THINKING AND CREATIVITY IN LEARNING MATHEMATICS TEACHING

BY

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DISSERTATION

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Abstract

Preparation to teach school mathematics should include developing an understanding of classroom mathematical interactions. The research literature and professional expertise agree that a teacher’s ability to respond to classroom mathematical interactions depends on her understanding of the subject matter within pedagogical situations. As a consequence, teacher education programs try to cultivate the novice teacher’s ability to respond to classroom mathematical interactions by developing the integration of mathematical and pedagogical thinking. This study re-evaluates that commitment, giving attention to current cognitive models of teachers’ integration of knowledge and to practice-based approaches to teacher learning. Unlike previous studies on teachers’ professional knowledge, this research provides a micro-perspective on teachers’ responses to student utterances. Using extended dialogues from the classrooms of two experienced and respected teachers, the study searched for a relationship between the teachers’ mathematical and pedagogical thoughts. Analysis drew attention to a complex dynamic: one of association of mathematical and pedagogical thoughts through the teacher’s perceptual and situational understanding of the dialogue. Clearly, classroom mathematical interactions were found to include metaphoric paraphrasing of what students said, but little integration. This finding highlights the need to understand better teachers’ comprehension of mathematical classroom dialogue from an interactional perspective. At the practical level, the study suggests a greater place in professional development for perceptual mediation of teachers’ understanding of classroom interactions.
To my parents
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There is a broad conception and there are specific thoughts when it comes to what teaching mathematics involves. The questions of this study have been in my mind long before I was aware of them, perhaps since I started studying three-dimensional geometric shapes in 8th grade. I used to ask my mother how people who understand Geometry would see and use geometrical concepts when they explained them to others. Listening to my mother, a mathematics teacher, I learned the difference between explaining and teaching. As I grew up and decided to pursue a Mathematics Degree and later a Mathematics Education doctorate, I persistently recalled my mother’s dedication to teaching mathematics and my father’s belief in life-long learning. I became more alert to the peculiarity and intertwining of mathematical and pedagogical thoughts, especially for those who teach mathematics. I owe the possibility of writing this thesis to my parents more than anyone else. They have been my constant support at all times. This thesis is dedicated to them.

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Chapter 1

Introduction

Mathematics teaching entails the coexistence of mathematical and pedagogical thinking. Still, mathematical thoughts are hardly conducive to pedagogical thoughts. Pedagogical thoughts do not necessarily encourage mathematical thinking. Teachers are challenged to exploit this coexistence during classroom mathematical interactions and learning to teach mathematics can be overwhelming given the abundance of mathematical and pedagogical thoughts. Responding to this challenge, teacher education programs aim to developing practices for steering the learning of mathematics teaching so that teachers, especially novices of teaching, are able to coordinate their mathematical and pedagogical discourses.

Among currently preferred practices in teacher education are such activities as: examining practice-based materials, placing mathematics in pedagogical contexts, and reflective thinking. Here, the composite nature of teacher knowledge is made apparent for novices. Most present-day teacher education programs (Borko, Peressini, Romagnano, & Knuth, 2000; Carpenter, Fennema, & Franke, 1996; Cooney, Shealy, & Arvold, 1998; Lampert & Ball, 1998; Schifter, 1995; Steele, 2005, 2008) are oriented towards situative learning activities integrating mathematical discourses into a pedagogical reasoning aligned with the new conceptualizations of classroom mathematics learning.

Research studies (Ball, 2000; Cooney, 1994, 1999) have argued for the necessity that teacher education programs highlight the specificity of mathematical discourses in pedagogical contexts. They have emphasized that the thinking in mathematics teaching is different from the thinking of mathematicians (Ball, 1992; Ball & Bass, 2003; Kahan, Cooper, & Bethea, 2003) and commented that novices enter their formal learning of teaching with problematic perceptions of
mathematics teaching and specifically of classroom interactions practices (Ball, 1988, 1992; Cooney, 1999; Cooney et al., 1998). Teacher education activities would consequently unfold the necessity of inserting mathematical discourse into the context of teaching (Ball, 1988, 2000; Cooney, 1999; Davis & Simmt, 2006; Simon, 2006). This task could be accomplished by treating pedagogical knowledge not only as a repertoire of strategies and tactics of teaching, but as a repertoire finely tuned to the uniqueness of the key tasks of mathematics teaching situations (Ball, Hill, & Bass, 2005; Silverman & Thompson, 2008; Simon, 1995; Thompson, 2008; Watson & Mason, 2007) and, more recently, to specific high-leverage instructional practices (Leinhardt & Steele, 2005).

Clearly, teachers’ ways of thinking about teaching practices involve implicit relationships between mathematical and pedagogical knowledge. Manifestations of these relationships have been prominent in reports of teachers engaged in the reflective practices of teaching (Heaton, 2000; Lampert, 1990). Additionally, research on how experienced teachers hold custody of their knowledge has indicated that teachers commonly develop pedagogical frameworks to almost continuously transform their content and methods knowledge, each into the other (Sherin, 1996; Sherin, Mandez, & Louis, 2004; Sherin, Sherin, & Madanes, 2000; Shulman, 1986, 1987). Separating mathematics and pedagogy in learning mathematics teaching became a supervisory format. Given these views, the integration of mathematics and pedagogy became a foundational idea and a sine-qua-non condition for learning mathematics teaching. More recently, research discourse extended from the integration of mathematical and pedagogical discourses into specific professional knowledge base to “high-leverage instructional practices,” where integration stands as the necessary assumption (Ball & Forzani, 2009; Borko & Whitcomb, 2008; Liston, Borko, & Whitcomb, 2008).
To some extent, teacher education has begun to relate preservice teachers’ mathematical and pedagogical discourses to the “visibility” of mathematics teaching (Lewis, 2007). To support this focus, the intrinsic relationship between mathematics and pedagogy became the main assumption in teacher interactive thinking. To learn particular instructional practices, future teachers need to “visualize” the individuality of these practices within their discipline. “High-leverage practices” have been identified, like “learning about student understanding and orchestrating classroom discussions.” As Borko and Whitcomb (2008) suggest, “Each of these core practices is composed of more fine-grained components and plays out differently in each subject matter” (p. 570). It was thus made evident that particular connections and integration should be part of teacher education discourse, relating the specificity of mathematics to such practices. Without integrating the discourse, learning about the practices may not stand to understand the individuality of each within the knowing subject matter for teaching.

An argument to avoid a fragmented practice of mathematics teaching (Ball, 2000) boosted the conviction that good teaching practice involves integration of mathematical and pedagogical knowledge. It claimed that teacher education should accomplish this integration by bridging mathematical and pedagogical discourses in the context of high-leverage practices. But the argument remains problematic. Such integration is hard to learn. It is not a natural product in a student’s study of mathematics teaching. Ball and Forzani (2009) mentioned it as part of the “unnatural” work of teaching. By nature, this integration is perspectival, interpretative, and situational (Wineburg, 2008). Professional development activities need to be anchored in practices of teaching (Ball & Forzani, 2009) and research has to identify the instances of mathematics for teaching “in the practices- the embodied and enacted understandings - of experienced teachers” (Davis & Simmt, 2006).
The Problem of Integrating Mathematical and Pedagogical Discourses in Teacher Education

Although the presence of mathematical and pedagogical discourses in mathematics teaching is uncontestable, the relationship between teacher’s mathematical and pedagogical discourses during classroom interactions is not a matter of simple observation. It is especially not apparent to novices in mathematics teaching. Their inclination is to keep the mathematical and pedagogical discourses separate. And as it has been illustrated (Steele, 2005), merging mathematical and pedagogical reasoning in teacher education combines two discourses with two different reasoning structures.

At an analytical level, engaged in the examination of real classroom episodes, pre-service teachers can accept idea of integrating their mathematical discourse into teaching. But rationalist analysis of teacher action based exclusively on conceptual recognition makes integration highly intangible during the dynamics of classroom interactions, threatening “analytical paralysis” (Lampert & Ball, 1998) in practice. Few novice teachers seem able to connect the integrating analysis with future enactments of their mathematics teaching. Although novices are willing to employ specific mathematical tasks, the discourses in teacher education and in mathematics teaching engage teachers differently and pose different challenges.

To claim a necessity for integrating novices’ mathematical and pedagogical discourses as preparation for classroom interactions requires empirical support. We need support from teaching research (Davis & Simms, 2006), specifically on teacher classroom dialogues. Teaching involves a dynamic participation within particular dialogic teacher-students situations. Such dialogic situations involve teachers’ ability to cope with interactional thoughts. Contemporary teacher education literature claims that the integration of mathematical and pedagogical thinking is necessary for a good, efficient teacher response during such dialogues.
Paradoxically, although the influence of integration of teacher’s mathematical and pedagogical thinking on teacher’s classroom actions is intuitively accepted, integration research discourse became separated from the classroom interactions research discourse. This separateness became part of early teacher learning. They appear independent to each other. Integration remains sustained only by rationalist advocacy for the quality of mathematics teaching actions.

As Watson and Mason (2007) remark about the difficulty of designing activities in teacher education, this could be a matter of the nature of the teacher education milieu:

Unless the phenomena of teaching and learning can become real for teachers, they are unlikely to make sense to them, just as mathematics has to become real for their students. The notion of ‘what is real’ for someone refers to their inner state and not necessarily to actions in the material world. An additional problem […] is that the institutionalization of teacher education requires that the sessions described are not taking place within the relevant environment: the tasks and activity are not the tasks and activity of teaching; the teacher education milieu is not the classroom; ‘reality’ has to be about internalization rather than immediate action. (p.210)

To overcome how constrains of the teacher learning milieu shape tasks of mathematics and patterns of discourse differently, it is necessary for the teacher education curriculum to have empirical support on how teachers use integration in classroom interactions. Teacher educators act mostly on a logical argument about the relationship between integration and classroom interactions (Hill et al., 2008; Silverman & Thompson, 2008) with little, at best hypothetical, understanding of how integrating mathematical and pedagogical thinking influences teachers’ responses during classroom interactions.

These notions [distinctions between subject content knowledge and pedagogic content knowledge] need to be augmented by, among other things, understanding how being knowledgeable about mathematics teaching influences classroom actions and knowing to act in the moment through having pertinent possibilities come to mind. (Watson and Mason, 2007, p. 209)
The Purpose of the Study

The current study had both a theoretical and practical purpose, thus proving to be both analytical and descriptive in nature. At the practical level, it aimed to understand how a teacher’s mathematical and pedagogical discourses are represented and related in classroom mathematics interactions, thus complementing previous studies focused exclusively to explain teachers’ integrating discourses and thus possibly to provide teacher education with descriptions of a teacher’s discursive mechanisms during classroom mathematical interactions. Made observable the association between a teacher’s mathematical and pedagogical thinking could clarify its role in classroom interactions. At the theoretical level, the study aimed to understand how research in teacher education conceptualizes activities and practices based on what is known about mathematics teaching, especially during classroom mathematical interactions.

Over a period of three years, I observed a small number of excellent mathematics teachers in their classrooms. Eventually, I concentrated on the practices of two experienced, respected teachers, one in elementary school and one in secondary school. My audio recordings permitted a conversation analysis of classroom episodes. I analyzed and interpreted these records, regularly looking for the presence of patterns of related mathematical and pedagogical thinking. I contemplated cognitive models, and especially those encouraging a situative perspective of teacher learning, and the role they might play for teacher’s ways of thinking of classroom dialogues. The perspectives of this study were constructed from the data of classroom interaction. I paid attention especially to teacher’s mathematical understandings as used almost instantaneously in classroom teaching and interaction.

To unpack observable relationships, I selected inferences of teacher categorizations of student utterances during interactional dialogues. Through a progressive combination of
observations, interviews, and, ultimately, analysis of transcripts, I sought to understand such inferences and the ways they were related with various conceptualizations of teacher mathematical and pedagogical discourses. I looked for any patterns of categorization the teachers might have used to respond to students’ utterances. To understand how the relationship between a teacher’s mathematical and pedagogical thinking functions in classroom teaching, I took as my unit of analysis a teacher’s dialogic action. By focusing on the immediate contexts of teaching situations to analyze teacher categorizations of their students’ actions, I did conversation analysis of classroom interactions. This study focused on the verbatim dialogue of students and a mathematics teacher to understand what possible actions reveal relationships of teacher mathematical and pedagogical discourses. I sought possible positioning and patterning in the conversational turns. Reported here in this dissertation, it is a flow of ideas that might inform teacher education both from a constructivist and an interactionist view. I saw various enactments of particular mathematics teaching practices by the same teacher. This provided a special understanding of the different mechanics behind how a teacher uses her mathematical and pedagogical discourses.

As a secondary matter, I brought in the analysis teachers’ pre and post-lesson thoughts as well. Not so much the content, but the description of the classroom discursive action helped me see the ways that the teachers experienced dialogue. I found teacher interactions with students, in some ways, challenging teacher pre- and post- lesson thoughts especially those relating mathematical and pedagogical thoughts with certain situations. In the dialogues with teachers I sought information on the events happening in classroom and how such events are coordinated with the ways teachers relate their mathematical and pedagogical discourses in pre- and post-lesson reasoning for their actions. Such was intended to enlarge how a teacher uses her
mathematical and pedagogical discourses while teaching, and, most importantly, subsequently to inform teacher education practices.

**Studying Teacher Mathematical and Pedagogical Thinking in Classroom Interactions**

Current research efforts to recognize and promote the integration of teacher’s mathematical and pedagogical discourses are greatly an effect of reformist views advocacies. They seek to improve mathematics education, helping teachers shape their actions according to new conceptualizations of classroom mathematics learning. Although the phenomenon of shaping content knowledge within a pedagogical reasoning has been observed and conceptualized as characteristic to teaching, the question came up as how one could use the characteristics of teachers’ discourse to support teachers adjusting their practices to new ways of learning mathematics.

Research also claims that teacher education programs should prepare teachers to work efficiently in classroom by promoting the integration of novices’ mathematical and pedagogical discourses. But teachers’ actions in classroom teaching are part of their engagement with students. How a teacher responds is also tuned to how interactions shape the classroom discourse and how this discourse molds teacher thinking. Consequently, to know how to prepare future teachers requires understandings of how classroom dialogues are shaped by interactional classroom dynamics.

Most current discussions about integrating mathematical and pedagogical discourses of teachers and especially of novices of mathematics teaching have been based on theoretical rationalities about how teachers reason. The researchers’ foci have been mostly on the effects of teachers’ lacunary knowledge, specifically what is missing from their mathematical knowledge.
Concern has been expressed about their lack of preparation to respond to particular views of teaching mathematics (Ball, Lubienski, & Mewborn, 2001). More recent studies have been searching for characteristics in the exemplary practices of mathematics teaching to accord with reformist learning views. Such properties have been searched and advocated as a means to correlate teaching and learning of mathematics and teaching and instruction of mathematics (Hill & al., 2008) and as an argument to avoid a fragmented practice of teaching mathematics.

There is little empirical research specifically showing how integration of mathematical and pedagogical thinking influences and informs teachers’ interactions with students. As mentioned above, most of the literature points instead to its necessity as reasoned through the establishing of quality of mathematics teaching actions. One reason for promoting integration in teacher learning claims that it is difficult for teachers to realize such integration of mathematical and pedagogical thinking during classroom interactions. This argument finds that teacher education activities need to prepare for a baggage of professional knowledge in which instances of integration are recognized and their use based on a quality of teaching enactments. Complementary to this argument is also the observation that integration is an experiencing process and thus formed along teaching experiences, facilitated by a reflective awareness of student learning of mathematics. In this case classroom interactions are resources for such awareness and farther reflection. These views of integration hold that novices can be prepared to reason and reflect over their actions.

To shape perceptual understanding during classroom interactions, teacher education activities became based on the initiative that pre-service teachers should learn to engage in reflective thinking of practice-based activities. It would help them pay attention to such details of teaching they were not spotting. Concepts like reflection on practice and in practice (Schön,
1987) have been considered and approached in various teacher education practices. Such reflective engagements are not easy because meaningful reflection requires also experience. Novices have experiences as learners of mathematics, but with little insight into what teaching mathematics involves. There is little that can be anticipated by novices about the difficulty of interpreting mathematics for learners.

Additionally, researchers hypothesized that the kinds of teacher’s thinking during classroom interaction differ from the kinds of thinking that teachers do before and after classroom interactions (Clark & Peterson, 1986). We have amassed findings based on integrating discourses exclusively based on teachers’ ways of thinking before and after classroom interactions. It should be questioned that the same ways of integrating show up in the ways teachers actually think during classroom interaction. The matter is of importance. Having different ways of relating mathematical and pedagogical thinking during classroom interactions and in after-class thinking has important implications for how we construct ways novices should learn to teach. It relates as well to how we evaluate mathematics teaching.

If our training assumes that teacher action during classroom interactions is greatly based on integrated mathematical and pedagogical knowledge as envisaged in post-lesson reflections, then trainers may also presume that the way teachers integrate their mathematical and pedagogical thinking during classroom interactions is similar to the way they reason out their action after the lesson. If this assumption is true then we have good reason to encourage further rationalist analysis of classroom episodes followed by reflective entries to improve teaching practices. But if the process of relating mathematical and pedagogical knowledge during classroom interaction is different from the process teachers use to reason through their actions after lessons, then we may be misleading our future teachers. We risk building a problematic
training apparatus based on hypothetical relationships between how teacher knowledge and actions manifest in the teaching of mathematics.

Because most research studies have claimed that experienced teachers explicitly rely on integrated mathematical and pedagogical knowledge (Gess-Newsome, 1999) I designed a qualitative study to analyze the coexistence of mathematical and pedagogical thinking in the classrooms of two experienced teachers, highly respected teachers. In the immediacy of the moment and in the local context of the classroom, I looked at these teachers’ mathematical and pedagogical discourses. From utterances, actions, and recollections, and relying heavily on the teaching experiences of the two, I tried to account for their meanings of integration. I looked specifically at how teacher-students classroom interactions confronted the integration of mathematical and pedagogical discourses. I paid more attention to the teachers’ naturalistic thinking about their teaching actions and questioned more rationalist representations of the quality of teaching actions. I sought to understand the nature of the coexistence of mathematical and pedagogical thinking as teachers make sense of their mathematical interactions with students. Few research studies considered that classroom interactions could be regarded as a learning environment for teachers in which teacher classroom discourse is dynamically transformed not just simply enacted. I also tried to see the potential of such an environment.
The arguments for the integration rationale are: (a) Teachers cannot integrate in classroom under interactional demands pressure. It requires experience and reflection to understand integration relationships in mathematics teaching. (b) Without integrating teachers have difficulties in responding classroom interactions and deficient instructional actions.

*Figure 1.* Studying Mathematical and Pedagogical Discourses in Classroom Interactions.
The Research Question

As stated in the rationale section, teacher education relies on the assumptions that, to carry on their teaching activities, teachers relate their mathematics knowledge with their knowledge and beliefs about teaching. Experienced teachers may be able to reason pedagogically for their classroom mathematical interactions with students, leading us to believe that particular integration occurred and has been used and had a correspondent in their classroom discourse. However, some research has conceptualized as different mechanisms the thoughts teachers have after teaching versus those during classroom interactions (Clark and Peterson, 1986). Other researchers have wondered about a possible mismatch between teachers’ reasoning discourse about their actions and their discourse during classroom interactions.

If we are to overlook the epistemological differences between mathematical and pedagogical discourses, we find the questions “How is it that novices of mathematics teaching keep separate their mathematical and pedagogical discourses?” and “Why is it so difficult to observe mathematics teachers integrating their mathematical and pedagogical discourses during classroom interactions?” We would like integration to be a phenomenon observed by novices of mathematics teaching. Only traces of the events are observable. Novices and trainers seem to prefer the rationalizations more than they need analytical integration and perceptual support. It is to these questions that this study turns. It is a search for understanding the issues of integrating mathematical and pedagogical discourses in teacher’s learning of mathematics teaching.

Most teacher educators seem to think there is a necessary relationship between mathematical and pedagogical thinking. Most resist the idea that mathematical thinking is the one which governs classroom teaching action. But to improve upon these ideas we need to
understand better the nature of the relationship between teachers’ mathematical and pedagogical discourses in the classroom. Thus I started the study with following research question:

*How do teachers relate and use mathematical and pedagogical discourses (exhibiting thoughts and thinking) in classroom mathematical interactions?*

A most recent convenient answer has been that if proper pedagogical translations of mathematics have been made available and that teachers just have to understand them, implement, and enact them properly. But, how do teachers integrate mathematics and pedagogical thoughts while implementing pedagogical translations of mathematics? This question may appear pleonastic: if one understood pedagogical translations of mathematics, one would be expected to interactively integrate mathematics and its pedagogical counterparts and work from there. In the perspective of reflective practice, the teacher does not just apply a solution – as a retrospective experiential construction of the integration of mathematics and pedagogy- to a recognizable problem. If we recognize that teaching involves choices and alternatives of action, then teaching involves active use of thoughts in constructing teaching action. The nature of these thoughts and the relationships between them stir the final judgment about the classroom situations.

Through this question I attempted to reevaluate the ways conceptualizations of mathematics teaching are embedded in learning to mathematics teaching. One such conceptualization is that for their teaching action teachers rely on integrated mathematical and pedagogical discourses. The focus of this study was to evaluate the mechanism through which experienced teachers used their integrated discourses, could I find such integrated discourses related to particular teaching action. At the time this study was started, teacher education theories expressed a challenge to what was practiced as “pedagogies of investigations and reflections.” Eventually, such efforts were shaped towards what are called “pedagogies of enactment.” There
are efforts to conceptualize teacher education through such rationalizations. Specific teaching actions and the meanings behind them sustain belief in particular relationships between mathematical and pedagogical discourses. To conceptualize teachers’ ways of knowing and thus to assist mathematics teaching, this study advances this research question as important for both teacher educators’ practices and programs of teacher education.

**Overview of the Dissertation**

In Chapter two I review the research literature on the relationship between teacher’s mathematical and pedagogical knowledge. I particularly consider how researchers gave more attention to the concept of integration and describe how research on both knowledge and classroom interactions provide various representations of integration. I conclude restating the missing link: the empirical grounds for the rationale that integration supports teacher actions. The missing link is the research topic for this study.

In chapter three I detail the conceptual framework of the thesis briefly mentioned in the rationale of the study. I highlight the differences between macro- and micro-research and interpretations. I consider in this picture also the various ways research positioned teacher action and thoughts and the difference between the nature of teacher’s thinking after-lesson and during classroom interactions. I show how this difference is problematic in the development of various conceptualizations regarding teachers’ knowledge and actions. I view different perspectives on how mathematical and pedagogical discourses are practiced by teachers during classroom interaction. I show how integration has been borrowed as a metaphor in research discourse and how gradually the term was objectified in the descriptions of various key domains of mathematics teaching. I identify some of the researchers who developed the contemporary
rationalistic relationship between learning and teaching of mathematics. I close with a formulation of issues for the study.

In Chapter four I explain the methods and analysis of the study. I turn back to the differences between micro- and macro-research, specifically regarding the micro-analysis, describing the choices of participants as well as methodological issues to consider in analysis of teachers’ thinking in classroom teaching. I explain the concept of patches (Stake, 2010) as critical to my analysis of teacher experiences. These episodic patches illustrate unexpected lines of action in teachers’ classroom experiences, particularly challenging predetermined relationships between mathematical and pedagogical reasoning. Patches need be neither random nor extraordinary episodes; on the contrary they are carefully selected encounters in mathematics teaching experience. They avoid what might be called accidental deviation from the discourse of mathematics teaching. In this case, my patches pointed to the “negotiated conventions” or “spontaneous improvisations on basic patterns of interactions” (Cazden, 2001, p.39) I thus referred to the teacher’s creativity to be a “skillful rule follower” (Sfard, 2008, p. 216) Once such patches were identified, I explain how -- through conversation analysis -- I looked at them to find specific patterns to understand the mechanism of relating teacher’s mathematical and pedagogical discourses during classroom interactions. But I also explain here what I came to see as mathematical and pedagogical.

In Chapter five, with specific patches from mathematics classroom interactions and post-lesson interviews, I illustrate how the teacher-participants acknowledged that they did not act according with any anticipated reasoning that would assure the enactment of a particular pedagogical thinking. I show analysis of eight such “patches” of data and exemplify the various forms of “interrupted reasoning” they represent. I compare “interrupted reasoning” as a sort of
break within teacher practical rationality and a different sort of interruption, an episodic interruption, which is additional and unplanned by teacher’s analytical rationality. I found out that the phenomenon of interrupted reasoning may illuminate our perspectives on the nature of integration mathematical thinking in pedagogical thinking as normative for teaching enactments. Chapter five illustrations include interrupted reasoning in both the elementary and the secondary teacher cases. I highlight that interrupted reasoning is not a shortcoming in teacher planning or action, but displays the complex teacher interactional thoughts. This complexity needs to be recognized when researchers make recommendations for teacher actions. I summarize with a cross-analysis of all eight episodes.

Chapter six discusses the four patches with an emphasis on focal and conversational analysis. The scope of the chapter is to highlight certain patterns of teachers’ categorizations of students’ utterances. The patterns show similarities and a difference in teacher’s thinking between post-lesson and during classroom interactions situations. The teachers’ categorizations and analysis bring forward two different mechanisms the two teachers used to associate their mathematical and pedagogical discourses. The first mechanism pertained to their after-lesson reasoning in which rules for teaching actions were highlighted as normative of the mathematics teaching domain. The second mechanism happened during classroom interactions and was based on analogy between the two discourses. The two classroom interactional mechanisms, based on analogy, demonstrate how teachers use both an association mechanism, which explains teaching actions in a local context and a similarity mechanism, in which teachers’ mathematical and pedagogical discourses are actively alternated during classroom mathematical interactions. These two mechanisms show the difference I remark between the static and dynamic relationships between teacher mathematical and pedagogical thinking as described in Chapter 3.
In Chapter seven I reevaluate the two ways of thinking characteristic to teaching: reflective and metaphorical. I retake the patches analyzed in Chapter 5 and Chapter 6 in the context of teacher learning and evidence how such episodes could be developed as workables\(^1\) for teacher education through alternative perspectives on teacher action and student reactions; thus preparing the novices for the disruptions in integration of mathematical thinking and pedagogical thinking. I return to propose the exercise of the interrupted reasoning to disrupt the action sequence of classroom conversations and highlight the difference in the perceptions of situations at the macro-level and at the micro-level.

Chapter eight takes this discussion farther in research conception of teacher learning and claims that while we may provide rules for building a mathematics teaching discourse, what we engage in as routines of teacher learning need to be reevaluated. I discuss what could be redefined as learnable mathematics teaching and mathematics teachability and how these influence what I define as teachable mathematics. Perceptions of specific episodes do not need to lead to only one form of interpretation but many. Ignoring the analogical awareness with which teachers engage and the different mechanics of relating mathematical and pedagogical discourses in after-lesson and during classroom teaching we place high expectations on teacher education practices which by nature are suitable only for a way of practical thinking.

\(^1\) Part of a classroom dialogue, a workable is more suitable for novice teacher learning. Developed from a research patch, it aims to the alternative procedural meanings captured in an interactional episode of teaching. Instead of guiding towards particular explanatory understanding of teacher action, a workable leaves space for alternative actions and directs attention to the immediacy of conversational context.
Chapter 2

Background: Teacher Action in Classroom Interactions and the Relationship Between the Teacher’s Mathematical and Pedagogical Thinking

This chapter will explore how the rationale for practicing the integration of mathematical and pedagogical thinking in teacher education finds support in the research literature of mathematics teaching. Here are grounds on which contemporary arguments in teacher education have been set. One of the rationales for teacher learning claims that integration of mathematical and pedagogical thinking supports the quality of teacher’s actions in classroom. I have limited this review to those studies that account for teacher actions and try to spell out the existence and structure of the integration. In the first section, I present various conclusions of authors of different studies of teachers’ knowledge on how events during classroom interactions reveal changes in established relationships between a teacher’s mathematical and pedagogical thoughts.

In the second section I consider how research on classroom interaction illustrates teacher’s use of pedagogical and mathematical discourses. In the third section, I evaluate how the rationale for encouraging integrated forms of knowledge develop from both kinds of research, teacher’s knowledge and classroom interactions, but in both cases based on a logical argument that integration influences teacher action. The last section of the chapter evaluates how such efforts have been reflected in the approaches to model teacher learning activities. It is thus the purpose of this chapter to illustrate that – in studies from mathematics teaching research -- the rationale for integrating mathematical and pedagogical thinking is supported by following assumptions:

1. Modifications in teacher action due to interventions in curriculum and pedagogy of mathematics teaching result in structural modification of the established relationships between the teacher’s mathematical and pedagogical thoughts.
2. Teachers’ reflective reasoning, when stimulated by teaching dilemmas, evaluates established relationships between teachers’ mathematical and pedagogical thoughts.

3. Missing connections and having lacunar knowledge of mathematics results in problematic pedagogical action. Conversely, inability to act pedagogically leads to poor communication of mathematical ideas.

All three assumptions lie behind the rationale that integration influences and even determines teacher classroom actions. They are supported with examples from research literature in first section of the chapter. However, these arguments provide only logical explanations why integration of teacher’s mathematical and pedagogical thinking is necessary for teaching actions. In other words, the three assumptions could completely support the rationale for practicing integration of mathematical and pedagogical thinking, only if a fourth assumption is warranted:

4. Teacher action in the classroom interactions displays how teachers use forms of integration of mathematics and pedagogy and how such interaction reflects the necessity of integration for teacher’s actions.

It is for this reason that (in the second section of the chapter) I analyze research on classroom interactions to look for this integrated knowledge. Research on classroom interaction includes an alternative perspective on the relationship between mathematical and pedagogical thoughts: the conceptualization of integration through context, with an improvisational nature, and with the identification of patterns in interaction that lead to affordances and recognition of constraint on teacher action.

Both kinds of research, the one focused on teacher’s knowledge and the one on classroom interactions, suggest that integration of teacher’s thoughts is necessary for teacher’s actions, but from different venues. However, they rely exclusively on a logical argument as to why it is necessary, rather than showing how it is actually enacted and authenticated during classroom interactions. In the first case, of cognitive psychological perspectives, the influence of thoughts
on actions is axiomatic. This created hypothetical cognitive models in which thoughts influence actions through the principles of cognition.

From the very beginning, Lee Shulman (1986) argued that “teachers’ cognitive understanding of subject matter content and the relationships between such understanding and the instruction teachers provide for students” (p. 25) may be the “missing paradigm” in educational research. He carries farther the idea that such integrated cognitive model must be guided by the teachability of a body of mathematical knowledge. Teachability preserves the importance of pedagogical reasoning in practice.

In the second kind of modeling approach, the distinction between thought and action is abandoned, as the intimate relationship of teacher’s practical thinking, to give priority to situations and activities of teaching. Mathematics is seen to require specific forms of understanding tailored by such teaching situations and activities. In this case a different kind of integration is brought forth. The authors claimed to encompass Shulman’s concept. However, both cases lack empirical evidence for integrated use of teacher’s mathematics and pedagogical thoughts, but rather a building of hypothesis about such integrated use of knowledge, a logical argument that integration increases the quality of teacher action.

The last section of the chapter analyzes the implications for teachers learning activities for both approaches. Although the two approaches see teaching activity from two perspectives, one theoretical and another practical, they both face the novice of teaching with the same gap between learning activities to stimulate rationalist integration and learning activities for practicing classroom interactions.
Changes, Dilemmas, and Deficiencies in Mathematics Teaching Action

Most of the research on teacher knowledge highlights how we should model the teacher’s knowledge base to correspond with specific thoughts a teacher would have in classroom teaching. What kind of questions and discursive patterns bring insight into student learning? In this section I will show how certain phenomena of teaching lead to the identification of integration of mathematical and pedagogical thinking in classroom interaction. These phenomena show modifications in relationship between teacher’s mathematical and pedagogical thoughts that could contribute to a long-term process of creation of integration.

At one extreme, PCK does not exist and teacher knowledge can be most readily explained by the intersection of three constructs: subject matter, pedagogy and context. Teaching, then, is the act of integrating knowledge across these domains. For convenience, I will call this the Integrative model. At the other extreme, PCK is the synthesis of all knowledge needed in order to be an effective teacher. In this case PCK is the transformation of subject matter, pedagogical, and contextual knowledge into a unique form – the only form of knowledge that impacts teaching practice. I will call this the Transformative model. (Gess-Newsome, 1999, p. 10)

Pedagogical and curriculum interventions. Starting with Shulman’s concept of pedagogical content knowledge (1987) and Sherin’s (1996) research on how teachers develop “complexes” of knowledge, the relationship between teacher’s mathematical and pedagogical knowledge has been remarked as evident in the case of pedagogical and curriculum interventions. In her dissertation, Sherin (1996) discussing the nature and the dynamics of teacher’s content knowledge argues that “as teachers develop expertise, elements of subject matter knowledge and pedagogical content knowledge become tightly connected in knowledge structures called content knowledge complexes.” (p. 245)

In a more recent study on pedagogical reform “Fostering a Community of Learners,” in different disciplines, Sherin et al. (2004) remark how, in the case of mathematics the participant
structure component of the pedagogy needed to be modified, and they tried to identify the pedagogical principle:

They (authors of pedagogical innovation) did not simply transfer the activities that had been proved successful in one domain or another. Rather, they went back to the principles on which the structure was developed, and then looked for strategies that would be appropriate for new domain. This suggests two underlying assumptions: it is important to consider whether a participant structure is appropriate for a given domain; and it is the application of FCL principles that should remain intact and not necessarily the participant structures. (p.211)

Interaction with student thinking has the teacher reevaluating his own conceptions about mathematics knowledge in relationship to the new pedagogical principles. This dynamical communication between teachers’ mathematical and pedagogical understandings suggests that as teachers to be able act in mathematics classroom according with FCL need to revise the relationships between their understanding of mathematics and pedagogy.

Analysis of data suggests that applying FCL pedagogy with mathematics was facilitated by developments in three related areas of the teacher’s knowledge: his understanding of mathematics, his views of implementing mathematics-education reform, and his ideas about FCL pedagogy. (Sherin et al., 2004, p. 214)

When required to implement new pedagogies and curricula, teachers need to re-think their classroom teaching. This happens in planning, when teachers need to think strategically. Equally, during classroom interaction teachers need to review what they know about mathematics and how this is related to the new pedagogical context. Notice that such modifications in teachers’ mathematical and pedagogical thoughts make a strong argument that teachers use their “complexes” of knowledge in practice. Teachers talk about such changes in their perceptions of classroom interactions, as happened in Sherin et al. (2004) study. I will show in the next subsection how such relationships between the teachers’ mathematical and pedagogical thoughts are influenced by teachers’ perceptions of classroom events and their interactions with students.
Dilemmas of mathematics teaching. Based on active research methods or self-studies, using reflective thinking as the main venue, various reflective studies have indicated that teachers’ reasons for their action are based on a close relationship between their mathematical and pedagogical knowledge. Most of the times, they would realize, after deliberations and reflection, that future action needs to be modified. Similarly as in the case of interventions this would have effect on the already established relationships between teacher’s mathematical and pedagogical thoughts.

In an attempt to consider specific social interactions “appropriate to making mathematical arguments in response to students’ conjectures,” from mathematical, pedagogical, and sociolinguistic perspectives, Lampert (1990) interpreted her own teaching practices. She was studying her teaching on exponents. Her study represents one of the first reported attempts to intertwine content and classroom interaction. She worked to reduce students beholding the teacher as authority, but rather “the act of individual thinking” (p. 55) and she remarks how her role both in the mathematical discourse and in the teaching discourse is rather conflicting on how to have students interpret the norm of the discourse:

What does change is that the class group, as a learning community, comes to regard mathematical discourse, rather than more typical forms of school interaction, as the norm. But as with other forms of socially destructive student activity, like passing notes or fighting on the playground, it continued to be my responsibility as the teacher to remind students of the norm. (p. 58)

Heaton (1992) study on teaching mathematics to the new standards shows how she became more aware of the conceptual understanding of mathematics. In one episode she remarks that it was the concept not the context that mattered. She does this reflection on an intertwined argument of matching the context of activity with that of the mathematical construct. It was a dilemma if a certain activity may be more appropriate than another for students learning, but also for her confidence with explaining linear measurement.
I wanted to get as far away from linear measurement as possible. After switching contexts from inches of rain to cup of flour and seeing what students did with the muffin problem, I realized the mathematics that students were grappling with was not something resolved by attending to flour rather than rain. The difficulty was with shifting units and it did not matter whether the context was rain or flour. (p. 131)

This section showed how experienced teachers, when talking about their teaching, display an explanatory framework, one based on a continuous relationship between mathematical and pedagogical thinking. In this case, alternatives of mathematics teaching actions are considered and pondered according with different learning contexts. Still, the nature of reflective thinking is something based on experience, thus provoking a cognitive dilemma. Max van Manen (1995) took issue with advocates of self-criticism in teaching:

If my allusion to the practical tact of teaching is indeed in keeping with how thoughtful teachers actually experience their practice then the requirement for critical-reflection-in-action may need reconsideration. Why should we demand that everything one does as a teacher requires critical reflection, reasons or justifications? Molander (1992) and Socket (1987) have made provocative counter suggestion. They have suggested that it is doubt and distrust in certain practices that may require reasons and justification. Indeed we may sometimes put a misplaced emphasis on critical reflection on teaching. The aim of critical reflection is to create doubt and critique on ongoing actions. But it is obviously not possible to act thoughtfully and self-confidently while doubting oneself at the same time. If teachers were to try to be constantly critically aware of what they were doing and why they were doing these things, they would inevitably become artificial and flounder. It would disturb the functional epistemology of practice that animates everything that they do. (p. 13)

**Lacunar knowledge of mathematics and deficiencies in instruction.** Based on fine-grained analysis in extensive and elaborate studies on teaching, Leinhardt (1988) revealed how teachers did not know or lacked understanding of mathematics and proper pedagogical principles to shape classroom activities. In such cases, deficiencies in teachers’ actions have been associated with what was missing from teachers’ knowledge. Some of the studies collected concurrent measures of teacher knowledge and classroom instruction. For example, Heaton (1992) told how the participant teachers in her study used inappropriate metaphors for teaching inverse functions and related this to the teacher’s lack of knowledge. Cohen (1990) described
how, in a lesson on estimation, the teacher did not see and accepted inaccurate guesses rather than pressing for what would be called more “reasonable” answers and estimation strategies.

There are two resources which provide indication for integration of mathematical and pedagogical thinking: reflective thinking after dilemmas of teaching inter-action and hypothetical conceptualizations of what is missing from teacher’s knowledge to justify poor actions of teaching. Both of these approaches take indirect tracks from teaching quality to teacher’s integrating mathematical and pedagogical thinking. For the first argument, it is a necessary component of experiencing teaching and is based on a teacher’s discourse for justifying her actions. In the second argument, it hypothesizes what is missing from knowledge and that integration is useful.

Teachers’ difficulty to respond during classroom mathematical interactions, especially when reformist views of learning mathematics have been involved, has been ample described in previous case studies (Ball, 1988; Heaton, 2000; Lampert, 1990; Leinhardt, 1988). Some of these studies tried to illuminate issues that appeared during classroom interactions set by pedagogies promoting reformist views of mathematics learning. One of these issues referred to how teachers’ mathematical and pedagogical discourses could be aligned to the same tune of the new ways of students’ learning of mathematics (Cooney, 1999; National Council of Teachers of Mathematics [NCTM], 2000). Particular practices and investigative pedagogies have been proposed (Ball, 1988).

The teacher lacked knowledge of several key mathematical ideas about functions and graphing, and his knowledge was organized in a superficial way that did not include deep connections among ideas. These limitations led to an overemphasis on rules and procedures and missed opportunities for fostering meaningful connections between key concepts and representations. Similarly in science, Hashweh (1987) found that teachers teaching outside areas of their own expertise (physics and biology) tended to treat material in the science textbooks mechanically and missed errors in the texts. (Borko & Putnam, 1995, p. 44)
In this section, I presented examples of research literature trying to connect problems in teacher action and omitted links among various mathematical and pedagogical concepts. Much of the argument says that if teachers lack such relationship, their action would be defective. I have described how research observations have led to conclusions that a relationship between mathematical and pedagogical is evident during teaching. These arguments led to claims of the necessity of integration and to calling for integration in teacher learning activities. This initiative is considered in the next section.

**Research on Classroom Interactions**

Research on classroom interaction has brought little insight into how teachers’ interactive dialogic thinking is related with their integration of mathematical and pedagogical thinking. Still, most interactive thinking is acknowledged as having an analogical character, as well as improvisational, and in context, when placed in dialogues with students. It also can be considered a process of decision taking and identification of alternatives.

Most of the research on classroom interaction has brought insight into the teachers’ role and into participative frameworks (Empson, 2003). Equal attention has been given to norms (Rasmunsen, Yackel, & King, 2004) discourse perspectives on mathematics situations (Cobb, Wood, & Yackel, 1992) and mathematics instruction. Because this interactional aspect of thinking has been made visible in discourse, recommendations for teacher learning have been made at the level of shaping classroom norms and discourse. I divided the research on classroom interaction into two approaches: cognitive models in context and situative analysis of teaching.
**Contexts and knowledge.** Researched based on an extension of what it means to transform or model knowledge has been advanced as important (Borko & Putnam, 1995). Much of the research on classroom interaction and teacher action does not show how integration might be visible in the process of dialogical thinking. I will illustrate this aspect in the next section. Research on teacher learning activities has regularly been divided between: cognitive (Silverman & Thompson, 2008; Simon, 2006) and practice-based activities (via situated cognition).

Researchers faced the quandary of integrating the demands of classroom interactions and the conditions for the quality of teaching action. First, cognitive psychological research perspective on professional knowledge, Borko and Putnam (1995) explain the relationship between knowledge, thinking and actions:

Virtually all cognitive psychologists share a fundamental assumption that an individual’s knowledge structures and mental representations of the world play a central role in perceiving, thinking, and acting (Putnam, Lampert, & Peterson, 1990). Teachers’ thinking is directly influenced by their knowledge. Their thinking, in turn, determines their actions in the classroom. Thus, to understand teaching, we must study teachers’ knowledge systems; their thoughts, judgments, and decisions; the relationships between teachers’ knowledge systems and their cognitions; and how these cognitions are translated into action. (p. 36-37)

First cognitive research looked for a knowledge base approach. Later, research on practice-based activities, ignoring the lack of empirical support for integration advanced the idea of practices that warrant the quality in teaching actions.

One major influence on teacher decision-making is the knowledge that drives teachers’ actions and decisions and provides them with the flexibility that enables them to reason, to judge to weigh alternatives, to reflect, and to act (Clark & Yinger, 1978). Ultimately, how a teacher behaves and the potential effectiveness of a teacher rest on the knowledge that a teacher possesses. Most of the work with decision – making and how it is influenced by knowledge has to do with personal and practical (general) knowledge of teachers. (Leinhardt, Young, & Merriman, 1995, p. 407)

In one of his studies, Hashweh (2005) spoke about pedagogical constructions and claims that “teacher pedagogical constructions result mainly from planning, but also from the interactive
and post-active phases of teaching” (p. 277). In his conceptualization of pedagogical transformations Hashweh remarks (ibid.) of pedagogical content knowledge:

If anything among all teacher knowledge categories is truly constructed, it is definitely the PCK category. Of course, these constructions are further developed as a result of interactive teaching and post-active reflection. A teacher might invent an analogy during interactive teaching when she realizes she needs one more representation to explain a certain concept. (p. 279)

One conceptualization of research on teacher knowledge regarding the transformation was emphasized by Shulman (1987):

The key to distinguishing the knowledge base of teaching lies at the intersection of content and pedagogy, in the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students. (p.15)

Teacher knowledge develops in context, Fennema and Franke (1992) conceptualized in their model:

The critical word here is transform. Teachers have to take their complex knowledge and somehow change it so that their students are able to interact with the material and learn. This transformation is not simple, nor does it occur at one point in time. Instead, it is continuous and must change as the students who are being taught change. In other words, teachers’ use of their knowledge must change as the context in which they work changes.

Examination of teachers is beginning to indicate that knowledge can be and is transformed through classroom interaction. (p. 162)

Their conceptualization recognizes the limitations in their model:

The transforming of knowledge in action is understandably complex. Little research is available that explains the relationship between the components of knowledge as new knowledge develops in teaching, nor is information available regarding the parameters of knowledge being transformed through teacher implementation. (p.163)

**Situative analysis of teaching: affordances and constraints.** The newer perspectives on learning considered the analysis of teaching and learning in a different way from that of the cognitive approaches. An important difference was brought by the situative analysis of teaching. One nice comparison between cognitive and situative analyses was brought in Greeno’s (1998)
response to Schoenfeld’s model of teaching process. Schoenfeld model (2000) of teaching process was analytic:

> An analytic model that is used to characterize, in detail, the decisions and actions of teachers as they teach. In simplest terms, (representations of) teachers’ decisions and actions in the model are a function of (representations of) the teacher’s knowledge, goals, and beliefs. (p. 243)

In reaction to Schoenfeld’s model, Greeno described a situative analysis of teaching as having “focus, primarily on classroom interactions as the systems that would be investigated and to trajectories of participation by the teacher and students.” What Greeno notes is that teacher actions are not predictable based exclusively on their knowledge, goals, and beliefs.

For Greeno (1998), the theory of affordances is an extension of situative analysis of teaching interactive activities:

> In our theorizing, my colleagues and I are using concepts that we draw from philosophical situation theory (e.g., Barwise & Perry, 1983) and from ecological psychology (e.g. Reed, 1996). We conceptualize social practices, including ways that people interact with material and informational systems, in terms of constraints and affordances for activity. Individuals who participate in the practices are attuned to these constraints and affordances differently, and more complete attunement generally results in more successful participation […] Therefore, situative analyses of teaching and learning focus especially on regularities in patterns of discourse (constraints and affordances, in our terms) that structure the opportunities…Successful teaching includes organizing and managing the discourse in which students participate in these processes, and students’ learning includes becoming attuned to the constraints and affordances of participation in the practices of the classroom communities of which they are members. Cobb and his associates (e.g. Cobb et al, 1992) have emphasized these aspects of teaching in their discussion of classroom norms. (p.5)

For the situative perspective, integration of teacher mathematical and pedagogical thinking became an assumption to support teacher discourse such that the teacher would be able facilitate “students’ learning” and students “becoming attuned to the constraints and affordances of participation in the practices of the classroom communities of which they are members.”
Integration - A Necessary Construct for Mathematics Teacher Learning

One of the main important reasons for promoting integration as a practice in teacher education is that classroom practices require it.

Subject matter and pedagogy have been peculiarly and persistently divided in the conceptualization and curriculum of teacher education and learning to teach. This fragmentation of practice leaves teachers on their own with challenge of integrating subject matter knowledge and pedagogy in the context of their work. Yet, being able to do this is fundamental to engaging in the core tasks of teaching, and it is critical to being able to teach all students well. (Ball, 2000 p. 241)

In “bridging the gap,” Ball proposes three problems to be “solved:”

The first problem concerns identifying the content knowledge that matters for teaching, the second regards understanding how such knowledge needs to be held, and the third centers on what it takes to learn to use such knowledge in practice. (Ball, 2000, p. 241)

In the next section I will discuss the attributes and metaphors that helped researchers give integrated thinking a main role in teacher classroom interaction. The arguments above suggest a tight correlation between teacher action and integration. It should be necessary to show how integration influences classroom dialogues through the unfolding of the dialogue.

There is little empirical research in this sense. We have records arguing that something seems to shift in the structures of knowledge. It seems that there are modifications in these structures that I liken to Escher’s metamorphoses. Based on such changes we assume that teacher thinking is integrated. However we do not know if such integrated thinking is a driving force within classroom interactions. We have little information about what makes a teacher approach a certain happening during classroom interactions with or without integrated knowledge.

The argument that integration affects teaching actions are captured in following research which still keeps the argument at a logical level not at the level to illustrate how integration manifests in classroom interactions and how thus influences the efficiency and quality of teacher action.
**Teachability.** It was also Lee Shulman (1987) who claimed that a teacher acts on a repertoire of analogies and how this is represented and referred to as “teachability.”

Pedagogical content knowledge was defined by him (Shulman, 1986) including the word:

That special amalgam of content and pedagogy that is uniquely the providence of teachers, their own special form of professional understanding … Pedagogical content knowledge … identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue. (p. 8)

Previously he said:

A second kind of content knowledge is pedagogical content knowledge, which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching. I still speak of content knowledge here, but of the particular form of content knowledge that embodies the aspects of content most germane to its teachability.

(Shulman, p. 9).

Related to teachability, it is very interesting to remark how in the case of the Fostering Communities of Learning pedagogy mentioned earlier and which was part of Shulman’s study, the implementation of mathematics brings out a teacher’s comments and his role for promoting teachable moments:

Once the class has encountered a discussion situation or a teachable moment, I, as teacher, need to be thoughtful about the kinds of impromptu comments I make and the probing questions I ask so that I can help promote student learning. Since it is difficult to predict exactly what methods students will use and what questions they will have, the way I react to student discussion seems to have an impact on how “teachable” those moments actually turn out to be. (Louis 1997a, p.51)

Shulman’s (1986) construct of pedagogical content knowledge emphasizing teachability has been far from a mechanical, procedural integration of pedagogy and mathematics. Wineberg (2008) discusses such futile attempt in the context of teacher evaluation. He emphasizes how pedagogical content knowledge is hard to capture in multiple choice items as Carlson does in the example below of simply aligning principles of pedagogy with mathematics. In this example,
Carlson (1990, as cited in Wineburg, 2008) advances the possibility to capture the complex integrated practical thinking of teaching mathematics in a multiple choice evaluation.

A test item in mathematics that requires examinees to choose the most appropriate sequence in which to present three measurement activities to students requires not only knowledge of the general principle but also mathematical knowledge about measurement. The examinee will have to use the mathematical knowledge to analyze each measurement activity in light of the general pedagogical principle of concrete before abstract. In other words, the examinee will have to have knowledge of mathematics in order to recognize the correct application of the pedagogical principle. This type taps pedagogical content knowledge. (Carlson, 1990, p. 160)

In response to Carlson, Wineburg (2008) invokes the interpretative nature of pedagogical content knowledge and brings forth the argument that it is difficult to “tap pedagogical reasoning” simply because teaching is the quintessential “ill-structured problem”:

Carlson fundamentally misunderstand Shulman’s concept of pedagogical content knowledge. The above item is “nowhere near what I meant by pedagogical content knowledge” (Lee Shulman, personal communication, August 7, 1994). Carlson has construed pedagogical content knowledge as a kind of algorithmic process by which general (and in this case, highly dubious) pedagogical principle is applied to a specific content area. But pedagogical content knowledge is the integration of knowledge of subject matter and how best to teach it given (a) the specific nature of the subject matter, (b) a particular group of students, (c) the context in which one teaches, (d) what was already come before the curriculum and what will come after, (e) the prior knowledge students bring to instruction, and (f) the community in which one dwells (cf Willson, 1995)

It is quite evident that teachability of mathematics has been seen in the context of integrating teacher mathematical and pedagogical thinking more from a pedagogical umbrella and perspective of how it happens in classroom. A different, more mathematical approach has been advanced in the last 10 years to shape a specific pedagogy tuned to the mathematics practices.

**Knowing mathematics for teaching.** Trying to approach the issue of integration from a different perspective, of the specific nature of the subject matter, another important research conclusion is that understanding mathematics for teaching is different than understanding
mathematics in general. This idea was somewhat a reaction to seeking a specific pedagogy tuned to the nature of the discipline.

A recent analysis provides a glimpse of the importance of the distinction between knowing how to do math and knowing it in ways that enable its use in practice. This distinction is key to understanding how mathematics knowledge matter in good teaching. (Ball, 2000, p. 243)

In the attempt to align pedagogy with teaching practices, hypotheses about relationships between mathematics and pedagogy have been anchored in the situations of mathematics teaching. This was an assumption on which to respond to the question of why integration of teachers’ mathematical and pedagogical thinking is necessary for teachers’ actions. It was anchored in the study of classroom instruction and resisted the idea of improvisation and contextual creation of mathematical and pedagogical relationships as marked in the quote by Franke and Fennema (1992). Much of this is based on the research on lacunar knowledge and lack of connection, but the essence of it stands on a repertoire of situations and an established analysis of the relationships between tasks and activities of mathematics teaching.

The underlying epistemological assumption of this body of research is that teachers need to understand and use mathematics in ways that are specific to the work of teaching and that often differ from the ways in which mathematics is attuned to the needs of other workplaces such as nursing and engineering physics. (Stylianides & Ball, 2008, p. 308)

The construct has been also recognized by other researchers with other names and more refined:

We follow the European usage and refer to mathematical pedagogy as the collection of strategies and detailed ways of working with learners on mathematics across topics, and mathematical didactics as the collection of strategies, cultural pedagogic tools and associated psychology to do with learning particular mathematical topics at the level of individual concepts, techniques and properties. (Watson and Mason, 2007, p.210)

Attempting to identify the mathematical demands for teaching and important ideas about proving in mathematics learning, Stylianides and Ball (2008) remarked:
We explain that existing research informs the knowledge about the logico-linguistic aspects of proof that teachers might need, and we argue that this knowledge should be complemented by what we call knowledge of situations for proving. This form of knowledge is essential as teachers mobilize proving opportunities for their students in mathematics classroom. We identify two sub-components of the knowledge of situations for proving: knowledge of different kinds of proving tasks and knowledge of the relationship between proving tasks and proving activity. (p.307)

As in the case of pedagogical content knowledge, teacher evaluation proves to be a tempting line of action:

The second strand develops measures of mathematical knowledge for teaching based on hypothesis formulated by the first strand, and tests these hypotheses by tracing the effect of mathematical knowledge for teaching on student achievement. […] Thus, the second strand informs the first by validating or suggesting revisions of formulated hypotheses. The third strand, which is just beginning, capitalizes on the findings of the other two and aims to develop and refine ways to effectively promote mathematical knowledge for teaching in teacher education and teacher professional development programs. (ibid., p. 308)

In this case, the nature of this integration is recognized as hypothetical and thus needing validation, but such validation only recently has been approach in Heather Hill’s study of “usable knowledge” for mathematics teaching.

As with the cognitive approach, there is again the question how such discipline-integrated knowledge would support and align with particular ideas about the quality of mathematics instruction:

However, we believe that only developing grounded theory about the elements and definitions of mathematical knowledge for teaching is not enough. If we argue for professional knowledge for teaching mathematics, the burden is on us to demonstrate that improving this knowledge also enhances student achievement. And, as the current debates over teacher preparation demonstrate, there are legitimate competing definitions of mathematical knowledge for teaching and, by extension, what “teacher quality” means for mathematics instruction. (Ball & al., 2005, p. 22)

Hill et al. (2008) in their study connected teacher mathematical knowledge for teaching with what they call elements of mathematical quality of instruction. They reasoned for their study, claiming that only “Two study provide important addenda […] about the connections
between mathematical knowledge and teaching.” (p. 435) They quote Thompson and Thompson (1994) study of a teacher who although was well-prepared in mathematics, he had trouble explaining ratio concepts, and Borko et al. (2000) study of a preservice teacher who also was prepared on upper level mathematics was not able to explain division of fractions in response to a student question. Hill et al. (ibid.) conclude: “These two cases, along with the affordances literature already mentioned, suggest that there is knowledge used in classroom beyond formal subject matter knowledge, a contention also supported by Shulman’s (1986) notion of pedagogical content knowledge.” (p. 435)

According with the study of Hill et al. (2008), in a situative perspective of student learning, teachers who are prepared in mathematical knowledge perspective would be able to create more affordances for their students’ perceptual learning. In a slightly different way Hill and colleagues differ in approaching what would be the affordances of teaching practices and how to proceed with them. They rather base teacher evaluation on such theory of affordances, and thus replace a previous one of deficit.

In this section, I have described how integration of mathematics and pedagogical thinking has been advanced and various hypothetically conceptualizations of it developed as a necessary construct for high quality of teacher action. I have noted how it can be approached from a pedagogical or a mathematical perspective. In the next section I will show how these ideas are reflected in the case of designing and thinking activities of learning to teach mathematics.
The Quandary of Designing Mathematics Teacher Education Curriculum

The quandary of teacher education is to relate what to know with how it is done in classroom teaching. On the one hand we have the suggestion to consider integration of mathematical and pedagogical thinking and on the other hand we have analogical, improvisational, interactive thinking for classroom teaching. In this section I will review literature informing and providing recommendations for teacher education activities, much of it trying to avoid the problems of interactional thinking and acting in classroom teaching. This will be reviewed from the perspective of integrating mathematical and pedagogical thinking and its role for and in classroom interactions. I show that both cognitive and the more practice oriented approaches of teacher education curriculum literature cannot support to the question of how integration supports teacher action in classroom. It keeps separated the two approaches in teacher education: those oriented towards integration and those oriented towards classroom interaction. The separation shows the problem (which preserves the quandary) of approaching experiential teaching dilemmas with rationalist forms of explaining the quality of teacher action. In the first subsection I will review mostly cognitive approaches of teacher education, and in the second subsection I will refer to more recent views of teacher education anchored in practices of teaching.

In this section I will consider the assumptions of teacher educators favoring an integration of mathematical and pedagogical thinking, rather than a dynamic relationship or simply a mixing of it. First I will discuss it as a repertoire of analogies and second I will show how frustration occurred. It suggested a specific form of knowing mathematics for teaching that led to further understanding of how to form such knowledge. I conclude with the observation that these attempts shifted from a knowledge base to drive the teacher’s function on the spot, to a
professional practical knowledge base as a repertoire all of it changing the nature of how a teacher thinks during classroom interactions. This shift has also conceptualized differently the teacher perception in classroom interaction and its role for learning.

**Cognitive approaches: knowledge base and actions.** In the case of cognitive models, the efforts have been towards understanding what kind of knowledge base would prove useful for teachers to act in classroom, especially to guide their interactional thinking. Most of these efforts have moved gradually to how teachers construct their integrated knowledge and to what is the role of the interactions. Two models have been considered here: a transformative model and an integrative model.

The research of Borko and Putnam’ (1995) showed efforts to build such knowledge. One important project (Cognitive Guided Instruction, 1996) illustrated how teachers who were aware of the various understanding of learners could deliberate flexibly and make decisions about their future actions. Research on the knowledge base was the springboard for subsequent emphasis on situative analysis and the more recent idea of affordances. In part of mathematical knowledge for teaching, specific understandings of mathematics, prospective and forewarning, tailored were considered affordances for teaching situations.

The problem of integrating mathematical and pedagogical thinking has came in researchers attention also as a matter of how teachers come to transform their mathematical knowledge and construct an integrated one. Silverman and Thompson (2008) gave special attention in their cognitive framework to what has been called by Simon (2006) key developmental understandings in mathematics and remarked that, first teachers need to make a transition from mathematics understanding to such key mathematical understanding. Eventually to make connections with the pedagogical context and bring that to teaching action, teachers
need a reflective awareness towards students’ understanding. Based on a Piagetian conceptual cognitive model, Silverman and Thompson’s framework are among the few models to support pedagogical content knowledge formation in teacher education. Still they do not address teacher response in classroom interactions and, as they say, “is axiomatic that teachers’ knowledge of mathematics alone is insufficient to support their attempts to teacher for understanding” (p. 499).

**Situative approaches: high-leverage practices.** Pedagogical content knowledge has proved to be a good conceptual ground for understanding the knowledge necessary for teaching. Difficulties in trying to transfer it in teacher education showed that it is not easy to transfer it to teacher education and it requires a more refined version:

Framing such issues, Ball explained:

Here lies a fundamental difficulty in learning to teach, for despite its centrality, usable content knowledge is not something teacher education, in the main, provides effectively. Although some teachers have important understandings of the content, they often do not know it in ways that help them hear students, select good tasks, or help all their students learn. (Ball, 2000, p. 243)

In relation with learning mathematics teaching, Ball spoke of a second problem:

This is the problem of how subject matter must be understood to be usable in teaching. [] Viewed from the perspective of practice and the actual work of teaching, at least two aspects seem central. First is the capacity to deconstruct one’s own knowledge into a less polished and final form, where critical components are accessible and visible. This feature of teaching means that, paradoxically, expert personal knowledge of subject matter is often ironically inadequate for teaching. Because teachers must be able to work with content for students in its growing unfinished state, they must be able to do something perverse: work backward from mature and compressed understanding of the content to unpack its constituent elements. Knowing for teaching requires a transcendence of the tacit understanding that characterizes and is sufficient for personal knowledge and performance. (Ball, 2000, p. 245)

To be able to engage and experience such aspects, teacher education should situate novices’ experiences in settings in which mathematical knowledge is situated in pedagogical contexts. Cooney (1999) in his study of teachers’ ways of knowing makes one of the first arguments to situate teachers’ learning in a pedagogical context which may give teachers the
opportunity to experience an integrated form of knowing. Positioning teachers learning in a setting which brings mathematics within pedagogical contexts has not been a sufficient condition. Issues exists such as differences in teacher’s role in classroom discourse, a compatibility of settings, (Borko et al., 2000) and conceptualizations and understanding of how teachers learn in teacher education (Davis & Simmt, 2006)

Leinhardt et al. (1995) remark on how practices and a specific analysis of it could be helpful for student teachers and specifically for teacher educators approach. When describing one way to engage in specific cognitive actions of learning mathematics teaching they conclude:

Our example shows how student teachers might integrate and transform their knowledge by focusing on multiple examples of real practice by observing, predicting, critiquing, generating, and analyzing various components of practice. While doing so, student teachers are asked to attend and to integrate conceptual, procedural, pragmatic, and theoretical ideas and issues. They are called on to see theory in practice and to see practice in theory. The knowledge annotations and integrations that result serve as affordances in their professional knowledge which will help them to generate, understand, and revise practice in the future. (p.407)

In their study, Hill & al. (2008) explained why fore-knowledge as Ball advances is crucial for teacher instruction and action.

And critically, many focused on illuminating the construct if teacher mathematical knowledge rather than on quantitatively examining the strength of the relationship between teacher knowledge and the quality of instruction. In fact, many influential studies, including Ball (1990), Ma (1999), Post et al. (1991), and Simon (1993) used logical claims rather than direct observations to make the case that this knowledge is critical to instruction. (p. 433)

Training should “highlight the affordances strong mathematical (and related) understandings create for classroom culture and instruction” (Hill & al., 2008, p. 433).

The above constructs refer to the influence of integration on teacher’s actions excluding the position that teacher’s actions are part of classroom interactions, but treating them exclusively as individual forms of enactments. But the approach as affordances which Hill advances is rather a transfer from situative theory of affordances in student learning space.
Affordances are rather properties of the environment which stimulate and are in connection with student perceptual learning. Teacher integrated knowledge supports teacher activity to create such affordances in the learning environment for students learning (as part of the perceptual learning). The theory of affordances is consequently extended to studies of teacher learning. At this level, novices should need to be aware of the mathematical and pedagogical affordances to be better prepared for later teaching.

Summary – What is Missing From the Arguments for the Prevailing Rationale?

This chapter provided various arguments for quality of teaching action being linked to a necessity to integrate the teacher’s mathematical and pedagogical thinking. However, the three main arguments leave uncovered certain empirical space as to how the integration of mathematical and pedagogical thinking is present during the teacher’s interactive thinking in classroom mathematical interactions. In fact, one of the arguments for integrated thinking in teacher education refers directly to how hard it is to realize such integration during classroom interaction and how it is better to provide teachers and future teachers with already integrated forms of thinking, which could be used in teaching practice.

The nature of interactive thinking and the ways integrated knowledge is conceived need to be aligned with the ways teachers act during dialogic interactions. More recently with such integrated knowledge it is assumed that teachers would offer a classroom environment with affordances for students learning, ensuring the quality of teaching actions. Still how teachers would use such knowledge to provide conditions for learning remains a research question. Just a simple integration of mathematics within pedagogical context could prove limiting to teachers perceptions of classroom interactions or could provoke a paralysis because of the strangeness of
the new teacher of classroom interactions. A recent approach considers equipping novices with training on high-leverage practices in which mathematical and pedagogical integration are implicitly exercised. Liston et al (2008) mark on the importance of the enactment of the core practices for teacher education:

For several years, a number of researchers and teacher educators have explored shifting the focus of teacher education curriculum from one built around knowledge domains to one organized around interactive teaching practices. They have begun the work of identifying high-leverage practices – such as probing students’ thinking or giving instructional explanations – that might be at the center of a teacher education curriculum. Conversations about core practices for effective mathematics instruction, and about how best to prepare teachers to enact them, offer a fresh approach to program coherence. (p. 387)

Regarding the enactment of high-leverage practices Grossman and McDonald (2008) and Lampert (2010) remark how it is a matter of craft knowledge and of what it means to learn from practices. In her study, Ghousseini (2009) emphasizes the importance of classroom discourse routines and the reorientation of teacher education curriculum on pedagogies of enactment. The existent gap in teacher learning between integrated knowledge and acting in classroom interactions has claimed more and more attention from the research community.

More specific teacher education approaches are considered by Leinhardt, Young, and Merriman (1995) in their analysis of a unit of Margaret Lampert’s lesson, attempting to make specific what was called: “Selection of focus components can address salient student teacher concerns, overlooked pedagogical actions, or driving theoretical issues” (p.407). They detail farther:

A task for researchers, such as those contributing to this issue, is to determine the critical components of practice in their respective domains, to further examine not only the content of professional knowledge but also how it is organized in use, and to consider the particular location that resist or would most benefit from knowledge integration” (p. 408-409).
We could interpret a tendency of current research on situative approaches of teacher learning to address and understand better the learnability of mathematics teaching. How much could a setting similar to that of classroom teaching support teachers’ transfer of what they learn in the setting of teacher education?
Chapter 3

Conceptual Framework and Issues

In Chapter 2, I reviewed the literature in mathematics teaching research to illustrate how the relationship between teacher mathematical and pedagogical thinking has been directly or indirectly investigated within the context of classroom instruction. Through the aims and methods of each mentioned study, researchers advanced the idea that the relationship between mathematical and pedagogical thinking is a necessary condition for teachers’ classroom actions.

The proposal became so pervasive and intuitively appealing that research reoriented to identify situations and associate them with specific relationships between the teacher’s mathematical and pedagogical thinking. Most of the ground of such initiatives exists in observations such as that particular pedagogical strategies need adaptation to mathematics specificity. Still, certain general principles of pedagogy could be preserved as was the case with the Sherin et al. (2004) model of Fostering Communities of Learning. When we look closely at this amalgam of pedagogical strategies and mathematics we see the temptation to apply particular constraints to particular pedagogies, as in the case with the participatory framework in the Sherin et al. study.

The arguments for integration of teacher thinking have not shown that it benefits teacher actions during interaction with students. We have had propositions in which instances of integration are envisioned as teacher abilities attuned to specific constraints in the environments of mathematics teaching situations. In such cases, the particularities of situations are made explicit and the specific action required responding. It is to this point that I turned my attention, to attempting to clarify the rationale for integrating the teacher’s mathematical and pedagogical
thinking in teacher education, particularly as necessary (implicitly determinative) for the
teacher’s subsequent classroom action.

The arguments researchers voiced about the relationship between teacher mathematical
and pedagogical thinking made this a matter of both practical thinking and interactive thinking.
However, to make it a usable relationship they introduced the metaphor of integration to
investigate its usability in teacher’s practical thinking.

In this Chapter I will show that this kind of perspective on the relationship between
teacher mathematical and pedagogical thinking is more attuned to a macro-perspective on
teaching situations. It is characterized by an objectification of the “integration” metaphor and a
rationalization of the meaning of teacher action. Due to the gap in teacher comprehension
between knowing and acting in classroom interactions, this macro-perspective poses a dilemma
for novice thinking and teacher educator activities: Is the relationship between teacher
mathematical and pedagogical thinking a static or a dynamic relationship? And how does this
relationship adjust to the teacher learning?

Even when based on qualitative methods, much of the research has been oriented to a
macro-perspective for interpreting teacher situations. It provides little understanding of the
compatibility between the experiential nature of the teacher’s thinking and hypothetical
integration of mathematics and pedagogy. I offer here a micro-perspective of interactional
situations that gives us a different insight into the existence of integration. I close the chapter
with issues of these conceptualizations of teacher classroom thinking, issues that guided the
research design of the study.
The Macro- and Micro-Research Perspectives of Mathematics Teaching

A macro-perspective of what one says and how it is said means avoiding the individual and particular details that can derail explanation. A macro-perspective is a perspective that offers an understanding of how a feature works in general, a larger perception of the whole mechanism. Such perception of the mechanism can have powerful explanatory value. We say that a macro-perspective of a mechanism provides general meaning of what is happening. “Macro-interpretation is making meaning in terms of what large groups of people (or machines or other bodies) do, such as choosing a president, preparing for college, or nursing infants.” (Stake, 2010, p. 39)

In contrast, a micro-perspective zooms to a part of the mechanism, not to understand better the larger mechanism but because of interest in the small details in that mechanism. It works especially to see how the larger mechanism is represented in individual experience. Sometimes it seeks what the individual experience tells us about the larger mechanism.

Micro-interpretation is giving meaning in terms of what an individual person can experience, such as climbing a particular tree, or listening to the opening movement of a concerto while driving home, or becoming acquainted with the cooking course your friend took. You might think of it as a single instance, something like a single “measurement,” however complicated, in the form of human experience. (Stake, 2010, p. 39)

Most of the time, a macro-perspective engages general meanings of things. It is concerned with explaining why things happen in general and the consequences. It often is a perspective oriented toward providing guidelines for future actions. It is a more realist approach of research, “Moreover, the kind of outputs that this research approach seeks to deliver are precisely those demanded by ‘users’ in the community, who seek immediate practical payoffs from social science research” (Silverman, 2003, p. 345).
A micro perspective is oriented to understanding the detailed experience of each structure and thus can seldom offer broad direction for action. It accounts for the experience in particular action. The benefit of it stands especially in highlighting individual experiences, close-in perspectives, and thus diversity in the experiencing of a situation.

There is a dialectic between the macro and the micro perspective, and researchers would like to explore it, but we should do so with caution. Robert Stake (2010) tried to make us aware of problems of aligning the micro and macro interpretation of social and interactional problems:

It is easy to think of these two, the micro and macro, as shading into each other, from small number of experiences to large, but it is difficult to get to the general knowledge from particular knowledge, no matter the number of people involved. Patterns of immigration are not easy to learn by studying individual immigrants. Is there gradual shading or a discrete change from general to particular knowledge? And from particular to general? Something to think about. (p. 40)

If we are to think about the relationship between teacher mathematical and pedagogical thinking we definitely imagine it a personal aspect, but mostly experienced in a public space. When viewed as quality of teaching action, we are tempted to think about it from a macro perspective, rather than from the experience of interactional thinking. What does quality of teaching demand from a teacher? Broader meanings attributed to teaching action and subscribed to a general way push more toward a macro perspective on mathematics teaching. What kind of knowledge and how to hold such knowledge (Ball, 2000)? The macro-view accounts for issues and problems of interactional thinking; it may offer solutions for them, and propose initiatives for a teacher education based on practice, but the experiential immediacy of dialogic interaction, is indistinct in the macro-perspective. The macro perspective stands to see quality of teaching through a rational network of consequences and effects due to knowledge. One holds it to be little affected by the experience of the situation, little developed by classroom interaction. Some may think to relate teacher knowledge with the developing situation. In doing so, they usually
keep within a macro perspective because they are devoted to the mechanism of the large picture of mathematics teaching and what may explain it (thus the rationalist view) as a good mechanism. This study is more concerned with probing, in micro perspective, the fine grain relationship between teachers’ mathematical and pedagogical thinking instead of simply acting on the unsubstantiated premise (or rather rationalistically done) that such a relationship is determinative for the quality of teaching action.

It is parallel to the Guida deAbreu (2000) micro perspective on social and cultural contexts. The micro is characterized by the immediacy of the interaction: “the immediate interactional setting where face-to-face interactions take place. The macro-context is used to refer to non-immediate interactional settings.” (p.2) He explained how this position is important for learning and knowledge:

In my interpretation various of these theories show a concern with the issues I raised in this paper. That is, they call for conceptions of human learning and uses of knowledge that go beyond the mastering of skills to incorporate a construction of self. In doing this they put also into perspective the need to pay more attention to the constitutive role of social order, in the valuing and establishing rules for the use of cultural knowledge. And, more attention to the fact that these can shape human action and interaction by being present either physically or symbolically. (p.23)

There is thus an important difference between seeing classroom interactions out of the immediacy of their interactional setting and it is important to see the relevance of a sequential analysis of classroom dialogues. The immediacy of interaction is especially important for understanding the meaning and quality of individual teaching action. One example is in the way we understand the utterance, “I am sick,” as a response to the question “Can you meet me tomorrow?” It changes the direction of the dialogue, reducing the possibility of meeting, but also provides a certain perspective on how to elaborate upon the quality of action. This immediacy of interaction—corroborated in the occasioned context of individual experience—illustrates the meaning and the quality of mathematics teaching action.
Much research on mathematics teaching has seen the relationship between teachers’ mathematical and pedagogical thinking in a macro-interpretative view. Often the findings of the study have been advanced as particular activities of teacher education, incorporated with teacher classroom action. They have found expression and articulation into conceptualizations aligned with a macro perspective on how such relationships find correspondence in teachers’ interactional thinking during teacher-students dialogues.

Next I’ll describe how two aspects which characterize the macro perspective have been omnipresent in the conceptualizations of this relationship and played a major role in the design of teacher education. I refer to the objectification of thinking by using the word integration, and how the relationship is explained through the rationalization of a teacher’s action.

**The objectification of teacher mathematical and pedagogical thinking.** It is interesting to note how the relationship between teacher mathematical and pedagogical thinking, either as a pedagogical content knowledge construct or mathematical knowledge for teaching construct, became an object in the vocabulary of teachers and researchers. Gess-Newsome (1999) remarked in her discussion of pedagogical content knowledge: “In addition, PCK is an intuitively appealing construct, one that has been actively incorporated into the vocabulary of many teachers and researchers alike.” (p. 10)

The relationship between teacher mathematical and pedagogical thinking started to be conceived as something that teachers have: a mathematical knowledge for teaching and a pedagogical content knowledge. The metaphor of integration gradually took the place of the coexistence of teacher mathematical and pedagogical thoughts. Novices’ failure to respond to particular teaching situations became a lacking of pedagogical content knowledge or mathematical content knowledge. This objectification postulated a correspondence between such
kind of knowledge and the teaching situation. As I claimed in Chapter 2, such a correspondence has had little empirical support.

Although the process which develops such knowledge is studied and recognized, often it is detached from what any integration realized at the very end, an integration experienced teachers would have. The relationship between mathematical and pedagogical discourses became an object of learning to be acquired to make teaching more effective. The relationship between teacher mathematical and pedagogical thinking thus became an object.

Such a hypothetical relationship between teacher mathematical and pedagogical knowledge does not resolve or even advance queries as simple as, “How do teachers use such knowledge in their practices?” From such research studies how could one inform the activities of teacher learning?

Even when mathematical and pedagogical knowledge are conceived as a matter of private individual knowledge and understanding, the conception of relationship becomes an object as a result of teacher thinking. In a study of “Teaching in Context,” (2000) Schoenfeld’s model of predicting teacher action from goals, mathematical preparation, and beliefs explains shortcomings in novice teaching as a lack of pedagogical content knowledge that would be mostly acquired in the experience of teaching. Using metaphor seems a natural consequence of using cognitive models of thinking.

In the case of situative perspectives, integration is postulated as those abilities which respond or attune to particularities and constraints in the situation. What in the cognitive perspective was about thoughts influencing actions, in the situative perspective is about abilities dependent on “attunement to constraints.” Both perspectives framed in the macro picture transform the relationship between mathematical and pedagogical thinking into an object outside
classroom interactive discourse. We could think of environmental constraints as a result of such classroom interactions, but the object of integration is still outside the classroom interaction. It is simply a matter of those abilities in activity. One tendency in this situation is to make visible such “awareness of particulars” (Polyanyi, 1958), and that means the special skills required to respond and thus the object of integrating mathematical and pedagogical thinking:

It is ironic that attending more to the detailed and intricate nature of practice is seen as in tension with respecting the entailed professional skill. Part of this resistance is due to the view of teaching as improvisational. But, part is inherent in the nature of expertise: At least some of the knowledge and skill wielded by experts is tacit, and not all practitioners are able to make the understanding and reasoning that guide their actions visible to others (Polyanyi, 1958). One challenge involved in centering teacher education in practice is careful deconstruction and articulation of the work of teaching with an eye toward making the most detailed elements of instruction learnable without reducing teaching practice to an atomized collection of discrete and unconnected tiny acts. (Ball & Forzani, 2009, p. 507)

**The rationalist conception of mathematics teaching action.** Another aspect of seeing mathematics teaching in a macro-research perspective is to understand teacher action through a rationalist perspective. I will use here Hindess’s example (as cited in Kemmis, 2004) to describe a rationalist perspective of action:

Rationalist epistemology conceives of the world as a rational order in the sense that its parts and the relations between them conform to concepts and the relations between them, the concept giving the essence of the real. Where rationalist epistemology presupposes an a priori correspondence, a pre-given harmony, between ideas and the world, the rationalist conception of action postulates a mechanism of the realization of ideas. For example, in Weber’s conception of action as ‘oriented in its course’ by meanings, the relation between action and its meaning is one coherence and logical consistency: the action realizes the logical consequences of its meaning. (p. 8)

As cognitive aspects of thinking postulated in the relationship, actions reflect teacher thoughts. And its situative perspective attempts to assign relationship between mathematical and pedagogical knowledge to a kind of “practical professional knowledge.” Encountering an individualistic perspective the rationalist conception of action disregards those parts that are consequences of an individual teacher’s meaning. They are seen as part of the ‘store’ of
knowing, happening to be engaged in the interactional classroom discourse. That meaning has been distorted. It is exactly to this challenge that the teacher needs to respond, especially the novice teacher. One can imagine such distortions as constraints to which teacher abilities need to be attuned. The macro picture has a tendency to see such abilities as skills and as part of thinking, somewhat integrated thinking, and their objectification as affordances in the environment. It comes to be a matter of valuing how one makes concrete something in continuous change according to the nature of environment.

It is to this point that I turn to explain a forced compatibility between the milieu of teaching and milieu of learning mathematics teaching. This compatibility imposes a static image of the relationship between mathematical and pedagogical thinking, excluding the reflexivity of the participant engaged in practice.

**Mathematical and Pedagogical Thinking as a Static and Dynamic Relationship**

One of the challenges of modeling curriculum in teacher education and especially enacting activities for learning mathematics teaching exists in the nature of the action of teaching, especially to imagine it in all demands of the interactional context. Activities in teacher education make sense for teacher learning if they become relevant to the teacher’s experiences. But there is more than that. They become relevant especially when it is more than just solving a problem. It is about the compatibility of the teacher learning milieu with the experience of teaching.

We can imagine the relationship between teacher mathematical and pedagogical thinking due to its objectification and rationalist ways of teaching action--to be a matter of static enactment. Still, we have the question that enactments in teaching use the relationship as a
resource, as part of the enactment, something which has not been defined in a way to influence
teacher action.

The teachability of mathematics (those pedagogical particulars in the ways mathematics
could be taught) is not altogether compatible with the learnability of mathematics teaching. I
refer to learnability in the sense Ball and Forzani (2009) described it in the above quote.
Certainly we might avoid the shortcoming Ball and Forzani referred to as “atomized collection,”
if we see the relationship in a static way, in the sense that those relationships between the
teacher’s mathematical and pedagogical thinking could be inflexibly carried on. To create such
affordances for teacher learning means to ignore teachers’

reading of the situation as it unfolds in and through practice, in the light of changing
perceptions, observations and ways of seeing the situation, and in the light of changes
brought about by seeing how others see it, and how they are reacting and responding to
changes as the situation unfolds. (Kemmis, 2004, p. 2)

It is to the last two Kemmis points that I say that the relationship between an individual
teacher’s mathematical and pedagogical thinking requires more investigation. Assuming that
this relationship influences teacher action means also that it has a dynamic nature. The
mathematical and pedagogical discourses of novices make us wonder how classroom discourse
could be characterized in an integrated perspective.

The milieu in teacher education does not allow an experience of independent actions.
When rationalistically approached in the milieu of teacher learning, integration may face a static
or dynamic milieu of learning.

If we speak about teachability of mathematics, then learning means the learnability of
mathematics teaching and the dynamic aspect is generative for teacher learning. However when
we start talking about learning mathematical knowledge for teaching the character of
understanding is subsumed by the norms of mathematical discourse. As Steele (2005) pointed in
his article, the nature of novices’ discourse is different and influences differently the milieu of learning mathematics teaching.

“Discussions about pedagogy were similar in many ways to the discussions of mathematics, with similar percentages of claims and evidence and similar amounts of support for claim. However, pedagogical discussions contained conversational turns that were denser…” while “Discussions about mathematics featured greater depth, more challenges and echoes …” (Steele, p. 326)

Why a Micro-Perspective on Teacher Classroom Action?

In the previous sections I illustrated how most of the research integration of teachers’ mathematical and pedagogical thinking has been characterized by two aspects: integration has been constructed outside the classroom and thus objectified, and most of the discussion about interactional thinking and integration has relied on rationalistic interpretations of teaching action. This attribution however is problematic because research on classroom interaction also conceptualizes interactive thinking during classroom exchanges as based on metaphors and analogies (Hashweh, 2005).

I will pose in this section a different argument about why a micro-perspective may bring a different understanding of teacher interactional thinking in classroom dialogue, one departing from the macro-perspective on classroom and its disposition to provide explanations. Such a micro-perspective may respond to the quandary of teacher learning raised by the two approaches: structural and improvisational thinking in classroom teaching.

**Experiencing classroom mathematical situations.** One of the critiques of the macro-perspective on classroom teaching situations has been that it sees a situation through its general features instead of attending to individualized personal experience. It is partly because of this absence of experience that teacher learning is mostly based and tentatively designed on general concepts of teaching situations (as affordances and constraints). But it is also because of
disregard of the new situations. In taking a micro-perspective of teaching situation, I was trying to understand the experience of a teacher during classroom interactions; I was trying to recognize situations of teaching that can be seen only through a closer look at this experiential complexity of the situation, not shading into potential interference from characteristics and variables that may describe situations in general.

The micro-view on teacher’s classroom actions departs from the tendency to see the meaning of a teacher’s actions exclusively through its consequential aspects, but tries to bring into the picture more of an experiential knowledge and a perception of the situation.

To find out if integration of mathematical and pedagogical thinking is actually present in classroom interactional thinking, such a micro-perspective departs from the tendency to invent rationalities and possibly superficial connections between teacher knowledge and instruction. In this way, I see the quality of teacher action given not by an external standard but by the occasioned context as interactions unfold in the situation. Stylianides and Ball’s (2008) study has been concerned with relationships between situations of teaching and mathematics, but their focus was not for teacher thinking as much as for hypothesizing particular important relationships between teaching mathematical topics and the situations involved.

It is interesting to note that such identifications are an:

approach to analyze the mathematical demands of mathematics teaching, but also core mathematical ideas potentially important for teachers to know, aiming to understand, describe, and generate hypotheses about how, where, and in what ways mathematical knowledge may be useful in teachers’ practices. (p. 307)

To reveal if integration of mathematical and pedagogical thinking is present in classroom interactional thinking such a micro perspective departs from the tendency to invent rationalities and superficial connections between teacher’s knowledge and instruction. In this way, I see the
quality of teachers’ actions given not by an external standard but by the occasioned context as interactions unfold within the situation of the activity.

Teacher education practices emphasizing the integration of pedagogical and mathematical discourse lack the support that should exist according to current conceptualizations. We have a gap which novices need to address. Most often this gap as Bauersfeld (1988) remarked is a tension between a mathematical and a social logic for an “ideal teaching-learning process”:

The covert social structure of classroom action masks or supersedes the mathematical structures, which the teacher has in mind and which he has tried to stage, and which the student can construct only through the regularities of his own (internal and external) actions. In these situations, the learner’s adaptive efforts towards an acceptable use of mathematical symbols and language are bound to generate context- and problem-specific routines and skills rather than insight, self-confidence, flexible strategies, and autonomy. *The mathematical logic of an ideal teaching-learning process thus becomes replaced by the social logic of this type of interactions.* This, perhaps, is the core of the notorious school-generated failure of so many mathematical school careers. (p.37-38)

Achieving quality in mathematics teaching is a difficult matter. In this study, it is considered more complex than a function of variables independent of the situation. It needs to include a portrait image of what the teacher accounts for between the learning experience and recognition of student development, a portrait image in which descriptions are corroborated by actions and situations.

**Focusing the Research Question: Issues**

The scope of this section is to illustrate how we push certain assumptions in teacher education without a strong basis from research on teaching practice. In other words, a certain theoretical/technical stereotyping may be problematic for mathematics teaching when complex situations of teaching are encountered.
Actions in teaching flow from different sources, but the thinking individual teachers are faced with (thus forced to think themselves as students do) during those actions invites a micro-perspective of teaching. An action could be triggered by lots of forces such as reasoning why that analogy is appropriate or not in that place and time, but in the realm of structures and rational spaces, how the teacher moves from one to another has been insufficiently covered in the literature. At a macro level, researchers develop rationales for certain pathways of teaching. At a micro level, teachers react but also engage in thinking necessary to respond to student utterances. In this micro-perspective, the current study re-evaluates the idea of integration of different forms of teaching knowledge and raises the following issues:

*Does the integration of mathematical and pedagogical thinking support or compete with teachers’ personal experience of teacher-student interactions?*

*Is discipline-based integration (relationships captured in mathematics knowledge for teaching) determinative of teacher action or merely explanatory after the fact?*

*Is a teacher’s response guided more by listening to what students say or by what teachers are disposed to hear?*

*How does listening determine a teacher’s action?*

*Are instructional moves (actions) determinatively related to the integration of teachers’ mathematical and pedagogical discourses?*

**Determinative role of integration for teacher action.** The argument to have integration in teacher education is that it has a logical and determinative role in the quality of teacher action. A determinative role is based on rationalities and reasons for specific teaching, especially for what represents the quality in mathematics instruction. In the case of mathematical knowledge for teaching such relationships take the form of definitions of mathematical knowledge for teaching, mostly as the rules and norms accepted in mathematical discourse. Few pedagogical
rules are recognized. Those that are subscribe to a pedagogy created by mathematics as a discipline.

The norms of pedagogy, and its problems and issues, are severe, especially because a specific pedagogy influences the learning of individual learners. In his Foucaldian analysis, Popkewitz (2004) illustrated how such pedagogies tend towards specific social groups.

Integration as agent of explanation and reflection. I wonder about this dual character of integration, its explanatory use and its metaphorical one. We define integration, and then ask whether or not it is present in teacher action. Some see this tension as part of range: at one part some accept that integration is a matter of reflective thinking and thus is embedded in the explanatory nature of teacher action. Others attribute a different nature to integration. Is it participating in interactive thinking in a metonymical memory or a metaphorical combination?

Understanding this issue should tell us how teachers think in classroom interactions. If they support it, it means they offer resources for action, as in affordances. If they oppose it, we can see a dilemma of teaching, with solutions resolved in the rationalization about the practice of mathematics teaching and also in the reflective perspective.

It is interesting to think about how deliberation in active teaching takes place. The argument that novices need to integrate mathematical and pedagogical reasoning discourses is based on the assumption that such preparation helps them in future classrooms to ponder and deliberate. In this sense two aspects are considered, integration helps support the understanding of the teaching situation, and second, it allows flexible interactional thinking at the moment of decision. In the last case one can see a luggage of routines that change and adapt, at the same time seeing norms for understanding mathematical teaching situations.
I explained how the macro-perspective of classroom interactional discourse poses problems for teacher training because it objectifies and stands apart from the individual experience of real classroom discourse. Thus it diminishes the experience of the relationship between mathematical and pedagogical thinking. It is the tension between the two that needs to be acknowledged. It needs resolution for teacher education activities. But activities include interactions which in turn are influenced by specific perceptions. We could say that certain experiences are triggered by specific actions, (for example, discursive reactions). One may conclude that current efforts to overcome the avalanche of mandated constraints on teaching actions in turn affect discussions about the actions that trigger individual experience.

Summary

In Chapter 3, I have illustrated how researchers conceptualize the relationship between teacher action and thought. They bring different perspectives on how to develop activities for teacher learning. In the first case, teacher educators consider how to influence teacher action. In the second case, the nature of the milieu of learning brought a different perspective on what it means to internalize and enact mathematics teaching.

I showed that in both cases we have the quandary of teacher education maintained in teacher education. The tendency is to objectify something that is instead a matter of process and being. This objectification derives from macro-interpretation of classroom situations. Features of activities are poorly seen in individual situations. The meaning of teaching actions is not seen through individual experiences but through the conceptualizations of relationships between tasks and activities. This places the meaning of teaching action in the sphere of rationalism, where the meaning of teaching action is seen through association with the inevitability of its consequences.
I spoke of this approach as problematic. Research on actual teaching shows a different nature of interactional thinking. I have argued that it takes a micro-perspective to understand how teachers use their mathematical and pedagogical thinking during classroom interaction. The micro-perspective is characterized by an experiential perspective and is more individualized. The macro-perspective fails to give attention to the interaction of teacher thinking with student thinking. This interaction influences teacher action. I claimed that a micro-perspective will provide a closer understanding of teacher’s dialogues than that guided by rationalistic explanation.
Chapter 4

Methodology

This study questions a number of recent reform efforts of mathematics teacher education to support dialogue in the classroom, particularly those pursuing integration of the mathematical and pedagogical thinking of novices. These questions start with the premise that teacher action is grounded in specifics of the classroom situations. Some of these specifics probably can be captured and exercised in teacher education activities to improve the quality of mathematics instruction. Using a micro-interpretative perspective, I searched for insight into the assumption that integration of these two lines of thinking enhances the teaching of mathematics. To attain the micro-perspective described in Chapter 3, I used case study methods. I took the main units of analysis to be teacher-student mathematical dialogue. The center of attention was the problematic, complex, intrinsic relationship between teacher thought and action, between teacher knowing and doing.

In this report I present a qualitative study of teacher action, an investigation of the meanings behind the action, specifically as Thomas Schwanot called it, “the system of meanings to which belongs”:

To say that human action is meaningful is to claim that either its has a certain intentional content that indicates the kind of actions it is and/or that an action means can be grasped only in terms of the system of meanings to which it belongs. (Schwanot, 2000, p. 191)

In this chapter’s final section entitled Interpretive Frameworks, I illustrate my approach to understanding teacher action, particularly dialogic action in the classroom. I have drawn on three forms of qualitative inquiry: (a) interactionism, (b) conversation analysis, and (c) focal analysis. Instead of relying primarily on the descriptions teachers or observers give to their pedagogical activity, and the choices behind it, I aggregated their utterances and worked indirectly into their immediate intentions. I did not presume that what they said was what they
intended to say. Surely it often was, but sometimes surely, intentions were ill-formed and correspond to a deeper commitment. I wanted to see how good teaching occurred whether or not it was a rational construction. I wanted to consider whether or not teacher educators put too much stock in preparing teachers for rational dialogue in the classroom.

In this study I was using an alternative way to attribute meaning to teacher action, drawing inferences—subjectively but critically—from utterances. I was questioning contemporary objectification of teacher thinking. As I illustrated in Chapter 2, instructional dilemmas lacking experiential base can be problematic. And I claimed in Chapter 3, spurious description results from such objectification, ultimately diminishing opportunities for lifelong learning about teaching.

The questions raised in this study were based on how teacher thinking can be inferred by interaction among teachers, students, tasks, activities, mathematics and pedagogical strategy. For now, the key interactions are between thinking and action. I did not use dialogic utterances as a variable of a larger perspective on teacher knowledge, but considered them as interactions shaped by the immediate situation. I tried to avoid the more common rationalizations of teacher action, those often called “professional practical knowledge,” those macro perspectives used in labeling teacher action. I shied away from idealizing teaching as integrating mathematical discourse into pedagogical discourses in the context of classroom interactions—even so, I looked for evidence of such integration.

The issues developed in the last section of the previous chapter indicate how such rationalizations happen in mathematics teaching. The issues showed up partly as teachers build their narratives about teaching activity. The few studies I found of experienced teachers using “integrated” knowledge accompanied an interventionist approach. In this study I took a
naturalistic non-interventionist stance. I sought relationships between mathematical and pedagogical discourses in examples of superior mathematics teaching.

I tried to focus on empirical, directly visible, easily recognizable, ordinary experiences, thus to avoid hypothetical explanations of teaching action, and to avoid rationalistic description of the quality of teaching.

I decided to work with two teachers, one each at elementary and secondary levels, not for comparative purposes as much as for a variation of the complexity of situations. I had an interest in the secondary classroom especially because much of the research on mathematics teaching had been from elementary mathematics illustrating integrated, discipline-based teaching.

Below, it is a diagram describing the design of the study, followed by a description of methods. Recorded activities of teaching were the most important data collected. The activities were what the teacher did: interactions with students, interactions with tasks, and interactions with subject-matter (mathematics). And, of course, the ways these three interrelated and communicated with each other during mathematics classed. The issues were identified in the last section of the previous Chapter.
Research Question:
How can teachers use and relate their pedagogical and mathematical discourses in classroom interactions?

Issues: Does the integration of mathematical and pedagogical thinking support or compete with teachers’ personal experience of teacher-student interactions?

Is discipline-based integration (relationships captured in mathematics knowledge for teaching) determinative of teacher action or merely explanatory after the fact?

Is a teacher’s response guided more by listening to what students say or by what teachers are disposed to hear?

How does listening determine a teacher’s action?

Are instructional moves (actions) determinatively related to the integration of teacher’s mathematical and pedagogical discourses?

Figure 2. Research Design
Methods and Procedures

The research was first oriented towards the case studies of two mathematics teachers. My micro interpretive perspective of case work indicated a necessity for revising the local events in terms of social procedures. As indicated earlier, it was important to avoid constructing rationales for teacher actions and decisions, which I saw as the main characteristic of what I labeled as the research macro-view of teacher knowledge. I saw the teachers’ temptation to explain what happens in classroom teaching through general accounts of what justifies good teaching actions. I needed both methods of observations and analysis to give me access to teachers’ thoughts. The main artifacts were audio records of classroom interactions.

Toward the end of the study, for each teacher, I re-focused analysis on several lessons richest in dialogic excerpts. As explained in Chapter 3, the focus of this study was to maintain a macro perspective of classroom interactions and then to move to micro interpretation. The latter interpretation featured utterances as teacher actions. Socially, the students individually and as a group were given extensive attention.

Initially I considered the classroom as the case study because of the specific, unique learning context. Studying the classroom learning situation, I had the opportunity to see different facets of the interaction, instructor-students and student-student as well as to contemplate initial problems that novice teachers encounter.

Eventually, the necessity to have more detailed descriptions of how teachers exhibit mathematical and pedagogical thoughts in their classroom discourse made me realize the

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2 In the first stage of the study and the beginning of the second stage I had extensive discussion with teachers about their actions in classrooms. They sometimes “explained” their actions as interaction of their mathematical knowledge and their teaching knowledge and beliefs. But these explanations brought little insight into interactive thinking as determinative of classroom discourse. Subsequently, I used my interviews as a source for modification and challenge to what was expected. I hoped that interviews would offer also a perspective if interactions were a challenge in terms of how teachers interact their mathematical and teaching knowledge. Instead, as experienced teachers, they recognized modifications of their planned interactions, but did not necessarily connect those with a necessary change in their mathematical and teaching interaction.
necessity to audio record and transcribe classroom talk. The shift to these data was partly because pre and post interviews proved problematic, often offering a general rationale for teaching. It was important to know what they thought, but I had to consider it possibly misleading as to how the teachers were actually integrating mathematical and pedagogical thinking.

My research procedure was thus focused on the idea that communication manifested teacher thoughts more than that thinking was the drive forced behind communication. Departing from the macro-picture, I examined what the communication might tell us about teachers’ classroom organization of knowledge. I did not pay attention to how such objects of focus are created but how they are initiated and revealed by dialogic interaction.

**Classroom observations.** In the first stage of the data collection, I mainly observed the mathematics teachers. In case study research, observations are the main method for capturing experiences. Different from other case studies, however, this research did not intend to illustrate stories of teachers. However, I did find small stories which would be called “patches.”

I used the patches to record peculiarities of teacher action, the relationships within their specific situations and how I might subsequently describe “occasioned contexts.” Thus my final assertions about the issues developed around a highly selected collection of patches captured from classroom activities.

Patches were then my main way of capturing the observations of teaching. I placed them within a micro-interpretation perspective because I wanted “interpretive data.” I recognized that such patches do not make a conventional aggregation of data. They were purposive rather than random sample, selected because they were relevant to my issues. In his book, “Qualitative Research: Studying How Things Work,” Robert Stake (2010) described such patches:
In the field, some observation data are immediately seen as valuable. At an Indianapolis federal office, I asked an official if she thought of herself as a bureaucrat. She said, “I’m not a bureaucrat.” “What are you?” “I’m a Hoosier.” I knew almost immediately I would repeat that patch in my report. As I said earlier, I call those data that immediately, by themselves, seem relevant “interpretive data.” (p.91)

In my analysis, patches provided intriguing teacher action, illustrating dialogic interaction with students, tasks, the mathematics and the planned instructional strategy. The patches steered me to specific meanings of teachers’ actions, shaped by the commonalities and uniqueness of situations. Patches helped me capture a complex intrinsic relationship in teacher action.

**Interviews.** I started the data-gathering using field observations and pre and post lesson discussions with teachers. I intended to make both the teachers and myself comfortable. Although they were aware of my interest in their mathematical and pedagogical knowledge, I did not indicate to them my specific concern about the relationships between the two and their often-mentioned integration. I wanted them to talk about what they found important in the details of the day-by-day teaching. I assumed I would be able to record those moments when experienced teachers ponder over dilemmas and tensions. I wanted such moments to happen naturally without me prompting or probing. I wanted to minimize abstract rationalizations or explanations.

I specifically worked on moments when the mathematical discourse seemed to take a specific shape. Most of the time, my questions before or after the lesson would be general: “What would you teach about today?” “What do you think happened?” “Do you usually have specific concerns about this topic or lesson or students’ misconceptions?” Gradually we took deeper looks. I would point to specific moments in the lesson. I sometimes would ask what the teacher thought at that moment. In general, my first two years of observations were amicable, a joint effort at analysis. These data were complemented by student artifacts and my participation in two professional events.
Several of my interviews with teachers were audio-taped. For the less formal before and after class discussions I took notes. For the secondary teacher I audio-recorded, transcribed, and kept for analysis a total of 16 interviews and 25 lessons. With the elementary teacher, I kept 4 interviews and 7 lessons. The records of audio taped mathematics classrooms were rich in empirical detail. It allowed the repeated and detailed examination of particular events – sometimes, the interruptions of pedagogical reasoning- in classroom interaction.

I need to mention here that although my main interest was to see natural dialogues and observe what happens in such situations, I had a particular interest in what teachers would say about nonverbal happenings during classroom interactions. I tried to follow what Witz (2007) pointed as:

The interviewer is ever sensitive to the new aspects that the participant brings up, content or feeling that either goes beyond what is currently going on in the conversation, or beyond what would be normally expected as an answer to a question. It is these new elements that originate from the interviewee (or that the interviewee originates, not the interviewer) that lead to the ‘existing structuring’ that the investigator begins to see in the participant’s subjective world. (p. 91)

Below is a list of the discussions and recorded interviews I had with teachers before and after their lesson. The table indicates the pseudonym for the teacher and the subject taught. I named the tasks involved and if specific “patches” were noted at the end of the lesson. Keeping track of these patches helped me connect them with subsequent and previous year observations of possibly the same lessons. Most often the teacher used similar strategies and mathematical tasks for teaching.

Interviews were divided by two approaches: informal discussions and semi-structured interviews. Informal discussions were aimed, especially in the first phase of the study, to build bonds with teachers and better understand their mental activities. Semi-structured interviews aimed to a more precise and refined dialogue. I attempted to re-evaluate the boundaries of the
spaces in which teacher thinking moved. The semi-structured interviews were conversations trying to recall teacher’ interactive thoughts with their students: “How did you realize that it was a mistake in the way students reasoned?” The question is a question triggering both teacher reasoning and thinking. Teachers’ reasons for next action were also considered to keep connected with the larger view, but the focus of this study was on the thinking with which teachers engaged while teaching and if that required or was based on integration. The new relationships the teacher had established, while derailing his pedagogical analogies, were active. This indicated not only the effectiveness of his pedagogical analogy but also teacher capability to engage his thinking in teaching.

Audio- and video-recordings. In the third stage of data collection, which was also the third year, I decided to triangulate my data and also audio and video record lessons. I asked permission from both teachers. With letters of consent approved and signed (see Appendix F) I engaged the teachers in deciding which lessons they would feel comfortable having taped.

During the audio recording both the elementary and the secondary teacher carried a small audio-recording device in their pocket or around the neck. The purpose was also to record quiet conversations with students, especially those working in groups. Most of the times during the first two stages of the data, I was limited in my data collection because I was in another part of the room from where the teacher was talking. Later I would find out he used what those students said but I had not captured the dialogue. I would ask the teacher about that particular context still I was limited in my understanding of the situation. Asking the teacher afterwards was problematic because most of the times the teacher would say that there was nothing more than what he would later tell all students. It was a question of accuracy of my data and my
interpretations. This was especially for the case of the secondary teacher, who often would spend almost half of lesson interacting with students in groups.

Video recording gave me more chance to observe how the teacher moved about and used the board. It made a difference to what I was studying. Although video brings a better picture of the instructional setting, I was interested in teacher and students interactions. Having already analyzed mainly verbal dialogues, I decided to include their analysis, too. Still, the video brought another level and helped me triangulate data.

Through the process of transcription, I had access to a wide range of interactional episodes, to be inspected for alternatives and similarities. The system of transcription (see Appendix A) was used to reveal the sequential features of the talk, and the sequential production of talk-in-interaction. Sometimes it was important for the record to show that student and teacher talked at the same time, paused, looked away, or laughed.

Table 1

List of the Audio- and Video-Recorded Lessons

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Topic and Activity Observed and Recorded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mike:</td>
<td>Refresher: factoring polynomials; “algorithm” for factoring; tie-in to zero product property and factor theorem</td>
</tr>
<tr>
<td></td>
<td>Introduction to Rational Functions; definition, domain, asymptotes. Activities: 1) students examine three functions, answering questions about domain, zeroes, end behavior, then graph; 2) students examine 6 functions, exploring relationship between degrees of numerator/denominator, ldg coeff of num/den, and horizontal asymptotes, try to write down explicit rules for determining whether a rational function has vertical and/or horizontal asymptotes [spoiler: one function has a removable discontinuity]</td>
</tr>
<tr>
<td></td>
<td>Review rational function “theory” from 2/14; begin work with fractions (multiplication/division); “Pizzazz” activity</td>
</tr>
<tr>
<td></td>
<td>Review activity Modeling with rational functions/expressions; finance option</td>
</tr>
</tbody>
</table>

(continued)
<table>
<thead>
<tr>
<th>Teacher:</th>
<th>Topic and Activity Observed and Recorded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intro to periodic functions: <strong>“Bouncing Into Trig”</strong> (Activity 1)</td>
<td></td>
</tr>
<tr>
<td>Brief recap: period, amplitude, axis of oscillation. Physical aspects of circular motion that affect each. Translate from circular motion to linear motion (speed).</td>
<td></td>
</tr>
<tr>
<td>Data collection: <strong>“Rolling Into Trig.”</strong> Students collect data about height of “dot” on a can vs. distance rolled; divide by the radius (to begin the idea of radian measure)</td>
<td></td>
</tr>
<tr>
<td>Radian measure; introduction to unit circle, with horizontal and vertical positions. Introduction to sine and cosine functions. Activity: <strong>“Ferris Wheel Frolics”</strong></td>
<td></td>
</tr>
<tr>
<td>More with radian measure; angular velocity. Students begin study of “basic” sinusoid graphs, then explore parameters in ( f(x) = A \sin(B(x - C)) + D ); relate parameters to vocabulary (axis of oscillation, amplitude, period, introduce <strong>phase shift</strong>).</td>
<td></td>
</tr>
<tr>
<td>Recap of yesterday’s activity, radian measure. Intro to amusement park project.</td>
<td></td>
</tr>
<tr>
<td>Review of parameter work; “fitting” sine functions; function aspects of trig functions (based on unit circle definitions).</td>
<td></td>
</tr>
<tr>
<td>Review function aspects; discuss symmetry issues. Amusement park project: students choose projects. Work on exact values of trig functions for “reference angles” in all four quadrants.</td>
<td></td>
</tr>
<tr>
<td>Graphing reciprocals</td>
<td></td>
</tr>
<tr>
<td>Law of Sines</td>
<td></td>
</tr>
<tr>
<td>Proving trig identities: examples, then two levels of student work</td>
<td></td>
</tr>
<tr>
<td>Trigonometry of sound waves</td>
<td></td>
</tr>
<tr>
<td>Superposition and Damping</td>
<td></td>
</tr>
<tr>
<td>Harvey:</td>
<td>Ratio topic – Gas per mileage activity</td>
</tr>
<tr>
<td>Carnival activity</td>
<td></td>
</tr>
<tr>
<td>Binary system – measurements</td>
<td></td>
</tr>
<tr>
<td>Geometric Shapes</td>
<td></td>
</tr>
<tr>
<td>Fractions and Decimals</td>
<td></td>
</tr>
<tr>
<td>Mystery Activity</td>
<td></td>
</tr>
<tr>
<td>Valentine Day activity</td>
<td></td>
</tr>
<tr>
<td>The problem of the day</td>
<td></td>
</tr>
<tr>
<td>What is the Fraction?</td>
<td></td>
</tr>
<tr>
<td>How would you explain to a kindergarten student?</td>
<td></td>
</tr>
</tbody>
</table>
Participants Selection and Data Collection

Data collection and analysis of this study occurred in several stages, each marked by a particular position against the data and how I oriented the analysis. In the first stage, I was mainly interested to work with experienced teachers and focused on brief visits to the classrooms of about twenty different mathematics teachers.

Table 2

List of Visited Teachers, Topics of Their Lessons, and Activities During the Participant Selection Process

<table>
<thead>
<tr>
<th>Teacher School Level</th>
<th>Subject</th>
<th>Topic/Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Secondary</td>
<td>Algebra</td>
<td>Solving System of Equations</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Assignment Exercises</td>
</tr>
<tr>
<td>Secondary</td>
<td>Algebra</td>
<td>Solving Equations Problems</td>
</tr>
<tr>
<td>Secondary</td>
<td>Algebra</td>
<td>Binomial Theorem/Group working</td>
</tr>
<tr>
<td>Secondary</td>
<td>High-Level Calculus</td>
<td>Vectors and Integrals</td>
</tr>
<tr>
<td>Elementary</td>
<td>Ratio</td>
<td>Millage per Gallon Activity</td>
</tr>
<tr>
<td>Elementary</td>
<td>Division</td>
<td>Solving Problems/Proof</td>
</tr>
<tr>
<td>Elementary</td>
<td>Multiplication</td>
<td>Commercial Setting Activity</td>
</tr>
</tbody>
</table>
The second stage of data collection brought everyday routines of the classrooms of two teachers. The emphasis shifted to their way of reasoning through teaching practices, with interaction between their mathematical and pedagogical discourses. This gave me the possibility in the second and third years of observation of the very same lessons to remark on similarities and differences in these two teachers’ practices, also to relate such events to successive student cohorts. Later in this study I will present transcripts of fours such lessons: two audio-taped and two videotaped. The greater parts of the transcripts are presented in four appendices.

In the first year of the study, I visited several teachers’ classrooms trying repeatedly to see how I might gain access to their mathematics thinking. The main criteria for selecting the participants were twofold: a strong mathematical understanding. I especially looked here at a variety of mathematical understandings and also at any inclination to elaborate on their mathematical understanding. I expected that these teachers would be aware of the mathematical formalism behind the pedagogical transformation.

In regards to pedagogical knowledge, perhaps surprisingly, I did not want to study someone with pedagogical fluency as much as someone who would not shy away from possible incongruity between mathematical procedure and pedagogical strategy. I was looking for teachers who might “improvise” when a student did not respond to instructional routine or pedagogical metaphor as expected. I wanted the teachers to be articulated about the tensions in their teaching, anticipating issues and student misunderstandings.

I selected two experienced and respected teachers, one a secondary teacher, the other an elementary teacher. I tried to avoid the “analysis paralysis” a novice might experience facing a need to integrate mathematical and pedagogical thoughts during classroom interactions. The data collection for the second stage of the research occurred over a three year period. One reason for
being engaged with the study of the two teachers for almost three years was to develop a sense of predicting the immediate action of these two teachers. Comparing with previous teacher’s performances I also had the chance to notice differences in teacher’s actions and situations.

I saw two needs in the data collection: One a detailed and accurate record of the dialogues between teacher and students in the classroom; the other a record of the happenings behind the utterances. My specimens reflected complex lessons. They varied in mathematical topic and pedagogical strategy at both the elementary and secondary levels. I had some interest in how different the elementary and secondary procedures might be, but I assumed that the mathematical thoughts in both places would address the same difficulties in teacher thinking.

The elementary teacher. In May 2006, I asked a few faculty members I knew if they could recommend an experienced elementary teacher with a good reputation in mathematics teaching. Several of them named the elementary teachers I ultimately chose to work with. He was said to be a willing collaborator.

In our first meeting, after I described my research interests, he suggested that I observe his next mathematics lesson. He said the students were having difficulty using various meanings of "ratio" in everyday situations. He had designed a new activity to stimulate students' mathematical thinking in the context of car performance. I knew it would be unusual to find an experienced elementary teacher who not only mastered the mathematical content, but also elaborated his teaching routine.

The next day, I was surprised by his teaching experiment. In the milieu of his classroom, the lesson had students articulating their reasoning, arguing different views of a complex mathematical problem, and working collaboratively. The teacher impressed me with his ability to sustain a teachable moment. Any student withdrawal during a mathematical situation reoriented
his teaching discourse. His intervention focused the student's reasoning to the mathematical idea. By the end of the lesson, students were evaluating car quality as to gas consumption, speed, and price.

In an amicable discussion about a new cohort he would comment:

Luisa: Tell me about student misunderstandings.

Harvey: We are now in Unit 9, a unit on percents. This is a good unit for them. When they come to me in the fall they think: All correct on a test means 100%. Miss one, 99%. Miss two 98%. They are far beyond that now.

On one of my next visits, I invited a colleague, Robert Stake, to observe with me one of the mathematics classes. We witnessed students sustaining interest and effort, "doing mathematics," although this was the last period on a Friday. In subsequent conversation, I remarked to Stake how the teacher almost effortlessly transformed the routines of teaching into novelties of mathematics instruction. Analyzing this ability, we postulated three important qualities of his mathematics teaching: continuous awareness of his students' mathematical understandings, a large repertoire of instructional moves, and spontaneous creation of new experiences for mathematics learning.

I had the opportunity to observe him teaching three cohorts of third graders. They were students with a variety of learning styles. I could see how, in group situations, he engaged all the students. Simultaneously he challenged a gifted student and encouraged another anxious about adding fractions. In addition to his expertise in elementary mathematics, in early childhood pedagogy, and extensive experiential knowledge of teaching mathematics, he was particularly interested in developing new forms of inquiry into mathematics learning. Here is an example of his remarks in a small group of colleagues, anticipating a new learning experience:

Harvey: I thought I'd share with you some thoughts about the lesson for next Friday. You remember how I posed the question: If every square on a cube has an equal chance of falling on any side, will a pyramid with a
square base have the same likelihood of falling on its square base? If so, we should expect the probability of landing on the square to be 1 in 5, as there are 5 faces in all.

Harvey: I will review the circumstances as I define the problem. The basic question is: What are the chances it will land on the square base? I've made enough copies of the polyhedra and the children will work in pairs to gather our data. We will set a fair procedure for flipping the "dice" and the students will keep tallies until given a signal to stop. I'll collect the pieces and the groups will report their findings. We'll compile the data at the chalkboard. I will express our findings as a ratio represented as a fraction. Likely, I will do some rounding to make it a friendly fraction that can be simplified. From my point of view, this is where I either ruin the data or make it meaningful for them.

Harvey: A follow-up inquiry question is to ask them to check this at home by continuing their own trials with the pyramids they made last week. Some other questions I may pose for discussion:
Does it matter how they are flipped?
Would this yield the same results if the pyramid were taller or shorter?
What are the chances that it will land on a triangle?
What are the chances that it will land on a particular triangle? (What would you have to do to test this question? - Label it.)

Harvey: I have no idea what the answer to the basic question is. I would term this activity "inquiry" because I don't know the answer. The question is of interest to them, and they don't know the answer either. It's inquiry because they can understand the question, the question arose from prior activity, and we have a way to test the question and analyze the data to bring closure to our query. It's also a question worthy of pursuit because it leads to other questions. What is the likelihood we will answer follow-up questions?

Harvey: As I have mentioned, it will be a lesson about probability. I know where this lesson should go. But I am so busy preparing other things that I have decided not to structure too many aspects of the lesson. The framework is there and already enough preparation is done to accomplish our goals. But I am trying not to organize their record keeping. I plan to tell them what to do, but not set up all the charts they should organize. Ideally the only thing I should have ready is the graph paper.
**The secondary teacher.** The secondary teacher I worked with volunteered to participate in the study after my application for IRB approval. The person in the IRB office had sent a short copy of my proposal (see Appendix G) to schools and this teacher sent back word that he would like to participate. At our first meeting he made it clear that his pedagogical preparation was not formal. Instead he had 15 years of teaching experience and a mathematical background. He had, at a prestigious university, satisfied his doctoral requirements except for the dissertation. He proudly said that “he was bitten by the teaching bug” and decided to teach high-school. His entire teaching experience had been, for the last 12 years at the same high school where I observed his teaching.

Although he mentioned his lack of pedagogical preparation, after I first observing his class, I could not have been more interested in his teaching. His relationship with students could be thought of as authoritarian pedagogy. He wanted students to solve the tasks with minimal deviation from instruction. In this regards he would say.

Mike: So, sometime would it happen that students would say something and I’ll misunderstand and I’ll think the mathematical thinking is wrong and then I’ll discuss it with them, and I’ll figure out they were right after all...sometimes, I think, I don’t really listen, they start down a path and I think oh I know the conclusion of this and therefore I don’t quite listen that happens I try not to but it happens but it does...

His teaching was based on group work, interventions during student work and discussion following. He was enthusiastic about how mathematical patterns appear in nature. I saw this as characterizing his disposition to mathematics. He talked with enthusiasm about the origin of a word like “sinus” and also about huge elliptic patterns on land observed from above and related them with how mathematics “appears” in nature. He talked concise and to the point. Most of his emails were brief, except for with elaboration revealing his enthusiasm, concern, or interest for a problem. He motivated students to learn mathematics especially arranging assignments showing
mathematics in day-by-day activities. He would pay attention to details in the students’ mathematical thinking and remark especially on alternatives in approach. I was impressed with his capacity to follow deeper into a student’s affirmation, working toward understanding of concepts and allowing space for variation. This last feature showed a contradiction with research who would characterize authoritarian teachers to have limited mathematics understanding and lack flexibility in accepting various students’ mathematical discourses.

When I asked him why he volunteered to participate in my study he said that he had always been disposed to support research. Early in graduate study, when portable computers were just available, he participated in a pilot study about using laptops in mathematics work. While I was working with him during the third year, he also worked with a graduate student in computer science who created a software program to support teacher work with students. Because of this, many student artifacts were coming from recorded work of notebook laptops. During one amicable conversation I inquired what he expected from research on mathematics education. He said that he would like to find out more about why students forget instruction. It was obvious they could reproduce solutions in certain lessons, but then could not recall during testing or in other applications. Probing his interests, I found that he was interested in improving his methods of teaching to insure “mathematics learning transfer” in various situations. He saw the purpose of testing as different from those of daily classroom activities.

In my attempt to understand the thinking he engaged with in classroom interactions, he characterized himself and being engaged with mathematical thinking:

Mike: I guess, that’s probably mathematical thinking, thinking of how the aaa, how the pieces fit together and maybe verbally the pieces weren’t described in the way I was expecting but when I parse what was said and I think about how it should …how looks…I mean I …I am visual thinker, So I think about how staff looks, then I can put pieces together…
Analysis of Cases

The analysis for these data became microanalytic. Patches were recorded and transcribed. To pursue the relationship between teacher action and the knowledge of teaching, I especially sought alternatives in teaching. With patches reflecting what happens in teacher and student dialogues, I complemented the analysis with a description of the task and the mathematical and pedagogical roots of teaching. Alternatives led to specificity of situation and the experience of classroom interactions.

For triangulation purposes I balanced the dynamics of interaction to see if commonalities were present. Most of the times the issues in mathematics reveal that the social logic of interaction takes place gradually.

Previously I said that the focus of this study was on naturalistic perspectives in the classroom, thus my un-interventionist ways of collecting data and research. Here I will explain how I selected what I called “natural challenges.” These challenges were seen in patches supporting description and portrayal of the four classroom cases included in this research. I gave attention especially to those lessons where -- in my notes and observations -- I experienced a certain relationship with the ways teachers reconsidered their response to students. I went back through my notes and observation to find such patches, combined them with post-lesson discussions and came to understand them better. In a similar vein, I looked to see if the same appeared in the interactional analysis.

To repeat, this was differently from the usual narrative study of mathematics teaching studies. In this study I went over “patches” of teaching again and again, trying to discern the compatibility between novice teacher learning policies and teacher conceptualization. What I focused in my analysis was the ways in which teachers implemented classroom mathematical
tasks. In particular, I search to find procedural foundations for the teacher’s carrying out of planned implementation. I expected that through these procedural foundations I could gain a glimpse of how these teachers use their mathematical and pedagogical thoughts. However, what I focused on was not the procedure that guided their action, one step after another, with a prior action determining the next, but the way they categorized what was said. What features delineated their categories? How did they talk about what students said?

An important aspect of research methodology in this case, as indicated earlier, was to avoid eliciting self-perceived rationales for teacher actions or decisions. Such formalism was the main characteristic of what I called the macro-view of teacher knowledge. I needed it methods of observation and analysis that might give me more experiential and yet behaviorist access to teachers’ thoughts. The main medium became audio records of classroom interaction. Then, “conversational analysis” became the main tool to obtain information about the features (mathematical or pedagogical) used by teachers to categorize student responses. The interpretive methods to elaborate my understandings of the observations were, to be sure, subjective.

Analyzing the dialogic text, I looked for pedagogical metaphors and transformations potentially in tension with mathematical accuracy. I tried to analyze them through the eyes of the teacher, not so much what they meant for an external observer about good mathematics or a proper act of teaching. Instead I looked at how the teacher, in classroom conversation, seemed to “categorize” his thoughts. By “categorize” I mean the ways he organized or talked about the incongruity between mathematical and pedagogical reasoning. How did he extend his pedagogical metaphor?

Pedagogical metaphor here is not a generalization about teaching, not a common process of classroom teachers, and not a syndrome of activity inferred from pedagogical reasoning.
Instead, pedagogical metaphor here is a situated happening, giving local meaning by the particular interaction between these students and that teacher. It is important to know how the teacher continues that pedagogical metaphor in the classroom conversation.

And how did he talk about mathematical accuracy? Mathematical accuracy here is not concerned so much with the rightness of the teacher’s conception long-term reasoning, but with any immediate local conflict in mathematical terms. Classroom interaction needs to sustain the teacher’s sense of integrity in the curriculum. How did the teacher continue what he perceived as mathematical accuracy? In what “activities” did the teacher’s mind engage to respond in way that repairs incongruity between the teacher’s pedagogical metaphor and his mathematical accuracy? Do we have similar activities belonging to pedagogical or mathematical domain?

Could they be used also during the teacher’s improvisations? If so, what features of the teaching situation determined the teacher’s thinking? The concepts arising would became apparent through the categorization the teacher created at the moment for the particular teaching situation. This categorization offered two kinds of information: first, it showed the “discursive activities” teacher engaged with to respond to the situation; second, it allowed a chance to see the concepts the teacher employed.

**Episodes selection.** For episodes of teaching I selected ones that displayed “what is observably-the-case” about the dialogues. I analyzed what I called “tagging” and the versions called “labeling” (Leinhardt & Steele, 2005), “signifiers” (Bauersfeld, 1988), and “symbolic signifiers” (Sfard, 2001). The episodes illustrating the analysis of this study are separated, because they have been part of a larger unit of analysis, the mathematics lesson. Most of these episodes have been a part of the lessons of which transcripts are considered in Appendices.

Each set of data was to aid the interpretation of the classroom dialogue “specimens”.

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“Unlike data from the factist perspective, a specimen as a form of research material is not treated as either a statement about or a reflection of reality; instead, a specimen is seen as part of the reality being studied.” (Alasuutari, 1995)

What Alasuutari calls “factist,” David Silverman (2003) may refer as “realist perspective” on analyzing data, which mainly characterizes the macro-perspective of situations:

An alternative approach treats interview data as accessing various stories or narratives through which people describe their worlds (see Holstein & Gubrium, 1997). This narrative approach claims that, by abandoning the attempt to treat respondents’ accounts as potentially “true” pictures of “reality,” we open up for analysis the culturally rich methods through which interviewers and interviewees, in concert generate plausible accounts of the world (e.g., Gubrium, 1993’ Voysey, 1975)

It was with this approach that I attended to my analysis of the data.

**Triangulation.** The assertions for the issues raised in this study were triangulated by using different methods of data collection in the very same situations: observations, interviews, and recordings of classroom dialogues. I also engaged three analytical perspectives, especially because I wanted to ponder the alternatives for action represented by each of these analytical perspectives.

As for member checking, most of the paragraphs and reports were discussed with the teachers as I recorded the data. Later on, draft assertions were also discussed. When the secondary teacher asked me the purpose of my study, I said, “teacher learning.” He immediately commented that he would like to have student teachers come and assist in his classroom and if not immediately, afterwards to ask questions like I asked pre and post lesson discussion. It was interesting to encounter contradictory positions on how teacher actions were “determined” in classroom teaching.

On a different task, the elementary teacher commented that my presence in his classroom and my comments made him “stand on tiptoes” while teaching. He explained that he was continuously and eagerly anticipating that something would happen.
In general, both teachers indicated that they did the same routines and recognized differences among cohorts of students. The elementary teacher would pay special interest to differences in experiencing new activities with a mathematical highly demanding cognitive task. They both agreed that they planned an activity, laying out general steps of the activity, but planned no dialogue. They said that most of the time they expected dialogues to be shaped by student responses.

**My knowledge of teaching.** As a teacher myself, I did not think of a separation or a relationship between my mathematical and pedagogical thoughts. They just existed. In a fraction of a second, unconsciously, they lingered among my other thoughts to guide or question my teaching actions. I acted consistent with my beliefs. As any other teacher I had habits of action. Now, I realize that, in my analysis of a lesson or a teaching issue, I sometimes separated the pedagogical from the mathematical thinking. At other times, I was motivated by a connection between them. Sometimes thinking pedagogically served a mathematical purpose. Sometimes thinking mathematically served a pedagogical purpose.

Traditionally trained in mathematics, I judged most child-centered teaching of mathematics as lacking depth, superficial, limited in content, without representing the substance of mathematical thinking. After eight years of encounters with child-centered pedagogies, I am intrigued by reasoning that transforms the student to an abstract conceptualization. Being familiar with both mathematical and pedagogical contexts of mathematics teaching, I learned to develop an empathic understanding for the situations they separately brought. Instead of looking for instruction weaknesses I attended to those teaching moments that participant eyes defined as meaningful. I found myself developing a different understanding of mathematics teaching. I balanced the intentions, values, frame of references of the participants, not “getting inside the
head of an actor,” but that “understanding comes from the act of looking over the shoulders of actors and trying up (by observing and by conversing) what the actors think they are up to.” (Schwandt, 2000, p. 192) An empathic identification allowed me to grasp participant intentions in their actions, but it also provided me with a specific analytical mentality.

I departed from what would have been an ethnometodological perspective to concentrate on dialogue. I chose those episodes for analytical descriptions of interactional episodes based on my observations and categorizations. I looked for what would be different narratives of the same instructional action as well as for instructional action of a particular kind. Engaged in this study, I followed a micro-perception of the mathematics teaching, not for an analytical account of mathematics teaching situations, but for understanding alternatives from what teacher mathematical and pedagogical thinking may derive. The view offered a perspective on how meanings emerge from inside particular teaching practices. I recognized what Silverman (2003) said about the status we attach to our data:

How far is it appropriate to think that people attach single meanings to their experiences? May there not be multiple meanings of a situation (e.g., living in a community home) or of an activity (e.g., being a male football fan) represented by what people say to the researcher, to each other, to caregivers, and so on (Gubrium, 1975/1997)?

This raises the important methodological issue of whether interview responses are to be treated as giving access to “experience” or as actively constructed “narratives” involving activities that themselves demand analysis (Holstein & Gubrium, 1997; Silverman, 1993). (p.346)

I was primarily interested in the second position, in the activities people engage in to construct their narratives, but I also kept an eye for participants experiences, especially because I needed to see how people construct their narratives and the activities associated with them in two different situations: outside of the classroom contexts and inside the classroom interactions.
Interpretive Frameworks

Constructionist and interactionist perspectives. The relationship between mathematical and pedagogical thinking has been largely recognized as one constructed and continuously constructed. Most frameworks in teacher learning wonder and conceptualize about this construction, but definitely considered it a construction of teachers as they experienced their teaching activities. But most of them stop short of their cognitive and individual aspect, while the relationship between teacher’s mathematical and pedagogical thinking is argued to be activated and active in classroom interactions.

It is definitely a particular tension between this long term construction even as simple as understanding those mathematics knowledge for teaching, which definitely require some positioning into a teaching situation.

According to an interactionist view, learning is socially constituted. Social interaction is necessary for learning to take place. Some also call it a micro-sociological perspective on research of learning. As put forth in Chapter 3, this study was to create a micro-perspective on how teachers use their mathematical and pedagogical thoughts during classroom interaction. Interactionism as a sociological perspective also aims for a micro picture of teaching experiences mainly focused on the interactions between students and teachers. I considered interactionism as one of the interpretive framework because I wanted to complement the more dominant perspective of teaching as “instructional action.” I wanted to problematize instructional action as it changes during teaching. Some changes occur from what I referred as “conversational action.”

One observation that anchored this study was that novices start with strong assumptions about classroom teaching. Although aware of what is happening (and not lacking mathematical knowledge) the trainees see sidetracks in the developing of classroom conversations that divert
the planned instruction. Using interactionism as a main interpretive framework, I intended to highlight and understand how such classroom interactions could be determined by and related to teachers’ integration of mathematical and pedagogical thinking. I thus confronted two ways of thinking about any one situation, as having a particular construction of the moment, a specific meaning associated with an instructional action, and as having mathematical meanings emerge from the classroom interaction. Each way challenges or supports a particular constructed meaning of the instructional action.

Engaging the two perspectives on how teachers build meanings in classroom situations, I also encountered the difference between listening to what the student says and listening to what teacher wants to hear, thus placing in opposition what otherwise might be an integrated resource.

It has been through relating the constructionism (actively engaging to construct meanings) and interactionism (interpretive perspectives on classroom action) that the tension emerges between what is constructed as a discourse and how it takes shape during classroom interactions.

It is critical for interactionists that interpretations drawn from data do not lose their footing. Since in the interactionist’s view, the course of a mathematics lesson is contingent upon the actions of students and teachers, the reflexivity of these actions is highly important. Reflexivity, here, refers to the fact that in the process of social interaction, participants make their actions understandable. They use linguistic markers to make themselves understood, and these markers may serve as starting points within micro-sociological analyses. Interactionists generally dedicate plenty of time for the reconstruction of the emergence of shared meanings among the students (or teacher and students). As a consequence of the close relationship between data and interpretation in interactionist research, the validity of its results is high. Idealistically, theoretical concepts are developed through analyses of empirical data. (Gellert, 2007, p. 1674-1675)
**Conversation analysis.** While the focus of this study of dialogical units was not on the meanings teachers attach to students’ responses, such meanings helped achieve a particular focal analysis, described later in this section. Conversation analysis supported a focus on what is called the communicational action of the teacher and how this has been developed within classroom conversation. I thus reached a triangulation of the meanings attributed to teacher action as part of the talk-in-interaction.

According with Paul ten Have (2007) "Conversational Analysis is based on 'analytical mentality' that seeks to explore the connections between the particularities in the details of human action and the generalities of shared organizational problems and resources" (p. 307). From micro-analysis of particularities and details, the researcher works to deduce thinking and intention. Conversations are seen to rely on categorizations that a speaker, perhaps unconsciously, visualizes as options for his next response. Some are membership categorizations, that is, insider talk, where persons of a group or situation think of categories that outsiders would not. Conversational analysis guided some of my analysis. Talk-in-interaction was, in fact, my main focus of analysis.

Harvey Sacks (1992) said, “How is it that people can produce sets of actions that allow others to see such things … [as:] persons during intimacy … persons lying, etc.? (p.119)” “to find out how they [members] go about choosing among the available sets of categories for grasping some event. (p. 41)” I assumed that through such analysis I would gain a certain access to a teacher's thinking in the form of a categorization of local events. Teachers code student utterances and draw upon their own categorizations of potentialities to make a response. The teachers' "coding" follows a sequential and inferential order. The categories they select in the classroom are indicated audibly. What a teacher's utterance actually accomplishes (Housley &
Fitzgerald, 2002) was not pursued in this study. Instead, I paid attention to the relationship between the situation in which the teacher utter some sentence and the situation “about which uttering the sentence conveys” (Greeno, 1994, p. 338).

In such conversational analysis, one of my interests was the detail of the teachers' social action, hoping to gain further understanding of what the teachers were trying to accomplish. Through the membership categorizations approach, I had access to two pieces of information: the teacher deciding on an event during the interaction and the teacher categorizing that event. I see such an analysis differing greatly from the researcher identifying, in a post observation interview, certain events observed, than asking the teacher to explain. And different from the case of a teacher providing a general picture of what happened, but not using specific dialogue.

When engaged in transcript analysis, I had opportunity to make sweeping observations. With them, as if solving a puzzle, I tried to hypothesize what those observations led to. In this way I interpreted the events of the interactions and the procedures the teachers engaged in, often feeling I had gained access to their thoughts.

Focusing on the categorizations of teachers rather than on my own categories provided a detailed picture, hypothetical but triangulated, of what the teacher thought at that moment. The data documented the teacher's action at that moment and thus brought a sharper picture of what the teacher thought: pedagogical features, mathematical features, instructional objectives, instructional strategy, and learner uniqueness. Behind action there is a thought patterning that described as based mainly on “deviant case analysis.” With it, I felt I could embrace a larger pattern in the social logic of teacher-student interaction.
**Focal analysis.** I also used focal analysis (Sfard, 2001). With Anna Sfard’s three types of discursive foci, namely: the pronounced focus, the attended focus, and the intended focus, I selected the objects of analysis from the transcripts of teacher-student dialogue. I combined my understanding of conversational analysis with the types of the discursive foci. I saw this giving my opportunity to access the way teacher’s discourse molds his thinking.

I gave emphasis to this new term, focus, as an interpretive concept, working with the debatable idea that it is up to the interpreter to decide what should count as focus of a given utterance. In making her decision, the researcher needs to consider the aims of her analysis and help herself with the general context of the conversation. In this interpretive activity, one should probably be particularly attentive to what seems to constitute 'the common thread' of successive utterances.

Although the idea of discursive focus seems intuitively clear, less clear is whether the term denotes the words used by an interlocutor to identify the object of attention, or refers to what and how we are attending-looking at, listening to, and so forth-when speaking. Because both of these discursive components, the words and the process, seem indispensable for effective communication, one should probably consider them in tandem.

Sfard’s third component, intended focus, should be considered along with the pronounced and attended focus. The intended focus is the interlocutor's interpretation of the pronounced and attended foci: “It is the whole cluster of experiences evoked by these other focal components as well as all the statements he or she would be able make on the entity of the question, even if they have not appeared in the present exchange.” (Sfard, 2001) As Sfard acknowledges, a researcher cannot access a teacher’s cognitive process, but can try to capture the focal object of classroom conversation and see how the teacher discursively communicates that focus.
It was in the teacher's intended focus that I looked for, perhaps equivalent to his pronounced focus and attended focus. But focal analysis mainly helped me to see how the specific “objects” in the communication and thus offered me access to teacher’s thoughts while teaching.

I will use below Anna Sfard’s example (2001) to illustrate the tripartite foci of the conversation.

Episode 1 (“Apples”): Children choose apples for a school excursion:

1. **Casey:** I think we should take those, the green ones. They are very sweet.
2. **Brad:** I prefer the pinkies. See the black spots on this one? The greens have worms.
3. **James:** Which one?
4. **Brad:** Up there in the first row, to your right.
5. **Janice:** Yeah, the pinkies are better. With the green one you never know. It is not consistent; sometimes you have it big, and sometimes you end up with a very small one. (p. 301)

First, of all what it has to be remarked here is the effectiveness of the communication. As Sfard claims if they were asked children probably they all would agree that they “are speaking about the same thing.” As Sfard explains:

> One may object to the above analysis saying that focus cannot become a well-defined term because it will never be unique. For example, when one says, “I like green apples,” is the sentence primarily about the speaker herself or about the green apples? I therefore emphasize that this new term, “focus,” is an interpretive concept and that it is up to an interpreter to decide what should count as a focus of a given utterance. In making the decision, one must consider the aims of her analysis and help herself with the general context of the conversation. In this interpretive activity, one should be particularly attentive to what seems to constitute “the common thread” of successive utterances. (p. 302)

Farther remains to be understood what a discursive focus refers to: the words or the way we attend to that focus. They both are in tandem and thus we call one “pronounced focus” (as for
green apples in line 1) and another “attended focus” (as for the way the green apple is referred to in line 4). But it is more to the way we refer to this focus. There is a third component: the intended focus which in this dialogue could be the claim about the sweetness of the green apple in utterance or the assertion about the worms.

Although both attended and intended foci are less tangible than the pronounced focus, their presence if signaled through discursive clues: the green ones, greens, they.

Sfard brings farther a mathematical example. If one looks at 3(x+2) and 3x+6 they are convinced they talk about the same mathematical object. Preserving the focus seems to offer a basis for the effectiveness of communication, still it is a matter of immediate context and relevance “unless proved otherwise” that makes the communication effective. We understand the meaning of an answer as “I am sick” in the context of the previous question: “Can you meet me today?” The relevance of the answer is that the responder refers to the issue of meeting. This has been related with what it is referred as the sequential analysis of the dialogue and which proved how those answers have meaning in relation with the previous response. Consequently I used this way to analyze dialogue, especially to escape the rationalist perspective on teacher action. I did not interpret teacher answer as bad or good in terms of the general consequences, but in terms of what students previous said. Later I placed the two kinds of taking meaning out of the teacher’s action against each other to understand better the tension for teacher and how they placed their mathematical and pedagogical thinking in this context.

I claim that we should understand the teacher-student dialogue through the same perspective. The construction of tripartite foci helps to bring together what would be the private space (intended focus) in the public discursive space. In creating these symbolic devices to anchor student attention, the teacher works from certain categories and predicates. It is an
inferential path connecting categories. In the case of this study it provides predicates for the particular device that may lead to a conjectured relationship between the teacher’s mathematical and pedagogical thoughts. However, a relationship that is rather described through its immediate context not through an external rationality of relating ideas as corresponding to a demanded reality of mathematics classroom. Important to repeat here is that conversation and teacher action are not thus consequences of rationalistic relationships, but rather understood in the immediacy of the interaction and in the occasioned context.

One aspect of my analysis tried to capture both the social and experiential organization of mathematical knowledge in classroom teaching. I referred to Bauersfeld’s words in Chapter 3, to show how the social logic is a very important aspect when referring to an interactional thinking, but this was not to imply that I will not consider the mathematical or pedagogical reasons.

Thinking and how researchers understand it. I need to mention here that most of research on thinking especially the cognitive approaches of thinking refers to a kind of thinking instead of ways of thinking. It is definitely hard to reach the particular private inner space of thinking, but we could have access to teacher’s discourse. Language molds people thinking and through communication people engage in a formation of their thinking. In my interpretations I recognize that I cannot have access to teacher’s thinking (and that may seem a behaviorist approach of research) but that did not mean that I abandoned the role of the relationship between teacher mathematical and pedagogical thoughts for teacher actions.

Instead I looked at teacher utterances and the discursive actions on students and at the communicational channels (as described by Anna Sfard). Instead of claiming that the teacher thought mathematically or pedagogically I focused on what would be identified as norms and rules governing teacher actions. Were they pedagogical or mathematical? Certainly we would
like to think that teacher actions are guided by pedagogical rules and students’ actions by mathematical norms. It follows immediately that to establish an effective classroom communication the teacher must master and understand both. And in having such expertise the teacher would also gradually develop a repertoire of integrated responses (Shulman, 1987) to certain features of certain mathematics teaching situations. And it is more tempting to find out such features (Ball, 2000) and farther work on making them “visible”. But that imposes a relationship of these features with situations more than with the teachers thinking.

It was within this kind of analysis to see rather the game between what I would interpret as being the application of a mathematical discourse and pedagogical norm. Certainly deciding if there was a matter of pedagogical or mathematical norm behind was a matter of understanding the situation as it has been shaped within by the activity. I also paid attention to regularities (if they happened): as the teacher offered for a specific response a specific norm constantly in a routine manner, and if later when inquired they rather simply rationalized (using similar reasoning) or rather offered an alternative view or where even just simply surprised by my question as in “I did not think in that way!” This sort of changing the “attributes” showed a difference not only in the mechanisms but also in when and where the mechanism has been used.

The following excerpt illustrates the connection between certain particularities in teacher's discursive actions and the general organization of mathematical knowledge, manifest in student-teacher interaction. The organization is followed through the categories found in a sequential order. The transcript conventions and notations for the excerpts used are available in Appendix A.

One important step in understanding the integration of mathematical and pedagogical thought was to uncover the ways in which the teacher shaped the dialogue. Finding a perceived
interactional reality, I sought how mathematical and pedagogical thoughts played a role in these perceptions. The interactional reality was reproduced in the teacher's enactment of his interpretation of it. And that is what makes the difference in the micro perspective. Before an enactment there is an interpretative stance of what supposed to be enacted and that interpretative stance is not only a matter of the individual macro perspective on a teaching situation, but also a matter of interpreting during enactment. Consequently it is very important to understand what interactional mechanisms guide the enactment as well as the sequential interaction which shapes the mechanism.

Excerpt example:

1. **Teacher:** So, the second function of the homework yesterday.
2. **Teacher:** You looked at: \( \frac{1}{e^{1-.1}} + e^{-x} \).
3. **Teacher:** Is that a rational function?
4. **Teacher:** (pause)
5. **Teacher:** Student D?
6. **Student D:** Yeah
7. **Teacher:** Student B, what do you think?
8. **Student C:** (responding before Student B) No!
9. **Teacher:** Why not?
10. **Student C:** (Because) \( e^{-x} \) is not a polynomial!
11. **Teacher:** 1 plus \( e^{-x} \) is not a polynomial!
12. **Teacher:** OK?
13. **Teacher:** So, in your homework last night I asked you to look at two functions that were fractions, that were ratios, but the second one wasn't what's called a rational function because it wasn't a ratio of polynomials.

Lines 7-9 indicate the three discursive foci identified earlier: the intended focus, attended focus, and the pronounced focus. The teacher linked them together, rather than developing them. It strongly seemed it was the occasioned context, the classroom situation, which made the teacher shape his instructional action this way.

My interest in this particular excerpt was mainly the use of "according to this definition," as an explicative, as a pronounced focus, where the focus would be the criterion for deciding if a
function is a rational function or not. But it equally counts as an attended focus. The teacher did not repeat the function to help Student D who made an "un-preferred" response. He did not imply that Student B did not pay attention. He moved to a means by which either student could have provided the "preferred" response. Additionally he made meaning of student response within the immediate context. Finding if a function is rational meant to refer to a certain particular definition in the common space teacher-student. The definition here became the affordance for student learning. It is also the affordance for teacher construction.

What the teacher did in this excerpt was to construct an interactional episode, indicating the mathematical procedure to decide if a fraction of functions could be called a rational function. The teacher expected the student to understand his (the teacher's) previous words. In another episode with a similar task, the same teacher used a similar definition in a different conversational action. Thus, the occasioned context was different. My analysis in this study examined such alternatives and variations in response to what seemed to be essentially the same task with a teacher having the same knowledge.

In this episode we see a routine introductory activity of the lesson on Rational Functions or RATS (see Appendix B). The apparent mathematical object here was the rational function, but more exactly, a criterion for defining rational functions. Mathematics textbooks abound in definitions of rational functions -- but this dialogue defines a specific learning experience in which students might recognize a rational function using as a prop in the polynomial function. Here, the criterion refers to two mathematical “affordances” for teacher action: ratio and polynomials. These criteria seem to provide an incomplete and, in fact, inaccurate definition of a function. Mathematically, a function is a relationship, but this excerpt illustrates the intention of
the teacher for a learning experience more than for a mathematical discussion. The necessary relationship as a function is thus somewhere in the polynomials.

The pedagogical experience sets clear criteria for the learning experience, offering with example and counter example, what is or not a rational function. The criteria set forth an incomplete learning experience, leaving unpredictable spaces for misunderstanding, even for alternative understandings! It is problematic in that the learning affordances reduce the significant and accurate understandings of mathematics for students. It remains the teacher’s work to make sure this does not happen. One argument here is that the greater the number of representations of mathematics, the more chance the teacher has to respond to those misunderstandings. But it also sets out the alternatives. The criteria in this example make the concept of rational function learnable. It is not an abstract definition, but gives features of rational functions that make the functions accessible for students. In such a learning experience, the routine activity set by the teacher aims to repair students' mathematical misunderstandings generated by the setting of the learning experience. Mathematics teaching regularly involves using such features. This teaching aligns the references to a function.

The way this learning experience was enacted by the teacher was based pedagogically on two examples. The students would become able to spot the mathematical criteria and thus recognize a situation in which there is no rational function. Some may consider this pedagogical approach mathematically as “example” and “counter-example.” However, a counter-example in mathematics usually involves proving that a statement is not true.

Were the teacher's thoughts structured by the mathematical or the pedagogical features of this learning situation? Were the teacher's thoughts following mathematical features of counter-example or pedagogical features of an example that is not a rational function? Are teachers'
thoughts serving a pedagogical purpose—to create an illustrative example of something that is not a rational function, but expressed as a ratio—or a mathematical purpose, which would be to prove that the statement is not true?

Or is it either? Had the teacher decided on a purpose? It would be just as wrong to suppose the teacher had fixed on a purpose as to conclude it was the wrong purpose. We do not have to suppose the utterances of a teacher are generated by prior purposes.

The teacher started with a question: "Is this a rational function?" Recognition thus serves a pedagogical purpose. I used this episode to point out that I did not see in the teacher’s discourse a reflection of his thoughts, but a mode of action:

Treating language as a “mode of social action rather than a mere reflection of thought” (Malinovski 1959, pp. 312-313) necessitates investigation of competent members of a society use language to deal with each other. This requires first, methods of data collection that maintain the sequential structure of indigenous interactive events (i.e., ones that exclude the ethnographer’s intervention through elicitation) and make visible the process that these events are both embedded within and constitute; and second, a mode of analysis that, rather than treating talk as either a means for observing information about other phenomena or a special type of verbal performance, focuses on how competent members use talk socially to act out the ordinary scenes of everyday life. (Goodwin, 1990, p. 286)

My purpose was not to reinvent behaviorism. I did want to learn what the teachers proposed to do and what they meant to accomplish. But, finding them disposed to use hypothetical constructs sometimes was not consistent with their action; I tried, as Goodwin advised, to treat talk as revealing the meanings of their teaching.

Summary

In this chapter I described the main methods of data collection, data analysis, and interpretation. I identified the teachers participating. I showed how being interested in the meanings of teacher’s actions called for a micro-interpretation to analyze teacher-students
dialogues. Some meanings were given by the complexity of local situations. Meanings came indirectly, with much pondering over the alternatives of teacher’s actions.

The data was collected from two experienced teachers’ mathematics classrooms. Most of the data analysis centered on particular excerpts of the classroom dialogues and teacher interview. I referred to the most thought-provoking ones as patches. A collection of patches helped me compose a representation of the complex issues surrounding the relationships between teacher action and what may be integrated mathematical and pedagogical thinking.

I also pointed out that to capture the ways teachers attend to their action, I analyzed their discourse in classroom interactions, complemented with pre- and post-discussions after class, as well as with observations. The data analysis was primarily of the discursive activities, indirectly interpreting the thinking of the teachers in the classroom. This analysis of thinking can then guide the preparation of mathematics teachers.
Chapter 5

Teacher Interrupted Reasoning

This chapter will illustrate how teachers' reasoning during classroom interaction is interrupted reasoning. During my three years of observation, I found a most interesting phenomenon: when I asked the teachers what they thought about the lesson just taught, they would say that everything occurred according to plan, nothing extraordinary happened. However, in most of the cases they would add that some line of action deviated, not intended. I called these instances "interrupted reasoning." It was a different kind of action. The teacher's reasoning changed. This phenomenon has been little studied in the cognitive research.

The intent of teachers’ use of mathematical and pedagogical discourses was challenged in classroom interaction. I followed the teachers' discourse before and after lessons and remarked how these discourses have been or not reflected in classroom action. When the teacher seemed to follow a rather different action than anticipated I noted the situations and local contexts. I presented a selection of those episodes in the Table 3 below for four analyzed lessons of which transcripts are in Appendices. They were not unusual episodes for classroom discourse, but deviated from what had been anticipated or from in the pre-lesson and after-lesson explanation.

This chapter aims to respond to the issues which came from the study of the two teachers' classes, examining the reasoning for a particular action. When the action changed track, did the teacher have a reason? Would his reasoning be backed up by additional mathematical and pedagogical thinking? I found this phenomenon helpful for understanding the issues of how teacher thoughts and reasoning influence teacher action.
What is Teacher Interrupted Reasoning?

It should be useful to explain that "interrupted" means change from episodic reasoning to broken-up reasoning. It is not just stop and go. It is change in direction, often to multiple directions. I will illustrate how the break-up of teacher reasoning for their action is apparent at the micro-level of their actions. It illuminates the issues teacher education has in not responding to reasoning and enactment. One can see here a general reasoning in the sense that Leinhardt (1988) developed for planning and scripts, but also more detailed reasoning for the quality of mathematics instruction (Hill & al., 2008). Reasoning there was referred to in the context of integrated mathematical and pedagogical thinking.

In this chapter, I will illustrate several episodes from four different audio-recorded classes and the changed lines of actions, I will focus on the relationship between mathematical and pedagogical thinking that I consider has been breached. The description and analysis of episodes were aiming to reveal the mechanism of metaphor, which takes one feature and carries its properties to another domain, but preserves the domains separated, and especially their rules.

This analysis considered the active way of engaging students within classroom interactional situations. My position became that it is not a situation to which the teacher transfers knowledge but, with each response to the student, with each discursive initiative, the teacher together with students created a situation. For example, in teaching radians, the teacher talked about angles even though he had not intended to do so. I saw actions shaping the classroom situation, rather than a pre-established situation in which the teacher could offer anticipated responses. I saw the teacher's cognition as situated. The dialogue that formed was not only a product of the teacher but also a product of the interaction with students.
The most important aspect is that the teacher did not say that something happened or the situation was complex and contextual. The perspective was: how is it that the teacher took up a different action without registering that it is different or special? He referred instead to his previous reasoning. What in teacher-student interaction mediates the observed action without it being perceptible in the after-class reasoning? I moved toward a different perspective of the relationship between mathematical and pedagogical reasoning in classroom teaching. Rather than trying to directly identify the integrative, I looked at how such unintended things happened. I concentrated for a period on breaches in the norms of integrated teaching action. I aimed then to understand how teachers actually enact their norms and how they are formed in classroom interactions at the micro-level; I sought a better understanding of the mechanism which relates mathematical and pedagogical thinking of the teacher to offer novices a better apparatus.

The idea of breaching is used methodologically when interviewing mathematics teachers and reconstructing their rationalities (Herbst & Chazan, 2003) to reveal their norms of practice. In my analysis, I used breaching to see how assumptions become manifest in the classroom and make visible the potential relationship between teacher mathematical and pedagogical thoughts influencing teacher actions.

**Patches.** My episodic patches showed the tasks per se and the features of what was developing during instruction. They brought awareness of what to think about when teaching mathematics: what has become of what the lesson plan was. We should imagine practice partly as enactment of interruptions. Lesson plans are not scripts for ballet or symphony. The enactments are composed partly of interferences in intent. How is it that teachers breach their previous reasoning? The following chapter describes how teachers aimed to do something different than they intended.
Table 3

*List of Patches for the Four Transcribed Lessons in Appendices.*

<table>
<thead>
<tr>
<th>Lessons and Patches</th>
<th>Interrupted Reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rational Functions</strong></td>
<td></td>
</tr>
<tr>
<td>According with this definition</td>
<td>Definition/Example</td>
</tr>
<tr>
<td>Can x be zero?</td>
<td>Definition</td>
</tr>
<tr>
<td>Number sense</td>
<td>Pedagogical creativity</td>
</tr>
<tr>
<td>For large values of x</td>
<td>Different situational context</td>
</tr>
<tr>
<td></td>
<td>Definition</td>
</tr>
<tr>
<td><strong>The Joy of Radians</strong></td>
<td></td>
</tr>
<tr>
<td>Radians definition</td>
<td>Different formulation</td>
</tr>
<tr>
<td>A bad formulae</td>
<td>Pedagogy</td>
</tr>
<tr>
<td>Proportion</td>
<td>That’s we don’t know</td>
</tr>
<tr>
<td><strong>Fractions and Tiles</strong></td>
<td></td>
</tr>
<tr>
<td>The biggest in history</td>
<td>Pedagogical</td>
</tr>
<tr>
<td>How about three</td>
<td>Tagging</td>
</tr>
<tr>
<td>This is your last question</td>
<td>I did not say that this time</td>
</tr>
<tr>
<td>What is the whole?</td>
<td>Student Intervention</td>
</tr>
<tr>
<td></td>
<td>Mistakes</td>
</tr>
<tr>
<td><strong>Mystery Activity</strong></td>
<td></td>
</tr>
<tr>
<td>You’re confusing the number</td>
<td>Example</td>
</tr>
<tr>
<td>Multiply by five</td>
<td>Planning mistake</td>
</tr>
</tbody>
</table>

My intention here was to understand how teachers manage the relationship between their mathematical and pedagogical reasoning (as rules of mathematical and pedagogical discourse) in particular instances in which teachers seemed not to follow their planned reasoning. For example
a teacher said, "I did not want to tell them about sines and cosines, but to develop the relationship between angles and chords. [But I had to help them grasp the functions.]" We might say he breached his micro-view while maintaining his macro-view. It is important to try to understand how the teacher uses his mathematical and pedagogical reasoning to respond at the moment. How do teachers use their mathematical and pedagogical thoughts differently for micro and macro viewing? This one claimed that the students said something that made them think this topic is important. So he talked about sine and cosine.

I was then interested to capture certain practices situations, as described in Table 3 of patches and see how and what happens for such generally acknowledged as encountered situations, corresponding to a macro view of teaching, in the micro view of teaching.

**Harvey Case Study**

The next three sections focus on the activities of the elementary teacher (pseudonym: Harvey). I describe five episodes from three observed classes two of which were audio recorded. The transcripts from the audio-recorded are included in Appendices B and C. The first two episodes are examples of discovery pedagogy. The third class is a routine class mainly following the textbook. The second lesson was a teaching experiment, however, unsuccessful with several missing elements which gave me an opportunity to analyze the unexpected aspects of interaction for and otherwise very successful teacher.

Harvey knew how to handle the pedagogical discourse in the class. His discourse preserved particular features across the four lessons such as constantly probing students’ answers and avoiding the funneling of a specific mathematical answer. It was, however, the differences
and likenesses of the mathematical and pedagogical contexts of these lessons that I could observe and analyze how Harvey carried his mathematical and pedagogical thoughts in his teaching.

As illustrated in Table 3, each of the episodes was puzzling regarding a certain consistency between the reasoning before and after-lesson thoughts and during classroom interactions.

**Carnival Activity**

One of the many interesting lessons I attended during my second year of observations in Harvey's classroom was Carnival Activity. Harvey said that it might be the tenth time he had taught the lesson. He was quite confident it would be successful. The analyses of this episode together with the ones in Mystery Activity, offered me the chance to observe a teacher engaging in a teaching experiment. He faced variations of unexpected moments: in a familiar teaching context and in one with which he was less familiar. Complemented with another teaching experiment and his routine activities, I had the chance to see Harvey confronting the unexpected in four complex contexts.

Before the lesson Harvey sent me a message regarding his experience teaching this lesson. Most of his pedagogical thoughts were oriented towards students' engagement with the activity. In his email, he also seemed to favor his instructional aims and actions, expecting that students would experience this activity and would realize that something does not always happen as it should. The mathematical explanation is not a prerequisite of this activity as Harvey mentions in his description: "If time permits…"

On Thursday I'm planning to run a probability game that's a pretty involved lesson. It will take all afternoon. So if you can come we'd probably start earlier, like 12:15. I have to run the game itself for at least 35 minutes to make sure the house wins conclusively. I'll try to describe the game:
I adopt 4 kids to run the game booths. The rest are players. Each play has $10 in play money that they have to pay back at the end of the game if they can. At the booth they flip five pennies after they place a dollar bet. If they flip all heads, all tails, one head, or one tail they win. If they flip 2 heads or two tails they lose. It seems on the surface to favor the player. After all, there seem to be four ways to win but only two ways to lose. If they run out of money they can borrow more. They can quit at any time and any money they have left over after the debt has been settled is theirs to keep. This is a lively time and the emotions run high. I warn them that it's just a game and that someone will end up in tears, usually a boy. No one HAS to play, though it would be boring to sit out the activity.

When we close up the carnival we take an accounting at the board. Few players will have money after their debt has been settled. I ask them to write an explanation for why things turned out as they did. A discussion usually will yield the idea that though it seemed like there were more ways to win than lose, that was not really the case. If time permits we look at Pascal's Triangle to sort out the probabilities.

Note his pre-lesson intentions to have students write explanations for why things turned out the way they did and only “if time permits” to look at Pascal Triangle to sort out probabilities. This established a relationship between his mathematical and pedagogical thoughts. One macro-interpretation would be that it will influence his instructional actions. Consequently, it is a general line of action appropriate for the general situation of teaching this activity.

Harvey’s comments illustrate that he knows what means for the activity to reach its goals (e.g., it has to run for at least 35 minutes; it is emotionally-involving). Harvey handles the mathematics behind this activity and what should be students’ tendency to consider it as mathematically important and what would be necessary to be emphasized so students would take the mathematics out of this activity. As his post-lesson thoughts show he considers student questioning before the activity “if the game is fair” “the core of its lesson”. He notices it and incorporated it into his classroom discourse.

I will analyze the way Harvey relates the mathematical explanation behind this activity illustrated by the Pascal triangle and his pedagogical intent to have students explain what happens. Harvey is a very good teacher and in many situations he masterfully led students to a
particularly discovery in a certain activity. Carnival activity is, however, very complex and requires time to have students engage with a proper explanation for what happens. The activity is consequently rich in teaching dilemmas on how is better to act and plan the lesson, and its complexity inevitably leads the teacher to various insights.

**Chances and Pascal's triangle.** We will see in this next episode first that Harvey hurried to consider the mathematical explanation for what happens in the activity and that offered a natural breach of his teaching intentions. The analysis of this episode offered me the chance to see if the teacher's between mathematical thoughts and pedagogical thoughts was preserved at the moment of breaching. This is the dialogue between teacher and students that closed the activity: (Double parentheses provide description of non-verbal behavior and the ➔ points to the moment during dialogue that should be paid attention to)

**Episode 5.1**

Harvey: I am goanna call you off. You will pay off your bet…If you count…
Harvey: Audrey… ((Students start counting money and teacher writes on the board as a two columns table + and -))
Audrey: ((Table included the money spent.)) At the end, clearly the Carnival wins.
Harvey: So, what happened?
Maria: It was not fair?
Harvey: How can you tell?
Robert: I think it is fair!
Harvey: Why?
Robert: Because there are 4 chances and 2 off.
Audrey: Pretty rare to see all or 4,1 But a lot for 2,3!
Helen: There were more ways to lose than to win…
Harvey: ➔ I am going to show you how this happens! I will tell you about a French guy who loved to play with dice and cards. “Who was Pascal?” ((Reading about him from dictionary, mentioning that he could not get on internet to find out more about Pascal.))
Harvey: He came with this (the triangle)
((Flipping 1 coin H, T))
((Flipping 2 coins TH (2), TT(1), HH (1)))
((With 3 coins: TTH (3), HHT (3), HHH (1), TTT (1)))
((Harvey does not insist on counting and trying all the situations out, but rather goes and fill in with the structure/patterns of the triangle.))
Harvey: See how they are generated? Going with the triangle by 5 coins…
Harvey: Why was easier for carnival?
Students: There are bigger chances (students explain for situations when 2 heads and 3 tails or 2 tails and 3 heads show up)
Harvey: There were how many ways to win for the Carnival?
Students: 20!
Harvey: But how many to lose?
Student: Aaaa…12!
Harvey: How many happened in all? 32!
Writing down the probabilities: 12/32  20/32”
Harvey: Can you reduce fractions to keep in simpler terms?
Harvey: What’s half? 12/32=6/16=3/8
5/8 loose because 3/8 +5/8 = 1
Harvey: If you were to play 8 times how many times should you lose?
Students: 5!
Harvey: Did that happened all the time?
Students: Are we going to play again? Who will be the booth person, the player?

It is interesting to note how, during the activity, Harvey did not make clear to students that he would be interested in the mathematics behind the activity. Students however had an expectation, as the below episode shows, and Harvey took their expectations provisionally within his instructional interactive intentions it seemed. Observing students engaged in the activity I heard the following:

"This is not math!" "We take money and give money!" I looked at the girl who made this statement, waiting for more. She added: "Well, we may have done some calculations, like five minus one, but this is not mathematics." I recalled she was the student who asked Harvey if the game was fair (Harvey will mention her question as the core of the lesson in his reflections). I asked her, "Do you think there is a way to win?" "No, because this is not fair!" "Why?"
"Because, it is not fair! I just keep losing money!" She amusingly had taken on the exasperated
attitude of a gambler. She definitely was engaged with the emotional aspect of the activity as Harvey intended, and also anticipated all students would be.

After the lesson I could not stop and wonder why Harvey breached his previous mathematical and pedagogical relationship and have priority to mathematics in his pedagogical intentions. Also I tried to respond the issues of this study and wondered if this breach affected the quality of his actions and if his mathematical thoughts have been influencing the ongoing of his teaching actions. It clearly has not. It may not have appeared as a discovery lesson, in the sense that students would find out the mathematical explanation, but the activity provided with a solid discovery ground on which students to build their understanding about how probabilities could explain unintuitive aspects of our life experience. The mathematics was anchored in a lively learning experience and the activity supported a strong connection between mathematics and particular situations. The teacher allowed students to engage in the activity however they found to be occasioned by the moment of interactions.

**Teacher post-lesson comments.** Later during the night, after a stormy afternoon with power failure, I received Harvey's thoughts about the lesson. I could not stop wondering about the switch in the priority Harvey offered to his intentions, without even realizing what he seemed to be favoring in his pre-lesson thoughts. In these post-lesson reflections he mentions as to “If we had more time” to have students write about an explanation for what happened. This switch made me reflect both on what would be teacher’s acting on planning, enacting, and also questioning a possible developing his integration of mathematical and pedagogical thoughts:

Jenny's question at the beginning is worth noting, "Is this a fair game?" It was the heart of the lesson. It was interesting that Alice commented early in the discussion that there were 4 ways to win versus 2 ways to lose. I will want to follow up the lesson with some flipping trials use two or three coins and have the children graph the results. I should also have a lesson using a question about the number of different pizzas that can be made with 6 ingredients. We can use Pascal's Triangle again to count. The lesson and
experience is a long one but I have found that it is better to try to do it in one day without a break. If we followed up the next day with the discussion much would be lost. If we had more time I might have had them write about it before the discussion and read their answers to this question: "Why did the carnival win and most players lose?"

From his comments it is clear that Harvey was aware of what would be the most important aspects of this activity: was the game fair and that students would need to account there would seem to be four ways to win against two. Harvey recognized immediately when students included in classroom discourse these two observations. However, these aspects of his mathematics thoughts were not taken within his classroom discourse explicitly and transformed into instructional actions as one would have expected. In a novice teacher education activity these thoughts would be part of the reasoning behind a certain instructional action. For Harvey, however, these comments seemed that tacitly have become part of his thinking during the lesson, but not visible in his instructional moves. His instructional actions have been guided by more complex relationships anchored in the interactional context.

Harvey's thoughts in his post-lesson reflections to organize mathematical and pedagogical strategies seemed to preserve the same relationship with different priority based on the same macro-view intentions as before the lesson: students would observe what happens and relate easily with the mathematical explanation behind. Still, the clear breach in the classroom episode and the switch on giving priority to mathematical explanation or pedagogical norm of discovery showed a problematic relationship between teacher’s thoughts influencing his actions.

**Two years earlier post-lesson comments.** Before the observed lesson, Harvey shared with me his reflections on the same activity when working with another student cohort, two years earlier. My reflections here, of course, are about a moment which did not happen during my observations. However, Harvey’s comments seemed to address the problem I noticed with both his classroom interactional breach and his pre- and post-lesson reflections. Harvey mentioned his
intent in his reflection: to have students write their explanations before the discussion. It is interesting to note how Harvey associated his students' responses with their learning experiences. These reflections however offer the opportunity to see what happened when student wrote their explanations and what the teacher thought about it.

The writing they did was hard to prompt. After an hour a play (split between before and after lunch) we settled our debts and did an accounting on the board. Anna was at the board because she had the chore this week that I call student scribe. She chalked the results in a plus/minus column array as the kids paid up or kept their winning. Three kids broke even, five had a modest profit and the rest lost. Once it was established how the experiment went I asked them to write and tell what happened. I was careful to say that they should not tell the story of what we did and what happened. They had to explain why they think the carnival won most of the money. The kids didn’t write responses at first. They were somewhat perplexed. Don’t tell me the story, tell me why it had to end the way it did and use mathematical thinking and language to do it.

I read aloud what they had written. This is always a difficult moment. Sometimes they become preoccupied with trying to guess who has written the piece rather than think about the ideas being shared. Time was growing short so I didn’t invite them to comment on the soundness of the ideas. I shared my thoughts and tried to honor all ideas, even when clearly many were barking up the wrong tree.

Discovery is a great way to make learning meaningful but there is no way that everyone will discover what is true. So after my kids raw impressions and theories have been explored it is time to delve into a way to count. We looked at Pascal’s triangle one bit at a time. By the time I was ready to ask what would happen with five coins, students were suggesting how to fill out the next row. I didn’t want this lesson to get sidetracked with looking for patterns and simply pouring out new rows. So I kept the focus on relating it to what we had done in our flipping.

You asked what will happen next, where do we take this up again. Good question. I’ll probably slog on ahead with the timeline so I don’t get transferred. The kids have already suggested through their behavior where this could go next. At dismissal an anonymous student added a row to Pascal’s Triangle. Two kids were already exploring the idea of a new game where it would be more fair.

Clearly, these reflective comments belong to a different classroom context with different interactional details and they all are noted by Harvey. The reflections offered me a perspective for analyzing the alternatives in teacher action, as well as the reasoning behind the action. Certainly, I also question if the relationship between Harvey’s mathematical and pedagogical
thoughts was not an established integrated one. I wondered what would make such relationship
to be integrated and determinative for teacher actions. I point Harvey's last paragraph, spoken
before teaching the lesson I observed. The detailed description of what happened told me more
about teacher perception of classroom interactions and students’ responses rather than a concern
with a breach in an established relationship of his thoughts.

The lesson I observed had many facets of interruption and discovery. After the lesson, I
wanted to pay even more attention to teacher discursive routines. Especially intriguing was how
Harvey responded, several months after me observing his lesson, to these questions: "What's
next? What should students take from this? Why talk about Pascal's triangle?"

To my surprise, Harvey replied that he would not be interested in having students recall
Pascal's triangle correctly, or in a mathematical explanation why the Carnival wins. He said he
preferred to have his students recall this learning experience later. He sought a context to help
them make sense of mathematics encountered later in their studies. In a sense, Harvey's remark
was related to his note that a student added one more line in the Pascal triangle or that students
wondered about what would be "more fair." It explicated his intentions, the way they work in
classroom interaction. But I wondered to what may be the integrated nature of his mathematical
and pedagogical thoughts regarding this activity and how the relationship between his
mathematical and pedagogical thoughts would affect the quality of his instructional actions.

Through this experience I became more sensitive to how teacher mathematical and
pedagogical thoughts are guided by the immediacy of the context. It was reasonable to wonder if
the relationship between his mathematical and pedagogical thinking was preserved in all these
different situations. Harvey referred to the Pascal Triangle in different ways with different
intentions at various moments during his immediate teaching, but also along the teaching year.
The variation and differences in his discourse offered me a good opportunity to reflect on the effect of his thoughts on his actions, in various situations: pre-lesson, during lesson, and after-lesson discussions.

**Mystery Activity**

This was a probability lesson meant to be sent in for a Teacher Award competition. It was one of the several teaching experiments I observed by this teacher. All except this one were successful lessons and the teacher declared to be pleased with the way lessons went. This lesson, however, his confidence was not high. I found it an interesting opportunity to explore his teaching in such a case. Until this lesson, I had seen him teaching a successful discovery lesson, several after-textbook lessons, and some following traditional instructional sections and episodes of the lesson.

**Teacher pre-lesson comments.** Like for the Carnival lesson I will use Harvey’s pre-lesson written thoughts as description for the activity and the goals of the planned lesson. I had the chance to see how he offered his lesson plan for discussion to other colleagues and incorporated their comments. Here is the plan:

**Lesson Plan- The Law of Large Numbers**

Rationale and Focus: The students have encountered probability explorations and questions that have exposed them to probability ratios and viewing probability as the likelihood of an event. The focus of this exploration is the Law of Large Numbers. Small groups will draw colored cubes from identical collections in paper bags and try to determine from their data how many of each color are in the bag. We will compile the results from the entire group and the small groups will get another chance to refine their “guesses” based upon a larger pool of data.

Objectives:

Students will express the likelihood of an event as a probability ratio.
Students will convert fractions to decimals or percents and use the data to determine how many of each color cubes are in the mystery bags.

Harvey also related the activity with specific standards:

10.A.2a  Organize and display data using pictures, tallies, tables, charts, bar graphs, line graphs, line plots and stem-and-leaf graphs.

10.A.2c  Make predictions and decisions based on data and communicate their reasoning.

6.C.3b  Show evidence that computational results using whole numbers, fractions, decimals, percents and proportions are correct and/or that estimates are reasonable.

Procedure:

A warm-up of mental math that includes oral drill in percents.

In groups of three, students will draw a cube from a sack without looking. After each draw they will tally the result and replace the cube. After 20 draws they will total their tallies, multiply that by 5 and record a percentage. Groups will be asked to try to determine how many of each color cubes are in their bag. It will be stated that each bag contains 16 cubes and all bags contain the same cubes. A recorder from each group will surrender their bag without having looked in it then enter their group’s data on the chalkboard table. We will combine the data for a class total, then the groups will try to determine how many of each color cubes are in a bag. They will be encouraged to check their “guesses” using a calculator.

The bags will all contain the same collection of 16 cubes: 8 red, 4 green, 3 blue, 1 yellow.

Our class totals should yield something close to 50% red, 25% green, 18.25% blue, 6.25% yellow

The combined tallies should total 160. There’s no certainty that our results will yield the expected results but the following totals should approximate our results: 80 red, 40 green, 30 blue, 10 yellow.

Before groups attempt to ascertain the number of each color of cubes I will check for understanding. “What are you being asked to do?” The desired response is, “Try to figure out how many of each color there are.” I will ask, “How might you do that?” I will listen to response and if I don’t hear a good strategy I’ll suggest that they use their calculators to convert fractions to percents or decimals (or percents) then use that figure to determine how many out of 16 that might be. For instance, if we find that collectively we found that 48% of the time red cubes were drawn then groups may decide that 48% is close to 50%, which means one half. One half of 16 is 8, so there are probably 8 red cubes.
I expect that in our discussion and my encounters with individual groups I will ask, “Which color is least likely to be drawn and which color most likely to be drawn.” Those should serve as good starting points in solving the puzzle.

After preparing bags for the activity, I decided to test one. The first 6 cubes drawn were red. At first I panicked. Then I realized that we will be looking at a larger sample when we combine our results. In this may emerge the crux of the lesson. I will ask, “After we looked at the data from the whole class, did your group change their findings?” And, “How did the new results affect your analysis?”

Evaluation:

This is a small group and whole class exploration. Getting the “right” answer is not as critical as gains the groups are likely to make by interacting with other group members. So some of my observations during and after the activity will center on social concerns. Did the groups use their time well to solve the mystery? Have I structured the exploration to help them achieve meaningful and reliable results? There’s no telling what numbers we’ll be working with. But group work can be deemed successful if they decide that most of the cubes must be red and yellow must be pretty rare.

Materials:

Each group’s tub should contain a recording sheet, an opaque mystery bag with 16 cubes in the planned numbers, a calculator, a small tablet, and 2 pencils.

Group Assignments:

In the interest of conserving time roles are pre-assigned: Recorder, Bag Holder, Cube Puller

When Harvey sent me the plan I saw a great opportunity to see if I could observe and note general teaching situations which would be applied across various activities relating the law of large numbers. In the case of this activity the goal would have been achieved if students realize by the end the difference between the tallies record in their group and the tallies record as a whole class. This was the law of large numbers (which has not been necessarily stated and mentioned during classroom discourse). The lesson offered me a good opportunity to see the effect of teacher’s mathematical knowledge for teaching and of the “mental chunks” of his knowledge and understanding for the quality of teacher actions in the immediacy of the interactional context. In their article, Stylianides and Ball (2008) claimed the possibility to
identify situations of classroom teaching, which would represent specific mathematical knowledge for teaching, as a form of integrated knowledge. These researchers acknowledged that variations exist in activities, but their belief has been that there are general situations, which could be identified for teacher’s knowledge of mathematics. In this lesson, Harvey seemed knowledgeable of the general situations that would apply for this topic and also of this activity, still the enactment of the teaching proved that the teacher had difficulties although his knowledge informed him of the conflicts with his planned or intended actions. Equally he was also knowledgeable of the way to enact discursive routines to probe students thinking. I tried to compare how the specific knowledge of mathematics and the routines of classroom discourse would support his actions during interactions with students. I was also interested to see if there was a characteristic relationship between his mathematical and pedagogical thoughts and his discursive actions to support his interactions with students, especially this was a fresh lesson. I had the chance to see how the assumptions carried about teacher knowledge to support novices actions are happening with an experienced teacher.

“I think you are confusing.” In this episode, which followed immediately after Harvey explained to students the procedure and asked for questions I had the opportunity to note the first encounter of a possible general situation which Harvey tried to recall in his action to respond to a student’s misunderstanding. It is very interesting to observe how although Harvey perfectly understands the student’s mistake, and he had a potential example, he also realizes that the situational misunderstanding requires time to be processed by the student and possibly could have been responded to differently and more efficiently.
**Episode 5.2**  
Harvey is the teacher and Marteen is the student. Single parenthesis () marks barely audible response, <> mark that the speaker spoke slower, and <> the speaker spoke faster:

Marteen:  
Do we have to aaa…write the number of cubes…((inaudible)) the cubes will be off

Harvey:  
So… What do you mean the number of cubes? (Cubes) will be off?

Marteen:  
I mean…aaa…because if you when you get 15 blocks and you draw them each one and one, then you have to get it over…

Harvey:  
Yeah…You know what I think you're doing? You're confusing the number of cubes that we <have> in the <bag> with the numbers of trials that we're gonna take.

So, if I hand you three coins and said: Flip these three coins all together 25 times, it wouldn’t matter <at all> that you were not just flipping them three times. (Or something)

> You know what I mean? I'm not sure you do.<

Talk to your partners about that.

Marteen:  
Ok

Harvey:  
And we’re gonna back up if still are questions. All right now remember what I said about move where most of the people in your group sit?

So, recorders go ahead and get your ((noise increases)) material,

Harvey hurried into a mathematical discourse about the activity. It was immediately followed by his pedagogical intentions, some conflicting. I had the chance to analyze his pre-lesson reflections. It seemed an integrated activity, but during enactment, under non-routine situations, it unexpectedly split.

This episode 5.2 is part of the beginning of a lesson in which the teacher had just finished explaining the task. When asked to pose questions about the task, one student produced a response to which the teacher needed to respond. However, it seemed to be a matter interfering with both the teacher's pedagogical intentions and the mathematics that the student should perform. Harvey’s response also was responding to student mathematics and only after responding he realizes the instructional effects of his interpretation. He knows how to prompt the student until the student details his question.
The student is curious if it should count the cubes they draw off with the numbers of trials. Harvey immediately accounts what that means for the activity. He also deliberates a confusion that may lead to a mathematical interpretation. It is quite clear that he recognizes a potential situation. He also hurries on the moment to also offer an example why student question could be misleading for farther mathematical understanding of the activity. Still he seems to realize that it may require specific practice for the students to grasp his examples and he switches to his next instructional move suggesting students to “talk with their partners about it”. Still he seems to have doubts if this would support or not the purpose and goal of activities.

In a sense this breach in teacher reasoning about his actions had to do with his pre-lesson question: “Have I structured the exploration to help them achieve meaningful results?” Still student question seemed to direct teacher thoughts on a different topic terrain, although still in the area of probability. The teacher recognizes and identifies confusion with mathematics understanding consequences but there is no specific trigger for an instructional action to fit the situation. A novice who would come with such an understanding and action would be probably evaluated as offering particular affordances for student learning. In this case, the teacher, however, seems unsatisfied with his response and the alternation of mathematical and pedagogical thoughts reveal an interesting jump from a previous intention to another, as specifically seem to be anchored in the local interactional context. If this episode seems dependable on local context and rather detached from the lesson topic, emphasizing potential mathematical relationships of this activity and pedagogical interventions, the next episode seems anchored in a different situation in which Harvey realizes that something was wrong with his lesson planning and it may affect the overall result and the success of the activity.
“I made a big mistake in there.” This episode relates with Harvey’s plan to show the percentage of each color and have students engage with corresponding calculations. In his discourse and presentation he however realizes he made a calculation mistake which does not necessarily affect the activity as much as students’ anticipation of what is supposed to be seen in this activity.

**Episode 5.3**

Harvey: OK, so … five, ten five … How could that be twenty? I think that’s …

Student: ((Inaudible))

Harvey: How many do you have here? Is it twenty? … So, five there … you know what? I take my … Yeah, I made a mistake here. Oh … get out. Noooo. I think, yeah, that’s the recorder of the time. So, that’ll be 25% there.

Student: You need to divide the number by fifteen?

Harvey: I think you have to find the percent here on your sheet then multiply the number you got here by five, and that’s where you …

Student: Ok, but by five …

Harvey: Come here, come here … During the … You have to go ahead and … the bag.

Harvey: OK … Is this your group? You did three. Three times five is fifteen. That would be fifteen percent of the time it came up that way. Oh, think about it. I made a big mistake there. So, yeah, ten out of twenty, multiply that by five and you get fifty percent. It looks like you can divide by five because twenty turns into a fraction. It will be some [part] of a hundred. So, multiply by five to have it …

I illustrated through the analysis of these selected episodes that the teacher uses specific conversational devices to project the next action for students. The teacher "creates" instantly a local context for students’ responses. Students, by producing their action show, or do not, an understanding of the action before. These three excerpts illustrate different situations in which the teacher initiates the action. All three examples exemplify how the teacher not only initiates
student actions during his turn taking but also is influenced in his action by what students say and do. The interaction attends to both teacher and students expectations.

The main plan for this study was to see how the teacher's thoughts are shaped through the interactional meaning. The sequential order of teacher-student exchanges influences teacher procedure to categorize the next student action. In the next chapter we will use these categorizations to identify the cluster of concepts behind the categorizations and reveal the relationships behind the teacher's mathematical and pedagogical thoughts.

Fractions and Decimals

Compared with the previous two lessons based on discovery pedagogy, Harvey considered the lesson “Fractions and Decimals” a routine lesson. At some point, during the second year, he suggested I should see a routine lesson, considering that mostly I had observed activities based on discovery learning.

"Here is your last question." This next excerpt shows interruption, a jumping from one idea to another, from one teaching action to another. Interrupted reasoning--as I described in the previous section--is expressed not only in the tension with the pre and post-lesson thoughts, but also how gradually it becomes more dynamic. Teacher interactive thinking jumps from one thought to another. It shows a flow of thoughts, not only as a response to students, but also as initiating the discourse. In their study, Borko et al. (2000) noted on the difference between teacher education discourse as passive, and the necessity that in actual teaching the teacher to have an initiating role. Similarly, in his study comparing novice and experienced teachers, Leinhardt (1988) remarked that experienced teachers would be more aware in their planning about the instructional actions they would engage with. In this lesson excerpt, that was not the
case. In a span of less than 5 minutes the teacher took about eight instructional initiatives, mostly rolling from one thought to another, hardly part of a plan.

The dialogue has been part of an activity in which the teacher asked students to write fractions respecting certain conditions using the ten digits: 0,1,2,3,4,5,6,7,8,9.

*Episode 5.4:*

Harvey: Here is your last question.  
For this I would say there is a variety of correct responses.  
So you and your neighbors may not *show* that same thing.  
What is the smallest fraction you can write with any two of these digits?

Student: You can get nine …

Harvey: Nope! You can get smaller than that.  
Do you remember the first one, the first one I asked you, “What’s the smallest fraction you can make that is greater than zero?”  
I didn’t say that this time!

Student: Aaaaah …

My fraction …

It’s like …

You have one …

Harvey: (talking with a student) Are you using more than two digits?

Student: (talking) All right. Now … Look at your own answers.  
There could be a variety of answers, ‘cause Marteen has actually shown them all!

Student: I wrote it also!

Harvey: That’s all right.  
So … Show me. Explain it to me.

Student: (arguing) Cool!

Harvey: (talking with a student) So, tell me. Explain. Yeah, I ask tough questions.  
How does it work? Why are any of these the right response? Ben, could you sit down, stop interacting, but interact with me a little. …Dresiana?

Dresiana: Because they all zero!

Harvey: What do you mean?

Dresiana: (inaudible)

Harvey: Do we have to move those people back into rows, even rows, with two and half feet between each child?  
Dressiana, what is it about these fractions that makes them the smallest they can be?

Dresiana: Each fraction is zero.

Harvey: OK, each of these fractions is equal to zero. Then who can prove it?  
Which words?

Jo, you probably could …
What do they mean?

Jo:   ((Inaudible))

Harvey: OK, Remember the denominator is the bottom number. That tells how many equal size pieces there are. If we had a physical model and the number on top, the numerator, tells how many of those pieces there are, so if there are zero of those pieces, well that sure’s not many, is it?

All right, clear your board.

Now I want you to write a fraction that’s not possible.

Students:          ((Noise))

Harvey:  Don’t say anything … Just write it. Did you use the same digits twice? … They were in the rules so …

All right! Show us! Yeah, if you have zero as your denominator, you are right … That’s a fraction that doesn’t exist … and why not, Renata?

Renata:  (Inaudible)

Harvey: There isn’t such a thing as zero? Yeah, but why can’t you do that?

I can not hear you.

Students:  Yeah, I think they represent a part that doesn’t exist …

Harvey: And remember what I said about calculators? What a calculator basically does if you try to divide by zero? Yeah, you cannot divide by zero. It is gonna tell you, yes, an error or something of the kind …

The teacher said that this was the last question, then he used the moment to launch a review of other mathematical concepts. I would add that Harvey attempted to keep these learning experiences alive in the students' memory. It was an extension of the situation of interrupted reasoning. If in the previous lessons, I pointed to specific cases such as a student unexpected intervention, or teacher in his interaction with student realizes a necessary switch of thought, this five minutes exchange dialogue shows the complexity of classroom interactional discourse and how the teacher had to keep up with the three interrupted reasoning situations mentioned previously. In my attention for the way Harvey handles his discursive routines I had the chance to see a different interrupted reasoning, mostly focused on the alternatives the teacher ponders for student understanding of a routine situation. The macro-view sees such episode in the rich knowledge of representations, still research has pointed how difficult is to use such
representations and make a good use of them in classroom teaching. In a sense I related this difficulty at the discursive level with the difficulty in using an appropriate example.

"Let me give you a different question." In the previous episode the teacher moved from one student to another, leading the dialogue from one mathematical idea to another. The following classroom dialogue is mathematically focused but directed in various pedagogical and instructional directions. In this episode of the same lesson the teacher used tiles to model fractions and their understanding. Teacher displayed a geometric figure on the overhead projector and students had to imagine the one, or the whole. Immediately after the tiles activity, the teacher replaced the geometric tile with a specific number of counters. Students had to respond how many counters were included in a “whole” or a “one.” Much of the teaching routine has been focused on the procedures used to think about the whole or the one. Still at some point a student, Colby, has problems to respond. The teacher uses this case and reinforces not the procedures but the variety of contexts for the procedure.

Episode 5.5:

Harvey: If this is one third, what is the one?
Colby, if that’s a third, what does the one tell me? How many counters will make the one?
Are you telling me that the whole is a four that will be a three fourth this is a third…
If twelve is the whole, what is a fourth?
What do you do to find a whole if you know what a third of it is?

Harvey: Yeah, you need to triple it, you need to times by three,
So let me give you a different question, Colby: If that is a third, what is whole?

Colby: (inaudible)

Harvey: Fifteen is right.
If this is two fifths, what is the whole?
Jerry?

Harvey: How did you figure that out?
OK. Your way to figure that out, if this is two fifths, then what must be one fifth, so that’s two fifths and here is another fifth and another fifth …
If these are two thirds, what would three thirds look like?
Or what is equal to one?
Show me with your fingers. If you showing me six, I agree with you…

Student: I knew it!

This episode illustrates how the teacher based on student difficulty to carry a procedure, re-directs his intentions in various mathematical contexts and moves from counters to fingers as part probably of the recalling routine. The episode certainly made me reanalyze how teachers use chunks of their teaching experiences and how the relationship between mathematical and pedagogical thoughts play a role in this recall and if the relationship is important for teachers actions. I remarked once more the interrupted reasoning which this time has been transparent in the variations of his pedagogical directions and mathematical knowledge clearly one independent of another.

If at the elementary level, teachers seem more inclined to ponder their activities and pedagogical aspects of mathematics thus giving me the opportunity to assume that there is more liberty and space in teacher pedagogical thinking I wanted to consider if there is different at the secondary level. Would the richer and more complex context of mathematical relationships influencing the integration of teacher mathematical and pedagogical thoughts? Would this in turn prove influencing teacher actions?

Mike Case Study

The pseudonym for the secondary teacher is Mike. In one of our casual discussions about thinking mathematically and pedagogically, Mike offered me an interesting way to see the relationship between his mathematical and pedagogical thinking:

Luisa: I know it is quite difficult to be aware of your thoughts in the moment of teaching. Much is experience, preparation of the design of the lesson and
routine in teaching, but how would you describe your thinking while teaching?

Mike: Thinking of what questions to have, when I say too much, what else I have to say, etc. … Pedagogy is very much like mathematics, many patterns out there. So practically, it is just an optimization of the connection between the patterns (numbers) of mathematics and patterns of pedagogy. … It is the content that dictates. It is the priority for teaching mathematics, not the method of teaching.

Earlier, Mike had mentioned that interactions with students are most difficult because they require "appropriation" and "attunement" with students and with what happens in that moment. He said he would recall planning the lesson just sufficiently to want to make changes and have ideas about what such changes would involve.

"The joy of radians." This unit was last in the functions section of the algebra class. The teacher had several commitments in mind and among them was to make students think "unitless" before approaching a more detailed analysis of trigonometric functions. I described how the teacher prepared his lesson for a new unit: Radians. But before that, he engaged with this "unitless" idea to get students focused on the conceptual aspects of the models offered. Here it dealt with the transformation of linear motion into circular.

Interview pre-lessons on Radian and Trigonometric unit

Mike: I do not think that today will be much about misconceptions although there will be some start minutes as they will think about revolutions per minutes, feet per minute, and miles per hour. So units and conversions are issues that will arise today.

Luisa: OK.

Mike: And I hope we will get those done.

Luisa: Any awkward feeling about how linear motion transforms to circular?

Mike: Oh, the most important thing this week happens tomorrow. That’s going to be actually collecting data about the height of the dot on a can as the can rolls and the position of the can on the ground. So, if they’re careful in their data collection, they will produce a sine curve They will divide all of
the measurements by the radius, then everything will be unitless. That will be an introduction to the radian measure. So, it’s the data collection tomorrow both introducing the use of the height of the dot unit circle and introducing radian measure.

Luisa: Yeah, I recall it from last year.

Mike: So, today, it’s just to get them to think about motion around the circle and how the speed of the motion around the circle is related to the how fast it is spinning and to the distance traveled.

This interview helped me understand the reasoning and intent behind his actions for this unit goal. I had also the opportunity to record his knowledge of what seems to be “the core” of the unit. It was not a simple activity, but one on which Mike would have students understand later the trigonometric functions and why mathematicians prefer radians to other units of measurement. I was interested to see if he would make this reasoning explicit during his teaching. If yes, how that would be accepted by students. If not, what would correspond to this macro perspective on a general intention for a mathematics teaching situation? As I will illustrate with the below episode, Mike also interrupted his reasoning as happened in Harvey’s case. In this case, as in the case of Carnival activity, the teacher jumps from listening to students’ thinking to a mathematical norm.

**Radian definition.** In this episode Mike continues the discussion with students on the radian definition. He required students to offer a definition of radian which was not just a simple expression of a formula, but rather a definition to show that the student understands the reasoning behind the definition of the radian.

*Episode 5.6 Formulation is bad.*

Mike: Yes, Bridget?

Bridget: (Inaudible)

Mike: Why?
Bridget: Because the number of radians tells you how many radians you are and you think about the number of radians times the radians …

Mike: So, Bridget’s formulation is bad. Radians times the number of radians, but if we think: as we go around, the proportion of the circle p then, that would be the same as the proportion of the circumference. What’s the circumference?

Sophie: $2\pi r$

Mike: $2\pi r$ … So, we’ve had p times $2\pi r$. If you think about the proportion of the circle as being p and the circumference as being $2\pi r$ then Bridget’s idea is exactly right … because $2\pi p$ represents the number of radians. So, you don’t have to do all that reasoning that Sophie talked about, although it is absolutely correct. It’s much easier just to note that, from the definition of what a radian is, the number of radius lengths you go around outside the circle. That’s what a radian is. So, if we go $\pi$ radians, that means we go $\pi$ radians lengths around the outside of the circle. Well, that’s what we want, that’s what arc length is: $\pi$ times 15. So 15 $\pi$ inches would be the arc length.

The teacher reasoning seems to be broken by how the student explains. Still he recognizes student thinking and makes a distinction with a poor formulation. Definitely the teacher is aware of the mathematical knowledge required for teaching this moment. More than this he perceives such thinking. His discursive moves suffer, possibly because students have a difficulty in communicating the idea of radian. It seems that they want to use something they know in general mathematics terms. I expected Mike would use previous lesson when he pointed to the dimensionless nature of radian, but it seemed that Mike was rather driven in his action by the activity students had to accomplish and the performance students were assumed to display.

“I am thinking that’s really useful.” This was a dialogue I had with Mike one day after teaching the lesson on radians and the episode with student who struggled to offer a radian definition. I was very interested to understand how Mike moved his actions from one direction to another. He particularly wanted students to fill in a table. The activity could have been easily accomplished by the means of a formula. Mike pushes towards a more than just using formula
approach (as he points in the comments below) but had a conflict with the realization of the activity. It was difficult to adjust the conceptual aspect behind radian understanding and the formula use. Both issues related to a definition of a mathematical object. I was not sure how I could obtain such information and I tried to avoid as much as possible to ask why he acted in a way or another.

Luisa: What would be the most challenging mathematical thinking in all this process? I mean, the way they have to think, mathematically …

Mike: The most difficult thing in my experience with teaching trigonometry is to move away from the idea that when you take the sine of something that something has to be an angle. That’s why, in the first activity, there is nothing about geometry. There is nothing about angles. The periods are all in time, not angles in the can rolling. The period was a distance not an angle. … I got it inside out in the classroom you observed Wednesday, where they had the table and they were supposed to think about the heights of things. … My goal was not to have them think about the angles at all until they pretty much completed the table, and then mention the sine and the cosine. but one student mentioned the sine and the cosine and I jumped on it. And then I realized I shouldn’t have, but …

Luisa: OK … That goes to my next question. What do you do if a students asked you what … I’m trying to remember, she saw in degrees to some point when you made the transformations.

Mike: Right.

Luisa: The way she put it, practically taking the circle and dividing the number … but, yes, what would you say … I know it is hard. Would you try not to tell her to make the connections? Where would be that border be when you would tell to stay with the degrees and not come back …

Mike: It really depends on the student … I mean, if I see how a student thinks, there are some students who want to see the formula. If I said, to get from radians to degrees, all you have to do is multiply by 180 and divide by $\pi$. You’re set. Some people would say, OK, I’m done. I don’t have to think anymore, that’s good, I am happy, I can do that. But I didn’t say that to anybody because I don’t want them to think about converting from radians to degrees just in terms of a formula, I wanted them to think about radians as part of a circle. So the way she described it actually was good thinking. I tried to see if measuring might be a shortcut to the thinking process. I mean, she reasoned it out, but some of the algebra in the reasoning wasn’t quite there. So, some of the simplifying steps … I wouldn’t necessarily
have explained it the way she talked about it but it was right. I didn’t want
to mess with that construct in her mind, but I did want to point out that if
you had that construct, then you can maybe simplify the algebra and come
up with a new construct that maybe is little bit simpler to think about.

Luisa: When listening to her and responding to the entire class, what got them
into mathematical thinking much more, or student thinking, or both of
them? Listening to her response and, after that, getting back to the class,
was that forcing the thinking toward how she thought or more to the
mathematical thought itself?

Mike: Yeah ... Little things like that happen frequently. Students will say
something in a slightly different way and I’m thinking that’s really useful.

Mike’s declaration to use students thinking, particularly in this case when a discussion
like this would not necessarily have been the point of the activity, made me reflect once more on
teacher interrupted reasoning during classroom interactions. Instead of interpreting this moment
as teacher problematic discursive movement I preferred to attend to teacher initiative on thinking
during the interactional moment differently. Schoenfeld (2000) in their study on how teachers’
actions could be predicated by a composite of cognition would have said that an experienced
teacher knows how to act in the context of teaching. But I paid more attention to the interactional
situation than to the general mathematical situation as pedagogical.

**Not thinking about angles.** At a similar post-lesson interview about a week later, Mike
reveals a most interesting aspect about talking in classroom about something that actually his
reasoning tries to avoid. Still in the moment the situation calls a different action and he although
seems unpleased the situation appears he is satisfied that he had to respond such a situation.

Mike: As you were talking about the physical model and motion around the
circle, I was thinking I wanted them to have more than that.

Luisa: OK.

Mike: On the very first day of this unit, I gave them some data about the
illumination of the moon, the hours of daylight, those sorts of things. I was
thinking that the way I’d like them to start today was by looking at the
percent illumination of the moon. I’ll see if I can find some data.

Luisa: It’s not necessary.

Mike: So, find some data for this year to see where the first full moon of the year
was and … So, see if we can model the percent illumination for every day
in this year but then, to talk about the domain and range of that function.
So, not only to be able write down a sinusoidal model for it, which is what
we did yesterday, but to think about that formula as a function, to think
about what the domain and range are in context, and then use that and start
looking back at the unit circle, and get domain and range for all six of the
trig functions …

Luisa: So, I use that circle … and this periodic model … Yesterday, when you
showed them the sinusoidal example on the phase shift, you had three
kinds of function.

Mike: Right.

Luisa: For the second one, I guess you could have had two kinds of function?

Mike: I had the sine, the opposite of sine, and the cosine. I could have had an
opposite of cosine.

Luisa: Yeah, I think it was the opposite of sine. Yeah (laughing). It was the
opposite of sine but there were two possible ways to write that function …

Mike: Right, because they were on the graph that I’ve shown. There were two
obvious starting places.

Luisa: Did you purposefully choose …

Mike: No, two different students gave those two answers.

Luisa: So, it was actually in that moment yesterday that you realized you have
two. I was just thinking in terms of preparing this lesson and purposefully
choosing that way …

Mike: It just happened. I did plan to talk about how many representations I have
there. There were infinitely many more you can get, each just adding the
period of 2 π to the phase shift, and on. The students are already given
two different examples with the same function with different phase shifts.
It was a useful thing. I did plan to talk about that.
The lessons on radians showed a complex mathematical concept, radian, which the teacher took it in an apparently integrated manner. I found the teacher very knowledgeable of both the general mathematical situations as well as the mistakes students may make. I saw no direct influence on teacher action due to this knowledge although the focus of teacher discourse seems to address this mistake. Still his actions were driven by what students said, as he recognizes, and his reasoning was interrupted by various relations he found within the local context.

**Rational Functions**

This episode is part of the lesson on the rational functions unit. It was the first lesson and the teacher’s intent was to establish particular grounds for further analysis of rational functions. At this point during the year, students knew general concepts about functions and Mike used and built on several of them. He divided the lesson in two parts. The first part was more concerned with mathematical routines. The second part of the lesson pushed towards what Mike called “number sense.” I found his use of number sense particularly special at this mathematical level.

**Pre-lesson teacher instructional actions.**

Mike: One issue is “number sense.” What happens if you add/subtract one from a really big number? What happens if you divide [two] really big numbers that are really close together? What happens if you divide a “not so big” number by a really big number? What happens if you divide by a very small number?

Mike: A more subtle, but related issue is signed numbers. Can students recognize/predict the sign of a rational expression at different values of x? With the “making rules” page, will students recognize the “leading coefficient” of the numerator of k(x) as -3?”

Mike: Planned Instructional Actions to Preserve the Pedagogical Reasoning Behind: “I’ll probably have to intervene to guide discussion through the “end behavior” analysis (large absolute value x). I may have to prompt
students to think about what happens “near” where the denominator is zero.”

He does not talk about exactly about what happens “near” as much he suggests to them what to do. When I asked him what “mistakes” he anticipates he answered:

Luisa: What student mistakes do you anticipate?

Mike: Not so much a mistake, but with the "graphs" page, students will have difficulty explaining how f(x) acts for large values of x. For the third example, they'll struggle with what happens near x = 1 as well.

In the "making rules" portion, I set the first three examples to have the same domain and same vertical asymptotes, but with three different "end" behaviors. I expect students to be able to see what happens, hopefully make connections between degrees of numerator and denominator and existence of horizontal asymptotes, and connect leading coefficients (for g) with location of the horizontal asymptote. They should be able to confirm their hypotheses, with functions j and k. Function m is tricky. It has no vertical asymptote, even though the denominator goes to zero. Some students may get stuck here (though I expect most won't make it that far.)

Teacher moves are rather independent of the relationship between pedagogical translation and mathematics. Instead, they are part of a repertoire to preserve the norms of the classroom and especially of the learning philosophy. It is interesting to observe here how the teacher has attention for so many issues that, along the classroom discourse, will be approached only for several seconds without much developing, as they are captured in the lines above. Below however we could see that in the interaction with a student he attempts to preserve the relationship between a particular mathematics and a pedagogical approach, but abandons the relationship in favor of the mathematical norm, of the definition. Apparently the student tried to copy his modeling of a previous example, but without a proper understanding student performance was problematic. The teacher does not attempt to refer to his previous example and build on it. Instead jumps to the definition and its use in this case. The teacher is aware of the student confusion. In Harvey’s case we encountered a similar situation when the student makes a
certain mistake and the teacher attempts to identify it as part of student thinking. In the below episode the case is also identifying the mistake (relatively easily) but again the teacher faces a delicate pedagogical aspect on how to respond. Harvey offered an instructional analysis. Mike refers to a definition as was stated and has to be respected.

**Definitions and Examples**

The episode is interesting because in the first part of the dialogue one could observe what seems to be the relationship between the mathematical and pedagogical thoughts. The teacher uses examples to support the assimilation of the particularities of a definition. Still the examples do not provide a sufficient support for his teaching action.

*Episode 5.8 All real numbers except real numbers*

**Mike:** A rational function is a ratio of polynomials where the denominator can’t be zero. So the roots of the denominator are not in the domain of the rational function. I have some examples for you. My first example: 3x+2 divided by x²+1. It is a rational function because it is a ratio of polynomials. The denominator is x²+1 and that is never zero. Therefore, the domain of that first rational function is all real numbers. OK?
The second rational function (pointing to function (3x+2)/(x²+x-6) on the PowerPoint Presentation) You should be able to factor this denominator. Yes what are the factors?

**SSs:** x+3, x-2
**Mike:** x+3, x-2? Ok, so this is all … The domain of this is all real numbers except those…except… 2 and -3 which are roots of this polynomial. OK! Rational functions even though they are hybrid, a sort of mixture, … they are a composition of polynomial functions. … They show up a lot … Average production cost … where you look at the cost of producing x items divided by the numbers of items produced … is often a rational function … We will see some other uses for rational functions. In photography, rational functions show up in the law of optics. It shows you how far … how to focus the camera … So the focal length … It’s the
equation that equates focal lengths and the distance to the object. That’s a rational function.
Here is a lovely rational function. (points to the function, \( \frac{3}{x^2+1} \))
What is the domain of this beast, S.K.?
S.K.: All real numbers.
Mike: Why?
S.K.: Because there is no value for \( x \) be negative … (inaudible)
Mike: OK. There is no value for \( x \) that can do what?
S.K.: Be negative for the \( x \) …
Mike: It has nothing to do with negative numbers or zero … or …
What is the domain of every rational function?
It’s the set of all real numbers.
Except …
Roots of …. denominator?
So … when is the denominator of this thing zero?
For what real numbers?
For all real numbers … I said all real numbers except real numbers when the denominator is zero. When is the denominator zero?
SSs: Never!
Mike: Never! OK? So the domain is all real numbers.
Is this function ever zero? When is a fraction equal to zero?

We can follow how the definition appears in three ways to support certain flexibility in its application, but how, in this flexibility, the student still misses important aspects. The way the teacher tries to turn back is not an integration of his mathematical and pedagogical thoughts, but simply a mathematical norm, which directed his thoughts towards his instructional action.

"Number Sense"

In the following episode, the teacher relies on his pedagogical metaphor, "number sense," to respond to the students. The interesting aspect of this is that in the mechanism of transforming mathematics into mathematical tasks and mathematical tasks into instructional moves, pedagogical metaphor offers "loose" connection to the implementation of mathematical tasks in the classroom. The pedagogical metaphor offers an alternative resource for the teacher to create meanings from students' responses. These two episodes take up similar tasks: analysis of
functions behavior without seeing the graphs of the functions. The first episode is dominated by student utterances, while the second is dominated by teacher utterances. The first episode is thus student-driven, whereas the second is teacher-driven.

In this episode students worked on the analysis of the function \((x^2 - 1)/(x^2 + 1)\). I was trying to capture how in this dialogue the channels of communication between teacher and student are particularly evasive. I could not stop thinking how teacher reasoning about number sense is used in this case. The teacher accepts a part of his reasoning and he is happy to see that the student used just that part. In the following episode, however, is evidence that Mike tries to turn back to his intent and reasoning and elaborates it.

*Number Sense Episode 5.9:*

1. Mike: S.D., can you share your discussion? (addressing a student) (addressing the group he was talking to) You guys did a good job. (addressing the entire class) Everybody listen to Mr S.L.
2. S.L.: OK. The domain is all real numbers.
3. Zero at 1 and negative 1 … For very large values of x, \(f(x)\) is getting closer to 1.
4. Mike: \(\Rightarrow\) Why do you say that?
5. S.L.: Because…
6. If the fraction keeps growing … all….
7. Mike: OK … (With a sigh).
8. That’s the …
9. S.L: If you have like 10 and then it is 99 and then you get 101
10. Mike: (with an enthusiastic tone) OK!
11. S.L: Which is closer to one, then you get 102.
12. Mike: Aha!
13. S: So, … the more and more … the higher the number the fraction gets closer to one … smaller and smaller…
14. Mike: OK, that’s good reasoning …That’s a kind of number sense I hope that you all have! So, that’s good reasoning.

In this episode the teacher takes the lead of the conversation also with a point to focus the attention on what is important in terms of mathematical particularities. Compared with the episode on definition, this is a more complex aspect for student understanding and similarly with
Harvey’s case after the explanation of the problematic understanding, Mike is turning to students to have them think more about such a possibility. Compared with the previous episode he does not seem to rely on what happened with the student and how his answer in previous examples may have been influenced by other students’ understandings. He gets back to the number sense and works on it, accordingly with his plan, but still he brings an interesting example (line 17) again and realizes it may not be his best pedagogical tool for the case.

**Episode 5.10 Number sense again:**

1. Teacher:  What happens if $x$ gets very large?
2. Students:  It gets closer to 1.
3. Teacher:  It doesn’t get closer to 1 … ’cause if you had $x^2$, $x$ is …
4. If $x$ is huge, $x^2-4$ is not going to be very different from $x^2$, right?
5. With $x-1$, if $x$ is huge isn’t going to be very different from $x$, right?
6. So what is $x^2$ divided by $x$?
7. Students:  I don’t know… $x$?
8. Teacher:  $x$!
9. So if $x$ gets very large what happens with $f(x)$?
10. Student:  $x$ …
11. Teacher:  It’s gonna get very large, too.
12. Do you understand the reasoning I used there?
13. When $x$ is very large the numerator is approximately $x^2$.
14. When $x$ is very large the denominator is approximately $x$.
15. So, when $x$ gets very large this fraction is approximately $x^2$ over $x$.
16. And $x^2$ divided by $x$ is just $x$.
17. So, as $x$ gets very large, this function’s graph looks like the line $y$ equals $x$ … More or less …

I focus on the sequence here, how students and teacher departed from the preceding talk and how "the meaning" is at some point "tagged" by the teacher. It is interesting to see especially in Excerpt 5.10 how the previous turn "projects" the actions of what is happening in the next. The teacher's role in this projection is extremely important. In this case, he does not leave the students to develop much if he does not know where the student talk is going. By the end, practically what the students say is what the teacher "thinks." The two excerpts show very nicely
what the teacher mentioned in one of the interviews: "I do not leave them to develop much, but when I get a new solution, I welcome it."

Episode 5.9 shows little intervention from the teacher. It shows how he "follows" student reasoning. When in Line 6 the student says, "if the fraction keeps growing" the teacher practically "jumps in," but the student does not let him. He realizes the mistake and continues. Excerpt 10, although having little substantial intervention from teacher, except to "guide" student answers, shows here how this time the teacher "tags" the situation with "reasoning" compatible with his metaphor. This excerpt will be discussed with another excerpt from a different lesson, in the next chapter.

Episode 5.10, to the contrary, is dominated by the teacher. In this dialogue, the teacher is not talking about "number sense," but makes the connection with the mathematical mode. These "approximations" of mathematics are interesting, showcasing the way the teacher "tags" a particular sequence with a mathematical or pedagogical feature of the sequence.

The teacher's metaphor may seem to be just a product of pedagogical content knowledge. From his teaching experience, the teacher realizes students have trouble when understanding rational functions. Based on his own mathematical knowledge and understanding, he builds this metaphor to help him guide his instructional actions. His metaphor is backed up by strong pedagogical reasoning to explain why it should be employed in teaching this topic. The metaphor may have variations across other teachers' practices. Having his metaphor, he could "recognize" student misunderstandings. The problem with such metaphors is the same as with any received constructs by novices: they leap with any meticulous teacher's perception of the classroom situation. However, here I depart from the belief's theory and pedagogical reasoning: This pedagogical metaphor is adapted to the situation. In other words, the perception in the classroom
not only adds up to the teacher's experience and develops this pedagogical metaphor, but also
describes how the teacher employs his mathematical thoughts. The productive aspect emerges
when the pedagogical transformation is used in classroom. How does the teacher carry his
metaphor into the classroom under the perception of students' understandings?

**Perceptions and Extending Recognition**

How one learns and passes through practice teaching is mediated by perception. The role
of perception matures with classroom mathematics interaction and as part of the cognitive loop.
The different aspects of intention and of understanding situations are influential not so much
through their permanent features but through experiential understanding. How perception is
differentially shaped is partly based on the ever-unique experience of discursive action.

Teachers' perceptual knowledge remains the key element in which the teacher connects
mathematical and pedagogical thoughts. In his analysis of psychometric and pedagogical
perceptions of student assessment, Stake, (1992) graphically displayed some of the complexity of
teacher thoughts:

> Complexity of teacher thinking was illustrated earlier in Figure 2 by seven mathematics
> items. To a testing person, these items are points on a single scale; they measure
> essentially the same thing. To a teacher, each is unique. The statistical correlation among
> the seven would run high but each item requires its own understanding of terms and
> operations. The mathematics teacher extends instruction to the details of each item.
> Getting any six of the items right does not assure getting the seventh right. To a teacher,
> mathematics achievement is not just getting the best score on the test, it is understanding
> and performing the work. (p.10)

We need to better understand how the ways mathematics teachers "visualize the
experience" (Stake, 2002, p.303) of their students learning of mathematics as much as to make
visible in teacher education aspects of teaching we want novices to see. The reason I took up the
analysis of classroom conversation is that mainly in teacher education the analysis and
recognition of various conceptions of mathematics teaching is based on readings and discussions of real classroom excerpts. I wanted to be engaged in a similar process and to understand what could be seen more than rationales and reasons for teachers’ actions. That learning should enhance the quality of mathematics teaching. I tried to achieve two aims: on the one hand, I analyzed transcripts again and again, trying to understand the nature of the relationships between the teacher's mathematical and pedagogical thoughts behind their instructional moves. On the other hand I tried to begin a future path to convey this understanding in teacher education through scenarios or "workables" which would offer alternative procedural meanings of interactional teaching practices. While certain aspects of teaching may not be teachable, they may mark indirect paths to teachability.

**Summary**

In this chapter I presented a number of episodes, dialogues or workables that illustrate continual interruptions in fine-grain classroom exchanges. They provide a rough idea of the immediate mathematics and pedagogy the teacher was thinking about. These episodes remind us that classroom exchanges between teacher and students are a panorama of interrupted thinking. The talk is more than doglegs and backtracks along an instructional path. It is branching from target to target in content and jumping from pedagogical ploy to ploy. These exchanges sometimes are extended discourse, but even while the teacher alone is talking, they are marked by multiple purposes and interrupted logic.

I was not studying the students here, but let us notice that it would be a mistake to presume the teacher's interrupted thinking was ineffective or undercut the student learning process, although both could be true. The content of mathematics may be regular and smooth-
flowing in its ideal disposition, but even in the thinking of master teachers, it branches from scheme to scheme and jumps from relationship to relationship. Experience with ill-formed instructional dialogue may be important for learners than the well-edited logic of textbooks.

The episodes were laid out in transcript form to provide the reader opportunity to recognize mathematical and pedagogical features. This was a continuation of the effort of Chapter 4 to clarify the methods by which micro-analytic detail might differ from macro-analytic representations of professional practice knowledge. The teachers' use of such features for dealing with unexpected response or for changing immediate purposes was demonstrated.

When the teacher changed direction and said, "Can you reduce fractions to make simpler terms?," followed immediately by "What's half?," the interrupted thought about representing fractional parts was in turn interrupted to resume extrapolating unity from a fractional part. Both were important in the teacher's thinking. Any dialogue could be composed differently to reduce the jolt of interruption, but dialogue is seldom well composed and the thinking behind it must be also seldom well composed. In this chapter my interest was not so much in the integrity of sequential utterances but on the simultaneous presence and disharmony of mathematical and pedagogical thought.

This is not to say that a disharmony and apparent illogic would miss a deeper meaningful structure in teacher thinking. Contrary, I would like to point out that such “failure” to stay with planning is an essential part of an effective process for efficient communication and that an apparent strength in a well-composed discourse is rather ineffective at building shared meaning.
Ringle and Bruce’s study (1982), which analyzes the prevalence of “conversation failure” and the essentially creative response that interlocutors apply to deal with it points towards:

Speakers frequently misunderstand one another but are somehow able to detect and repair one another’s errors. Conversation failure, in fact, appears to be the rule rather than the exception. The reason that dialogue is such an effective means of communication is not because the thoughts of the participants are in such perfect harmony, but rather because the lack of harmony can be discovered and addressed when it is necessary (p. 203).
Chapter 6

Two Mechanisms of Relating Mathematical and Pedagogical Discourses in Classroom Interactions

“if we provide only apparatus and tasks, novices can abdicate from teaching mathematics because they are unaware of how to make sense of the environment by imposing teaching relations upon it. Somehow, tasks offered to teachers need to afford opportunity for working on mathematical relations and forimagining how they might enable others to develop similar awareness.” (Watson and Mason, 2007, p. 211)

Watson and Mason point to the cognitive effort of a teacher to develop an awareness of the mathematical relations in his students. This awareness takes in both mathematics learnability and teachability. The teacher needs to be ready for the mathematical and pedagogical challenges. Chapter 5 illustrated how teachers, although knowledgeable of what would be identified as general situations of mathematics teaching and discursive routines, have not displayed a guarantee for their teaching actions. The previous chapter also illustrated that what seems to be behind teachers instructional actions is the interrupted teaching reasoning in the immediacy of the interactional context and in the occasioned context of interaction. It was interesting to focus on a closer analysis (as I mentioned in Chapter 4) of the teacher discursive moves both for a better understanding of the mechanisms of teacher interactive thinking, but also for the purpose of triangulation of the previous observation on interruptions as recognized alternatives, especially for understanding the mechanism to recall previous teaching experiences and particular situations of teaching specific topics.

Instead of seeking good teaching, this study attempted also to understand better how teacher and students create a sequence of discursive actions. It also paid attention to what it tells us about their thinking. The study focused on how, mutually, teacher and students respond to and build the foci of their dialogue. In this process, I followed detailed teacher use of mathematical and pedagogical norms and tried to understand specific relationships among them. How were
they preserved during the dialogues with students? As I illustrated in Chapter 5, most of the time the dialogic paths were repeatedly interrupted. Action was possible among multiple alternatives. In this chapter, I focus on the patterns and mechanisms characterizing the deviations from plan. How did the deviations fit with already established relations between teacher mathematical and pedagogical thoughts? In this way, I tried to capture patterns of mechanisms that could be made "workable" for novice teachers.

Similar with “stories” which can be very effective at communicating complex information, because they jump from point to point and invite the reader to fill in the gaps, I tried to understand a mechanism illustrating enthymematic entailment. Premises are omitted, allusions are called upon, but in so doing, the teacher and students co-construct complex understandings.

Current researchers raise questions about teacher use of metaphors and analogies. Which ones are good for preserving the integrity of mathematics? Teacher educators have sought more refined and usable forms of such indirect logic. That teachers use metaphors in their teaching has been largely recognized as a way of thinking. This study focused on how analogical ways of teacher thinking are good for developing mathematical understanding.

This study also re-evaluates how such models of teacher thinking in classroom teaching are represented in teacher learning and what they mean in terms of thinking and creative use. I thus focused on how teachers associate action with experience. What discursive forms of action influence perception? Two mechanisms for relating mathematical and pedagogical discourses are the metaphorical and metonymical. I will focus on them next as I discuss what each brings to learning to teach mathematics.

First, I will explain these mechanisms with examples, to show how they work and how they express the issues of my study. I also show how they differ as domains of professional
knowledge. They are alike in that they each borrow a term form one domain to use in another
domain. I will provide examples of how they augment thinking in classroom teaching.

In a previous chapter I illustrated how analogy is the mechanism used by teachers when
substituting one representation for another. It is more than a simple application of technique in a
particular dialogue. It is not merely to symbolize something, but to make familiar symbols
substitute for unfamiliar ones. The enactment is made relevant by codifying the mathematical
and pedagogical features of the assigned tasks. One can analyze the combination of such
features, yet scarcely see what the developing activity is.

The Primacy of Analogy

One important aspect in terms of communication has been the object of interrupted
reasoning. I tried to follow how such an object seemed to be related to other aspects of teaching
discourse or classroom interactional discourse. I found analogy to be a particular feature of
teacher interactional thinking. Analogy does not refer to what the teacher will say using
resemblance properties. My use of analogy here draws from Hofstadter’s (1999, 2001)
conception of analogy. It relates with the way one person recalls certain memories and associates
them with the present moment. This has been a way of thinking analyzed by many authors. John
Dewey (1910, 1938) in his writing on thinking refers to several examples in this sense. That is,
however, what would be called an individual aspect of thinking. In this study, I was more
concerned with how analogy is present during interactional moments. I will illustrate with two
examples. One is about an unexpected hexagon and another refers to the repetition of a question
that leads to different meanings and has different intents. I tried thus to counter two situations:
one, the unexpected hexagon, in which analogy seems clear, the other, can x be zero, where the
question seems more relevant to the mathematical understanding and thus carrying little pedagogical power. Still as I will analyze, the question seems to be exclusively pedagogical and has been “created” within the local context of the lesson, due to hesitation by students. In both cases the focus seems abandoned, which an interrupted reasoning may suggest, but, in fact, the interrupted reasoning participates in the construction of the new conversational focus.

**An unexpected hexagon.** This episode appeared in a lesson on fractions. It was interesting to see it as an episode that recalls ideas from other lessons. I analyzed this episode to illustrate four recalls in a row: one refers to the shape analogy, the second refers to the definition, then the number of sides analogy, and the lasts brings up an unusual case of a hexagon. The teacher called out his repeated question: "What's the whole?" but then interrupted himself, to attend to the relation between his pedagogical and mathematical norms, thus moving on a different track. In Chapter five, Episode 5.3 (based on teacher use of counters) I referred to a dialogue continuing the below episode. In this episode, however, the teacher uses geometrical tiles to stimulate students’ understanding of various shapes, configurations, and representations of the whole or the one. In this particular episode, one can imagine how a regular hexagon has been broken into four equal parts.

I called this episode an unexpected hexagon not because the shape of the “whole” or the “one” was unexpected. Mathematically it was the correct answer. It was unexpected because Harvey, the teacher, found an interesting opportunity to refer to shapes and connect with other mathematical features. These features were not required for structural mathematical understandings and they also were not related through a pedagogical reasoning to justify teacher action. However, as the dialogue shows it was occasioned by a discussion on a “shape” topic, as
a sort of perceptual mediator for teacher knowledge. It was an alternative to the procedural
meaning of interactional teaching practice.

Episode 6.1 Un unexpected hexagon

Harvey: Sure, I could put two of these together then to show that is the whole…if
this different situation, if this is three fourths (Oh my god)
((3))
What is the whole?
How could I draw that it may not have an easy name?
((2))
How would you draw it?
Take what you need and drop it and later on it…ok yeah…
Let’s get it up here…
So this three fourth and then this must be the whole,
((1))
→By the way if this were a polygon what would the name may be?
Don’t tell me baby care or anything like that so, it were a polygon what is
it? Sophie?
Sophie: A quadrangle?
Harvey: No, a quadrangle,
→Remember? has four sides…
I’m really talking about all the way around this thing …
→How many sides does it have?
Jonathan?
Jonathan: Six
Harvey: Six sides, so it’s properly called a hexagon…
Students: [yeah]
Harvey: Yeah somebody called it a concave hexagon over here, too…that’s
hexagon isn’t it? Yeah?
→But a hexagon doesn’t have to look like that because we saw one that
doesn’t look like that
All right, back on the subject though,„,

The teacher found the hexagonal shape feature so powerful that he accepted and
embraced the interruption in order to draw the analogy. It was illustrated earlier that the analogy,
on the other hand, seemed tightly related with the interrupted reasoning with which the teacher
engaged. It seemed to be an immediate response to teacher need for action. The interesting part is
that there seemed to be other possibilities to create a “one” or a “whole” as mathematical
conceptions and the alternatives may have produced different shapes. This occurrence in the

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activity was occasioned by the classroom interactional discourse regarding shapes. It occasioned an “unexpected” episode of geometry during fractions’ discussion.

"Can x be zero?" This episode is recalled from the Rational Functions lesson. It was interesting in that it showed how the teacher used this question for various intentions, not only to signal something or probe students thinking, but also to locally tag for students a way of looking at the domain of functions. This situation relates to Deborah Ball's macro-analysis in that it refers to a grand way of looking at the domain of functions, but also how to look using a formal logic.

In this next episode is a recollection from that same lesson mentioned in Chapter 5 and which appears in Appendix D. The question of the teacher, "Can x be zero?" is not the same kind of question as in previous episodes which I discussed in Chapter 5. This time it points to a connection with the domain of the function. Notice how the teacher builds upon this question of unity. It is based on previous aspects of the lesson, but not as an already established relationship between mathematical and pedagogical norms of discourse. The reference to the domain mentioned already is clear, but this time there is the possibility that one can calculate the value of the function at point 0. Mike's remark on how better to see what happens close to zero rather than to see what happens when x is zero, requires the question "Can x be zero?" from a mathematical point of view, respecting the mathematical norm. The teacher repeats and it is a routine. However, it is an analogy, building on a specific understanding. What happens is that the teacher is taking the previous experience and bringing it into a new one with a slight twist, barely perceptible from a pedagogical perspective. However, it is necessary to respect the mathematical norm: x cannot take other values. The dialogues between teacher and students in mathematics classrooms require continuously building up categories and conversational devices in which to place such categories. Sometimes these are only temporary devices to support teacher efforts for
a specific analogy. At other times they are more durable. Still, at no point in the case of the two teachers I worked with, did it seem that they restricted their actions to a particular relation between mathematical and pedagogical thinking. Instead it was necessary to respond to the particular teaching action as demanded and constructed through the interactions and dialogue with students.

*Episode 6.2.1. Can x be zero?*

In this episode, the teacher discusses the function $3/(x^2+1)$.

Mike: (addressing students) What happens if x is very small? X gets closer to zero. What happens?  
(No Student response)  
$\Rightarrow$ Can x be zero?  
SSs: No (choir)  
Mike: Why can’t x be zero?  
SSs: (chuckle) It can …  
Mike: Then why did you say no?  
(the class had sounded like a hard piece choir)  
Can x be zero?  
SSs: Yes (choir)  
Mike: What happens … when x is zero?  
St: 3 over one.  
Mike: You got 3 over one!  
What happens if x is close to zero?  
Then x square is …  
If x is close to zero, then x square is?  
SSs: Small  
Mike: $\Rightarrow$ Also close to zero, so the denominator is  
SSs: Close to one (Choir)  
Mike: Close to one. So the all fraction should be close to …  
St: Three…  
Mike: Three! OK.

Later during the lesson, Mike reiterates the question, but the local situation is different:

*Episode 6.2.2. Can x be zero?* In this case the fraction to be discussed is $(x^2-1)/(x^2+1)$:

The episode happens after Episode 6.2.1.

Mike: Oh, Neil told me!
Victor, can you tell what happens when \( x \) gets small? Close to zero…

Victor: (No response, trying to respond)

Mike: \( \rightarrow \text{Can } x \text{ be zero?} \)

Victor: ((Commenting with her colleague Brigitte on the reasoning))

Mike: Brigitte! What was the answer to the first question?

What is the domain?

Student: All real numbers.

Mike: \( \rightarrow \text{Can } x \text{ be zero?} \)

SSs: Yes!

Mike: \( \rightarrow \text{Then how better to answer the question about what happens if that gets close to zero then to see what happens when } x \text{ is equal to zero?} \)

SSs: Close to zero is negative one

Mike: \( \rightarrow \text{What fraction will it be at zero?} \)

St: \(-1\) over \(1\). \(-1\)

Mike: \(-1\) Ok?

In the sense that the teacher responds and understands what happened based on the previous perception, the analogy required an understanding of two aspects: Are these analogies a matter of how the teacher refers to the immediacy of the context? Or are they simply general, macro-connections? It seems to be the general case in the episode of "An unexpected hexagon." That is a general situation, that things are related occasionally, not necessarily through a mathematical connection. The two episodes required further understanding if such analogies in the teacher's mind are to be considered due to integration between teacher mathematical and pedagogical thoughts. It seems that they make more sense in the local context of dialogue than in a general domain of knowledge. The "Can \( x \) be zero?" seems to suggest a general macro-mathematical norm to be respected, but how the teacher refers to this norm is what makes it most interesting for the analysis. The question is played in a variety of learning contexts and although it seems that the general situation regards the domain of the function as a condition to discuss the behavior of the function in a certain point, something which is a matter of mathematical norm, and possibly of a discipline-based integration of teacher knowledge, the two pedagogical situations are different. In the first case the teacher is using a connection with mathematical
norm; in the second situation, however, the question seems challenging because it details the use of the same norm for farther cases. Apparently the teacher had already used a previous experience developed in the same lesson.

**The Sequential Organization**

The previous two patches show how important it is to build meaning of what teachers say in their classrooms from the structure of the dialogues. In this sense, all the examples illustrate that we do not have simply a general situation in which the teacher enacts a particular teaching and thus should act in a certain way. It is a matter of developing within the dialogue and the situation. I continued to pay attention to the sequential structure of the dialogue with students.

Borrowing from conversation analysis, I paid attention how teacher actions were occasioned and what seemed to lead to teacher construction of a specific “device” and what had been the predicates as possibilities to be applied that the teacher attached to the specific categories created during the teacher student dialogues. In the case of the episode regarding the unexpected hexagon, the teacher seemed to recall a previous device on geometric shape with specific particularities for the number of sides. In the case of the “x can be zero” episode the question seemed to play various roles along the lesson and for the efficiency of the communication the teacher extended a mathematical connection with the domain of the function.

I was thus closer to engage with a focal analysis in the way I described in Chapter 4 to be able follow teacher metaphorical construction but also what seemed to provide with a certain efficiency of the communication between teacher and student.
"What is the whole? What is the one?" The following excerpt immediately preceded the excerpt on page 3 of this chapter. I analyzed how the teacher used not only the same question but a consistency in the way the students and teacher thought.

Episode 6.3 What is the whole? What is the one?

Harvey: I’m gonna show you some pattern blocks and I’m gonna ask you <what is the one?> I’m gonna figure out what that rattling is? think that the ??? is broken why would it suddenly ??? when it never had before? All right if this (), what I’m projecting is one half…what is the one look like?

Student: []

Harvey: Look up here? ignore that thing … if that’s the half what is the one? Sunkia?

Sunkia: If that’s half?

Harvey: yeah if that’s half, what is the whole, what’s the one? Show!

Student: Add it?

Harvey: sure, I could put two of these together then to show that is the whole…if this different situation, if this is three fourths (Oh my god) what is the whole?

I found interesting the pronounced focus in this discussion “the whole” “the one” which was the teacher’s basis for calling students mathematical perception on fractions. Sometime students had a different intended focus due to the variations in forms.

The teacher continued the interruption of his reasoning. The teacher, however, uses a certain pronounced focus as Sfard (2001) remarks to be necessary when one needs to construct an efficient communication. Still in this episode, the question is more than a task question. There are many variations, and the teacher prefers to keep it as constant within his discourse. The question seems to also be more appropriate to a macro analysis, a general instructional situation. Still it was interesting to follow how the teacher carries this question along the lesson and the variations it had in its meaning and how students negotiate the meaning. In this case, the student
responds with what she is supposed to do to obtain the one: “add it” than showing what the one as a geometric shape is. I saw the question “what is the whole? What is the one?” as a tag for the variation of situation of the dialogue teacher-student. This perspective offered a better understanding of teacher interruptions. Equally it provided a different way of relating teacher thoughts than rationalistic relationships attempting to simply provide rationales for the quality of a teaching action.

"According to this definition". This definition episode has proved interesting because I could see in it a different lesson on radians, one where the definition did not work out the same way. The teacher created a different kind of dialogue with the students and it provided more insight about various possibilities, about certain categorizations and how they work. It showed the interpretative nature of the dialogue, especially when one expects that a definition is simply a definition. But as the radians lesson showed, it may be the case that a definition requires more attention and pondering and thus may not provide the tagging attribute.

Episode 6.4 According with this definition

Mike: So, the second function that you looked on homework yesterday. We’re look at … You looked at 1 over e to the… 1 minus ….1 plus… e to negative x. Is that a rational function?
(7)
SD?
Mike: =Yea…
SD: =Student B, what do you think? <
>Is it a rational function? < ()
⇒According with this definition?
SC: ((SC responds before SB)) No!
Mike: >Wh:.y not? <
SC: (Because)... e to the negative x is not a polynomial!
Mike: <L plus e to the negative x is not a polynomial! >
Mike: Ok?
SC: So: in your homework last night I asked you to look at two functions that were fractions, that were ratios, but the second one wasn’t what’s call a rational function, because it wasn’t a ratio of polynomials.>
Similarly with Episode 6.3, in this case the teacher also uses a pronounced focus “according with this definition” for supporting his interrupted reasoning. The interesting aspect has been teacher reference to a specific definition, particularly corresponding to a general instructional situation. Again there was an interruption in which the teacher felt it necessary, for an instant, to refer exclusively to a mathematical norm and abandoned the pedagogical intent.

The Two Mechanisms

"Integrated" is a frequently-used word in education, in various domains: curriculum, teacher knowledge, situated learning, child-centered education, and many others. Often the term has only an intuitive, experiential, a rather indistinct meaning. We need to have a deeper understanding of how differently the word is used in educational situations. When the concept of integration is employed in deciding what should be essential teacher knowledge--and if the concept becomes an assumption of teacher education and evaluation--the meaning of integration needs to be elaborated.

This study focused on understanding how teachers use and relate their mathematical and pedagogical thinking during classroom interactions. It was driven partly by the claim that the relationship between mathematical and pedagogical discourses has been rather explanatory for teacher actions, but not necessarily a vernacular for the way things happened during classroom interactions. In the sections below I explain my analysis of how teachers’ pedagogical and mathematical discourses are associated, sometime through contiguity, sometimes simply through similarity. The first case happened through tagging and the second through analogy. I thus provided two mechanisms of interactive thinking that activated mathematical and pedagogical discourses: a metonymic mechanism and a metaphorical one.
This study has been concerned with the ways teachers used their mathematical and pedagogical thoughts, especially their already integrated contemplations during classroom teaching. In the previous chapter, I illustrated that teachers, as in the case with students, could appear to have the same performances but we do not know if they enacted the same routines and norms in both. That raised an important issue because we should then be more cautious in trying to understand teachers' performances and enactments. I will analyze this issue in the next chapter. In this section, I will consider the issues of this study: if, when explaining their action, teachers speak of using integrated mathematical and pedagogical thinking, does that mean this integration actually occurred? Is the nature of interactive thinking in classroom teaching as a whole and in dialogic interactions one by one, the same?

As I illustrated in previous sections, the two teachers I worked with mainly use tagging and analogies in their interactions with students. These forms of communication with students have been also remarked by other researchers (Leinhardt & Steele, 2005; Bauersfeled, 1988). What I will consider in my final analysis is what the constant presence of these patterns in teacher-students communication may reveal about the way in which teachers use their mathematical and pedagogical thoughts. I have considered especially those episodes in which I had expected that teachers would most likely use integrated forms of mathematical and pedagogical discourses, considering their pre and post-lesson thoughts. I showed in Chapter 5 how the teachers both in pre and post-lesson discussions maintained the same line of reasoning about their actions, although in classroom interactions, their reasoning was interrupted. What it was more interesting was that they did not recall conflicts between these interruptions and their out-of-class reasoning. They equally recognized the differences from their expectations of how the action in class went. These observations made me conclude that teachers definitely have a
different way to think in the classroom than out of the classroom about their actions. It was in this chapter that I reanalyzed data to understand what would characterize and mark the differences especially those differences that affected teachers' way of relating their mathematical and pedagogical thoughts. It is to this question I turn to understand further.

**The association of classroom situations.** Some mathematical objects are just part of other complex parts. We refer to a function and think of something related with a function. And the teacher needs to work with that representation in the student’s mind. When a teacher thinks of a particular mathematics issue he may associate that mathematics with a particular teaching situation. He may think function in general and think of it as a mathematical definition, or he may simply think function and associate it with a student mistake. These cases refer to general situations, they also encompass a macro perspective of mathematics teaching.

On the other hand, during mathematics class, the teacher develops and establishes with students certain links. I called them tagging. They may differ from one student to another, from one class to another, from one mathematics topic to another. They may be part of the teacher out-of-class related and integrated mathematical and pedagogical thoughts. But the mathematical and pedagogical norms under which they function are part of the way they developed within the moment. They may remain in teacher integrated knowledge as stable, recognized ways of thinking or they just may be exceptions to the case.

In the episodes I presented in the first part of this chapter tagging relates especially to mathematics, specifically with some mathematics understanding. I used focal analysis to understand those conversations and thus I also tracked whether this would be eventually used in other situations, as pronounced focus. Tagging becomes a pronounced focus and it helps the teacher associate a mathematical thought within that communication with student. It seems a
pedagogical tactic, and also a matter of communication. But it also tells that there is a different way of thinking when teaching than when communicating about teaching. Tagging does not appear in after-class discussion, although I asked the teacher to explain what happened in that moment.

What Leinhardt and Steele (2005) called labeling in their lesson analysis of Lampert’s instructional explanations, finds no mentioning in Lampert's reflections. The teacher is more concerned about the student and the instructional situation.

Still, tagging indicates an association of certain mathematics with another for the student, but for the teacher this association of a mathematical thinking with pedagogical thinking. When the teacher asked "Can x be zero?" he not only marked a point to find out an answer, he also indicated to the student, tagged the situation, momentarily for a better understanding, to indicate that this is related to the domain of the function and finding out if the function has asymptotes. In the next lessons this tagging could disappear or the question may have a different value depending on the local context.

I illustrated how this is certainly a matter of situation. The same tagging could work across several situations. Teacher and student categorizations of certain short-momentary situations could be carried later, could be improperly used, could be simply routines, or could be a matter of spontaneity. However, the categorizations certainly are a matter of momentarily situation. Such associations certainly may prove useful and can be a basis for integrated mathematical and pedagogical thinking. Sill in their incipient phase, in classroom teaching, they are simply a matter of communication and perceptual learning. Stake shows the case of bubble gum in his case study where the teacher mistakes the definition of mean for the sake of such tagging.
A tag is not developed by a teacher without a certain perception of the student response or of the development of the teacher-student dialogue. Sometimes it may just be guessing students’ thinking. In that moment, still, the teacher is not acting for a perfect performance or for respecting specific norms; those moments are moments in which the teacher develops in the moment as a matter of anticipating student thinking.

The similarity of classroom situations. The second mechanism refers especially at replacing objects of mathematical discourse with pedagogical transformations. This aspect involves pedagogical thinking, not just as alignment with instruction (as pedagogical strategies). Some refer to those as representations of mathematics. Some mathematical objects find easily a correspondent in a phenomenon. When a teacher thinks of mathematics he tries to find similarities between what the student says and what the teacher wanted to say. We can thus assist at two kinds of relationships between mathematical and pedagogical discourses, both equally practiced by mathematics teachers.

The similarity of classroom situations is not simply a matter of similar combination between instructional strategy, specific pedagogy and mathematics topics. The perceived similarity in particular classroom interactions is mostly encountered when teachers instantly identify the specific features of the interactional context, and transfer those to the mathematical understandings or pedagogical intentions. The interesting aspect about similarity is not the recognition and application of a discursive routine, but when the teacher enacts such routine.
Summary

Tagging as referred in this analysis is not simply the labeling of a situation. It could resemble the communicational "tagging" as in the cloud tagging. It is a classroom interactional mechanism. It shows a different mechanics between mathematical and pedagogical discourses in classroom interactions from the discussions of justifications of actions. The tagging refers to particular details present during the lesson or later. The details are part of teacher categorization of certain aspects of learning mathematics specifically for certain students or according to previous experiences during the lesson or previous lessons. But what I followed in my analysis was not the teacher's intent to a memory of the student, but rather the teacher's own memory for that particular understanding and participation, (such as a specific attitude towards a specific detail which marked the student's learning experience and something the teacher remembered). It is different from what it is called collective memory or labeling in that those constructs refer to what is aimed to support students' memory during an efficient and proficient instructional action and I inquired to understand what sustains teacher's memory when responding to students' interactions.

The two mechanisms of thinking in teaching refer to similarity and association. So, I followed teacher intent, but was mostly interested in the mechanism the teacher developed for his relations between mathematical and pedagogical norms of discourse.
Chapter 7

Reflective and Metaphorical Thinking in Classroom Teaching

I noted in Chapter 2 that one way to facilitate prospective teachers' adjustments to their future classroom practices focuses on situative perspectives of learning in teacher education (Putnam & Borko, 2000). Each classroom moment has its own situation. Where is the situative perspective in prospective teacher learning? Among the preferred activities are discussions and analysis of: practice-based materials, records of practice, and mathematical tasks. Since the very beginning, teacher educator action has been oriented to the analysis of such practices. One of the goals of such analysis has been to support pre-service teacher integration of mathematical and pedagogical thinking, supposing that experienced teachers use their mathematical and pedagogical knowledge in an integrated manner.

Analysis of practice-based materials and records of practices has encouraged what would be called pedagogies of investigation. A rational analysis of teacher action and reflective thinking should prepare the future teacher's professional practical knowledge and support practice-based thinking.

How teacher education activities can be improved by connecting knowledge with practice remains a challenge for the teacher education curriculum. The shift from what teachers should know to ways of knowing teaching added an important dimension to the preparation of prospective teachers.
Deborah Ball and Magdalene Lampert (1998) said:

Instead of taking a position in the argument about what prospective teachers need to know, we would like to enter the fray at another point. Asking instead how they should know those things. To take this up, we return to what we know about the nature of teaching practice. [...] How much can knowledge, and an understanding of what it means to 'use' it, be generated outside of the situations of action? How could we design our work in teacher education to give prospective teachers opportunities to develop knowledge as well as an appreciation of what it takes to use it wisely in context? (p.33)

Recently, further analysis of teaching practices indicated that teacher education has few approximations of practices by which pre-service teachers can "paddle" in the "calm waters" of the teacher education environment. "University classrooms, on the other hand, can provide learning opportunities that are absent in fieldwork, allowing novices greater freedom to experiment, falter, regroup, and reflect." (Grossman et al., 2009. p. 27)

The use of scenarios-based analysis (Bruce & Reynolds, 2010) was found to be a good framework for human-computer interactions and may show a compatibility with Ball's call for usable knowledge:

Representing the use of a system or application with a set of user interaction scenarios makes the use explicit… It can help designers and analysts to focus attention on the assumptions about people and their tasks that are implicit in systems and applications. (Carroll, 1999, p.2)

Although we have evidence of teacher-integrated knowledge, this evidence has been almost exclusively based on how teachers think while not teaching. Research shows that teacher thinking is different during classroom interaction and outside of the classroom. We may have a problem in teacher education. Based largely on post-lesson comments of teaching, we try to prepare future teachers to use their knowledge during classroom interactions in an integrated way, but that may raise expectations unrealistically.

To claim that the masterful teacher thinking is characterized by integration requires new evidence. This study looked but failed to find evidence that, during their classroom teaching, two
expert teachers used their mathematical and pedagogical thoughts in an integrated manner. It may well have been the case that they saw mathematical tasks through pedagogical lenses and their classroom interactional actions may well have been guided by specific pedagogical intentions. It might well have been the case that, during classroom interactions, the teachers pondered the options to compose a mathematical response. Their mathematical thinking required checking mathematical norms. Among the recorded lessons, I did find the two teachers explaining their thinking after class in a way that was consistent with their actions. But, integration? No.

Since this study started, research in teacher education has focused more and more on how pre-service teachers put what they know about mathematics teaching into practice. What I saw as the gap between knowledge and action remained. The gap was apparent in the rationalistic way for teacher enactment of discursive routines.

In teacher education, the rationalistic way has been called pedagogies of investigation and pedagogies of enactment. Modeling and rehearsal of discursive routines of teaching were identified by Ghousseini (2009) as: "revoicing a student contribution, orienting students to each other's ideas, pressing students for explanations, connecting students' ideas, and making the structure of mathematical discourse visible" (p. 206). Although apparently useful, they continue to keep problematic the teacher roles in such discourses as well as in the different milieus in which they are applied (Borko et al., 2000; Perissini & Knuth, 1998). If we are to prepare teachers "to know how to do things in practice and do them interactively" (Ghousseini, 2009, p. 206), the findings in this study illustrated that here is another dimension that needs to be prioritized: teachers' perception of the local dialogical context. When pondering and engaged with spontaneous judgments for alternatives of actions, the teacher not only enacts a discursive
routine but makes it specific to the situation. His thinking is engaged not in a rationalistic way of deliberating alternatives of actions, but adapting his discourse to the immediacy of the interactional context.

Alternatives in Mathematics Teaching Thinking

Alternatives in mathematics teacher thinking are not a multiple-choice task. The tasks of a mathematics teacher involve situations in which there are different understandings of the task and thus involve alternative perspectives. The teacher who responds to the task may not use more than mathematical thinking to respond.

This issue captures the most important aspect of teaching mathematics, teacher engagement with pedagogical intention during classroom teaching. It is the relationship between teacher mathematical discourse and pedagogical intention in the local context. It brings a new perspective into the mathematical discourse. However, as this study illustrated there is no evidence that the teacher would work to engage more in integrated thinking. They simply use both, more or less at the same time.

Leinhardt et al. (1995) spoke of efforts at "integration of professional knowledge:"

By examining several effective ways in which a component practice is carried out and by generating alternatives to that practice, the student teacher makes salient and open for inspection the microscopic aspects of practice that tend to go unexamined in real time. […] What we suggest instead is for student teachers to strive for a degree of specificity during the observational stage - to describe how long discussion activity lasts, how many students talk, what types of thing they say, what cues extend or end the discussion, how the meaning of the problem evolves during discussion. (p. 406)

The question here is how a novice can generate alternatives for such practices when the most pressing question is: the repertoire of ways a novice would generate knowledge of teaching.

Maxine Greene pointed out, "It is a matter of imagination in order to be able to perceive the alternatives" (Greene, 2001, p.6). This is not to say that we are devoid of teacher
development of courses of actions or analysis of them. Instead it is to emphasize that these
courses of action have different meanings in classroom situations. Attaching to them a qualitative
attribute does not guarantee matching the situation with the appropriate action. What does
thinking interactively mean to the student? I tried to move current goals for teacher preparation a
little bit further along. We would not want to reduce teachers' interactive thoughts to the scope of
thinking of their students, making rigid the relationship between action and diagnosis.

I will discuss the use of scenarios in teacher education for both rationalistic and situative
discussion of practice. Scenarios are eminently contextual. The difference between how one talks
and acts in context is well recognized. How can they be used in teacher education? What can
scenario exercises do to change perception? Perception should be a main issue in practice-based
activities. We should reconsider the tension between how one talks about actions and performs
actions. I thus claim the role of perception of classroom interactions to be important. What
Ghousseini (2009) and Lampert (2010) attempt is about enactment and practices, also, about
“rules of engagement,” but less about interactional thinking. The scenarios below will be called
"workables" and will act as perceptual mediators for different kinds of thinking involved in
classroom interactions.

In a sense, this dissertation has been focused on the discrepancy between how teaching
research visualizes the quality of mathematics teaching and teacher education. Learning research
shows incompatible mechanisms for teachers' learning processes. The two mechanisms
highlighted discussed in previous chapter illustrated a critical pair: two mechanisms, one based
on association and another on similarity between the immediacy of the classroom situations.
They are all too much a novelty in teacher learning. They need to be clarified for teacher
education activities. How to bring perception of classroom mathematical interactions into this
picture and into teacher education activities will be prominent in the remainder of the dissertation.

We have different perceptions of life. They depend on what we see and how we interpret the world. We need to be careful how our interpretations are mediated by perceptions of the situations. As William James remarked long ago, a paper could have many meanings according to its purpose. In similar fashion, I will show below how we should make teachers sensitive to the various meanings of the same "object of practice." But we try to bridge it in two ways: rationalization and creative thinking. Using their analogical capacity (at the core of teacher cognition) opens the terrain to a more complex perception of what teacher-students classroom interactions are. Theories of mathematics teaching knowledge have enriched their content with more situative meanings of mathematics (Stylianides & Ball, 2008). They show a more complex and intricate business of mathematics teaching. I use to say that the difference between mathematical knowledge and mathematical knowledge for teaching is like the difference between meaning in a macro perspective and meaning in a micro perspective.

The next chapter will take up the relationship between learning and teaching in terms of learnable mathematics teaching and teachable mathematics. It studies what in “communication” terms is called "the learning-teaching agreement." I refer to this relationship as essential for the study of the teacher's pedagogical and mathematical discourses. In terms of current reforms studies it goes into what discourse becomes dominant in classroom. I re-analyze how the teacher could experience various challenges in learning-teaching agreements and particularly the events in which the teacher's pedagogical and mathematical thinking is exposed.

It starts with the expectation of teacher enactments of mathematical thinking integrated into pedagogical thinking. I argued earlier that integration is not only a matter of transformation.
of cognition through such experiences, but especially a matter of engagement in classroom
discourse. I pointed out that a relationship between teachers’ mathematical and pedagogical
thinking needs to be understood better, maybe trying to understand the ways teachers perceive
the conditions for action in their classroom teaching environments. Following Sfard’s (2008)
analysis of discursive routines we need teacher enactments for both the "how" and "when"
routines of discourse are enacted. The workables analysis aims to show that while in the process
of learning we may be concerned with the “how” of specific routines; we tend to ignore the
“when.” The data analysis in this study paid specific attention to the "when" moments. They
illustrate that certain moments in classroom discourse are critical for a teacher's integration of
mathematical and pedagogical thinking. I showed that such moments mark what I identified as
"interrupted reasoning.” As routines of communication they identify turning points in the
teacher's utterances. The objects of discourse have "different indications" for learners. I pointed
to the episodic nature of the teacher's classroom discourse and questioned even the possibility
that teachers would be able to carry on uninterrupted reasoning. More than that, this
characteristic of mathematics classroom discourse reminds us of an important point of
mathematics teaching practice: Teachers rely separately on their mathematical and pedagogical
thinking, thus treating them as two separate disciplines.

There appear to be no stable forms of integration of mathematical thinking in
pedagogical reasoning. Each form may cause passive teaching or analytical dilemmas. The
question remains whether the discursive routines of mathematics teaching enactments may be
treated as learnable mathematics teaching. This chapter concerns especially the ways teachers
become engaged in their classroom interactions. The issues of learnable mathematics teaching
will be revisited in the next chapter.
In the next section, three cases will show how workables may contribute to sharpening teacher perception of the dialogic interaction: immediate context, imagining student responses within specific immediate contexts, and the alternation of pedagogical and mathematical meanings. These exercises in communication are oriented to novice teacher perception, association and similarity, rather than to rationalistic explanations for the quality of teaching actions.

**Exercises in Mathematics Teaching: Workables**

This section concerns the opportunity to transform episodes of classroom interaction in potential episodes supporting teacher learning. Compared with situative approaches, these episodes extend novices’ analytical abilities beyond rationalistic explanations for teacher actions. Instead, they prompt novices to imagine possibilities to respond and to become aware of such possibilities. Three important features are discussed: immediacy of context, imagining students answers, and alternation of meanings.

**Immediacy of context.** I have already analyzed the episode 7.1 in chapter 5 on interrupted reasoning. I showed how in the course of responding to the student, Harvey hurried into a mathematical discourse about the activity. It was immediately followed by his pedagogical intentions, some conflicting. I had also the chance to analyze his pre-lesson reflections. It seemed an integrated activity, but during enactment, under non-routine situations, it unexpectedly split. It offered the chance to pay attention to when a routine of teaching is enacted.

This episode was part of the beginning of a lesson in which the teacher had just finished explaining the task. When asked to pose questions about the task, one student produced a
response to which the teacher needed to respond. However, it seemed to be a matter interfering with both teacher pedagogical intentions and the mathematics that the student should perform.

Episode 7.1:

Marteen: Do we have to aaa…write the number of cubes…((inaudible)) the cubes will be off
Harvey: So… What do you mean the number of cubes? (Cubes) will be off?
Marteen: I mean…aaa…because if you when you get 15 blocks and you draw them each one and one, then you have to get it over…
Harvey: Yeah…You know what I think you're doing? You're confusing the number of cubes that we <have> in the <bag> with the numbers of trials that we're gonna take. So, if I hand you three coins and said: Flip these three coins all together 25 times, it wouldn’t matter <at all> that you were not just flipping them three times. (Or something) You know what I mean? I'm not sure you do. Talk to your partners about that.

A feature of the analysis is trying not only to explain why the student responded in that particular way, but also to stretch that response into a specific frame of reference for the student. Usually the "why students respond in a certain way" comes out with a mathematical explanation, (which I provided it in Chapter five), whereas "how to act in the course of teaching" becomes a matter of "putting on the student's shoes" and creating affordances for the student. This shows once more that the nature of integration as mathematics within pedagogy and pedagogy within mathematics is essentially a matter of the local situation and the immediacy of the context.

I tried thus to point to the importance of the immediacy of context. The teacher would have extended his examples, but changed his instructional action towards a pedagogical move: “Talk to your partner about that.” The episode indicated how the teacher also moved from a mathematical meaning to a pedagogical meaning of the situation. But one did not necessarily influence the other. Certainly one would say that there is a mathematical knowledge for teaching in this particular confusion of the student and thus the teacher was able to inform his pedagogical
action. That is true. But we do not know if the same mathematical thinking could not have
informed a different instructional action. In Chapter 5, I showed how, during the same class, the
teacher had three different situations, with the same line of reasoning, and the same routines, and
for which the teacher attached at the end three different pedagogical actions. Pedagogical actions
were rather related and associated with the immediacy of the context than with a situation of
mathematics in general. Teacher action was not part of an integrated way of relationships
between a mathematical thinking of the situation and the instructional action.

Instead of trying to reason if the teacher action was good and why, a better approach may
be to stretch a novice’s perceptual understanding of the situation and of what would make the
situation different: How would you have the teacher respond? What do you think about the
teacher's response? Can you imagine what made the student think that way? Do you think that
the teacher changed his pedagogical or instructional intentions? Would you have changed yours?
Try to modify the activity or how you would explain it? Is there mathematical thinking in this
situation? Would you think that teacher's example was helpful to the students? Is it good that the
teacher explained the mistake or should he have probed more with the student?

**Imagining student answers.** Another important point of this study is that teachers need
to be able to imagine students' answers, not only why they answer a certain way. The previous
section showed the importance to consider not only as an explanation, which may seem to be at
the very first analysis.

"RATS” Episode 7.2: Mathematical Norms of Discourse and Instructional Actions:

Teacher: All right! Do I have all the assignments? All of them?

Teacher: That looks like a huge number! What did we talk about last time?

Teacher: RATS? Annoying, huh? What does RATS stand for?
Teacher: Rational function? Then let’s warm up a little bit! Would $1/(x^2+1)$ be a rational function?

Teacher: Correct. It is a rational function, because it is a ratio of polynomials. How about $1/(1+e^{-x})$?

Teacher: Other opinions? According with this definition, is $1/1+e^x$ a rational function?

Teacher: Why?

Teacher: $1 + e^{-x}$ is an exponential function. Therefore, it is not a rational function. What is the domain of the function $1/(x^2+1)$?

Teacher: All real numbers, because $x^2+1$ has no real roots. Is this function ever zero?

Teacher: Why? Why $1/(x^2+1)$ is never zero?

Teacher: You are right, the numerator is never zero! What happens with the function when x gets large?

Teacher: What happens to a fraction when the denominator is large?

Teacher: Good! If x goes large in the negative direction, what happens?

Teacher: The same, the fraction gets closer and closer to zero. What happens when x gets smaller and smaller?

Teacher: Can x be zero?

Teacher: Why you say so? Ah! You changed your mind! Then what happens with the function when x is zero?

Teacher: You are correct, so $1/(x^2+1)$ is a rational function, it never crosses the x axis because it has no roots, it is always positive, when x is very small, it gets closer and closer to 1, as x gets larger and larger, as gets farther and farther from the origin, the graph of the functions gets closer to x axis. How is x axis called?

Teacher: Look in your graph calculator and tell me if there is any other asymptote for this function.

Episode 7.3: Pedagogical Norms of Discourse and Instructional Actions:

Teacher: Do you guys have any rules established?
Teacher: Not yet? Why is that?
Teacher: Yeah, that’s not quite true, but it’s close. Did you look at the first three cases?
Teacher: Ok, that works for the first two cases, but what about the third case?
Teacher: You should make a rubric with what is common to those cases and what is not!
Teacher: Yes, you are right! It is important to know what you write from your observations!
Teacher: What counts as common and what is not quite common? Can you give an example?
Teacher: It may be more revealing if you enlarge the picture. It is because your calculator is ignorant. But what can you say about this?
Teacher: Yes, they are different, but if you look at their pictures all graphs have a common property. What don’t you see?
Teacher: It is a mistake there. But it works pretty nice! See what you can say in the sixth case! Does it work the same?
Teacher: Well, if it doesn’t work…maybe the way it doesn’t work will tell you something!

Try to respond the following questions: (a) What kind of answers did the teacher expect from her students? (b) Have you used mathematical or pedagogical thinking to figure students’ responses? (c) How about if you do not use teachers’ following questions? (d) Do you think is the same kind of thoughts students have been involved with? Why?

I will try to refer to how to "define" mathematical and pedagogical meanings as induced in relationship with mathematical or pedagogical norms of discourse. A relationship with mathematics and pedagogy had been determined through the features a teacher uses to formulate his instructional turns, particularly his implementation. Such formulations indicate how the teacher categorizes the respective questions to obtain responses from students. These two
dialogues show clearly how the teacher's questions lead either to mathematical or pedagogical characterization. What is important about them is their nature. In the first case, we had questions with declarative mathematical meaning. In the second case, the teacher had questions with declarative pedagogical meaning. This means that to provide the corresponding answer, the student needs to call from memory the corresponding thought. In other words, to respond to the teacher's conversational turn, the student does not have to engage in some sort of procedure. When I attach mathematical or pedagogical meanings to these questions I do not refer to either the teacher's intentions (which is already clear as being pedagogical) or the teacher's reasons to ask those questions (which are mainly determined by circumstances in the situation: lesson planning or contextual factors, for example). When I talk about mathematical and pedagogical meaning, I refer to what the teacher's questions speak: mathematics or instructional transformation of mathematics. It is not that the line between the two is a clear-cut. But it is good to see the dual nature of student-teacher conversation: mathematical questions lead to mathematical answers and instructional questions lead to ways of thinking. Some would say that in the first case the teacher follows mathematical reasoning, while in the second, pedagogical reasoning. Others might argue that pedagogical reasoning appears in both episodes.

Few observe the interrupted character of these dialogues. It is the nature of these two dialogues that I would draw attention to. I illustrated that they become important to trigger the teacher's thoughts. How is this possible? How is it possible that teachers jump from one topical question to another or from one instructional line to another? It may be the point of general conversational analysis, but this may be the pedagogical reasoning for the teacher's actions. The interrupted nature of teacher-student conversation (in terms of thoughts) provides certain borders represented by the flow along the conversation. The interrupted character of the conversation
demarcates the sequence of turns between students and teacher. One characterization given to a task appears in the next episode, with a new event. It is in this micro-space between two events that I will focus my attention to the teacher's inter-play between mathematical and pedagogical thoughts.

**Alternation of meanings.** Somewhere in the analysis I asked, could we alternate events from one dialogue with events from the other? What does it tell us if the conversation still makes sense in general lines? More integrated lines. Let us look at the dialogue below:

*Episode 7.4:*

Teacher: Do you guys have any rules established?

Teacher: Ok, that works for function $1/(x^2-1)$, but does the denominator of $x/(x^2+1)$ have any roots?

Teacher: Have you looked at the graph? What does that tell you about the rule?

Teacher: Yes, the roots of the denominator indicate the vertical asymptotes.

We could distinguish a mathematical and a pedagogical code there. Although they seem to participate in the same pedagogical goal-establishing a rule-the teacher alternates a mathematical and pedagogical code in his conversation. It makes sense. I alternated events from one dialogue to another. This alternation of procedural meanings illustrates that during classroom interactions, the teacher engages both mathematical and pedagogical thoughts, not necessarily in a specific routine way. What does this scenario tell us about the teacher's integrated knowledge? The teacher would have used both kinds of thoughts, but with little dependence one on the other. In other words, although there are rationales which may explain why the teacher would act in a certain way--perhaps related to the student's comment--it is difficult to attribute a relationship among the corresponding thoughts to link the lines of the dialogue. They could be guided overall
by a purpose, but there is an interrupted character to this short conversation that makes it easy to replace the turns, but allows for keeping the purpose.

We can have as potential students' responses for a better image for how student responses modify the immediacy of the context:

*Episode 7.5:*

Student: No
Student: No, there are no roots.
Student: We know the vertical asymptotes by the roots of the denominator.

As an alternation, we can consider the following teacher questions:

*Episode 7.6:*

Teacher: Did you look at the roots of the denominator?
Teacher: Does it have any roots?
Teacher: You can look at the graph then to find vertical asymptotes!
Teacher: That's a correct observation.

If we intertwine the questions from Episode 7.6 with the student's previous responses, from Episode 7.5, we obtain a different episode, in which the student does not have to come up with a rule, but in which the student uses this rule. Was the character of alternation preserved? Not exactly, as first we had two mathematical questions where the student must perform some mathematics to be able to respond, and next we had other two telling the student what to do. This tells us enough that we know what it means for mathematical and pedagogical knowledge to be integrated in mathematics teaching.
Summary

This chapter discussed mathematical and pedagogical features and mathematical and pedagogical meanings. It was the purpose of the chapter to elucidate how the micro-view is different from the macro-view on what it means for teachers to use their knowledge in mathematics classrooms. I reviewed in Chapter 2 how research in the macro-view mainly builds mathematical and pedagogical meaning on what teachers say as being mathematics and pedagogy. It is the point of the micro-view to show a different perspective. In this sense, the mathematical or pedagogical features embedded in the implementation of the mathematical tasks are categorized by the teacher. Consequently, the associations that I used to declare the mathematical or pedagogical meanings are my assumptions or my membership knowledge. The teachers' use of such features for dialogue is of highest importance. This analysis moved the balance as to how teachers use their thoughts in classroom teaching from rationalistic to situative. I used a formalist reformulation of the mathematics embedded in the mathematical task to connect, as closely as I could, to the mathematical culture.

In this chapter I pointed to how important it is not only to engage pre-service teachers with specific discursive routines and investigation of practice, but also to engage them in interruptions of reasoning, especially during discourses with students. Rules of engagement for teaching practice and particularly for teacher-student interactions simultaneously are rules of mathematics and pedagogy, and rules of interruption and response. Thinking for discourse improves with disciplined experience in instantaneous reflection and respect for outside perspectives.
Chapter 8
Thinking and Creativity in Learning Mathematics Teaching

In their book, Bertrand Bruce and A. Rubin (1993) remarked:

The training model misses the most salient fact about implementation: that it is a *creative process* involving critical analysis of the innovation's potential in the light of institutional and socio-cultural context, physical resources, student needs, and pedagogical goals. The innovation process doesn't end, but begins, with the teacher. (p. 140)

In this study I questioned the empirical basis on which current initiatives of teacher education curricula include practice-based activities to develop the integration of mathematical and pedagogical thinking in novice teachers. One of the rationales has been that integrated thinking enhances teaching during mathematics classroom interactions. The purpose of my study was to analyze teacher thinking during classroom dialogue, seeking evidence of integrating mathematical and pedagogical discourses. For this purpose, I used specific analyses and tried to discern the norms, pedagogical and mathematical, that the teacher used.

In the USA since 2000, teacher education activities have increasingly accommodated the situative nature of teacher learning. Effective teaching responds to the perceived flow of life around it. This study brings forward one problematic side of teacher perceptual learning. Situative viewpoints can have advantages but cannot overcome the problems of perception itself. Recent initiatives, sometimes articulated as pedagogies of investigation and enactments (Ball & Forzani, 2009; Ghousseini, 2009; Lampert, 2010), use records of practice for both analytical and enactment purposes. The gap between enactment and analysis stands. Understanding the interrupted and personal nature of classroom interactional discourse is needed to bridge the gap.

This study illustrated how easy it is for rational discussions to ignore the importance of teacher perception and to overlook such perception in teacher education. Current research efforts have been focused on preconditions for teacher education activities. Based on the analysis of
experienced teachers' reflections, they have identified one precondition as the integration of mathematical and pedagogical thinking. They claim that this precondition attunes to the milieu of mathematics teaching. But the patterns of discourse in teacher education and those in mathematics teaching reveal significant differences from milieu to milieu.

Another pre-condition is affordances. In his review of Gibson's theory of affordances, James Greeno (1994) spoke of the importance of understanding how affordances--as preconditions for action--are perceived. Understanding the use of discursive options in informal teaching situations is greatly important in working through the complexity of mathematics teaching. This study found no evidence to support teacher integration of mathematical and pedagogical thinking as an affordance for learning in teacher education. It may, in fact, place too much burden on novice teachers' experience and preparation.

In this chapter, I discuss two important aspects to be considered in the design of the mathematics teacher education curriculum: current constraints in the milieu of teacher education and constraints on the perception of novice teacher thinking. Both teacher educators and researchers may be overly optimistic that perception of the actual classroom situation can be approximated during training.

I return to two points in my analysis. One refers to the learnability. For students in the classroom, learnability is the gradient of comprehension for different levels of readiness of learners. For novice teachers, learnability of mathematics teaching is that gradient of pre-conditions that support learning to teach mathematics. I discussed this learnability in relation to another condition of teacher knowledge, the teachability of mathematics. This in turn brought up a problem in current designing teacher education curricula, almost a shutting off the inclination of pre-service teachers to see teaching on their own terms. Clearly their own terms are
inadequate; they do not have experience at teaching. They cannot devise an effective way of learning to teach. But part of the professional understanding comes from individualistic perception. A sophisticated perception of their own should be nourished by the teacher education program.

Lack of teaching experience was one of the factors in this study. Reflective thinking is difficult to develop. Integration requires reflective thinking. In this study I ventured to understand the ways in which novices see teaching practice through new eyes. It is important to prepare teachers to carry a substantial load of professional practical knowledge, but we also need to prepare them as continuing learners. Some of teacher learning starts in classroom enactments as perceptual learning. And perception is closely related with being engaged in the creation of subsequent environments for learning.

I find some approaches to teacher education troubling, in particular those oriented toward high-leverage teaching practices, allowing but modest artisanal contribution. Their outlook for integrating mathematics and pedagogical thinking takes too much away from creativity. The outlook is fixed on aspirations. Creativity in teaching and in mathematics in particular requires openness to seeing things differently. As claimed in previous chapters, the possibility for alternatives is a matter of imagination. Teacher educators need to orient their efforts and encourage pre-service teachers to be imaginative in understanding classroom situations and interactions. Novices should not be diverted from manifesting their creativity. Personal innovation will incorporate new understandings of teaching.
Learnability of Mathematics Teaching and Mathematics Teachability

In one of her articles about the importance of communication in mathematics teaching, Anna Sfard (2001) marked this difficulty of mathematics teaching:

The scarcity of perceptual mediation in mathematical discourse may be a principal reason many people find mathematics prohibitively difficult, almost inaccessible. The students' task is further complicated by the fact that most of the mathematical objects discussed at school, instead of being known in advance and tightly related to children's former experiences, are built through the discursive activity itself. (p. 67)

This difficulty marks one of the most frequently encountered issues of understanding mathematics teaching. From a teaching perspective, this "scarcity of perceptual mediation in mathematical discourse" (ibid.) is the source of how teachers establish relationships between their mathematical and pedagogical thoughts. It happens in classroom interactional dynamics. It shapes the possibility of subsequent perceptions to be captured as part of curriculum. Identifying relationships between mathematical and pedagogical thought in classroom teaching becomes a matter of being able to respond to real instructional problems, of the kinds shown in previous chapters, showing connection and disconnection from analytical perspectives of the same relationships.

On a pedagogical level, as Schoenfeld (1988) remarked in his discussion of learning about teaching geometrical constructions, the need for perceptual mediation indicates that the job of mathematics teaching is to create opportunities to work on students' awareness to understand their use of mathematical knowledge:

Students' inability to solve the problem was not simply because they did not know the relevant information. Rather, the students did not access that information because it did not occur to them that this information would help them solve the given problem. (p.34)

As a response to the ever-growing challenges of mathematics teaching, most teacher education programs try to power up activities reflecting the integrated perspective of pedagogy and mathematics. The purpose of teacher educators generally has been to offer pre-service
teachers opportunities to explore the coexistence of pedagogical and mathematical thoughts. They discuss "practice-based materials" and "real practices". One topic has been the issue of transferability of knowledge. But there is still little understanding of where and how integrated knowledge is used in interaction with students. Another issue refers to the bipolar analytic stance of prospective teachers engaged in discussions of practice-based material. This study supports one more issue, the discursive routines of the classroom, the situative perspective, and the simultaneous purposes of the teacher.

Broadly speaking, researchers of teacher education have switched focus from what preservice teachers should know to ways of knowing about teaching practice. They have shown that we in teacher education operate with a limited understanding of the relationship between mathematical and pedagogical thought. It does not take research to see the difficulty of recognizing such relationships in teacher education classrooms.

Focusing on teacher discursive actions in this study, I pointed to the importance of understanding the nature of perceptual learning in classroom teaching. While efforts in teacher education attempt to prepare prospective teachers and support their learning based on similar conditions and interactional environments, little is known about how such an interactional environment affects teacher perceptual learning.

The need for redirected experience (from being a student to being a teacher) has led research and development efforts toward what could be made visible, what could be made recognizable, as necessary features of the teaching situation. But the situations are diverse. And the recognition of metaphors and logos is not necessarily a good way to learn teaching. Pre-service practice is short. Should those hours be dominated by formal conceptualizations of enactment? The learnability of mathematics teaching remains an issue for the teacher education
curriculum. Within diminishing resources, the specific activities for teaching have to be reset. It remains difficult to identify the conditions to facilitate the teacher’s use of classroom interactions.

**Different Forms of Creativity in Learning Mathematics Teaching**

**Artisanal thinking.** Most current researchers emphasize attainment of professional practical knowledge. They claim that teachers have room for individualized and creative teaching, but try not to pressure novice teachers toward spontaneous improvisation, fearing personalistic teaching to be neither proper nor efficient. They prioritize skills, calculating that they might be assembled creatively. Creative teaching becomes perfecting technical skills. Teaching is seen as using skills. Along this line, Lampert (2010) remarked:

> To consider the practicing of technical skills as a mode of learning the relational and contextually situated aspects of teaching, we must overcome the idea that creating teaching in response to observations of what particular students know and are able to do is entirely a matter of inventing action on the spot. Organizational learning researcher Dvora Yanow (2001) participated in a class on theatre improvisation and made the following assertion, based on that experience:

> Possibly the most egregious misunderstanding about improvisation - whether in a theatre setting or in an organization - is the notion that improvised activities are invented on the spot, from scratch, as if in a void, without any preparation and without context. What became clear to me in both the improv and the scene classes is the extent to which improv teams practices together, and observe one another extensively, over time. Improvised activity, invented "in the moment" in response to some provocation…builds on extended, prior conjoint experience and mutual, collective, inter-knowing…There is extended preparation (training or apprenticeship) in the rules of engagement, the rules of practice.

> Building on Yanow's observation, we could define the goal of teacher education to be preparing novice teachers in using the "rules of engagement." Even though that work must be created by knowing how to make particular productive relationships, it is not necessarily inconsistent with practicing as a way of learning to do the work of teaching. (p. 28)
In a similar vein she continued:

…we could construe the repeated efforts at productive interaction that occur until everyone settles into a routine as a kind of repeated practice from which a teacher learns what Yanow called "the rules of engagement" for teaching a particular class. Both the teacher and the students learn, from their experiences with one another, what to expect and what the results of particular actions can be predicted to be. This perspective on learning from practice focuses attention on the personal uniqueness of expertise for teaching, because a particular teacher is getting better at teaching a particular group of students the longer the teacher works with them. (p.28)

Teacher creativity is envisioned as inside the specific practice. Artisanal crafting actually is part of teaching. As with any craft knowledge, artisanal work is functional, more than just stylish. The enactments of certain teaching practices work toward getting a sense of social engagement. It is perceived in its interactional local context. Teaching is not simply a play and replay of the same enactments. Sets of improvisation can sometimes be found, but constantly through the day, there is creativity during teaching. Creativity is drawn upon as the teacher finds himself: learning more about students, about content, about class, about relationships. Practical thinking requires more than simply a best practices implementation or a bold artistically-mastered interpretation of enactment. Teaching is an interactional practice. Discursive routines are central to the finest moments. For teachers to be truly responsive, discursive routines emerge in what Kemmis (2004) referred to as "local public spaces." In these public spaces, teacher thinking is not only a matter of craft, but a matter of perceptual sensitivity, to see a different possibility for that moment. I showed in Chapter 6 teachers’ work with this kind of thinking in their response to students: they use previous experiences and they, sometimes for short period of times, tag such experiences in their communication. It is professionalism, more than craft.

As I said before, in teacher education we cannot rely much on pre-service teachers’ reflective thinking and experiential knowledge. That does not mean we should avoid having
them encounter particular scenarios specific to teaching situations. Several examples were mentioned in the previous chapter. The interpretation of a teaching action depends on the temporal connection, the sequential order of the communication. It is not only about explaining why an action is good according to particular criteria, but also about imagining and being expectant of the likely student answers.

This study problematized the assumption that teacher education activities should strongly encourage novices to integrate their mathematical and pedagogical thinking, in order to respond to classroom interaction. The possibility that classroom interaction does not fit within the advocacy of integration set the rationale and scope of this study. I argued that most of the research sees the integration as a basis/ground/source of good and quality action in mathematics teaching; in a sense, claiming it an indispensable condition of an effective teaching action. And as part of this claim, classroom interactions are left out of the picture. The claim does not fully relate teachers' knowledge of practice, pedagogy, and their enactments. The study showed that in the case of two experienced teachers, several mechanisms were functioning in classroom teaching, not only one. Such mechanisms were shaped by classroom talk-in-interaction and, strangely brought teachers' intentions in a different direction. Still the quality of the teaching action was preserved.

We see in such episodes more than just the use of knowledge, even professional knowledge. We see the way that classroom interaction shapes the discourse and how student moves have a strong influence on teachers' discursive actions. The question then comes to how we emphasize thinking, especially reflective thinking, as a mode to encourage integration, and what novices need. Detail and context shape learning opportunities for students. With practice
being a matter of "instructional explanation," how is integration of mathematics and pedagogy necessary or even preferable, say, to situated problem solving?

One very important aspect of simply learning a better enactment is that it focuses teacher action on her acting performance. It puts teachers in tension with input from students. In teacher learning, we need to address more than teaching sharp skills for making good performance. The quality of teaching resides partly in the teacher's perceptual ability to capture what in the moment is important. It is found in what motivates student learning of the specific mathematical topic and adding to a continuing learning experience.

Teaching is more than perfect enactment. As Higgins (2001) put it in his short critique of Schön's reflective practice: "someone may be said to do something well without necessarily acting for the good." (p. 10)

**Teachable mathematics.** The second case of teacher thinking refers to what is teachable mathematics for teaching. Teachable mathematics for teaching refers to what the teacher decides, partly by herself, to be necessary for understanding particular mathematical concepts. Teachable mathematics refers to those aspects of mathematics that the teacher finds important to be understood. They differ from learnable mathematics (Kaput, Hoyles, & Noss, 2002; Noss, 2001), which instead point to what makes mathematical concepts accessible to students’ cognition.

There are various pedagogical transformations of a mathematical object and there is a variety of philosophies behind those objects. In classroom interactions most of these pedagogical theories and transformations of mathematics take a particular shape through the interactional context. In this context we understand more about teachable mathematics. This study indicated that teachers use analogy, but it also showed how in the local context, the teacher and the students tag mathematical objects during the creation of particular metaphors. They also create a
local immediate environment. This could be a perceptual medium for both teacher and student, which may facilitate or not pedagogical transformations of mathematics, the rationalized integration. In this milieu, the teacher creates teachable mathematics. Teachable mathematics is perceived by the teacher in this interactional context.

Again, the aim of this study was to understand how teachers play and interplay their mathematical and pedagogical concepts amid their discourse moves. I held their discourse utterances to be a reflection of their thinking in instructional situations. I pondered how the play and interplay between mathematical and pedagogical concepts corresponded with the teacher's intentions?

In my first analysis, I illustrated the emerging theme of how the teacher's pedagogical metaphors were dependent on the mathematical structure that the “teachable” student chooses to think about. Could a teacher identify the features of the structure? I think so, getting the student to talk about them, point to them on paper or screen--but doing so, we wonder if the mathematics is teachable any more, and if the student is teachable as well?

We may miss aspects of understanding certain phenomena by focusing exclusively on certain features and disregarding a broader purview. As long as we think that somebody is teachable, we conceive of being teachable as a permanent state. Permanency, as the feature of being teachable, helps us conceive of being teachable in certain ways. But if we think of being teachable as temporary, we conceive of being teachable differently. Our thinking directs us to other ways of being interested in that subject.

Features showing up in classroom discourse are used by teachers to develop their pedagogical metaphors. The quality of teaching rests on the teacher’s ability to recognize how the students conceptualize those objects. Current research on mathematics learning shows that
the quality of teaching also rests on the teacher's ability to recognize the features of student understanding. It then became an object of research to typify such understandings and pack them as objects of teacher knowledge. The features of mathematical objects are part of the teacher's thinking in the classroom and are recognizable in teacher discourse.

Certain moral and ethical concepts seem not to be teachable because the features of those concepts bring into play situations that contradict each other. It is difficult to conceive of a concept that lives in two contradictory structures. Mathematical concepts are, however, teachable. A multitude of features may be related to a certain mathematical object and offer different perspectives of the object and its situation. There will be no contradictory situations without a feature missing in what was conceived to be school mathematics or teachable mathematics.

At the moment the student conceives a mathematical object, the teacher sets a variety of learning situations in which the student "experiences" the features, facilitating the conceptualization of mathematical objects. The teacher needs to be "in dialogue" with those features of teachable mathematics, working a lesson preparation or deconstructing known pedagogical metaphors. Both dialogues are episodic. Although, later, they may be part of reflective thinking, during the teaching, the episodes can be reconstituted, still keeping the rationale intact. The density and contextuality of the episodes are important. They need to become part of a teacher's repertoire. Given creativity, there are very few scripted moves in classroom discourse if one wants to address the teachable student. We can borrow from aesthetics. Chris Higgins (2009) urged us:

I have suggested that imagination is a central aim of education, a core component of our ideals of the educated person and visions of human flourishing. For in the end, what could be less optional than learning to think beyond received ideas, learning to feel what is really happening, learning to see what is really there? (p.12)
We want future teachers to engage repeatedly in simple and complex teaching situations. We want perceptual recognition of real and quality teaching. We want personal experience of what teaching means to a teacher. Given advances in educational media, we can realize some of it in teacher education. In particular, we need to know more about the interactional mathematics teaching classroom milieu as teacher learning milieu. We need to assure that the creativity of pre-service teachers is supported.
References


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Appendix A

Transcript Notations

Notations:

The following conventions were used for transcripts of the audio recordings. I considered
Gail Jefferson’s transcript notations. When doing the transcript I tried to mark: overlapping talk,
pace, instances of pitch and stress as well as lapses in time. Conversation analysis has an interest
for more detailed description in the transcripts like pronunciations etc, especial to illustrate the
temporal order in the social organization. My interest was, however, exclusively for sequential
and inferential order of the talk-in-interaction.

(1.00) Numbers in parentheses denote the approximate duration pauses or gaps
between utterances in seconds or tenths of seconds.

(.) Point in parenthesis indicates a micro-pause of less than two tenths of a
second

(because) Letters, words in parentheses represent sounds, words or activities that are
difficult to locate to a particular interlocutor, in general guessing who the
interlocutor was and what was said.

((Moving)) Double parenthesis provides description of non-verbal behavior

[ ] Square brackets indicate the points where talk overlaps

(::) Full colons denote an extension in the vowel or consonant sound in the
utterance of a word

word Underlined word indicates change of pitch

<> The speaker spoke slower

>< The speaker spoke faster

= Equals sign identifies a “latching” between utterances, whereby
utterances follow each other rapidly after a preceding utterance
Anonymising Data

Names have been changed according with the following rules:

1. I preserved syllable length and stress pattern
2. Maintain gender
3. Made sure contractions were possible
4. Preserved ethnicity
5. Use replacements of similar commonness or rarity
6. Preserved probable conventions of age, class, and locality
Appendix B

Fractions Lesson Transcript

Harvey: All right… Show me…
(9)
Students: It’s nineteen..
Harvey: Oh I see a variety of things.
(3)
I am seeing some people say this is the largest possible fraction you can make nine eighth
Students: Noooo…[nine one
Harvey: [I’ll give you this. It’s grater than one, but you can get bigger than that—far bigger.
(2)
Student: Nine, one
Harvey: So, if you were showing this <fraction> (.) <nine over one>
Students: I got that!
Harvey: That’s the greatest fraction you can make!
Students (talking all over): That’s what I did
(9)
Harvey: ↑What is (2) (waiting for students) <the largest fraction> you can make that is (2) still less than one?
Students: Aaaaa
Student 1: In history? =
Harvey: =Yes (. ) in history, but using these digits, by our rules…
Students: Oh! Oh!)
(6)
Harvey: It has to be less than one but it has to be < as large as you can make it.
Students: ((talking))
(7)
Harvey: ((addressing a student in a group)): Are you sure (there’ something)? They’ll right.
Students: ((in group work)) eight ninth…eight, nine…
(13)
Harvey: You can make it much larger than that!
Students: no…I already…
Harvey: All right! Show me!
(5)
If you wrote (.) If you wrote eight ninth you are fine (4)
And here is the tough one for you:
(4)
What is the largest fraction that you can…
No, no I’m sorry The <smallest> fraction that you can write that is greater than one half?]?
Students: Oh…Oh mine…Oh
Harvey: And after you make your answer I want you to hide it from your neighbors so they have to think it for themselves< ok?
Students: What was? What was the question?
Harvey: The question is: what is the smallest fraction you can make that is still greater than one half?
Students: Oh greater! Aaaa…
(12)
Harvey: Let me see…your…
Students: Can I go to the bathroom?
Harvey: Yeah…we’ve been you know…So, in the future you know try to do that, when we do that…
Students: (talking): I:::s Greater than one…
Are you?
Harvey: [Greater than a half! (.) But the smallest one you can make that is grater than a half!
((Students comment and work together))
Harvey: All right! If you wrote this:
(10) ((Harvey writes on board 5/9))
Harvey: And clear. Here is your la::st question:
(3)
And for this I would say there are a variety of correct responses!
(2)
So you and your neighbors may not show that same thing.
(2)
What is the smallest fraction you can write with any two of these digits?
(9)
Students: You can get nine…
Harvey: =Nope! You can get smaller than th::t !
(2)
You remember the first one? The first one I asked you what’s the smallest fraction you can make that is greater than (. ) ze:::ro? (.)
I didn’t say that this time! ↓
Students: Aaaaa…
My fraction …
It’s like …
You have one…
Harvey: ( (talking with a student)) So are you using more than two digits?
Students: (talking)
(13)
Harvey: All right↑ No::w… Look at your own answers (3)
There could be a variety of answers ‘cause Martin is actually shown them all!
Students: I wrote also!
Harvey: That’s all right
So:: Show me. Explain to me
Students: (arguing) Cool!
Harvey: (talking with a student) So, tell me and explain yeah tough I ask questions.
How this it works? Why are any of these the right response? Ben could you sit down, stop interacting, but interact with me a little bit more…Dresiana?
Dresiana: Because they all, because they all zero!
Harvey: What do you mean?
((raising noise))
Harvey: All my words, do we have to move those people back into rows, even rows with two and half feet between each child…I hope not, next year in much the same way but you can get they move you into the spare room because no child should interfere with the lesson…
I Dressiana repeat: what is about these fractions that makes them all the smallest they can be made?
Dressiana: (inaudible)
Harvey: ok, each of these fractions is equal to zero than who can help prove that for ourselves? Which words? (5)
Jo, you probably could…
[mmmm]
What do they mean?
Jo: (inaudible)
Harvey: ok, and remember the denominator the bottom number, that tells how many equal size pieces there are:: if we had a physical model and the number on top the numerator tells how many of those pieces there are so if there are zero of those pieces well that sure’s not many is it?
All right and clear your board and now I want you to write a fraction that’s not possible…
Students: (choir sound)
Harvey: don’t say anything…just write it and make it did you used the same digits twice,,they were in the rules so
All right! Show! Yeah if you have zero as your denominator you are right…that’s a fraction that don’t exist…and why not renata?
Dresiana: No zero
Harvey: there isn’t such a thing as zero, yeah but why can’t you do that?
I can not hear
Students: [] yeah, I think so they represent a part that doesn’t exist
Harvey: And remember what I said about calculators? [] what a calculator basically do if you try to divide by zero? [] yeah you can not divide by zero is gonna tell you, yes error or something of a kind…
All right, clear off…I have different kinds of questions now for you…
If I write a fraction, you write a decimal fraction that’s equal to it [] ???
time to show…
If you wrote zero and three tenths you’re correct.
Clear your board.  
I’m writing a decimal number you write a fraction which is the same thing or is equivalent to it…  
I am seeing a variety of things the simplest one I see to are four tenths but I’m also seeing stuff like forty hundredths or four hundred thousands …All are correct …  
I wrote a decimal you write the fraction.  
All right show and I already showing it, you are upside down! All right good! Clea::r! And let’s see if it si possible for any particular group to reassemble these plates, erasers and pens and get them put away with the captain’s help <without saying a word>…  
I’m timing it  
Starting now…  
(10)  
and those people who are at your seats take out your math journals and opened it up to page 217  
Oh call this group off ?  
(10)  
and Issablele I’m gonna need a little bit of assistance and Sophie I’m gonna need a little bit of electricity here …  
Boys and girls that was an excellent transition  
Shows me that you are serious about our work and that’s goona save time (inaudible)  
Your mathematics this morning it said if the triangle below is one thir::d then what is the whole? The one! and then it asked you to draw it on the grid…let me come around and see your drawing that works too…some people arranged it some a little differently…is this the whole, ben? You added it on to show the whole? Ok…some people drew the whole in a different position and other people add it on to to make it if what you made is the equivalent of three equilateral triangles then you are correct  

Students: yes  
Harvey: I’m sorry you spoke out I will answer to that question but you spoke out I’mm not gonna (inaudible) with that  
The second question said, if one quarter of Miss Chin class is 8 students then how many students that’s she have all together?  
What did you do to figure that out? Erika?  
Erica: I multiplied by four  
Harvey: Yeah…and a good plan, too..ops I’m gonna need that tomorrow …  
Because one fourth of it is 8 students then what we will have is four groups of eight that’s why she multiplied by four or added it eight so eight plus eight plus eight …  
⇒ I’m gonna show you some pattern blocks and I’m gonna ask you <what is the one?> I’m gonna figure out what that rattling is ? think that the ??? is broken why would it suddenly ?? when it never had before?  
All right if this (inaudible), what I’m projecting is one half…what is the one look like?
((students comments))

Harvey: Look up here? ignore that thing …
if that’s the half what is the one? Sunkia?
Sunkia: If that’s half?
Harvey: yeah if that’s half, what is the whole, what’s the one? Show!
Students: add it?
Harvey: sure, I could put two of these together then to show that is the whole…
if this different situation, if this is three fourths (Oh my god) what is the whole?
how could I draw that
>it may not have an easy name<
how would you draw it?
Take what you need and drop it and later on it…ok yeah…
let’s get it up here…
so this three fourth and then this must be the whole,
→By the way if this were a polygon what would the name may be?
Don’t tell me baby care or anything like that so,
→it were a polygon what is it? Sophie?
Sophie: A quadrangle?
Harvey: No, a quadrangle, remember? has four sides..
i’m really talking about all the way around this thing …
how many sides does it have?
Jonathan?
Jonathan: Six sides
Harvey: Six sides, so it’s properly called a hexagon…
Students: yeah
Harvey: yeah somebody called it a concave hexagon over here, too…that’s hexagon isn’t it? Yeah? But a hexagon doesn’t have to look like that because we saw one that doesn’t look like that
→All right, back on the subject though,,
If this is two thirds what si the whole look like? () Jamie?
Jamie: ((asking if he can show))
Harvey: You’re gonna come over and make it and use these tiles to help you I’ve got tiles over here.
Ok yeah you can see that! How about now? All right if this is two thirds he figured that a third may be a half of one of these so the whole thing may look like that…it doesn’t really matter how we arrange them.
If oh I finally get to use this
If that’s a third what is the whole look like?
Student: ((
Harvey: No, I;m saying if that is a third what is the one?
What is the whole? Erika?
Erica: ((
Harvey: All the hands hooked because they are gonna play at the overhead …
Students: ((
Harvey: yeah if two of those pieces out of…
Harvey: Well if two of those pieces out of six that’s what’s a third so you added it not fix it an whole and last question of this sort If this is a ha::If what is the whole look like? Martin? By the way you folks exercising a lot of good self-control I appreciate that all right I would’ve picked out two more of these and just add on But that works because all of these are equivalent in size any way. Let’s take a look at some chips instead. (Changing geometric shapes with chips/counters) Can you back those up? If that is a half …what is one and a half? Harvey: How many counters you are just gonna count them si you are not gonna make that reach all? Students: () Harvey: That’s a half and that’s not one and half! You said six if that’s half what is the one and half Ben… Ben: () Harvey: Oh add six to that, ok so how many of these in all, tell me how many in all is the one? Nine, ok…barely hear… If this is a third , what is the one? Students: (talking) Harvey: Colby? If that’s a third what is the one tell me how many counters will make the on:e? Colby: () Harvey: If that is a third …you telling me that the whole is a four that will be a three fourth this is a third… Students: () Harvey: if twelve is whole that is a fourth () Harvey: what do you need to do to find a whole if you know what a third of it is? Colby: () Harvey: yeah, you need to triple it, you need to times by three, So let me give you a different question Colby, If that is a third what is the whole? Colby: Fifteen Harvey: Fifteen is right if this uuh if ut’s all right if this is two fifths () what is the whole what is the one? () Jerry Jerry: () Harvey: How did you figure that out? Student: () Harvey: ok he your way to help figure that out if this is two fifths then just ?? must be one fifth so that’s two fifths then here is another fifth another fifth … If the (inaudible) are two thirds what three thirds look like, or what is the one? Show me with your fingers
Harvey: If you showing me six I agree with you…
Student: I knew it!
()
Harvey: If what you see up there with put on your 3-d glass
If that’s a tenth
What is the one?
Students: (showing and talking)
Harvey: Show me with your fingers
Students: ()
Harvey: You can’t just hold up there
show me with your fingers and holds (up…ok, put your fingers awa:y…)
Students: No I can (inaudible) doing like this!
Harvey: On page 217 at the bottom it says use your geometry templates to draw the
answers to problems 2 to size I’m gonna give you captains the templates
to distribute, Ben I don’t want you to pull your drawing down
I’ve got them already!
And on p 218 in your group please answer your question! Talk about it,
but talk quietly please
Students: (starting to organize their work in groups)
No you’re gonna need to put shapes on here
(20)
Harvey: Hang on, you got a captain he is supposed to do something about it…
All the shapes on your template are the same size as the shapes in your
book [yeah] good they better be…
Harvey: Why? Is that important?
Students: No
Harvey: Why si it important Renata that the shapes on here match the size of the
shapes there, the scale be the same…
Renata: ()
Harvey: yeah …I do think you can make it correctly if they [] or would it be
smaller
Harvey: (checking group work)
…let me see your answer here
if that’s quarter then this shows three quarters …
how many of you solve a 1 let’s what’s up…
Oh no 7 is the same question that Colby answered turn into next page and
solve (inaudible) of the different kind…
() Harvey: yeah…
(Students working)
Harvey: (checking student work)
do you want me to have counters to help you?
Student: (inaudible, taking the question as a joke)
Harvey: no:: you don’t have to use counters …you can figure that all in your head
that’s two thirds ad if that’s two fifths because now you filling four
fifths…
Students: (working in groups, asking Harvey a question, inaudible)
Harvey: No, hey boys and girls just a little few more minutes.
By now you should be on page 218 and finish it work with that
(inaudible question)
Harvey: No, I don’t think so
(inaudible question)
No. Which?
Students: (8) (working in groups)
Harvey: Hold your pencil still that’s what (.3) think of like at
if you have two fifths
then half of that is one fifth then one fifth plus one fifths is two fifths like half of that
Student: (asking about a different method)
Harvey: That needs the rules of fraction topics ‘cause you got to play some games too …
(Rings bell)
I can tell some of you are done ‘cause the level of the noise if all the way up here is what you need to do. When you finish then you’re ready for a game, the game is described in terms of its rules and objectives on page one hundred ninety seven on the student reference book…you may organize yourself as you finish into the groups of two or three study the rules and get set to play
Harvey: You mind help collecting the geometry templates
Students: ok
Harvey: That’s one () under the desk
Two hundred eighteen is part of your assignment
Student: ()
Harvey: One ninety seven
Students: (ah)
Harvey: ((He collects))…Have you finished? I cannot hear you!
You need to get set up with people to play fractions
Happened I may be the only person to know?
Students: no!
Harvey: Oh where did they say that? (talking with students who prepare to play the fractions game)
You can’t have a fore
I don’t want you to exclude somebody who wants to play it.
How many decks of cards you need to play?
That’s too much!
Oh! Is that all of them?
Remember how you feel?
Students: (Inaudible)
Are you playing already?
Harvey: I’ll take it…
It’s a little high
(6)
(Teacher walks around the room)
(Bell rings.)
Who can put it in your own words?
How you play it and how you start it?
No other voices except the students I call on!
Great! I’ll call
Victor, tell us in a loud voice!

Victor: You have to yield the number of the cards that each player of fraction can
side up sixteen if two players, 12 if three players eight if four players Then
you play the cards fractions side up. And the person with the (inaudible)

Harvey: Ok, did you hear that part? That’s really important! If you or your
opponent have a dispute, you turn a card over and compare the blue sides
Then you know it. But you’re supposed to play it with the fraction side up.
Greatest fraction takes the trick or the hand, at the end you add up who got
the most cards that is the winner. As soon as you play a game of that if
you want to come to me and get another game to play that’s a little
different see me after your first game.
Another version?
Oh…

Students: (inaudible)
Harvey: If you tight? Still have to figure that out!
Student:()
Harvey: The fractions side? Look at the one with numbers!
Students: How would you know which cards to pick?
Harvey: Get your deck and put them all in your hand!
Students: Oh!
Harvey: Pretty much if you want to know and you want to get watching the other
guys is not gonna matter as long as you are not trying to cheat.
(Students start playing and Harvey walks around)
Harvey: (addresses a student in a pair) Got of mix them up a little bit. Ok?
(starting to play with a student)
Is your deck ready?
All right!
Put them face up, the fractions side up or…Oh I see what we should
doing!
You can either go like this, in your hand you have it ready to play, or you
can out it on the board and turn it that way, but if you do that you gonna
pick that this way!
So, hold them in your hand I’d say…
(looking at another pair how they play, waiting for his student to move)
And now, how do you guys doing over there?
(student asking)
How can you even see everybody’s else card?
Put them in your hand like that!
(students talk)
Otherwise you just take up the one you want, which is you know ok, for who knows the fraction! All right…how do you break a tie?

Student: You do it again!
Harvey: This one. You get a tie. Each player takes another card, that’s right. Ha: you’re back!

Student: (makes a move on his card) (this is the student Harvey plays with)
Harvey: What? You would rather play this? ((amusing mimicking to engage the student))
Yeah…
There is another game to play. Yeah. Come on over here. You set a …to me. You need a set of paper to write on. ()
This player gets three turns in all, you set up your board to play like this …Oppsie …

Students: ((asking to play a different game))
Harvey: It’s a different game. We are not using spinner this time.
What you’re doing on a tie.
Is it the end of the game?
Oh you guys come up over here to see what’s the next game is like…and see…
(1:01)

Harvey: (stopping the play and initiating discussion)
Time to record any discover that you have ma::de.
Did anybody make an interesting discovery?= =I saw one made I was interested to see and share it.
Did you discover anything while playing?
((students make comments, bell rings))
Appendix C

Probability Lesson Transcript

Harvey: Let’s have a quick warm up. Mental math first. Get you in the mood.
Aaaa…
Three times four (.4) plus eight (1) divided by two (1.2) Tell me what 10% of that is.
Polly: Five!
Harvey: Yes.
Put six times eight plus two take away four . tell me what 10% of that is.
Student: (Inaudible)
Harvey: Yes
Four times four divided by four times ten plus two. Tell me what twenty percent of that is.
Students: (Inaudible)
Did you say 8.4? Yeah…
Harvey: Today, boys and girls we’re going to solve a mystery. You are going to solve a mystery because I already know what’s in the bag.
Each on of you, your group will have a mystery bag that contains sixteen cubes.
They are four different colors, as you can see: blue, green, red, and yellow. Among the sixteen cubes.
However, we don’t know how many of each. You don’t.
So, what we are going to is have you in your groups roll 20 times for the bag.
Each time you draw them and you must be careful never to look in the bag Each time you draw one, pull it out all in your group to see, the recorder will tally them in the tally chart that you have then you’ll put the cube back in the bag
Turn it in the bag in your group, gave it a little shake to mixed up then you’ll draw yet another one and another one.
You do that until you draw all twenty times.
Hold your question because it may be an answer in the course of the directions.
You have three roles in the group. One person is the recorder. One person is the drawer, who holds the cubes bad. And the other person is the group bagger.
Now, couple of people are absent from your group So, if you are in a group that has only two people you can figure between yourself to get the whole job done.
It will be really important as you are working to keep track of how many times you have drawn.
Because you have drawn 30 times you of twenty or not enough times I am not sure what
I am going to do with that information. So once you have drawn twenty
times and kept the tally marks up you crumble over the top of the bag
and the recorder will take the bag over the here and it comes to the chalk
board find your group letter name and answer the data that you’ve derived
You’re gonna try to figure out
How many you think are in the bag once you’ve twenty draws.
How many of each among the sixteen are in the bag.
So, I think, even before you turn the bag in that would be a good time in
your group to finish filling out the form that you’ll find.
Do you wanna talk about it. You wanna talk about the strategy of filling
out how many were in the bag but in no time should you peek.
The very end of the lesson we’ll take a peek one by one we’ll do all the
same in a various group
We’ll see if you’re able figure it out.
Now there’re couple of parts on the sheet The top part is what you’re
gonna do in your small group at first and the very last column you’ll
answer how many of each you think are in the bag
We’re gonna take all of these data and we’re gonna put it on the chalk and
then were gonna have a much larger sample.
Then you’ll have another chance back in your groups to take anew guess
So a new analysis with the largest sample and see if you’re gonna change
your answers of what is in the bag.
Are there any questions about what you’re gonna do? Even?

Even: I have two questions
Harvey: All right
Even: Aaa. Why? if you have…
Harvey: They will have labels on them.
Even: Oh
Harvey: So, the name and the group letter-name is on there
Maya: Which order one of us is first?
Harvey: All right…Are you the recorder? So, the recorder is first . first we’re
gonna use this one. So, what group are you in?
Maya: D(inaudible)
Harvey: So, you’ll come, how many blues did you have? How many greens, how
many reds, how many yellows and that should just take a minute. All
right? Yes?
Paul: So, it’s how many times you picked it up
Harvey: No, It’s how many times you picked it up. That’s a good question. So, this
will relate with the tally that you’ll make in your chart. Yeah?
Victor: Do we write the number or we write the tallies?
Harvey: So, you write the number. The tally is just to keep as you’re gathering
data. More questions? Yes?
Caroline: The total is as…
Harvey: You know what? This total and the fraction part we’ll do together as a
class once we’ll have all the data yet
Maya: What do we write in that table?]
Harvey: Nothing yet. We’ll talk about that later.
Marteen?
Marteen: Do we have to aaaa…write the number of cubes…((inaudible)) the cubes will be off
Harvey: So… What do you mean the number of cubes? (Cubes) will be off?
Marteen: I mean…aaa…because if you when you get 15 blocks and you draw them each one and one, then you have to get it over…
Harvey: You know what I think you’re doing? You’re confusing the number of cubes that we have in the bag with the numbers of trials that we’re gonna take. So, if I handle you three coins and said “Flip these three coins all together 25 times it’ll matter at all that you just flipping them three times. (Or something). You know what I mean? I’m not sure you do. Talk to your partners about that.”
Marteen: Ok
Harvey: And we’re gonna back up if still are questions. All right now remember what I said about move where most of the [eople in your group sit?
So, recorders go ahead and get your ((noise increases)) material,
Student: (starting the activity)
Harvey: All right. Boys and girls stop for just a second. Make sure that=
Make sure that if you are the bagger you’re holding the bag. You know I don’t think you’re hearing me out! You’re holding the bag in such a way that nobody including the bagger can see in there So keep it up high give it a shake
Harvey: ((talking in a group)) Are you a bagger?
Harvey: Eighteen more to go. Keep track of that.
((bags shaking))
Students: Green…
Harvey: Ok good way to shake. Don’t look in the bag!
Student: (in group) You know it may be only one yellow
Harvey: How many times you’ve done it? How many to do? Fifteen keep track. You’re trying to get here or something. Yeah. You don’t try to get here until we get all the tally in!
(6)
Harvey: Keep on going. Don’t look in it.
Student: ((trying to guess) I doubt is yellow
(6)
Harvey: Oh there is some of each.
(6)
Student: ((playing))
Harvey: Yes. I liked the way you look away. So, that’s fair. ((Talking within another group)) How many more to go?
Students: Five
Harvey: Ok. Keep track of that.
((Addressing the class))
Keep track of how many to go!
(3)
Harvey: (addressing a student in a group)
How many trials you have to go here?
Students: Fifteen
Harvey: Ok, that’s =
(.3)
Harvey: You got eight, right? Wait…
Students: No, nine
Harvey: Is six here?
Ok, nine
(looking at trials)
Harvey: Six, seven, eight, nine, eleven more to go. Ten more to go.
((moving to another group))
How many more to go?
Students: Aaaaa...
(.3)
Harvey: Stop and see.
Students: One more
Harvey: One more? All right.
(4)
Harvey: (addressing the class)
Ok, now, Finish filling out talk about what you think is the bag, but at
some point you’re gonna get up on the chalk board what your group did.
Students: (inaudible)
Harvey: Yeah, why won’t you turn your bag in Bag there on the table.
(4)
Harvey: Ok, So…five, ten five…
(making a count in a group)
How could that be twenty ???
⇒ I think that’s …
Students: (trying to explain)
Harvey: How many do you have here?
It is twenty ??? Here? So, five there…you know what?
I take my …Yeah,
⇒ I made a mistake here
Oh…get out. Noooo.
I think…(responding a student)Yeah, that’s the recorder of the time.
So, that’ll be 25% there
(3)
Student: You need to divide the number by fifteen?
(.2)
Harvey: I think all you have to find the percent here in your sheet then multiply the number you got here by five and that’s where you [Ok but by five…] Come here, come here…During the You have to go ahead and (inaudible) The bag (checking once more the results, moving to another group)

Harvey: Ok…Is this your group? You did three. Three times five is fifteen, That would be fifteen percent of the times it came up that. Oh think about I made a big mistake in there So, yeah ten out of twenty…Multiply that by five and you get fifteen percent. It looks like you can divide by five because twenty to turn into a fraction will some number of hundred. ➔ So, multiply by five to have it.

(The lesson continues. Harvey compares the results of the groups and the entire class’s results as recorded in the table written on the board)

*The rest of the transcript is available upon request.*
Appendix D

Rational Functions Lesson Transcript

Mike: All right! So do I have all the homework?
    All right! All of them? Ok!
That looks like a huge number
I have some Valentines for you, but I would like for you to…
Let’s see…I need to get stopped?...Aaahh
I don’t see many people have login in yet…only ten people so far…
What’s not showing up?
If it’s not a file…Did you…did you…
(Pause, checking with the technician desk)
(Turning back to students)
Did you have Algebra 2 there? (12)
(Teacher commenting with the technician) Let’s see if it works with an update!
Th: (technician Input) It looks like you don’t even see it, isn’t it?
Mike: Yes! (2)That’s it!
((Addressing students)) Ok!
So…Aaa…Go ahead and log out and then log back in…
(20)
So today…I haven’t even look at your test yesterday except for one.
So, I don’t have, for some reason…Yet, all the corrections that need to be turned in on Monday. So, if that applies to you, think about what you may do about that…Aaa…The rest of this unit is about rational functions.
So, a rational function…
The reason I had you do I had you look at…
What you looked at in the homework yesterday…
A rational function is a function that’s a ratio of polynomials.
Yes? SP?
SP: Inaudible
(Several students complaining the tablets are not showing Algebra2 file)
Mike: I just turned on the power…
Ok…
In the homework, yesterday, you looked at… (Checking in the papers)
Number one, you looked at… I think was it x+1. Was that the problem?
SSs: Yes (Confirm that was the HW problem, nodding.)
Mike: And then you looked at 1 over x+1?!
SSs: Yes.
Mike: That was a rational function!
In number two, you looked at 1 plus e to the aaa…
SSs: Negative one…
Mike: […]e to the negative x] and then you looked at 1 over 1 plus e to the negative x.
Is that a rational function? (The teacher returns to the desk with the computers.)

SSs: It’s still blank (Commenting they do not see the files)
Mike: It’s still showing up blank. (The teacher turns back to the technician) Tablets are all blank. This is frustrating.
Ok…Please log out and try log in and …
S: What about if we just go in these places before we logout
Mike: That shouldn’t make a difference …
S: It shouldn’t make: ?
Mike: Is it: ?
So:
Finish and then start over…
((Addressing one of the students)) >The show-in up is there, right? <
S: >Yes<
((Addressing one of the students)) The second function that you looked on homework yesterday. We’re look at …
You looked at 1 over e to the… 1 minus ….1 plus… e to negative x.
Is that a rational function?
(7)
SD?
SD: =Yea…
Mike: >=Student B, what do you think? <
>Is it a rational function? < (.)
→ According with this definition?
SC: ((SC responds before SB)) No!
Mike: >Wh::y not? <
SC: (Because)… e to the negative x is not a polynomial!
Mike: <1 plus e to the negative x is not a polynomial! >
Ok?
So: in your homework last night I asked you to look at two functions that were fractions, that were ratios, but the second one wasn’t what’s call a rational function, because it wasn’t a ratio of polynomials.>
We still have blank files, ah? ((Teacher is checking students’ tablets))
SSs: Yep…
Mike: Okay:
Did you say something on it…?
S: ((Inaudible))
Mike: Ok …The…it’s fine, it’s fine…so it’s fine about being blank.
You have to take notes about what I am doing up here.
(.)
Ok…The domain of the rational function
(.)
Is… the set of all real numbers:
So that the denominator of the rational function is not zero.

(The teacher points to the PowerPoint presentation where this statement is written.)

(The teacher takes attention from the PowerPoint and addresses the class)

So, the domain of the rational function is the set of all real numbers except that the denominator is nonzero.

So, rational number…rational function is a ratio of polynomials where the denominator can’t be zero. So the roots of the denominator are not in the domain of the rational function.

So, I have some examples for you.

My first example! $3x+2$ divided by $x^2+1$.

It’s a rational function because it is a ratio of polynomials. The denominator is $x^2+1$ and that…and x square plus 1 is never zero. Therefore …The domain of that first rational function is all real numbers.

Ok?

The second rational function ((Pointing to function $3x+2/x^2+x-6$ on the PowerPoint Presentation))

You should be able to factor this denominator.

The denominator. Yes what are the factors?

SSs: $x+3$, $x-2$

Mike: $x+3$, $x-2$…?

Ok, so this is all…The all…

The domain of this is all real numbers except those…except… 2 and -3 which are roots of this polynomial.

Ok! (.) Rational functions even though they are hybrid, they are… a sort of mixture… they are a composition of polynomial functions aaaa…they show up a lot….Average production cost … where you look at aaa the cost of producing x items divided by the numbers of items produced…is often a rational function…and we will see some other uses for rational functions.

In photography, rational functions show in the in the law optics it tells you show you how far…how to focus the cameras…So the focal length…It’s the equation that equates the focal lengths and the distance to the object that’s a rational function.

Here is a lovely rational function. (Points to the function $3/x^2+1$)

What is the domain of this beast?

SK?

SK: All real numbers.

Mike: Why?

SK: Because there is no value for x be negative… (Inaudible)

Mike: Ok. There is no value for x that can do what?

SK: Be negative for the x…

Mike: It’s not…Has nothing to do with negative numbers or zee…or…aaa What… What is the domain of every rational function?

It’s the set of all real numbers
Except …
Roots of …. denominator?
So …When is the denominator of this thing zero?  
For what real numbers?
For all real numbers …I said all real numbers except real numbers when
the denominator is zero. When is the denominator zero?
SSs:  Never!
Mike:  Never! Ok? So the domain is all real numbers.  
Is this function ever zero? When is a fraction equal to zero?  
(Talking with the supervisor technical person) So, the file never showed
up…Aaa
Mike:  When is a function…When is a fraction ever zero? SM?  
SM:  A fraction…when the numerator is zero
Mike:  The only time a fraction can be zero is when the numerator can be
zero. So what’s the numerator of that fraction?
SSs:  Three?!
Th:  There is no file…This has nothing to do…
Mike:  There is a file is just not showing up
((Supervisor technician talking with the person who uploaded the file))
Mike:  I just put it here (shows on the screen)
Mike:  ((Turning back to class)) Aaaa…So, the numerator is …three. How
often is that zero?
S:  Never!
Mike:  Pretty much never …ok? (Teacher turns to the computer desk)
S:  Please don’t click….
((The comment refers on what is written at the bottom of the slide, which
was supposed to be on their tablets))
Mike:  What?
S:  (( Repeats))…Inaudible
Mike:  I know, I am not going to click.
Mike:  So this fraction is never zero.
This function is never zero.
The function has no roots.
The domain is all real numbers and the function is never zero.
What happens if x is very large?
What happens?
What happens to a fraction if the denominator is large?
SA?
SA:  The fraction gets closer to zero?
Mike:  If the fraction of the denominator is large, the fraction itself is small.
So we have three over big number closer and closer to zero. Is the
denominator ever negative?
SSs:  ((Commenting, but not a clear response))
Mike:  Is this denominator ever negative?
SSs:  No
Mike: No…
So, this fraction... f(x)...
This rational function is always positive and as x gets very large it gets
closer and closer to zero.
What about when x large gets in negative direction?
X goes up to the negative infinity, what happens?
SSs: Same (Inaudible)
Mike: Same thing. It still gets closer and closer to zero. Ok?
Th: (Inaudible)
Mike: (Responds to technician) No, we’re ok… //
(Addressing students) What happens if x is very small? x gets closer to zero.
What happens?
((No Student response))
Can x be zero?
SSs: No // Choir
Mike: ➔ Why x can’t be zero?
SSs: ((chuckle)) It can…
Mike: Then why did you say no? The all choir…
The all class sounded like a hard piece choir…
Can x be zero?
SSs: Yes ((Choir))
Mike: What happens… when x is zero?
St: 3 over one
Mike: You got 3 over one!
What happens if x is close to zero?
Then x square is…
If x is close to zero, then x square is::
SSs: Small
Mike: Also () Close to zero, so the denominator is
SSs: Close to one ((Choir))
Mike: Close to one. So the all fraction should be close to …
St: Three…
Mike: Three! Ok
So, now, we have a function that let’s see () x zero the value is three, it
ever crosses the x axis because it has no roots, and as x gets larger and
larger, as x gets further and further from the origin the function the graph
gets closer and closer to the x axis.
What’s that called?
SSs: Asymptote!
Mike: Asymptote!
Now look at the graph of that function in your calculator. The idea that
you just expressed should show up in your calculator.
((Students taking time to take calculators out from their backpacks))
And it’s still…
I guess it shouldn’t amazed me because it happens every day, but when I ask you to get a calculator it shouldn’t take you forever because this is what kind of class?

SSs: Math

Mike: Math…you should have your calculator out when you sit down people!

Geez…

Your standard view window zoom six if you are on CA.

Negative ten to ten in both directions if you are not on the CA

Th: I just got on my cell phone…

Mike: Ok…show me!

((Technician gives directions to upload the file in a different place.))

Mike: Ok.

Mike: ((Back to class)) All right. We… We set? Is everybody set?

Ok do the same analysis with this function. ((Points to the function $x^2 - \frac{1}{x^2 + 1}$))

Don’t graph this function yet! So, put your calculator aside.

See if you can figure out what’s the domain of this green function?

Figure out when is it zero…

Figure out… if you can … what happens for very large …

Talk about it with your neighbors but don’t look at the graph… don’t use your calculators…

Talk about those questions…

I’ll call on some answers in a minute

So SN, I want you talk about those questions with your colleagues and you don’t need to play with the computer right now. Ok…?

((Talking…Students working in groups at their tables…for about 2 minutes))

((Teacher works on the file for his computers.))

Mike: ((Addressing students in a group))

What you all… did you …what you decided here?

SSs: We decided that the domain all real numbers and its zero to be 1 and -1 and it’s gonna grow closer to 1 and then gets larger numbers back

Mike: Ok…have you all towards 1?

S: Aaa

Mike: For very small absolute values of x how does f act?

SSs: That will get closer to negative one

Mike: Really?

SSs: Yeah very small like in negative and

Very small… like just for zero?

Mike: Close to zero! You say small absolute values! Ok…?

SD, can you share your discussion? ((Addressing to a student, in another group))

((Addressing to the group he was talking)) You guys did a good job…

((Addressing to the entire class)) Everybody listen to Mr SL

SL: Ok…the domain is all real numbers…
Zero at 1 and negative 1…for very large values of x, f (x) is getting closer to 1

Mike: Why do you say that?
SL.: Because… aaaa… (4)
   If the fraction keeps growing … all….
Mike: \( \rightarrow \text{Ok} \quad ((\text{With a sigh})) \)
   That’s the [ =
SL: =[If you like () if you have like 10 and then is 99 and then you get 101
Mike: ((With an enthusiastic tone)) Oke::::y
SL: Which it’s closer to one, then you get 102
Mike: Aha!
S: So…aa…the more and more …the higher the number the fraction gets closer to one…smaller and smaller…
Mike: Ok, that’s some good reasoning…That’s that’s that’s a kind of number sense I hope that you all have! So, that’s good reasoning
What about for small absolute values of x?
You’re on SD….
SD: ((Inaudible))
Mike: x close to zero
SD: Yeah! Oh…I don’t know…
Mike: You told me when I stand in there!...
SD: Non, SN. told you!
Mike: Oh, SN. told me!
   SV., can you:: you tell what happens when x gets small? Close to zero…
SV: (No response, trying to respond)
Mike: Can x be zero?
SV. ((commenting with her colleague SB. on the reasoning))
Mike: SB.! What was the answer to the first question?
   What is the domain?
SX: All real numbers.
Mike: Can x be zero?
SSs: Yes!
Mike: Then how better to answer the question about what happens if that gets close to zero then to see what happens when x is equal to zero?
SSs: Close to zero is negative one
Mike: What fraction will it be at zero?
St: -1 over 1. -1
Mike: -1 Ok?
Look at the graph in your calculator now.
Recall we said there were two places where the function crossed the x axis at 1 and negative 1 We know that the value of the function is negative 1 when x is equal to zero and that the function gets closer and closer to… 1.
If you have a graph of the first function fill in your calculator … you may notice some similarities or if you have one planted in your memory you may notice some similarities in the few graphs…
Everybody see what happens?
SSs: Yes…
Mike: Ok same stuff different day different function!
((The teacher points to f(x) = x^2-4/(x-1)))
Same questions use the green function.
Answer the questions for the green function now.
The answers will be slightly different. The graph when you get to will be different but answer the questions first without looking at the graph.
St: (Inaudible)
Mike: Yea. That’s right…
The answer to the first question is different for this for…the third example!
Stop playing with the computer…
St: I didn’t play
(Group discussion. Teacher at their table listening)
St1: There is a large size of the…
St2: If you farther away…
So if x is 10 it gives us ninety …ninety nine…for larger values
((Mixing voices))
And you can get it 99 goes over 1…yeah it gets on the one…
Mike: What’s x-1?
St: It’s 9 so the higher the number …
Ts: Ha…
(Moving to another table)
How would you doing here ladies and … gent?
Ss: Aaa
St1: We close to the asymptotes….
Mike: Are you?
S1: I don’t know…they don’t know what it be
Mike: Ok…That’s not a horizontal asymptote…
St2: Ok
Mike: ((Addressing to the class)) So, what’s the domain of this function?
Ss: All real numbers except …
Mike: All real numbers except for 1!
Mike: Ok? Aaaa… Is this function ever zero?
Ss: Yes…yes
Mike: A::t…
Ss: +or-2!
Mike: + or – 2
What happens if x gets very large?
Ss: It gets closer to 1
Mike: It doesn’t get closer to 1…’cause if you had x^2 x is …if x is huge x^2-4 is not going to be very different from x^2 isn’t it?
x-1 if x is huge isn’t going to be very different from x isn’t it?
So what is x^2 divided by x?
Ss: I don’t know… x?
Mike: X!
So if x gets very large what happens with f(x)?

St:  X...
Mike:  It’s gonna get very large, too.
You understand the reasoning I used…there?
When x is very large the numerator is approximately $x^2$.
When x is very large the denominator is approximately x
So…when x gets very large this fraction is approximately $x^2$ over x.
And $x^2$ divided by x is just x.
So, as x gets very large, this function’s graph actually looks like the line y equals x…  (2)
More or Less…
Aaaa…what happens for small values of x?
Is x 0? Can the …can x equal zero?

St1:  Yeah…
St2:  Yeah…
Mike:  Thank you!
What happens as x gets close to 1 though?
So, I am going to talk about that carefully.
What …what about in numerator…
As x gets close to 1 what is the numerator gets closer to?
St3:  -1… wait that will be… when x gets closer to 1
St 4:  -3
Mike:  As x gets closer to the 1 the numerator gets closer to….
Ss:  -3 …
Mike:  - 3
As x gets close to 1…if x is a little bit bigger than 1…what’s the denominator?
St:  1…small …
Mike:  A small number, but still positive
Numerator close to negative 3, denominator small and positive …
What happens when you divide negative 3 by small positive numbers?

Ss:  Negative number…
Mike:  You get a really big negative number.
What if x is a little bit less than 1? The denominator…
The numerator is still close to numerator -3
The denominator is x is less than then one x-1 will be less than zero, but if
x is still close to 1, x -1 will very close to zero but negative.
What happens when you divide a number close to…
aaa.. 3 by a negative number close to zero?
You get…

Ss:  Positive
Mike:  **Big** positive number…
Ok? Look at your graph on your calculators
You could see that behavior…
So, I heard Britney using and Nora using the A word…
Asymptotes!
Aaaa. We saw with power functions. Power functions have either no asymptotes or they have horizontal and vertical asymptotes…

Mike: Shhh…
Exponential functions all have horizontal asymptotes on one side…
Logarithmic functions all have vertical asymptotes …
But with rational functions you can get different sort of things …
You can have horizontal asymptotes…
The first two examples we looked at had horizontal asymptotes the first one had horizontal asymptotes
y=0, the second one had an horizontal asymptote that y=1,
The third function you looked at had a vertical asymptote at x=1 when the denominator was zero…
Aaaa. In general, an asymptote is a line that the graph of the function approaches and I say ()
It approaches at extremes for large values of x or y or possible large value of x and y and…with rational functions a lot of time you can reason out just by thinking about how the numbers work just as you did when you did what happens for large x what happens for x close to zero.
Aaa…You can reason out where the asymptotes will going to lay I have that confidence in you!
Now, I like you to reward my confidence by showing that you can do it. I like for you… The goal is for you to be able to write down some rules that you can use that you be able to construct.
Ok!
Mike: I can look at this function here and I can tell you that this got this asymptote and make a list of the equations for the asymptotes. That’s the goal.
I want you to be able write down a rule that will tell me …tell you how to do that. So, look at these six functions.
Look at one at a time. I’d recommend to put no more than one graph on any calculator at your table, but you may look at like the first three graphs. Have different calculators with each graph and see if you can make some conjectures some things to look for are:
The degrees of the numerator and the denominator the leading coefficient of the numerator and the denominator and the zeros of the numerator and the denominator …
If you can look at those clues and come up with a list of rules
While you work I’ll pass out my little Valentines for you
Ss: Yohoo
Mike: Which you may not think that they your valentines once you open them up
Ss: Ooooh
Mike: They are your current grade!
Ss: Oh no!
Ts: So…I want you working on this…You work on this… We missing Kelly.
Are you counting revisions?

This includes the test revisions turned out on Monday
Some of you were successful with your tests revisions…
And as usual you have the usual opportunity to retest …
And I would suggest you see me during the next week
Next week so you can pick up some new material so you can do that…
Work on this! Your grade is something you need to worry about at some other time right now you need to worry about this!

((While passing the “Valentines” teacher checks on students work at tables))
That’s not true for any of those.

Because, because it has to be …
Ok so if this is \( f = -1 \) because was…

So if you’re not looking at the graphs of the first three functions as your talking then you’re wasting your graphs …Please look…
Have a different calculator at each table…
Graph \( f \) and \( g \) and \( h \) and look at those three graphs and see if you can figure out what’s the relationship between the degrees, leading coefficients and roots of the numerator, and the denominator is …
What the relationships between those and the asymptotes are.

((Commenting at their table)) \( g \) is never reaching possible one beside

Hmmm. Change your window go from -5 to 5 for the \( x \) the \( y \) is ok…
That picture may be more revealing …
So you’re looking at this graph.
Can you see any asymptotes on that graph?

Yeah

Aaa…it looks like there is a mistake in entering the function

Yeah

The \( x^2 - 1 \) not the \( x^2 + 1 \) the denominator …
Ok… What about it?

We did a mistake about that?

No! That’s correct! That looks good

Thank a lot!

Anytime!

So, think about what’s the domain for the function all three \( f \), \( g \) and \( h \) all have the same domain

Ok

What’s the domain?

All real numbers except for 1 and -1

Ok, so…What happens when \( x \) is 1 and -1 and all three of the graph you have ok?

What’s that called ?

Asymptote? No
Mike: Well, it is an asymptote. It gets closer and closer to that vertical line here. It’s getting closer on the positive side and here its’ getting closer on the negative side. Yeah because you can’t divide by zero…ok?

St: I got… I got two of that…

Mike: You might change the x window to go from negative 5 to 5 to go in both directions for all three those graphs.

St: Two vertical asymptotes

Mike: Ok now SK. What happens there is that the calculator is drawing all those vertical lines which shouldn’t be there those and the asymptotes. So should be asymptotes, but the calculator is just ignorantly connecting the dots at fixed values if you do that what happens. There

Ss: Ohoho

Mike: That when you zoom decimals zoom 4 oh I see what it is

Mike: Interesting It is different

Mike: Which one you have? h, g…g yeah that’s ok!

St: So it should be different!

Mike: Yeah it’s not quite symmetric…I mean ahhhaa

Mike: More than not quite

Mike: I think let’s let’s look at what you had there…

Mike: So -1 and 1

St: Yeah that’s ok

Mike: (Talking with a different group) So do you guys have some rules established?

St: A rule

Mike: A rule … sort of? Shouldn’t pass me see what happens what about the flag poles see if everybody so loose…

St: At the extremes of x

Mike: For large values of x, ok

St: If x like a polynomial of degree, the degree of the numerator minus the degree of the denominator … leading coefficient

Mike: Ok, how would that be? That would …what’s of that different of f ?

St: That…The asymptote… can you have a polynomial degree that is less]

Mike: ] Polynomials always have degrees greater than are at least zero…

St: Oh…but it worked pretty nice though

Mike: Yeah it works nice for the other ones

St: See what happens what rule can you say in the case of what’re the

Mike: I got as the power functions

Mike: So, where is the horizontal asymptote?

S: I guess we could say I am not sure

Mike: What?
St: Negative five and five …that
Mike: Look at the graph… Look at the graph
((At another table))
You guys have any rules established?
St: Roots of the denominator
Roots give the vertical asymptote
Mike: (. ) Ok how about the horizontal asymptotes?
St: That’s none
Mike: Ok
((Moving to another table))
You guys have any rules established?
St1: We trying to but it’s… it’s well…
If the degree of the denominator and the numerator is the same and they
have the same sign then the asymptote will be a 1
St2: I don’t think so!
St1: You don’t think that’s right?
Mike: Ok that works for g…
See if that works for j ‘cause the degrees are the same on g and j
St: It probably will not work for j
Mike: Well if it doesn’t work
Maybe the way maybe it doesn’t work will tell you something…ok?
((Moving to another table))
No, do you guys have some rules?
St: Not quite yet
Mike: Not yet
St: Yet
Mike: Ok … obviously
St: Horizontal asymptote depend on … maybe here will depend on
Mike: ((Moving to another table)) What’s your rule, SV?
SV: I have no rule at all because so far
as x equal I am working for the domain]
Mike: Ok
St: So in this case for all real numbers except 1 that
Mike: All right …ok… how about the horizontal asymptotes of those three
functions two of them have horizontal asymptotes and one doesn’t …
Where did it used to have a horizontal asymptote?
These two do, and this one doesn’t
St:
Mike: What if this had a horizontal asymptote?
Can you tell that if you zoom out
If you take your window
Here SM change your x to go from make it
Go from -50 to 50 for the x but leave the ys alone…
SM: So here its’ like at whatever that
Mike: So figure out what that is…
Use trace ..
Why what use ok

St: One point

Mike: Ok see if you can figure out why is 1
Can you come up with...here is g so see if you can tell what about f causes your horizontal asymptote be zero what about g causes it to be a 1 and what about h causes it not to have one at all ok?

St: Ok

Mike: See if you can figure out what about those functions cause those different kinds of horizontal asymptote behavior

(Moving back to the table where students compared functions j and g)

Did you guys make any discoveries?

St: It doesn’t work

Mike: It doesn’t work for j,

Does j have any horizontal asymptotes?

St: Yes

Mike: Yes! What if you change your window

SN: change your window ...go have say -50 to 50 and leave the ys alone

St: All right I made that

Mike: Ok that’s fine...maybe make the y be a zero also it looks it appear to be positive...

Ok now look at the graph

See if you can figure out what that asymptote is...maybe by tracing

St: Yeah

Mike: Hmmm ...What about that function may tell you

St: 1.5

Is g having a horizontal asymptote? Because mine is going along the side

Mike: Which one is yours? f(x)?

St: I don’t know what to do I’m

Mike: Because you haven’t graph it correctly

St: I didn’t?

Mike: Nope! See that 3x+2? See that big line near 3x+2?

That means that are grouped together in order to graph those you have to put the direct 3x+2 in parenthesis

St: 1...

Mike: Here is graphed correctly

You...It doesn’t have a horizontal asymptote

You are correct and that’s some SM1

Here is one what SM2

SM2: (Inaudible)

Mike: Oh ok Now do you see one? Go to zoom six...

St: Yeah

Mike: Ok... can you tell where it is

St: Think like at 1

Mike: Ok let me suggest that ok ...

You think that is a one... ok so ok ...

You have a horizontal asymptote y=1 you have one?
St: Not that I see
Mike: What’s happening as x gets very large?
St: ()
Mike: As x gets large
St: []
Mike: Ok so that’s a horizontal asymptote
St: y=2
Mike: and this one doesn’t have a horizontal asymptote at all
So look at those three functions f, g and h
Look at f, g and h and see if you can figure out what causes what aspects
of those functions cause a horizontal asymptote at zero or at one or not at
all ((Addressing to another group))
What you guys come up with?
St:
Mike: Close
St: Now I’m afraid to make any rules being
[is not a line]
Mike: What’s the domain of it?
St: Oh everything except 1
Mike: Aha…so it’s a line with a hole in it
St: Mine doesn’t have a hole in it!
Mike: That’s because your graph is ignorant
Your calculator is ignorant…do zoom decimals to see a hole
St: ((totally surprised)) Ha!ha!
Mike: So what did you come up with as rules?
St: Asymptotes that
Mike: hmmm not quite not quite aaaa
Mike: ((Addressing the entire class)) Can I have a volunteer and determine
for me how could I look at a function and determine for me if or whether it
has and where they are vertical asymptotes?
Whether it ash vertical asymptotes and where they are…yes Mr SW
SW: If the denominator has a rational root it has a vertical asymptote
Mike: If the denominator has a rational root will it have a vertical asymptote!
Aaa if the denominator has a real root,
It doesn’t have to just be rational then is likely to have a vertical
asymptote
Function m doesn’t have a vertical asymptote
So that’s not quite true… but it’s close
Reasonably close. Did people observe that?
Where these three functions, all three, have the vertical asymptotes?
Ss: (talking)
Mike: At x equal + or – 1 Ok
When I say a vertical asymptote or when I say an asymptote and you say +
or – 1 that’s not right
St: y equal + or -1
Mike: x= + or -1 you need to give an equation for of a line
x = + or – 1 are vertical asymptotes for f and g and h
Anybody I had a nice succinct rule for when a function for where
function has a horizontal asymptote
Yeahs SD. I wanna hear from SM.

St: The leading coefficient of the numerator divided by the leading coefficient
of the denominator

Mike: Aha…he says when the degrees are the same …
The leading coefficient of the numerator divided by the leading coefficient
of the denominator gives you the horizontal asymptote Six over…y equals
6 over 4 here
What are the leading coefficients for these two? Not zero right?

St: One
Mike: So the horizontal asymptote for this function is 1
Ok continue
That’s only if in the special case when the degrees are the same
What about in this case? ((Pointing to function h(x )))

St: Aaaa
Mike: What about … this one had a horizontal asymptote and this one didn’t! SB
So I heard this table had a rule about horizontal asymptotes

SB: SA has one
Mike: Ok SA. read as yours
SA: The horizontal asymptote…
Mike: Yes
SA: If the degree of the numerator is less than the degree of the denominator
Mike: Ok As in this case (pointing to function f)
SA: It has a horizontal asymptote close to zero
Mike: If the degree of the denominator is less than the degree of the numerator
then the asymptote is y equal zero ok…
So that’s degree of the denominator greater and you covered that.
SM covered if the degree of the denominator is the same as the degree of
the numerator
What about in this case?
If the degree of the numerator is greater than the degree of the
denominator? SS?

SS: That gets close to one? Though that’s not here…
Mike: No… not quite SL?
SL: No asymptote?
Mike: There is no horizontal asymptote in this case.
So, in this particular case there is a horizontal asymptote
It’s a line because x^3 is approximately 1 mean x^3 -3x is approximately x^3
x^2 -1 is approximately x^2 so this will be approximately x.
Ok?
Those rules work except in this case in this special case ok…
Finish…
You have homework the day due after this vacation week …
Don’t forget about (directions about test and online quiz))
Appendix E

Radians Lesson Transcript

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Radians</th>
<th>Radius</th>
<th>Arc Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>180°</td>
<td>π</td>
<td>15”</td>
<td></td>
</tr>
<tr>
<td>60°</td>
<td></td>
<td></td>
<td>9 ft.</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>6 m.</td>
<td></td>
</tr>
<tr>
<td>5π/4</td>
<td></td>
<td>40 cm.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>15”</td>
<td>4”</td>
<td></td>
</tr>
<tr>
<td>1°</td>
<td></td>
<td>4000 mi.</td>
<td>4.3 km</td>
</tr>
<tr>
<td></td>
<td>6400 km</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Figure E1. Radians Activity Table*

Mike: So, you have this table (see Figure 4) for you to fill in, let us remember, (.4) What is radian measure? What is radian measure? Yes, Sophie…

Sophie: Oh, well it’s type of how many radius-lengths the radian measures. How much π is. So, how many times π goes around the (.3) How many times you go around half the circle times π

Mike: Ok, ((with a sigh))… aaa… Yeah… It is how many times you go around (.2) half of the circle times π, but that’s…that’s sort of a formulae idea. I really want to have a more gutsy how to measure with a piece of string radiant measure idea, because that’s all you need to measure radius, ((correcting)) radians. Yes, Bridget?

Bridget: ((Inaudible))

(.5)

Mike: Ok, So radian measure is the number of the radius-lengths you go around=, =Stop talking Norman,
Radian measure is the number of radius-lengths you measure around the
circumference of the circle, around the outside of the circle measured counter-clockwise, starting from (1,0).
So, number of radius-lengths measured around the circle.
So, in this table, that I’ve got you to fill out, here, when I say 2 radians I am not talking 2 $\pi$ radians.
I am not talking 2 times around the circle.
I am talking 2 radius-lengths measured around the circle.
Radian measure, really doesn’t correspond to the angles in a natural way, radian measure is a distance along an arc, but, you know from Geometry, that the distance along an arc is often measured in terms of the central angle, which is the relationship between the radians and the degrees and Sophie’s idea about you go half a circle that’s $\pi$ radians.
It’s a way to think about how to convert from radians to degrees and vice-versa.
So, fill in as much as of this table as you can, and here’s the hand you should be able to fill in, but the table, without leaving any blanks in it.

Mike: Here it’s a little help. 180 degrees $\pi$ radians
Mike: How do we get arc length?
Mike: Yes, Sophie?
Sophie: The length will be the fraction of the circumference.
Mike: The fraction of the circumference?
Sophie: Right. If the radius is the distance on the circumference
If the length corresponds to the 180 degrees
that’s half of the circumference

Mike: So, Sophie ok, everybody listen there is some chattering on there on back at the basketball team table …aaa…
Sophie said to get the arc length you can take, figure out what portion of the circle this is and therefore figure out what proportion of the circumference would be the arc length…
but, this, maybe…aaa…let’s …but if we think about that…
I’m goanna write on this board over here. What’s the circumference?

Students: $2 \pi r$
Mike: $2 \pi r$? ((Writing on the board))
Mike: Ok, How much of the circle do we go around?
Student: $\pi$
Mike: For a given arc.
Mike: That’s what we don’t know!
Proportion! I’m goanna let p be the proportion of the circle. ((Writing on the board))
Mike: So, p is the proportion of the circle, how many degrees have we gone?
Students: Inaudible
Mike: Let’s say $p$ is half of the circle, we’ve gone
Students: =180
Mike: 180 degrees, so
how can you convert the $p$,
the proportion into a number of degrees? Yes, Jim?
Student: You gone, a 180 is half of it half away around it so you divided by 2
Mike: So, If $p$ it’s what’s gone, the $p$ fraction around it, the $p$ is the proportion
how many degrees have we gone?
Student: No audible response
Mike: What if we go one time around it? How many degrees?
Student: 180
Mike: What if we go, what if $p$ is three quarters how many degrees?
Student: 270
Mike: How did you get 270?
Students: ((Collective response with different responses)) Divide it…Take three
quarters of 360.
Mike: Three quarters of 360.
What if you go $p$ fraction around it?
Student: 360 times $p$.
Mike: 360 times $p$? So, 360 $p$ is the number of degrees. ((Writing on the board))
If the circumference, if we go a proportion of the …of $p$ around the circle
how many radians have we gone?
Student: ((() Mike: If we go 360 $p$ degrees, how many radians do we go?
How many radians if $p$ is one?
Student: $2\pi$
Mike: How many radians if $p$ is a half?
Student: $\pi$
Mike: One times $\pi$?
Mike: So, how many radians if $p$ is $p$?
Student: more students responding
Mike: What would you say?
Student: $p \pi$
Mike: It’s not quite $p*\pi$ ‘cause if $p$ is one what is it?
Student: Oh
Mike: $2\pi$. If $p$ is a half?
Students: ((Collective response)) $2\pi p$
Mike: $2\pi p$ Right? So, I’m goanna write $2\pi p$ as the number of radians. So, let’s
say $p$ is the proportion of the circle, what is the arc length that we’ve gone
around?
Mike: So, yes, Bridget?
Student: Inaudible
Mike: Why?
Student: Because the number of radians tells you how many radians you are and
you think about the number of radians times the radians
Mike: So, Bridget’s formulation is bad: Radians times the number of radians,
but if we think if we go around,
the proportion of the circle is $p$ then,
that would be the same as the proportion of the circumference.
What’s the circumference?

Student: $2 \pi r$

Mike: $2 \pi r$... So, we’d had $p$ times $2 \pi r$.
So, if you think about the proportion of the circle as being $p$ and the
circumference as being $2 \pi r$ then Bridget’s idea is exactly right;
because $2 \pi p$ represents the number of radians.
So, you don’t have to do all of that reasoning that Sophie talked about
although is absolutely correct.
It’s much easier just to note that, from the definition of what a radian is,
the number of radius lengths you go around outside the circle.
That’s what a radian is.
So, we go $\pi$ radians, that means we go $\pi$ radians lengths around the
outside of the circle.
Well, that’s what we want,
that’s what arc length is: $\pi$ times 15 so
15 $\pi$ inches would be the arc length.

Mike: Ok? Now you should be successful in filling out the rest of the table…

*The activity continued. After group work the discussion moved to sine and cosine functions.*

*The rest of the transcript is available upon request.*
Appendix F

Consent Forms

Consent Form for Mathematics Teachers Participants

You are invited to participate in a study that is intended to understand mathematics teachers’ dilemmas in the process of learning mathematics teaching.

If you agree to take part in this research, I will take notes during your mathematics classroom meetings. With your consent I will also look at your students’ assignments, projects, portfolios, and tests and I will possibly video-tape (with audio) you and your students during your mathematics regular classroom. The recording will last approximately 50 minutes. You will be recorded once during my period of observation. The observations will take place for at least one-half of your classroom meetings. I may also engage with you in informal conversations about the dilemmas and issues of mathematics teaching. If you agree I will audio-tape one of this conversation. The conversations will take around one hour.

Your participation in this research is voluntary. You may refuse to participate, discontinue participation at any time without penalty or loss of the benefits to which you are otherwise entitled. Your decision will not affect your status or position in the school. In the case of recording you are free to (a) request that the video or audio-recorder be turned off at any time, and (b) request that a recorded session be destroyed and excluded from the study. During our informal conversations you may skip any topics you don’t wish to discuss.

There are no known risks in this study beyond those of ordinary life. However, since this study investigates during your classroom participation and our informal conversations, it is possible that you might experience some discomfort talking about your concerns in regards with teaching mathematics as they refer to your feelings and attitudes towards future mathematics teaching career, especially during audio or video recording. Keep in mind that the potential benefit of this research is to deepen the understanding of the learning process of mathematics teaching.

Please note that any information that is obtained in connection with this study and that can be identified with you will remain confidential and will be disclosed only with your permission. All video and audio-cassettes will be transcribed using codes and pseudonyms so that no personally identifying information other than your image is on the cassettes. I will keep all cassettes in a secure place such as a locked file cabinet. Only the principal researcher will have access to research findings associated with your identity. In the event of publication or dissemination of this research, no personally identifying information will be disclosed without your permission. *In the case that some video or audio clips will be used for presentations, facial images and names will be blurred so that they won’t be identifiable.*
Parent Consent Letter

Dear Madam/Sir,

Your child is invited to participate in the research project “Thinking and Creativity in Learning Mathematics Teaching.”

If you agree as your child to take part in this research, I will take notes during his/her mathematics classes. The time framework for your child's participation is 2007-2008 school year. The observations will take place at least one-half of the classroom meetings. With your consent I will also look at your child’s assignments, projects, portfolios, and tests and I will possibly video-tape (with audio) the instructor activity during regular classroom discussions. The recording will last approximately 1 hour. It will be no more than two video recordings during the semester. I may also engage with your child in informal mathematical conversations about the topic of the day class.

There are no known risks in this study beyond those of ordinary life. This study is solely focused on mathematics teaching process. However, since this study investigates teacher’s activity your child mathematics classroom participation will be observed and recorded as part of teacher-student interaction. Your child’s assignments, projects, portfolios and tests will be looked upon only to understand teacher thinking and planning future activity. Keep in mind that the potential benefit of this research is to deepen and improve the understanding of mathematics teaching process.

Your child participation in this research is voluntary. You may refuse to participate, discontinue your child’s participation at any time without penalty or loss of the benefits to which you are otherwise entitled. Your decision will not affect your child’s grades or status at his/her school, or your family’s relationship with the University of Illinois. In the case of recording, the teacher, you or your child are free to (a) request that the video or audio-recorder be turned off at any time, and (b) request that a recorded session be destroyed and excluded from the study. During informal mathematical conversations your child may skip any topics he/she doesn’t wish to discuss.

Please note that any information that is obtained in connection with this study and that can be identified with your child will remain confidential and will be disclosed only with your permission. All video and audio-cassettes will be transcribed using codes and pseudonyms so that no personally identifying information other than your child’s image is on the cassettes. I will keep all cassettes and recordings in a secure place such as a locked file cabinet. Only the principal researcher will have access to research findings associated with your child’s identity. In the event of publication or dissemination of this research, no personally identifying information will be disclosed without your permission. In the case that some video or audio clips will be used for presentations, facial images and names will be blurred so that they won’t be identifiable.
Student Assent Letter

(May be read to students)

Hi! We are here from the University of Illinois to do a project in your classroom about your mathematics teacher. If you want to participate in this project, we will take notes and observe your activity during mathematics classes. We will audiotape and videotape some lessons so that we can look at them later. We also may talk with you about different mathematical topics you learned with your teacher.

Your participation in this project is voluntary—this means that you can decide whether or not you want to do this project. If you want to stop doing the project at any time, you can stop. The audiotapes and videotapes and all the other information from this project will be kept private and secure. The audiotapes and videotapes will be kept in a locked file cabinet and only people who work on this project will be able to look at them. The videotapes will be coded to remove your names and will be erased after the project is finished. In the case that some video or audio clips will be used for presentations, faces and names will be blurred so that they won’t be identifiable.

This project won’t go on your school record or count toward your math grade. If you decide not to do this project, we will not ask you to discuss with us during your mathematics class and we will not include you in the videotape.
Appendix G

Research in School Short Proposal Form

Brief Summary of Project and Abstract of Procedure

The researcher will be an observer in the school classroom meetings and have after-class informal discussions with the teachers of the classroom pertinent to the practical situation that may show the teaching liaison between mathematics thinking and pedagogical thinking. The observations will take one hour per meeting. Meetings will be scheduled bi-weekly for one semester. The observations will follow with short written reports which will be shared with the teacher for member checking. I will possible have one audio-taped discussion with the teacher about the pertinent topics of this study: questioning, creativity, mathematical thinking, and pedagogical thinking. The audio-taped discussion will take no longer than an hour. The experience of these classes will be used by the researcher for a better understanding of the questions novice teachers have in their mathematics teacher education programs.