GENERALIZED AND RESTRICTED MULTIPLICATION TABLES OF INTEGERS

BY

DIMITRIOS KOUKOULOPOULOS

DISSERTATION

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Doctoral Committee:

Professor Bruce C. Berndt, Chair
Professor Kevin Ford, Director of Research
Professor A.J. Hildebrand, Contingent Chair
Associate Professor Scott Ahlgren

Abstract

In 1955 Erdős posed the *multiplication table problem*: Given a large integer N, how many distinct products of the form ab with $a \leq N$ and $b \leq N$ are there? The order of magnitude of the above quantity was determined by Ford. The purpose of this thesis is to study generalizations of Erdős's question in two different directions. The first one concerns the k-dimensional version of the multiplication table problem: for a fixed integer $k \geq 3$ and a large parameter N, we establish the order of magnitude of the number of distinct products $n_1 \cdots n_k$ with $n_i \leq N$ for all $i \in \{1, \ldots, k\}$. The second question we shall discuss is the restricted multiplication table problem. More precisely, for $\mathscr{B} \subset \mathbb{N}$ we seek estimates on the number of distinct products $ab \in \mathscr{B}$ with $a \leq N$ and $b \leq N$. This problem is intimately connected with the available information on the distribution of $\mathscr{B}$ in arithmetic progressions. We focus on the special and important case when $\mathscr{B} = P_s = \{p + s : p \text{ prime}\}$ for some fixed $s \in \mathbb{Z} \setminus \{0\}$. Ford established upper bounds of the expected order of magnitude for $|\{ab \in P_s : a \leq N, b \leq N\}|$. We prove the corresponding lower bounds thus determining the size of the quantity in question up to multiplicative constants.

To my parents, Dimitra and Paris

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Chapter 1

Introduction

1.1 The Erdős multiplication table problem

When we learn to multiply in base 10 we memorize the following table.

Table 1.1: The 10×10 multiplication table

	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Even though this multiplication table has 100 entries, only 42 distinct numbers appear in it. In 1955 Erdős [Erd55, Erd60] asked what happens if one considers larger tables, that is for a large integer N what is the asymptotic behavior of

$$A(N) := |\{ab : a \leq N, b \leq N\}|?$$

An argument based on the number of prime factors of a 'typical' integer quickly reveals that

$$A(N) = o(N^2) \quad (N \to \infty).$$

Indeed, we have that

$$\omega(n) := |\{p \text{ prime} : p|n\}| \sim \log\log n$$

on a sequence of integers n of density 1 (see Theorem 1.1 below). So for most pairs of integers (a, b) with $a \le N$ and $b \le N$ the product ab has about $2\log\log N$ prime factors and hence the density of such products in $[1, N^2]$ tends to 0 as $N \to \infty$. Even though this argument may seem a bit naive, a simple generalization of it quickly leads to relatively sharp upper bounds on $A(N)$. Before we proceed we state a well-known result due to Hardy and Ramanujan.

Theorem 1.1 (Hardy-Ramanujan [HarR]). *There are absolute constants C_1 and C_2 such that for all $x \ge 2$ and all $r \in \mathbb{N}$ we have*

$$\pi_r(x) := |\{n \le x : \omega(n) = r\}| \le \frac{C_1 x}{\log x}\frac{(\log\log x + C_2)^{r-1}}{(r-1)!}.$$

Fix now a parameter $\lambda > 1$ and set $L = \lfloor \lambda \log\log N \rfloor$ and

$$Q(\lambda) = \lambda\log\lambda - \lambda + 1.$$

Then

$$\begin{aligned}
A^*(N) &:= |\{ab : a \le N, b \le N, (a,b) = 1\}| \\
&\le |\{n \le N^2 : \omega(n) > L\}| + |\{(a,b) : a \le N, b \le N, \omega(a) + \omega(b) \le L\}| \\
&= \sum_{r > L} \pi_r(N^2) + \sum_{r+s \le L} \pi_r(N)\pi_s(N) \\
&\ll_\lambda \left(1 + (\log N)^{\lambda\log 2 - 1}\right)\frac{N^2}{(\log N)^{Q(\lambda)}(\log\log N)^{1/2}},
\end{aligned}$$

by Theorem 1.1 and Stirling's formula. Choosing $\lambda = 1/\log 2$ in order to optimize the above estimate yields

$$A^*(N) \ll \frac{N^2}{(\log N)^{Q(1/\log 2)}(\log\log N)^{1/2}}.$$

Consequently,

$$A(N) \le \sum_{d\le N} A^*(N/d) \ll \frac{N^2}{(\log N)^{Q(1/\log 2)}(\log\log N)^{1/2}}. \tag{1.1.1}$$

The above argument, which is due to Erdős [Erd60], suggests that most of the distinct entries in the $N \times N$ multiplication table have about $\log\log N/\log 2$ prime factors. Determining the order of magnitude of $A(N)$ boils down to understanding the number of representations of such integers as products ab with $a \le N$ and $b \le N$. This was carried out by Ford in [For08a, For08b], who improved upon estimates of Tenenbaum [Ten84].

Theorem 1.2 (Ford [For08a, For08b])**.** *For $N \ge 3$ we have*

$$A(N) \asymp \frac{N^2}{(\log N)^{Q(1/\log 2)}(\log\log N)^{3/2}}.$$

The main new ingredient in Ford's work was the realization that most of the contribution to $A(N)$ comes from integers n with $\omega(n) = m = \lfloor \log\log N/\log 2\rfloor$ whose sequence of prime factors $p_1 < \cdots < p_m$ satisfies

$$\log\log p_j \ge \frac{j}{m}\log\log N - O(1) \quad (1 \le j \le m). \tag{1.1.2}$$

Furthermore, such integers appear at most a bounded number of times in the multiplication table, at least in an average sense. Via standard probabilistic heuristics we may reduce the probability that condition (1.1.2) holds to the estimation of

$$\mathbf{Prob}\Big(\xi_j \ge \frac{j-O(1)}{m} \;\Big|\; 0 \le \xi_1 \le \cdots \le \xi_m \le 1\Big),$$

which was proven to be about $1/m \asymp 1/\log\log N$ by Ford [For08c]. This estimate together with (1.1.1) gives a rough heuristic explanation of Theorem 1.2.

1.2 The $(k+1)$-dimensional multiplication table problem

A natural generalization of the Erdős multiplication table problem comes from looking at products of more than two integers. More precisely, for a fixed integer $k \geq 2$ and a large integer N we seek estimates for

$$A_{k+1}(N) := |\{n_1 \cdots n_{k+1} : n_i \leq N \ (1 \leq i \leq k+1)\}|.$$

A similar argument with the one leading to (1.1.1) implies

$$A_{k+1}(N) \ll_k \frac{N^{k+1}}{(\log N)^{Q(k/\log(k+1))}(\log\log N)^{1/2}}. \tag{1.2.1}$$

This estimate suggests that most of the distinct entries in the $\underbrace{N \times \cdots \times N}_{k+1 \text{ times}}$ multiplication table have about

$$m = \left\lfloor \frac{k \log\log N}{\log(k+1)} \right\rfloor$$

prime factors. Further analysis of the multiplicative structure of such integers indicates that most of the contribution to $A_{k+1}(N)$ comes from integers n with $\omega(n) = m$ whose prime factors $p_1 < \cdots < p_m$ satisfy

$$\log\log p_j \geq \frac{j}{m} \log\log N - O(1) \quad (1 \leq j \leq m). \tag{1.2.2}$$

As in Ford's work when $k = 1$, this suggests that the order of magnitude of $A_{k+1}(N)$ is the right hand side of (1.2.1) multiplied by $1/\log\log N$. Indeed, we have the following theorem, which was proven in [Kou10a].

Theorem 1.3. *Fix $k \geq 2$. For all $N \geq 3$ we have*

$$A_{k+1}(N) \asymp_k \frac{N^{k+1}}{(\log N)^{Q(k/\log(k+1))}(\log\log N)^{3/2}}.$$

In Section 2.3 we shall give a more precise heuristic explanation of the above theorem. The proof of Theorem 1.3 is based on the methods developed by Ford in [For08a, For08b] to handle the case $k = 1$. The hardest part of the argument consists of showing that the average number of representations in the $(k+1)$-dimensional multiplication table of integers that satisfy (1.2.2) is bounded. We shall elaborate further on this in Section 2.4.

1.3 Shifted primes in the multiplication table

In the previous section we discussed the analogue of the Erdős multiplication table for products of three or more integers. However, even when we consider products of just two integers there are still unresolved questions. For example, given an arithmetic sequence $\mathscr{B} \subset \mathbb{N}$, how many elements of $\mathscr{B}$ appear in the $N \times N$ multiplication table, that is what is the size of

$$A(N; \mathscr{B}) := |\{ab \in \mathscr{B} : a \leq N, b \leq N\}|$$

as $N \to \infty$? We call this the *restricted multiplication table problem.* If $\mathscr{B}$ is reasonably well-distributed in arithmetic progressions $0 \pmod d$, then a relatively straightforward heuristic argument shows that we should have

$$A(N; \mathscr{B}) \approx \frac{|\mathscr{B} \cap [1, N^2]|}{N^2} A(N).$$

We focus on the special and important case when $\mathscr{B} = P_s := \{p + s : p \text{ prime}\}$ for some fixed $s \in \mathbb{Z} \setminus \{0\}$. In [For08b] Ford proved the expected upper bound on $A(N; P_s)$ using the techniques he developed to handle $A(N)$ together with upper sieve estimates.

Theorem 1.4 (Ford [For08b]). *Fix $s \in \mathbb{Z} \setminus \{0\}$. For all $N \geq 3$ we have*

$$A(N; P_s) \ll_s \frac{A(N)}{\log N}.$$

Lower bounds on $A(N; P_s)$ are harder because they need as input more precise information on primes in arithmetic progressions, a problem which is notoriously difficult. The most straightforward way to bound $A(N; P_s)$ from below is to use a *linear sieve*, whose successful application is vitally dependent on having good control of the counting function of primes in arithmetic progressions on average. The standard way of obtaining such control is via the Bombieri-Vinogradov theorem [Dav, p. 161]. However, in this setting this theorem is inapplicable. Indeed, the function $A(N; P_s)$ counts shifted primes of the form $p + s = ab$ with $a \leq N$ and $b \leq N$, which means that in order to bound $A(N; P_s)$ we need control of the number of primes $p \leq N^2 - s$ in arithmetic progressions $-s \pmod a$ of modulus a that can be as large as $N \sim \sqrt{N^2 - s}$. The Bombieri-Vinogradov theorem can only handle arithmetic progressions of modulus $a \leq N^{1-\epsilon}$ for an arbitrarily small, but nevertheless fixed, positive ϵ. To overcome this problem we appeal to a result proven by Bombieri, Friedlander and Iwaniec, which is Theorem 9 in [BFI].

Theorem 1.5 (Bombieri, Friedlander, Iwaniec [BFI]). *Fix $a \in \mathbb{Z} \setminus \{0\}$, $C > 0$ and $\epsilon > 0$. There exists a constant C' depending at most on C such that*

$$\sum_{r \leq R} \left| \sum_{q \leq Q} \left(\pi(x; rq, a) - \frac{\mathrm{li}(x)}{\phi(rq)} \right) \right| \ll_{a,C,\epsilon} \frac{x}{(\log x)^C}$$

uniformly in $R \leq x^{1/10-\epsilon}$ and $RQ \leq x(\log x)^{-C'}$.

Remark 1.3.1. In fact, Theorem 9 in [BFI] is stated in terms of

$$\psi(x; d, a) := \sum_{\substack{p^m \leq x \\ p^m \equiv a \pmod d}} \log p,$$

but a standard partial summation argument can easily convert it to the above form.

Using Theorem 1.5 together with a preliminary sieve, via the fundamental lemma of sieve methods (cf. Lemma 3.1.2) to smoothen certain summands [1], we establish the expected lower bound for $A(N; P_s)$, a result which appeared in [Kou10b].

Theorem 1.6. *Fix $s \in \mathbb{Z} \setminus \{0\}$. For all $N \geq 3$ we have*

$$A(N; P_s) \gg_s \frac{A(N)}{\log N}.$$

1.4 Outline of the dissertation

In Chapter 2 we introduce certain divisor functions, which are the main objects of investigation of this work, and show how to pass from them to the results of Chapter 1. Also, we state our main results about these divisor functions and comment on some of the methods and ideas that are central in their study. In Chapter 3 we list several preliminary results from number theory, analysis and statistics that will be used in subsequent chapters. The first result of Chapter 4 is a reduction theorem that is the starting point towards the proof of our main results. Also, we demonstrate how to reduce the problem of bounding $A(N; P_s)$ to the problem of bounding $A(N)$ and prove Theorem 1.6. Chapter 5 is dedicated to the $(k+1)$-dimensional problem, translated in the language of divisor functions. Finally, in Chapter 6 we comment on some work still in progress and state some preliminary results which generalize our estimates for $A_{k+1}(N)$.

[1] A way to view the fundamental lemma, which lies at the heart of classical sieve methods, is as an attempt to approximate the characteristic function of integers n whose prime factors are greater than z with a 'smooth' function using combinatorial and other methods. Here the role of the smooth approximation is played by a convolution $\lambda * 1$, where λ has small support. The adjective 'smooth' is justified because, by opening the summation in $\lambda * 1$, a single sum weighted with $\lambda * 1$ can be converted to a double sum whose inner sum is weighted with the smooth function 1 and the outer sum has small support

1.5 Notation

We make use of some standard notation. The symbol S_k stands for the set of permutations of $\{1, \ldots, k\}$. If $a(n)$, $b(n)$ are two arithmetic functions, then we denote with $a*b$ their Dirichlet convolution. For $n \in \mathbb{N}$ we use $P^+(n)$ and $P^-(n)$ to denote the largest and smallest prime factor of n, respectively, with the notational conventions that $P^+(1) = 0$ and $P^-(1) = +\infty$. Furthermore, $\tau(n)$ stands for the number of divisors of n, $\omega(n)$ for the number of distinct prime factors of n and $\Omega(n)$ for the total number of prime factors of n. Given $1 \le y < z$, $\mathscr{P}(y, z)$ denotes the set of all integers n such that $P^+(n) \le z$ and $P^-(n) > y$. Finally, $\pi(x; q, a)$ stands for the number of primes up to x in the arithmetic progression $a \pmod q$ and $\mathrm{li}(x)$ for the logarithmic integral $\int_2^x dt/\log t$.

Constants implied by $\ll$, $\gg$ and $\asymp$ are absolute unless otherwise specified, e.g. by a subscript. Also, we use the letters c and C to denote constants, not necessarily the same ones in every place, possibly depending on certain parameters that will be specified by subscripts and other means. Also, bold letters always denote vectors whose coordinates are indexed by the same letter with subscripts, e.g. $\boldsymbol{a} = (a_1, \ldots, a_k)$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_r)$. The dimension of the vectors will not be explicitly specified if it is clear by the context.

Finally, we give a table of some basic non-standard notation that we will be using with references to page numbers for its definition.

Symbol	Page
$Q(\lambda)$	2
P_s	5
η	11
$H(x, y, z)$	10
$H(x, y, z; P_s)$	12
$H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$	13
$\mathcal{L}(a; \sigma)$	15
$L(a; \sigma)$	15
$\mathcal{L}^{(k+1)}(\boldsymbol{a})$	16
$L^{(k+1)}(\boldsymbol{a})$	16
$L^{(k+1)}(a)$	19
$S^{(k+1)}(\boldsymbol{t})$	38
$\tau_{k+1}(\boldsymbol{a})$	17
$\tau_{k+1}(a)$	15
$\tau_{k+1}(a, \boldsymbol{y}, 2\boldsymbol{y})$	18
$\mathscr{P}_*(y, z)$	16
$\mathscr{P}_*^k(\boldsymbol{t})$	16
$\boldsymbol{e}_k,\ e_{k,i}$	15
ρ	62

Chapter 2

Main results

In this chapter we shift our focus from the multiplication table to certain divisor functions which will be the main technical objects of investigation.

2.1 Local divisor functions

In [For08b] Ford deduced Theorem 1.2 via his bounds on a closely related function: For positive real numbers x, y and z define

$$H(x, y, z) = |\{n \leq x : \exists\, d|n \text{ with } y < d \leq z\}|.$$

Using dyadic decomposition we can relate $A(N)$ to the size of $H(x, y, 2y)$. Indeed, we have that

$$H\Big(\frac{N^2}{2}, \frac{N}{2}, N\Big) \leq A(N) \leq \sum_{m \geq 0} H\Big(\frac{N^2}{2^m}, \frac{N}{2^{m+1}}, \frac{N}{2^m}\Big). \tag{2.1.1}$$

There are two main advantages in working with $H(x, y, 2y)$ - and, more generally, with $H(x, y, z)$ - instead of $A(N)$. Firstly, bounds on $H(x, y, 2y)$ are applicable to problems beyond the $N \times N$ multiplication table; we refer the reader to [For08b] for several such applications. Secondly, bounding $H(x, y, 2y)$ is technically slightly easier than bounding $A(N)$.

In [For08b] Ford determined the order of magnitude of $H(x, y, z)$ uniformly for all choices of parameters x, y, z. In order to state his result we introduce some notation. For a given pair (y,z) with $2 \le y < z$ define η, u, β and ξ by

$$z = e^{\eta} y = y^{1+u}, \quad \eta = (\log y)^{-\beta}, \quad \beta = \log 4 - 1 + \frac{\xi}{\sqrt{\log\log y}}.$$

Furthermore, set

$$z_0(y) = y \exp\{(\log y)^{-\log 4 + 1}\} \approx y + y(\log y)^{-\log 4+1}$$

and

$$G(\beta) = \begin{cases} Q\Big(\dfrac{1+\beta}{\log 2}\Big), & 0 \le \beta \le \log 4 - 1, \\ \beta, & \log 4 - 1 \le \beta. \end{cases}$$

Theorem 2.1 (Ford [For08b]). *Let* $3 \le y + 1 \le z \le x$.

(a) If $y \le \sqrt{x}$, *then*

$$\frac{H(x,y,z)}{x} \asymp \begin{cases} \log(z/y) = \eta, & y+1 \le z \le z_0(y), \\ \dfrac{\beta}{\max\{1, -\xi\}(\log y)^{G(\beta)}}, & z_0(y) \le z \le 2y, \\ u^{Q(1/\log 2)}(\log \frac{2}{u})^{-3/2}, & 2y \le z \le y^2, \\ 1, & z \ge y^2. \end{cases}$$

(b) If $y > \sqrt{x}$, *then*

$$H(x,y,z) \asymp \begin{cases} H(x, \frac{x}{z}, \frac{x}{y}), & \text{if } \frac{x}{y} \ge \frac{x}{z} + 1, \\ \eta x, & \textit{else}. \end{cases}$$

Theorem 1.2 then follows as an immediate corollary of the above theorem and inequality (2.1.1).

In a similar fashion, instead of estimating $A(N; P_s)$ we work with the function

$$H(x, y, z; P_s) := |\{p + s \le x : \exists\, d | p + s \text{ with } y < d \le z\}|.$$

This function was studied in [For08b], where it was shown to satisfy the expected upper bound.

Theorem 2.2 (Ford [For08b])**.** *Fix $s \in \mathbb{Z} \setminus \{0\}$. For $3 \le y + 1 \le z \le x$ with $y \le \sqrt{x}$ we have*

$$H(x, y, z; P_s) \ll_s \begin{cases} \dfrac{H(x, y, z)}{\log x}, & \text{if } z \ge y + (\log y)^{2/3}, \\ \dfrac{x}{\log x} \displaystyle\sum_{y < d \le z} \dfrac{1}{\phi(d)}, & \text{else.} \end{cases}$$

Remark 2.1.1. The reason that the upper bound in Theorem 2.2 has this particular shape is due to our incomplete knowledge about the sum $\sum_{y<d\le z} \frac{1}{\phi(d)}$ when the interval $(y, z]$ is very short. The main theorem in [Sit] implies that

$$\sum_{y<d\le z} \frac{1}{\phi(d)} \asymp \log(z/y) \quad (z \ge y + (\log y)^{2/3}),$$

whereas standard conjectures on Weyl sums would yield that

$$\sum_{y<d\le z} \frac{1}{\phi(d)} \asymp \log(z/y) \quad (z \ge y + \log\log y). \tag{2.1.2}$$

The range of y and z in (2.1.2) is the best possible one can hope for, since it is well-known that the order of $n/\phi(n)$ can be as large as $\log\log n$ if n has many small prime factors.

In addition to Theorem 2.2, Ford proved a lower bound of the expected size for $H(x, y, z; P_s)$ in a special case of the parameters.

Theorem 2.3 (Ford [For08b]). *Fix $s \in \mathbb{Z} \setminus \{0\}$, $0 \le a < b \le 1$. For $x \ge 2$ we have*

$$H(x, x^a, x^b; P_s) \gg_{s,a,b} \frac{x}{\log x}.$$

In [Kou10b] we extended the range of validity of the above theorem significantly. We state below a weak form of Theorem 6 in [Kou10b].

Theorem 2.4. *Fix $s \in \mathbb{Z} \setminus \{0\}$ and $C \ge 2$. For $3 \le y + 1 \le z \le x$ with $y \le \sqrt{x}$ and $z \ge y + y(\log y)^{-C}$ we have*

$$H(x, y, z; P_s) \gg_{s,C} \frac{H(x, y, z)}{\log x}.$$

Remark 2.1.2. In [Kou10b] more general results were proven, which partially cover the range $z \le y + y(\log y)^{-C}$ as well. However, for the sake of the economy of the exposition we shall not state or prove these results, since the main motivation of this dissertation is the multiplication table and its generalizations for which Theorem 2.4 is sufficient.

Theorem 2.4 will be shown in Section 4.4. Combining Theorems 2.2 and 2.4 with an inequality similar to (2.1.1) we immediately obtain Theorems 1.2 and 1.6.

Finally, continuing in the above spirit, instead of studying $A_{k+1}(N)$ directly, we focus on the counting function of localized factorizations of integers, which is defined for $x \ge 1$, $\boldsymbol{y} \in [0, +\infty)^k$ and $\boldsymbol{z} \in [0, +\infty)^k$ by

$$H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y}) = |\{n \le x : \exists\, d_1 \cdots d_k | n \text{ with } y_i < d_i \le z_i \ (1 \le i \le k)\}|.$$

Theorem 2.5 establishes the expected quantitative relation between $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ and

$$A_{k+1}(N_1, \ldots, N_{k+1}) := |\{n_1 \cdots n_{k+1} : n_i \le N_i \ (1 \le i \le k+1)\}|,$$

where $N_1, \ldots, N_{k+1}$ are large integers.

Theorem 2.5. *Fix $k \geq 2$. For $3 \leq N_1 \leq N_2 \leq \cdots \leq N_{k+1}$ we have*

$$A_{k+1}(N_1, \dots, N_{k+1}) \asymp_k H^{(k+1)}\Big(N_1 \cdots N_{k+1}, \Big(\frac{N_1}{2}, \dots, \frac{N_k}{2}\Big), (N_1, \dots, N_k)\Big).$$

Remark 2.1.3. We call the problem of estimating $A_{k+1}(N_1, \dots, N_{k+1})$ for arbitrary choices of $N_1, \dots, N_{k+1}$ the *generalized multiplication table problem.*

The proof of Theorem 2.5 will be given in Section 4.3. It is worth noticing that its proof does not depend on knowing the exact size of $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$; rather, we deduce it from a reduction result for $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ (cf. Theorem 2.8). In view of Theorem 2.5, in order to bound $A_{k+1}(N)$ it suffices to bound $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ when $y_1 = \cdots = y_k$, uniformly in x and y_1. Thus the following estimate, which appeared in [Kou10a], completes the proof of Theorem 1.3.

Theorem 2.6. *Fix $k \geq 2$ and $c \geq 1$. Let $x \geq 1$ and $3 \leq y_1 \leq \cdots \leq y_k \leq y_1^c$ with $2^k y_1 \cdots y_k \leq x/y_k$. Then*

$$H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y}) \asymp_{k,c} \frac{x}{(\log y_1)^{Q(k/\log(k+1))}(\log\log y_1)^{3/2}}.$$

Theorem 2.6 will be proven in Chapter 5.

Remark 2.1.4. The condition $2^k y_1 \cdots y_k \leq x/y_k$ causes essentially no harm to generality because of the following elementary reason: if $d_1 \cdots d_k | n$ and we set $d_{k+1} = n/(d_1 \cdots d_k)$, then $d_1 \cdots d_{k-1} d_{k+1} | n$.

2.2 From local to global divisor functions

In [For08b] the first important step in the study of $H(x, y, z)$ is the reduction of the counting in $H(x, y, z)$, which contains information about the local distribution of the divisors of an integer, to the estimation of certain quantities that carry information about the global distribution of the divisors of integers. More precisely, for $a \in \mathbb{N}$ and $\sigma > 0$ define

$$\mathcal{L}(a;\sigma) = \bigcup_{d|a} [\log d - \sigma, \log d)$$

and

$$L(a;\sigma) = \mathrm{Vol}(\mathcal{L}(a;\sigma)).$$

Then we have the following theorem.

Theorem 2.7 (Ford [For08b]). *Fix $\epsilon > 0$ and $B > 0$. For $3 \le y+1 \le z \le x$ with $y \le \sqrt{x}$ and*

$$y + \frac{y}{(\log y)^B} \le z \le y^{101/100}$$

we have

$$H(x,y,z) \asymp_{\epsilon,B} \frac{x}{\log^2 y} \sum_{\substack{a \le y^\epsilon \\ \mu^2(a)=1}} \frac{L(a;\eta)}{a}.$$

Remark 2.2.1. Even though the above theorem is not stated explicitly in [For08b], it is a direct corollary of the methods there: see Theorem 1 and Lemmas 4.1, 4.2, 4.5, 4.8 and 4.9 in [For08b].

As we will demonstrate in Section 4.4, the proof of Theorem 2.4 passes through the proof of a reduction result for $H(x,y,z;P_s)$ analogous to Theorem 2.7 for $H(x,y,z)$.

Similarly, the first step towards the proof of Theorem 2.5 consists of showing a generalization of Theorem 2.7. First, let

$$\boldsymbol{e}_k = (e_{k,1},\dots,e_{k,k}) = (1,1,\dots,1,2) \in \mathbb{R}^k.$$

For $a \in \mathbb{N}$ and $\boldsymbol{a} \in \mathbb{N}^k$ define

$$\tau_{k+1}(a) = |\{(d_1,\dots,d_k) \in \mathbb{N}^k : d_1 \cdots d_k | a\}|,$$

$$\mathcal{L}^{(k+1)}(\boldsymbol{a}) = \bigcup_{\substack{d_1\cdots d_i | a_1\cdots a_i \\ 1\le i\le k}} [\log(d_1/2), \log d_1) \times \cdots \times [\log(d_k/2), \log d_k)$$

and

$$L^{(k+1)}(\boldsymbol{a}) = \mathrm{Vol}(\mathcal{L}^{(k+1)}(\boldsymbol{a})).$$

Also, for $1 \le y < z$ set

$$\mathscr{P}_*(y,z) = \{n \in \mathscr{P}(y,z) : \mu^2(n) = 1\}$$

and for $\boldsymbol{t} = (t_1, \dots, t_k)$ with $1 = t_0 \le t_1 \le \cdots \le t_k$ set

$$\mathscr{P}_*^k(\boldsymbol{t}) = \{\boldsymbol{a} \in \mathbb{N}^k : a_i \in \mathscr{P}_*(t_{i-1}, t_i) \ (1 \le i \le k)\}.$$

Theorem 2.8. *Fix $k \ge 1$. For $x \ge 1$ and $3 \le y_1 \le \cdots \le y_k$ with $2^k y_1 \cdots y_k \le x/y_k$ we have*

$$\frac{H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})}{x} \asymp_k \left(\prod_{i=1}^{k} \log^{-e_{k,i}} y_i\right) \sum_{\boldsymbol{a}\in\mathscr{P}_*^k(\boldsymbol{y})} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k}.$$

Theorem 2.8 will be proven in Sections 4.1 and 4.2. As an immediate consequence of it, we have the following result.

Corollary 2.1. *Let $k \ge 2$ and for $i \in \{1,2\}$ consider $x_i \ge 1$ and $\boldsymbol{y}_i = (y_{i,1}, \dots, y_{i,k}) \in [1, +\infty)^k$. Assume that $2^k y_{i,1}\cdots y_{i,k} \le x_i/y_{i,k}$ for $i \in \{1,2\}$ and that there exist constants c and C such that $y_{1,j}^c \le y_{2,j} \le y_{1,j}^C$ for all $j \in \{1, \dots, k\}$. Then*

$$\frac{H^{(k+1)}(x_1, \boldsymbol{y}_1, 2\boldsymbol{y}_1)}{x_1} \asymp_{k,c,C} \frac{H^{(k+1)}(x_2, \boldsymbol{y}_2, 2\boldsymbol{y}_2)}{x_2}.$$

Proof. The result follows easily by Theorem 2.8, Lemma 2.3.1(b) below and the standard estimate

$$\sum_{a\in\mathscr{P}_*(t,t^B)} \frac{\tau_m(a)}{a} \asymp_{m,B} 1 \quad (t \ge 1),$$

which holds for every fixed $m \geq 1$ and $B \geq 1$. $\square$

When $k = 1$, a stronger version of the above corollary is known to be true: see Corollary 1 in [For08b].

2.3 A heuristic argument for $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$

In this section we develop a heuristic argument which gives a rough explanation of Theorem 2.6 as well as how condition (1.2.2) makes its appearance in the study of $A_{k+1}(N)$. It is a generalization of an argument given by Ford in [For08a] for the case $k = 1$. Before we delve into the details of this argument, we state a simple but basic result we will be using extensively throughout this dissertation. With a slight abuse of notation, for $\boldsymbol{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$ set

$$\tau_{k+1}(\boldsymbol{a}) = |\{(d_1, \ldots, d_k) \in \mathbb{N}^k : d_1 \cdots d_i | a_1 \cdots a_i \ (1 \leq i \leq k)\}|.$$

Then we have the following lemma.

Lemma 2.3.1. *(a) For $\boldsymbol{a} \in \mathbb{N}^k$ we have*

$$L^{(k+1)}(\boldsymbol{a}) \leq \min\Big\{\tau_{k+1}(\boldsymbol{a})(\log 2)^k, \prod_{i=1}^{k}(\log a_1 + \cdots + \log a_i + \log 2)\Big\}.$$

(b) If $(a_1 \cdots a_k, b_1 \cdots b_k) = 1$, then

$$L^{(k+1)}(a_1 b_1, \ldots, a_k b_k) \leq \tau_{k+1}(\boldsymbol{a}) L^{(k+1)}(\boldsymbol{b}).$$

(c) For $(a, b) = 1$ and $\sigma > 0$ we have

$$L(ab; \sigma) \leq \tau(a) L(b; \sigma).$$

Proof. Parts (a) and (b) have very similar proofs with items (i) and (ii) of Lemma 3.1 in [For08b], respectively. Part (c) is item (ii) of Lemma 3.1 in [For08b]. □

Consider real numbers $x \geq 1$ and $3 \leq y_1 \leq \cdots \leq y_k$ as in Theorem 2.6. Given $n \in \mathbb{N} \cap [1, x]$ we decompose it as $n = ab$, where

$$a = \prod_{p^l \| n, p \leq 2y_k} p^l.$$

For simplicity assume that a is square-free and that $\log a \asymp \log y_1$. The integer n is counted by $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ if, and only if,

$$\tau_{k+1}(n, \boldsymbol{y}, 2\boldsymbol{y}) := |\{(d_1, \ldots, d_k) \in \mathbb{N}^k : d_1 \cdots d_k | n, y_i < d_i \leq 2y_i \ (1 \leq i \leq k)\}| \geq 1.$$

Consider the set

$$D_{k+1}(a) = \{(\log d_1, \ldots, \log d_k) : d_1 \cdots d_k | a\}$$

and assume for the moment that $D_{k+1}(a)$ is well-distributed in $[0, \log a]^k$. Then we would expect that

$$\tau_{k+1}(n, \boldsymbol{y}, 2\boldsymbol{y}) = \tau_{k+1}(a, \boldsymbol{y}, 2\boldsymbol{y}) \approx \frac{(\log 2)^k}{(\log a)^k} \tau_{k+1}(a) \approx \frac{(k+1)^{\omega(a)}}{(\log y_1)^k}.$$

Therefore we should have that $\tau_{k+1}(n, \boldsymbol{y}, 2\boldsymbol{y}) \geq 1$ precisely when

$$\omega(a) \geq m = \left\lfloor \frac{k \log \log y_1}{\log(k+1)} + O(1) \right\rfloor.$$

Since

$$|\{n \leq x : \omega(a) = r\}| \approx \frac{x}{\log y_1} \frac{(\log \log y_1)^{r-1}}{(r-1)!}$$

(see [Ten, Theorem 4, p. 205]), we arrive at the heuristic estimate

$$\begin{aligned} H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y}) &\approx \frac{x}{\log y_1} \sum_{r \geq m} \frac{(\log\log y_1)^{r-1}}{(r-1)!} \\ &\asymp \frac{x}{(\log y_1)^{Q(k/\log(k+1))}(\log\log y_1)^{1/2}}. \end{aligned}$$

Comparing this estimate with Theorem 2.6 we see that we are off by a factor of $\log\log y_1$. The reason for this discrepancy lies in our assumption that $D_{k+1}(a)$ is well-distributed. Actually, most of the time the elements of $D_{k+1}(a)$ form big clumps. A way to measure this clustering is the quantity

$$\begin{aligned} L^{(k+1)}(a) &:= L^{(k+1)}(a, 1, 1, \dots, 1) \\ &= \mathrm{Vol}\Big(\bigcup_{d_1 \cdots d_k | a} [\log(d_1/2), \log d_1) \times \cdots \times [\log(d_k/2), \log d_k) \Big). \end{aligned}$$

Consider n with $\omega(a) = m$ and let $p_1 < \cdots < p_m$ be the sequence of prime factors of a. We expect that the numbers $p_1, \dots, p_m$ are uniformly distributed on a $\log\log$ scale, that is

$$\log\log p_j \sim \frac{j}{m} \log\log y_1 \quad (1 \leq j \leq m).$$

But we also expect that the quantities $\log\log p_j$ deviate from their mean value $j \log\log y_1/m$. In particular, the Law of the Iterated Logarithm [HT, Theorem 11] implies that if $C = O(\sqrt{\log\log y_1})$, then with probability tending to 1 as $y_1 \to \infty$ there is some $j \in \{1, \dots, m\}$ such that

$$\log\log p_j \leq \frac{j}{m} \log\log y_1 - C.$$

Therefore Lemma 2.3.1 yields

$$\begin{aligned} L^{(k+1)}(a) \le \tau_{k+1}(p_{j+1}\cdots p_k)L^{(k+1)}(p_1\cdots p_j) &\le (k+1)^{m-j}\log^k(2p_1\cdots p_j) \\ &\lessapprox (k+1)^{m-j}\log^k p_j \\ &\le (k+1)^{m-j}e^{kj\log\log y_1/m-kC} \\ &\asymp (k+1)^m e^{-kC}, \end{aligned}$$

which is much less than $\tau_{k+1}(a) = (k+1)^m$ if $C \to \infty$ as $y_1 \to \infty$. This suggests that we should focus on integers n for which

$$\log\log p_j \ge \frac{j}{m}\log\log y_1 - O(1) \quad (1 \le j \le m). \tag{2.3.1}$$

As we mentioned in Chapter 1, the probability that the above condition holds is about $1/m \asymp 1/\log\log y_1$ [For07]. So we deduce the refined heuristic estimate

$$H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y}) \approx \frac{x}{(\log y_1)^{Q(k/\log(k+1))}(\log\log y_1)^{3/2}},$$

which turns out to be the correct one.

2.4 Some comments about the proof of Theorem 2.6

The hardest part in the proof of Theorem 1.3 is showing that if n satisfies (2.3.1), then $D_{k+1}(a)$ is well-distributed in the sense that $L^{(k+1)}(a) \gg (k+1)^m \asymp (\log a)^k$ on average. One way to bound $L^{(k+1)}(a)$ from below is to use Hölder's inequality. In turn, this reduces to estimating sums of the form

$$\sum_a M_p(a) w_a,$$

where w_a are certain weights and

$$M_p(a) = \int \tau_{k+1}(a, e^{\boldsymbol{u}}, 2e^{\boldsymbol{u}})^p d\boldsymbol{u}$$

with the notational convention that $e^{\boldsymbol{u}} = (e^{u_1}, \ldots, e^{u_k})$ for $\boldsymbol{u} = (u_1, \ldots, u_k)$. Indeed, this approach with $p = 2$ is used in [For08a, For08b] in order to bound $H(x, y, 2y)$ and can be generalized to show Theorem 2.6 when $k \leq 3$. However, when $k > 3$ this method breaks down because the L^2 norm under consideration is too big. To overcome this problem we are forced to consider L^p norms for some fixed $p \in (1, 2)$. The main difficulty in this modified approach can be described as follows. A straightforward computation shows that

$$M_p(a) \approx \sum_{d_1 \cdots d_k | a} \Bigg(\sum_{\substack{e_1 \cdots e_k | a \\ |\log(e_i/d_i)| < \log 2 \\ 1 \leq i \leq k}} 1 \Bigg)^{p-1}. \tag{2.4.1}$$

A key feature of the L^2 norm, which is taken advantage of in [For08a, For08b], is its combinatorial interpretation: as (2.4.1) indicates, it can be viewed as counting pairs of points under certain constraints. However, when one considers L^p norms for $p \in (1, 2)$, this combinatorial interpretation is lost due to the fractional exponent $p - 1$ in the right hand side of (2.4.1). In order to circumvent this problem we perform a special type of interpolation between L^1 and L^2 estimates. We have

$$\begin{aligned} \sum_a M_p(a) w_a &\approx \sum_a w_a \sum_{d_1 \cdots d_k | a} \Bigg(\sum_{\substack{e_1 \cdots e_k | a \\ |\log(e_i/d_i)| < \log 2 \\ 1 \leq i \leq k}} 1 \Bigg)^{p-1} \\ &= \sum_{d_1, \ldots, d_k} \sum_{a \equiv 0 \pmod{d_1 \cdots d_k}} w_a \Bigg(\sum_{\substack{e_1 \cdots e_k | a \\ |\log(e_i/d_i)| < \log 2 \\ 1 \leq i \leq k}} 1 \Bigg)^{p-1}. \end{aligned}$$

Hence Hölder's inequality implies that

$$\sum_a M_p(a) w_a \lessapprox \sum_{d_1,\dots,d_k} \Bigg(\sum_{a \equiv 0 \, (\mathrm{mod}\, d_1 \cdots d_k)} w_a \Bigg)^{2-p} \Bigg(\sum_{a \equiv 0 \, (\mathrm{mod}\, d_1 \cdots d_k)} w_a \sum_{\substack{e_1 \cdots e_k | a \\ |\log(e_i/d_i)| < \log 2 \\ 1 \le i \le k}} 1 \Bigg)^{p-1}.$$

Note that the sums

$$\sum_{a \equiv 0 \, (\mathrm{mod}\, d_1 \cdots d_k)} w_a \quad \text{and} \quad \sum_{a \equiv 0 \, (\mathrm{mod}\, d_1 \cdots d_k)} w_a \sum_{\substack{e_1 \cdots e_k | a \\ |\log(e_i/d_i)| < \log 2 \\ 1 \le i \le k}} 1$$

can be viewed as incomplete first and second moments, respectively. The crucial feature of the interpolation described above is that Hölder's inequality is applied at a point where it is essentially sharp: the contribution from the incomplete second moment is tamed by the small exponent $p - 1$. Lastly, the incomplete first and second moments are estimated via combinatorial means. We shall describe this argument rigorously in Sections 5.3 and 5.4. See also Remark 5.2.1.

Chapter 3

Auxiliary results

In this chapter we list various results from number theory, analysis and statistics we will be using throughout the rest of this work.

3.1 Number theoretic results

We shall need various results from number theory, predominantly from sieve methods. We start with the following standard estimate [HR, Theorem 8.4].

Lemma 3.1.1. *Uniformly in $4 \leq 2z \leq x$ we have*

$$|\{n \leq x : P^-(n) > z\}| \asymp \frac{x}{\log z}.$$

Next, we state a result known as the 'fundamental lemma' of sieve methods. It has appeared in the literature in several different forms (see for example [HR, Theorem 2.5, p. 82]). We need a version of it that can be found in [FI] and [Iwa80b].

Lemma 3.1.2. *Let $D \geq 2$, $D = Z^t$ with $t \geq 3$.*

(a) Fix $\kappa > 0$. There exist two sequences $\{\lambda^+(d)\}_{d \leq D}$, and $\{\lambda^-(d)\}_{d \leq D}$ such that

$$|\lambda^{\pm}(d)| \leq 1,$$

$$\begin{cases} (\lambda^- * 1)(n) = (\lambda^+ * 1)(n) = 1 & \text{if } P^-(n) > Z, \\ (\lambda^- * 1)(n) \leq 0 \leq (\lambda^+ * 1)(n) & \text{otherwise}, \end{cases}$$

and, for any multiplicative function $f(d)$ with $0 \le f(p) \le \min\{\kappa, p-1\}$,

$$\sum_{d \le D} \lambda^{\pm}(d) \frac{f(d)}{d} = \prod_{p \le Z} \Big(1 - \frac{f(p)}{p}\Big)(1 + O_\kappa(e^{-t})).$$

(b) There exists a sequence $\{\lambda_0(d)\}_{d \le D}$ such that

$$|\lambda_0(d)| \le 1, \tag{3.1.1}$$

$$\begin{cases} (\lambda_0 * 1)(n) = 1 & \textit{if } P^-(n) > Z, \\ (\lambda_0 * 1)(n) \le 0 & \textit{otherwise}, \end{cases} \tag{3.1.2}$$

and, for any multiplicative function $f(d)$ satisfying $0 \le f(p) \le p-1$ and

$$\prod_{y < p \le w} \Big(1 - \frac{f(p)}{p}\Big)^{-1} \le \frac{\log w}{\log y}\Big(1 + \frac{C}{\log y}\Big) \quad (3/2 \le y \le w), \tag{3.1.3}$$

we have

$$\sum_{d \le D} \lambda_0(d) \frac{f(d)}{d} \gg \prod_{p \le Z} \Big(1 - \frac{f(p)}{p}\Big), \tag{3.1.4}$$

provided that $D \ge D_0(C)$, where $D_0(C)$ is a constant depending only on C.

Proof. (a) The result follows by [FI, Lemma 5, p. 732].

(b) The construction of the sequence $\{\lambda_0(d)\}_{d \le D}$ and the proof that it satisfies the desired properties is based on [FI, Lemma 5] and [Iwa80b, Lemma 3]. We sketch the proof below. Without loss of generality we may assume that $Z \notin \mathbb{N}$. Set $P(Z) = \prod_{p<Z} p$ and $\lambda_0(d) = \mu(d)\mathbf{1}_{\mathscr{A}}(d)$, where $\mathbf{1}_{\mathscr{A}}$ is the characteristic function of the set

$$\mathscr{A} = \{d | P(Z) : d = p_1 \cdots p_r,\ p_r < \cdots < p_1 < Z,\ p_{2l}^3 p_{2l-1} \cdots p_1 < D\ (1 \le l \le r/2)\}.$$

By the proof of Lemma 5 in [FI], the sequence $\{\lambda_0(d)\}_{d=1}^{\infty}$ is supported in $\{d \in \mathbb{N} : d < D\}$

and satisfies (3.1.1) and (3.1.2). Finally, by Lemma 3 in [Iwa80b], there exists a function h, independent of the parameters D, Z and K, such that

$$\sum_{d\le D}\lambda_0(d)\frac{f(d)}{d}\ge\big(h(t)+O(e^{\sqrt{C}-t}(\log D)^{-1/3})\big)\prod_{p<Z}\Big(1-\frac{f(p)}{p}\Big)$$

for all multiplicative functions $f(d)$ that satisfy $0\le f(p)\le p-1$ and (3.1.3). In addition, h is increasing and $h(3)>0$, by [Iwa80a, p. 172-173]. This proves that (3.1.4) holds too and completes the proof of the lemma. □

The next two lemmas are concerned with estimates of functions that satisfy certain growth conditions of multiplicative nature.

Lemma 3.1.3. *Let $f:\mathbb{N}\to[0,+\infty)$ be an arithmetic function. Assume that there exists a constant C_f depending only on f such that $f(ap)\le C_f f(a)$ for all $a\in\mathbb{N}$ and all primes p with $(a,p)=1$.*

(a) For $3/2\le y\le x$ and $n\in\mathbb{N}\cup\{0\}$ we have

$$\sum_{a\in\mathscr{P}_*(y,x)}\frac{f(a)(\log a)^n}{a}\ll_{C_f}\frac{n!(n+1)^{c_f}}{2^n}(\log x+1)^n\sum_{a\in\mathscr{P}_*(y,x)}\frac{f(a)}{a},$$

where c_f is a constant depending only on C_f.

(b) Let $A\in\mathbb{R}$ and $3/2\le y\le x\le z^C$ for some $C>0$. Then

$$\sum_{\substack{a\in\mathscr{P}_*(y,x)\\ a>z}}\frac{f(a)}{a}\log^A(P^+(a))\ll_{C_f,A,C}\exp\Big\{-\frac{\log z}{2\log x}\Big\}(\log x)^A\sum_{a\in\mathscr{P}_*(y,x)}\frac{f(a)}{a}.$$

Proof. (a) We claim that for all $n\ge0$ and every number $x>0$,

$$\sum_{m=0}^{n}\binom{n}{m}\frac{1}{n-m+1}\prod_{i=0}^{m-1}\Big(x+\frac{i}{2}\Big)\le\prod_{i=1}^{n}\Big(x+\frac{i}{2}\Big).\tag{3.1.5}$$

Observe that each side of (3.1.5) is a polynomial of degree n in x. Therefore it suffices to compare the coefficients of x^r of the two sides. Note that the coefficient of x^r of the right hand side of (3.1.5) is equal to

$$\frac{1}{2^{n-r}} \sum_{1\le i_1<\cdots<i_{n-r}\le n} i_1\cdots i_{n-r}, \tag{3.1.6}$$

where the sum is interpreted to be 1 if $r = n$. For each summand $i_1 \cdots i_{n-r}$ in (3.1.6) there is a unique $j \in \{0, 1, \ldots, n-r\}$ such that $i_{n-r} = n,\ i_{n-r-1} = n-1, \ldots, i_{n-r-j+1} = n-j+1$ and $i_{n-r-j} < n-j$. So

$$\sum_{1\le i_1<\cdots<i_{n-r}\le n} i_1\cdots i_{n-r} = \sum_{j=0}^{n-r} n(n-1)\cdots(n-j+1) \sum_{1\le i_1<\cdots<i_{n-r-j}<n-j} i_1\cdots i_{n-r-j}. \tag{3.1.7}$$

But the coefficient of x^r of the left hand side of (3.1.5) is equal to

$$\begin{aligned}
&\sum_{m=r}^{n} \binom{n}{m} \frac{1}{n-m+1}\frac{1}{2^{m-r}} \sum_{1\le i_1<\cdots<i_{m-r}\le m-1} i_1\cdots i_{m-r}\\
&= \sum_{j=0}^{n-r} \binom{n}{j} \frac{1}{j+1}\frac{1}{2^{n-r-j}} \sum_{1\le i_1<\cdots<i_{n-r-j}<n-j} i_1\cdots i_{n-r-j}\\
&= \frac{1}{2^{n-r}} \sum_{j=0}^{n-r} \frac{2^j}{(j+1)!} n(n-1)\cdots(n-j+1) \sum_{1\le i_1<\cdots<i_{n-r-j}<n-j} i_1\cdots i_{n-r-j}\\
&\le \frac{1}{2^{n-r}} \sum_{1\le i_1<\cdots<i_{n-r}\le n} i_1\cdots i_{n-r},
\end{aligned}$$

by (3.1.7) and the inequality $2^j \le (j+1)!$ for $j \in \mathbb{N} \cup \{0\}$. This shows (3.1.5). Also, for every $r \ge 0$ Mertens's estimate on the sum $\sum_{p\le t} \log p/p$ and partial summation imply that

$$\sum_{p\le x} \frac{(\log p)^{r+1}}{p} = \frac{(\log x)^{r+1}}{r+1} + O((\log x)^r) \le M\frac{(\log x+1)^{r+1}}{r+1} \tag{3.1.8}$$

for some absolute constant M. Set $c = C_f M$. We shall prove the lemma with $c_f = 2c-1$.

In fact, we are going to prove that

$$\sum_{a\in\mathscr{P}_*(y,x)} \frac{f(a)(\log a)^n}{a} \le (\log x+1)^n \prod_{i=0}^{n-1}\Big(c+\frac{i}{2}\Big) \sum_{a\in\mathscr{P}_*(y,x)} \frac{f(a)}{a} \tag{3.1.9}$$

for all $n \ge 0$. We argue inductively. If $n=0$, it is clear that (3.1.9) is true. Fix now $n \ge 0$ and suppose that (3.1.9) holds for all $m \le n$. Then

$$\begin{aligned}
\sum_{a\in\mathscr{P}_*(y,x)} \frac{f(a)(\log a)^{n+1}}{a} &= \sum_{a\in\mathscr{P}_*(y,x)} \frac{f(a)(\log a)^n}{a} \sum_{p|a} \log p \\
&= \sum_{y<p\le x} \frac{\log p}{p} \sum_{\substack{b\in\mathscr{P}_*(y,x) \\ p\nmid b}} \frac{f(bp)}{b} (\log b+\log p)^n \\
&\le C_f \sum_{p\le x} \frac{\log p}{p} \sum_{b\in\mathscr{P}_*(y,x)} \frac{f(b)}{b} \sum_{m=0}^{n} \binom{n}{m} (\log b)^m (\log p)^{n-m} \\
&= C_f \sum_{m=0}^{n} \binom{n}{m} \sum_{p\le x} \frac{(\log p)^{1+n-m}}{p} \sum_{b\in\mathscr{P}_*(y,x)} \frac{f(b)(\log b)^m}{b}.
\end{aligned}$$

So, by the induction hypothesis, (3.1.5) and (3.1.8), we find that

$$\begin{aligned}
&\sum_{a\in\mathscr{P}_*(y,x)} \frac{f(a)(\log a)^{n+1}}{a} \\
&\quad\le C_f M (\log x+1)^{n+1} \sum_{m=0}^{n} \binom{n}{m} \frac{1}{n-m+1} \prod_{i=0}^{m-1}\Big(c+\frac{i}{2}\Big) \sum_{a\in\mathscr{P}_*(y,x)} \frac{f(a)}{a} \\
&\le c(\log x+1)^{n+1} \prod_{i=1}^{n}\Big(c+\frac{i}{2}\Big) \sum_{a\in\mathscr{P}_*(y,x)} \frac{f(a)}{a}.
\end{aligned}$$

This completes the inductive step and thus the proof of (3.1.9). Finally, observe that

$$\prod_{i=0}^{n-1}\Big(c+\frac{i}{2}\Big) = \frac{1}{2^n}\frac{\Gamma(2c+n)}{\Gamma(2c)} \asymp_c \frac{(n+1)^{2c-1}n!}{2^n},$$

by Stirling's formula.

(b) By part (a) we have that

$$\sum_{a\in\mathscr{P}_*(y,x)}\frac{f(a)}{a^{1-1/(2\log x)}}=\sum_{a\in\mathscr{P}_*(y,x)}\frac{f(a)}{a}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(\log a)^n}{(2\log x)^n}$$
$$\ll_{C_f}\sum_{n=0}^{\infty}\frac{(n+1)^{c_f}}{4^n}\Big(1+\frac{1}{\log x}\Big)^n\sum_{a\in\mathscr{P}_*(y,x)}\frac{f(a)}{a}\tag{3.1.10}$$
$$\ll_{C_f}\sum_{a\in\mathscr{P}_*(y,x)}\frac{f(a)}{a}$$

for all $x\geq 2$, since $1+1/\log 2<3$. Thus we have

$$\sum_{\substack{a\in\mathscr{P}_*(y,x)\\ a>z}}\frac{f(a)}{a}\log^A(P^+(a))=\sum_{y<p\leq x}\frac{(\log p)^A}{p}\sum_{\substack{b\in\mathscr{P}_*(y,p)\\ p\nmid b,b>z/p}}\frac{f(bp)}{b}$$
$$\leq C_f\sum_{p\leq\min\{x,z\}}\frac{(\log p)^A}{p}\exp\Big\{-\frac{\log(z/p)}{2\log p}\Big\}\sum_{b\in\mathscr{P}_*(y,p)}\frac{f(b)}{b^{1-1/(2\log p)}}$$
$$+C_f\sum_{\min\{x,z\}<p\leq x}\frac{(\log p)^A}{p}\sum_{b\in\mathscr{P}_*(y,x)}\frac{f(b)}{b}\tag{3.1.11}$$
$$\ll_{C_f}\sum_{a\in\mathscr{P}_*(y,x)}\frac{f(a)}{a}\left(\sum_{p\leq\min\{x,z\}}\frac{(\log p)^A}{p}\exp\Big\{-\frac{\log z}{2\log p}\Big\}+\sum_{\min\{x,z\}<p\leq x}\frac{(\log p)^A}{p}\right),$$

by (3.1.10). Moreover, if $z<x$, then

$$\sum_{\min\{x,z\}<p\leq x}\frac{(\log p)^A}{p}\leq e^{1/2}\sum_{\min\{x,z\}<p\leq x}\frac{(\log p)^A}{p}\exp\Big\{-\frac{\log z}{2\log p}\Big\}.\tag{3.1.12}$$

On the other hand, if $z\geq x$, then both sides of (3.1.12) are equal to zero. In any case, (3.1.12) holds. Combining this with (3.1.11) we find that

$$\sum_{\substack{a\in\mathscr{P}_*(y,x)\\ a>z}}\frac{f(a)}{a}\log^A(P^+(a))\ll_{C_f}\sum_{a\in\mathscr{P}_*(y,x)}\frac{f(a)}{a}\sum_{p\leq x}\frac{(\log p)^A}{p}\exp\Big\{-\frac{\log z}{2\log p}\Big\}.$$

So it suffices to show that

$$T := \sum_{p\le x} \frac{(\log p)^A}{p} \exp\Big\{-\frac{\log z}{2\log p}\Big\} \ll_{A,C} \exp\Big\{-\frac{\log z}{2\log x}\Big\}(\log x)^A.$$

Set $\mu = \exp\{\frac{\log z}{2\log x}\}$. Note that $\mu \ge e^{1/(2C)} > 1$. Thus for $x \ge 100$ we have that

$$\begin{aligned} T &\le \sum_{1\le n\le\sqrt{\log x}} \mu^{-n} \sum_{x^{1/(n+1)}<p\le x^{1/n}} \frac{(\log p)^A}{p} + \exp\Big\{-\frac{\log z}{2\sqrt{\log x}}\Big\} \sum_{p\le e^{\sqrt{\log x}}} \frac{(\log p)^A}{p} \\ &\ll_A (\log x)^A \sum_{n=1}^{\infty} \frac{1}{\mu^n n^{A+1}} + \exp\Big\{-\frac{\log z}{2\sqrt{\log x}}\Big\}(\log x)^{|A|/2} \log\log x \\ &\ll_{A,C} (\log x)^A \exp\Big\{-\frac{\log z}{2\log x}\Big\}, \end{aligned}$$

which completes the proof of the lemma. □

Let $\mathcal{M}$ denote the class of functions $f : \mathbb{N} \to [0,+\infty)$ for which there exist constants A_f and $B_{f,\epsilon}$, $\epsilon > 0$, such that

$$f(nm) \le \min\{A_f^{\Omega(m)}, B_{f,\epsilon} m^{\epsilon}\} f(n)$$

for all $(m,n) = 1$ and all $\epsilon > 0$. The following lemma is an easy application of the results and methods in [NT].

Lemma 3.1.4. *Let $f \in \mathcal{M}$, $a \in \mathbb{Z}\setminus\{0\}$ and $1 \le q \le h \le x$ such that $(a,q) = 1$ and $x > |a|$. If $q \le x^{1-\epsilon}$ and $h/q \ge ((x-a)/q)^{\epsilon}$ for some $\epsilon > 0$, then*

$$\sum_{\substack{x-h<p\le x \\ p\equiv a \pmod q}} f\Big(\frac{p-a}{q}\Big) \ll_{a,\epsilon,f} \frac{h}{\phi(q)(\log x)^2} \sum_{n\le x} \frac{f(n)}{n};$$

the implied constant depends on f only via the constants A_f and $B_{f,\alpha}$, $\alpha > 0$.

Proof. Observe that it suffices to show the lemma for the function $\widetilde{f}$ defined for $n = 2^r m$

with $(m,2)=1$ by

$$\widetilde{f}(n)=\min\{A_f^r,\min_{\epsilon>0}(B_{f,\epsilon}2^{r\epsilon})\}f(m).$$

We have that $\widetilde{f}\in\mathcal{M}$ with parameters A_f and $B_{f,\alpha}^2$, $\alpha>0$. Without loss of generality we may assume that $\widetilde{f}(1)=1$. Also, suppose that $x\geq x_0(\epsilon,a,f)$, where $x_0(a,\epsilon,f)$ is a sufficiently large constant; otherwise, the result is trivial. Set

$$q_1=\begin{cases} q, & \text{if } 2|aq \\ 2q, & \text{if } 2\nmid aq, \end{cases}$$

and note that if $p\equiv a \pmod q$ and $p>2$, then $p\equiv a \pmod{q_1}$. So if we set $p=q_1m+a$ for $p>2$, then

$$\begin{aligned}\sum_{\substack{x-h<p\leq x\\ p\equiv a \pmod q}}\widetilde{f}\Big(\frac{p-a}{q}\Big) &\leq \sum_{\substack{X-H<m\leq X\\ P^-(q_1m+a)>\sqrt{X}}}\widetilde{f}\Big(\frac{q_1}{q}m\Big)+\sum_{\substack{X-H<m\leq X\\ 3\leq q_1m+a\leq\sqrt{X}}}\widetilde{f}\Big(\frac{q_1}{q}m\Big)+O_{a,f}(1)\\ &\ll_{a,f}\sum_{\substack{X-H<m\leq X\\ P^-(q_1m+a)>\sqrt{X}}}\widetilde{f}(m)+\sum_{\substack{X-H<m\leq X\\ m\leq\sqrt{X}-a}}\widetilde{f}(m)+1,\end{aligned}$$

since $q_1/q\in\{1,2\}$ and $\widetilde{f}(2n)\ll_f\widetilde{f}(n)$ for all $n\in\mathbb{N}$. Let $f_1(n)=\widetilde{f}(n)$ and $f_2(n)$ be the characteristic function of integers n such that $P^-(n)>\sqrt{X}$. Let $Q_1(x)=x$, $Q_2(x)=q_1x+a$ and $Q=Q_1Q_2$. Also, if $P(x)\in\mathbb{Z}[x]$, then let $\rho_P(m)$ be the number of solution of the congruence $P(x)\equiv 0 \pmod m$. By Corollary 3 in [NT], we have that

$$\begin{aligned}\sum_{\substack{X-H<m\leq X\\ P^-(q_1m+a)>\sqrt{X}}}\widetilde{f}(m) &= \sum_{X-H<m\leq X}f_1(m)f_2(q_1m+a)\\ &\ll_{a,\epsilon,f} H\prod_{p\leq X}\Big(1-\frac{\rho_Q(p)}{p}\Big)\prod_{j=1}^{2}\sum_{n\leq X}\frac{f_j(n)\rho_{Q_j}(n)}{n} \qquad (3.1.13)\\ &\ll_{a,\epsilon}\frac{h}{\phi(q)}\frac{1}{\log^2 x}\sum_{n\leq X}\frac{\widetilde{f}(n)}{n},\end{aligned}$$

since $2|aq_1$, $H \geq X^\epsilon$, $q \leq x^{1-\epsilon}$ and the discriminant of Q depends only on a. Also, if the sum

$$\sum_{\substack{X-H<m\leq X \\ m\leq\sqrt{X}-a}} \widetilde{f}(m)$$

is non-zero, then $H \geq X/2$. In this case, Corollary 3 in [NT] implies that

$$\sum_{\substack{X-H<m\leq X \\ m\leq\sqrt{X}-a}} \widetilde{f}(m) \ll_{a,\epsilon,F} \frac{\sqrt{X}}{\log X} \sum_{n\leq X} \frac{\widetilde{f}(n)}{n} \ll_{a,\epsilon} \frac{h}{q\log^2 x} \sum_{n\leq X} \frac{\widetilde{f}(n)}{n},$$

which, combined with (3.1.13), completes the proof of the lemma. □

Lastly, we state an estimate on the summatory function of the reciprocals of Euler's ϕ function and other closely related quantities. Such a result was proved by Sitaramachandra Rao [Sit]. Using the methods of [Sit] we extend this result according to our needs.

Lemma 3.1.5. *Let $a \in \mathbb{N}$, $s \in \mathbb{N}$ and $x \geq 1$ with $s \leq x$. Then*

$$\sum_{\substack{n\leq x \\ (n,s)=1}} \frac{\phi(a)}{\phi(an)} = \frac{315\zeta(3)}{2\pi^4} \frac{\phi(s)}{s} g(as)\Big(\log x + \gamma - \sum_{p\nmid as} \frac{\log p}{p^2-p+1} + \sum_{p|s} \frac{\log p}{p-1}\Big) + O\Big(\tau(s)\frac{as}{\phi(as)}\frac{(\log 2x)^{2/3}}{x}\Big),$$

where $g(as) = \prod_{p|as} \frac{p(p-1)}{p^2-p+1}$.

Proof. Since the proof of this part is along the same lines with the proof of the main result in [Sit], we simply sketch it. Let $P(x) = \{x\} - 1/2$, where $\{x\}$ denotes the fractional part of x. Then using the estimate

$$\sum_{n\leq x} \frac{P(x/n)}{n} \ll (\log 2x)^{2/3},$$

which was proved in [Wal, p. 98], along with a similar argument with the one leading to

Lemma 2.2 in [Sit], we find that

$$\sum_{\substack{n\le x\\(n,r)=1}}\frac{\mu^2(n)}{\phi(n)}P(x/n)\ll\frac{r}{\phi(r)}(\log 2x)^{2/3} \tag{3.1.14}$$

for every $r\in\mathbb{N}$. Also, by the Euler-McLaurin summation formula we have

$$\sum_{n\le x}\frac{1}{n}=\log x+\gamma-\frac{P(x)}{x}+O\Big(\frac{1}{x^2}\Big). \tag{3.1.15}$$

Observe that the arithmetic function $n\to\phi(a)/\phi(an)$ is multiplicative. In particular, we have

$$\frac{\phi(a)}{\phi(an)}=\sum_{\substack{ml=n\\(m,a)=1}}\frac{\mu^2(m)}{m\phi(m)l}. \tag{3.1.16}$$

Using relations (3.1.14), (3.1.15) and (3.1.16) and estimating the error terms as in [Sit] gives us

$$\begin{aligned}\sum_{\substack{n\le x\\(n,s)=1}}\frac{\phi(a)}{\phi(an)}&=\sum_{\substack{m\le x\\(m,as)=1}}\frac{\mu^2(m)}{m\phi(m)}\sum_{\substack{l\le x/m\\(l,s)=1}}\frac{1}{l}=\sum_{d|s}\frac{\mu(d)}{d}\sum_{\substack{m\le x/d\\(m,as)=1}}\frac{\mu^2(m)}{m\phi(m)}\sum_{b\le x/dm}\frac{1}{b}\\&=\sum_{d|s}\frac{\mu(d)}{d}\sum_{\substack{m\le x/d\\(m,as)=1}}\frac{\mu^2(m)}{m\phi(m)}\Big(\log\frac{x/d}{m}+\gamma-\frac{m}{x/d}P\Big(\frac{x/d}{m}\Big)+O\Big(\frac{m^2}{(x/d)^2}\Big)\Big)\\&=\sum_{d|s}\frac{\mu(d)}{d}\sum_{\substack{m=1\\(m,as)=1}}^{\infty}\frac{\mu^2(m)}{m\phi(m)}\Big(\log\frac{x/d}{m}+\gamma\Big)+O\Big(\frac{\tau(s)as}{\phi(as)}\frac{(\log 2x)^{2/3}}{x}\Big),\end{aligned}$$

since $s\le x$. Finally, a simple calculation and the identity

$$\sum_{m=1}^{\infty}\frac{\mu^2(m)}{m\phi(m)}=\frac{315\zeta(3)}{2\pi^4}$$

complete the proof. □

3.2 The Vitali covering lemma

We state below a simple but very useful covering lemma, which is a variation of the Vitali covering lemma [Fol, Lemma 3.15]. For a positive real number r and a k-dimensional rectangle I we denote with rI the rectangle which has the same center with I and r times its diameter. More formally, if $\boldsymbol{x_0}$ is the center of I, then $rI := \{r(\boldsymbol{x} - \boldsymbol{x_0}) + \boldsymbol{x_0} : \boldsymbol{x} \in I\}$.

Lemma 3.2.1. *Let $I_1, ..., I_N$ be k-dimensional cubes of the form $[a_1, b_1) \times \cdots \times [a_k, b_k)$ $(b_1 - a_1 = \cdots = b_k - a_k > 0)$. Then there exists a sub-collection $I_{i_1}, \ldots, I_{i_M}$ of mutually disjoint cubes such that*

$$\bigcup_{n=1}^{N} I_n \subset \bigcup_{m=1}^{M} 3I_{i_m}.$$

3.3 Estimates from order statistics

In this section we extend some results about uniform order statistics proven in [For08b]. Set

$$S_r(u, v) = \left\{\boldsymbol{\xi} \in \mathbb{R}^r : 0 \le \xi_1 \le \cdots \le \xi_r \le 1,\ \xi_i \ge \frac{i-u}{v}\ (1 \le i \le r)\right\}$$

and

$$\begin{aligned} Q_r(u, v) &= \mathbf{Prob}\Big(\xi_i \ge \frac{i-u}{v}\ (1 \le i \le k)\ \Big|\ 0 \le \xi_1 \le \cdots \le \xi_r \le 1\Big) \\ &= r!\,\mathrm{Vol}(S_r(u, v)). \end{aligned}$$

Combining Theorem 1 in [For08c] and Lemma 11.1 in [For08b] we have the following result.

Lemma 3.3.1. *Let $w = u + v - r$. Uniformly in $u > 0$, $w > 0$ and $r \in \mathbb{N}$, we have*

$$Q_r(u, v) \ll \frac{(u+1)(w+1)}{r}.$$

Furthermore, if $1 \leq u \leq r$*, then*

$$Q_r(u, r+1-u) \geq \frac{u-1/2}{r+1/2}.$$

Next, we state a slightly stronger version of Lemma 4.3 in [For08a]. The proof is very similar; the only difference is that we use Lemma 3.3.1 in place of [For08a, Lemma 4.1].

Lemma 3.3.2. *Suppose* $j, h, r, u, v \in \mathbb{N}$ *satisfy*

$$2 \leq j \leq r/2, \ h \geq 0, \ r \leq 10v, \ u \geq 0, \ w = u + v - r \geq 1.$$

Let R *be the set of* $\boldsymbol{\xi} \in S_r(u, v)$ *such that for some* $l \geq j+1$ *we have*

$$\frac{l-u}{v} \leq \xi_l \leq \frac{l-u+1}{v}, \quad \xi_{l-j} \geq \frac{l-u-h}{v}.$$

Then

$$\mathrm{Vol}(R) \ll \frac{(10(h+1))^j}{(j-2)!} \frac{(u+1)w}{(r+1)!}.$$

For $\mu > 1$ define

$$\mathcal{T}_\mu(r, v, \gamma) = \{0 \leq \xi_1 \leq \cdots \leq \xi_r \leq 1 : \mu^{v\xi_1} + \cdots + \mu^{v\xi_j} \geq \mu^{j-\gamma} \ (1 \leq j \leq r)\}. \tag{3.3.1}$$

Using Lemmas 3.3.1 and 3.3.2 we estimate $\mathrm{Vol}(\mathcal{T}_\mu(r, v, \gamma))$ using a similar argument with the one leading to Lemma 4.4 in [For08a].

Lemma 3.3.3. *Let* $u \geq 0$, $v \geq 1$ *and* $r \in \mathbb{N}$ *such that* $w = u + v - r \geq -C$*, where* $C \geq 0$ *is a constant. Then*

$$\mathrm{Vol}(\mathcal{T}_\mu(r, v, u)) \ll_{C,\mu} \frac{(u+1)(w+C+1)}{(r+1)!}.$$

Proof. Note that if $r \geq 2v$, then the result follows by the trivial bound $\mathrm{Vol}(\mathcal{T}_\mu(r, v, u)) \leq 1/r!$, since in this case $u \geq r/2 - C$. So assume that $1 \leq r \leq 2v$. Moreover, suppose that $C \in \mathbb{N}$

and $C \geq 2 + \log 32/\log \mu$. For every $\boldsymbol{\xi} \in \mathcal{T}_\mu(r, v, u)$ either

$$\xi_j > \frac{j - u - C}{v} \quad (1 \leq j \leq r) \tag{3.3.2}$$

or there are integers $h \geq C + 1$ and $1 \leq l \leq r$ such that

$$\min_{1 \leq j \leq r} \Big(\xi_j - \frac{j - u}{v}\Big) = \xi_l - \frac{l - u}{v} \in \Big[\frac{-h}{v}, \frac{-h + 1}{v}\Big]. \tag{3.3.3}$$

Let V_1 be the volume of $\boldsymbol{\xi} \in \mathcal{T}_\mu(r, v, u)$ that satisfy (3.3.2) and let V_2 be the volume of $\boldsymbol{\xi} \in \mathcal{T}_\mu(r, v, u)$ that satisfy (3.3.3) for some integers $h \geq C + 1$ and $1 \leq l \leq r$. Then Lemma 3.3.1 implies that

$$V_1 \leq \frac{Q_r(u + C, v)}{r!} \ll_C \frac{(u + 1)(w + C + 1)}{(r + 1)!},$$

which is admissible. To bound V_2 fix $h \geq C + 1$ and $1 \leq l \leq r$ and consider $\boldsymbol{\xi} \in \mathcal{T}_\mu(r, v, u)$ that satisfies (3.3.3). Then

$$-\frac{l - u}{v} \leq \xi_l - \frac{l - u}{v} \leq \frac{-h + 1}{v}$$

and consequently

$$l \geq u + h - 1 \geq u + C > 2.$$

Set

$$h_0 = h - 1 - \Big\lceil \frac{\log 4}{\log \mu} \Big\rceil \geq C - \Big(\frac{\log 4}{\log \mu} + 1\Big) \geq 1 + \frac{\log 8}{\log \mu}. \tag{3.3.4}$$

We claim that there exists some $m \in \mathbb{N}$ with $m \geq h_0$, $\lfloor \mu^m \rfloor < l/2$ and

$$\xi_{l - \lfloor \mu^m \rfloor} \geq \frac{l - u - 2m}{v}. \tag{3.3.5}$$

Indeed, note that

$$\begin{aligned}
\mu^{v\xi_1} + \cdots + \mu^{v\xi_l} &\le 2 \sum_{l/2<j\le l} \mu^{v\xi_j} \\
&\le 2\Big(\mu^{v\xi_l} + \sum_{\substack{m\ge 0\\ \lfloor\mu^m\rfloor<l/2}} \sum_{\lfloor\mu^m\rfloor\le j<\lfloor\mu^{m+1}\rfloor} \mu^{v\xi_{l-j}}\Big) \\
&\le 2\Big(\mu^{h_0}\mu^{v\xi_l} + \sum_{\substack{m\ge h_0\\ \lfloor\mu^m\rfloor<l/2}} (\lfloor\mu^{m+1}\rfloor - \lfloor\mu^m\rfloor)\mu^{v\xi_{l-\lfloor\mu^m\rfloor}}\Big).
\end{aligned} \tag{3.3.6}$$

So if (3.3.5) failed for all $m \ge h_0$ with $\lfloor\mu^m\rfloor < l/2$, then (3.3.3) and (3.3.6) would imply that

$$\begin{aligned}
\mu^{v\xi_1} + \cdots + \mu^{v\xi_l} &< 2\Big(\mu^{h_0}\mu^{l-u-h+1} + \sum_{m\ge h_0} (\lfloor\mu^{m+1}\rfloor - \lfloor\mu^m\rfloor)\mu^{l-u-2m}\Big) \\
&= 2\mu^{l-u}\Big(\mu^{-\lceil\frac{\log 4}{\log\mu}\rceil} + (\mu^2-1)\sum_{m\ge h_0+1} \lfloor\mu^m\rfloor\mu^{-2m} - \lfloor\mu^{h_0}\rfloor\mu^{-2h_0}\Big) \\
&\le 2\mu^{l-u}\Big(\frac{1}{4} + \frac{\mu^{h_0+1}+1}{\mu^{2h_0}}\Big) \le 2\mu^{l-u}\Big(\frac{1}{4} + \frac{2}{\mu^{h_0-1}}\Big) \le \mu^{l-u},
\end{aligned}$$

by (3.3.4), which is a contradiction. Hence (3.3.5) does hold and Lemma 3.3.2 applied with $u+h$, $\lfloor\mu^m\rfloor$ and $2m$ in place of u, j and h, respectively, implies that

$$V_2 \ll \sum_{h\ge C+1} \sum_{m\ge h_0} \frac{(u+h)(w+h)}{(r+1)!} \frac{(10(2m+1))^{\lfloor\mu^m\rfloor}}{(\lfloor\mu^m\rfloor-1)!} \ll_{C,\mu} \frac{(u+1)(w+C+1)}{(r+1)!},$$

which completes the proof. □

We conclude this section with the following lemma.

Lemma 3.3.4. *Let $\mu > 1$, $r \in \mathbb{N}$, u and v with $1 \le v \le r \le 100(v-1)$ and $u + v = r + 1$. If r is large enough, then*

$$\int_{S_r(u,v)} \sum_{j=1}^{r} \mu^{j-v\xi_j} d\boldsymbol{\xi} \ll_\mu \frac{\mu^u u}{(r+1)!}.$$

Proof. In [For08b, Lemma 4.9, p. 423-424] it is shown that

$$\int_{S_r(u,v)} \sum_{j=1}^{r} 2^{j-v\xi_j} d\boldsymbol{\xi} \ll \frac{2^u u}{(r+1)!}$$

under the same conditions for u, v, r. Following the same argument we deduce the desired result; the only thing we need to check is that $\int_0^\infty (y+1)^3 \mu^{-y} dy < +\infty$. $\square$

Chapter 4

Local-to-global estimates

In the first two sections of this chapter we reduce the counting in $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ to the estimation of

$$S^{(k+1)}(\boldsymbol{a}) := \sum_{\boldsymbol{a} \in \mathscr{P}_*^k(\boldsymbol{t})} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k}$$

and prove Theorem 2.8. The basic ideas behind this reduction can be found in [For08a, For08b, Kou10a]. However, the details are more complicated, especially in the proof of the upper bound implicit in Theorem 2.4, because of the presence of more parameters. Finally, in Sections 4.3 and 4.4 we show Theorems 2.5 and 2.4, respectively.

Remark 4.0.1. In order to show Theorem 2.8 we may assume without loss of generality that $y_1 > C_k$, where C_k is a sufficiently large constant. Indeed, suppose for the moment that Theorem 2.8 holds for all $k \geq 1$ if $y_1 > C_k$ and consider the case when $y_1 \leq C_k$. Then either $y_k \leq C_k$, in which case Theorem 2.8 follows immediately, or there exists $l \in \{1, \dots, k-1\}$ such that $y_l \leq C_k < y_{l+1}$. In the latter case let $\boldsymbol{y}' = (y_{l+1}, \dots, y_k)$ and $d = \lfloor 2y_1 \rfloor \cdots \lfloor 2y_l \rfloor \leq 2^l y_1 \cdots y_l \leq (2C_k)^k$ and note that

$$H^{(k-l+1)}\Big(\frac{x}{d}, \boldsymbol{y}', 2\boldsymbol{y}'\Big) \leq H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y}) \leq H^{(k-l+1)}(x, \boldsymbol{y}', 2\boldsymbol{y}'),$$

Moreover,

$$\frac{x/d}{y_{l+1} \cdots y_k} \geq \frac{x}{2^l y_1 \cdots y_k} \geq 2^{k-l} y_k.$$

So the desired bound on $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ follows by Theorem 2.8 applied to $H^{(k-l+1)}(x, \boldsymbol{y}', 2\boldsymbol{y}')$ and $H^{(k-l+1)}(x/d, \boldsymbol{y}', 2\boldsymbol{y}')$, which holds since $y_{l+1} > C_k$.

4.1 The lower bound in Theorem 2.8

We start with the proof of the lower bound implicit in Theorem 2.4, which is simpler. First, we prove a weaker result; then we use Lemma 3.1.3 to complete the proof. Note that the lemma below is similar to Lemma 2.1 in [For08a], Lemma 4.1 in [For08b] and Lemma 3.2 in [Kou10a].

Lemma 4.1.1. *Fix $k \ge 1$. For $x \ge 1$ and $3 \le y_1 \le y_2 \le \cdots \le y_k$ with $2^k y_1 \cdots y_k \le x/y_k$ and $y_1 > C_k$, we have that*

$$\frac{H_{k+1}(x, \boldsymbol{y}, 2\boldsymbol{y})}{x} \gg_k \Big(\prod_{i=1}^{k} \log^{-e_{k,i}} y_i\Big) \sum_{\substack{\boldsymbol{a} \in \mathscr{P}_*^k(2\boldsymbol{y}) \\ a_i \le y_i^{1/8k}\ (1 \le i \le k)}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k}.$$

Proof. Consider integers $n = a_1 \cdots a_k p_1 \cdots p_k b \in (x/2, x]$ such that

(1) $\boldsymbol{a} \in \mathscr{P}_*^k(2\boldsymbol{y})$ and $a_i \le y_i^{1/8k}$ for $i = 1, \dots, k$;

(2) $p_1, \dots, p_k$ are prime numbers with $(\log(y_1/p_1), \dots, \log(y_k/p_k)) \in \mathcal{L}^{(k+1)}(\boldsymbol{a})$;

(3) $P^-(b) > y_k^{1/8}$ and b has at most one prime factor in $(y_k^{1/8}, 2y_k]$.

Note that for every $i \in \{1, \dots, k\}$ all prime factors of a_i lie in $(y_{i-1}, y_i^{1/8k}]$. Also, condition (2) is equivalent to the existence of integers $d_1, \dots, d_k$ such that $d_1 \cdots d_i | a_1 \cdots a_i$ and $y_i/p_i < d_i \le 2y_i/p_i$, $i = 1, \dots k$. In particular, $\tau_{k+1}(n, \boldsymbol{y}, 2\boldsymbol{y}) \ge 1$. Furthermore,

$$y_i^{7/8} \le \frac{y_i}{a_1 \cdots a_i} \le \frac{y_i}{d_i} < p_i \le 2\frac{y_i}{d_i} \le 2y_i.$$

So $(a_1 \cdots a_k, p_1 \cdots p_k b) = 1$ and hence this representation of n, if it exists, is unique up to a possible permutation of $p_1, \dots, p_k$ and the prime factors of b that lie in $(y_1^{7/8}, 2y_k]$. Since b has at most one such prime factor, n has a bounded number of such representations. Fix

$a_1, \ldots, a_k$ and $p_1, \ldots, p_k$ and note that

$$X := \frac{x}{a_1 \cdots a_k p_1 \cdots p_k} \geq \frac{x}{2^k y_1 \cdots y_k} \frac{1}{y_k^{1/8}} \geq y_k^{7/8}.$$

So Lemma 3.1.1 and the Prime Number Theorem yield

$$\sum_{b \text{ admissible}} 1 \geq \frac{1}{2}\Bigg(\sum_{X/2 < p \leq X} 1 + \sum_{\substack{m \leq X \\ P^-(m) > 2y_k}} 1 \Bigg) \gg_k \frac{X}{\log y_k}$$

and consequently

$$H_{k+1}(x, \boldsymbol{y}, 2\boldsymbol{y}) \gg_k \frac{x}{\log y_k} \sum_{\substack{\boldsymbol{a} \in \mathscr{P}_*^k(2\boldsymbol{y}) \\ a_i \leq y_i^{1/8k}\ (1 \leq i \leq k)}} \frac{1}{a_1 \cdots a_k} \sum_{(\log \frac{y_1}{p_1}, \ldots, \log \frac{y_k}{p_k}) \in \mathcal{L}^{(k+1)}(\boldsymbol{a})} \frac{1}{p_1 \cdots p_k}. \tag{4.1.1}$$

Fix $\boldsymbol{a} \in \mathscr{P}_*^k(2\boldsymbol{y})$ with $a_i \leq y_i^{1/8}$ for $i = 1, \ldots, k$. Let $\{I_r\}_{r=1}^R$ be the collection of cubes $[\log(d_1/2), \log d_1) \times \cdots \times [\log(d_k/2), \log d_k)$ with $d_1 \cdots d_i | a_1 \cdots a_i$, $1 \leq i \leq k$. Then for $I = [\log(d_1/2), \log d_1) \times \cdots \times [\log(d_k/2), \log d_k)$ in this collection we have

$$\sum_{(\log \frac{y_1}{p_1}, \ldots, \log \frac{y_k}{p_k}) \in I} \frac{1}{p_1 \cdots p_k} = \prod_{i=1}^k \sum_{y_i/d_i < p_i \leq 2y_i/d_i} \frac{1}{p_i} \gg_k \frac{1}{\log y_1 \cdots \log y_k}$$

because $d_i \leq a_1 \cdots a_i \leq y_i^{1/8}$ for $1 \leq i \leq k$. By Lemma 3.2.1, there exists a sub-collection $\{I_{r_j}\}_{j=1}^J$ of mutually disjoint cubes such that

$$J(\log 2)^k = \mathrm{Vol}\Big(\bigcup_{j=1}^J I_{r_j} \Big) \geq \frac{1}{3^k} \mathrm{Vol}\Big(\bigcup_{r=1}^R I_r \Big) = \frac{L^{(k+1)}(\boldsymbol{a})}{3^k}.$$

Hence

$$\sum_{(\log \frac{y_1}{p_1}, \ldots, \log \frac{y_k}{p_k}) \in \mathcal{L}^{(k+1)}(\boldsymbol{a})} \frac{1}{p_1 \cdots p_k} \geq \sum_{j=1}^J \sum_{(\log \frac{y_1}{p_1}, \ldots, \log \frac{y_k}{p_k}) \in I_{r_j}} \frac{1}{p_1 \cdots p_k} \gg_k \frac{L^{(k+1)}(\boldsymbol{a})}{\log y_1 \cdots \log y_k}.$$

Combining the above estimate with (4.1.1) completes the proof of the lemma. □

Proof of Theorem 2.8(lower bound). For fixed $i \in \{1, \dots, k\}$ as well as integers $a_1, \dots, a_{i-1}$ and $a_{i+1}, \dots, a_k$, the function $a_i \to L^{(k+1)}(\boldsymbol{a})$ satisfies the hypothesis of Lemma 3.1.3 with $C_f = k - i + 2 \le k + 1$, by Lemma 2.3.1(b). So if we set

$$\mathcal{P} = \{\boldsymbol{a} \in \mathbb{N}^k : a_i \in \mathscr{P}_*(2y_{i-1}, y_i^{1/C}) \ (1 \le i \le k)\}$$

for some sufficiently large $C = C(k)$, then

$$\sum_{\substack{\boldsymbol{a} \in \mathscr{P}_*^k(2\boldsymbol{y}) \\ a_i \le y_i^{1/8k} \\ 1 \le i \le k}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k} \ge \sum_{\substack{\boldsymbol{a} \in \mathcal{P} \\ a_i \le y_i^{1/8k} \\ 1 \le i \le k}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k} = \sum_{\boldsymbol{a} \in \mathcal{P}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k} \left(1 + O_k\left(e^{-\frac{C}{16k}}\right)\right) \ge \frac{1}{2} \sum_{\boldsymbol{a} \in \mathcal{P}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k}.$$

By the above inequality and Lemma 2.3.1(b), we deduce that

$$S^{(k+1)}(\boldsymbol{y}) \le \sum_{\boldsymbol{a} \in \mathcal{P}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k} \prod_{i=1}^{k} \sum_{\substack{b_i \in \mathscr{P}_*(y_{i-1}, 2y_{i-1}) \\ \text{or } b_i \in \mathscr{P}_*(y_i^{1/C}, y_i)}} \frac{\tau_{k-i+2}(b_i)}{b_i} \ll_k \sum_{\substack{\boldsymbol{a} \in \mathscr{P}_*^k(2\boldsymbol{y}) \\ a_i \le y_i^{1/8k} \\ 1 \le i \le k}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k}.$$

Combining the above estimate with Lemma 4.1.1 completes the proof of the lower bound in Theorem 2.8. □

4.2 The upper bound in Theorem 2.8

In this section we complete the proof of Theorem 2.8. Before we proceed to the proof, we need to define some auxiliary notation. For $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^k$ and $x \ge 1$ set

$$H_*^{(k+1)}(x, \boldsymbol{y}, \boldsymbol{z}) = |\{n \le x : \mu^2(n) = 1, \ \tau_{k+1}(n, \boldsymbol{y}, \boldsymbol{z}) \ge 1\}|.$$

Also, for $\boldsymbol{t} \in [1, +\infty)^k$, $\boldsymbol{h} \in [0, +\infty)^k$ and $\epsilon > 0$ define

$$\mathscr{P}_*^k(\boldsymbol{t}; \epsilon) = \Big\{ \boldsymbol{a} \in \mathbb{N}^k : P^+(a_{i-1}) < P^-(a_i) \ (2 \le i \le k), \ a_i \in \mathscr{P}_*\Big(\frac{t_{i-1}^\epsilon}{a_1 \dots a_{i-1}}, t_i \Big) \ (1 \le i \le k) \Big\},$$

where $t_0 = 1$, and

$$S^{(k+1)}(\boldsymbol{t}; \boldsymbol{h}, \epsilon) = \sum_{\boldsymbol{a} \in \mathscr{P}_*^k(\boldsymbol{t};\epsilon)} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \dots a_k} \prod_{i=1}^k \log^{-h_i}\Big(P^+(a_1 \cdots a_i) + \frac{t_i^\epsilon}{a_1 \cdots a_i} \Big).$$

Then we have the following estimate.

Lemma 4.2.1. *Let $3 \le y_1 \le \cdots \le y_k \le x$ with $2^{k+1} y_1 \cdots y_k \le x/(2y_k)^{7/8}$. Then*

$$H_*^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y}) - H_*^{(k+1)}(x/2, \boldsymbol{y}, 2\boldsymbol{y}) \ll_k x S^{(k+1)}(2\boldsymbol{y}; \boldsymbol{e}_k, 7/8).$$

Proof. Let $n \in (x/2, x]$ be a square-free integer such that $\tau_{k+1}(n, \boldsymbol{y}, 2\boldsymbol{y}) \ge 1$. Then we may write $n = d_1 \cdots d_{k+1}$ with $y_i < d_i \le 2y_i$ for $i = 1, \dots, k$. So if we set $y_{k+1} = x/(2^{k+1} y_1 \cdots y_k)$, then $y_i < d_i \le 2^{k+1} y_i$ for $1 \le i \le k+1$. Let $z_1, \dots, z_{k+1}$ be the sequence $y_1, \dots, y_{k+1}$ ordered in increasing order. For a unique permutation $\sigma \in S_{k+1}$ we have that $P^+(d_{\sigma(1)}) < \cdots < P^+(d_{\sigma(k+1)})$. Set $p_j = P^+(d_{\sigma(j)})$ for $1 \le j \le k+1$ and $p_0 = 1$ and write $n = a_1 \cdots a_k p_1 \cdots p_k b$ with $P^-(b) > p_k$ and $a_i \in \mathscr{P}_*(p_{i-1}, p_i)$ for all $1 \le i \le k$. We claim that

$$p_i > Q_i := \max\Big\{ P^+(a_1 \cdots a_i), \frac{(2y_i)^{7/8}}{a_1 \dots a_i} \Big\} \quad (1 \le i \le k). \tag{4.2.1}$$

Indeed, we have that $y_{\sigma(i)} < d_{\sigma(i)} = p_i d$ for some $d | a_1 \dots a_i$. Hence $y_{\sigma(i)} < p_i a_1 \dots a_i$ for all $i \in \{1, \dots, k\}$ and consequently

$$p_i > \max_{1 \le j \le i} \frac{y_{\sigma(j)}}{a_1 \cdots a_j} \ge \frac{\max_{1 \le j \le i} y_{\sigma(j)}}{a_1 \cdots a_i} \ge \frac{z_i}{a_1 \cdots a_i} \ge \frac{(2y_i)^{7/8}}{a_1 \cdots a_i} \quad (1 \le i \le k),$$

by our assumption that $y_1 \le \cdots \le y_k \le \frac{1}{2} y_{k+1}^{8/7}$ and the definition of $z_1, \dots, z_{k+1}$. Since we also have that $p_i > \max_{1\le j\le i} P^+(a_j) = P^+(a_1\cdots a_i)$ by definition, (4.2.1) follows. Moreover,

$$P^+(a_i) < p_i = P^+(d_{\sigma(i)}) \le \max_{1\le j\le i} P^+(d_j) \le 2y_i \quad (1\le i\le k),$$

by the choice of σ, and

$$P^-(a_i) > p_{i-1} > Q_{i-1}.$$

In particular, $\boldsymbol{a} = (a_1,\dots,a_k) \in \mathscr{P}_*(2\boldsymbol{y}; 7/8)$. Next, note that $p_{k+1}|b$ and consequently $b \ge p_{k+1} > p_k$. So for fixed $a_1,\dots,a_k$ and $p_1,\dots,p_k$ the number of possibilities for b is at most

$$\sum_{\substack{p_k<b\le x/(a_1\cdots a_kp_1\cdots p_k)\\ P^-(b)>p_k}} 1 \ll \frac{x}{a_1\cdots a_kp_1\cdots p_k\log p_k} \le \frac{x}{a_1\cdots a_kp_1\cdots p_k\log Q_k},$$

by Lemma 3.1.1 and relation (4.2.1). Therefore

$$H_*^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y}) - H_*^{(k+1)}(x/2,\boldsymbol{y},2\boldsymbol{y}) \ll_k x \sum_{\sigma\in S_{k+1}} \sum_{\substack{a_1,\dots,a_k\\ p_1,\dots,p_k}} \frac{1}{a_1\cdots a_kp_1\cdots p_k\log Q_k}. \quad (4.2.2)$$

Fix $a_1,\dots,a_k$ and $\sigma\in S_{k+1}$ as above and note that

$$(d_{\sigma(1)}/p_1)\cdots(d_{\sigma(i)}/p_i)|a_1\cdots a_i \quad\text{and}\quad \frac{y_{\sigma(i)}}{p_i} < \frac{d_{\sigma(i)}}{p_i} \le \frac{2^{k+1}y_{\sigma(i)}}{p_i} \quad (1\le i\le k).$$

The above relation implies that for some $l_1,\dots,l_k\in\{1,2,2^2,\dots,2^k\}$ we have that

$$\boldsymbol{y'} := \left(\log\frac{l_1y_{\sigma(1)}}{p_1},\dots,\log\frac{l_ky_{\sigma(k)}}{p_k}\right) \in \mathcal{L}^{(k+1)}(\boldsymbol{a}).$$

Let $m_1,\dots,m_k$ be integers with $m_1\cdots m_i|a_1\dots a_i$ for all $i=1,\dots,k$. Set

$$I = [\log(m_1/2),\log m_1)\times\cdots\times[\log(m_k/2),\log m_k)$$

and

$$U_i = \frac{l_i y_{\sigma(i)}}{2m_i} \quad (1 \le i \le k).$$

Then we find that $\boldsymbol{y}' \in 3I = [\log(m_1/4), \log(2m_1)) \times \cdots \times [\log(m_k/4), \log(2m_k))$ if, and only if, $U_i < p_i \le 8U_i$ for all $i = 1, \ldots, k$. So

$$\sum_{\substack{p_1 < \cdots < p_k \\ p_i > Q_i \ (1 \le i \le k) \\ \boldsymbol{y}' \in 3I}} \frac{1}{p_1 \cdots p_k} \le \prod_{i=1}^{k} \sum_{\substack{U_i < p_i \le 8U_i \\ p_i > Q_i}} \frac{1}{p_i} \ll_k \prod_{i=1}^{k} \frac{1}{\log(\max\{U_i, Q_i\})} \le \prod_{i=1}^{k} \frac{1}{\log Q_i}.$$

Combine the above estimate with Lemma 3.2.1 to deduce that

$$\sum_{\substack{p_1 < \cdots < p_k \\ p_i > Q_i \ (1 \le i \le k) \\ \boldsymbol{y}' \in \mathcal{L}^{(k+1)}(a_1, \ldots, a_k)}} \frac{1}{p_1 \cdots p_k} \ll_k \frac{L^{(k+1)}(\boldsymbol{a})}{\log Q_1 \cdots \log Q_k}.$$

Inserting the above estimate into (4.2.2) and summing over all permutations $\sigma \in S_{k+1}$ and all $l_1, \ldots, l_k \in \{1, 2, 2^2, \ldots, 2^k\}$ completes the proof of the lemma. □

Next, we bound the sum $S^{(k+1)}(\boldsymbol{t}; \boldsymbol{h}, \epsilon)$ from above in terms of $S^{(k+1)}(\boldsymbol{t})$, by establishing an iterative inequality that simplifies the complicated range of summation $\mathscr{P}_*^k(\boldsymbol{t}; \epsilon)$ by gradually reducing it to the much simpler set $\mathscr{P}_*^k(\boldsymbol{t})$ and, at the same time, eliminates the complicated logarithms that appear in the summands of $S^{(k+1)}(\boldsymbol{t}; \boldsymbol{h}, \epsilon)$. Lemma 3.1.3 plays a crucial role in the proof of this inequality

Lemma 4.2.2. *Fix $k \ge 1$, $\epsilon > 0$ and $\boldsymbol{h} = (h_1, \ldots, h_k) \in [0, +\infty)^k$. For $\boldsymbol{t} = (t_1, \ldots, t_k)$ with $3 \le t_1 \le \cdots \le t_k$ we have*

$$S^{(k+1)}(\boldsymbol{t}; \boldsymbol{h}, \epsilon) \ll_{k, \boldsymbol{h}, \epsilon} \Big(\prod_{i=1}^{k} \log^{-h_i} t_i \Big) S^{(k+1)}(\boldsymbol{t}).$$

Proof. Set $\delta = \epsilon/2k$ and $t_0 = 1$. For $l \in \{1, \dots, k\}$, define

$$h_{l,i} = \begin{cases} h_i & \text{if } i \in \{1, \dots, l-1\} \cup \{k\}, \\ h_i + k - i + 1 & \text{if } l \leq i \leq k-1 \end{cases}$$

and

$$\mathscr{P}_l(\boldsymbol{t}) = \Bigg\{ \boldsymbol{a} \in \mathbb{N}^k : a_i \in \mathscr{P}_*\Bigg(\max\Bigg\{ P^+(a_1 \cdots a_{i-1}), \frac{t_{i-1}^{\epsilon/2 + l\delta}}{a_1 \cdots a_{i-1}} \Bigg\}, t_i \Bigg) \; (1 \leq i \leq l), \\ a_i \in \mathscr{P}_*(t_{i-1}, t_i) \; (l+1 \leq i \leq k) \Bigg\}.$$

Also, let $h_{0,i} = h_{1,i}$ for $i \in \{1, \dots, k\}$ and $\mathscr{P}_0(\boldsymbol{t}) = \mathscr{P}_1(\boldsymbol{t})$. Lastly, for $l \in \{0, \dots, k\}$ set $\boldsymbol{h}_l = (h_{l,1}, \dots, h_{l,k})$ and

$$\widetilde{S}_l^{(k+1)}(\boldsymbol{t}; \boldsymbol{h}_l) = \sum_{\boldsymbol{a} \in \mathscr{P}_l(\boldsymbol{t})} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1 \cdots a_k} \prod_{i=1}^{l} \log^{-h_{l,i}} \Bigg(P^+(a_1 \cdots a_i) + \frac{t_i^{\epsilon/2 + l\delta}}{a_1 \cdots a_i} \Bigg) \\ \times \prod_{i=l+1}^{k} \log^{-h_{l,i}} \Big(P^+(a_1 \cdots a_l) + \frac{t_i^{\epsilon/2 + l\delta}}{a_1 \cdots a_l} \Big).$$

We shall prove that

$$\widetilde{S}_l^{(k+1)}(\boldsymbol{t}; \boldsymbol{h}_l) \ll_{k, \boldsymbol{h}, \epsilon} (\log 2t_{l-1})^{k-l+2} \widetilde{S}_{l-1}^{(k+1)}(\boldsymbol{t}; \boldsymbol{h}_{l-1}) \quad (1 \leq l \leq k). \tag{4.2.3}$$

Fix $l \in \{1, \dots, k\}$. Consider integers $a_1, \dots, a_{l-1}$ such that

$$a_i \in \mathscr{P}_*\Bigg(\max\Bigg\{ P^+(a_1 \cdots a_{i-1}), \frac{t_{i-1}^{\epsilon/2 + l\delta}}{a_1 \cdots a_{i-1}} \Bigg\}, t_i \Bigg) \quad (1 \leq i \leq l-1)$$

and $a_{l+1}, \dots, a_k$ such that

$$a_i \in \mathscr{P}_*(t_{i-1}, t_i) \quad (l+1 \leq i \leq k)$$

and set

$$t'_{l-1} = \max\left\{P^+(a_1\cdots a_{l-1}), \frac{t_{l-1}^{\epsilon/2+l\delta}}{a_1\cdots a_{l-1}}\right\}.$$

Observe that in order to show (4.2.3) it suffices to prove that

$$\begin{aligned} T :&= \sum_{a_l\in\mathscr{P}_*(t'_{l-1},t_l)} \frac{L^{(k+1)}(\boldsymbol{a})}{a_l} \prod_{i=l}^{k} \log^{-h_{l,i}}\Big(P^+(a_1\cdots a_l) + \frac{t_i^{\epsilon/2+l\delta}}{a_1\cdots a_l}\Big) \\ &\ll_{k,\boldsymbol{h},\epsilon} \sum_{a_l\in\mathscr{P}_*(t'_{l-1},t_l)} \frac{L^{(k+1)}(\boldsymbol{a})}{a_l} \prod_{i=l}^{k} \log^{-h_{l,i}}\Big(P^+(a_1\cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1\cdots a_{l-1}}\Big). \end{aligned} \tag{4.2.4}$$

Indeed, if (4.2.4) holds, then Lemma 2.3.1(b) and the relation

$$\sum_{a\in\mathscr{P}_*(t'_{l-1},t_{l-1})} \frac{\tau_{k-l+2}(a)}{a} = \prod_{t'_{l-1}<p\le t_{l-1}} \Big(1 + \frac{k-l+2}{p}\Big) \ll_k \Big(\frac{\log 2t_{l-1}}{\log 2t'_{l-1}}\Big)^{k-l+2}$$

complete the proof of (4.2.3). To prove (4.2.4) we decompose T into the sums

$$T_m = \sum_{\substack{a_l\in\mathscr{P}_*(t'_{l-1},t_l)\\ a_l\in I_m}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_l} \prod_{i=l}^{k} \log^{-h_{l,i}}\Big(P^+(a_1\cdots a_l) + \frac{t_i^{\epsilon/2+l\delta}}{a_1\cdots a_l}\Big) \quad (l\le m\le k+1),$$

where $I_l = (0, t_l^\delta]$, $I_m = (t_{m-1}^\delta, t_m^\delta]$ if $m\in\{l+1,\dots,k\}$ and $I_{k+1} = (t_k^\delta, +\infty)$. First, we estimate T_l. If $a_l\in I_l$, then

$$P^+(a_1\cdots a_l) + \frac{t_i^{\epsilon/2+l\delta}}{a_1\cdots a_l} \ge P^+(a_1\cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1\cdots a_{l-1}} \quad (l\le i\le k)$$

and thus we immediately deduce that

$$T_l \le \sum_{a_l\in\mathscr{P}_*(t'_{l-1},t_l)} \frac{L^{(k+1)}(\boldsymbol{a})}{a_l} \prod_{i=l}^{k} \log^{-h_{l,i}}\Big(P^+(a_1\cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1\cdots a_{l-1}}\Big). \tag{4.2.5}$$

Next, we fix $m \in \{l+1, \dots, k+1\}$ and bound T_m. For every $a_l \in I_m$ we have that

$$P^+(a_1 \cdots a_l) + \frac{t_i^{\epsilon/2+l\delta}}{a_1 \cdots a_l} \geq \begin{cases} P^+(a_l) & \text{if } l \leq i < m, \\ P^+(a_1 \cdots a_{l-1}) + \dfrac{t_i^{\epsilon/2+(l-1)\delta}}{a_1 \cdots a_{l-1}} & \text{if } m \leq i \leq k. \end{cases}$$

Moreover, the function $a_l \to L^{(k+1)}(\boldsymbol{a})$ satisfies the hypothesis of Lemma 3.1.3 with $C_f = k - l + 2$, by Lemma 2.3.1(b). Hence

$$\begin{aligned} T_m &\leq \left(\prod_{i=m}^{k} \log^{-h_{l,i}} \Big(P^+(a_1 \cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1 \cdots a_{l-1}} \Big) \right) \sum_{\substack{a_l \in \mathscr{P}_*(t'_{l-1}, t_l) \\ a_l > t_{m-1}^{\delta}}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_l (\log P^+(a_l))^{h_{l,l}+\cdots+h_{l,m-1}}} \\ &\ll_{k,\boldsymbol{h},\epsilon} \left(\prod_{i=l}^{k} \log^{-h_{l,i}} \Big(P^+(a_1 \cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-2)\delta}}{a_1 \cdots a_{l-1}} \Big) \right) \left(\prod_{i=l}^{m-1} \log^{h_{l,i}} t_i \right) \\ &\qquad \times \exp\Big\{ -\frac{\delta \log t_{m-1}}{2 \log t_l} \Big\} (\log t_l)^{-(h_{l,l}+\cdots+h_{l,m-1})} \sum_{a_l \in \mathscr{P}_*(t'_{l-1}, t_l)} \frac{L^{(k+1)}(\boldsymbol{a})}{a_l} \\ &\ll_{k,\boldsymbol{h},\epsilon} \sum_{a_l \in \mathscr{P}_*(t'_{l-1}, t_l)} \frac{L^{(k+1)}(\boldsymbol{a})}{a_l} \prod_{i=l}^{k} \log^{-h_{l,i}} \Big(P^+(a_1 \cdots a_{l-1}) + \frac{t_i^{\epsilon/2+(l-1)\delta}}{a_1 \cdots a_{l-1}} \Big). \end{aligned}$$

Combining the above estimate with (4.2.5) shows (4.2.4) and hence (4.2.3). Finally, iterating (4.2.3) completes the proof of the lemma. □

Before we prove the upper bound in Theorem 2.8, we need one last intermediate result.

Lemma 4.2.3. *Let $1 \leq l \leq k-1$ and $3 \leq t_1 \leq \cdots \leq t_k$. Then*

$$S^{(k-l+1)}(t_{l+1}, \dots, t_k) \leq (\log 2)^{-l} S^{(k+1)}(t_1, \dots, t_k).$$

Proof. Note that

$$\mathcal{L}^{(k+1)}(\boldsymbol{a}) \supset \bigcup_{\substack{d_1\cdots d_i | a_1\cdots a_i \ (1\le i\le k) \\ d_i=1 \ (1\le i\le l)}} [\log(d_1/2), \log d_1) \times \cdots \times [\log(d_k/2), \log d_k)$$
$$= [-\log 2, 0)^l \times \mathcal{L}^{(k-l+1)}(a_1\cdots a_{l+1}, a_{l+2}, \ldots, a_k)$$

and consequently

$$L^{(k+1)}(\boldsymbol{a}) \ge (\log 2)^l L^{(k-l+1)}(a_1\cdots a_{l+1}, a_{l+2}, \ldots, a_k).$$

The desired result then follows immediately. □

We are now in position to show the upper bound in Theorem 2.8. In fact, we shall prove a slightly stronger estimate, which will be useful in the proof of Theorem 2.6.

Theorem 4.1. *Fix $k \ge 1$. Let $x \ge 1$ and $3 \le y_1 \le \cdots \le y_k$ with $2^k y_1 \cdots y_k \le x/y_k$. There exists a constant C_k such that*

$$\frac{H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})}{x} \ll_k \Big(\prod_{i=1}^{k} \log^{-e_{k,i}} y_i\Big) \sum_{\substack{\boldsymbol{a}\in\mathscr{P}^k_*(\boldsymbol{y}) \\ a_i \le y_i^{C_k} \ (1\le i\le k)}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k}.$$

Proof. Observe that it suffices to show that

$$H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y}) \ll_k x\Big(\prod_{i=1}^{k} \log^{-e_{k,i}} y_i\Big) T, \tag{4.2.6}$$

where

$$T := \max\{S^{(k+1)}(\boldsymbol{t}) : 1 \le t_1 \le \cdots \le t_k, \ \sqrt{y_i} \le t_i \le 2y_i \ (m \le i \le k)\}.$$

Indeed, assume for the moment that (4.2.6) holds. Note that

$$T \ll_k S^{(k+1)}(\boldsymbol{y}),$$

by Lemma 2.3.1(b) and the inequality

$$\sum_{a\in\mathscr{P}_*(t,t^c)} \frac{\tau_l(a)}{a} \ll_{l,c} 1.$$

Also,

$$\sum_{\substack{\boldsymbol{a}\in\mathscr{P}_*^k(\boldsymbol{y}) \\ a_i>y_i^{C_k}}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k} \ll_k e^{-1/(2C_k)} S^{(k+1)}(\boldsymbol{y}) \quad (1\le i\le k),$$

by Lemma 3.1.3(b) applied to the arithmetic function $a_i \to L^{(k+1)}(\boldsymbol{a})$. Hence if C_k is large enough, then we find that

$$T \ll_k S^{(k+1)}(\boldsymbol{y}) \le 2 \sum_{\substack{\boldsymbol{a}\in\mathscr{P}_*^k(\boldsymbol{y}) \\ a_i\le y_i^{C_k} \ (1\le i\le k)}} \frac{L^{(k+1)}(\boldsymbol{a})}{a_1\cdots a_k},$$

which together with (4.2.6) completes the proof of the theorem.

In order to prove (4.2.6) we first reduce the counting in $H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})$ to square-free integers. Let $n\le x$ be such that $\tau_{k+1}(n,\boldsymbol{y},2\boldsymbol{y})\ge 1$. Write $n=ab$ with a being square-full, b square-free and $(a,b)=1$. The number of $n\le x$ with $a>(\log y_k)^{2k+2}$ is at most

$$x \sum_{\substack{a>(\log y_k)^{2k+2} \\ a \text{ square-full}}} \frac{1}{a} \ll_k \frac{x}{(\log y_k)^{k+1}}.$$

Assume now that $a\le(\log y_k)^{2k+2}$. Set $y_0=1$ and

$$I_j = \{a\in\mathbb{N}\cap((\log y_{j-1})^{2k+2},(\log y_j)^{2k+2}] : a \text{ square}-\text{full}\} \quad (1\le j\le k).$$

Let $d_1\cdots d_k|n$ with $y_i<d_i\le 2y_i$ for $1\le i\le k$. Then we may uniquely write $d_i=f_ie_i$ with

$f_1\cdots f_k|a$ and $e_1\cdots e_k|b$. Therefore

$$H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y}) \le \sum_{m=1}^{k}\sum_{a\in I_m}\sum_{f_1\cdots f_k|a} H_*^{(k+1)}\Big(\frac{x}{a},\Big(\frac{y_1}{f_1},\dots,\frac{y_k}{f_k}\Big),2\Big(\frac{y_1}{f_1},\dots,\frac{y_k}{f_k}\Big)\Big) + O\Big(\frac{x}{(\log y_k)^{k+1}}\Big). \tag{4.2.7}$$

Fix $m\in\{1,\dots,k\}$, $a\in I_m$ and positive integers $f_1,\dots,f_k$ such that $f_1\cdots f_k|a$. Set $x'=x/a$, and $y_i'=y_i/f_i$ for $1\le i\le k$. Let $z_1,\dots,z_k$ be the sequence $y_1',\dots,y_k'$ in increasing order. Define a permutation $\sigma\in S_k$ by $z_i=y'_{\sigma(i)}$ for $i=1,\dots,k$. Set $\boldsymbol{z}'=(z_m,\dots,z_k)$ and note that

$$H_*^{(k+1)}(x',\boldsymbol{y}',2\boldsymbol{y}') \le H_*^{(k-m+2)}(x',\boldsymbol{z}',2\boldsymbol{z}'). \tag{4.2.8}$$

Also,

$$f_i \le a \le (\log y_m)^{2k+2} \le \sqrt{y_i} \quad (m\le i\le k), \tag{4.2.9}$$

provided that y_1 is large enough. Let $i\in\{m,\dots,k\}$. By the pigeonhole principle, there exists some $j\in\{1,\dots,i\}$ such that $\sigma(j)\ge i\ge m$. So

$$z_i \ge z_j = \frac{y_{\sigma(j)}}{f_{\sigma(j)}} \ge \sqrt{y_{\sigma(j)}} \ge \sqrt{y_i}, \tag{4.2.10}$$

by (4.2.9). Similarly, there exists some $j'\in\{i,\dots,k\}$ such that $\sigma(j')\le i$ and consequently

$$z_i \le z_{j'} = \frac{y_{\sigma(j')}}{f_{\sigma(j')}} \le y_{\sigma(j')} \le y_i. \tag{4.2.11}$$

Each $n\in(x'/(\log y_k)^{k+1},x']$ lies in a interval $(2^{-r-1}x',2^{-r}x']$ for some integer $0\le r\le (k+1)\log\log y_k/\log 2$. Note that

$$\frac{x'2^{-r}}{2^{k-m+2}z_m\cdots z_k} \ge \frac{x'2^{-r}}{2^{k+1}z_1\cdots z_k} = \frac{x}{2^k y_1\cdots y_k}\frac{f_1\cdots f_k}{2^{r+1}a} \ge (2y_k)^{7/8} \ge (2z_k)^{7/8}.$$

Thus Lemma 4.2.1 with $k-m+1$ in place of k, $2^{-r}x'$ in place of x and $z_m,\dots,z_k$ in place of $y_1, \ldots, y_k$, combined with Lemmas 4.2.2 and 4.2.3, relations (4.2.8), (4.2.10) and (4.2.11) and the observation that

$$T \geq L^{(k+1)}(1, \ldots, 1) = (\log 2)^k, \tag{4.2.12}$$

yields

$$\begin{aligned} H_*^{(k+1)}(x', \boldsymbol{y}', 2\boldsymbol{y}') &\ll_k \sum_{r\geq 0} \frac{x'}{2^r} \Big(\prod_{i=m}^{k} (\log y_i)^{-e_{k,i}} \Big) S^{(k-m+2)}(2\boldsymbol{z}') + \frac{x'}{(\log y_k)^{k+1}} \\ &\ll_k x' \Big(\prod_{i=m}^{k} \log^{-e_{k,i}} y_i \Big) T \ll_k x' (\log 2y_{m-1})^{m-1} \Big(\prod_{i=1}^{k} \log^{-e_{k,i}} y_i \Big) T. \end{aligned} \tag{4.2.13}$$

Furthermore,

$$\sum_{\substack{a > (\log y_{m-1})^{2k+2} \\ a \text{ square-full}}} \frac{\tau_{k+1}(a)}{a} \ll_k \frac{1}{(\log 2y_{m-1})^k},$$

which together with (4.2.13) implies

$$\sum_{a\in I_m} \sum_{f_1\cdots f_k | a} H_*^{(k+1)} \Big(\frac{x}{a}, \Big(\frac{y_1}{f_1}, \ldots, \frac{y_k}{f_k} \Big), 2\Big(\frac{y_1}{f_1}, \ldots, \frac{y_k}{f_k} \Big) \Big) \ll_k \frac{x(\prod_{i=1}^{k} \log^{-e_{k,i}} y_i)T}{(\log 2y_{m-1})^{k-m+1}}.$$

Inserting the above estimate into (4.2.7) and combining the resulting inequality with (4.2.12) shows (4.2.6) and therefore concludes the proof of the theorem. □

4.3 Proof of Theorem 2.5

In this section we prove Theorem 2.5. Let $3 = N_0 \leq N_1 \leq \cdots \leq N_{k+1}$. We have that

$$\begin{aligned} A_{k+1}(N_1, \ldots, N_{k+1}) &\geq H^{(k+1)} \Big(\frac{N_1 \cdots N_{k+1}}{2^{k^2}}, \Big(\frac{N_1}{2^k}, \ldots, \frac{N_k}{2^k} \Big), \Big(\frac{N_1}{2^{k-1}}, \ldots, \frac{N_k}{2^{k-1}} \Big) \Big) \\ &\asymp_k H^{(k+1)} \Big(N_1 \cdots N_{k+1}, \Big(\frac{N_1}{2}, \ldots, \frac{N_k}{2} \Big), (N_1, \ldots, N_{k+1}) \Big), \end{aligned} \tag{4.3.1}$$

by Corollary 2.1. Also,

$$A_{k+1}(N_1,\dots,N_{k+1}) \le \sum_{\substack{1\le 2^{m_i}\le N_i\\ 1\le i\le k}} H^{(k+1)}\Big(\frac{N_1\cdots N_{k+1}}{2^{m_1+\cdots+m_k}}, \Big(\frac{N_1}{2^{m_1+1}},\dots,\frac{N_k}{2^{m_k}}\Big), \Big(\frac{N_1}{2^{m_1}},\dots,\frac{N_k}{2^{m_k}}\Big)\Big). \tag{4.3.2}$$

For fixed $i\in\{0,1,\dots,k\}$, let $\mathcal{M}_i$ be the set of vectors $\boldsymbol{m}\in(\mathbb{N}\cup\{0\})^k$ such that $2^{m_j}\le\sqrt{N_j}$ for $i<j\le k$ and $\sqrt{N_i}<2^{m_i}\le N_i$ and set

$$T_i = \sum_{\boldsymbol{m}\in\mathcal{M}_i} H^{(k+1)}\Big(\frac{N_1\cdots N_{k+1}}{2^{m_1+\cdots+m_k}}, \Big(\frac{N_1}{2^{m_1+1}},\dots,\frac{N_k}{2^{m_k+1}}\Big), \Big(\frac{N_1}{2^{m_1}},\dots,\frac{N_k}{2^{m_k+1}}\Big)\Big).$$

To bound T_i consider $\boldsymbol{m}\in\mathcal{M}_i$ and let $\boldsymbol{N'}=(N'_{i+1},\dots,N'_k)$ be the vector whose coordinates are the sequence $\{N_j/2^{m_j+1}\}_{j=i+1}^k$ in increasing order. Then Corollary 2.1, Theorem 2.4, Lemma 4.2.3 and the fact that $\sqrt{N_j}\le N'_j\le N_j$ for $j\in\{i+1,\dots,k\}$ imply that

$$\begin{aligned}
&H^{(k+1)}\Big(\frac{N_1\cdots N_{k+1}}{2^{m_1+\cdots+m_k}}, \Big(\frac{N_1}{2^{m_1+1}},\dots,\frac{N_k}{2^{m_k+1}}\Big), \Big(\frac{N_1}{2^{m_1}},\dots,\frac{N_k}{2^{m_k}}\Big)\Big)\\
&\le H^{(k-i+1)}\Big(\frac{N_1\cdots N_{k+1}}{2^{m_1+\cdots+m_k}}, \boldsymbol{N'}, 2\boldsymbol{N'}\Big)\\
&\asymp_k \frac{N_1\cdots N_{k+1}}{2^{m_1+\cdots+m_k}} S^{(k-i+1)}(\boldsymbol{N'}) \prod_{j=i+1}^{k} (\log N'_j)^{-e_{k,j}}\\
&\ll_k \frac{N_1\cdots N_{k+1}}{2^{m_1+\cdots+m_k}} S^{(k+1)}(\sqrt{N_1},\dots,\sqrt{N_i},\boldsymbol{N'}) \prod_{j=i+1}^{k} (\log N_j)^{-e_{k,j}}\\
&\asymp_k \frac{H^{(k+1)}(N_1\cdots N_{k+1},(N_1,\dots,N_k)/2,(N_1,\dots,N_k))}{2^{m_1+\cdots+m_k}} \prod_{j=1}^{i}(\log N_j)^{e_{k,j}}.
\end{aligned}$$

Summing the above inequality over $\boldsymbol{m}\in\mathcal{M}_i$ gives us that

$$T_i \ll_k \frac{H^{(k+1)}(N_1\cdots N_{k+1},(N_1,\dots,N_k)/2,(N_1,\dots,N_k)\Big)}{\sqrt{N_i}} \prod_{j=1}^{i}(\log N_j)^{e_{k,j}},$$

which together with (4.3.1) and (4.3.2) completes the proof of Theorem 2.5.

4.4 Divisors of shifted primes

We conclude this chapter with the proof of Theorem 2.4. Fix $C \geq 2$ and $s \in \mathbb{Z} \setminus \{0\}$ and let x, y, z be as in the statement of Theorem 2.4. Without loss of generality, we may assume that

$$z \leq y^{101/100}. \tag{4.4.1}$$

Indeed, if Theorem 2.4 is true when (4.4.1) holds, then for $z > y^{101/100}$ we have

$$H(x,y,z;P_s) \geq H(x,y,y^{101/100};P_s) \gg_s \frac{H(x,y,y^{101/100})}{\log x} \asymp \frac{H(x,y,z)}{\log x},$$

by Theorem 2.1, and consequently Theorem 2.4 is true for $z > y^{101/100}$ as well. So assume that (4.4.1) does hold. Let $y_0 = y_0(s,C)$ be a large positive constant. If $y \leq y_0$, then

$$H(x,y,z;P_s) \geq \max_{\substack{y<d\leq z \\ (d,s)=1}} \pi(x-s;d,-s) \gg_{y_0} \frac{x}{\log x} \asymp_{y_0} \frac{H(x,y,z)}{\log x},$$

by the Prime Number Theorem for arithmetic progressions [Dav, p. 123] and our assumption that $\{y < d \leq z : (d,s)=1\} \neq \emptyset$. Suppose now that $y > y_0$. Fix an integer $t = t(s) \geq 3$ and set $w = z^{1/20t}$. We will choose t later; till then, all implied constants will be independent of t. Consider integers $n = aqb_1b_2s_1 \leq x$ such that

(1) $s_1 = 2/(s,2)$;

(2) $a \leq w$, $\mu^2(a) = 1$ and $(a,2s) = 1$;

(3) $\log(y/q) \in \mathcal{L}(a;\eta)$, $P^-(q) > w$ and $(q,2s) = 1$;

(4) $b_1 \in \mathscr{P}(w,z)$ and $\tau(b_1) \leq t^2$;

(5) $P^-(b_2) > z$;

(6) $n \in P_s$.

Condition (3) implies that there exists $d|a$ such that $y/d < q \le z/d$; in particular, we have that $\tau(n, y, z) \ge 1$ and thus n is counted by $H(x, y, z; P_s)$. Also, $\Omega(q) \le \log z/\log w = 20t$ and therefore

$$\tau(qb_1) \le 2^{\Omega(q)}\tau(b_1) \le 2^{20t}t^2.$$

Since each n has at most $\tau(qb_1) \le 2^{20t}t^2$ representations of the above form, we find that

$$2^{20t}t^2 H(x, y, z; P_s) \ge \sum_{\substack{a\le w\\ \mu^2(a)=1\\ (a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ P^-(q)>w\\ (q,2s)=1}} \sum_{\substack{b_1b_2\le x/aqs_1\\ b_1\in\mathscr{P}(w,z),\ P^-(b_2)>z\\ \tau(b_1)\le t^2\\ aqb_1b_2s_1-s \text{ prime}}} 1 =: \sum_{\substack{a\le w\\ \mu^2(a)=1\\ (a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ P^-(q)>w\\ (q,2s)=1}} B_0(a, q). \tag{4.4.2}$$

For a and q as above let

$$B(a, q) = \sum_{\substack{b\le x/aqs_1\\ P^-(b)>w\\ aqbs_1-s \text{ prime}}} 1 \quad \text{and} \quad R(a, q) = B(a, q) - B_0(a, q).$$

Given b with $P^-(b) > w$, write $b = b_1b_2$ with $b_1 \in \mathscr{P}(w, z)$ and $P^-(b_2) > z$ and put $f(b) = \tau(b_1)$. Then, for fixed a and q with $(aq, 2s) = 1$, we have that

$$R(a, q) \le \frac{1}{t^2} \sum_{\substack{b\le x/aqs_1\\ P^-(b)>w\\ aqbs_1-s \text{ prime}}} f(b) = \frac{1}{t^2} \sum_{\substack{p+s\le x\\ p\equiv -s \ (\mathrm{mod}\ aqs_1)\\ P^-(\frac{p+s}{aqs_1})>w}} f\Big(\frac{p+s}{aqs_1}\Big) \ll_s \frac{1}{t}\frac{x}{\phi(aq)\log x\log w},$$

by Lemma 3.1.4. Inserting the above estimate into (4.4.2) yields that

$$\begin{aligned} 2^{20t}t^2 H(x, y, z; P_s) \ge & \sum_{\substack{a\le w\\ \mu^2(a)=1\\ (a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ P^-(q)>w\\ (q,2s)=1}} B(a, q) \\ & - O_s\left(\frac{1}{t}\frac{x}{\log x\log w} \sum_{\substack{a\le w\\ \mu^2(a)=1\\ (a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ P^-(q)>w\\ (q,2s)=1}} \frac{1}{\phi(aq)}\right). \end{aligned} \tag{4.4.3}$$

Next, we need to approximate the characteristic function of integers n with $P^-(n) > w$ with a 'smoother' function, the reason being that the error term $\pi(x; rq, a) - \mathrm{li}(x)/\phi(rq)$ in Theorem 1.5 is weighted with the smooth function 1 as q runs through $[1, Q] \cap \mathbb{N}$. To do this we appeal to Lemma 3.1.2(a) with $Z = w$, $D = z^{1/20}$ and $\kappa = 2$. Then

$$\begin{aligned} 2^{20t} t^2 H(x, y, z; P_s) &\geq \sum_{\substack{a \leq w \\ \mu^2(a)=1, (a,2s)=1}} \sum_{\substack{\log(y/q) \in \mathcal{L}(a;\eta) \\ (q,2s)=1}} (\lambda^- * 1)(q) B(a, q) - O_s(\mathscr{R}_1) \\ &= \sum_{\substack{a \leq w \\ \mu^2(a)=1, (a,2s)=1}} \sum_{\substack{\log(y/q) \in \mathcal{L}(a;\eta) \\ (q,2s)=1}} (\lambda^+ * 1)(q) B(a, q) - O_s(\mathscr{R}_1 + \mathscr{R}_2), \end{aligned} \tag{4.4.4}$$

where

$$\mathscr{R}_1 := \frac{1}{t} \frac{x}{\log x \log w} \sum_{\substack{a \leq w \\ \mu^2(a)=1, (a,2s)=1}} \sum_{\substack{\log(y/q) \in \mathcal{L}(a;\eta) \\ (q,2s)=1}} \frac{(\lambda^+ * 1)(q)}{\phi(aq)}$$

and

$$\mathscr{R}_2 := \sum_{\substack{a \leq w \\ \mu^2(a)=1, (a,2s)=1}} \sum_{\substack{\log(y/q) \in \mathcal{L}(a;\eta) \\ (q,2s)=1}} \big((\lambda^+ * 1)(q) - (\lambda^- * 1)(q)\big) B(a, q).$$

First, we bound $\mathscr{R}_2$ from above. For fixed a and q with $(aq, 2s) = 1$ we have

$$B(a, q) \ll_s \frac{x}{\phi(aq) \log x \log w},$$

by Lemma 3.1.4. Since $\lambda^+ * 1 - \lambda^- * 1$ is always non-negative, we get that

$$\mathscr{R}_2 \ll_s \frac{x}{\log x \log w} \sum_{\substack{a \leq w \\ \mu^2(a)=1, (a,2s)=1}} \sum_{\substack{\log(y/q) \in \mathcal{L}(a;\eta) \\ (q,2s)=1}} \frac{(\lambda^+ * 1)(q) - (\lambda^- * 1)(q)}{\phi(aq)}. \tag{4.4.5}$$

Fix $a \leq w$ with $(a, 2s) = 1$ and let $\{I_r\}_{r=1}^R$ be the collection of the intervals $[\log d - \eta, \log d)$ with $d|a$. Then for $I = [\log d - \eta, \log d)$ in this collection Lemmas 3.1.5 and 3.1.2(a) imply that

$$\sum_{\substack{\log(y/q)\in 3I\\(q,2s)=1}} \frac{(\lambda^+*1)(q)-(\lambda^-*1)(q)}{\phi(aq)}$$

$$= \sum_{\substack{h\le z^{1/20}\\(h,2s)=1}} (\lambda^+(h)-\lambda^-(h)) \sum_{\substack{e^{-\eta}y/hd<m\le e^{2\eta}y/hd\\(m,2s)=1}} \frac{1}{\phi(ahm)}$$

$$= \frac{315\zeta(3)}{2\pi^4}\frac{g(2as)\phi(2s)}{2|s|\phi(a)} \sum_{\substack{h\le z^{1/20}\\(c,2s)=1}} \frac{\lambda^+(h)-\lambda^-(h)}{h}\frac{g(ah)}{g(a)}\frac{h\phi(a)}{\phi(ah)}\Big(3\eta+O_s(y^{-2/3})\Big)$$

$$\ll_s \frac{\eta}{e^t\phi(a)} \prod_{\substack{p\le w\\p\nmid 2s,p|a}}\Big(1-\frac{1}{p}\Big)\prod_{\substack{p\le w\\p\nmid 2sa}}\Big(1-\frac{g(p)}{p-1}\Big) + \frac{1}{\phi(a)\sqrt{y}} \asymp_s \frac{1}{e^t}\frac{\eta}{\phi(a)\log w},$$

provided that y_0 is large enough, since $g(p)p/(p-1)\le \min\{p-1,2\}$ for $p\ge 3$, $g(p) = 1+O(p^{-2})$ and $g(a)\asymp 1$. Hence Lemma 3.2.1 implies that

$$\sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\(q,2s)=1}} \frac{(\lambda^+*1)(q)-(\lambda^-*1)(q)}{\phi(aq)} \ll_s \frac{1}{e^t}\frac{L(a;\eta)}{\phi(a)\log w}, \tag{4.4.6}$$

since $\lambda^+*1-\lambda^-*1$ is always non-negative. By the above inequality and (4.4.5) we deduce that

$$\mathscr{R}_2 \ll_s \frac{1}{e^t}\frac{x}{\log x\log^2 w} \sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}} \frac{L(a;\eta)}{\phi(a)}. \tag{4.4.7}$$

Next, we bound from below the sum

$$\mathscr{S} := \sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\(q,2s)=1}} (\lambda^+*1)(q)B(a,q).$$

We fix a and q with $(aq,2s)=1$ and seek a lower bound on $B(a,q)$. By Lemma 3.1.2(b) applied with $Z=w$ and $D=w^3$, there exists a sequence $\{\lambda_0(d)\}_{d\le w^3}$ such that λ_0*1 is bounded above by the characteristic function of integers b with $P^-(b) > w$. Thus

$$B(a,q) = \sum_{\substack{p \le x-x \\ p \equiv -s \pmod{aqs_1} \\ P^-((p+s)/aqs_1) > w}} 1 \ge \sum_{\substack{p \le x-s \\ p \equiv -s \pmod{aqs_1} \\ p \nmid s}} (\lambda_0 * 1)\Big(\frac{p+s}{aqs_1}\Big)$$

$$= \sum_{\substack{m \le w^3 \\ (m,s)=1}} \lambda_0(m)\pi(x-s; aqs_1m, -s) + O_s(1).$$

So, if we set

$$E(x;m,a) = \pi(x-s;m,a) - \frac{\mathrm{li}(x-s)}{\phi(m)},$$

then Lemma 3.1.2(b) and the fact that $2|s_1s$ imply that

$$B(a,q) \ge \mathrm{li}(x-s) \sum_{\substack{m \le w^3 \\ (m,s)=1}} \frac{\lambda_0(m)}{\phi(aqs_1m)} + O_s(1) + \mathscr{R}'_{aqs_1} \ge C_1(s)\frac{x}{\phi(aq)\log x \log w} + \mathscr{R}'_{aqs_1}$$

for some positive constant $C_1(s)$, where

$$\mathscr{R}'_{aqs_1} = \sum_{\substack{m \le w^3 \\ (m,s)=1}} \lambda_0(m)E(x; aqs_1m, -s).$$

Since $\lambda^+ * 1$ is always non-negative, we deduce that

$$\mathscr{S} \ge C_1(s)\frac{x}{\log x \log w} \sum_{\substack{a \le w \\ \mu^2(a)=1,(a,2s)=1}} \sum_{\substack{\log(y/q) \in \mathcal{L}(a;\eta) \\ (q,2s)=1}} \frac{(\lambda^+ * 1)(q)}{\phi(aq)} + \mathscr{R}', \tag{4.4.8}$$

where

$$\mathscr{R}' = \sum_{\substack{a \le w \\ \mu^2(a)=1,(a,2s)=1}} \sum_{\substack{\log(y/q) \in \mathcal{L}(a;\eta) \\ (q,2s)=1}} (\lambda^+ * 1)(q)\mathscr{R}'_{aqs_1}.$$

Combining (4.4.4), (4.4.7) and (4.4.8) we get that

$$
\begin{aligned}
2^{20t}t^2 H(x,y,z;P_s) \ge \frac{C_1(s)}{2}\frac{x}{\log x\log w} & \sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}} \sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}} \frac{(\lambda^+ * 1)(q)}{\phi(aq)} \\
& - O_s\Bigg(|\mathscr{R}'| + \frac{1}{e^t}\frac{x}{\log x\log^2 w}\sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}}\frac{L(a;\eta)}{\phi(a)}\Bigg),
\end{aligned}
\tag{4.4.9}
$$

provided that t is large enough. Following a similar argument with the one leading to (4.4.6) and using the fact that $\lambda^+ * 1$ is non-negative, we find that for every $a \le w$ with $(a,2s)=1$ we have

$$
\sum_{\substack{\log(y/q)\in\mathcal{L}(a;\eta)\\ (q,2s)=1}} \frac{(\lambda^+ * 1)(q)}{\phi(aq)} \gg_s \frac{L(a;\eta)}{\phi(a)\log w},
$$

provided that y_0 and t are large enough. Inserting this inequality into (4.4.9) and choosing a large enough t we conclude that

$$
H(x,y,z;P_s) \ge C_2(s)\frac{x}{\log x\log^2 y}\sum_{\substack{a\le w\\ \mu^2(a)=1,(a,2s)=1}}\frac{L(a;\eta)}{\phi(a)} - O_s(|\mathscr{R}'|) \tag{4.4.10}
$$

for some positive constant $C_2(s)$. Furthermore, note that if a is squarefree, we may uniquely write $a = db$, where $d|2s$, $\mu^2(d) = \mu^2(b) = 1$ and $(b,2s)=1$, in which case $L(a;\eta) \le \tau(d)L(b;\eta)$, by Lemma 2.3.1(c). Thus

$$
\sum_{\substack{a\le w\\ \mu^2(a)=1}}\frac{L(a;\eta)}{\phi(a)} \le \sum_{d|2s,\mu^2(d)=1}\frac{\tau(d)}{\phi(d)}\sum_{\substack{b\le w/d\\ \mu^2(b)=1\\ (b,2s)=1}}\frac{L(b;\eta)}{\phi(b)} \le \Bigg(\sum_{d|2s}\frac{\tau(d)}{\phi(d)}\Bigg)\sum_{\substack{b\le w\\ \mu^2(b)=1\\ (b,2s)=1}}\frac{L(b;\eta)}{\phi(b)},
$$

which, combined with (4.4.10), Theorem 2.7 and the trivial inequality $\phi(a)\le a$, implies that

$$
H(x,y,z;P_s) \ge C_3(s)\frac{H(x,y,z)}{\log x} - O_s(|\mathscr{R}'|)
$$

for some positive constant $C_3(s)$. In addition, observe that

$$H(x,y,z) \gg \frac{x}{(\log y)^C},$$

by Theorem 2.1 and our assumption that $z - y \geq y(\log y)^{-C}$. Hence

$$H(x,y,z;P_s) \gg_s \frac{H(x,y,z)}{\log x}\Big(1 - O_s\Big(\frac{(\log x)(\log y)^C|\mathscr{R}'|}{x}\Big)\Big).$$

So in order to prove Theorem 2.4 it suffices to show that

$$|\mathscr{R}'| \ll_{s,C} \frac{x}{(\log x)(\log y)^{C+1}}. \tag{4.4.11}$$

For every $a \in \mathbb{N}$ there is a unique set D_a of pairs (d,d') with $d \leq d'$, $d|a$ and $d'|a$ such that

$$\mathcal{L}(a;\eta) = \bigcup_{(d,d')\in D_a} [\log d - \eta, \log d')$$

and the intervals $[\log d - \eta, \log d')$ for $(d,d') \in D_a$ are mutually disjoint. With this notation we have that

$$\begin{aligned}
|\mathscr{R}'| &= \Big|\sum_a \sum_m \lambda_0(m) \sum_{(d,d')\in D_a} \sum_{\substack{y/d'<q\leq z/d \\ (q,2s)=1}} (\lambda^+ * 1)(q) E(x; ams_1q, -s)\Big| \\
&= \Big|\sum_a \sum_m \lambda_0(m) \sum_{(d,d')\in D_a} \sum_h \lambda^+(h) \sum_{\substack{y/hd'<l\leq z/hd \\ (l,2s)=1}} E(x; ams_1hl, -s)\Big| \\
&\leq \sum_{\substack{a\leq w \\ (a,2s)=1}} \sum_{\substack{m\leq w^3 \\ (m,s)=1}} \sum_{\substack{h\leq z^{1/20} \\ (h,2s)=1}} \sum_{(d,d')\in D_a} \Big| \sum_{\substack{y/hd'<l\leq z/hd \\ (l,2s)=1}} E(x; ams_1hl, -s)\Big|.
\end{aligned}$$

So writing the inner sum as a difference of two sums we obtain that

$$
\begin{aligned}
|\mathscr{R}'| &\le 2 \sup_{y\le T\le z}\Bigg\{ \sum_{\substack{a\le w\\(a,2s)=1}} \sum_{\substack{m\le w^3\\(m,s)=1}} \sum_{\substack{h\le z^{1/20}\\(h,2s)=1}} \sum_{b|ams_1h} \Bigg| \sum_{\substack{l\le T/b\\(l,2s)=1}} E(x;ams_1hl,-s)\Bigg|\Bigg\} \\
&\le 2 \sup_{y\le T\le z}\Bigg\{ \sum_{\substack{r\le 2z^{7/60}\\(r,s)=1}} \tau_3(r)\sum_{b|r}\Bigg| \sum_{\substack{l\le T/b\\(l,2s)=1}} E(x;rl,-s)\Bigg|\Bigg\} \\
&\le 4 \sup_{y\le T\le z}\Bigg\{ \sum_{\substack{r\le z^{1/8}\\(r,s)=1}} \tau_3(r)\sum_{b|r}\Bigg| \sum_{\substack{l\le T/b\\(l,s)=1}} E(x;rl,-s)\Bigg|\Bigg\},
\end{aligned}
\tag{4.4.12}
$$

since $w^4z^{1/20} \le z^{7/60} \le z^{1/8}/4$ for all $t \ge 3$. Put $\mu = 1 + (\log y)^{-C-7}$ and cover the interval $[1, z^{1/8}]$ by intervals of the form $[\mu^n, \mu^{n+1})$ for $n = 0, 1, \ldots, N$. We may take $N \ll (\log y)^{C+8}$. Since $|E(x;m,-s)| \ll_s x/(\phi(m)\log x)$ for $m \le z^{9/8} \le x^{3/4}$ with $(m,s) = 1$ by Lemma 3.1.4, we have that

$$
\begin{aligned}
&\sum_{\substack{r\le z^{1/8}\\(r,s)=1}} \tau_3(r)\sum_{n=0}^{N} \sum_{\substack{b|r\\ \mu^n\le b<\mu^{n+1}}} \Bigg| \sum_{\substack{l\le T/b\\(l,s)=1}} E(x;rl,-s) - \sum_{\substack{l\le T/\mu^n\\(l,s)=1}} E(x;rl,-s)\Bigg| \\
&\ll_s \sum_{\substack{r\le z^{1/8}\\(r,s)=1}} \tau_3(r)\sum_{n=0}^{N} \sum_{\substack{b|r\\ \mu^n\le b<\mu^{n+1}}} \sum_{T/\mu^{n+1}<l\le T/\mu^n} \frac{x}{\phi(rl)\log x} \\
&\ll \frac{x\log\mu}{\log x}\sum_{r\le z^{1/8}} \frac{\tau_3(r)}{\phi(r)}\sum_{b|r} 1 \ll \frac{x}{(\log x)(\log y)^{C+1}}
\end{aligned}
$$

for all $T \in [y,z]$, by Lemma 3.1.5, which is admissible. Combining the above estimate with (4.4.12) we find that

$$
|\mathscr{R}'| \ll_s \sup_{y\le T\le z}\Bigg\{\sum_{n=0}^{N} \sum_{\substack{r\le z^{1/8}\\(r,s)=1}} \tau_3(r)\tau(r)\Bigg| \sum_{\substack{l\le T/\mu^n\\(l,s)=1}} E(x;rl,-s)\Bigg|\Bigg\} + \frac{x}{(\log x)(\log y)^{C+1}}. \tag{4.4.13}
$$

Finally, Theorem 1.5 applied with $4C+56$ in place of C and the Cauchy-Schwarz inequality yield

$$\begin{aligned}
&\sum_{\substack{r\le z^{1/8}\\(r,s)=1}} \tau_3(r)\tau(r)\Big|\sum_{\substack{l\le T/\mu^n\\(g,s)=1}} E(x;rl,-s)\Big| \\
&\ll_s \left(\frac{x}{\log x}\sum_{r\le z^{1/8}}\sum_{l\le T/\mu^n}\frac{(\tau_3(r)\tau(r))^2}{\phi(rl)}\right)^{1/2}\left(\sum_{\substack{r\le z^{1/8}\\(r,s)=1}}\Big|\sum_{\substack{l\le T/\mu^n\\(l,s)=1}} E(x;rl,-s)\Big|\right)^{1/2} \\
&\ll_{s,C} \frac{x}{(\log x)^{2C+10}}
\end{aligned}$$

for all $T\in[y,z]$ and all $n\in\{0,1,\dots,N\}$, since $z^{1/8}\le x^{1/12}$ and $z^{9/8}\le x^{3/4}$. Plugging this estimate into (4.4.13) gives us

$$|\mathscr{R}'| \ll_{s,C} N\frac{x}{(\log x)^{2C+10}}+\frac{x}{(\log x)(\log y)^{C+1}} \ll \frac{x}{(\log x)(\log y)^{C+1}},$$

which shows (4.4.11) and thus completes the proof of Theorem 2.4.

Chapter 5

Localized factorizations of integers

In this chapter we prove Theorem 2.6.

5.1 The upper bound in Theorem 2.6

We start with the proof of the upper bound in Theorem 2.6, which is easier. In view of Corollary 2.1 and Theorem 4.1, it suffices to show that

$$S^{(k+1)}(y_1) := \sum_{\substack{P^+(a)\le y_1 \\ a\le y_1^{C_k},\, \mu^2(a)=1}} \frac{L^{(k+1)}(a)}{a} \ll_k \frac{(\log y_1)^{k+1-Q(k/\log(k+1))}}{(\log\log y_1)^{3/2}}, \tag{5.1.1}$$

where C_k is some sufficiently large constant. Before we proceed to the proof, we make some definitions. Set

$$\rho = (k+1)^{1/k}.$$

We start with the construction of a sequence of primes $\ell_1, \ell_2, ...,$ as in [For08a, For08b, Kou10a]. Set $\ell_0 = \min\{p \text{ prime} : p \ge k+1\} - 1$ and then define inductively ℓ_j as the largest prime such that

$$\sum_{\ell_{j-1}<p\le\ell_j} \frac{1}{p} \le \log\rho. \tag{5.1.2}$$

Note that $1/(\ell_0+1) \le 1/(k+1) < \log\rho$ because $(k+1)\log\rho = (k+1)\log(k+1)/k$ is an increasing function of k and $\log 4 > 1$. Thus the sequence $\{\ell_j\}_{j=1}^{\infty}$ is well-defined. Set

$$D_j = \{p \text{ prime} : \ell_{j-1} < p \le \ell_j\} \quad (j \in \mathbb{N}).$$

We have the following lemma.

Lemma 5.1.1. *There exists a positive integer L_k such that*

$$\rho^{j-L_k} \le \log \ell_j \le \rho^{j+L_k} \quad (j \in \mathbb{N}).$$

Proof. By the Prime Number Theorem with de la Valee Poussin error term [Dav, p. 111], there exists some positive constant c_1 such that

$$\log\log \ell_j - \log\log \ell_{j-1} = \log \rho + O(e^{-c_1\sqrt{\log \ell_{j-1}}}) \tag{5.1.3}$$

for all $j \in \mathbb{N}$. In addition, $\ell_j \to \infty$ as $j \to \infty$, by construction. So if we fix $\rho' \in (1,\rho)$, then (5.1.3) implies that

$$\frac{\log \ell_j}{\log \ell_{j-1}} \ge \rho'$$

for sufficiently large j, which in turn implies that that the series $\sum_j e^{-c_1\sqrt{\log \ell_j}}$ converges. Hence telescoping the summation of (5.1.3) completes the proof of the lemma. □

We are now ready to start our course towards the proof of (5.1.1). Set

$$\omega_k(a) = |\{p|a : p > k\}|$$

and

$$S_r^{(k+1)}(y_1) = \sum_{\substack{P^+(a)\le y_1, a\le y_1^{C_k} \\ \omega_k(a)=r, \mu^2(a)=1}} \frac{L^{(k+1)}(a)}{a}.$$

Lemma 5.1.2. *Let $y_1 \ge 3$ and set $v = \lfloor \log\log y_1/\log \rho \rfloor$. For $r \ge 0$ we have that*

$$S_r^{(k+1)}(y_1) \ll_k \frac{(\log\log y_1 + O_k(1))^r (k+1)^{\min\{r,v\}}(1+|r-v|)}{(r+1)!}.$$

Proof. First, note that

$$\sum_{\substack{P^+(n)\le k\\ \mu^2(n)=1}} \frac{L^{(k+1)}(n)}{n} \le (\log 2)^k \sum_{\substack{P^+(n)\le k\\ \mu^2(n)=1}} \frac{\tau_{k+1}(n)}{n} \ll_k 1, \tag{5.1.4}$$

by Lemma 2.3.1(a). This completes the proof if $r = 0$. So from now on we assume that $r \ge 1$. For the sets D_j constructed above we have

$$\{p \text{ prime} : k < p \le y_1\} \subset \bigcup_{j=1}^{v+L_k+1} D_j,$$

by Lemma 5.1.1. Consider a square-free integer $a = bp_1 \cdots p_r$ with $a \le y_1^{C_k}$ and $P^+(b) \le k < p_1 < \cdots < p_r$ and define j_i by $p_i \in D_{j_i}$, $1 \le i \le r$. By Lemmas 2.3.1 and 5.1.1, we have

$$\begin{aligned} L^{(k+1)}(a) &\le \tau_{k+1}(b) L^{(k+1)}(p_1 \cdots p_r) \\ &\le \tau_{k+1}(b) \min_{0\le i\le r} (k+1)^{r-i} (\log p_1 + \cdots + \log p_i + \log 2)^k \\ &\ll_k \tau_{k+1}(b)(k+1)^r \min\{1, F(\boldsymbol{j})\}^k, \end{aligned} \tag{5.1.5}$$

where

$$F(\boldsymbol{j}) = \min_{1\le i\le r} \rho^{-i}(\rho^{j_1} + \cdots + \rho^{j_i}).$$

Observe that

$$F(\boldsymbol{j}) \le \rho^{-r}(\rho^{j_1} + \cdots + \rho^{j_r}) \ll_k \rho^r \log a \ll_k \rho^{v-r}, \tag{5.1.6}$$

by Lemma 5.1.1. Let $\mathcal{J}$ denote the set of vectors $\boldsymbol{j} = (j_1, \ldots, j_r)$ satisfying $1 \le j_1 \le \cdots \le j_r \le v + L_k + 1$ and (5.1.6). Then (5.1.4) and (5.1.5) imply that

$$S_r^{(k+1)}(y_1) \ll_k (k+1)^r \sum_{\boldsymbol{j}\in\mathcal{J}} \min\{1, F(\boldsymbol{j})\}^k \sum_{\substack{p_1<\cdots<p_r\\ p_i\in D_{j_i}\ (1\le i\le r)}} \frac{1}{p_1 \cdots p_r}. \tag{5.1.7}$$

Fix $\boldsymbol{j} = (j_1, \ldots, j_r) \in \mathcal{J}$ and let $b_m = |\{1 \le i \le r : j_i = m\}|$ for $1 \le m \le v + L_k + 1$. By

(5.1.2) , the sum over $p_1, \dots, p_r$ in (5.1.7) is at most

$$\prod_{m=1}^{v+L_k+1} \frac{1}{b_m!}\Big(\sum_{p\in D_m} \frac{1}{p}\Big)^{b_m} \le \frac{(\log\rho)^r}{b_1!\cdots b_{v+L_k+1}!} = ((v+L_k+1)\log\rho)^r \operatorname{Vol}(I(\boldsymbol{j})) \\ = (\log\log y_1 + O_k(1))^r \operatorname{Vol}(I(\boldsymbol{j})), \tag{5.1.8}$$

where

$$I(\boldsymbol{j}) := \{0 \le \xi_1 \le \cdots \le \xi_r \le 1 : j_i - 1 \le (v+L_k+1)\xi_i < j_i \ (1 \le i \le r)\}.$$

Inserting (5.1.8) into (5.1.7) we deduce that

$$S_r^{(k+1)}(y_1) \ll_k (\log\log y_1 + O_k(1))^r (k+1)^r \sum_{\boldsymbol{j}\in\mathcal{J}} \min\{1, F(\boldsymbol{j})\}^k \operatorname{Vol}(I(\boldsymbol{j})). \tag{5.1.9}$$

Note that for every $\boldsymbol{\xi} \in I(\boldsymbol{j})$ we have that

$$\rho^{j_i} \le \rho^{1+(v+L_k+1)\xi_i} \le \rho^{L_k+2}\rho^{v\xi_i} \le \rho^{L_k+2}\rho^{j_i}$$

and thus

$$F(\boldsymbol{j}) \ll_k \min_{1\le i\le r} \rho^{r-i}(\rho^{v\xi_1} + \cdots + \rho^{v\xi_i}) =: \widetilde{F}(\boldsymbol{\xi}) \le F(\boldsymbol{j}),$$

which in turn implies that

$$\sum_{\boldsymbol{j}\in\mathcal{J}} \min\{1, F(\boldsymbol{j})\}^k \operatorname{Vol}(I(\boldsymbol{j})) \ll_k \int\cdots\int_{\substack{0\le\xi_1\le\cdots\le\xi_r\le 1\\ \widetilde{F}(\boldsymbol{\xi})\le\rho^{v-r+c_k}}} \min\{1, \widetilde{F}(\boldsymbol{\xi})\}^k d\boldsymbol{\xi} \tag{5.1.10}$$

for some sufficiently large constant c_k. Finally, note that

$$\begin{aligned}
&\underset{\substack{0\le\xi_1\le\cdots\le\xi_r\le 1\\ \widetilde{F}(\boldsymbol{\xi})\le\rho^{v-r+c_k}}}{\int\cdots\int} \min\{1,\widetilde{F}(\boldsymbol{\xi})\}^k d\boldsymbol{\xi}\\
&= \int_0^{\min\{1,\rho^{v-r+c_k}\}} k\alpha^{k-1}\operatorname{Vol}(0\le\xi_1\le\cdots\le\xi_r\le 1:\widetilde{F}(\boldsymbol{\xi})>\alpha)d\alpha\\
&= \int_0^{\min\{1,\rho^{v-r+c_k}\}} k\alpha^{k-1}\operatorname{Vol}\Big(\mathcal{T}_\rho\Big(r,v,-\frac{\log\alpha}{\log\rho}\Big)\Big)d\alpha,
\end{aligned}$$

where $\mathcal{T}_\rho(r,v,-\log\alpha/\log\rho)$ is defined by (3.3.1). Hence making the change of variable $\alpha=\rho^{-u}$ and applying Lemma 3.3.4 yields

$$\begin{aligned}
\underset{\substack{0\le\xi_1\le\cdots\le\xi_r\le 1\\ \widetilde{F}(\boldsymbol{\xi})\le\rho^{v-r+c_k}}}{\int\cdots\int} \min\{1,\widetilde{F}(\boldsymbol{\xi})\}^k d\boldsymbol{\xi} &= \log(k+1)\int_{\max\{0,r-v-c_k\}}^{\infty}(k+1)^{-u}\operatorname{Vol}(\mathcal{T}_\rho(r,v,u))du\\
&\ll_k \int_{\max\{0,r-v-c_k\}}^{\infty}\frac{(u+1)(u+v-r+c_k+1)}{(r+1)!(k+1)^u}du\\
&\ll_k \frac{1+|r-v|}{(r+1)!(1+(k+1)^{r-v})}.
\end{aligned}$$

Combining the above inequality with (5.1.9) and (5.1.10) completes the proof of the lemma.

□

Having proven the above result, it easy to bound $S^{(k+1)}(y_1)$ from above. Indeed, if we set $v=\lfloor\log\log y_1/\log\rho\rfloor$, then

$$\begin{aligned}
S^{(k+1)}(y_1)=\sum_{r=0}^{\infty}S_r^{(k+1)}(y_1) &\ll_k \sum_{r=0}^{\infty}\frac{(k+1)^{\min\{v,r\}}(\log\log y_1+O_k(1))^r(1+|v-r|)}{(r+1)!}\\
&\ll_k \frac{(k+1)^v(\log\log y_1+O_k(1))^v}{(v+1)!}\\
&\asymp_k \frac{(\log y_1)^{k+1-Q(1/\log\rho)}}{(\log\log y_1)^{3/2}},
\end{aligned}$$

by Lemma 5.1.2, Stirling's formula and the inequalities

$$\frac{1}{k+1} < \log \rho < 1.$$

This establishes (5.1.1) and thus completes the proof of the upper bound in Theorem 2.6.

5.2 The lower bound in Theorem 2.6: outline of the proof

In this section we give the main steps towards the proof of the lower bound in Theorem 2.6. As in the previous section, observe that it is sufficient to show that

$$\sum_{\substack{P^+(a)\le y_1\\ \mu^2(a)=1}} \frac{L^{(k+1)}(a)}{a} \gg_k \frac{(\log y_1)^{k+1-Q(k/\log(k+1))}}{(\log\log y_1)^{3/2}}; \tag{5.2.1}$$

then Corollary 2.1 and Theorem 2.8 yield the lower bound in Theorem 2.6 immediately.

As we mentioned in Section 2.4, the main tool we use in order to bound $L^{(k+1)}(a)$ from below is Hölder's inequality. To this end, given $P \in (1,+\infty)$ and $a \in \mathbb{N}$ set

$$W_{k+1}^P(a) = \sum_{d_1\cdots d_k|a} \left|\left\{(e_1,\dots,e_k)\in\mathbb{N}^k : e_1\cdots e_k|a,\ \left|\log\frac{e_i}{d_i}\right| < \log 2\ (1\le i\le k)\right\}\right|^{P-1}.$$

Lemma 5.2.1. *Let $\mathcal{A}$ be a finite set of positive integers and $P \in (1,+\infty)$. Then*

$$\sum_{a\in\mathcal{A}} \frac{\tau_{k+1}(a)}{a} \le \left(\sum_{a\in\mathcal{A}} \frac{W_{k+1}^P(a)}{a}\right)^{1/P} \left(\frac{1}{(\log 2)^k}\sum_{a\in\mathcal{A}} \frac{L^{(k+1)}(a)}{a}\right)^{1-1/P}.$$

Proof. For $\boldsymbol{d} = (d_1,\dots,d_k) \in \mathbb{R}^k$ let $\chi_{\boldsymbol{d}}$ be the characteristic function of the k-dimensional cube $[\log(d_1/2), \log d_1) \times \cdots \times [\log(d_k/2), \log d_k)$. Then it is easy to see that

$$\tau_{k+1}(a, e^{\boldsymbol{u}}, 2e^{\boldsymbol{u}}) = \sum_{d_1 \cdots d_k | a} \chi_{\boldsymbol{d}}(\boldsymbol{u})$$

for all $a \in \mathbb{N}$, where $e^{\boldsymbol{u}} = (e^{u_1}, \ldots, e^{u_k})$ for $\boldsymbol{u} = (u_1, \ldots, u_k) \in \mathbb{R}^k$. Hence

$$\int_{\mathbb{R}^k} \tau_{k+1}(a, e^{\boldsymbol{u}}, 2e^{\boldsymbol{u}}) d\boldsymbol{u} = \tau_{k+1}(a)(\log 2)^k$$

and a double application of Hölder's inequality yields

$$(\log 2)^k \sum_{a \in \mathcal{A}} \frac{\tau_{k+1}(a)}{a} \le \left(\sum_{a \in \mathcal{A}} \frac{1}{a} \int_{\mathbb{R}^k} \tau_{k+1}(a, e^{\boldsymbol{u}}, 2e^{\boldsymbol{u}})^P d\boldsymbol{u} \right)^{1/P} \left(\sum_{a \in \mathcal{A}} \frac{L^{(k+1)}(a)}{a} \right)^{1-1/P}. \quad (5.2.2)$$

Finally, note that

$$\begin{aligned} \tau_{k+1}(a, e^{\boldsymbol{u}}, 2e^{\boldsymbol{u}})^P &= \sum_{\substack{d_1 \cdots d_k | a \\ u_i < \log d_i \le u_i + \log 2 \\ 1 \le i \le k}} \left(\sum_{\substack{e_1 \cdots e_k | a \\ u_i < \log e_i \le u_i + \log 2 \\ 1 \le i \le k}} 1 \right)^{P-1} \\ &\le \sum_{\substack{d_1 \cdots d_k | a \\ u_i < \log d_i \le u_i + \log 2 \\ 1 \le i \le k}} \left(\sum_{\substack{e_1 \cdots e_k | a \\ |\log(e_i/d_i)| < \log 2 \\ 1 \le i \le k}} 1 \right)^{P-1}. \end{aligned}$$

So

$$\int_{\mathbb{R}^k} \tau(a, e^{\boldsymbol{u}}, 2e^{\boldsymbol{u}})^P d\boldsymbol{u} \le \sum_{d_1 \cdots d_k | a} \left(\sum_{\substack{e_1 \cdots e_k | a \\ |\log(e_i/d_i)| < \log 2 \\ 1 \le i \le k}} 1 \right)^{P-1} \int_{\mathbb{R}^k} \chi_{\boldsymbol{d}}(\boldsymbol{u}) d\boldsymbol{u} = (\log 2)^k W_{k+1}^P(a),$$

which together with (5.2.2) completes the proof of the lemma. □

Our next goal is to estimate

$$\sum_{a \in \mathcal{A}} \frac{W_{k+1}^P(a)}{a}$$

for suitably chosen sets $\mathcal{A}$. Recall the sets D_j constructed in Section 5.1. For $\boldsymbol{b} = (b_1, \ldots, b_H) \in (\mathbb{N} \cup \{0\})^H$ let $\mathcal{A}(\boldsymbol{b})$ be the set of square-free integers composed of exactly b_j prime factors from D_j for each j. Set $B = b_1 + \cdots + b_H$, $B_0 = 0$ and $B_i = b_1 + \cdots + b_i$ for all $i = 1, \ldots, H$.

Lemma 5.2.2. *Let $P \in (1, 2]$ and $\boldsymbol{b} = (b_1, \ldots, b_H) \in (\mathbb{N} \cup \{0\})^H$. Then*

$$\sum_{a \in \mathcal{A}(\boldsymbol{b})} \frac{W_{k+1}^P(a)}{a} \ll_k \frac{((k+1)\log\rho)^B}{b_1! \cdots b_H!} \sum_{0 \le j_1 \le \cdots \le j_k \le H} (\rho^{P-1})^{-(j_1 + \cdots + j_k)} \times \prod_{i=1}^{k} \Big(\frac{i - 1 + (k-i+2)^P}{i + (k-i+1)^P} \Big)^{B_{j_i}}.$$

The proof of Lemma 5.2.2 will be given in three steps: the first one is carried out in Section 5.3, the second one in Section 5.4 and the last one in Section 5.5.

Remark 5.2.1. [1] Lemma 5.2.2 is essentially sharp. To see this $H = \lfloor \log\log y_1 / \log \rho \rfloor$ and note that for every fixed $l \in \{0, 1, \ldots, k\}$ we have

$$(k+1)^B \prod_{i=1}^{k} (\rho^{P-1})^{-j_i} \Big(\frac{i - 1 + (k-i+2)^P}{i + (k-i+1)^P} \Big)^{B_{j_i}} \asymp \frac{(l + (k-l+1)^P)^B}{(\log y_1)^{(P-1)(k-l)}}, \tag{5.2.3}$$

when $j_i = 0$ for $i \le l$, and $j_i = H$ for $i > l$. Moreover, we claim that if $a \in \mathbb{N}$ is such that $\mu^2(a) = 1$, $\omega(a) = B$ and $\log a \ll \log y_1$, then

$$W_{k+1}^P(a) \gg \frac{(l + (k-l+1)^P)^B}{(\log y_1)^{(P-1)(k-l)}}. \tag{5.2.4}$$

Indeed, recall that

$$W_{k+1}^P(a) = \sum_{d_1 \cdots d_k | a} \Bigg(\sum_{\substack{e_1 \cdots e_k | a \\ |\log(e_i/d_i)| < \log 2 \\ 1 \le i \le k}} \Bigg)^{P-1}.$$

[1] The argument given here was discovered in conversations with Kevin Ford

So setting $e_i = d_i$ for $1 \le i \le l$ in the inner sum yields

$$W_{k+1}^P(a) \ge \sum_{d|a} \tau_l(d) W_{k-l+1}^P(a/d)$$

for every $a \in \mathbb{N}$. By Lemma 5.2.1, we get that

$$W_{k-l+1}^P(b) \ge \frac{(\log 2)^{(k-l+1)(P-1)} \tau_{k-l+1}(b)^P}{L^{(k-l+1)}(b)^{P-1}} \gg \frac{(k-l+1)^{P\omega(b)}}{(\log y_1)^{(P-1)(k-l)}}$$

for all $b \in \mathbb{N}$. So

$$W_{k+1}^P(a) \gg (\log y_1)^{-(P-1)(k-l)} \sum_{d|a} l^{\omega(d)} (k-l+1)^{P\omega(a/d)} = \frac{(l + (k-l+1)^P)^{\omega(a)}}{(\log y_1)^{(P-1)(k-l)}}, \quad (5.2.5)$$

which proves (5.2.4). In view of Lemma (5.2.2) and relations (5.2.3) and (5.2.4), it is reasonable to assume that

$$W_{k+1}^P(a) \approx \sum_{l=0}^{k} \frac{(l + (k-l+1)^P)^B}{(\log y_1)^{(P-1)(k-l)}}$$

for $a \in \mathbb{N}$ with $\omega(a) = B$ and $\log a \asymp \log y_1$. In order to make Hölder's inequality (cf. Lemma 5.2.1) sharp, we would like to show that

$$W_{k+1}^P(a) \approx (k+1)^B, \quad (5.2.6)$$

which essentially reduces to showing that

$$\frac{l + (k-l+1)^P}{k+1} \le (\rho^{P-1})^{k-l} \quad (0 \le l < k).$$

This is accomplished by choosing P small enough (see Lemmas 5.2.3 and 5.5.1 below).

Next, we impose some conditions on $\boldsymbol{b}$ and P to simplify the upper bound in Lemma 5.2.2.

More precisely, set

$$P = \min\Big\{2, \frac{(k+1)^2 \log^2 \rho}{(k+1)^2 \log^2 \rho - 1}\Big\} \tag{5.2.7}$$

and let $\mathcal{B}$ be the set of vectors $(b_1, \ldots, b_H)$ such that $B_i \leq i$ for all $i \in \{1, \ldots, H\}$. Lastly, set

$$\nu = \frac{(k+1)^P}{k^P + 1} > 1.$$

Lemma 5.2.3. *Let $k \geq 2$ and $\boldsymbol{b} = (b_1, \ldots, b_H) \in \mathcal{B}$. If P is defined by (5.2.7), then*

$$\sum_{a \in \mathcal{A}(\boldsymbol{b})} \frac{W_{k+1}^P(a)}{a} \ll_k \frac{((k+1)\log \rho)^B}{b_1! \cdots b_H!} \Big(1 + \sum_{j=1}^{H} \nu^{B_j - j}\Big).$$

The proof of Lemma 5.2.3 will be given in Section 5.5. Using this result, we complete the proof of (5.2.1) - and consequently of the lower bound in Theorem 2.6 - in Section 5.6.

5.3 The method of low moments: interpolating between L^1 and L^2 estimates

In this section we carry out the first step towards the proof of Lemma 5.2.2. Before we proceed, we introduce some notation. Given $\boldsymbol{b} \in (\mathbb{N} \cup \{0\})^H$ and $I \in \{0, 1, \ldots, B\}$, define $E_{\boldsymbol{b}}(I)$ by $B_{E_{\boldsymbol{b}}(I)-1} < I \leq B_{E_{\boldsymbol{b}}(I)}$ if $I > 0$ and set $E_{\boldsymbol{b}}(I) = 0$ if $I = 0$. Also, for $R \in \mathbb{N}$ let

$$\mathscr{P}_R^k = \{(Y_1, \ldots, Y_k) : Y_i \subset \{1, \ldots, R\}, Y_i \cap Y_j = \emptyset \text{ if } i \neq j\}.$$

For $\boldsymbol{Y} \in \mathscr{P}_R^k$ and $\boldsymbol{I} \in \{0, 1, \ldots, R\}^k$ set

$$M_R^k(\boldsymbol{Y}; \boldsymbol{I}) = \Big|\Big\{\boldsymbol{Z} \in \mathscr{P}_R : \bigcup_{r=j}^{k} \big(Z_r \cap (I_j, R]\big) = \bigcup_{r=j}^{k} \big(Y_r \cap (I_j, R]\big) \ (1 \leq j \leq k)\Big\}\Big|.$$

Lastly, for a family of sets $\{X_j\}_{j\in J}$ define

$$\mathcal{U}(\{X_j : j \in J\}) = \Big\{x \in \bigcup_{j\in J} X_j : |\{i \in J : x \in X_i\}| = 1\Big\}.$$

In particular,

$$\mathcal{U}(\{X_1, X_2\}) = X_1 \triangle X_2,$$

the symmetric difference of X_1 and X_2, and

$$\mathcal{U}(\emptyset) = \emptyset.$$

Remark 5.3.1. Assume that $Y_1, \ldots, Y_n$ and $Z_1, \ldots, Z_n$ satisfy $Y_i \cap Y_j = Z_i \cap Z_j = \emptyset$ for $i \neq j$. Then the condition

$$\mathcal{U}(\{Y_j \triangle Z_j : 1 \leq j \leq n\}) = \emptyset$$

is equivalent to

$$\bigcup_{j=1}^{n} Y_j = \bigcup_{j=1}^{n} Z_j.$$

Lemma 5.3.1. *Let $k \geq 2$, $P \in (1, 2]$ and $\boldsymbol{b} = (b_1, \ldots, b_H) \in (\mathbb{N} \cup \{0\})^H$. Then*

$$\sum_{a\in\mathcal{A}(\boldsymbol{b})} \frac{W_{k+1}^P(a)}{a} \ll_k \frac{(\log\rho)^B}{b_1! \cdots b_H!} \sum_{0\leq I_1,\ldots,I_k\leq B} \prod_{j=1}^{k} (\rho^{P-1})^{-E_{\boldsymbol{b}}(I_j)} \sum_{\boldsymbol{Y}\in\mathscr{P}_B^k} \big(M_B^k(\boldsymbol{Y};\boldsymbol{I})\big)^{P-1}.$$

Proof. Let $a = p_1 \cdots p_B \in \mathcal{A}(\boldsymbol{b})$, where

$$p_{B_{i-1}+1}, \ldots, p_{B_i} \in D_i \quad (1 \leq i \leq H), \tag{5.3.1}$$

and the primes in each interval D_j for $j = 1, \ldots, H$ are unordered. Observe that, since $a = p_1 \cdots p_B$ is square-free and has precisely B prime factors, the k-tuples $(d_1, \ldots, d_k) \in \mathbb{N}^k$ with $d_1 \cdots d_k | a$ are in one to one correspondence with k-tuples $(Y_1, \ldots, Y_k) \in \mathscr{P}_B^k$; this correspondence is given by

$$d_j = \prod_{i \in Y_j} p_i \quad (1 \le j \le k).$$

Using this observation twice we find that

$$\begin{aligned} W_{k+1}^P(a) = \sum_{(Y_1,\ldots,Y_k) \in \mathscr{P}_B^k} \Bigg| \Bigg\{ &(e_1, \ldots, e_k) \in \mathbb{N}^k : e_1 \cdots e_k | a, \\ &\Big| \log e_j - \sum_{i \in Y_j} \log p_i \Big| < \log 2 \ (1 \le j \le k) \Bigg\} \Bigg|^{P-1} \\ = \sum_{(Y_1,\ldots,Y_k) \in \mathscr{P}_B^k} &\Bigg(\sum_{\substack{(Z_1,\ldots,Z_k) \in \mathscr{P}_B^k \\ (5.3.2)}} 1 \Bigg)^{P-1}, \end{aligned}$$

where for two k-tuples $(Y_1, \ldots, Y_k)$ and $(Z_1, \ldots, Z_k)$ in $\mathscr{P}_B^k$ condition (5.3.2) is defined by

$$-\log 2 < \sum_{i \in Y_j} \log p_i - \sum_{i \in Z_j} \log p_i < \log 2 \quad (1 \le j \le k). \tag{5.3.2}$$

Moreover, every integer $a \in \mathcal{A}(\boldsymbol{b})$ has exactly $b_1! \cdots b_H!$ representations of the form $a = p_1 \cdots p_B$, corresponding to the possible permutations of the primes $p_1, \ldots, p_B$ under condition (5.3.1). Thus

$$\begin{aligned} \sum_{a \in \mathcal{A}(\boldsymbol{b})} \frac{W_{k+1}^P(a)}{a} &= \frac{1}{b_1! \cdots b_H!} \sum_{\substack{p_1,\ldots,p_B \\ (5.3.1)}} \frac{1}{p_1 \cdots p_B} \sum_{\boldsymbol{Y} \in \mathscr{P}_B^k} \Bigg(\sum_{\substack{\boldsymbol{Z} \in \mathscr{P}_B^k \\ (5.3.2)}} 1 \Bigg)^{P-1} \\ &= \frac{1}{b_1! \cdots b_H!} \sum_{\boldsymbol{Y} \in \mathscr{P}_B^k} \sum_{\substack{p_1,\ldots,p_B \\ (5.3.1)}} \frac{1}{p_1 \cdots p_B} \Bigg(\sum_{\substack{\boldsymbol{Z} \in \mathscr{P}_B^k \\ (5.3.2)}} 1 \Bigg)^{P-1} \\ &\le \frac{1}{b_1! \cdots b_H!} \sum_{\boldsymbol{Y} \in \mathscr{P}_B^k} \Bigg(\sum_{\substack{p_1,\ldots,p_B \\ (5.3.1)}} \frac{1}{p_1 \cdots p_B} \Bigg)^{2-P} \Bigg(\sum_{\substack{p_1,\ldots,p_B \\ (5.3.1)}} \frac{1}{p_1 \cdots p_B} \sum_{\substack{\boldsymbol{Z} \in \mathscr{P}_B^k \\ (5.3.2)}} 1 \Bigg)^{P-1}, \end{aligned}$$

by Hölder's inequality if $P < 2$ and trivially if $P = 2$. Observe that

$$\sum_{\substack{p_1,\ldots,p_B \\ (5.3.1)}} \frac{1}{p_1\cdots p_B} \le \prod_{j=1}^{H}\Big(\sum_{p\in D_j}\frac{1}{p}\Big)^{b_j} \le (\log\rho)^B,$$

by (5.1.2). Consequently,

$$\begin{aligned}\sum_{a\in\mathcal{A}(\boldsymbol{b})}\frac{W_{k+1}^P(a)}{a} &\le \frac{(\log\rho)^{(2-P)B}}{b_1!\cdots b_H!}\sum_{\boldsymbol{Y}\in\mathscr{P}_B^k}\Bigg(\sum_{\substack{p_1,\ldots,p_B \\ (5.3.1)}}\frac{1}{p_1\cdots p_B}\sum_{\substack{\boldsymbol{Z}\in\mathscr{P}_B^k \\ (5.3.2)}} 1\Bigg)^{P-1} \\ &= \frac{(\log\rho)^{(2-P)B}}{b_1!\cdots b_H!}\sum_{\boldsymbol{Y}\in\mathscr{P}_B^k}\Bigg(\sum_{\boldsymbol{Z}\in\mathscr{P}_B^k}\sum_{\substack{p_1,\ldots,p_B \\ (5.3.1),(5.3.2)}}\frac{1}{p_1\cdots p_B}\Bigg)^{P-1}.\end{aligned} \tag{5.3.3}$$

Next, we fix $\boldsymbol{Y}\in\mathscr{P}_B^k$ and $\boldsymbol{Z}\in\mathscr{P}_B^k$ and proceed to the estimation of the sum

$$\sum_{\substack{p_1,\ldots,p_B \\ (5.3.1),(5.3.2)}}\frac{1}{p_1\cdots p_B}.$$

Note that (5.3.2) is equivalent to

$$-\log 2 < \sum_{i\in Y_j\setminus Z_j}\log p_i - \sum_{i\in Z_j\setminus Y_j}\log p_i < \log 2 \quad (1\le j\le k). \tag{5.3.4}$$

Conditions (5.3.4), $1\le j\le k$, are a system of k inequalities. For every $j\in\{1,\ldots,k\}$ and every $I_j\in Y_j\triangle Z_j$ (5.3.4) implies that $p_{I_j}\in[X_j,4X_j]$, where X_j is a constant depending only on the primes p_i for $i\in Y_j\triangle Z_j\setminus\{I_j\}$. In order to exploit this simple observation to its full potential we need to choose $I_1,\ldots,I_k$ as large as possible. After this is done, we fix the primes p_i for $i\in\{1,\ldots,B\}\setminus\{I_1,\ldots,I_k\}$ and estimate the sum over $p_{I_1},\ldots,p_{I_k}$. The obvious choice is to set $I_j=\max Y_j\triangle Z_j$, $1\le j\le k$. However, in this case the indices $I_1,\ldots,I_k$ and the numbers $X_1,\ldots,X_k$ might be interdependent in a complicated way, which would make the estimation of the sum over $p_{I_1},\ldots,p_{I_k}$ very hard. So it is important to choose large $I_1,\ldots,I_k$ for which at the same time the dependence of $X_1,\ldots,X_k$ is simple enough to allow the estimation of the sum over $p_{I_1}, \ldots, p_{I_k}$. What we will do is to construct large $I_1, \ldots, I_k$ such that if we fix the primes p_i for $i \in \{1, \ldots, B\} \setminus \{I_1, \ldots, I_k\}$, then (5.3.4) becomes a linear system of inequalities with respect to $\log p_{I_1}, \ldots, \log p_{I_k}$ that corresponds to a triangular matrix and hence is easily solvable (actually, we have to be slightly more careful, but this is the main idea).

Define $I_1, \ldots, I_k$ and $m_1, \ldots, m_k$ with $I_i \in (Y_{m_i} \triangle Z_{m_i}) \cup \{0\}$ for all $i \in \{1, \ldots, k\}$ inductively, as follows. Let

$$I_1 = \max\big\{\mathcal{U}(Y_1 \triangle Z_1, \ldots, Y_k \triangle Z_k) \cup \{0\}\big\}.$$

If $I_1 = 0$, set $m_1 = 1$. Else, define m_1 to be the unique element of $\{1, \ldots, k\}$ such that $I_1 \in Y_{m_1} \triangle Z_{m_1}$. Assume we have defined $I_1, \ldots, I_i$ and $m_1, \ldots, m_i$ for some $i \in \{1, \ldots, k-1\}$ with $I_r \in (Y_{m_r} \triangle Z_{m_r}) \cup \{0\}$ for $r = 1, \ldots, i$. Then set

$$I_{i+1} = \max\big\{\mathcal{U}\big(\{Y_j \triangle Z_j : j \in \{1, \ldots, k\} \setminus \{m_1, \ldots, m_i\}\}\big) \cup \{0\}\big\}.$$

If $I_{i+1} = 0$, set $m_{i+1} = \min\big(\{1, \ldots, k\} \setminus \{m_1, \ldots, m_i\}\big)$. Otherwise, define m_{i+1} to be the unique element of $\{1, \ldots, k\} \setminus \{m_1, \ldots, m_i\}$ such that $I_{i+1} \in Y_{m_{i+1}} \triangle Z_{m_{i+1}}$. This completes the inductive step. Let $\{1 \le j \le k : I_j > 0\} = \{j_1, \ldots, j_n\}$, where $j_1 < \cdots < j_n$, and put $\mathscr{J} = \{m_{j_r} : 1 \le r \le n\}$. Notice that, by construction, we have that $\{m_1, \ldots, m_k\} = \{1, \ldots, k\}$ and $I_{j_r} \ne I_{j_s}$ for $1 \le r < s \le n$.

Fix the primes p_i for $i \in \mathcal{I} = \{1, \ldots, B\} \setminus \{I_{j_1}, \ldots, I_{j_n}\}$. By the definition of the indices $I_1, \ldots, I_k$, for every $r \in \{1, \ldots, n\}$ the prime number $p_{I_{j_r}}$ appears in (5.3.4) for $j = m_{j_r}$, but does not appear in (5.3.4) for $j \in \{m_{j_{r+1}}, \ldots, m_{j_n}\}$. So (5.3.4), $j \in \mathscr{J}$, is a linear system with respect to $\log p_{I_{j_1}}, \ldots, \log p_{I_{j_n}}$ corresponding to a triangular matrix (up to a permutation of its rows) and a straightforward manipulation of its rows implies that $p_{I_{j_r}} \in [V_r, 4^k V_r]$, $1 \le r \le n$, for some numbers V_r that depend only on the primes p_i for $i \in \mathcal{I}$

and the k-tuples $\boldsymbol{Y}$ and $\boldsymbol{Z}$, which we have fixed. Therefore

$$\sum_{\substack{p_{I_{j_1}},\dots,p_{I_{j_n}}\\ (5.3.1),(5.3.2)}} \frac{1}{p_{I_{j_1}}\cdots p_{I_{j_n}}} \le \prod_{r=1}^{n} \sum_{\substack{V_r\le p_{I_{j_r}}\le 4^k V_r\\ p_{I_{j_r}}\in D_{E_{\boldsymbol{b}}(I_{j_r})}}} \frac{1}{p_{I_{j_r}}} \ll_k \prod_{r=1}^{n} \frac{1}{\log(\max\{V_r,\ell_{E_{\boldsymbol{b}}(I_{j_r})-1}\})} \ll_k \prod_{r=1}^{n} \rho^{-E_{\boldsymbol{b}}(I_{j_r})},$$

by Lemma 5.1.1, and consequently

$$\sum_{\substack{p_1,\dots,p_B\\ (5.3.1),(5.3.2)}} \frac{1}{p_1\cdots p_B} \ll_k \prod_{r=1}^{n} \rho^{-E_{\boldsymbol{b}}(I_{j_r})} \sum_{\substack{p_i,\ i\in\mathcal{I}\\ (5.3.1)}} \prod_{i\in\mathcal{I}} \frac{1}{p_i} \le (\log\rho)^{B-n} \prod_{j=1}^{k} \rho^{-E_{\boldsymbol{b}}(I_j)},$$

by (5.1.2). Inserting the above estimate into (5.3.3) we deduce that

$$\sum_{a\in\mathcal{A}(\boldsymbol{b})} \frac{W_{k+1}^P(a)}{a} \ll_k \frac{(\log\rho)^B}{b_1!\cdots b_H!} \sum_{\boldsymbol{Y}\in\mathscr{P}_B^k} \Bigg(\sum_{\boldsymbol{Z}\in\mathscr{P}_B^k} \prod_{j=1}^{k} \rho^{-E_{\boldsymbol{b}}(I_j)} \Bigg)^{P-1}. \tag{5.3.5}$$

Next, observe that the definition of $I_1,\dots,I_k$ implies that

$$(I_j,B]\cap\mathcal{U}\big(\{Y_{m_r}\triangle Z_{m_r} : j\le r\le k\}\big)=\emptyset \quad (1\le j\le k)$$

or, equivalently,

$$\bigcup_{r=j}^{k}\big(Z_{m_r}\cap(I_j,B]\big)=\bigcup_{r=j}^{k}\big(Y_{m_r}\cap(I_j,B]\big) \quad (1\le j\le k),$$

by Remark 5.3.1. Hence for fixed $\boldsymbol{Y}\in\mathscr{P}_B^k$, $I_1,\dots,I_k\in\{0,1,\dots,B\}$ and $\boldsymbol{m}=(m_1,\dots,m_k)$ with $\{m_1,\dots,m_k\}=\{1,\dots,k\}$, the number of admissible k-tuples $\boldsymbol{Z}\in\mathscr{P}_B^k$ is at most $M_B^k(\boldsymbol{Y}_{\boldsymbol{m}};\boldsymbol{I})$, where $\boldsymbol{Y}_{\boldsymbol{m}}=(Y_{m_1},\dots,Y_{m_k})$, which together with (5.3.5) yields that

$$\sum_{a\in\mathcal{A}(\boldsymbol{b})} \frac{W_{k+1}^P(a)}{a} \ll_k \frac{(\log\rho)^B}{b_1!\cdots b_H!} \sum_{\boldsymbol{Y}\in\mathscr{P}_B^k} \Bigg(\sum_{0\le I_1,\dots,I_k\le B} \sum_{\boldsymbol{m}} M_B^k(\boldsymbol{Y}_{\boldsymbol{m}};\boldsymbol{I}) \prod_{j=1}^{k} \rho^{-E_{\boldsymbol{b}}(I_j)} \Bigg)^{P-1}.$$

So, by the inequality inequality $(a+b)^{P-1}\le a^{P-1}+b^{P-1}$ for $a\ge0$ and $b\ge0$, which holds precisely when $1 < P \leq 2$, we find that

$$\sum_{a\in\mathcal{A}(\boldsymbol{b})} \frac{W_{k+1}^P(a)}{a} \ll_k \frac{(\log\rho)^B}{b_1!\cdots b_H!} \sum_{\boldsymbol{m}} \sum_{0\leq I_1,\ldots,I_k\leq B} \prod_{j=1}^{k} (\rho^{P-1})^{-E_{\boldsymbol{b}}(I_j)} \sum_{\boldsymbol{Y}\in\mathscr{P}_B^k} \big(M_B^k(\boldsymbol{Y}_{\boldsymbol{m}};\boldsymbol{I})\big)^{P-1}.$$

Finally, note that

$$\sum_{\boldsymbol{Y}\in\mathscr{P}_B^k} \big(M_B^k(\boldsymbol{Y}_{\boldsymbol{m}};\boldsymbol{I})\big)^{P-1} = \sum_{\boldsymbol{Y}\in\mathscr{P}_B^k} \big(M_B^k(\boldsymbol{Y};\boldsymbol{I})\big)^{P-1}$$

for every $\boldsymbol{m} = (m_1,\ldots,m_k)$ with $\{m_1,\ldots,m_k\} = \{1,\ldots,k\}$, which completes the proof of the lemma. □

5.4 The method of low moments: combinatorial arguments

In this section we show the second step towards the proof of Lemma 5.2.2.

Lemma 5.4.1. *Let $P \in (1,+\infty)$ and $0 \leq I_1,\ldots,I_k \leq B$ so that $I_{\sigma(1)} \leq \cdots \leq I_{\sigma(k)}$ for some permutation $\sigma \in S_k$. Then*

$$\sum_{\boldsymbol{Y}\in\mathscr{P}_B^k} \big(M_B^k(\boldsymbol{Y};\boldsymbol{I})\big)^{P-1} \leq (k+1)^B \prod_{j=1}^{k} \Big(\frac{j-1+(k-j+2)^P}{j+(k-j+1)^P}\Big)^{I_{\sigma(j)}}.$$

Proof. First, we calculate $M_B^k(\boldsymbol{Y};\boldsymbol{I})$ for fixed $\boldsymbol{Y} = (Y_1,\ldots,Y_k) \in \mathscr{P}_B^k$. Set $I_0 = 0$, $I_{k+1} = B$, $\sigma(0) = 0$, $\sigma(k+1) = k+1$ and

$$\mathcal{N}_j = (I_{\sigma(j)}, I_{\sigma(j+1)}] \cap \{1,\ldots,B\} \quad (0 \leq j \leq k).$$

In addition, put

$$Y_0 = \{1,\ldots,B\} \setminus \bigcup_{j=1}^{k} Y_j$$

as well as

$$Y_{i,j} = \mathcal{N}_i \cap Y_j \quad \text{and} \quad y_{i,j} = |Y_{i,j}| \quad (0 \le i \le k,\ 0 \le j \le k). \tag{5.4.1}$$

A k-tuple $(Z_1, \dots, Z_k) \in \mathscr{P}_B^k$ is counted by $M_B^k(\boldsymbol{Y}; \boldsymbol{I})$ if, and only if,

$$\bigcup_{r=j}^{k} \big(Z_r \cap (I_j, B]\big) = \bigcup_{r=j}^{k} \big(Y_r \cap (I_j, B]\big) \quad (1 \le j \le k). \tag{5.4.2}$$

If we set

$$Z_0 = \{1, \dots, B\} \setminus \bigcup_{j=1}^{k} Z_j$$

and

$$Z_{i,j} = \mathcal{N}_i \cap Z_j \quad (0 \le i \le k,\ 0 \le j \le k),$$

then (5.4.2) is equivalent to

$$\bigcup_{r=\sigma(j)}^{k} Z_{i,r} = \bigcup_{r=\sigma(j)}^{k} Y_{i,r} \quad (0 \le i \le k,\ 0 \le j \le i). \tag{5.4.3}$$

For every $i \in \{0, 1, \dots, k\}$ let $\chi_i(0), \dots, \chi_i(i+1)$ be the sequence $\sigma(0), \sigma(1), \dots, \sigma(i), \sigma(k+1)$ ordered increasingly. In particular, $\chi_i(0) = \sigma(0) = 0$ and $\chi_i(i+1) = \sigma(k+1) = k+1$. With this notation (5.4.3) becomes

$$\bigcup_{r=\chi_i(j)}^{k} Z_{i,r} = \bigcup_{r=\chi_i(j)}^{k} Y_{i,r} \quad (0 \le i \le k,\ 0 \le j \le i),$$

which is equivalent to

$$\bigcup_{r=\chi_i(j)}^{\chi_i(j+1)-1} Z_{i,r} = \bigcup_{r=\chi_i(j)}^{\chi_i(j+1)-1} Y_{i,r} \quad (0 \le i \le k,\ 0 \le j \le i).$$

For each $i \in \{0, 1, \dots, k\}$ let M_i denote the total number of mutually disjoint $(k+1)$-tuples

$(Z_{i,0}, Z_{i,1}, \ldots, Z_{i,k})$ such that

$$\bigcup_{r=\chi_i(j)}^{\chi_i(j+1)-1} Z_{i,r} = \bigcup_{r=\chi_i(j)}^{\chi_i(j+1)-1} Y_{i,r} \quad (0 \le j \le i).$$

Then

$$M_B^k(\boldsymbol{Y}; \boldsymbol{I}) = \prod_{i=0}^{k} M_i. \tag{5.4.4}$$

Moreover, it is immediate from the definition of M_i that

$$M_i = \prod_{j=0}^{i} (\chi_i(j+1) - \chi_i(j))^{y_{i,\chi_i(j)} + \cdots + y_{i,\chi_i(j+1)-1}}.$$

Set $v_{i,j+1} = \chi_i(j+1) - \chi_i(j)$ for $j \in \{0, \ldots, i\}$. Note that $v_{i,1} + \cdots + v_{i,i+1} = k+1$ and that $v_{i,j+1} \ge 1$ for all $j \in \{0, \ldots, i\}$. Let

$$W_{i,j} = \bigcup_{r=\chi_i(j)}^{\chi_i(j+1)-1} Y_{i,r}, \quad w_{i,j} = |W_{i,j}| \quad (0 \le j \le i). \tag{5.4.5}$$

With this notation we have that

$$M_i = \prod_{j=0}^{i} v_{i,j+1}^{w_{i,j}} \quad (0 \le i \le k). \tag{5.4.6}$$

Inserting (5.4.6) into (5.4.4) we deduce that

$$M_B^k(\boldsymbol{Y}; \boldsymbol{I}) = \prod_{i=0}^{k} \prod_{j=0}^{i} v_{i,j+1}^{w_{i,j}}. \tag{5.4.7}$$

Therefore

$$T := \sum_{\boldsymbol{Y} \in \mathscr{P}_B^k} \left(M_B^k(\boldsymbol{Y}; \boldsymbol{I}) \right)^{P-1} = \prod_{i=0}^{k} \sum_{Y_{i,0}, \ldots, Y_{i,k}} \prod_{j=0}^{i} (v_{i,j+1}^{P-1})^{w_{i,j}},$$

where the sets $Y_{i,j}$ are defined by (5.4.1). Fix $i \in \{0, 1, \ldots, k\}$. Given $W_{i,0}, \ldots, W_{i,i}$, a partition of $\mathcal{N}_i$, the number of $Y_{i,0}, \dots, Y_{i,k}$ satisfying (5.4.5) is

$$\prod_{j=0}^{i} (\chi_i(j+1) - \chi_i(j))^{|W_{i,j}|} = \prod_{j=0}^{i} v_{i,j+1}^{w_{i,j}}.$$

Hence

$$\sum_{Y_{i,0},\dots,Y_{i,k}} \prod_{j=0}^{i} (v_{i,j+1}^{P-1})^{w_{i,j}} = \sum_{W_{i,0},\dots,W_{i,i}} \prod_{j=0}^{i} (v_{i,j+1})^{Pw_{i,j}} = (v_{i,1}^P + \cdots + v_{i,i+1}^P)^{|\mathcal{N}_i|},$$

by the multinomial theorem. So

$$T = \prod_{i=0}^{k} (v_{i,1}^P + \cdots + v_{i,i+1}^P)^{|\mathcal{N}_i|} = \prod_{i=0}^{k} (v_{i,1}^P + \cdots + v_{i,i+1}^P)^{I_{\sigma(i+1)} - I_{\sigma(i)}}.$$

Finally, recall that $v_{i,1} + \cdots + v_{i,i+1} = k+1$ as well as $v_{i,j+1} \geq 1$ for all $0 \leq j \leq i \leq k$, and note that

$$\max\left\{ \sum_{j=1}^{i+1} x_j^P : \sum_{j=1}^{i+1} x_j = k+1, x_j \geq 1 \ (1 \leq j \leq i+1) \right\} = i + (k+1-i)^P \quad (0 \leq i \leq k),$$

since the maximum of a convex function in a simplex occurs at its vertices. Hence we conclude that

$$T \leq \prod_{i=0}^{k} \big(i + (k+1-i)^P\big)^{I_{\sigma(i+1)} - I_{\sigma(i)}} = (k+1)^B \prod_{i=1}^{k} \Big(\frac{i-1+(k-i+2)^P}{i+(k-i+1)^P} \Big)^{I_{\sigma(i)}},$$

which completes the proof of the lemma. □

5.5 The method of low moments: completion of the proof

In this section we prove Lemmas 5.2.2 and 5.2.3.

Proof of Lemma 5.2.2. Lemmas 5.3.1 and 5.4.1 imply that

$$\begin{aligned}\sum_{a\in\mathcal{A}(\boldsymbol{b})}\frac{W_{k+1}^{P}(a)}{a} &\ll_k \frac{((k+1)\log\rho)^B}{b_1!\cdots b_H!}\sum_{0\le I_1\le\cdots\le I_k\le B}\prod_{m=1}^{k}(\rho^{P-1})^{-E_{\boldsymbol{b}}(I_m)}\Big(\frac{m-1+(k-m+2)^P}{m+(k-m+1)^P}\Big)^{I_m}\\ &\ll_k \frac{((k+1)\log\rho)^B}{b_1!\cdots b_H!}\sum_{0=j_0\le j_1\le\cdots\le j_k\le H}(\rho^{P-1})^{-(j_1+\cdots+j_k)}\\ &\qquad\times\prod_{m=1}^{k}\sum_{B_{j_{m-1}}\le I_m\le B_{j_m}}\Big(\frac{m-1+(k-m+2)^P}{m+(k-m+1)^P}\Big)^{I_m}\\ &\ll_k \frac{((k+1)\log\rho)^B}{b_1!\cdots b_H!}\sum_{0\le j_1\le\cdots\le j_k\le H}\prod_{m=1}^{k}(\rho^{P-1})^{-j_m}\Big(\frac{m-1+(k-m+2)^P}{m+(k-m+1)^P}\Big)^{B_{j_m}},\end{aligned}$$

since the sequence $\{m-1+(k-m+2)^P\}_{m=1}^{k+1}$ is strictly decreasing. This proves the desired result. □

Before we prove Lemma 5.2.3, we establish the following crucial inequality.

Lemma 5.5.1. *Let $k\ge 2$ and P defined by* (5.2.7)*. Then*

$$\frac{i-1+(k-i+2)^P}{k+1}<(\rho^{P-1})^{k-i+1}\quad(2\le i\le k).$$

Proof. Set

$$f(x)=(k+1)(\rho^{P-1})^x+x-(x+1)^P-k,\quad x\in[0,k].$$

It suffices to show that $f(x)>0$ for $1\le x\le k-1$. Observe that $f(0)=f(k)=0$. Moreover, since $1<P\le 2$, $f'''(x)>0$ for all x. Hence f'' is strictly increasing. Note that

$$f''(k)=(P-1)^2(\log\rho)^2(k+1)^P-P(P-1)(k+1)^{P-2}\le 0,$$

by our choice of P. Hence $f''(x)<0$ for $x\in(0,k)$, that is f is a concave function and thus it is positive for $x\in(0,k)$. □

Proof of Lemma 5.2.3. Set

$$\nu_i = \frac{i-1+(k-i+2)^P}{k+1} \quad (1 \le i \le k+1).$$

Then Lemma 5.2.2 implies that

$$\sum_{a\in\mathcal{A}(\boldsymbol{b})} \frac{W_{k+1}^P(a)}{a} \ll_k \frac{((k+1)\log\rho)^B}{b_1!\cdots b_H!} \sum_{0\le j_1\le\cdots\le j_k\le H} \prod_{i=1}^{k} (\rho^{P-1})^{-j_i}\Big(\frac{\nu_i}{\nu_{i+1}}\Big)^{B_{j_i}}. \tag{5.5.1}$$

Moreover,

$$\prod_{i=1}^{k}\Big(\frac{\nu_i}{\nu_{i+1}}\Big)^{B_{j_i}} \le \Big(\frac{\nu_1}{\nu_2}\Big)^{B_{j_1}} \prod_{i=2}^{k}\Big(\frac{\nu_i}{\nu_{i+1}}\Big)^{j_i} = \Big(\frac{\nu_1}{\nu_2}\Big)^{B_{j_1}} \nu_2^{j_1} \prod_{i=2}^{k} \nu_i^{j_i-j_{i-1}},$$

by our assumption that $\boldsymbol{b}\in\mathcal{B}$. Thus, setting $r_1 = j_1$ and $r_i = j_i - j_{i-1}$ for $i = 2,\dots,k$ yields

$$\begin{aligned}
\prod_{i=1}^{k} (\rho^{P-1})^{-j_i}\Big(\frac{\nu_i}{\nu_{i+1}}\Big)^{B_{j_i}} &\le (\rho^{P-1})^{-(j_1+\cdots+j_k)} \Big(\frac{\nu_1}{\nu_2}\Big)^{B_{j_1}} \nu_2^{j_1} \prod_{i=2}^{k} \nu_i^{j_i-j_{i-1}} \\
&= \Big(\frac{\nu_1}{\nu_2}\Big)^{B_{r_1}} \Big(\frac{\nu_2}{\rho^{(P-1)k}}\Big)^{r_1} \prod_{i=2}^{k}\Big(\frac{\nu_i}{(\rho^{P-1})^{k-i+1}}\Big)^{r_i} \\
&= \nu^{B_{r_1}-r_1} \prod_{i=2}^{k}\Big(\frac{\nu_i}{(\rho^{P-1})^{k-i+1}}\Big)^{r_i},
\end{aligned}$$

since $\rho^{(P-1)k} = \nu_1$ and $\nu_1/\nu_2 = \nu$. Consequently,

$$\sum_{0\le j_1\le\cdots\le j_k\le H} \prod_{i=1}^{k} (\rho^{P-1})^{-j_i}\Big(\frac{\nu_i}{\nu_{i+1}}\Big)^{B_{j_i}} \le \sum_{\substack{0\le r_i\le H\\ 1\le i\le k}} \nu^{B_{r_1}-r_1} \prod_{i=2}^{k}\Big(\frac{\nu_i}{(\rho^{P-1})^{k-i+1}}\Big)^{r_i} \ll_k \sum_{r_1=0}^{H} \nu^{B_{r_1}-r_1}, \tag{5.5.2}$$

since $\nu_i < (\rho^{P-1})^{k-i+1}$ for $i = 2,\dots,k$ by Lemma 5.5.1. Inserting (5.5.2) into (5.5.1) completes the proof of the lemma. □

5.6 The lower bound in Theorem 2.6: completion of the proof

In this section we complete the proof of the lower bound in Theorem 2.6, by showing that (5.2.1) holds. We may assume that y_1 is large enough. Let $N = N(k)$ be a sufficiently large integer to be chosen later and set

$$H = \left\lfloor \frac{\log\log y_1}{\log \rho} - L_k \right\rfloor \quad \text{and} \quad B = H - N + 1.$$

Consider the set $\mathcal{B}^*$ of vectors $(b_1, \ldots, b_H) \in (\mathbb{N} \cup \{0\})^H$ such that $b_i = 0$ for $i < N$,

$$B_i \leq i - N + 1 \quad (N \leq i \leq H) \tag{5.6.1}$$

and

$$\sum_{m=N}^{H} \nu^{B_m - m} \leq \frac{\nu + \nu^{-N}}{1 - 1/\nu}. \tag{5.6.2}$$

Lemma 5.1.1 and the definition of H imply that $\log \ell_H \leq \rho^{H+L_k} \leq \log y_1$. Hence

$$\bigcup_{\boldsymbol{b} \in \mathcal{B}^*} \mathcal{A}(\boldsymbol{b}) \subset \{a \in \mathbb{N} : P^+(a) \leq y_1,\ \mu^2(a) = 1\}. \tag{5.6.3}$$

Fix for the moment $\boldsymbol{b} \in \mathcal{B}^* \subset \mathcal{B}$. By Lemma 5.2.3 and relation (5.6.2) we have that

$$\sum_{a \in \mathcal{A}(\boldsymbol{b})} \frac{W_{k+1}^P(a)}{a} \ll_k \frac{((k+1)\log\rho)^B}{b_N! \cdots b_H!} \left(1 + \sum_{m=N}^{H} \nu^{B_m - m}\right) \ll_k \frac{((k+1)\log\rho)^B}{b_N! \cdots b_H!}. \tag{5.6.4}$$

Also, if N is large enough, then Lemma 5.1.1 and relation (5.6.1) imply that

$$\begin{aligned}\sum_{a\in\mathcal{A}(\boldsymbol{b})}\frac{\tau_{k+1}(a)}{a} &= (k+1)^B\prod_{j=N}^{H}\frac{1}{b_j!}\Bigg(\sum_{p_1\in D_j}\frac{1}{p_1}\sum_{\substack{p_2\in D_j\\ p_2\neq p_1}}\frac{1}{p_2}\cdots\sum_{\substack{p_{b_j}\in D_j\\ p_{b_j}\notin\{p_1,\dots,p_{b_j-1}\}}}\frac{1}{p_{b_j}}\Bigg)\\ &\ge \frac{(k+1)^B}{b_N!\cdots b_H!}\prod_{j=N}^{H}\left(\log\rho-\frac{b_j}{\ell_{j-1}}\right)^{b_j}\\ &\ge \frac{((k+1)\log\rho)^B}{b_N!\cdots b_H!}\prod_{j=N}^{H}\left(1-\frac{j-N+1}{(\log\rho)\exp\{\rho^{j-L_k-1}\}}\right)^{j-N+1}\\ &\ge \frac{1}{2}\frac{((k+1)\log\rho)^B}{b_N!\cdots b_H!}.\end{aligned} \tag{5.6.5}$$

Combining Lemma 5.2.1 with relations (5.6.3), (5.6.4) and (5.6.5) we deduce that

$$\sum_{\substack{P^+(a)\le y_1\\ \mu^2(a)=1}}\frac{L^{(k+1)}(a)}{a}\ge\sum_{\boldsymbol{b}\in\mathcal{B}^*}\sum_{a\in\mathcal{A}(\boldsymbol{b})}\frac{L^{(k+1)}(a)}{a}\gg_k((k+1)\log\rho)^B\sum_{\boldsymbol{b}\in\mathscr{B}^*}\frac{1}{b_N!\cdots b_H!}.$$

For $i\in\{1,\dots,B\}$ set $g_i=b_{N-1+i}$ and let $G_i=g_1+\cdots+g_i$. Then

$$G_i=B_{i+N-1}\le i\quad(1\le i\le B) \tag{5.6.6}$$

and

$$\sum_{i=1}^{B}\nu^{G_i-i}=\nu^{N-1}\sum_{m=N}^{H}\nu^{B_m-m}\le\frac{\nu^N+1/\nu}{1-1/\nu}, \tag{5.6.7}$$

by (5.6.1) and (5.6.2), respectively. With this notation we have that

$$\sum_{\substack{P^+(a)\le t\\ \mu^2(a)=1}}\frac{L^{(k+1)}(a)}{a}\gg_k((k+1)\log\rho)^B\sum_{\boldsymbol{g}\in\mathcal{G}}\frac{1}{g_1!\cdots g_B!}, \tag{5.6.8}$$

where $\mathcal{G}$ is the set of vectors $\boldsymbol{g}\in(\mathbb{N}\cup\{0\})^B$ with $g_1+\cdots+g_B=B$ and such that (5.6.6) and (5.6.7) hold. For $\boldsymbol{g}\in\mathcal{G}$ let $R(\boldsymbol{g})$ be the set of $\boldsymbol{x}\in\mathbb{R}^B$ such that $0\le x_1\le\cdots\le x_B\le B$

and exactly g_i of the numbers x_j lie in $[i-1, i)$ for each i. Then

$$\sum_{\boldsymbol{g}\in\mathcal{G}} \frac{1}{g_1!\cdots g_B!} = \sum_{\boldsymbol{g}\in\mathcal{G}} \mathrm{Vol}(R(\boldsymbol{g})) = \mathrm{Vol}(\cup_{\boldsymbol{g}\in\mathcal{G}} R(\boldsymbol{g})). \tag{5.6.9}$$

We claim that

$$\mathrm{Vol}(\cup_{\boldsymbol{g}\in\mathcal{G}} R(\boldsymbol{g})) \geq B^B\,\mathrm{Vol}(Y_B(N)), \tag{5.6.10}$$

where $Y_B(N)$ is the set of $\boldsymbol{\xi} = (\xi_1, \dots, \xi_B) \in S_B(1, B)$ that satisfy

$$\sum_{j=1}^{B} \nu^{j-B\xi_j} \leq \nu^N \tag{5.6.11}$$

(see Section 3.3 for the definition of $S_B(1, B)$). Indeed, let $\boldsymbol{\xi} \in Y_B(N)$ with $\xi_B < 1$ and set $x_j = B\xi_j$. Let $g_i = |\{1 \leq j \leq B : i-1 \leq x_j < i\}|$ for $1 \leq i \leq B$. It suffices to show that $\boldsymbol{g} = (g_1, \dots, g_B) \in \mathcal{G}$. First, we have that

$$x_{i+1} \geq i \quad (1 \leq i \leq B-1),$$

which yields (5.6.6). Finally, inequality (5.6.11) implies

$$\begin{aligned}\frac{\nu^N}{1-1/\nu} &\geq \frac{1}{1-1/\nu}\sum_{j=1}^{B}\nu^{j-x_j} \geq \frac{1}{1-1/\nu}\sum_{i=1}^{B}\nu^{-i}\sum_{j:\,x_j\in[i-1,i)}\nu^j \geq \sum_{i=1}^{B}\sum_{m=i}^{B}\nu^{-m}\sum_{j:\,x_j\in[i-1,i)}\nu^j\\ &= \sum_{m=1}^{B}\nu^{-m}\sum_{j:\,x_j<m}\nu^j \geq \sum_{\substack{1\leq m\leq B\\ G_m>0}}\nu^{-m+G_m} \geq -\frac{1}{\nu-1}+\sum_{m=1}^{B}\nu^{-m+G_m},\end{aligned}$$

that is (5.6.7) holds. To conclude, we have showed that $\boldsymbol{g} \in \mathcal{G}$, which proves that inequality (5.6.10) does hold. To bound $\mathrm{Vol}(Y_B(N))$ from below set

$$f(\boldsymbol{\xi}) = \sum_{j=1}^{B} \nu^{j-B\xi_j}$$

and observe that

$$\begin{aligned}
\mathrm{Vol}(Y_B(N)) &= \mathrm{Vol}(S_B(1,B)) - \mathrm{Vol}(\{\boldsymbol{\xi} \in S_B(1,B) : f(\boldsymbol{\xi}) > \nu^N\}) \\
&\geq \frac{1}{(2B+1)B!} - \frac{1}{\nu^N} \int\limits_{S_B(1,B)} f(\boldsymbol{\xi}) d\boldsymbol{\xi} \\
&= \frac{1}{(2B+1)B!} - O_k\Big(\frac{\nu^{-N}}{(B+1)!}\Big) \geq \frac{1}{4(B+1)!},
\end{aligned} \tag{5.6.12}$$

by Lemmas 3.3.1 and 3.3.4, provided that N is large enough. The above inequality along with relations (5.6.8), (5.6.9) and (5.6.10) yields that

$$\sum_{\substack{P^+(a) \leq y_1 \\ \mu^2(a)=1}} \frac{L^{(k+1)}(a)}{a} \gg_k \frac{((k+1)B \log \rho)^B}{(B+1)!}.$$

Applying Stirling's formula to the right hand side of the above inequality completes the proof of (5.2.1) and thus of the lower bound in Theorem 2.6.

Chapter 6

Work in progress

In general, our knowledge on $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ is rather incomplete, especially when the sizes of $\log y_1, \dots, \log y_k$ are vastly different. We have made partial progress towards understanding the behavior of $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ beyond the range of validity of Theorem 2.5: in [Kou] we determine the order of magnitude of $H^{(k+1)}(x, \boldsymbol{y}, 2\boldsymbol{y})$ uniformly for all choices of $y_1, \dots, y_k$ when $k \le 5$. In order to state our result we need to introduce some notation. Given numbers $3 = y_0 \le y_1 \le \cdots \le y_k$, set

$$\mathscr{L}_i = \log \frac{3 \log y_i}{\log y_{i-1}} \quad (1 \le i \le k).$$

Also, let i_1 be the smallest element of $\{1, \dots, k\}$ such that

$$\mathscr{L}_{i_1} = \max\{\mathscr{L}_i : 1 \le i \le k\}$$

and define $\Theta = \Theta(k; \boldsymbol{y})$ by

$$\Theta = \min\Big\{1, \frac{(1 + \mathscr{L}_1 + \cdots + \mathscr{L}_{i_1 - 1})(1 + \mathscr{L}_{i_1+1} + \cdots + \mathscr{L}_k)}{\mathscr{L}_{i_1}}\Big\}.$$

Lastly, define $\vartheta = \vartheta(k; \boldsymbol{y})$ implicitly, via the equation

$$\sum_{i=1}^{k} (k - i + 2)^{\vartheta} \log(k - i + 2) \mathscr{L}_i = \sum_{i=1}^{k} (k - i + 1) \mathscr{L}_i.$$

Theorem 6.1. *Let $k \in \{2, 3, 4, 5\}$, $x \ge 3$ and $3 \le y_1 \le \cdots \le y_k$ such that $2^k y_1 \cdots y_k \le x / y_k$.*

Then

$$\frac{H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})}{x} \asymp \frac{\Theta}{\sqrt{\log\log y_k}} \prod_{i=1}^{k} \Big(\frac{\log y_i}{\log y_{i-1}}\Big)^{-Q((k-i+2)^{\vartheta})}.$$

Moreover, we show that if $k \geq 6$, then Theorem 6.1 does not hold in general, namely there are choices of $y_1, \ldots, y_k$ for which the size of $H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})$ is smaller than the one predicted by Theorem 6.1.

The ultimate goal of this project would be to determine the order of $H^{(k+1)}(x,\boldsymbol{y},2\boldsymbol{y})$ uniformly in x and $\boldsymbol{y}$ for all $k \geq 6$, or at least understand the case $k = 6$.

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