1, the chips from v_1 and v_5 cannot reach the same vertex. Therefore, no five chips in P_n can reach the same vertex, which implies that $a_u(P_n) \geq \lceil n/4 \rceil$. \square

Having established the unit acquisition number for paths, the unit acquisition number for cycles follows immediately.

Corollary 4.2.5. *If C_n is the cycle on n vertices, then $a_u(C) = \lceil n/4 \rceil$.*

Proof. Label the vertices of C_n in order so that $C_n = [v_1, \dots, v_n]$. Consider the first unit acquisition move in any protocol performed on C_n . By symmetry, we may assume that the move transfers a chip from v_1 to v_2 . After this move, it is impossible for any chip to move along the edge v_1v_n , and we did not use this edge on the first move. Therefore, the protocol (including the first move) is equivalent to a protocol on $C_n - v_1v_n$, which is isomorphic to P_n . By Proposition 4.2.4, it follows that $a_u(C_n) = \lceil n/4 \rceil$. \square

The path and cycle are not the only graphs for which the total acquisition and unit acquisition numbers are equal. Lemma 4.2.1 implies that the graph $T(\ell, m)$ from Theorem 3.2.5 with maximum degree $\ell + 2$ and diameter $2m + 4$ satisfies $a_u(T(\ell, m)) = a_t(T(\ell, m))$. Thus the maximum

unit acquisition number of an n -vertex graph is $\lfloor (n+1)/3 \rfloor$, and this is achieved on trees with any even diameter from 6 to $\frac{2}{3}(n+1)$ or any maximum degree at least 3.

In contrast to total acquisition for trees with diameter 4, it is much easier to prove a sharp upper bound for the unit acquisition number on such trees.

Proposition 4.2.6. *If T is an n -vertex tree with diameter 4, then $a_u(T) \leq \lfloor \sqrt{n-1} \rfloor$, and this is sharp.*

Proof. Let v be the central vertex of T , and let $\{v_1, \dots, v_k\}$ be the neighbors of v labeled in nonincreasing order of degree. If $k \leq \lfloor \sqrt{n-1} \rfloor$, transfer all of the weight in T to $N(v)$. If $k > \lfloor \sqrt{n-1} \rfloor$, then $d(v_1) \leq \lfloor \sqrt{n-1} \rfloor$. In this case, transfer the chip from v_1 to v , and all of the vertices except the children of v_1 are in a v -ascending tree.

Let T be the n -vertex tree in which $k = \lfloor \sqrt{n-1} \rfloor$, and the degree of each v_i is at least $\lfloor \sqrt{n-1} \rfloor$. Every optimal protocol must move weight along an edge incident to v . If the first move involving v takes weight off v , then weight remains on at least k components of $T - v$. Otherwise, the chip comes from some v_i , and weight remains on at least $d(v_i)$ components of $T - v_i$. Thus the bound is sharp. \square

We now establish an upper bound on the unit acquisition number of general graphs.

Proposition 4.2.7. *If G is a connected graph, then $a_u(G)$ is at most the size of the smallest maximal induced matching in G , and this bound is sharp on an infinite family of graphs. Thus $a_u(G) \leq |V(G)|/(\delta(G) + 1)$.*

Proof. If M is a the smallest maximal induced matching in G , then $V(G)$ can be partitioned into $|M|$ sets spanned by trees with diameter at most 3, each of which has unit acquisition number 1. Thus $a_u(G) \leq |M|$. Since the number of vertices adjacent to vertices of an edge in a maximal induced matching is at least $\delta(G) + 1$, the smallest such M satisfies $|M| \leq |V(G)|/(\delta(G) + 1)$. The resulting bound is sharp for complete graphs.

For each M there are connected graphs for which the bound in terms of $|M|$ is optimal. To build G_m , We begin with m copies of P_4 and label the vertices on the i th copy $v_{i,1}, \dots, v_{i,4}$ in

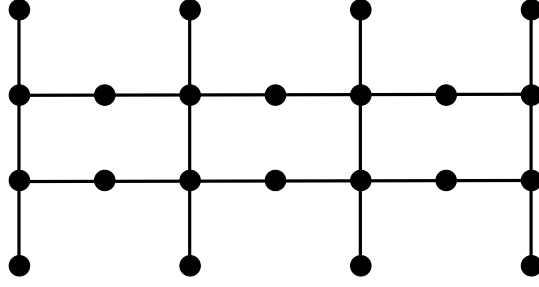


Figure 4.2: The graph G_4 .

order. For $i \in [m - 1]$, join $v_{i,2}$ to $v_{i+1,2}$ and $v_{i,3}$ to $v_{i+1,3}$ via paths of length 2. As an example, G_4 is shown in Figure 4.2. The edges $\{v_{i,2}v_{i,3} : 1 \leq i \leq m\}$ form a maximal induced matching.

To show that $a_u(G_m) = m$, consider $v_{i,1}$ and $v_{j,1}$ with $i < j$. The internal vertices on the paths of length 2 joining $v_{i,2}$ to $v_{i+1,2}$ and $v_{i,3}$ to $v_{i+1,3}$ form a minimal $v_{i,1}, v_{j,1}$ -cut in which every vertex has degree 2, and $v_{i,1}$ and $v_{j,1}$ have no neighbors in S . By Lemma 4.2.1, the chips from $v_{i,1}$ and $v_{j,1}$ cannot reach the same vertex. Therefore $a_u(G_m) = m$, and the bound is sharp. \square

In the case of total acquisition, Lampert and Slater proved that $a_t(G) \geq n(G)/2^{\Delta(G)}$ using the fact that the maximum amount of weight that a vertex can acquire via total acquisition moves in G is $2^{\Delta(G)}$. We show that a similar result is not possible for unit acquisition by constructing an infinite family of trees with maximum degree 5 and unit acquisition number 1. Thus there is no upper bound on the amount of weight that can be acquired by a vertex of degree 5.

Theorem 4.2.8. *For every positive integer M , there is a tree T with $\Delta(T) = 5$ such that, starting from the distribution $\mathbf{1}$, a vertex v in G can acquire weight at least M via unit acquisition moves.*

Proof. Let M be a positive integer. We inductively construct a rooted tree T_M of depth M with root r and a feasible weight assignment w_M on T_M . The weight assignment w_M is strictly ascending to the root r , gives weight 1 to at least one vertex at depth M , and can be attained from the weight distribution $\mathbf{1}$ on T_M via unit acquisition moves. It follows that $a_u(T_M) = 1$, and the root r can acquire weight at least M . In fact, the root r can acquire weight at least $\binom{M+1}{2}$.

We will construct T_M to be a rooted $(5, 1)$ -tree (a tree in which every vertex has degree 5 or 1)

of depth M with root r such that r is adjacent to a leaf, which we call x . It will be a subtree of the rooted complete 4-ary tree of depth M with an extra leaf adjacent to the root r . Therefore, we define a labeling l on the vertex set of the complete 4-ary tree of depth M , which we then apply to T_M . Arbitrarily assign the children of a vertex in the complete 4-ary tree indices 1 through 4. The label of a vertex at depth i is the i -tuple in $[4]^i$ corresponding to the indices describing the path from r to v .

Given our labeling, we now provide our inductive construction of T_M and w_M , starting with a two-step base case. Let T_0 to be a single vertex with weight 1, and let T_1 be $K_{1,5}$ with $w_1(r) = 2$, $w_1(x) = 0$, and $w_1(v_{(j)}) = 1$ for $1 \leq j \leq 4$. Note that the weight assignment w_1 is feasible.

Now consider $M \geq 2$. By our induction hypothesis, there exists a tree T_{M-1} with a feasible weight assignment w_{M-1} that is strictly ascending and gives weight 1 to at least one vertex at depth $M - 1$. We call a vertex v in T_{M-1} *active* if $w_{M-1}(v) \geq 1$. Let b_i denote the number of active vertices in T_{M-1} at depth i . Note that $b_0 = 1$ in both T_0 and T_1 , and $b_1 = 4$ in T_1 .

To construct T_M for $M \geq 3$, give each active vertex at depth $M - 1$ in T_{M-1} four children. Note that we can perform the unit acquisition protocol that attains the weight distribution w_{M-1} on the subtree of T_M that is isomorphic to T_{M-1} . This protocol does not move any chip that is on a vertex at depth M , and therefore it produces an ascending weight distribution w'_M on T_M that is not strictly ascending. We present a unit acquisition protocol that transfers chips from the vertices at depth M to form a strictly ascending weight distribution on T_M . We will show that this distribution satisfies the requirements of our induction hypothesis.

From the weight distribution w'_M we will produce w_M by applying various path protocols. If the weight on every vertex that is the ancestor of a vertex at depth M that remains active increases by 1, then the resulting weight distribution becomes strictly ascending. Proceed through the active vertices in T_M from depth 0 to depth $(M - 1)$, and within the levels by the lexicographic order on their labels. Let v be an active vertex at depth i , and let $U(v)$ be the currently active vertices at depth M whose labels are lexicographically at least $(l(v), 1, \dots, 1)$. Let u be the vertex in $U(v)$ with the least label. If u is a descendant of v , then the u, v -path in T_M ascends to v , and we transfer

the chip from u to v with the path protocol $\mathcal{A}(u, v)$. If u is not a descendant of v , then we transfer the chip at u to the root r using the path protocol $\mathcal{A}(u, r)$. If u is not a descendant of v , then the weight on v does not increase. However, if u is not a descendant of v , then v has no active descendants at depth M (such a descendant would occur before u in the lexicographic order), and it is not necessary to increase the weight on v to attain a strictly increasing weight assignment. Halt if every active vertex with depth at most $M - 1$ has been processed or if there are no remaining vertices at depth M with positive weight. Let the resulting weight distribution on T_M be w_M

If there are active vertices at depth M when the process halts, then w_M is a feasible, strictly ascending weight distribution on T_M with $b_M > 0$. The total number of vertices at depth M in T_M that lose their chip is equal to the number of active vertices in T_{M-1} . Thus we establish the following recursive formula for b_M :

$$b_M = 4b_{M-1} - \left(\sum_{i=0}^{M-1} b_i \right); \quad \text{for } M \geq 3.$$

Equivalently $\sum_{i=0}^M b_i = 4a_{M-1}$ for $M > 2$. Thus the recursion simplifies to

$$b_M = 4b_{M-1} - 4b_{M-2}; \quad b_0 = 1, b_1 = 4, b_2 = 9.$$

By the characteristic equation method, the solution to the recurrence for $M \geq 3$ is

$$b_M = (3M + 5)2^{M-2}.$$

Therefore b_M is positive, which completes the induction step. □

It is worth noting that 5 is the smallest maximum degree for which this method works. When we restrict to maximum degree 1, 2, or 3, a straightforward case check shows that the maximum amount of weight that a vertex can acquire is 1, 4 or 10 respectively. When the maximum degree

in the graph is 4, then the corresponding recurrence is

$$b_n = 3(b_{n-1} - b_{n-2}); \quad b_0 = 1, b_1 = 3, b_2 = 5.$$

In this recurrence, $a_5 = -9$, indicating that we cannot grow T_d past depth 4. However, there is a fair amount of waste in this method, so with more careful analysis it could be shown that the bound does not exist for $k = 4$. However, we do conjecture that the maximum amount of weight attainable by a vertex in a graph with maximum degree 4 is 239.

4.3 Diameter 2

We will show that all graphs with diameter 2 have unit acquisition number at most 2. Furthermore, C_5 and the Petersen graph are the only graphs for which equality holds.

Let S be a set of vertices in a graph G , and let $v \in S$. A vertex u in $G - S$ is a *solo-neighbor* of v if $uv \in E(G)$ and u has no other neighbors in S .

Lemma 4.3.1. *Let G be a graph with diameter 2. If G contains a clique Q with at least two vertices and some vertex $u \in Q$ has no solo-neighbors, then $a_u(G) = 1$.*

Proof. Let u be a vertex in Q with no solo-neighbors, and let v be another vertex in Q . Because u has no solo-neighbors, whenever $y \notin Q$ there is a y, v -path of length at most 2 that does not contain u . Therefore, moving the chip from u to v yields a weight assignment on G that ascends to v , and it follows that $a_u(G) = 1$. □

Theorem 4.3.2. *If $\text{diam}(G) = 2$ and G is not C_5 or the Petersen graph, then $a_u(G) = 1$.*

Proof. Let G be a graph with diameter 2 that is not C_5 or the Petersen graph. If G is a tree, then it is a star and $a_u(G) = 1$; we assume that G is not a tree. We provide a unit acquisition protocol that produces an ascending weight assignment on G . We break our argument into three cases based on the girth of G ; note that a graph with diameter 2 has girth at most 5.

Case 1: G has girth 3. Let Q be a largest clique in G . Because G has girth 3, $|Q| \geq 3$. Fix three vertices in Q and call them v , u , and x . Let $N(Q)$ be the set of neighbors of Q that are not in Q , and let $N^2(Q)$ be the set of vertices that are distance 2 from all the vertices in Q . If $N^2(Q) = \emptyset$, then we can transfer all chips to Q , and then $a_u(G) = 1$. Assume that $N^2(Q)$ is nonempty. By Lemma 4.3.1, we may assume that each vertex in Q has a solo-neighbor.

Let v' be a solo-neighbor of v . Move the chip on v' to v . For each solo-neighbor u' of u , apply the path protocol on $\langle u', u, v \rangle$ to move the chip from u' to v . Finally, move the chip from u to x . Call the current weight assignment w . Note that $w(v) \geq 3$ and $w(x) = 2$.

If y is a vertex in Q , then clearly there is an ascending y, v -path in w . If y is in $N(Q)$ and has positive weight, then y is adjacent to a vertex in Q with positive weight (y is not a solo-neighbor of u), and there is an ascending y, v -path. If y is in $N^2(Q)$, then there is a path P of length 2 joining y and x . Since the vertices of $N(Q)$ with weight 0 in w are solo-neighbors of v or u , the internal vertex of P still has weight 1. Thus via x there is an ascending y, v -path in G , and w is a v -ascending weight assignment. Therefore $a_u(G) = 1$.

Case 2: G has girth 4. Let $[v_1, \dots, v_4]$ be a 4-cycle in G . The remaining vertices in G can be partitioned into seven sets: $V_1, \dots, V_4, V_{1,3}, V_{2,4}$, and N , where $V_i = N(v_i) - N(v_{i+2})$ for $i \in [4]$, $V_{i,i+2} = N(v_i) \cap N(v_{i+2}) - \{v_{i+1}, v_{i-1}\}$ for $i \in [2]$, and $N = N^2(\{v_1, \dots, v_4\})$, where arithmetic of subscripts is performed modulo 4 (see Figure 4.3). The only vertices in G that are not adjacent to v_1 or v_4 are those in V_2, V_3 and N . Since any vertex u in V_2 is distance 2 from v_4 and G does not contain 3-cycles, u must be adjacent to a vertex in V_4 . Similarly, if $u \in V_3$, then u has a neighbor in V_1 . Because a vertex in N is distance 2 from v_1 , it must have a neighbor in V_1 or $V_{1,3}$. Thus there is a path of length at most 2 in $G - \{v_2, v_3\}$ joining every to v_1 or v_4 . Therefore, if we move the chip from v_2 to v_1 and the chip from v_3 to v_4 , every vertex is in an ascending path ending at v_1 or v_4 . It follows that all the chips in G can be transferred to $\{v_1, v_4\}$, and then all the chips can be moved to a single vertex.

Case 3: G has girth 5. Since G has diameter 2 and girth 5, G is a Moore graph. Hoffman and Singleton [16] determined that Moore graphs with diameter 2 are regular of degree 2, 3, 7 or 57.

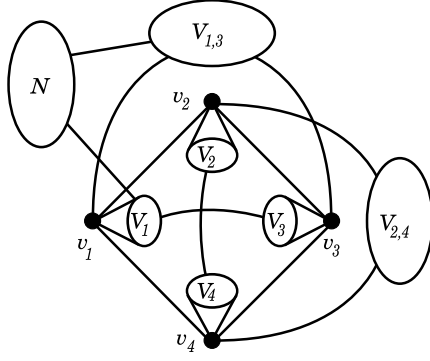


Figure 4.3: The structure of a diameter 2 graph with girth 4.

Furthermore, the only 2-regular Moore graph with diameter 2 is C_5 , and the only 3-regular Moore graph with diameter 2 is the Petersen graph. Therefore we know that G is a k -regular Moore graph with $k \in \{7, 57\}$. Theorem 3.5.5 implies that such graphs have total acquisition number 1. By the bound in Proposition 3.1.3, $a_u(G) = 1$. \square

Finally, we show that C_5 and the Petersen graph both have unit acquisition number 2. First we prove a generalization of Lemma 4.2.2.

Lemma 4.3.3. *If T is a weighted tree and v is a non-leaf with weight 1 that has no leaf neighbors with weight 1, then no unit protocol from this distribution can transfer all of the weight to a single vertex.*

Proof. Consider the first unit acquisition move involving v . If weight moves from v , then the weight at v is 0, and there is weight on multiple components of $T - v$. If weight moves onto v , then it came from a non-leaf neighbor v' of v , since all leaf neighbors of v have larger weight. Hence v' now has weight 0, and there is weight on multiple components of $T - v'$. \square

Theorem 4.3.4. *If G is C_5 or the Petersen graph, then $a_u(G) = 2$.*

Proof. We have already shown that $a_u(C_5) = 2$ in Proposition 4.2.4. The Petersen graph G can be decomposed into two 5-cycles and a matching. Starting by moving all weight onto one 5-cycle via the matching yields $a_u(G) \leq 2$.

Suppose that $a_u(G) = 1$. Given an optimal unit protocol \mathcal{A} , consider the set S of the last six vertices with positive weight. Since $a_u(G) = 1$, $G[S]$ is connected. We consider two cases: $G[S]$ is a tree or $G[S]$ contains a cycle. Note that any vertex of degree 3 in $G[S]$ has weight 1.

There are two 6-vertex trees in the Petersen graph: $K_{1,3}$ with two edges each subdivided once, and the double star containing two vertices of degree 3. By Lemma 4.3.3, every vertex of degree 3 must have an adjacent leaf with weight 1. There are three such weight assignments on the double star (see Figure 4.4). Only one of which has unit acquisition number 1, but it is not feasible.

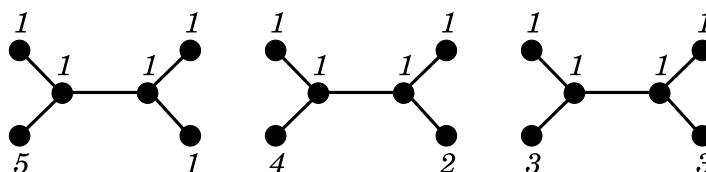


Figure 4.4: Weight assignments on the double star. The first is the only one satisfying $a_u(G) = 1$, but it is not feasible.

On $K_{1,3}$ with two subdivided edges there are ten weight assignments with total weight 10 not disqualified by Lemma 4.3.3 (Figure 4.5), five of which have unit acquisition number 1. However, the assignments with unit acquisition number 1 are not feasible.

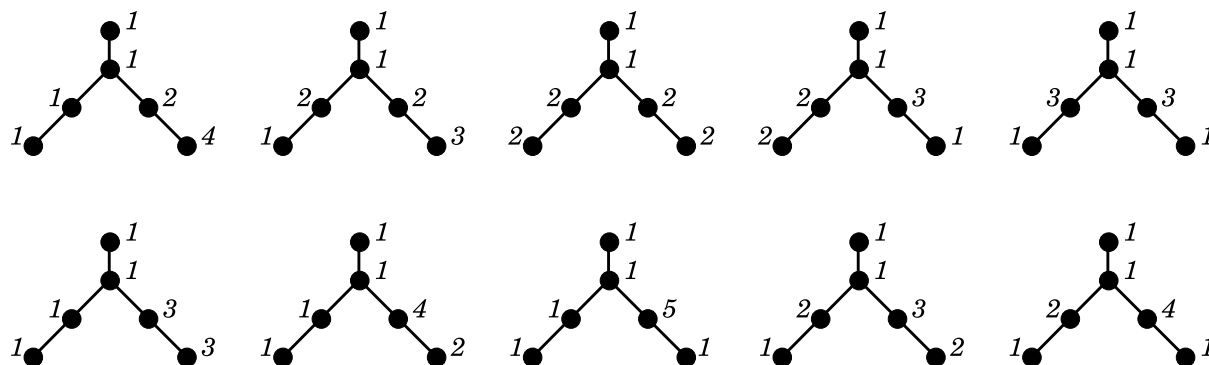


Figure 4.5: Weight assignments on the subdivided star not disqualified by Lemma 4.3.3. Only the bottom five have unit acquisition number 1, but they are not feasible.

We now consider the case when $G[S]$ contains a cycle. The only connected 6-vertex induced subgraphs of the Petersen graph containing a cycle are C_6 and the graph H consisting of C_5 with a pendant edge. It is not possible for all of the vertices of a 6-cycle to have positive weight while no

other vertices have positive weight. Therefore we only need to consider H . Every feasible weight assignment on H can be achieved from one of five weight assignments on the H (Figure 4.6). A straightforward check shows that the weight in none of these assignments can be moved to a single vertex by unit acquisition moves.

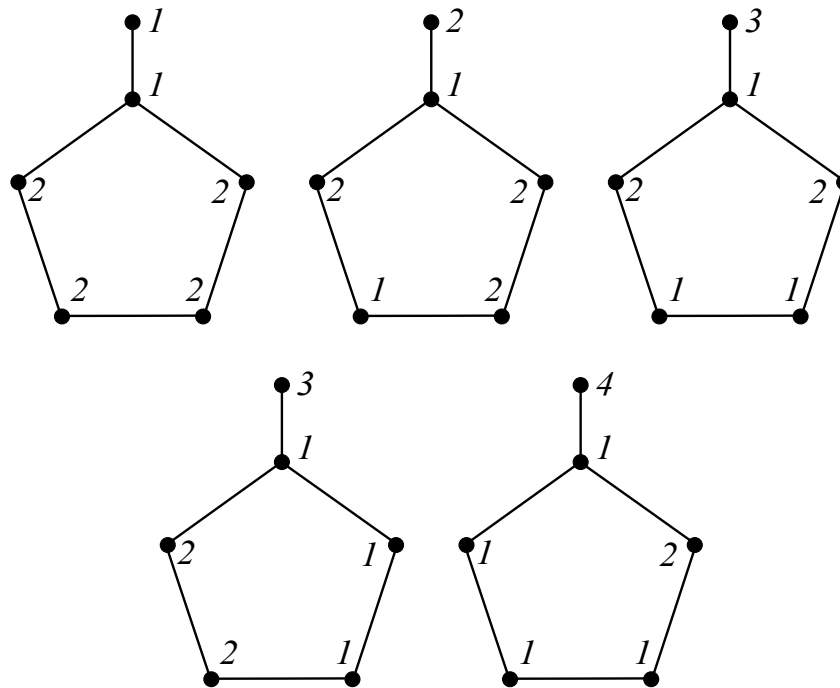


Figure 4.6: Feasible weight assignments on C_5 with a pendant.

Therefore the unit acquisition number of the Petersen graph is 2. □

Chapter 5

Fractional Acquisition in Graphs

5.1 Introduction

A *fractional acquisition move* moves any positive amount of weight from a vertex to a neighbor which has at least as much weight. For brevity we refer to a sequence of fractional acquisition moves as a *fractional protocol*. The *fractional acquisition number* of G , denoted $a_f(G)$, is the minimum number of vertices in G with positive weight after applying a fractional protocol.

In Section 5.2, we study fractional acquisition protocols on paths and cycles. We establish that the fractional acquisition number of the n -vertex path and n -vertex cycle is $\lceil n/4 \rceil$. Thus even with the freedom of fractional acquisition moves there is no protocol on paths or cycles that performs better than an optimal total acquisition protocol. However, we also prove that in the family of paths, the amount of weight that a single vertex can acquire is unbounded. This result is analogous to Theorem 4.2.8, however the degree sufficient to achieve unbounded weights is 2, not 5.

In Section 5.3, we prove our most surprising result: If G is a connected graph and $\Delta(G) \geq 3$, then $a_f(G) = 1$. We prove the result for trees using a modification of the fractional protocol that allows vertices in paths to acquire large amounts of weight. For arbitrary connected graphs with maximum degree at least 3 it is then sufficient to use the edge set of a spanning tree that contains a vertex of degree at least 3. Thus we have completely determined the fractional acquisition number of every graph.

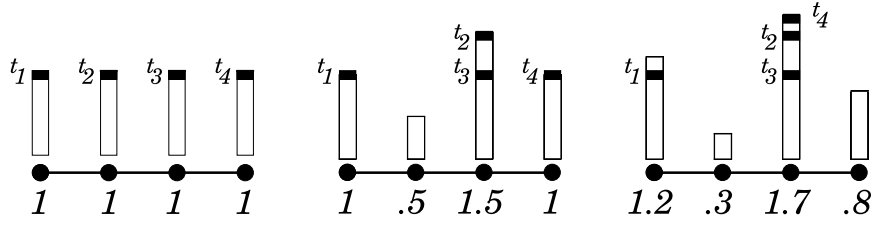


Figure 5.1: Two fractional acquisition moves performed on a graph, with the movement of the tips shown. The height of t_2 in the last two graphs is 1.5.

5.2 Paths and Cycles

We show that if G is an n -vertex path or cycle, then $a_f(G) = \lceil n/4 \rceil$; thus $a_f(G) = a_u(G) = a_t(G)$.

To prove this result, we use a model of fractional acquisition that is analogous to the moving of chips in unit acquisition. Model the weight on the vertex v as a vertical interval with length $w(v)$. When u acquires weight α from v , cut an interval of length α off of the top of the interval at v and attach it to the top of the interval at u . When all of the vertices have weight 1, define the top of each interval to be a *tip*. Throughout the course of a fractional acquisition protocol, define the *height* of a particular tip to be the length of the longest interval of which the tip is the top. In this model of fractional acquisition, the height of a tip is nondecreasing. Figure 5.1 provides an example of this model.

Theorem 5.2.1. *If P_n is the n -vertex path, then $a_f(P_n) = \lceil n/4 \rceil$.*

Proof. By Proposition 3.2.2, $a_t(P_n) = \lceil n/4 \rceil$, so $a_f(P_n) \leq \lceil n/4 \rceil$.

We will show that given five vertices v_1, \dots, v_5 , labeled in order, the tips from v_1 and v_5 cannot reach the same vertex. Because the height of a tip increases each time it is moved, it is not possible for a fractional acquisition move to transfer a tip to a vertex with weight less than 1. The first fractional acquisition move involving v_3 results in either v_3 or one of its neighbors having weight less than 1. The vertex with weight less than 1 lies between the tips from v_1 and v_5 . Any move involving a vertex v with weight less than 1 yields a vertex with weight less than 1 at v or a neighbor do v . Thus, after any subsequent move involving the vertex with weight less than 1, the tips from

v_1 and v_5 are still separated by a vertex with weight less than 1. It follows that the tips from v_1 and v_5 cannot reach the same vertex, so five tips cannot reach the same vertex. Thus $a_f(P) \geq \lceil n/4 \rceil$, and consequently $a_f(P) = \lceil n/4 \rceil$. \square

A similar proof works for the cycle C_n .

Proposition 5.2.2. *If G is the n -vertex cycle, then $a_f(G) = \lceil n/4 \rceil$.*

Proof. Because $a_t(C_n) = \lceil n/4 \rceil$, we know that $a_f(C_n) \leq \lceil n/4 \rceil$. Suppose that there are two tips t_1 and t_2 with heights h_1 and h_2 , respectively, and let $h_1 \geq h_2$. We claim that if there is a cutset between the tips containing vertices with weights less than h_2 , then the tips cannot reach the same vertex. Any fractional acquisition move involving a vertex v in the cutset leaves either v or a neighbor of v with weight less than h_2 . Thus the existence of a cutset with weights less than h_2 is preserved by all fractional protocols, and the tips cannot reach the same vertex; this depends on C_n being 2-regular.

Let v_1, \dots, v_5 be five vertices in C_n , labeled cyclically. For $i \in [5]$, let t_i be the tip from v_i , and let h_i be the height of t_i . We show that the tips from these vertices cannot all reach the same vertex. Consider the first vertex to lose its tip; by symmetry, we may assume that it is v_1 and that the tip moves towards v_2 . Now $h_1 > 1$ and there is a vertex (namely v_1) with weight less than 1 that lies on one of the two paths joining the vertices holding t_1 and t_5 . Note that t_1 cannot move further toward v_3 without creating another vertex with weight less than 1, thereby forming a cutset with weights less than 1 between t_1 and t_5 . If t_5 moves, then $h_5 > 1$ and there is a cutset with weights at most 1 between t_1 and t_5 . By the argument in the preceding paragraph, t_1 and t_5 cannot reach the same vertex. Therefore no five tips can reach the same vertex, and $a_u(C_n) = \lceil n/4 \rceil$. \square

We have determined the fractional acquisition numbers of both paths and cycles, and they are the same as the total and unit acquisition numbers. However, the freedom of fractional acquisition moves allows a vertex of degree 2 in a suitably long path to acquire an arbitrarily large amount of weight. This is analogous to Theorem 4.2.8, and the fractional protocol used shows up in a modified form in Section 5.3. Heuristically, this protocol depends on splitting the path into pairs

of equally weighted adjacent vertices. A vertex is alternately paired with the preceding vertex or the subsequent vertex, with the common weight alternating between values less than 1 and values greater than 1. When v_i and v_{i+1} are two adjacent vertices that have the same weight, it is possible to move weight between them so that $w(v_i) = w(v_{i-1})$, provided that the weights on $V(G)$ are sufficiently close to 1. In this way, the pairing shifts. With each iteration of this process, we add one vertex to an ascending path at the end of P_n .

We begin by producing an initial feasible weight distribution that begins the protocol.

Lemma 5.2.3. *There is a feasible weight distribution w on P_{2n+2} defined by*

- $w(v_1) = w(v_2) = 1$,
- $w(v_i) = 1 + \frac{(-1)^{\lceil i/2 \rceil} 2^{i-3}}{2^{2n-1}}$ for $3 \leq i \leq 2n+1$,
- $w(v_{2n+2}) > 0$ with the sign of $w(v_{2n+2}) - 1$ being the same as the sign of $(-1)^n$, and
- $|w(v_{2n+2}) - 1| \leq 1/2$.

Proof. We use induction on n . For $n = 1$, move weight $1/2$ from v_4 to v_3 to achieve the desired distribution.

By the induction hypothesis, there is a fractional protocol \mathcal{A} that achieves the desired weight distribution on P_{2n} . Repeat \mathcal{A} on the first $2n$ vertices of P_{2n+2} , but scale the amount of weight transferred in each move by $1/4$. This achieves the desired weights on the first $2n - 1$ vertices; furthermore, $|w(v_{2n}) - 1| < 1/8$ and $w(v_{2n+1}) = w(v_{2n+2}) = 1$. We consider two cases according to the parity of n .

If n is odd, then $w(v_{2n}) < 1$. We can move weight from v_{2n+1} to v_{2n+2} so that $w(v_{2n+1}) = w(v_{2n})$. The amount of weight moved to achieve this is less than $1/8$. The desired weight for v_{2n} is $1 + \frac{2^{2n-3}}{2^{2n-1}}$, which equals $\frac{5}{4}$, so we can move weight from v_{2n+1} to v_{2n} to achieve the desired weight on v_{2n} . The amount of weight moved to achieve this is less than $3/8$. Thus the total amount of weight that we have moved from v_{2n+1} is between $1/4$ and $1/2$. The desired weight on v_{2n+1} is $1/2$. We can now move the necessary amount of weight from v_{2n+1} to v_{2n+2} to achieve this desired

weight on v_{2n+1} . Any additional weight on v_{2n+2} came from v_{2n+1} . Since the total weight that moved from v_{2n+1} is $1/2$, with some going to v_{2n} , we have $1 < w(v_{2n+2}) < 3/2$.

If n is even, then $w(v_{2n}) > 1$. We can move weight from v_{2n+2} to v_{2n+1} so that $w(v_{2n+1}) = w(v_{2n})$. The amount of weight moved to achieve this is less than $1/8$. The desired weight for v_{2n} is $1 - \frac{2^{2n-3}}{2^{2n-1}}$, which equals $\frac{3}{4}$, so we can move weight from v_{2n} to v_{2n+1} to get the desired weight on v_{2n} . The amount of weight moved to achieve this is less than $3/8$. Thus the total amount of weight that we have moved onto v_{2n+1} is between $1/4$ and $1/2$. The desired weight on v_{2n+1} is $3/2$. We can now move the necessary amount of weight from v_{2n+2} to achieve this desired weight on v_{2n+1} . All of the original weight on v_{2n+2} is either on v_{2n+2} or v_{2n+1} . Since v_{2n+1} gained a total of $1/2$ with some coming from v_{2n} , we have $1/2 < w(v_{2n+2}) < 1$. \square

When P_{2n+2} has the weight distribution from Lemma 5.2.3, we will say that it is *initialized*. Given the initialized weight distribution, it is possible to transfer large amounts of weight to a single vertex in the path.

Theorem 5.2.4. *Given a positive real number M , there is a path P such that a vertex in P can acquire weight at least M .*

Proof. Let $P = P_{2n+2} = \langle v_1, \dots, v_{2n+2} \rangle$. We will prove that v_{n+2} can achieve weight

$$n + 2 + \frac{2(8^{\lceil \frac{n-1}{2} \rceil + 1} - 1)}{7 \cdot 2^{2n-1}} + \frac{4(8^{\lfloor \frac{n-1}{2} \rfloor + 1} - 1)}{7 \cdot 2^{2n-1}}.$$

Begin the process by initializing P . Initially, let $P' = \langle v_1, v_2, v_3, v_4 \rangle$, and let $Q = \langle v_5, \dots, v_{2n} \rangle$. Each step will move the first vertex of Q to the end of P' and delete the last vertex of Q . Let v_k denote the current first vertex of Q . We say that Q is in State 1 if:

- $|w(v_i) - 1| = 2|w(v_{i-1}) - 1|$;
- $w(v_i) < 1$ for $i \equiv k, k + 1 \pmod{4}$;
- $w(v_i) > 1$ for $i \equiv k + 2, k + 3 \pmod{4}$.

We say that Q is in State 2 if

- $|w(v_i) - 1| = 4|w(v_{i-1}) - 1|$ for $i \equiv k \pmod{2}$;
- $w(v_i) = w(v_{i-1})$ for $i \equiv k + 1 \pmod{2}$;
- $w(v_i) < 1$ for $i \equiv k \pmod{4}$;
- $w(v_i) > 1$ for $i \equiv k + 2 \pmod{4}$.

When P is initialized, Q is in State 1.

We now describe the fractional protocol on P . If Q is in State 1, then:

- move $1 - w(v_i)$ from v_i to v_{i-1} for $i \equiv k + 1 \pmod{4}$;
- move $2(w(v_i) - 1)$ from v_i to v_{i+1} for $i \equiv k + 2 \pmod{4}$.

If Q is in State 2, then:

- move $3(1 - w(v_i))$ from v_i to v_{i-1} for $i \equiv k + 1 \pmod{4}$;
- move $3(w(v_i) - 1)$ from v_i to v_{i+1} for $i \equiv k + 2 \pmod{4}$.

After doing one of these steps, add v_k to $P^{(1)}$, and let Q be the path $\langle v_{k+1}, v_{k+2}, \dots, v_{n-k+3} \rangle$.

We claim that when we perform the prescribed fractional acquisition moves, Q alternates between State 1 and State 2. Let $w(v_i)$ denote the weight on v_i before the set of moves, and let $w'(v_i)$ denote the weight on v_i afterward. Before performing these moves, Q starts at v_k . First we assume that Q is in State 1.

- If $i \equiv k + 1 \pmod{4}$, then we need $|w'(v_i) - 1| = 4|w'(v_{i-1}) - 1|$ and $w'(v_i) < 1$.

We have $w(v_i) < 1$. Thus $w'(v_i) = w(v_i) - (1 - w(v_i)) = 2w(v_i) - 1 < 1$.

We have $w(v_{i-1}) = 1 - \frac{1}{2}(1 - w(v_i)) = \frac{1}{2} + \frac{1}{2}w(v_i)$. Since $w'(v_{i-1}) = w(v_{i-1}) + (1 - w(v_i))$, we have

$$|w'(v_i) - 1| = |w(v_i) - (1 - w(v_i)) - 1| = |2w(v_i) - 2|.$$

This yields

$$\begin{aligned} |w'(v_{i-1})| &= |w(v_{i-1}) + (1 - w(v_i)) - 1| \\ &= \left| \frac{1}{2} + \frac{1}{2}w(v_i) + 1 - w(v_i) - 1 \right| = \left| \frac{1}{2} - \frac{1}{2}w(v_i) \right|. \end{aligned}$$

Thus $|w'(v_i) - 1| = 4|w'(v_{i-1}) - 1|$.

- If $i \equiv k + 2 \pmod{4}$, then we need $w'(v_i) = w'(v_{i-1})$.

We have $w'(v_i) = w(v_i) - 2(w(v_i) - 1)$, and $1 + 2(1 - w(v_{i-1})) = w(v_i)$. Therefore

$$\begin{aligned} w'(v_i) &= 1 + 2[1 - w(v_{i-1}) - 2((1 + 2(1 - w(v_{i-1}))) - 1)] \\ &= 3 - 2w(v_{i-1} - 4 + 4w(v_{i-1})) = 2w(v_{i-1}) - 1. \end{aligned}$$

This is the same value that we computed for $w(v_{i-1})$ in the previous item, so $w'(v_i) = w'(v_{i-1})$.

- If $i \equiv k + 3 \pmod{4}$, then we need $|w'(v_i) - 1| = 4|w'(v_{i-1}) - 1|$ and $w'(v_i) > 1$.

We have $w(v_i) > 1$. Thus $w'(v_i) = w(v_i) + 2(w(v_{i-1}) - 1) > 1$.

We have $w(v_i) = 1 + 2(w(v_{i-1}) - 1) = 2w(v_{i-1}) - 1$. Therefore

$$|w'(v_i) - 1| = |w(v_i) + 2(w(v_{i-1}) - 1) - 1| = |2w(v_{i-1}) - 1 + 2w(v_{i-1}) - 3| = |4w(v_{i-1}) - 4|,$$

and

$$|w'(v_{i-1}) - 1| = |w(v_{i-1}) - 2(w(v_{i-1}) - 1) - 1| = |-w(v_{i-1}) + 1|.$$

Thus $|w'(v_i) - 1| = 4|w'(v_{i-1}) - 1|$.

- If $1 \equiv k \pmod{4}$, then we need $w'(v_i) = w'(v_{i-1})$.

We have that v_{i+1} is in Q , and $w(v_{i+1}) = 1 - 2(1 - w(v_i)) = 2w(v_i) - 1$. Also, $w(v_{i-2}) =$

$1 + \frac{1}{4}(1 - w(v_i)) = \frac{5}{4} - \frac{1}{4}w(v_i)$ and $w(v_{i-1}) = 1 + \frac{1}{2}(1 - w(v_i)) = \frac{3}{2} - \frac{1}{2}w(v_i)$. Therefore

$$w'(v_i) = w(v_i) + (1 - w(v_{i+1})) = w(v_i) + (1 - (2w(v_i) - 1)) = 2 - w(v_i),$$

and

$$\begin{aligned} w'(v_{i-1}) - w(v_{i-1}) + 2(w(v_{i-2}) - 1) &= \frac{3}{2} - \frac{1}{2}w(v_i) + 2\left(\frac{5}{4} - \frac{1}{4}w(v_i) - 1\right) \\ &= 2 - w(v_i). \end{aligned}$$

Thus $w'(v_i) = w'(v_{i-1})$.

Thus Q switches from State 1 to State 2.

Now suppose that Q is in State 2.

- If $i \equiv k + 1 \pmod{4}$, then we need $|w'(v_i) - 1| = 2|w'(v_{i-1}) - 1|$ and $w'(v_i) < 1$.

We have $w(v_{i-1}) = w(v_i)$. We compute

$$|w'(v_i) - 1| = |w(v_i) - 3(1 - w(v_i)) - 1| = |4w(v_i) - 4| = 4|w(v_i) - 1|,$$

and

$$|w'(v_{i-1}) - 1| = |w(v_{i-1}) + 3(1 - w(v_i)) - 1| = |2 - 2w(v_i)| = 2|w(v_i) - 1|.$$

Thus $|w'(v_i) - 1| = 2|w'(v_{i-1}) - 1|$.

We have $w(v_i) < 1$, so $w'(v_i) = w(v_i) - 3(1 - w(v_i)) = 4w(v_i) - 3 < 1$.

- If $i \equiv k + 2 \pmod{4}$, then we need $|w'(v_i) - 1| = 2|w'(v_{i-1}) - 1|$ and $w'(v_i) < 1$.

We have $w(v_{i-1}) = 1 - \frac{1}{4}(w(v_i) - 1)$. We compute

$$|w'(v_i) - 1| = |w(v_i) - 3(1 - w(v_i))| = |2 - 2w(v_i)| = 2|1 - w(v_i)|,$$

and

$$|w'(v_{i-1}) - 1| = |w(v_{i-1}) - 3(1 - w(v_{i-1})) - 1| = |4w(v_{i-1}) - 4| = |1 - w(v_i)|.$$

Thus $|w'(v_i) - 1| = 2|w'(v_{i-1}) - 1|$.

We have $w(v_i) > 1$, so $w'(v_i) = 3 - 2w(v_i) = 1 - 2(w(v_i) - 1) < 1$.

- If $i \equiv k + 3 \pmod{4}$, then we need $|w'(v_i) - 1| = 2|w'(v_{i-1}) - 1|$ and $w'(v_i) > 1$.

We have $w(v_{i-1}) = w(v_i)$. We compute

$$|w'(v_i) - 1| = |w(v_i) + 3(w(v_i) - 1) - 1| = |4w(v_i) - 4| = 4|w(v_i) - 1|,$$

and

$$|w'(v_{i-1}) - 1| = |w(v_{i-1}) - 3(w(v_i) - 1) - 1| = |2 - 2w(v_i)| = 2|w(v_i) - 1|.$$

Thus $|w'(v_i) - 1| = 2|w'(v_{i-1}) - 1|$.

We have $w(v_i) > 1$, so $w'(v_i) = w(v_i) + 3(w(v_i) - 1) > 1$.

- If $i \equiv k \pmod{4}$, then we need $|w'(v_i) - 1| = 2|w'(v_{i-1}) - 1|$ and $w'(v_i) > 1$.

We have $w(v_{i-1}) = 1 + \frac{1}{4}(1 - w(v_i)) = \frac{5}{4} - \frac{1}{4}w(v_i)$ and $w(v_i) = w(v_{i+1})$. We compute

$$|w'(v_i) - 1| = |w(v_i) + 3(1 - w(v_i)) - 1| = |2 - 2w(v_i)| = 2|1 - w(v_i)|,$$

and

$$|w'(v_{i-1}) - 1| = |4w(v_{i-1}) - 3 - 1| = |5 - w(v_i) - 4| = |1 - w(v_i)|.$$

Thus $|w'(v_i) - 1| = 2|w'(v_{i-1}) - 1|$.

We have $w(v_i) < 1$, so $w'(v_i) = 1 + 2(1 - w(v_i)) > 1$.

Thus Q switches from State 2 to State 1.

The initial length of Q is $2n - 4$, and with each step Q loses two vertices, so we can iterate the process $n - 2$ times. Thus P' ends with length $n + 2$.

Now we claim that in P' , $w(v_i) > w(v_{i-1})$ for $i > 2$. This is clearly true when $P' = \langle v_1, v_2, v_3, v_4 \rangle$. Consider the j th set of fractional acquisition moves that we perform on P . This set adds one new vertex, namely v_{j+4} to P' . If the j th step switches Q from State 1 to State 2, then $w'(v_{j+4}) = 1 + (1 - w(v_{j+4}))$ and $w'(v_{j+3}) = w(v_{j+3})$. We have $1 - w(v_{j+4}) = 2(w(v_{j+3}) - 1)$, so $w'(v_{j+4}) > w'(v_{j+3})$. If the j th step switches Q from State 2 to State 1, then $w'(v_{j+4}) = 1 + 2(1 - w(v_{j+4}))$ and $w'(v_{j+3}) = w(v_{j+3})$. We have $1 - w(v_{j+4}) = 4(w(v_{j+3}) - 1)$, so $w'(v_{j+4}) > w'(v_{j+3})$.

Consider P' when the protocol terminates. It consists of $n+2$ vertices with $w(v_1) = w(v_2) = 1$ and $w(v_i) - w(v_{i-1}) \geq \frac{1}{2^{2n-3}}$ for $i \geq 3$. Thus P' has a weight assignment that ascends to v_{n+2} , and we can move all of the weight on this path to v_{n+2} . We now compute the total weight of P' .

Let $\hat{w}(v) = w(v) - 1$. The sum of the weights on P' is $n + 2 + \sum_{n=1}^{n+2} \hat{w}(v_i)$. We know that the values of $\hat{w}(v_i)$ increase by alternating factors of 2 and 4 from v_2 to v_{n+2} . Thus, by taking alternate vertices, we can split the sum of \hat{w} over P' into two finite geometric sums, both with ratio $\frac{8}{2^{2n-1}}$. One of the sums has length $\lceil (n+1)/2 \rceil$ with initial term 2 and the other has length $\lfloor (n+1)/2 \rfloor$ with initial term 3. Thus the sum of \hat{w} over P' can be expressed as $\frac{2(8^{\lceil (n+1)/2 \rceil} - 1)}{7 \cdot 2^{2n-1}} + \frac{4(8^{\lfloor (n+1)/2 \rfloor} - 1)}{7 \cdot 2^{2n-1}}$, and we can move a total of

$$n + 2 + \frac{2(8^{\lceil \frac{n-1}{2} \rceil + 1} - 1)}{7 \cdot 2^{2n-1}} + \frac{4(8^{\lfloor \frac{n-1}{2} \rfloor + 1} - 1)}{7 \cdot 2^{2n-1}}$$

to v_{n+2} . □

5.3 General Graphs

We now prove that if G is a connected graph with $\Delta(G) \geq 3$, then $a_f(G) = 1$. To prove this we need only fractional acquisition moves that transfer rational amounts of weight; therefore we introduce a new model of fractional acquisition, which we call the *normalized* model. Let each vertex start with weight 0, and move positive integer amounts of weight, allowing negative weights on vertices. As in the case of all acquisition moves, moving weight from u to v is valid only if the weight on v is at least the weight on u . We call a protocol of such moves a *normalized protocol*. In a normalized model, the vertex weights will always be integers that sum to 0. We obtain for any finite normalized protocol \mathcal{A} a corresponding fractional protocol \mathcal{A}' .

Lemma 5.3.1. *Let G be a graph. If \mathcal{A} is a finite normalized protocol on G , then there is a corresponding fractional protocol \mathcal{A}' on G such that a path in G is ascending after protocol \mathcal{A} if and only if it is ascending after protocol \mathcal{A}' .*

Proof. Let G be a weighted graph in which every vertex has weight 0. Suppose that a finite normalized protocol \mathcal{A} on G produces the weight assignment $w_{\mathcal{A}}$. Let m be the maximum absolute value of the weights on the vertices during the protocol \mathcal{A} . If step t in \mathcal{A} moves weight a from u to v , then step t in \mathcal{A}' moves weight a/m from u to v . The weight of any vertex v after t steps of \mathcal{A}' is 1 plus $1/m$ times its weight after t steps in \mathcal{A} . Through the course of \mathcal{A}' , no vertex ever has negative weight, and every move is a valid fractional acquisition move. Since a linear function has been applied to the weights, a path in G is ascending after protocol \mathcal{A} if and only if it is ascending after protocol \mathcal{A}' . □

We now prove a fractional acquisition analogue of Corollary 4.1.4. The definition of an ascending weight assignment extends to fractional and negative values.

Lemma 5.3.2. *If G is a graph and a normalized protocol \mathcal{A} yields an ascending weight assignment on a graph G , then $a_f(G) = 1$.*

Proof. By Lemma 5.3.1, there is a fractional protocol \mathcal{A}' that produces an ascending weight assignment w on G . Let α be the minimum nonzero difference between the weights of adjacent vertices in w . Because G is finite, α is bounded below. By the same argument as in Corollary 4.1.4, the fractional acquisition number of G is 1. \square

In light of Lemma 5.3.2, it is sufficient to show that on any graph G satisfying $\Delta(G) \geq 3$, there is a normalized protocol \mathcal{A} that produces an ascending weight distribution. A vertex is *heavy* if it has positive weight, and *neutral* if it has weight 0. If a vertex v is heavy in a normalized protocol, then in the corresponding fractional protocol v has weight greater than 1. If v is neutral, then v has weight 1 in the corresponding fractional protocol. Note that in the normalized model, the path protocol prescribed in Section 4.1 can also be used when the origin of the path has negative weight.

Lemma 5.3.3. *Let T be a weighted tree, and let v and u be vertices in T that are not leaves. Starting from a v -ascending weight assignment w , there is a normalized protocol that produces a u -ascending weight assignment.*

Proof. Let v' be the neighbor of v in $T(v, u)$, and let $w(v) - w(v') = a$. It suffices to show that we can produce a v' -ascending weight assignment. Let x be a leaf in T such that u lies on the unique x, v -path $T(x, v)$; the weights on that path ascend to v . Because we are allowed to drive the weight of x negative, we can apply the protocol $\mathcal{A}^{a+1}(x, v')$, producing a v' -ascending weight assignment. \square

We now present a modification of the protocol that moves large amounts of weight to vertices in paths. Our goal is to obtain a weight assignment on a subdivision of $K_{1,3}$ that ascends to the branch vertex. The presence of a vertex with degree 3 gives us the freedom necessary to achieve this. We will repeatedly use two protocols on strictly ascending paths. Suppose that $\langle v_1, \dots, v_m \rangle$ ascends to v_m . Let $\bar{\mathcal{A}}(v_1, v_m)$ denote the protocol that for i from 1 to $m - 1$ (in order) moves one chip from v_i to v_m . Let $\bar{\mathcal{A}}'(v_1, v_m)$ denote the protocol that for i from m to 2 (in order) moves one chip from v_1 to v_i .

weight assignment that ascends to v , the tree protocol, denoted $\mathcal{A}(T, v)$, consists of applying the path protocol $\mathcal{A}(u, v)$ once for each u in T , ordered so that if u' is in $T(u, v)$, then $\mathcal{A}(u, v)$ occurs before $\mathcal{A}(u', v)$. Thus weight 1 moves from each vertex in T to v .

Lemma 5.3.5. *Let T be a subdivided star with branch vertex v and at least three leaves. If all vertices are neutral except possibly one heavy leaf, then there is a normalized protocol that produces a v -ascending weight assignment.*

Proof. If T is a star, then the result holds using one move that transfers weight from a leaf with weight 0 to v . Thus we may assume that at least one path emanating from v has length at least 2. We use induction on the degree of v . Let u_1, \dots, u_k denote the leaves in T . Let v_i denote the neighbor of v in $T(v, u_i)$.

For the basis of the induction, consider the case $k = 3$. By symmetry, we may assume that u_3 is not adjacent to v . Furthermore, because T contains at most one heavy leaf and $d(v) = 3$, we may assume that v_2 is not a heavy leaf. Let T' be the subtree of T consisting of $T(u_3, v)$ and the edges vv_1 and vv_2 .

If $w(v_1) > 0$ (that is, v_1 is the heavy leaf in T), then apply the protocol $\mathcal{A}(v_2, v_1)$ to reduce the weight on v_2 . The resulting weight assignment on the subtree T' allows us to apply Lemma 5.3.4. When $w(v_1) = 0$, a more complicated protocol is necessary to produce a weight assignment on T' that allows the application of Lemma 5.3.4, as illustrated in Figure 5.3.

Move weight 1 from v to v_1 and from v_3 to its neighbor z in $T(v, u_3)$. Move weight 1 from v_3 to v_1 and then apply $\mathcal{A}(v_2, v_1)^3$, yielding $w(v, v_1, v_2, v_3, z) = (0, 4, -3, -2, 1)$ (Figure 5.3 (B)). Move weight 3 from v to v_1 and apply $\mathcal{A}(v_2, v_3)^3$; now $w(v, v_1, v_2, v_3, z) = (-3, 7, -6, 1, 1)$ (Figure 5.3 (C)). Move weight 5 from v_2 to v , and then apply $\mathcal{A}(z, v_1)$; now $w(v, v_1, v_2, v_3, z) = (2, 8, -11, 1, 0)$ (Figure 5.3 (D)). Finally, apply $\mathcal{A}(v_3, v_1)$ and then move weight 2 from v to v_1 ; now $w(v, v_1, v_2, v_3, z) = (0, 11, -11, 0, 0)$ (Figure 5.3 (E)).

By Lemma 5.3.4, there is a protocol that we can apply to the subtree of T consisting of $T(u_3, v)$ along with vv_1 and vv_2 so that $T(u_3, v)$ ascends to v and $w(v_1) > w(v) > w(v_2)$. Repeatedly apply

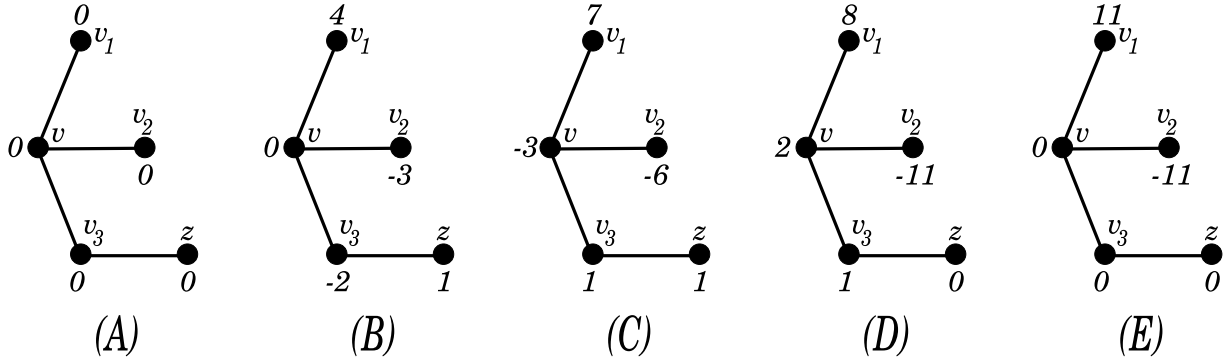


Figure 5.3: A progression of weight assignments for Lemma 5.3.5.

the protocol $\overline{\mathcal{A}}(u_3, v_1)$ so that the weights on $T(u_3, v_1)$ ascend to v_1 and $w(v) < w(v_2)$. Now repeatedly apply the path protocol $\mathcal{A}(u_3, v_2)$ until $w(v_2) = 0$, then apply $\mathcal{A}(u_3, v)$ until $w(v) = 0$. Now $w(v_3) < 0$, and the weights on the u_3, v_1 -path ascend to v_1 .

Applying the protocol of Lemma 5.3.4 with (u_1, u_2, v_1) as (u', z, u) , respectively, now produces a weight assignment on $T(u_2, v)$ such that $T(u_3, v_1)$ and $T(u_2, v_2)$ are ascending. We then repeatedly apply $\mathcal{A}(u_3, v)$ and then $\mathcal{A}(u_3, v_3)$ to ensure $w(v_3) > w(v) > w(v_1)$. We then apply the path protocol to $\langle v_1, v, v_3 \rangle$ and $\langle v, v_3 \rangle$ so that $w(v_1) = w(v) = 0$. Applying Lemma 5.3.4 again now yields a weight assignment on T that ascends to v_3 . Lemma 5.3.3 now produces a v -ascending weight assignment. Thus we have a weight assignment that ascends to v , and the base case is complete.

Now assume that $m \geq 4$ and T is a subdivision of $K_{1,m}$ with branch vertex v and leaves u_1, \dots, u_m . Let v_i be the neighbor of v in $T(u_i, v)$ for $i \in [m]$. Because T is not a star, we may assume that $u_1 \neq v_1$. Let B be the vertex set of $T(u_m, v_m)$. By the induction hypothesis, we can produce a weight assignment on $T - B$ that ascends to v . Since $u_1 \neq v_1$, we can apply the path protocol $\mathcal{A}(u_1, v_1)$ to achieve $w(v_1) > w(v)$. By applying Lemma 5.3.3, there is a protocol that produces a v_1 -ascending weight assignment on $T - B$; furthermore we require that v has weight 0. We now apply Lemma 5.3.4 to the subtree of T consisting of the paths joining v to u_m and u_2 along with the edge v, v_1 to attain a v_1 -ascending weight assignment on the subtree. Repeated application of the path protocol $\mathcal{A}(u_m, v)$ then yields a weight assignment on T that ascends to

v.

□

Lemma 5.3.5 will serve as the base case of an inductive proof that any tree T satisfying $\Delta(T) \geq 3$ has fractional acquisition number 1.

Theorem 5.3.6. *If T is a tree such that at most one leaf is heavy and all other vertices in T are neutral, then there is a normalized protocol that attains an ascending weight assignment on T .*

Proof. We proceed by induction on the number of branch vertices in T . If there is one branch vertex, then Lemma 5.3.5 suffices. Suppose that T has k branch vertices, where $k \geq 2$. If T has a heavy leaf, call it u ; otherwise, let u be an arbitrary leaf. Let x be the branch vertex closest to u , and let y be any other branch vertex in T .

If $d(x) \geq 4$, then let T' be the subtree of T consisting of x and all the components of $T - x$ that do not contain y . There are fewer than k branch vertices in T' , so by induction there is a normalized protocol on T' that produces an ascending weight assignment on T' . By Lemma 5.3.3, we can choose a protocol $\mathcal{A}_{T'}$ that produces an x -ascending weight assignment.

If $d(x) = 3$, then let T' be the subtree of T consisting of all branches at x that do not contain y and the first edge on the x, y path in T . Call this edge xx' . There are fewer than k branch vertices in T' , so by induction there is a normalized protocol on T' that produces an ascending weight assignment. By Lemma 5.3.3, we can choose an x -ascending weight assignment. To continue our argument, we wish to ensure that $w(x') = 0$. If $w(x') > 0$, move weight from x' to x to achieve this.

Otherwise, let x'' be the neighbor of x that is not x' and is not on the x, u -path in T . Using the path protocol, we can move enough weight from a leaf below x'' to x'' to ensure $w(x'') > w(x)$. We now repeatedly apply the protocol $\bar{\mathcal{A}}(u, x'')$ so that $\langle u, \dots, x, x'' \rangle$ is x'' -ascending and $w(x) < w(x')$. We then repeatedly apply the path protocol $\mathcal{A}(u, x')$ until $w(x') = 0$, and then $\mathcal{A}(u, x)$ until $w(x) > w(x'')$. The resulting protocol $\mathcal{A}_{T'}$ produces an x -ascending weight assignment on T' such that $w(x') = 0$.

Let $T'' = T - T' + xx'$. After performing the protocol on T' , we can treat x as a heavy leaf in

T'' . By our induction hypothesis, there is a normalized protocol $\mathcal{A}_{T''}$ that produces an ascending weight assignment on T'' . Note that because x is a leaf, the weight on x never increases in $\mathcal{A}_{T''}$. Thus a normalized acquisition move involves x only if it is part of a protocol of the form $\mathcal{A}(x, v)$, $\overline{\mathcal{A}}(x, v)$, or $\mathcal{A}(T'', v)$ for some vertex v . We replace these moves by $\mathcal{A}(x, u)$, $\mathcal{A}(T(x, v) \cup T', v)$ and $\mathcal{A}(T, v)$ respectively. Thus we obtain an ascending weight assignment on T . \square

We now show that every connected graph with maximum degree at least 3 has fractional acquisition number 1.

Theorem 5.3.7. *If G is a connected graph with $\Delta(G) \geq 3$, then $a_f(G) = 1$.*

Proof. If G has a vertex with degree at least 3, then G has a spanning tree T which has a vertex with degree at least 3. By Theorem 5.3.6, there is a fractional acquisition protocol on T that attains an ascending weight assignment, so $a_f(G) = 1$. \square

Chapter 6

Uniquely H -Saturated Graphs

6.1 Introduction

Extremal graph theory traces its origins to Turán's Theorem [27] proved in 1941. Turán proved that the largest n -vertex graph that does not contain a complete subgraph on r vertices is the complete $(r - 1)$ -partite graph in which the partite sets have sizes as equal as possible. Such a graph is called a *Turán graph*, denoted $T_{n,r}$. If an edge is added to $T_{n,r}$, then it must lie in a partite set, and the resulting graph contains many copies of K_r . Because of this property, $T_{n,r}$ is said to be K_r -saturated. Given a fixed graph H , a graph G is H -saturated if G does not contain H as a subgraph, but the addition of any edge from \overline{G} to G completes a copy of H . Note that if $n < |V(H)|$, then K_n vacuously satisfies the definition of H -saturated: H is not a subgraph of K_n , and there is no edge from $\overline{K_n}$ to add to K_n . For the remainder of this chapter, we say that a graph G is a *trivial* H -saturated graph when G is H -saturated and $|V(G)| < |V(H)|$.

We can now restate Turán's theorem: $T_{n,r}$ is the largest K_r -saturated graph with n vertices. The number of edges in the largest n -vertex H -saturated graph is the *extremal number* of H . Hence it is natural to ask for the minimum number of edges in an H -saturated graph; this is the *saturation number* of H . The saturation number of H exists because the empty graph is not H -saturated (provided $H \neq K_2$), and if edges are added to the empty graph in an arbitrary order, at some point a copy of H appears or the graph is complete. Erdős, Hajnal, and Moon [11] proved the first results on saturation numbers, determining the saturation number for K_r . For fixed n and r , the extremal graph in the Erdős-Hajnal-Moon result is obtained by adding all possible edges between a copy of K_{r-2} and $\overline{K_{n-r+2}}$; this graph is called a *generalized book*, and is denoted $B_{n-r+2,r-2}$.

Bollobás [4] determined the saturation number for complete bipartite graphs, and Kászonyi and Tuza [17] proved the best known general upper bound.

The Erdős-Hajnal-Moon result determined the saturation number of C_3 , and various authors have studied the saturation numbers of larger cycles. Ollmann [23] determined the minimum C_4 -saturated graphs, and in 2009 Chen [8] determined the minimum C_5 -saturated graphs. No other exact values of saturation numbers for cycles are known, though Barefoot, Clark, Entringer, Porter, Székely, and Tuza [3] showed that for sufficiently large n , the saturation number of C_t lies between $n + c_1 \frac{n}{t}$ and $n + c_2 \frac{n}{t}$ for constants c_1 and c_2 ; Gould, Łuczak, and Schmitt [14] provided constructions that improved the value of c_1 .

We have observed that adding an edge to $T_{n,r}$ completes many copies of K_r . However, adding an edge to the generalized book $B_{n-r+2,r-2}$ completes exactly one copy of K_r . For a fixed graph H , a graph G is *uniquely H -saturated* if G is H -saturated and the addition of any edge to G completes exactly one copy of H . In this chapter we study uniquely H -saturated graphs when H is a path or cycle.

The requirement that the addition of an edge to a graph G completes exactly one copy of some target graph H is a strong structural requirement. Thus our results are structural in nature, as opposed to the traditional extremal results on saturation. For instance, we know that the extremal number and saturation number are defined for all graphs, but it is not clear that uniquely H -saturated graphs exist for all H . Our first result is that nontrivial uniquely H -saturated graphs do not exist when H is a sufficiently long path. A graph is uniquely P_2 -saturated if and only if it contains no edges. It is also clear that the only uniquely P_3 -saturated graph is the graph on three vertices that has one edge. Again, a simple observation shows that a graph G is uniquely P_4 -saturated if and only if G is a perfect matching with at least two edges. However, we prove that there are no uniquely P_t -saturated graphs when $t \geq 5$.

Theorem 6.1.1. *If $t \geq 5$, then there are no uniquely P_t -saturated graphs.*

Proof. Let $t \geq 5$, let G be a candidate to be a uniquely P_t -saturated graph, and let x and y be a pair

of nonadjacent vertices in G . We refer to a copy of P_t in G as a t -path. The edge xy completes a unique t -path P in G . Let x' and y' be the endpoints of P . If x' has a neighbor that is not in $V(P)$, then $G + xy$ contains a $(t+1)$ -path, which contains two t -paths. If x' has a neighbor in $V(P)$ other than its neighbor along P , then $G + xy$ contains a cycle with a pendant path on a total of t vertices, which also contains two t -paths. Thus $d(x') = 1$, and by symmetry $d(y') = 1$. Because x' and y' are the endpoints of a t -path in $G + xy$, $x'y' \notin E(G)$. Therefore $G + x'y'$ contains a unique t -path P' . The endpoints of P' are x and y , and thus $d(x) = d(y) = 1$. Therefore, if a vertex v in G has a nonneighbor, then $d(v) = 1$. It follows that G is either a matching or a star. Stars and matchings are not uniquely C_t -saturated for $t \geq 5$, so we conclude that there are no uniquely P_t -saturated graphs for $t \geq 5$. \square

In the remainder of the chapter, we study cycle-saturated graphs in Section 6.2. The Erdős-Hajnal-Moon result shows that for every n there is an n -vertex uniquely C_3 -saturated graph, namely the star $K_{1,n-1}$. We prove that all other uniquely C_3 -saturated graphs have diameter 2 and girth 5. A *Moore graph* is a graph with diameter d and girth $2d + 1$; thus the uniquely C_3 -saturated graphs are stars and Moore Graphs with diameter 2. Hoffman and Singleton [16] proved that there are at most four Moore graphs with diameter 2

Theorem 6.1.2. *A graph G is uniquely C_3 -saturated if and only if G is a star or a Moore graph with diameter 2.*

Proof. Because the edge joining any two nonadjacent vertices completes a 3-cycle, G has diameter at most 2. If G is a tree, then G is either K_1 , K_2 , or a star. Clearly all of these graphs are uniquely C_3 -saturated.

We may assume that G contains a cycle and has diameter 2. By Lemma 6.2.5, C_4 is forbidden as a subgraph of G . Because G has diameter 2 and does not contain any 3-cycles or 4-cycles, G must have girth 5. Therefore G is a Moore graph with diameter 2; all such graphs are uniquely C_3 -saturated. \square

We can view uniquely C_k -saturated graphs as generalizing the Moore graphs of diameter 2.

This viewpoint is reflected in Section 6.3, where we determine the uniquely C_4 -saturated graphs; there are exactly ten. The structure and techniques of the result are very similar to the eigenvalue approach used to prove both the Hoffman-Singleton result on Moore graphs and also the “Friendship Theorem”, which states that a graph in which any two distinct vertices have exactly one common neighbor has a vertex adjacent to all others (see Wilf [30]). Structural arguments are used to show that under certain conditions the graphs in question are regular. Counting of walks then yields a polynomial equation involving the adjacency matrix, after which eigenvalue arguments exclude all but a few graphs.

In Section 6.4, we show that there are infinitely many uniquely C_5 -saturated graphs. The graphs that result from the Friendship Theorem consist of some number of triangles sharing a single vertex; these are the only nontrivial uniquely C_5 -saturated graphs. However, the proofs in this section do not require algebraic tools.

Finally, in Sections 6.5 and 6.6 we prove that there are no nontrivial uniquely C_k -saturated graphs when $k \in \{6, 7\}$. These proofs are purely combinatorial, and they rely extensively on the use of forbidden subgraphs.

6.2 Cycles

In this section we establish some general facts about uniquely C_t -saturated graphs for various values of t .

Lemma 6.2.1. *If G is uniquely C_t saturated, then $\text{diam}(G) \leq t - 1$.*

Proof. If x and y are nonadjacent vertices in G , then the edge xy completes a t -cycle in $G + xy$. Therefore there is an xy -path of length $t - 1$ in G . □

Lemma 6.2.2. *If G is a uniquely C_t -saturated graph for $t \geq 5$, and B is a block in G with at most $(t - 1)$ vertices, then B is a complete graph.*

Proof. Let B be a block in G , and let x and y be a pair of nonadjacent vertices in B . As B is a maximal 2-connected subgraph of G , any x, y -path in G is contained in B . Because there is a unique t -path joining x and y , B has at least t vertices. Therefore, if B has at most $t - 1$ vertices, then B is a complete graph. \square

If G is a nontrivial uniquely C_t -saturated graph for $t \geq 5$, we can bound the size of the largest complete block in G .

Lemma 6.2.3. *The largest complete block in a nontrivial uniquely C_t -saturated graph has at most three vertices.*

Proof. The result holds trivially if $t = 3$, so we assume $t \geq 4$. Let G be a uniquely C_t -saturated graph, and let B be a block in G that is a complete graph on at least four vertices. Let u and v be vertices in B , and let v be a cut-vertex of G . Let B' be another block in G that contains v and let x be a vertex in B' with $x \neq v$. As u and x are not adjacent, there is a unique t -path P joining u and x , which must contain v . If P contains more than two edges in B , then the subpath of P joining v and u is not unique. Therefore, P contains only one edge in B , namely uv . Let x' be the neighbor of x in P ($x' \neq v$ because $t \geq 4$). Because B contains at least four vertices, there are multiple 3-paths in B joining u and v . Thus there are multiple t -paths joining u and x' , and G is not uniquely C_t -saturated. We conclude that complete blocks in nontrivial uniquely C_t -saturated graphs have at most 3 vertices. \square

Given this bound on the size of complete blocks in nontrivial uniquely C_t -saturated graphs, we prove a lemma that allows us to restrict our attention to 2-connected graphs when $t \geq 6$.

Lemma 6.2.4. *If $t \geq 6$, then any nontrivial uniquely C_t -saturated graph contains a block that is not a complete graph.*

Proof. Let G be a nontrivial uniquely C_t saturated graph. If all blocks in G are complete, then by Lemma 6.2.3, all blocks have at most three vertices. Thus the longest cycle completed by an edge

joining two nonadjacent vertices from incident blocks has at most five vertices. Therefore, G must have a block that is not a complete graph. \square

Because a noncomplete block in a uniquely C_t -saturated graph is itself uniquely C_t -saturated, Lemma 6.2.4 implies that when $t \geq 6$ it is sufficient to study 2-connected noncomplete graphs. If no such graphs are uniquely C_t -saturated, then there are no nontrivial uniquely C_t -saturated graphs.

We now prove a general upper bounds on the girth of uniquely C_t saturated graphs for $t \geq 5$. To do so, we first introduce a family of graphs that are forbidden as subgraphs from uniquely C_t -saturated graphs for fixed t . Let $H_{m,\ell}$ be the graph that consists of an m -cycle with a pendant path of length ℓ . The graph $H_{6,3}$ is shown in Figure 6.1

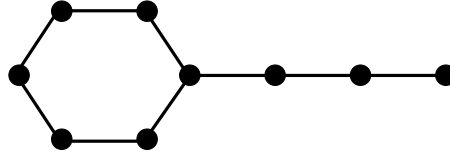


Figure 6.1: The graph $H_{6,3}$. It is forbidden as a subgraph of uniquely C_4 -, C_5 -, C_6 -, and C_7 -saturated graphs.

Lemma 6.2.5. *Let t be a fixed positive integer. If $k < t$ and $k + 1 + \ell \geq t$, then $H_{2k,\ell}$ is forbidden as a subgraph from the family of uniquely C_t -saturated graphs.*

Proof. Let v be the vertex that is shared by the cycle and the path in $H_{2k,\ell}$, and let v' be the vertex opposite v on the cycle. There are two $(k + 1)$ -paths joining v and v' . Thus there are two t -paths joining v' and the vertex on the path that is distance $t - (k + 1)$ from v . Therefore any graph containing $H_{2k,\ell}$ is not uniquely C_t -saturated. \square

Establishing $t + 1$ as an upper bound on the girth of uniquely C_t -saturated graphs is straight forward, and by utilizing the family of forbidden subgraphs from Lemma 6.2.5 we are able to improve this bound.

Lemma 6.2.6. *If $t \geq 5$ and G is a uniquely C_t -saturated graph, then G has girth at most $t - 1$.*

Proof. We first show that G has girth at most $t + 1$. We then show that if G contains a $(t + 1)$ -cycle, then G contains a cycle of length at most $t - 1$.

Any two nonadjacent vertices x and y in G are joined by a unique t -path P . Suppose $d(x, y) = 2$, and let z be a common neighbor of x and y . If P does not contain z , then the concatenation of P and the path $\langle x, z, y \rangle$ forms a $(t + 1)$ -cycle. If P contains z , then the union of P and $\langle x, z, y \rangle$ contains a cycle with length at most t . Therefore the girth of G is at most $t + 1$.

We now show that if G contains a $(t + 1)$ -cycle, then G contains a shorter cycle. Let C be a cycle of length $t + 1$ in G . Let $\langle u, v, x, y \rangle$ be four consecutive vertices in C . The edge uy must complete a unique t -cycle in G , so there is a unique t -path P joining u and y . If the internal vertices of P are disjoint from C , then G contains a $(t + 2)$ -cycle and a $(t + 1)$ -cycle that share four consecutive vertices. If t is even, then G contains $H_{t+2, t-4}$. By Lemma 6.2.5, this is a forbidden subgraph given that $t/2 + 2 + t - 4 \geq t$ when $t \geq 4$. If t is odd, then G contains $H_{t+1, t-2}$. Again this is forbidden because $\lceil t/2 \rceil + 1 + t - 2 \geq t$ when $t \geq 1$.

We consider the case when P has internal vertices that lie in C . Note that P is not contained in C . Thus there is a subpath P' of P that has no vertices in C and joins two vertices in C . If P' contains fewer than $\lfloor t/2 \rfloor$ vertices, then G contains a cycle having at most t vertices. Thus we assume that P' contains at least $\lfloor t/2 \rfloor$ vertices. Note that P can have at most one such subpath, otherwise P contains at least $2\lfloor t/2 \rfloor + 3$ vertices (P contains at least three vertices on C) which is greater than t . Therefore P consists of three subpaths, P_1 , P_2 , and P_3 , in order, where P_1 and P_3 are subgraphs of C and P_2 is a path in $G - V(C)$ whose endpoints are adjacent to vertices in C .

If t is odd, then G contains $H_{t+1, \lfloor t/2 \rfloor}$. Since $(t + 1)/2 + \lfloor t/2 \rfloor = t$, this is a forbidden subgraph. If t is even, then C_{t+1} is an odd cycle, and any additional path joining two vertices in C_{t+1} forms an even cycle. Thus $C + P_2$ contains an even cycle which we may assume has at least $t + 2$ vertices. If P_2 joins two vertices x and y on C that are at distance k in C with $k \leq t/2$, then P_2 must have at least $t + 1 - k$ edges. We have shown that P_2 contains at most $t - 1$ edges, so we conclude that $k \geq 2$. Let C_1 be the x, y -path of length k in C , and let C_2 be the x, y -path of length $t + 1 - k$ in C . If $P_2 + C_1$ is an even cycle, then G contains a member of $\{H_{t+2(i), t-k} : 1 \leq i \leq 2(\lfloor k/2 \rfloor - 1)\}$.

Since $2 \leq k \leq t/2$, it follows that $t + 2(i) \leq 2t$ and $t/2 + i + t - k \geq t$ when $i \leq 2(\lceil k/2 \rceil - 1)$.

By Lemma 6.2.5, these graphs are forbidden.

If $P_2 + C_2$ is an even cycle, then G contains a member of $\{H_{2(t-k+1)+i,k-1} : i \text{ even}, 0 \leq i \leq k-2\}$. Because $2 \leq k \leq t/2$, it follows that $2(t-k+1) + i \leq 2t$ and $t-k+1 + i/2 + k-1 \geq t$ when i is even and $0 \leq i \leq k-2$. By Lemma 6.2.5, these graphs are forbidden.

Therefore G contains a cycle with length less than $t + 1$. Because G cannot have girth t , the girth is at most $t - 1$. □

Two vertices that have the same neighborhood are called *twins*. We close this section by showing that uniquely C_t -saturated graphs cannot contain twins with degree at least 2.

Lemma 6.2.7. *If G is uniquely C_t -saturated for any t , then there are no twins in G with degree at least 2.*

Proof. Let x and y be twins in G with degree at least 2. The edge xy completes a t -cycle, and the cycle contains the subpath $\langle x', x, y, y' \rangle$. Because x and y are twins, $x'y$ and xy' are edges in G . Therefore there is a second t -cycle in $G + xy$ which contains $\langle x', y, x, y' \rangle$. □

6.3 Uniquely C_4 -Saturated Graphs

Determining the uniquely C_4 -saturated graphs requires the algebraic tools that Hoffman and Singleton used in their Moore graph result. We prove the following result.

Theorem 6.3.1. *There are precisely ten uniquely C_4 -saturated graphs.*

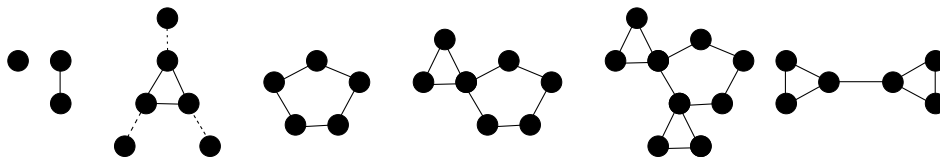


Figure 6.2: The ten uniquely C_4 -saturated graphs. The dotted edges indicate edges (hence leaves) that may or may not be present.

In the list, the only example with girth 5 is C_5 . The others are small trees or contain triangles; all have at most nine vertices. See Figure 6.2. We begin with basic observations about the structure of uniquely C_4 -saturated graphs.

Lemma 6.3.2. *The following properties hold for every uniquely C_4 -saturated graph G .*

- (a) G is connected and has diameter at most 3.
- (b) Any two nonadjacent vertices in G are the endpoints of exactly one 4-path.
- (c) G contains no 6-cycle and no two triangles sharing a vertex.

Proof. By Lemma 6.2.1, G has diameter at most 3. Since G is uniquely C_4 -saturated, x and y are the endpoints of exactly one 4-path. The 6-cycle is forbidden as a subgraph of G by Lemma 6.2.5. Two nonadjacent vertices in the union of two triangles sharing one vertex are the endpoints of two 4-paths. The union of two triangles sharing two vertices contains a 4-cycle. \square

Note that Lemma 6.2.6 states that a uniquely C_t -saturated graph has girth at most $t - 1$ if $t \geq 5$. The requirement that $t \geq 5$ is necessary, as C_5 is uniquely C_4 -saturated. We prove a similar lemma that restricts the girth of uniquely C_4 -saturated graphs.

Lemma 6.3.3. *If G is uniquely C_4 -saturated and $|V(G)| \geq 3$, then G has girth 3 or 5.*

Proof. If $|V(G)| = 3$, then G is K_3 . If $|V(G)| \geq 3$, then G is not a complete graph, and there are vertices x and y with $d(x, y) = 2$; let z be their unique common neighbor. By Lemma 6.3.2, there is a 4-path joining x and y . If it contains z , then G contains a triangle. Otherwise, x and y lie on a 5-cycle. Since G is C_4 -free, it follows that G has girth 3 or 5. \square

If G has maximum degree at most 1, then G is K_1 or K_2 , and these are uniquely C_4 -saturated. We may assume henceforth that $\Delta(G) \geq 2$. Lemma 6.3.3 then allows us to break the study of uniquely C_4 -saturated graphs into two cases: girth 3 and girth 5. We begin with the graphs with girth 5.

Lemma 6.3.4. *If G is a uniquely C_4 -saturated graph with girth 5, then G is regular.*

Proof. Let u and v be adjacent vertices with $d(u) \leq d(v)$. Since G is triangle-free, $N(v)$ is an independent set, and hence the 4-paths joining neighbors of v do not contain v . If $d(u) < d(v)$, then by the pigeonhole principle two of the unique 4-paths from u to the other $d(v) - 1$ neighbors of v begin along the same edge uu' incident to u . Each of these two paths continues along an edge of v to form distinct 4-paths from u' to v . Since $N(v)$ is independent, u' is not adjacent to v , so this contradicts Lemma 6.3.2.

We conclude that adjacent vertices in G have the same degree. Since G is connected, it follows that G is regular. \square

We now show that exactly one uniquely C_4 -saturated graph has girth 5.

Theorem 6.3.5. *The only uniquely C_4 -saturated graph with girth 5 is C_5 .*

Proof. Let G be a uniquely C_4 -saturated n -vertex graph with girth 5. By Lemma 6.3.4, G is regular; let k be the vertex degree. Let A be the adjacency matrix of G , let J be the n -by- n matrix with every entry 1, and let $\mathbf{1}$ be the n -dimensional vector with each coordinate 1. If x and y are nonadjacent vertices of G , then by Lemma 6.3.2 there is one x, y -path of length 3 and no other walk of length 3 joining x and y . If x and y are adjacent, then there are $2k - 1$ walks of length 3 joining them. If $x = y$, then no walk of length 3 joins x and y , because G is triangle-free. This yields $A^3 = (J - A - I) + (2k - 1)A$, or $J = A^3 - (2k - 2)A + I$.

Because J is a polynomial in A , every eigenvector of A is also an eigenvector of J . Since G is k -regular, $\mathbf{1}$ is an eigenvector of A with eigenvalue k . Also $\mathbf{1}$ is an eigenvector of J with eigenvalue n . This yields the following count of the vertices of G :

$$n = k^3 - (2k - 2)k + 1 = k^3 - 2k^2 + 2k + 1.$$

We have observed that every eigenvector of A is also an eigenvector of J . Since J has rank 1, we conclude that $Jx = 0x$ when x is an eigenvector of A other than $\mathbf{1}$. If λ is the corresponding

eigenvalue of A , then $J = A^3 - (2k - 2)A + I$ yields

$$0 = \lambda^3 - (2k - 2)\lambda + 1. \quad (6.1)$$

It follows that A has at most three eigenvalues other than k .

Let q denote the polynomial in (6.1). Being a cubic polynomial, it factors as

$$q(\lambda) = \lambda^3 - (2k - 2)\lambda + 1 = (\lambda - r_1)(\lambda - r_2)(\lambda - r_3). \quad (6.2)$$

It follows that

$$r_1 + r_2 + r_3 = 0. \quad (6.3)$$

Suppose first that two of these roots have a common value, r . From (6.3), the third is $-2r$, and we have

$$\lambda^3 - (2k - 2)\lambda + 1 = (\lambda - r)^2(\lambda + 2r) = \lambda^3 - 3r^2\lambda + 2r^3.$$

By equating coefficients, r equals both $(1/2)^{1/3}$ (irrational) and $(2k - 2)/3$ (rational). Hence q has three distinct roots.

Suppose next that q has a rational root. The Rational Root Theorem implies that 1 and -1 are the only possible rational roots of q . If -1 is a root, then $k = 1$ and G does not have girth 5. If 1 is a root, then $k = 2$ and $G = C_5$.

Hence we may assume that q has three distinct irrational roots. In this case we will obtain a contradiction. Index the eigenvalues so that the multiplicities a , b , and c of r_1 , r_2 , and r_3 (respectively) satisfy $a \leq b \leq c$. Letting p_A be the characteristic polynomial of A ,

$$p_A(\lambda) = (\lambda - k)(\lambda - r_1)^a(\lambda - r_2)^b(\lambda - r_3)^c. \quad (6.4)$$

Combining (6.2) and (6.4) yields

$$p_A(\lambda) = (\lambda - k)(\lambda^3 - (2k - 2)\lambda + 1)^a(\lambda - r_2)^{b-a}(\lambda - r_3)^{c-a}.$$

Because A has integer entries, $p_A(\lambda) \in \mathbb{Q}[\lambda]$. By applying the division algorithm, $p = rs$ and $p, r \in \mathbb{Q}[\lambda]$ imply $s \in \mathbb{Q}[\lambda]$. Hence $(\lambda - r_2)^{b-a}(\lambda - r_3)^{c-a} \in \mathbb{Q}[\lambda]$. Since $q(\lambda)$ is a monic cubic polynomial in $\mathbb{Q}[\lambda]$ with three irrational roots, it is irreducible and is the minimal polynomial of r_1, r_2 , and r_3 over \mathbb{Q} . Thus q divides $(\lambda - r_2)^{b-a}(\lambda - r_3)^{c-a}$ if $c > a$. In that case, since r_1 is a root of q , it is also a root of $(\lambda - r_2)^{b-a}(\lambda - r_3)^{c-a}$. We conclude that $c = a$, and all three eigenvalues have the same multiplicity.

The trace of A is 0, so

$$k + ar_1 + ar_2 + ar_3 = k + a(r_1 + r_2 + r_3) = \text{Tr}(A) = 0. \quad (6.5)$$

Together, (6.3) and (6.5) require $k = 0$. Thus q cannot have three distinct irrational roots when G has girth 5. \square

We now consider uniquely C_4 -saturated graphs with girth 3. The next lemma gives a structural decomposition. For a set $S \subset V(G)$, let $d(x, S) = \min\{d(x, v) : v \in S\}$, and recall that $N(S) = \{v \in V(G) : d(v, S) = 1\}$ and $N^k(S) = \{v \in V(G) : d(v, S) = k\}$.

Lemma 6.3.6. *Let S be the vertex set of a triangle in a graph G , with $S = \{v_1, v_2, v_3\}$. For $i \in \{1, 2, 3\}$, let $V_i = N(v_i) - S$, and let $V'_i = N^2(v_i) - N(S)$. Let $R = N^3(S)$. If G is uniquely C_4 -saturated, then G has the following structure:*

- (a) $V_i \cap V_j = \emptyset$ when $i \neq j$;
- (b) each vertex in V'_i has exactly one neighbor in V_i ;
- (c) $V'_i \cap V'_j = \emptyset$ when $i \neq j$;
- (d) no edges join V'_i and V'_j when $i \neq j$;
- (e) $N(S)$ is independent;

- (f) each V'_i induces a matching;
 (g) each vertex in R has exactly one neighbor in each V'_i .

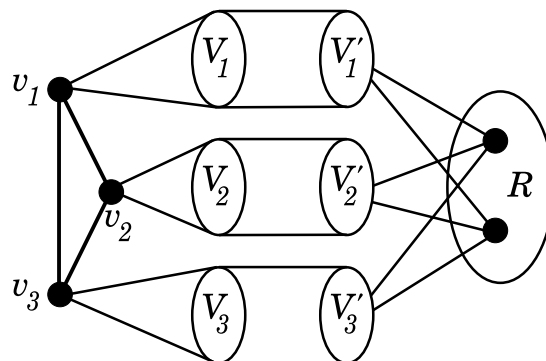


Figure 6.3: Structure of a uniquely C_4 -saturated graph with a triangle.

Proof. Since G has diameter 3, we have described all of $V(G)$. Figure 6.3 makes it easy to see most of the conclusions. The prohibition of 4-cycles and of triangles with common vertices implies (a), (b), and (e). The prohibition of 6-cycles implies (c) and (d).

Given these results, (f) is implied by the existence of a unique 4-path joining v_i to each vertex of V'_i . For (g), each vertex in R is joined by a unique 4-path to each vertex in S ; it can only reach v_i quickly enough by moving first to a vertex of V'_i , and uniqueness of the 4-path prohibits more than one such neighbor. \square

The main part of the argument is analogous to the regularity, walk-counting, and eigenvalue arguments in Lemma 6.3.4 and Theorem 6.3.5.

Theorem 6.3.7. *If G is a C_4 -saturated graph with a triangle, then $R = \emptyset$ in the partition of $V(G)$ given in Lemma 6.3.6.*

Proof. If $R \neq \emptyset$, then each set V_i and V'_i in the partition is nonempty. We show first that G is regular, then show that each vertex lies in one triangle, and finally count 4-paths to determine the cube of the adjacency matrix and obtain a contradiction using eigenvalues.

Consider V'_i and V_j with $i \neq j$. A vertex x in V'_i reaches each vertex of V_j by a unique 4-path, passing through R and V'_j . By Lemma 6.3.6(g), each vertex of R has one neighbor in V'_j , so each edge from x to R starts exactly one 4-path to V_j . By Lemma 6.3.6, the other neighbors of x are one each in V_i and V'_i , so $d(x) = |V_j| + 2$. Since the choice of i and j was arbitrary, we conclude that each vertex of $N^2(S) \cup S$ has degree $a + 2$, where $a = |V_1| = |V_2| = |V_3|$.

For $x \in V_i$ and $y \in V_j$ with $j \neq i$, the unique 4-path joining x to any neighbor of y in V'_j must pass through V'_i and R . By Lemma 6.3.6(g), these paths use distinct vertices in R ; since G has no 6-cycle through y , they also use distinct vertices in V'_i . Hence $d(x) \geq d(y)$. By symmetry, all vertices of $N(S)$ have the same degree; let this degree be $b + 1$.

Consider $r \in R$. By Lemma 6.3.6(g), 4-paths from r to V_i may visit another vertex in R and then reach V_i in exactly one way, or they may go directly to V'_i , traverse an edge within V'_i , and continue to V_i . The total number of such paths is $[d(r) - 3] + 1$, and this must equal $|V_i|$. Hence $d(r) = a + 2$. Since $|V_i| = a$ and $d(x) = b + 1$ for $x \in V_i$, Lemma 6.3.6 yields $|V'_i| = ab$.

Consider $x \in V'_i$ and $j \neq i$. Each 4-path from x to V'_j starts with an edge in V'_i , ends with an edge in V'_j , or uses two vertices in R . Since each vertex in $N^2(S)$ has a neighbors in R , there are a paths of each of the first two types. Since each vertex of R has degree $a + 2$, with three neighbors in $N^2(S)$, there are $a(a - 1)$ paths of the third type. Since these paths reach distinct vertices of V'_j , and every vertex of V'_j is reached, $|V'_j| = a(a + 1)$.

Hence $a(a + 1) = ab$, and $b = a + 1$. Since every vertex of G has degree $a + 2$ or $b + 1$, we conclude that G is k -regular, where $k = a + 2 = b + 1$.

We show next that every vertex of G lies in a triangle. If v lies in no triangle, then $N(v)$ is independent, and having unique 4-paths from $N^2(v)$ to v forces $N^2(v)$ to induce a 1-regular subgraph. Since $|N^2(v)| = k(k - 1)$, there are $\binom{k}{2}$ edges induced by $N^2(v)$. Each 4-path with both endpoints in $N(v)$ has internal vertices in $N^2(v)$. Since there are $\binom{k}{2}$ such pairs of endpoints and each edge within $N^2(v)$ extends to exactly one such path, no edge within $N^2(v)$ lies in a triangle with a vertex of $N(v)$. Thus each neighbor of v also lies in no triangle.

We conclude that neighboring vertices both do or both do not lie in triangles. By induction on

the distance from S , every vertex lies in a triangle. By Lemma 6.3.2, each vertex lies in exactly one triangle.

With A being the adjacency matrix of G , the matrix A^3 again counts walks of length 3. Since each vertex is on one triangle, each diagonal entry is 2. Since G is k -regular, entries for adjacent vertices are $2k - 1$, and by unique C_4 -saturation the remaining entries equal 1. Hence $A^3 = J + (2k - 2)A + I$, and again J is expressible as a polynomial in A :

$$J = A^3 - (2k - 2)A - I.$$

Again $\mathbf{1}$ is an eigenvector of A with eigenvalue k and of J with eigenvalue n . All other eigenvalues of A satisfy $p(\lambda) = 0$, where

$$p(\lambda) = \lambda^3 - (2k - 2)\lambda - 1.$$

Arguing as in the proof of Theorem 6.3.5, $p(\lambda)$ cannot be irreducible over \mathbb{Q} . If λ is rational, then $\lambda = \pm 1$, and $k \in \{1, 2\}$. However, $R \neq \emptyset$ requires $k \geq 3$. \square

Having shown that $R = \emptyset$, we now consider instances with $N^2(S) \neq \emptyset$.

Lemma 6.3.8. *Let G be a uniquely C_4 -saturated graph with a triangle, and let S be the vertex set of a triangle. If $N^2(S) \neq \emptyset$, then G is one of the three graphs in Figure 6.4.*

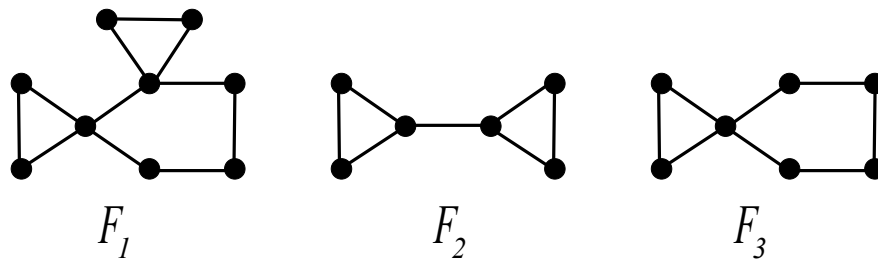


Figure 6.4: Examples having a vertex at distance 2 from a triangle.

Proof. Let $S = \{v_1, v_2, v_3\}$. In the partition defined in Lemma 6.3.6, a 4-path joining V_i' and V_j' must pass through R . Since $R = \emptyset$, we conclude that only one of $\{V_1', V_2', V_3'\}$ is nonempty; by

symmetry, let it be V'_1 . Since G has diameter 3, we have $V_2 = V_3 = \emptyset$.

By Lemma 6.3.6(f), V'_1 induces a matching. By Lemma 6.3.6(b), every vertex of V'_1 thus has degree 2. Consider $w \in V_1$ with neighbors u and v in V'_1 . If u and v are not adjacent, then a 4-path joining them must use w and the neighbor in V'_1 of one of them. Thus if w has three pairwise nonadjacent neighbors in V'_1 , then at least two of them have neighbors in V'_1 that are also neighbors of w . This yields two triangles containing w , contradicting Lemma 6.3.2. We conclude that w cannot have more than three neighbors in V'_1 .

If $w \in V_1$ has three neighbors in V'_1 , then two of them (say x and y) are adjacent. The only 4-paths that can leave x or y for other vertices of V'_1 end at the remaining neighbor of w or its mate in V'_1 . Hence $G = F_1$.

If $w \in V_1$ has two neighbors in V'_1 , then they are adjacent, and no 4-paths can join them to other vertices of V'_1 . Hence $G = F_2$.

In the remaining case, every vertex of V_1 has at most one neighbor in V'_1 . Since any two vertices of V_1 are joined by a 4-path through an edge within V'_1 , there can only be two vertices in V_1 , and $G = F_3$. □

One case remains.

Lemma 6.3.9. *If G is a uniquely C_4 -saturated graph having a triangle S adjacent to all vertices, then G consists of S and a matching joining S to the remaining (at most three) vertices.*

Proof. We have assumed $N^2(S) = \emptyset$. Since 4-paths joining vertices in V_i must pass through V'_i , each V_i has size 0 or 1. Since $V_i \cap V_j = \emptyset$ (Lemma 6.3.6(a)), G is as described. □

We can now prove Theorem 6.3.1.

Theorem 6.3.10. *There are exactly ten uniquely C_4 -saturated graphs.*

Proof. Trivially, K_1 , K_2 , and K_3 are uniquely C_4 -saturated. With girth 5, there is only C_5 , by Theorem 6.3.5. With girth 3, Lemma 6.3.8 provides three graphs when some vertex has distance 2 from a given triangle, and Lemma 6.3.9 provides three when there is no such vertex. □

6.4 Uniquely C_5 -Saturated Graphs

We show that the nontrivial uniquely C_5 -saturated graphs are precisely the “friendship graphs,” which form an infinite family. A *friendship graph* is a graph in which every two vertices have exactly one common neighbor. Erdős, Rényi, and Sós characterized these in 1966 [12], proving that these graphs consist of triangles sharing one vertex.

To complete our argument, we make use of a family of graphs that are forbidden as subgraphs of uniquely C_5 -saturated graphs. By Lemma 6.2.5, $H_{4,2}$ and $H_{6,1}$ are forbidden. We use three additional graphs: $K_{2,3}$; $F_{5,1}$, which is K_4 with a pendant edge; and $F_{5,2}$, consisting of 2-triangles that share a single vertex, and a pendant edge at another vertex. These graphs are shown in Figure 6.5.

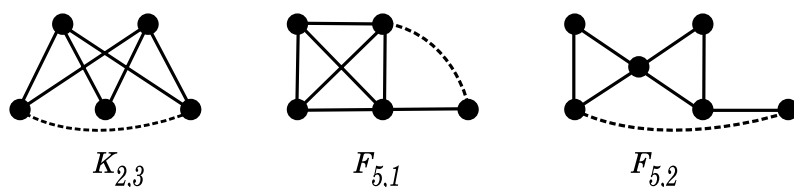


Figure 6.5: Three forbidden subgraphs in the family of uniquely C_5 -saturated subgraphs. The dotted edges indicate edges that complete multiple 5-cycles.

We now forbid 4-cycles in uniquely C_5 -saturated graphs.

Lemma 6.4.1. *The only uniquely C_5 -saturated graph that contains a 4-cycle is K_4 .*

Proof. Let G be a uniquely C_5 -saturated graph, and let S be the vertex set of a 4-cycle in G . Because $H_{4,2}$ is a forbidden subgraph (Lemma 6.2.5) and G is connected, each vertex in $G - S$ has a neighbor in S and $G - S$ has no edges. Because G contains no 5-cycles and $K_{2,3}$ is forbidden, each vertex in $G - S$ has exactly one neighbor in S . Therefore S is the vertex set of a block in G . By Lemma 6.2.2, S is a clique. Because $F_{5,1}$ is forbidden, $V(G - S) = \emptyset$ and $G = K_4$. \square

With 4-cycles forbidden as subgraphs, we can now also forbid 6-cycles from uniquely C_5 -saturated graphs.

Lemma 6.4.2. *If G is uniquely C_5 -saturated, then G does not contain a 6-cycle.*

Proof. Suppose that G is a uniquely C_5 -saturated graph that contains a 6-cycle. By Lemma 6.2.5, $H_{6,1}$ is a forbidden subgraph, so G has exactly six vertices. Every chord in a 6-cycle completes a 4-cycle or a 5-cycle, so by Lemma 6.4.1, the 6-cycle has no chord. Thus G is C_6 , which is not C_5 -saturated. \square

By Lemma 6.2.6, we know that a uniquely C_5 -saturated graph has girth at most 6. It then follows from Lemmas 6.4.1 and 6.4.2 that every uniquely C_5 -saturated graph has girth 3.

Theorem 6.4.3. *If G is a uniquely C_5 -saturated graph with at least five vertices, then G is a friendship graph.*

Proof. We have shown that G has girth 3, so let S be the vertices of a triangle, $S = \{x, y, z\}$. Let x have a neighbor x' that is not y or z . By Lemmas 6.4.1 and 6.4.2, G has no cycle of length 4, 5, or 6, and hence every path with at most five vertices that joins x' to y or z passes through x . In particular, the unique 5-path P joining x' and y contains x . Because there is no 4-path joining adjacent vertices in G (such a path would form a 4-cycle with the edge joining the vertices), x is not adjacent to x' or y in P . It follows that x' is the central vertex in P . Therefore, x and x' are also in a triangle with a third vertex x'' in G . Because $F_{5,2}$ is a forbidden subgraph, x' has no other neighbors in G (and neither do y, z or x''), and every vertex in G is adjacent to x . Because x' was chosen as an arbitrary neighbor of x , $G - x$ is a matching. Thus G is a friendship graph. \square

6.5 Uniquely C_6 -Saturated Graphs

We now prove that there are no nontrivial uniquely C_6 -saturated graphs. We proceed by proving that uniquely C_6 -saturated graphs cannot contain even cycles with length at most 10. Once we have forbidden these short even cycles, we show that nontrivial uniquely C_6 -saturated graphs cannot have girth 3 or 5. In light of the girth bound of Lemma 6.2.6, it follows that there are no nontrivial uniquely C_6 -saturated graphs.

Lemma 6.5.1. *If G is uniquely C_6 -saturated and contains a 4-cycle, then $G \in \{K_4, K_5\}$.*

Proof. Let G be uniquely C_6 -saturated, and let S be the set vertices of a 4-cycle in G . In addition to $H_{4,3}$, we begin with one forbidden subgraph, $F_{6,1}$, which is shown in Figure 6.6

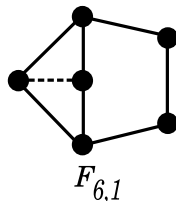


Figure 6.6: A forbidden subgraph in the family of uniquely C_6 -saturated subgraphs. The dotted edge indicates an edge that completes two 6-cycles.

Suppose that $G - S$ contains a 3-path $\langle u, z, v \rangle$, which we call P . By Lemma 6.2.4, we may assume that G is 2-connected, and there are two internally disjoint paths joining u and v to S . At most one of these paths contains z , so we may assume that the path P' joining u to S does not contain v or z . The concatenation of P' and P forms a pendant path of length at least 3 joined to a vertex in S , so $H_{4,3}$ is a subgraph of G . Thus we conclude that $G - S$ induces a matching.

If $G - S$ contains an edge uv , then u and v both have neighbors in S (since G is 2-connected). Because G does not contain 6-cycles or $F_{6,1}$, u and v both have exactly one neighbor in S , which is shared. This vertex in S is a cut vertex, as there is no other path joining u and v to S . Since G is assumed to be 2-connected, $V(G) - S$ is an independent set and each vertex in $V(G) - S$ has at least two neighbors in S .

Suppose that x and y are vertices in S that are not adjacent. Both x and y have at least two neighbors in S . The list of forbidden subgraphs in Figure 6.7 shows that u and v both have exactly two neighbors in S , and those neighbors are common (if x has more than 2 neighbors in S , then one of the graphs in the list is a subgraph). Therefore x and y are twins of degree 2, which is forbidden by Lemma 6.2.7. We conclude that G contains at most five vertices. Therefore G is K_4 or K_5 .

□

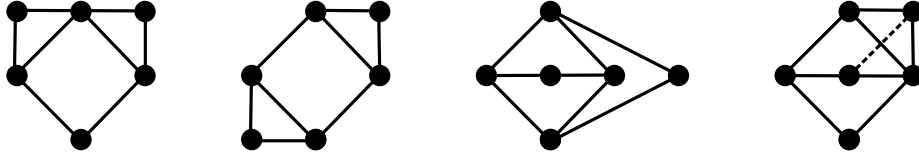


Figure 6.7: Forbidden subgraphs in the family of uniquely C_6 -saturated graphs. The first three contain a 6-cycle, the dotted edge in the fourth completes two 6-cycles.

Having forbidden 4-cycles as subgraphs, we now show that no uniquely C_6 -saturated graph contains an 8-cycle or a 10-cycle.

Lemma 6.5.2. *If G is uniquely C_6 -saturated, then G does not contain an 8-cycle or 10-cycle.*

Proof. By Lemma 6.2.5, $H_{8,1}$ is a forbidden subgraph, so if G contains an 8-cycle, then G has exactly eight vertices. If G contains an edge joining two vertices at distance 2 or 4 on the 8-cycle, then G is not uniquely C_6 -saturated, as is shown in Figure 6.8. If G contains an edge joining two vertices at distance 3 on the 8-cycle, then G contains a 6-cycle. Therefore if G contains an 8-cycle, then it has no additional edges and is not uniquely C_6 -saturated.

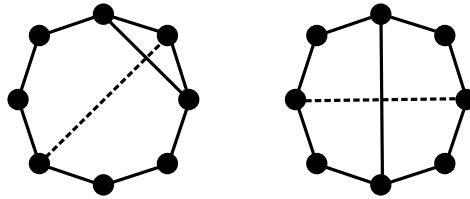


Figure 6.8: Forbidden subgraphs with an 8-cycle.

Lemma 6.2.5 also forbids C_{10} , so G cannot contain a 10-cycle. □

We have shown that all even cycles with length at most 10 are forbidden subgraphs in the family of uniquely C_6 -saturated graphs. We now show that a uniquely C_6 -saturated graph cannot have girth 3, 5, or 7. In light of Lemma 6.2.6, this proves that there are no uniquely C_6 -saturated graphs.

Lemma 6.5.3. *If G is uniquely C_6 -saturated and contains a triangle, then is a complete graph on at most five vertices.*

Proof. We make use of the forbidden subgraph $F_{6,2}$ pictured in Figure 6.9

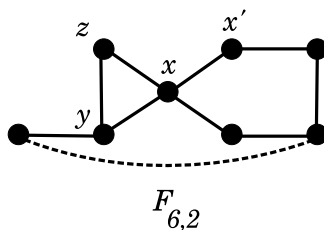


Figure 6.9: A forbidden subgraph in the family of uniquely C_6 -saturated subgraphs. The vertices are labeled as in Lemma 6.5.3.

Let x , y and z be the vertices in a triangle in G , and let $S = \{x, y, z\}$. Suppose that x has a neighbor x' that is not y or z . By Lemmas 6.5.1 and 6.5.2, there are no even cycles of length at most 10 in G , so any x, y -path that is not xy or $\langle x, z, y \rangle$ must contain at least eleven vertices. It follows that x' and y are not adjacent, and the unique 6-path P joining x' and y passes through x . Note that P and the edge xx' complete a cycle C in G with at most five vertices. Because G does not contain even cycles with fewer than twelve vertices, C is either a 3-cycle or a 5-cycle. As there is no 4-path joining x and y , P must be a 5-path. Since $F_{6,2}$ is a forbidden subgraph, it follows that z and y have no neighbors outside of S . Thus x is a cutvertex, a violation of Lemma 6.2.4. \square

Lemma 6.5.4. *There is no uniquely C_6 -saturated with girth 5.*

Proof. Let G be a graph with girth 5; G is shown in Figure 6.10. If G is uniquely C_6 -saturated, then Lemmas 6.5.1 and 6.5.2 imply that G does not contain any 4-, 8- or 10-cycles. Let C be a 5-cycle in G and let x and y be two vertices in C that are not adjacent. Because there are no even cycles of length 4, 6, 8 or 10 in G , any path joining two vertices of C that contains no edges of C has length at least 8. Suppose that x has a neighbor x' that is not in C . Because there are no 4-cycles in G , x' is not adjacent to y , and therefore there is a unique 6-path P joining x' and y . If P does not contain x , then there is a short even cycle in G . The edge $x'x$ is not in P , because this would require a 5-path joining x and y . Because the girth of G is at least 5, the subpath P' of P that joins x' and x contains at least four edges. However, the shortest path joining x and y contains two

We proceed by forbidding structures in $G - S$. First we show that every vertex in $G - S$ has a neighbor in S . Using this, we then show that there are no 3-paths in $G - S$. It follows that $G - S$ is a matching, so we then forbid $G - S$ from containing multiple edges. We then show that if $G - S$ contains a single edge, then G is not uniquely C_7 -saturated. The remaining case is that $G - S$ contains no edges, and a straightforward case check shows that no such graph is uniquely C_7 -saturated.

Clearly, if a vertex in $G - S$ has no neighbors in $G - S$, then it has neighbors in S . Let x and y be two adjacent vertices in $V(G) - S$. By Lemma 6.2.4 we assume that G is 2-connected and there are two internally disjoint paths joining $\{x, y\}$ and S . Because G cannot contain $H_{4,4}$, we know one of these paths contains only one edge, and the other contains at most two edges. If one of the paths contains two edges, then G is not uniquely C_7 -saturated (either G contains a 7-cycle, or a diagonal edge in S completes multiple 7-cycles as shown in Figure 6.11). Therefore x and y have distinct neighbors in S .

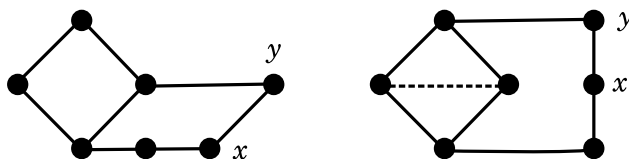


Figure 6.11: Graphs that show that x and y must have distinct neighbors in S .

Assume that $G - S$ contains a 3-path $\langle x, z, y \rangle$. Every vertex in $V(G) - S$ has a neighbor in S , and we have shown that if the endpoints of a 3-path in $G - S$ have distinct neighbors in S , then G is not uniquely C_7 -saturated. Therefore, x and y both have exactly one neighbor in S , which is a common neighbor of x and y . Because we have assumed that G is 2-connected, z has a distinct neighbor in S . Because $H_{4,4}$ is a forbidden subgraph, x and y have no neighbors in $G - S$ outside of $\{x, z, y\}$. If $xy \notin E(G)$, then x and y are twins with degree 2, and by Lemma 6.2.7 G is not uniquely C_7 -saturated. If $xy \in E(G)$, then $\langle x, y, z \rangle$ is a 3-path in $G - S$ whose endpoints have distinct neighbors in S , which is forbidden. Therefore $G - S$ contains no 3-paths.

We may now assume that $G - S$ is a matching. Suppose that there are two edges ux and yz in

$G - S$. Let v_1 and v_3 be an arbitrary pair of antipodal vertices in S . If v_1 has a neighbor in $\{u, x\}$ and v_3 has a neighbor in $\{y, z\}$ then G contains the forbidden subgraph $F_{7,1}$ shown in Figure 6.12.

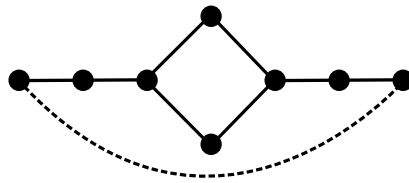


Figure 6.12: The forbidden subgraph $F_{7,1}$.

Since the endpoints of each edge in $G - S$ have distinct neighbors in S , there are two remaining ways that the edges can be joined to S . First, the edges may have two common neighbors that are adjacent in S . Second, the edges may have disjoint sets of neighbors in S , and each edge's neighbors are opposite in S . Both of these are forbidden subgraphs, as shown in Figure 6.12. Thus $G - S$ contains at most one edge.

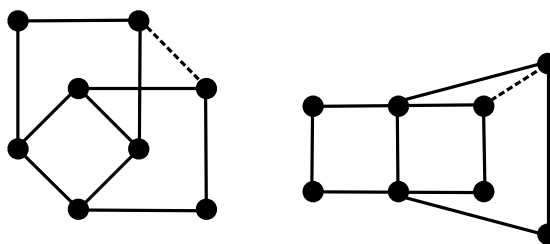


Figure 6.13: Two forbidden subgraphs.

Suppose that there is one edge, xy in $G - S$. If x and y are adjacent to two adjacent vertices v_1 and v_2 in the cycle in S , the G contains a second 4-cycle, $[v_1, v_2, x, y]$. By symmetry, there is at most one edge in $G - \{v_1, v_2, x, y\}$, namely v_3v_4 . Let z be another vertex in $V(G) - S$. Note that z cannot be in an edge in $G - \{v_1, v_2, x, y\}$, so z must have two neighbors in both S and $\{v_1, v_2, x, y\}$. Therefore z has degree 2 and its neighbors are v_1 and v_2 . Because G cannot contain twins of degree 2, there are no additional vertices in G .

If there are no additional edges in G , then G is not uniquely C_7 saturated. If G contains an edge joining antipodal vertices in one of the 4-cycles, then G is not uniquely saturated. If G contains

an edge joining $\{v_3, v_4\}$ and $\{x, y\}$, then G is not uniquely C_7 -saturated. These statements are justified by Figure 6.14. Therefore, the vertices of any edge in $G - S$ must be adjacent to antipodal vertices in S .

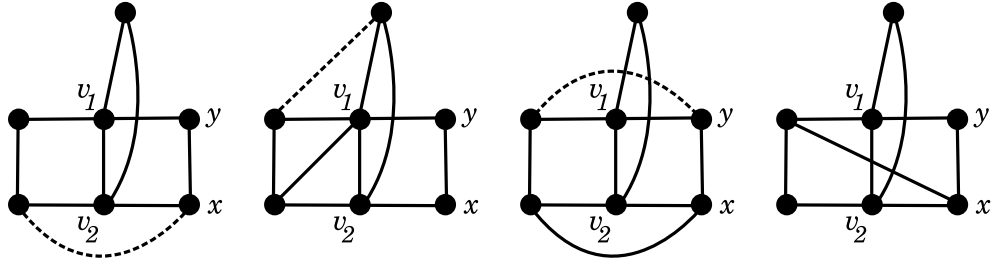


Figure 6.14: When two 4-cycles share an edge. The first graph has an edge that does not complete a 7-cycle, the next two have an edge that completes two 7-cycles, and the fourth contains a 7-cycle.

Now assume that $G - S$ contains an edge xy and it is the middle edge of a 4-path joining v_1 and v_3 . Let z be another vertex in G . It follows that $d(z) \geq 2$ and all neighbors of z lie in S (see Figure 6.15). If z is adjacent to v_2 and v_4 , then G contains a 7-cycle. If z has two neighbors in S that are adjacent in the cycle, then joining z to its nonneighbor in $\{v_2, v_4\}$ completes multiple 7-cycles. Therefore z has degree 2 and is adjacent to v_1 and v_3 . If v_2v_4 is not in G , then z, v_2 and v_4 are all twins with degree 2. Therefore, two of the three vertices in $\{z, v_2, v_4\}$ must have degree at least 3. The only such edge that can be added to G is v_2v_4 . However, zv_2 or zv_4 then completes multiple 7-cycles, a contradiction. Therefore there are no edges in $G - S$.

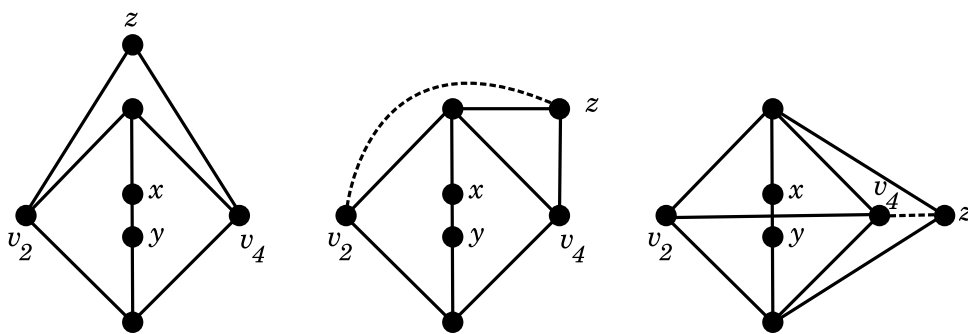


Figure 6.15: When $G - S$ contains an edge xy with antipodal neighbors in S .

We now assume that $V(G) - S$ is an independent set with size at least 3, as G must have at

least seven vertices. For each vertex in $V(G) - S$ we will select two of its incident edges and show that the resulting spanning subgraph G' either contains C_7 or that the addition of an edge to G' completes multiple 7-cycles; it follows that G is not uniquely C_7 -saturated. Note that at most two edges in S are in triangles with vertices in $V(G) - S$, otherwise G' contains a 7-cycle. The remaining cases are discussed below and pictured in Figure 6.16.

If two consecutive edges are in triangles with the vertices x and y , then a third vertex u has opposite vertices in S as its neighbors. If the neighborhoods of x and y cover the neighborhood of u , then an edge joining u to another vertex in S completes multiple 7-cycles. If the neighborhood of u is not covered by the neighborhoods of x and y , then the edge xy completes multiple 7-cycles. Thus two consecutive edges in S cannot be the neighborhoods of vertices in $V(G) - S$.

If two opposite edges are the neighborhoods of vertices x and y , then a third vertex u has opposite vertices in S as its neighbors, and the edge ux completes multiple 7-cycles. Therefore at most one edge is the neighborhood of a vertex in $V(G) - S$. This implies that there is at most one vertex of degree greater than 2 in $V(G) - S$, since the neighborhood of any vertex with degree greater than 2 in $V(G) - S$ includes the endpoints of at least two edges in S .

If there is a vertex z in $V(G) - S$ with degree greater than 2, then no other vertex in $V(G) - S$ can have a neighborhood that contains an edge in S . Therefore, every other vertex in $V(G) - S$ has degree 2, and its neighbors are antipodal vertices in S . Because a uniquely C_7 -saturated graph cannot have twins of degree 2, it follows that there are exactly two more vertices x and y in $G - S$, and these vertices have neighborhoods $\{v_1, v_3\}$ and $\{v_2, v_4\}$ respectively. However, G then contains a 7-cycle. Therefore, every vertex in $V(G) - S$ has degree 2.

Because there can be no twins of degree 2, it follows that there are three vertices in $V(G) - S$. We have shown that at most one of these vertices can be adjacent to the endpoints of an edge in S , so the remaining two vertices are adjacent to the pairs of antipodal vertices in S . However, then G contains a 7-cycle.

We have now shown that $V(G) - S$ is empty. A nontrivial uniquely C_7 -saturated graph must contain at least seven vertices, so G cannot contain a 4-cycle. □

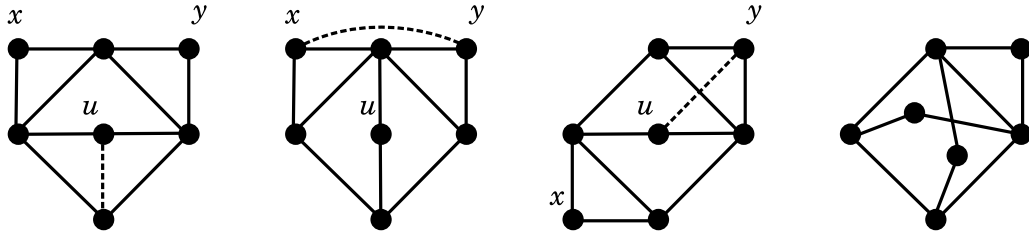


Figure 6.16: When $G - S$ contains three independent vertices. The first three graphs have an edge that completes multiple 7-cycles. The fourth graph contains C_7 .

Knowing that uniquely C_7 -saturated graphs cannot contain 4-cycles, we now show that they cannot contain 6-cycles. The proof has a similar feel to the proof of Lemma 6.6.1, but the case analysis is less extensive.

Lemma 6.6.2. *If G is a nontrivial uniquely C_7 -saturated graph, then G does not contain a 6-cycle.*

Proof. Let G be a uniquely C_7 -saturated graph with a 6-cycle C , and let S denote the set of vertices in C . Suppose that uv is an edge in $G - S$. Because G is 2-connected, there are two disjoint paths joining $\{u, v\}$ and S . If one of these paths contains at least two edges, then G contains $H_{6,3}$, which is forbidden by Lemma 6.2.5. Therefore both paths are single edges, and we can choose distinct neighbors x and y in S for u and v respectively. By Lemma 6.6.1, G does not contain a 4-cycle, so x and y are not adjacent. If x and y are distance 2 in C , then G contains a 7-cycle. Therefore, x and y are antipodal vertices in C . If x' is a neighbor of x in C , then the edge ux' then completes two 7-cycles. Therefore, $G - S$ contains no edges.

Let v be a vertex in $V(G) - S$. Because G is 2-connected, v has at least two neighbors in S . To avoid 7-cycles and 4-cycles, v must have exactly two neighbors, and they must be antipodal vertices in C . Because G cannot contain twins of degree 2, it follows that there are at most three vertices in G_S , and they all have distinct pairs of antipodal neighbors in C . If there are two such vertices, then the edge joining those vertices completes multiple 7-cycles (see Figure 6.17). Therefore $V(G) - S$ has exactly one vertex u . Let x and y be the neighbors of u . The edge xy is not in G because it completes a 4-cycle. However, adding xy to G does not complete a 7-cycle, since any 7-cycle must contain the path $\langle x, u, y \rangle$. Thus $V(G) - S$ is empty, and G cannot be a

nontrivial uniquely C_7 -saturated graph.

□

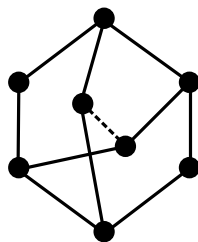


Figure 6.17: A uniquely C_7 -saturated graph cannot contain two vertices outside of a 6-cycle.

Having forbidden 4- and 6-cycles, we can now forbid 8-cycles.

Lemma 6.6.3. *If G is uniquely C_7 -saturated, then G does not contain an 8-cycle.*

Proof. Let C be an 8-cycle in G , and let S be the set of vertices in C . By Lemma 6.2.5, $H_{8,2}$ is a forbidden subgraph, so we conclude that $G - S$ contains no edges.

Let u be a vertex in $V(G) - S$ (see Figure 6.18). Because G is 2-connected, u has two neighbors in S ; call them x and y . In order to avoid 4-, 6-, and 7-cycles, x and y must be adjacent in C , and u has no other neighbors in S . Because G does not contain twins of degree 2, it follows that there are at most 8 vertices in $G - S$. Note that any chord in C now completes a 4-, 6- or 7-cycle, so C has no chords. Let x' be the neighbor of x in C that is not y . Any 7-path P joining x' and u contains a set of at most six consecutive vertices in C , beginning with x' ending with x or y , along with some vertices in $V(G) - S$. The only sets of vertices in C which suffice are $\{x', x\}$ and $\{x', x, y\}$. Because there are no edges in $G - S$, and the 7-path P has x' as an endpoint, there are at most three vertices from $V(G) - S$ in P . It follows that there cannot be seven vertices in any x', u -path in G , so G is not uniquely C_7 -saturated if $V(G) - S$ is nonempty.

We conclude that if G is uniquely C_7 -saturated and contains an 8-cycle, then $|V(G)| = 8$. A chord in an 8-cycle without avoids producing a 4- or 7-cycle only if it joins antipodal vertices. If two such chords are present, then G contains a 4-cycle or a 6-cycle, violating Lemma 6.6.1 or

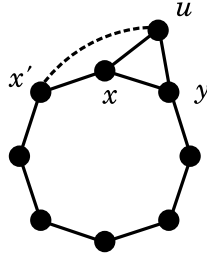


Figure 6.18: An 8-cycle in a uniquely C_7 -saturated graph. The edge $x'u$ cannot complete a 7-cycle, even if x and x' have a common neighbor.

6.6.2. Thus G is an 8-cycle with at most one chord, which joins a pair of antipodal vertices, and G is not C_7 -saturated. \square

We now forbid 10-cycles and 12-cycles from uniquely C_7 -saturated graphs.

Lemma 6.6.4. *If G is uniquely C_7 -saturated, then G does not contain a 10-cycle.*

Proof. By Lemma 6.2.5, $H_{10,1}$ is a forbidden subgraph in the family of uniquely C_7 -saturated graphs, so if G contains a 10-cycle C , then there are no other vertices in G . A chord xy in C with $d_C(x, y) \neq 2$ completes a cycle of length 4, 6 or 7. Therefore every chord in G joins vertices at distance 2 on C . If there are two such chords, then G contains an 8-cycle or a 4-cycle. Thus G is a 10-cycle with at most one chord, which joins two vertices at distance 2 along the cycle. Such a graph is not C_7 -saturated.

Lemma 6.2.5 also forbids C_{12} as a subgraph of a uniquely C_7 -saturated graph. \square

In Lemmas 6.6.1, 6.6.2, 6.6.3, and 6.6.4, we have shown that a nontrivial uniquely C_7 -saturated graph G cannot contain an even cycle on fewer than 14 vertices. We will now show that if G contains a short odd cycle, then G has a cut vertex. This violates Lemma 6.2.4, and completes the proof that there are no nontrivial uniquely C_7 -saturated graphs.

Lemma 6.6.5. *If G is a nontrivial uniquely C_7 -saturated graph, then G cannot have girth 3 or 5.*

Proof. Suppose that G has girth 3 and let S be the set of vertices in a 3-cycle, $S = \{v_1, v_2, v_3\}$ (see Figure 6.19). Suppose that v_1 has a neighbor x that is not v_2 or v_3 . Because G does not contain

any even cycles with length less than 14, x is not adjacent to v_2 or v_3 . Furthermore, there are no paths of length less than 14 joining two vertices in S . Therefore there is a unique 7-path P joining x and v_2 . To avoid short even cycles, P must consist of an odd path joining x and v_1 and then some edges contained in S . Thus there is a path of length 5 joining x and v_1 , and it follows that G contains a triangle and a 5-cycle that share a single vertex, namely v_1 . Let x' be the other neighbor of v_1 in the 5-cycle. If $xx' \in E(G)$, then G contains a 4-cycle. If $xx' \notin E(G)$, then there is a unique 7-path joining x and x' . Such a path has a subpath with at most seven edges joining two vertices on the 5-cycle. Thus G contains an even cycle with at most 12 edges, a contradiction.

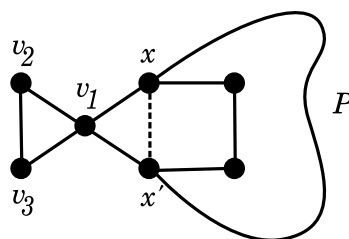


Figure 6.19: The local structure of a uniquely C_7 -saturated graph with a triangle. P is a possible 7-path joining x and x' .

Suppose that G has girth 5. Let x and y be two nonadjacent vertices in a 5-cycle C . If there is a 7-path joining x and y , then there is a path containing at most 7-edges joining two vertices in C with no internal vertices in C . Any such path completes an even cycle with at most 12 edges. Therefore G is not uniquely C_7 -saturated if G has girth 3 or 5. \square

We can now prove that there are no uniquely C_7 saturated graphs. By Lemma 6.2.6, a uniquely C_7 -saturated graph has girth at most 6. By Lemmas 6.6.1 and 6.6.2, nontrivial uniquely C_7 -saturated graphs cannot contain 4-cycles or 6-cycles and by Lemma 6.6.5, they cannot contain 3-cycles or 5-cycles. Therefore we conclude the following.

Theorem 6.6.6. *There are no nontrivial uniquely C_7 -saturated graphs.*

References

- [1] Thomas Andreae. Some results on visibility graphs. *Discrete Appl. Math.*, 40(1):5–17, 1992. Combinatorial methods in VLSI.
- [2] Gabor Bacsó and Zsolt Tuza. Graphs without induced P_5 and C_5 . *Discuss. Math. Graph Theory*, 24(3):503–507, 2004.
- [3] C. A. Barefoot, L. H. Clark, R. C. Entringer, T. D. Porter, L. A. Székely, and Zs. Tuza. Cycle-saturated graphs of minimum size. *Discrete Math.*, 150(1-3):31–48, 1996. Selected papers in honour of Paul Erdős on the occasion of his 80th birthday (Keszthely, 1993).
- [4] B. Bollobás. On a conjecture of Erdős, Hajnal and Moon. *Amer. Math. Monthly*, 74:178–179, 1967.
- [5] J. A. Bondy, P. Erdős, and S. Fajtlowicz. Graphs of diameter two with no 4-circuits. *Discrete Math.*, 200(1-3):21–25, 1999. Paul Erdős memorial collection.
- [6] Leizhen Cai. The recognition of union trees. *Inform. Process. Lett.*, 45(6):279–283, 1993.
- [7] Yi-Wu Chang, Joan P. Hutchinson, Michael S. Jacobson, Jenő Lehel, and Douglas B. West. The bar visibility number of a graph. *SIAM J. Discrete Math.*, 18(3):462–471 (electronic), 2004/05.
- [8] Ya-Chen Chen. Minimum C_5 -saturated graphs. *J. Graph Theory*, 61(2):111–126, 2009.
- [9] Alice M. Dean, William Evans, Ellen Gethner, Joshua D. Laison, Mohammad Ali Safari, and William T. Trotter. Bar k -visibility graphs: bounds on the number of edges, chromatic number, and thickness. In *Graph drawing*, volume 3843 of *Lecture Notes in Comput. Sci.*, pages 73–82. Springer, Berlin, 2006.
- [10] Alice M. Dean, William Evans, Ellen Gethner, Joshua D. Laison, Mohammad Ali Safari, and William T. Trotter. Bar k -visibility graphs. *J. Graph Algorithms Appl.*, 11(1):45–59 (electronic), 2007.
- [11] P. Erdős, A. Hajnal, and J. W. Moon. A problem in graph theory. *Amer. Math. Monthly*, 71:1107–1110, 1964.
- [12] P. Erdős, A. Rényi, and V. T. Sós. On a problem of graph theory. *Studia Sci. Math. Hungar.*, 1:215–235, 1966.

- [13] Michael R. Garey and David S. Johnson. *Computers and intractability*. W. H. Freeman and Co., San Francisco, Calif., 1979. A guide to the theory of NP-completeness, A Series of Books in the Mathematical Sciences.
- [14] Ronald Gould, Tomasz Łuczak, and John Schmitt. Constructive upper bounds for cycle-saturated graphs of minimum size. *Electron. J. Combin.*, 13(1):Research Paper 29, 19 pp. (electronic), 2006.
- [15] Stephen G. Hartke, Jennifer Vandenbussche, and Paul Wenger. Further results on bar k -visibility graphs. *SIAM J. Discrete Math.*, 21(2):523–531 (electronic), 2007.
- [16] A. J. Hoffman and R. R. Singleton. On Moore graphs with diameters 2 and 3. *IBM J. Res. Develop.*, 4:497–504, 1960.
- [17] L. Kászonyi and Zs. Tuza. Saturated graphs with minimal number of edges. *J. Graph Theory*, 10(2):203–210, 1986.
- [18] D. E. Lampert and P. J. Slater. The acquisition number of a graph. In *Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995)*, volume 109, pages 203–210, 1995.
- [19] C. G. Lekkerkerker and J. Ch. Boland. Representation of a finite graph by a set of intervals on the real line. *Fund. Math.*, 51:45–64, 1962/1963.
- [20] F. Luccio, S. Mazzone, and C. K. Wong. Visibility graphs. *University of Pisa, Prog. Naz. Teoria degli Algoritmi, Report 9*, 1983.
- [21] F. Luccio, S. Mazzone, and C. K. Wong. A note on visibility graphs. *Discrete Math.*, 64(2-3):209–219, 1987.
- [22] L. S. Melnikov. Problem at the 6th Hungarian Colloquium on Combinatorics, 1981.
- [23] L. Taylor Ollmann. $K_{2,2}$ saturated graphs with a minimal number of edges. In *Proceedings of the Third Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1972)*, pages 367–392, Boca Raton, Fla., 1972. Florida Atlantic Univ.
- [24] Oystein Ore. Note on Hamilton circuits. *Amer. Math. Monthly*, 67:55, 1960.
- [25] M. Schlag, F. Luccio, P. Maestrini, D. T. Lee, and C. K. Wong. *Advances in Computing Research*, volume 2, pages 259–282. JAI Press Inc., Greenwich, CT, 1985. A visibility problem in VLSI layout compaction.
- [26] Roberto Tamassia and Ioannis G. Tollis. A unified approach to visibility representations of planar graphs. *Discrete Comput. Geom.*, 1(4):321–341, 1986.
- [27] Paul Turán. Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok*, 48:436–452, 1941.

- [28] Douglas B. West. *Introduction to Graph Theory*. Prentice Hall, Upper Saddle River, NJ, 2000.
- [29] Douglas B. West. *Combinatorial Mathematics*. 2010.
- [30] Herbert S. Wilf. The friendship theorem. In *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, pages 307–309. Academic Press, London, 1971.
- [31] S. K. Wismath. Characterizing bar line-of-sight graphs. In *Proceedings of the ACM Symposium on Computational Geometry*, pages 147–152, Baltimore, MD, 1985.