# NONCOMMUTATIVE $L_P$-SPACES ASSOCIATED WITH LOCALLY COMPACT QUANTUM GROUPS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2010

Urbana, Illinois

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# Abstract

Results from abstract harmonic analysis are extended to locally compact quantum groups by considering the noncommutative $L_p$-spaces associated with the locally compact quantum groups.

Let $G$ be a locally compact abelian group with dual group $\hat{G}$. The Hausdorff–Young theorem states that if $f \in L^p(G)$, where $1 \leq p \leq 2$, then its Fourier transform $\mathcal{F}_p(f)$ belongs to $L^q(\hat{G})$ (where $\frac{1}{p} + \frac{1}{q} = 1$) and $\|\mathcal{F}_p(f)\|_q \leq \|f\|_p$. Kunze and Terp extended this to unimodular and locally compact groups, respectively. We further generalize this result to an arbitrary locally compact quantum group $\mathbb{G}$ by defining a Fourier transform $\mathcal{F}_p : L_p(\mathbb{G}) \to L_q(\hat{\mathbb{G}})$ and showing that this Fourier transform satisfies the Hausdorff–Young inequality.

Let $G$ be a locally compact group. Then $L_1(G)$ acts on $L_p(G)$ by convolution. We extend this result to Kac algebras and also discuss an operator space version of this result. Ruan and Junge showed that if $G$ is a discrete group with the approximation property, then $L_p(VN(G))$ has the operator space approximation property. Let $\mathbb{G}$ be a discrete Kac algebra with the approximation property. The aforementioned action of $L_1(\mathbb{G})$ is used to show that $L_p(\hat{\mathbb{G}})$ has the operator space approximation property. Similarly, if $\mathbb{G}$ is a weakly amenable discrete Kac algebra, then $L_p(\hat{\mathbb{G}})$ has the completely bounded approximation property.

# Acknowledgements

I would like to thank Professor Boca, Doctor Fima, and Professor Junge for serving on my dissertation committee and for having taught me so much during my time at the University of Illinois. I would especially like to thank my advisor Professor Ruan. Our discussions have contributed greatly to my understanding of this field and to the direction that this dissertation took. I am very grateful for his many explanations and suggestions.

Most of all, I would like to thank my parents for their many years of love and support, without which none of this would have been possible.

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# Chapter 1

# Introduction

In this dissertation we seek to extend concepts and results from abstract harmonic analysis to locally compact quantum groups. These locally compact quantum groups come equipped with Haar weights. Using these we can construct noncommutative $L_p$-spaces. As we construct these spaces using the Haar weights, there are relationships between the quantum group structure and the inclusions of $L_1(\mathbb{G}) \cap L_\infty(\mathbb{G})$ into $L_p(\mathbb{G})$, and the properties of the space $L_p(\mathbb{G})$.

We start by collecting those definitions and results that will be needed later in the dissertation. In Chapter 2, we discuss various approaches to noncommutative $L_p$-spaces. Particularly important will be that the noncommutative $L_p$-spaces are isometrically isomorphic to the interpolation spaces $(L_1(\mathbb{G}), L_\infty(\mathbb{G}))_{1/p}$ (for suitable inclusions of $L_1(\mathbb{G})$ and $L_\infty(\mathbb{G})$ into a Banach space $L_\alpha^*$). In this chapter, we will also record facts about Hilbert algebras and operator spaces that will be used later.

In Chapter 3, we recall the definitions of locally compact quantum groups and Kac algebras. Some of their properties will be briefly discussed. We also consider the completely bounded multipliers on a Kac algebra.

In Chapter 4, we consider the following situation. Let $G$ be a locally compact group with right Haar measure $ds$. Then $L_1(G)$ acts contractively by right convolution both on $L_1(G)$ and on $L_\infty(G)$:

$$\left(\Theta_1^r(f)(g)\right)(t) = \int_G g(ts)f(s)\;ds,$$
$$\left(\Theta^r(f)(h)\right)(t) = \int_G h(ts)f(s)\;ds,$$

for $f, g \in L_1(G)$, and $h \in L_\infty(G)$. Clearly these two actions agree on $L_1(G) \cap L_\infty(G)$. Interpolating between these two cases yields a right action of $L_1(G)$ on $L_p(G)$. We shall extend this result to Kac algebras. Junge, Neufang, and Ruan's paper [17] discusses how $L_1(\mathbb{G})$ (and indeed $M_0^r(L_1(\mathbb{G}))$) can be represented on $B(L_2(\mathbb{G}))$; their map $\Theta^r$ will be our starting point for showing how to represent $L_1(\mathbb{G})$ on $L_p(\mathbb{G})$.

We note that the dual Kac algebra to $L_\infty(G)$ is the group von Neumann algebra $VN(G)$ (with its usual comultiplication). Thus our treatment of the Kac algebra case will also generalize the action of the Fourier algebra $A(G)$ on the noncommutative $L_p$-spaces $L_p(VN(G))$.

We trace our way through the construction of the Haagerup noncommutative $L_p$-spaces and prove that the correct map in this context is given by

$$\Theta_p^r(f)(D^{1/2p}xD^{1/2p}) = D^{1/2p}\Theta^r(f)(x)D^{1/2p},$$

for $f \in L_1(\mathbb{G})$ and $x \in \mathfrak{M}_\psi$. We will show that in the case $p = 1$ this extends to a bounded map $L_1(\mathbb{G}) \to L_1(\mathbb{G})$. We then interpolate between the two cases $p = 1$ and $p = \infty$ to prove the general case.

We show how to alter the above argument to provide a completely contractive action of $L_1(\mathbb{G})$ on $L_p(\mathbb{G})$. We then adapt arguments due to Junge and Ruan in [18] and Kraus and Ruan in [24] to show that if $\mathbb{G}$ is a discrete Kac algebra with the approximation property, then $L_p(\hat{\mathbb{G}})$ has the operator space approximation property. If $\mathbb{G}$ is a weakly amenable discrete Kac algebra, then $L_p(\hat{\mathbb{G}})$ has the completely bounded approximation property.

In Chapter 5, we consider the following situation. Let $G$ be a locally compact abelian group with Haar measure $\mu$ and dual group $\hat{G}$. The Fourier transform of a function $f \in L_1(G, \mu)$ is defined by

$$\mathcal{F}_1(f)(\xi) = \hat{f}(\xi) = \int_G f(s)\overline{\xi(s)}d\mu(s), \qquad \xi \in \hat{G}.$$

Clearly, $\|\mathcal{F}_1(f)\|_\infty \le \|f\|_1$. For a suitably normalized Haar measure $\hat{\mu}$ on $\hat{G}$, if $f \in L_1(G,\mu) \cap L_2(G,\mu)$, then we have $\|\mathcal{F}_1(f)\|_2 = \|f\|_2$ and the Fourier transform extends to a unitary, $\mathcal{F}_2$, from $L_2(G,\mu)$ onto $L_2(\hat{G},\hat{\mu})$. Interpolating between these cases yields the Hausdorff–Young inequality: for $1 \le p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mathcal{F}_p : L_p(G,\mu) \to L_q(\hat{G},\hat{\mu})$ is a contraction, i.e., $\|\mathcal{F}_p(f)\|_q \le \|f\|_p$ (see, for example, [8]).

For $f \in L_1(G,\mu)$, let $\lambda(f)$ denote the operator on $L_2(G,\mu)$ given by $\lambda(f)(g) = f * g$. As $\widehat{f * g} = \hat{f}\hat{g}$, the operator $\lambda(f)$ is unitarily equivalent to the operator $\hat{f} \in L_\infty(\hat{G},\hat{\mu})$ acting by multiplication on $L_2(\hat{G},\hat{\mu})$. Inspired by this, Kunze dealt with the unimodular group case by making the definition $\mathcal{F}_1(f) = \lambda(f)$. The group von Neumann algebra $VN(G)$ is generated by $\{\lambda(f) \mid f \in L_1(G)\}$ and plays the role of $L_\infty(\hat{G})$. Using the noncommutative $L_p$ spaces associated with a trace on $VN(G)$, Kunze showed in [25] that the Hausdorff–Young inequality holds for unimodular groups.

Terp extended the Hausdorff–Young inequality to locally compact groups in [35] by using the following definition for the Fourier transform:

**Definition 1.0.1.** Let $G$ be a locally compact group with modular function $\Delta$. Let $f \in L_p(G)$, $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$. The *$L_p$-Fourier transform of $f$* is the operator $\mathcal{F}_p(f)$ on $L_2(G)$ given by

$$\mathcal{F}_p(f)\xi = f * \Delta^{1/q}\xi, \quad \xi \in \mathcal{D}(\mathcal{F}_p(f)),$$

where $\mathcal{D}(\mathcal{F}_p(f)) = \{\xi \in L_2(G) \mid f * \Delta^{1/q}\xi \in L_2(G)\}$.

Terp showed that $\mathcal{F}_p(f) \in L_q(VN(G))$, where $L_q(VN(G))$ is a non-commutative $L_q$ space constructed using the Plancherel weight on the group von Neumann algebra $VN(G)$, and showed that the Hausdorff–Young inequality holds. This is the definition that shall be extended to the locally compact quantum group case.

The main result of this chapter is as follows. Let $\mathbb{G}$ be a locally compact quantum group with dual locally compact quantum group $\hat{\mathbb{G}}$. Let $\varphi$ be the normal, semi-finite, faithful (nsf) Haar weight on $L_\infty(\mathbb{G})$ and let $\varphi'$ be a nsf weight on the commutant $L_\infty(\mathbb{G})'$. Let $d = \frac{d\varphi}{d\varphi'}$ be the spatial derivative of $\varphi$ with respect to $\varphi'$. Similarly, let $\hat{\varphi}$ be the Haar weight on $L_\infty(\hat{\mathbb{G}})$, let $\hat{\varphi}'$ be a nsf weight on $L_\infty(\hat{\mathbb{G}})'$, and let $\hat{d} = \frac{d\hat{\varphi}}{d\hat{\varphi}'}$. Let $L_p(\mathbb{G})$ and $L_p(\hat{\mathbb{G}})$ be the spatial non-commutative $L_p$-spaces constructed using $\varphi'$ and $\hat{\varphi}'$ respectively.

Let $\pi_l(\mathfrak{A}_0)$ be the set of entire elements $x$ such that $\sigma^{\varphi}_{\alpha}(x) \in \mathfrak{N}_\varphi \cap \mathfrak{N}^*_\varphi$ for all $\alpha \in \mathbb{C}$, and let $\pi_l(\mathfrak{A}_0^2) = \{xy \,:\, x, y \in \pi_l(\mathfrak{A}_0)\}$. Let $L$ be the set of $x \in L_\infty(\mathbb{G})$ such that there is a normal linear functional $\varphi_x$ in $L_1(\mathbb{G})$, the predual of $L_\infty(\mathbb{G})$, satisfying $\varphi_x(y) = \varphi(yx)$ for $y \in \pi_l(\mathfrak{A}_0^2)$. For each $x \in L$, there is a corresponding element $U_p(j^*(x))$ in $L_p(\mathbb{G})$. (For $x \in \mathfrak{A}_0^2$, we have $U_p(j^*(x)) = d^{1/2p}\sigma^{\varphi}_{i/2p}(x)d^{1/2p}$.)

**Theorem 5.1.9.** *The $L_p$-Fourier transform $\mathcal{F}_p$ is the map from $L_p(\mathbb{G})$ to $L_q(\hat{\mathbb{G}})$ such that*

$$\mathcal{F}_p\left(U_p(j^*(x)\right) = \lambda\left(\varphi_x\right)\hat{d}^{1/q},$$

*for $x \in L$. This map is a contraction, i.e., $\|\mathcal{F}_p\| \leq 1$.*

*For $x \in \pi_l(\mathfrak{A}_0^2)$, there is a more explicit description of $\mathcal{F}_p$:*

$$\mathcal{F}_p\left(d^{1/2p}\sigma^{\varphi}_{i/2p}(x)d^{1/2p}\right) = \lambda(\varphi_x)\hat{d}^{1/q}, \quad x \in \pi_l(\mathfrak{A}_0^2).$$

*If $\varphi$ is a state, we have the simpler expression:*

$$\mathcal{F}_p(xd^{1/p}) = \lambda(\varphi_x)\hat{d}^{1/q}, \qquad x \in L_\infty(\mathbb{G}).$$

We also note that a version of this chapter will be published as [4].

# Chapter 2

# Noncommutative $L_p$-spaces

## 2.1 Von Neumann Algebras

In this section we recall some basic results about von Neumann algebras and fix some notation that will be used throughout this dissertation.

Let $\mathcal{H}$ be a Hilbert space and let $B(\mathcal{H})$ denote the bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. A *$C^*$-algebra* is a self-adjoint norm-closed $^*$-algebra of $B(\mathcal{H})$. A *von Neumann algebra* $M$ is a C$^*$-algebra that is closed in the $\sigma$-weak operator topology on $B(\mathcal{H})$. Von Neumann algebras can also be characterized as the C$^*$-algebras $M$ for which there exists a Banach space $M_*$ such that $M = (M_*)^*$. The Banach space $M_*$ is necessarily unique (up to isomorphism) and is called the *predual* of $M$. The $\sigma$-weak operator topology on $M$ coincides with the weak$^*$-operator topology induced on $M$ by $M_*$.

A *weight* on a von Neumann algebra $M$ is a function $\varphi : M_+ \to [0, \infty]$ such that

$$\varphi(0) = 0, \quad \varphi(a+b) = \varphi(a) + \varphi(b), \text{ and } \varphi(\lambda a) = \lambda\varphi(a),$$

for all $a, b \in M_+$ and $\lambda \in \mathbb{R}_+$.

A weight $\varphi$ is said to be *normal* if, whenever $(x_i)$ is an increasing net in $M_+$ converging strongly to $x$, then $\varphi(x) = \lim \varphi(x_i)$. We write

$$\mathfrak{N}_\varphi = \{x \in M \,:\, \varphi(x^*x) < \infty\},$$
$$\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi = \operatorname{span}\{x^*y \,:\, x, y \in \mathfrak{N}_\varphi\}.$$

A normal weight $\varphi$ is said to be *semifinite* if $\mathfrak{M}_\varphi$ is $\sigma$-weakly dense in $M$. A weight $\varphi$ is *faithful* if $\varphi(a^*a) = 0$ implies that $a = 0$. A normal semifinite faithful weight will be called an *nsf* weight.

## 2.2 Hilbert Algebras

An involutive algebra $\mathfrak{A}$ over $\mathbb{C}$ with involution $\xi \in \mathfrak{A} \mapsto \xi^{\#} \in \mathfrak{A}$ is called a *left Hilbert algebra* if $\mathfrak{A}$ admits an inner product satisfying the following postulates:

1. Each fixed $\xi \in \mathfrak{A}$ gives rise to a bounded operator $\pi_l(\xi) : \eta \in \mathfrak{A} \mapsto \xi\eta \in \mathfrak{A}$ by multiplying from the left;

2. $(\xi\eta \mid \zeta) = (\eta \mid \xi^{\#}\zeta)$ for all $\xi, \eta, \zeta \in \mathfrak{A}$;

3. The involution $\xi \in \mathfrak{A} \to \xi^{\#} \in \mathfrak{A}$ is preclosed;

4. The subalgebra $\mathfrak{A}^2 = \text{span}\{\xi\eta : \xi, \eta \in \mathfrak{A}\}$ is dense in $\mathfrak{A}$ with respect to the norm induced by the inner product.

Let $M$ be a von Neumann algebra and let $\varphi$ be a nsf weight on $M$. Define a pre-inner product on $\mathfrak{N}_\varphi$ by $(x \mid y) = \varphi(y^*x)$, for $x, y \in \mathfrak{N}_\varphi$. The Hilbert space completion of $\mathfrak{N}_\varphi$ with respect to this inner product will be denoted by $\mathcal{H} = \mathcal{H}_\varphi$. The inclusion map from $\mathfrak{N}_\varphi \to \mathcal{H}$ will be denoted by $\Lambda = \Lambda_\varphi$.

Let $\mathfrak{A} = \mathfrak{A}_\varphi = \Lambda(\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*)$. This is a left Hilbert algebra with multiplication and adjoint given by

$$\Lambda(x)\Lambda(y) = \Lambda(xy), \qquad \Lambda(x)^{\#} = \Lambda(x^*).$$

If $x \in \mathfrak{A}$, then $\Lambda(y) \mapsto \Lambda(xy)$ extends to a bounded operator $\pi_l(\Lambda(x))$ on $\mathcal{H}$. The von Neumann algebra $M$ can and will be identified with $\pi_l(\mathfrak{A})'' \subset B(\mathcal{H})$. Under this identification, we have $\pi_l(\Lambda(x)) = x$.

Let $S = S_\varphi$ be the closure of the map $\Lambda(x) \mapsto \Lambda(x^*)$, with polar decomposition $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$. We denote the modular conjugation, modular operator, and modular automorphism group as follows:

$$J = J_\varphi, \qquad \Delta = \Delta_\varphi,$$
$$\sigma_t^\varphi(x) = \Delta^{it} x \Delta^{-it}, \text{ for } x \in M \text{ and } t \in \mathbb{R}.$$

In a similar fashion, we can define right Hilbert algebras, which will be used briefly later. We can obtain a right Hilbert algebra $\mathfrak{A}'_\varphi = \mathfrak{A}'$ from $\mathfrak{A}$ as follows. Let $F$ be the antilinear densely defined closed operator with domain $\mathcal{D}^\flat$ such that

1. $\mathcal{D}^\flat = \{\eta \in \mathcal{H} : \xi \in \mathcal{D}(S) \mapsto (\eta \mid S\xi) \text{ is bounded}\}$;

2. $(S\xi \mid \eta) = (F\eta \mid \xi)$, for $\xi \in \mathcal{D}(S)$ and $\eta \in \mathcal{D}^\flat$.

A vector $\eta \in \mathcal{H}$ is *right bounded* (written $\eta \in \mathfrak{B}'$) if

$$\sup\{\|\pi_l(\xi)\eta\| : \xi \in \mathfrak{A}, \|\xi\| \leq 1\} < \infty.$$

We now consider $\mathfrak{A}' = \mathfrak{B}' \cap \mathcal{D}^\flat$. We define an involution on $\mathfrak{A}'$ by $\eta \mapsto \eta^\flat = F\eta$. For each $\eta \in \mathfrak{B}'$, we have a bounded map $\pi_r(\eta)$ such that $\xi$ is mapped to $\pi_l(\xi)\eta$ for all $\xi \in \mathfrak{A}$. We can then define a multiplication on $\mathfrak{A}'$ by $\xi\eta = \pi_r(\eta)\xi$, for $\xi, \eta \in \mathfrak{A}'$. This makes $\mathfrak{A}'$ into a right Hilbert algebra such that $\pi_r(\mathfrak{A}')' = \pi_l(\mathfrak{A})''$.

We recall the following important results, which can be found as Theorem VI.1.19 in [33] and Corollary III.4.5.1 in [2].

**Theorem 2.2.1.** *Let $\varphi$ be a normal faithful semifinite weight with the left Hilbert Algebra described above with modular operator $\Delta$ and modular conjugation $J$. Then we have that*

$$JMJ = M' \quad \textit{and} \quad JM'J = M,$$

*and*

$$\Delta^{it} M \Delta^{-it} = M \quad \textit{and} \quad \Delta^{it} M' \Delta^{-it} = M',$$

*for $t \in \mathbb{R}$. This allows us to define a one parameter automorphism group $\sigma_t^\varphi(x) = \Delta^{it} x \Delta^{-it}$ for $t \in \mathbb{R}$ on $M$ and $M'$, respectively. The map $x \to Jx^*J$ is a $*$-antiautomorphism of $M$ onto $M'$.*

We can also consider the *Tomita algebra* $\mathfrak{A}_{0,\varphi} = \mathfrak{A}_0$. This is a left Hilbert algebra, which satisfies $\pi_l(\mathfrak{A}_0)'' = \pi_l(\mathfrak{A})''$, and consists of the elements

$$\mathfrak{A}_0 = \left\{ \xi \in \bigcap_{n \in \mathbb{Z}} \mathcal{D}(\Delta^n) : \Delta^n \xi \in \mathfrak{A}, n \in \mathbb{Z} \right\} \subset \Lambda(\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*).$$

An element $x \in M$ is an *entire* element if the map $t \mapsto \sigma_t(x)$ extends to an analytic function $\mathbb{C} \to M$, $\alpha \mapsto \sigma_\alpha^\varphi(x)$. The set of entire elements includes $\pi_l(\mathfrak{A}_0)$, and if $x \in \pi_l(\mathfrak{A}_0)$, then $\sigma_\alpha^\varphi(x) \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$, for all $\alpha \in \mathbb{C}$. We will denote $\operatorname{span}\{\Lambda(yx) : y, x \in \pi_l(\mathfrak{A}_0)\}$ by $\mathfrak{A}_0^2$.

The following standard result can be found in Section 2.18 in [31].

**Proposition 2.2.2.** *For all $\alpha \in \mathbb{C}$, $a \in \pi_l(\mathfrak{A}_0)$, and $z \in \mathfrak{M}_\varphi$,*

$$\varphi(z\sigma_\alpha^\varphi(a)) = \varphi(\sigma_{\alpha+i}^\varphi(a)z).$$

For further details concerning Hilbert algebras and Tomita algebras, see [31] and [33].

## 2.3 $\tau$-Measurable Operators

In this section, we recall some results about $\tau$-measurable operators that will be necessary for our discussion of noncommutative $L_p$-spaces. Further information about $\tau$-measurable operators can be found in [9], [27], [33], and [34].

Let $N$ be a von Neumann algebra acting on a Hilbert Space $\mathcal{H}$ and let $\tau$ be a normal semifinite faithful trace on $N$. We will denote the domain of an operator $x$ on the Hilbert space $\mathcal{H}$ by $\mathcal{D}(x)$. The closure of the operator $x$ will often also be denoted by $x$. When it is necessary to distinguish between the original operator and its closure, $[x]$ will be used to denote the closure of $x$.

**Definition 2.3.1.** A subspace $E$ of $\mathcal{H}$ is $\tau$-dense if for each $\delta > 0$ there exists a projection $p \in N$ such that

$$p\mathcal{H} \subset E \text{ and } \tau(1-p) \leq \delta.$$

A closed densely defined operator $x$ affiliated with $N$ is called *$\tau$-measurable* if $\mathcal{D}(x)$ is $\tau$-dense. The set of $\tau$-measurable operators is denoted by $\tilde{N}$.

A (not necessarily closed) operator $x$ on $\mathcal{H}$ is *$\tau$-premeasurable* if $x$ is affiliated to $N$ and for each $\delta > 0$, there exists a projection $p \in N$ such that

$$p\mathcal{H} \subset \mathcal{D}(x), \ \|xp\| < \infty, \text{ and } \tau(1-p) \leq \delta.$$

If $x$ is $\tau$-premeasurable with $\tau$-dense domain, then its closure $[x]$ is $\tau$-measurable.

We now collect some useful results about $\tau$-measurable operators that will be needed later. The following proposition summarizes Propositions 20 and 24 in [34] and Proposition 1.1 in [9].

**Proposition 2.3.2.** *1. Let $x, y$ be $\tau$-premeasurable operators. Then $x+y$ and $xy$ are also $\tau$-premeasurable.*

*2. Let $x, y$ be $\tau$-measurable operators. Then $x+y$ and $xy$ are $\tau$-premeasurable and $[x+y]$ and $[xy]$ are $\tau$-measurable operators. $\tilde{N}$ is a $*$-algebra with respect to the adjoint operation, strong sum and strong product (where the strong sum of two operators is the closure of the sum of the operators, and the strong product of two operators is the closure of the product of the two operators).*

*3. Let $x, y$ be closable $\tau$-premeasurable operators and let $E$ be a dense subspace of $\mathcal{H}$. If $x|_E = y|_E$, then $[x] = [y]$.*

## 2.4 Noncommutative $L_p$-spaces associated with a trace

Let $M$ be a von Neumann algebra with a normal faithful semifinite trace $\tau$. We can extend this trace $\tau$ from $M_+$ to $\tilde{M}_+$. For $1 \leq p \leq \infty$, we define the $p$-norm as follows:

$$\|x\|_p = \tau(|x|^p)^{1/p}, \qquad x \in \tilde{M}.$$

We then define the noncommutative $L_p$-space associated with the trace $\tau$ to be

$$L_p(M, \tau) = \left\{ x \in \tilde{M} \, : \, \|x\|_p < \infty \right\}.$$

Then $L_p(M, \tau)$ is a Banach space in which $M \cap L_p(M, \tau)$ is dense. The trace $\tau$ identifies $L_1(M, \tau)$ with $M_*$ by the bilinear form

$$(x, y) \in M \times L_1(M, \tau) \mapsto \tau(xy) \in \mathbb{C}.$$

If $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$, then the product of $L_p(M, \tau)$ and $L_q(M, \tau)$ agrees with $L_1(M, \tau)$ and satisfies the following Hölder's inequality:

$$|\tau(xy)| \leq \|x\|_p \, \|y\|_q \, ,$$

for $x \in L_p(M, \tau)$ and $y \in L_q(M, \tau)$. Furthermore, $L_p(M, \tau)$ and $L_q(M, \tau)$ are the conjugate spaces of each other.

For further information on noncommutative $L_p$-spaces associated with a trace, see [27] or [33].

## 2.5 The extended positive part of a von Neumann algebra

A von Neumann algebra is a generalization of the set of bounded measurable functions; its extended positive part corresponds to considering functions that take values in the extended reals. For details about the extended positive part of a von Neumann algebra, dual weights, and operator valued weights, see [11], [12], and [33].

**Definition 2.5.1.** Let $N \subset B(K)$ be a von Neumann algebra and $N_*$ its predual. A generalized positive operator affiliated with $N$ is a map $m : N_*^+ \to [0, \infty]$ satisfying

1. $m(\lambda\omega) = \lambda m(\omega)$, for $\omega \in N_*^+, \lambda \in \mathbb{R}_+$
2. $m(\omega_1 + \omega_2) = m(\omega_1) + m(\omega_2)$, for $\omega_i \in N_*^+$
3. $m$ is lower semicontinuous.

The set of such maps is called the *extended positive part of* $N$ and is denoted by $\hat{N}_+$.

Each $m \in \hat{N}_+$ corresponds to a unique pair $(K_1, A)$ where $K_1$ is a closed subspace of $H$ and $A$ is a positive self-adjoint operator affiliated with $N$, densely defined on $K_1$ such that

$$m(\omega_{\eta,\eta}) = \begin{cases} \left\|A^{1/2}\eta\right\|^2 & \eta \in \mathcal{D}(A^{1/2}) \\ \infty & \text{otherwise} \end{cases}$$

Each $m \in \hat{N}_+$ has a unique spectral resolution of the form

$$\int_0^\infty \lambda \, \mathrm{d}\omega(e_\lambda) + \infty \cdot p,$$

where $(e_\lambda)_{\lambda \in [0,\infty)}$ is an increasing family of projections in $N$ such that $\lambda \mapsto e_\lambda$ is strongly continuous from the right, and $p = 1 - \lim_{\lambda\to\infty} e_\lambda$. By this we mean that for all $\omega \in N_*^+$,

$$m(\omega) = \int_0^\infty \lambda \, \mathrm{d}\omega(e_\lambda) + \infty \cdot \omega(p),$$

where $\infty \cdot \omega(p)$ is taken to be 0 if $\omega(p) = 0$.

Any normal weight $\omega$ on $N$ has a unique extension $\hat{\omega}$ to $\hat{N}_+$ such that

$$\begin{aligned} \hat{\omega}(\lambda x) &= \lambda \hat{\omega}(x), \qquad x \in \hat{N}_+, \lambda \geq 0. \\ \hat{\omega}(x+y) &= \hat{\omega}(x) + \hat{\omega}(y), \qquad x, y \in \hat{N}_+ \\ x_i \nearrow x &\Rightarrow \hat{\omega}(x_i) \nearrow \hat{\omega}(x), \qquad x_i, x \in \hat{N}_+, \end{aligned}$$

where $x_i \nearrow x$ means that $\omega'(x_i) \nearrow \omega'(x)$ for all $\omega' \in N_*^+$.

The notion of a conditional expectation can be adapted to this situation as follows.

**Definition 2.5.2.** Let $M$ and $N$ be von Neumann algebras with $N \subseteq M$. An *operator valued weight from $M$ to $N$* is a map $T : M_+ \to \hat{N}_+$ which satisfies

1. $T(\lambda x) = \lambda T(x), \qquad \lambda \geq 0,\ x \in M_+;$
2. $T(x+y) = T(x) + T(y), \qquad x, y \in M_+;$
3. $T(a^* x a) = a^* T(x) a, \qquad x \in M_+,\ a \in N;$

   We say that $T$ is *normal* if

4. $x_i \nearrow x$ implies that $T(x_i) \nearrow T(x), \qquad x_i, x \in M_+.$

## 2.6 Haagerup's noncommutative $L_p$-spaces

In this section we shall recall the construction of and some basic results about Haagerup's non-commutative $L_p$ spaces. Further details can be found in [9] and [34]. This will be done in some detail as the approach taken in Section 4.2 follows the steps taken in constructing the Haagerup noncommutative $L_p$-spaces.

Let $M \subset B(\mathcal{H})$ be a von Neumann algebra and $\varphi$ a normal, faithful, semi-finite weight on $M$. The one parameter modular automorphism group $\sigma_t^\varphi$ provides a continuous action of $\mathbb{R}$ on $M$. We begin by constructing the crossed product of $M$ by this action of $\mathbb{R}$. We define a faithful normal representation $\pi$ of $M$ on $L_2(\mathbb{R}, \mathcal{H}) = L_2(\mathbb{R}) \otimes \mathcal{H}$ by

$$\pi(x)\xi(s) = \sigma^{\varphi}_{-s}(x)\xi(s)$$

for $x \in M$, and $\xi \in L_2(\mathbb{R}, \mathcal{H})$. Later on we shall frequently identify $M$ with $\pi(M)$ and write $x$ for $\pi(x)$.

For $t \in \mathbb{R}$, define the maps

$$\begin{aligned} l_t : L_2(\mathbb{R}) \to L_2(\mathbb{R}), \qquad & l_t(\xi(s)) = \xi(s-t), \\ \lambda_t : L_2(\mathbb{R}) \otimes \mathcal{H} \to L_2(\mathbb{R}) \otimes \mathcal{H}, \qquad & \lambda_t = l_t \otimes 1, \end{aligned}$$

It follows that $\lambda_t$ is then a unitary in $B(L_2(\mathbb{R}) \otimes \mathcal{H})$ and also that

$$\lambda_t \pi(x) \lambda_{-t} = \pi(\sigma_t(x)), \qquad \text{for } t \in \mathbb{R}.$$

The crossed product $N = M \rtimes_{\sigma^\varphi} \mathbb{R}$ is the von Neumann algebra generated by $\{\pi(x), \lambda_t : x \in M, t \in \mathbb{R}\}$ and $\operatorname{span}\{\pi(x)\lambda_t : x \in M, t \in \mathbb{R}\}$ is $\sigma$-weakly dense in $N$.

We define the dual action $\theta_t$ of $\mathbb{R}$ on $N = M \rtimes_{\sigma^\varphi} \mathbb{R}$ by

$$\begin{aligned} \theta_t(\pi(x)) &= \pi(x), \\ \theta_t(\lambda_s) &= e^{-ist}\lambda_s, \end{aligned}$$

for $x \in M$, and $s, t \in \mathbb{R}$.

Then $M = \{y \in N \mid \forall s \in \mathbb{R} : \theta_s(y) = y\}$. The maps $\theta_t$ extend to maps on $\hat{N}_+$, the extended positive part of $N$.

We define a normal, faithful, semifinite operator-valued weight $T$ from $N$ to $M$ by, for $\omega \in M_{*,+}, x \in N$,

$$\langle Tx, \omega \rangle = \int_{\mathbb{R}} \langle \theta_s(x), \omega \rangle \, \mathrm{d}s.$$

For each normal weight $\omega$ on $M$, we put $\tilde{\omega} = \hat{\omega} \circ T$, where $\hat{\omega}$ is the extension of $\omega$ to a normal weight on $\hat{M}_+$.

There exists a unique normal, faithful, semifinite trace $\tau$ on $N$ and an invertible, positive, self-adjoint operator $D$ on $L_2(\mathbb{R}) \otimes H$ such that

$$\forall t \in \mathbb{R}: \ D^{it} = (D\tilde{\varphi} : D\tau)_t = \lambda_t,$$
$$\forall s \in \mathbb{R}, x \in N: \ \tau(\theta_s(x)) = e^{-s}\tau(x).$$

We can now talk about the algebra of $\tau$-measurable operators $\tilde{N}$. The operator $D$ will be important in our discussion of the noncommutative $L_p$-spaces. However, the operator $D$ is not $\tau$-measurable (unless $\varphi$ is a state) and thus must be handled carefully. We use $\tilde{N}$ and the action $\theta$ to define the noncommutative $L_p$-spaces.

**Definition 2.6.1.** The *Haagerup noncommutative $L_p$-spaces* are defined to be

$$L_p(M, \varphi) = \{x \in \tilde{N} \, : \, \theta_s(x) = e^{-s/p}x, \forall s \in \mathbb{R}\}.$$

By Theorem 7 and Proposition 15 in Chapter 2 of [34], there is a one-to-one correspondence $\psi \in M_{*,+} \mapsto h_\psi = (D\tilde{\psi} : D\tau) \in \tilde{N}$; this allows $L_1(M, \varphi)$ to be identified with $M_*$. Using this identification, one defines a linear functional $Tr : L_1(M, \varphi) \to \mathbb{C}$ by $Tr(h_\psi) = \psi(1)$. The $L_p$-spaces then have norms given by

$$\|h\|_p = Tr(|h|^p)^{1/p}.$$

We note that the Haagerup noncommutative $L_p$-spaces are independent of the choice of normal faithful semifinite weight $\varphi$ (up to isometric isomorphism).

We also recall Proposition 12 and Theorem 32 from Chapter 2 of [34].

**Proposition 2.6.2.** *Let $a$ be a closed densely defined operator affiliated with $N$ with polar decomposition $a = u|a|$. For $1 \leq p < \infty$, $a \in L_p(M, \varphi)$ if and only if*

$$u \in M \text{ and } |a|^p \in L_1(M, \varphi).$$

*Suppose $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $a \in L_p(M, \varphi)$, $b \in L_q(M, \varphi)$. Then $ab, ba \in L_1(M, \varphi)$ and*

$$Tr(ab) = Tr(ba).$$

*Furthermore $a \mapsto Tr(a\,\cdot)$ is an isometric isomorphism of $L_q(M, \varphi)$ onto $L_p(M, \varphi)^*$.*

We will also recall some results from [9]. We consider the following subsets of $\mathfrak{N}_\varphi$ and $\mathfrak{M}_\varphi$ which consist of certain entire elements.

$$\begin{aligned}\mathfrak{N}_\infty =&\{a \in M \,:\, a \text{ is entire and } \sigma_\alpha(a) \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*, \forall \alpha \in \mathbb{C}\}\\ \mathfrak{M}_\infty =&\operatorname{span}\{a^*b \,:\, a, b \in \mathfrak{N}_\infty\}.\end{aligned}$$

These sets are used to define inclusions $\mu_p$ of $\mathfrak{M}_\varphi$ into $L_p(M, \varphi)$. Informally, we could write $\mu_p(x) = D^{1/2p}xD^{1/2p}$. We now discuss this inclusion at greater length as it will be used frequently later in this dissertation.

For each $x \in \mathfrak{N}_\varphi$, there is a normal linear functional $x\varphi x^*$ given by

$$\langle x\varphi x^*, y\rangle = \langle \varphi, x^*yx\rangle, \text{ for } y \in M.$$

This corresponds to the element $xDx^* = |D^{1/2}x^*|^2$ in the Haagerup $L_p$-space $L_1(M)$. This implies that $D^{1/2}x^*$ is $\tau$-measurable. It follows that $D^{1/q}x^*$ and $[xD^{1/q}]$ are $\tau$-measurable for all $q \geq 2$.

Recall that $\theta_s(\lambda_t) = \theta_s(D^{it}) = e^{-ist}D^{it}$ and $\theta_s(x) = x$ for $s, t \in \mathbb{R}$ and $x \in M$. We would then expect that $\theta_s(D^{1/q}) = e^{s/q}D^{1/q}$ and thus $\theta_s(D^{1/q}x^*) = e^{s/q}D^{1/q}x^*$. It follows that $D^{1/q}x^*$ and $[xD^{1/q}]$ are in $L_q(M)$. It is then clear that $D^{1/2p}x^*[xD^{1/2p}]$ is in $L_p(M)$ for all $p \geq 1$. For a more detailed and rigourous treatment of this point, see [9] and [34].

**Theorem 2.6.3.** *1. The sets $\mathfrak{N}_\infty$ and $\mathfrak{M}_\infty$ are $\sigma$-weakly dense in $M$.*

2. *For $2 \leq q \leq \infty$, the set $\{[aD^{1/q}] \,:\, a \in \mathfrak{N}_\infty\}$ is dense in $L_q(M, \varphi)$.*

3. *Let $\alpha \in \mathbb{C}$ with $0 \leq \operatorname{Re}\alpha \leq \frac{1}{2}$. Then we have*

$$[aD^\alpha] = D^\alpha \sigma_{i\alpha}(a),$$

*for all $a \in \mathfrak{N}_\infty$.*

4. *For $1 \leq p \leq \infty$, the maps*

$$\mu_p : \mathfrak{M}_{\infty,+} \to L_p(M, \varphi) \qquad \mu_p(x) = D^{1/2p}x^{1/2}[x^{1/2}D^{1/2p}]$$

*extend to $\sigma$-weakly densely defined operators that are closable. Their closures are injective and positivity preserving. Furthermore,*

$$\{\mu_p(a) \,:\, a \in \mathfrak{M}_\infty\} \text{ is dense in } L_p(M, \varphi).$$

5. *The following formula shows how the linear functional* $Tr$ *applies to* $\mu_1(\mathfrak{M}_\varphi)$*:*

$$\varphi(d) = Tr(\mu_1(d)), \qquad \textit{for all } d \in \mathfrak{M}_\varphi.$$

6. *The following formula shows how the inclusion* $\mu_1$ *is related to the duality between* $L_1(M, \varphi)$ *and* $M$*:*

$$\langle \mu_1(b), c \rangle_{L_1(M,\varphi),M} = Tr(\mu_1(b)c) = \varphi(\sigma_{i/2}(b)c),$$

*for all* $b \in \mathfrak{M}_\varphi \cap \mathfrak{N}_\infty$ *and* $c \in \mathfrak{N}_\varphi$*.*

Proposition 2.1.3 in [9] states that (6) holds for $b \in \mathfrak{M}_\infty$, but, examining the proof, we see that (6) is still valid if we instead only assume that $b \in \mathfrak{M}_\varphi \cap \mathfrak{N}_\infty$. This result will be used in this more general form later.

## 2.7 Spatial noncommutative $L_p$-spaces

An alternative approach to noncommutative $L_p$-spaces due to Connes [3] and Hilsum [15] will also be used. These spatial non-commutative $L_p$ spaces are isometrically isomorphic to those introduced by Haagerup (see [34] for further details).

Let $M$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\psi$ be a nsf weight on its commutant $M'$. Let $\mathfrak{N}_\psi = \{x \in M' : \psi(x^*x) < \infty\}$, let $\mathcal{H}_\psi$ be the Hilbert space completion of $\mathfrak{N}_\psi$, and $\Lambda_\psi$ the inclusion of $\mathfrak{N}_\psi$ into $\mathcal{H}_\psi$. The set of $\psi$-bounded vectors is

$$\mathcal{D}(\mathcal{H}, \psi) = \{\xi \in \mathcal{H} : \exists C > 0, \forall y \in \mathfrak{N}_\psi, \|y\xi\|_{\mathcal{H}} \leq C \|\Lambda_\psi(y)\|_{\mathcal{H}_\psi}\},$$

and for all $\xi \in \mathcal{D}(\mathcal{H}, \psi)$, $\Lambda_\psi(y) \mapsto y\xi$ extends to a bounded operator $R^\psi(\xi)$ from $\mathcal{H}_\psi$ to $\mathcal{H}$. If $\varphi$ is a normal semi-finite weight on $M$, then its spatial derivative $\frac{d\varphi}{d\psi}$ is the positive self-adjoint operator on $\mathcal{H}$ such that

$$(\frac{d\varphi}{d\psi}\xi \mid \xi) = \varphi(R^\psi(\xi)R^\psi(\xi)^*), \qquad \forall \xi \in \mathcal{D}(\mathcal{H}, \psi).$$

The spatial non-commutative $L_p$ spaces are defined as follows: $L_1(M, \psi)$ is the set of closed, densely defined operators with polar decomposition $x = u|x|$ such that $u \in M$ and there exists $\varphi \in M_{*,+}$ with $\frac{d\varphi}{d\psi} = |x|$. Setting

$\|x\|_1 = \varphi(1) = \|\varphi\|_{M_*}$ yields an isometric isomorphism between $L_1(M,\psi)$ and $M_*$. Similarly,

$$L_p(M,\psi) = \{x = u|x| \,:\, u \in M, \exists\varphi \in M_* \text{ with } |x|^p = \frac{d\varphi}{d\psi}\}, \quad \|x\|_p = \varphi(1)^{1/p}.$$

These spaces are isometrically isomorphic to the $L_p$ spaces of Haagerup and are thus independent of the choice of the weight $\psi$.

Using results of Terp's, we can write down dense subspaces of $L_p(M,\psi)$. The following can be found as Theorems 22, 26, and 27 from [36]. The following results are the spatial $L_p$-space version of Theorem 2.6.3.

**Theorem 2.7.1.** *Fix a normal semi-finite faithful weight $\varphi_0$ on $M$ and let $d = \frac{d\varphi_0}{d\psi}$.*

*Let $x \in \mathfrak{N}_\varphi$ and $2 \le p < \infty$. Then $xd^{1/p}$ is preclosed, its closure (also denoted by $xd^{1/p}$) is in $L_p(M,\psi)$ and the set of operators $\{xd^{1/p} : x \in \mathfrak{N}_\varphi\}$ is dense in $L_p(M,\psi)$. Furthermore,*

$$\left\|xd^{1/2}\right\|_2 = \|\Lambda(x)\|$$

*and the map $x \mapsto xd^{1/2} : \mathfrak{N}_\varphi \to L_2(M,\psi)$ extends to a linear isometry of $\mathcal{H}$ onto $L_2(M,\psi)$.*

*Similarly, if $1 \le p < \infty$, then $\{d^{1/2p}xd^{1/2p} : x \in \mathfrak{M}_\varphi\}$ is dense in $L_p(M,\psi)$.*

**Theorem 2.7.2.** *Let $x \in \pi_l(\mathfrak{A}_0)$. Then for $\alpha \in [0,\frac{1}{2}]$,*

$$xd^\alpha = d^\alpha \sigma^\varphi_{i\alpha}(x).$$

*Proof.* This has been proven in the context of Haagerup's construction of non-commutative $L^p$-spaces in [9]. It can be directly proved in the context of Hilsum's construction of non-commutative $L^p$-spaces by combining Lemma 2.6 from [35] with Lemma 22, Theorem 22, and Theorem 26 from [36]. □

## 2.8 Noncommutative $L_p$-spaces as interpolation spaces

Let $M$ be a von Neumann algebra with normal, faithful, semi-finite weight $\varphi$ acting faithfully on $\mathcal{H} = \mathcal{H}_\varphi$. Let $L_p(M, \varphi)$ be the non-commutative $L_p$-space constructed according to Hilsum's method, using a normal semi-finite weight on the commutant $M'$ (for example, $\varphi' = \varphi(J \cdot J)$, where $J$ is the modular conjugation). As in the commutative case, this noncommutative $L_p$-space can be viewed as an interpolation space, formed by interpolating between $M_* \simeq L_1(M, \varphi)$ and $M = L_\infty(M, \varphi)$. In this dissertation, results will often be proven first for the special cases $p = 1$, $p = 2$, and $p = \infty$; the general case will then follow by interpolation.

Kosaki showed in [22] that if $\varphi$ is a state, then the spaces $L_p(M, \varphi)$ can be viewed as interpolation spaces with $M \subset L_p(M, \varphi) \subset M_*$. For each $\eta \in [0, 1]$, his approach provides an embedding such that for $x \in M \subset L_p(M)$, the corresponding element of $L_p(M)$ is $d^{\eta/2p} x d^{(1-\eta)/2p}$.

Terp extended this to the case where $\varphi$ is a weight, but only for the symmeteric embedding ($\eta = \frac{1}{2}$). Her interpolation method is compatible with the inclusion $x \mapsto d^{1/2p} x d^{1/2p}$, where $x \in \mathfrak{M}_\varphi$ (see [36] for further details).

Izumi's construction in [16] simultaneously generalizes the work of both Terp and Kosaki. It works for a general normal, semi-finite, faithful weight, and provides a family of embeddings depending on a complex parameter $\alpha \in \mathbb{C}$. We first recall a few facts about the complex interpolation method.

A pair of Banach spaces $(V, W)$ is said to be a *compatible couple* if there is a topological vector space $X$ such that there are continuous inclusions of $V$ and $W$ into $X$. We can then discuss the spaces $V \cap W$ and $V + W$ inside $X$. These are Banach spaces when endowed with the norms

$$\|x\|_{V \cap W} = \max\{\|x\|_V, \|\|_W\}, \qquad x \in V \cap W, \text{ and}$$
$$\|x\|_{V+W} = \inf\{\|v\|_V + \|w\|_W : x = v + w\}.$$

Let $S$ denote the strip $\{z \in \mathbb{C} : 0 \le \mathrm{Re} z \le 1\}$ and let $S^0$ denote its interior. We let $\mathcal{F}$ denote the collection of all continuous and bounded functions $f : S \to V + W$ such that

1. $f$ is analytic on $S^0$,
2. $f(it) \in V$ and $f(1+it) \in W$ for all $t \in \mathbb{R}$,
3. $f(it) \to 0$ and $f(1+it) \to 0$ as $t \to \pm\infty$.

This is a Banach space when given the norm

$$\|f\|_{\mathcal{F}} = \max\{\sup\{\|f(it)\|_V : t \in \mathbb{R}\}, \sup\{\|f(1+it)\|_W : t \in \mathbb{R}\}\}.$$

For $0 \leq \theta \leq 1$, the space

$$(V,W)_\theta = \{a \in V + W : a = f(\theta) \text{ for some } f \in \mathcal{F}\}$$

is a Banach space with the norm given by

$$\|a\|_\theta = \inf\{\|f\|_{\mathcal{F}} : f(\theta) = a, f \in \mathcal{F}\}.$$

Returning to Izumi's interpolation method, for each $\alpha \in \mathbb{C}$ the intersection of $M_*$ and $M$ is given by

$$L_\alpha = \left\{ x \in M \;\middle|\; \begin{array}{l} \text{there exists a unique functional } \varphi_x^\alpha \in M_* \text{ such that} \\ \varphi_x^\alpha(y^*z) = (xJ\Delta^{\overline{\alpha}}\Lambda(y) \mid J\Delta^{-\alpha}\Lambda(z)) \\ \text{for all } y,z \in \pi_l(\mathfrak{A}_0) \end{array} \right\}.$$

This is a Banach space when considered with the norm

$$\|x\|_{L_\alpha} = \max\{\|x\|_\infty, \|\varphi_x^\alpha\|_1\},$$

where $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are the norms on $M$ and $M_*$, respectively.

We recall the following results, which can be found as Propositions 2.3, 2.4 and 2.6 in [16].

**Proposition 2.8.1.** *For any $\alpha \in \mathbb{C}$, we have*

$$\pi_l(\mathfrak{A}_0^2) \subset L_\alpha$$

*and for $y, z \in \pi_l(\mathfrak{A}_0)$, we have*

$$\varphi_{y^*z}^\alpha = \omega_{J\Delta^{-\overline{\alpha}}\Lambda(y), J\Delta^\alpha\Lambda(z)}$$

*where* $\omega_{\xi,\eta}$ *is the functional* $(\cdot\,\xi \mid \eta)$.

**Proposition 2.8.2.** *The maps*

$$\iota_\alpha : L_\alpha \to M, x \mapsto x \text{ and } j_\alpha : L_\alpha \to M_*, x \mapsto \varphi_x^\alpha,$$

*are norm-decreasing and injective. The set* $i_\alpha(\pi_l(\mathfrak{A}_0^2))$ *is* $\sigma$*-weakly dense in* $M$ *and the set* $j_\alpha(\pi_l(\mathfrak{A}_0^2))$ *is norm dense in* $M_*$.

**Proposition 2.8.3.** *For each* $\alpha \in \mathbb{C}$*, the Banach space* $L_\alpha$ *is a* $(\pi_l(\mathfrak{A}_0), \pi_l(\mathfrak{A}_0))$*-bimodule. If* $a, b \in \pi_l(\mathfrak{A}_0)$ *and* $x \in L_\alpha$*, then* $axb \in L_\alpha$ *and*

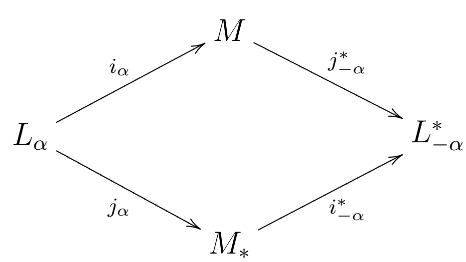
$$\varphi^\alpha_{axb} = \sigma^\varphi_{-i\alpha - i/2}(a)\, \varphi^\alpha_x\, \sigma^\varphi_{-i\alpha + i/2}(b),$$

*where for* $u, v \in M$*,* $\psi \in M_*$*, the symbol* $u\psi v$ *means an element of* $M_*$ *defined by the formula* $\langle u\psi v, a\rangle = \psi(vau)$*, for* $a \in M$.

To obtain a compatible pair in the sense of interpolation theory, the following embeddings are used: $i^*_{-\alpha} : M_* \hookrightarrow L^*_{-\alpha}$ and $j^*_{-\alpha} : M \hookrightarrow L^*_{-\alpha}$, where $i^*_{-\alpha}$ is the restriction of the usual adjoint map of $i_{-\alpha}$ to $M_*$. Explicitly,

$$\langle y, i^*_{-\alpha}(\psi)\rangle_{L_{-\alpha}, L^*_{-\alpha}} = \psi(y), \quad y \in L_{-\alpha}, \psi \in M_*;$$
$$\langle y, j^*_{-\alpha}(x)\rangle_{L_{-\alpha}, L^*_{-\alpha}} = \varphi^{-\alpha}_y(x), \quad y \in L_{-\alpha}, x \in M.$$

These maps are norm-decreasing and injective. These maps yield the following commutative diagram:




This compatible pair will be denoted $(M, M_*)_\alpha$ and is used to define noncommutative $L_p$-spaces

$$L_p(M, \varphi)_\alpha = C_{1/p}(M, M_*)_\alpha, \quad 1 < p < \infty, \alpha \in \mathbb{C},$$

using the complex interpolation method, which is an exact interpolation functor of exponent $\frac{1}{p}$. See [1] and [16] for more information about the complex interpolation method.

*Remark* 2.8.4. Only the two cases $\alpha = 0$ and $\alpha = -1/2$ will be used in this dissertation. It will always be clear from context which of these is being used; for this reason subscript and superscript $\alpha$'s will often be omitted.

Izumi showed (as Theorem 3.8 of [16]) that the $L_p$ spaces defined in this way are isometrically isomorphic to the interpolation spaces of Terp and thus to the $L_p$ spaces of Haagerup and Hilsum:

**Theorem 2.8.5.** *Let $M$ be a von Neumann algebra with normal, semi-finite, faithful weight $\varphi$. Let $\alpha, \beta \in \mathbb{C}$. There exists an isometric isomorphism*

$$U_{p,\beta,\alpha} : L_p(M,\varphi)_\alpha \to L_p(M,\varphi)_\beta, \quad 1 < p < \infty,$$

*such that*

$$U_{p,\beta,\alpha}(j^*_{-\alpha}(a)) = j^*_{-\beta}\left(\sigma^\varphi_{i\frac{r'-r}{p}-(s'-s)}(a)\right)$$

*for any $a \in \pi_l(\mathfrak{A}_0^2)$, where $\alpha = r + is$, $\beta = r' + is'$, $r, r', s, s' \in \mathbb{R}$.*

We also mention the following theorem from [18] that we will use later.

**Theorem 2.8.6.** *Let $R$ be a von Neumann algebra with a normal faithful tracial state $\tau$, and let $\mathcal{B}$ be a weak*-dense $C^*$-subalgebra of $R$. For $1 < p < \infty$, we have the complete isometry*

$$L_p(R) = (\mathcal{B}, L_1(R))_{1/p}.$$

## 2.9 Operator space structure of noncommutative $L_p$-spaces

An *operator space* $V$ is a linear space $V$ together with norms $\|\cdot\|_n$ on each linear space $M_n(V)$, the $n \times n$ matrices with entries in $V$, satisfying the conditions

1. $\|v \oplus w\|_{n+m} = \max\{\|v\|_m, \|w\|_n\}$, and
2. $\|\alpha v \beta\|_n \leq \|\alpha\| \, \|v\|_m \, \|\beta\|$,

for all $v \in M_m(V)$, $w \in M_n(V)$, $\alpha \in M_{n,m}(\mathbb{C})$, and $\beta \in M_{m,n}(\mathbb{C})$.

By Ruan's Theorem, this is equivalent to the existence of an inclusion of $V$ into $B(\mathcal{H})$ such that, for each $n \in \mathbb{N}$, the natural inclusion of $M_n(V)$ into $B(\mathcal{H}^n)$ induces the norm $\|\cdot\|_n$ on $M_n(V)$. There is a unique operator space structure on any $C^*$-algebra, which is given by any faithful representation on a Hilbert space. See [6] for further information about the theory of operator spaces.

Let $V$ and $W$ be operator spaces and let $T : V \to W$ be a linear map. The map $T$ is said to be *completely bounded* if

$$\sup_n \|T \otimes id_n : M_n(V) \to M_n(W)\| = \|T\|_{cb} < \infty.$$

Let $CB(V, W)$ denote the space of all completely bounded mappings from $V$ to $W$. There is a canonical operator space structure on $CB(V, W)$ given by identifying $M_n(CB(V, W))$ with $CB(V, M_n(W))$. In particular, we can define a dual operator space structure on $V^* = B(V, \mathbb{C}) = CB(V, \mathbb{C})$.

We can also define the *opposite* operator space structure, $V^{op}$, by

$$\left\| \left[ v_{ij}^{op} \right] \right\|_n = \|[v_{ji}]\|_n\,, \qquad \text{for } [v_{ij}] \in M_n(V^{op}).$$

Given a von Neumann algebra $M$, its predual has a canonical operator space structure given by its inclusion into $M^* = (M_*)^{**}$. The canonical operator space on the noncommutative $L_p$-space $L_p(M)$ is defined using Terp's inclusion for interpolation, yielding

$$M_n(L_p(M)) = (M_n(M), M_n(M_*^{op}))_{1/p}.$$

See [29] and [20] for further details.

In what follows, $\check{\otimes}$ denotes the operator space injective tensor product. The *operator space projective tensor product* will be denoted by $\widehat{\otimes}$. The operator spaces of bounded operators, compact operators, and trace class operators on a Hilbert space $\mathcal{H}$ will be denoted by $M_\infty$, $K_\infty$, and $T_\infty$, respectively.

**Definition 2.9.1.** An operator space $V$ is said to have the *completely bounded approximation property (CBAP)* if there exists a net of finite rank maps $T_\alpha : V \to V$ with $\|T_\alpha\|_{cb} \leq \lambda$ for some positive $\lambda$ such that $T_\alpha \to id_V$ in the *point-norm topology* on $V$; that is, we have $\|T_\alpha(x) - x\| \to 0$ for all $x \in V$.

An operator space $V$ is said to have the *operator space approximation property (OAP)* if there exists a net of finite rank maps $T_\alpha : V \to V$ such that $T_\alpha \to id_V$ in the *stable point-norm topology*; that is, we have $\|(T_\alpha \otimes id_\infty)(x) - x\| \to 0$ for all $x \in V \check{\otimes} K_\infty$.

# Chapter 3

# Locally Compact Quantum Groups

In this chapter we recall the definitions of locally compact quantum groups and Kac algebras. We also discuss how to construct the dual of a locally compact quantum group and gather some propositions about Kac algebras that will be needed later.

## 3.1 Locally compact quantum groups

A *locally compact quantum group* $\mathbb{G} = (M, \Gamma, \varphi, \varphi_r)$ consists of

- a von Neumann algebra $M$,
- a normal, unital, $*$-homomorphism $\Gamma$ from $M \to M \bar{\otimes} M$ such that

$$(\Gamma \otimes \iota) \circ \Gamma = (\iota \otimes \Gamma) \circ \Gamma;$$

- a nsf weight $\varphi$ on $M$ such that for all $\omega \in M_*^+$, $x \in \mathfrak{M}_\varphi^+$,

$$\varphi((\omega \otimes \iota)\Gamma(x)) = \varphi(x)\omega(1);$$

- a nsf weight $\varphi_r$ on $M$ such that for all $\omega \in M_*^+$, $x \in \mathfrak{M}_{\varphi_r}^+$,

$$\varphi_r((\iota \otimes \omega)\Gamma(x)) = \varphi_r(x)\omega(1).$$

Here $\bar{\otimes}$ denotes the von Neumann algebra tensor product. For further details on Kustermans and Vaes's definition of a locally compact quantum group and their construction of the dual locally compact quantum group, see [26].

If $G$ is a locally compact group, then the corresponding von Neumann algebra $M$ is $L_\infty(G, \mu)$, the comultiplication is the map $\Gamma : L_\infty(G) \to L_\infty(G \times G)$ given by $\Gamma(f)(s,t) = f(st)$, and the coassociativity condition corresponds to $f((st)u) = f(s(tu))$. The weights correspond to integrating against the left and right Haar measures on $G$.

The von Neumann algebra $M$ will often be denoted by $L_\infty(\mathbb{G})$, its predual $M_*$ by $L_1(\mathbb{G})$, and will be taken to be acting standardly on $\mathcal{H} = \mathcal{H}_\varphi$. The *left fundamental unitary operator* $W$ of $\mathbb{G}$ is the unitary $W$ on $\mathcal{H} \otimes \mathcal{H}$ (in fact, $W \in M \bar{\otimes} \hat{M}$) determined by

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Gamma(y)(x \otimes 1)),$$

for all $x, y \in \mathfrak{N}_\varphi$. This unitary implements the comultiplication: for $x \in L_\infty(\mathbb{G})$,

$$\Gamma(x) = W^*(1 \otimes x)W.$$

The unitary $W$ is a multiplicative unitary; it satisfies the *pentagonal equation*

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

The dual locally compact group will be denoted by $\hat{\mathbb{G}} = (L_\infty(\hat{\mathbb{G}}), \hat{\Gamma}, \hat{\varphi}, \hat{\psi})$. A symbol marked with $\hat{}$ denotes an object with the same properties as the unmarked symbol but now defined in the context of the dual locally compact quantum group. Locally compact quantum groups satisfy a Pontryagin duality theorem; we have the canonical identification $\hat{\hat{\mathbb{G}}} = \mathbb{G}$. We recall how to construct $L_\infty(\hat{\mathbb{G}})$ and $\hat{\varphi}$.

If $\omega_1, \omega_2 \in L_1(\mathbb{G})$, then $(\omega_1 * \omega_2)(x) = (\omega_1 \otimes \omega_2)(\Gamma(x))$ defines a multiplication on the predual $L_1(\mathbb{G})$ (which is convolution in the group case). Define the Fourier representation $\lambda$ by $\lambda(\omega) = (\omega \otimes 1)W$ for $\omega \in L_1(\mathbb{G})$; this satisfies $\lambda(\omega_1 * \omega_2) = \lambda(\omega_1)\lambda(\omega_2)$. The dual locally compact quantum group $\hat{\mathbb{G}}$ has as its von Neumann algebra $L_\infty(\hat{\mathbb{G}}) = \lambda(L_1(\mathbb{G}))''$, with the comultiplication $\hat{\Gamma}$ determined by the multiplicative unitary $\hat{W} = \Sigma W^* \Sigma$, where $\Sigma$ is the flip map on $\mathcal{H} \otimes \mathcal{H}$, that is $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$.

To construct a left Haar weight $\hat{\varphi}$, consider the set

$$\mathcal{I} = \{\omega \in L_1(\mathbb{G}) \mid \exists C \in \mathbb{R}^+ : |\omega(x^*)| \leq M \, \|\Lambda(x)\| \text{ for all } x \in \mathfrak{N}_\varphi\}.$$

By the Riesz Representation Theorem, there then exists $\xi(\omega) \in \mathcal{H}$ such that

$$\omega(x^*) = (\xi(\omega) \mid \Lambda(x)), \qquad x \in \mathfrak{N}_\varphi.$$

The set $\lambda(\mathcal{I})$ is a $\sigma$-strong*-norm core for the the unique $\sigma$-strong*-norm closed linear map $\hat{\Lambda}$ such that $\hat{\Lambda}(\lambda(\omega)) = \xi(\omega)$. The dual weight $\hat{\varphi}$ is the unique normal, semi-finite, faithful weight on $L_\infty(\hat{\mathbb{G}})$ having the triple $(\mathcal{H}, \iota, \hat{\Lambda})$ as its GNS construction. Here $\iota$ is the action $\iota(\lambda(\omega_1))\xi(\omega_2) = \xi(\omega_1 * \omega_2)$. Note that $\omega \in \mathcal{I}$ implies that $\lambda(\omega) \in \mathfrak{N}_{\hat{\varphi}}$. This identifies the Hilbert spaces $\mathcal{H}_\varphi$ and $\mathcal{H}_{\hat{\varphi}}$.

## 3.2 Kac Algebras

We shall also be interested in Kac algebras. These are certain "well-behaved" locally compact quantum groups. As noted in Section 3.2 of [37], Kac algebras can be characterized as those locally compact quantum groups for which the scaling group $(\tau_t)_{t\in\mathbb{R}}$ is trivial and the modular element $\delta_M$ is affiliated with the centre of $M$. We now recall some definitions and basic results concerning Kac algebras. For proofs and further details, please see Enock and Schwartz's book, [7], which provides a detailed discussion of the theory of Kac algebras.

**Definition 3.2.1.** A couple $(M, \Gamma)$ is called a *Hopf-von Neumann algebra* if

1. $M$ is a von Neumann algebra;

2. $\Gamma$ is an injective normal unital homomorphism from $M$ to the von Neumann algebra tensor product $M\bar{\otimes}M$, such that $\Gamma(1) = 1 \otimes 1$, and which has the co-associativity property, i.e.,

$$(\iota \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \iota) \circ \Gamma.$$

As the scaling group is trivial, the antipode of a Kac algebra is an involution.

**Definition 3.2.2.** A triple $(M, \Gamma, \kappa)$ is called a *co-involutive Hopf-von Neumann algebra* if

1. $(M, \Gamma)$ is a Hopf-von Neumann algebra;

2. $\kappa$ is an involutive anti-automorphism of $M$ (called the *co-involution* or *antipode*);

3. the co-product and the co-involution are such that

$$(\kappa \otimes \kappa) \circ \Gamma = \zeta \circ \Gamma \circ \kappa,$$

where $\zeta(a \otimes b) = b \otimes a$ for all $a, b \in M$.

**Definition 3.2.3.** Let $(M, \Gamma)$ be a Hopf-von Neumann algebra and $\psi$ be a faithful, semi-finite, normal weight on $M$. We shall say that $\psi$ is a *right-invariant weight with respect to* $\Gamma$ if

$$(\psi \otimes \iota)\Gamma(x) = \psi(x)1, \quad \text{for all } x \in M^+.$$

Let $(M, \Gamma, \kappa)$ be a co-involutive Hopf-von Neumann algebra. If $\psi$ is a right-invariant weight and also satisfies the following conditions

1. $(\psi \otimes \iota)((y^* \otimes 1)\Gamma(x)) = \kappa((\psi \otimes \iota)(\Gamma(y^*)(x \otimes 1))$ for all $x, y \in \mathfrak{N}_\varphi$;

2. $\kappa\sigma_t^\psi = \sigma_{-t}^\psi\kappa$ for all $t \in \mathbb{R}$;

then we shall say that $\psi$ is a *right Haar weight*.

**Definition 3.2.4.** Suppose that $(M, \gamma, \kappa)$ is a co-involutive Hopf-von Neumann algebra and $\varphi$ is a right Haar weight on $M$. The quadruple $(M, \Gamma, \kappa, \psi)$ is then called a *Kac algebra*.

We could clearly also define Kac algebras in terms of a left Haar weight. In fact, if $\psi$ is a right Haar weight, then $\psi \circ \kappa$ is a left Haar weight.

From now on, we shall assume we are working with a Kac algebra $(M, \Gamma, \kappa, \psi)$ acting standardly on $\mathcal{H}_\psi$. We have already mentioned the multiplicative unitary associated with the left Haar weight; we can also consider the unitary coming from the right Haar weight.

**Proposition 3.2.5.** *Let $\mathcal{H}_\psi$ denote the Hilbert space completion of $\mathfrak{N}_\psi$. There exists a* right fundamental unitary operator *$V$ on $\mathcal{H}_\psi \otimes \mathcal{H}_\psi$ defined by*

$$V(\Lambda_\psi(x) \otimes \Lambda_\psi(y)) = (\Lambda_\psi \otimes \Lambda_\psi)(\Gamma(x)(1 \otimes y)),$$

*for all $x, y \in \mathfrak{N}_\psi$. This operator $V$ satisifes the pentagonal relation*

$$V_{12}V_{13}V_{23} = V_{12}V_{23},$$

*and the co-multiplication* $\Gamma$ *on* $M$ *is given by*

$$\Gamma(x) = V(x \otimes 1)V^*.$$

The result below follows from the right invariance of $\psi$.

**Lemma 3.2.6.** *For all* $f \in M_*$*, we have*

1. $(\iota \otimes f)\Gamma(\mathfrak{M}_\psi) \subset \mathfrak{M}_\psi$,
2. $(\iota \otimes f)\Gamma(\mathfrak{N}_\psi) \subset \mathfrak{N}_\psi$.

As we are dealing with Kac algebras, the modular automorphism group and the comultiplication satisfy the following commutation relations (Theorem 2.7.6 and Corollary 2.5.7 in [7]):

**Theorem 3.2.7.** *For all* $t \in \mathbb{R}$ *and* $x \in M$*, we have*

$$\Gamma\sigma_t^\psi(x) = (\sigma_t^\psi \otimes \iota)\Gamma(x) = (\iota \otimes \sigma_t^\psi)\Gamma(x).$$

*Let* $x \in M$ *be analytic with respect to* $\psi$*. Then, for all* $f \in M_*$*, the element* $(\iota \otimes f)\Gamma(x)$ *is analytic with respect to* $\psi$*, and, for all* $\alpha \in \mathbb{C}$*, we have:*

$$\sigma_\alpha^\psi((\iota \otimes f)\Gamma(x)) = (\iota \otimes f)\Gamma(\sigma_\alpha^\psi(x)).$$

## 3.3 Multipliers of Kac algebras

For more information about multipliers of Kac algebras, see Kraus and Ruan's papers [23] and [24]. Let $\mathbb{G}$ be a Kac algebra. A *left multiplier* (respectively *right multiplier*) of $A(\mathbb{G}) = L_\infty(\hat{\mathbb{G}})_*$ is an element $a$ in $L_\infty(\mathbb{G})$ such that $a\hat{\lambda}(\hat{\omega}) \in \hat{\lambda}(A(\mathbb{G}))$ (respectively $\hat{\lambda}(\hat{\omega})a \in \hat{\lambda}(A(\mathbb{G}))$) for all $\hat{\omega} \in A(\mathbb{G})$. The set of all left (respectively right) multipliers will be denoted by $M^l(A(\mathbb{G})$ (respectively $M^r(A(\mathbb{G}))$). To each left (respectively right) multiplier $a$ of $A(\mathbb{G})$, we have a bounded map $m_a^l$ (respectively $m_a^r$) on $A(\mathbb{G})$ given by

$$m_a^l(\hat{\omega}) = \hat{\lambda}^{-1}(a\hat{\lambda}(\hat{\omega})) \text{ and } m_a^r(\hat{\omega}) = \hat{\lambda}^{-1}(\hat{\lambda}(\hat{\omega})a),$$

for all $\hat{\omega} \in A(\mathbb{G})$. A completely bounded left multiplier of $A(\mathbb{G})$ is a left multiplier that is a completely bounded as a map from $A(\mathbb{G})$ to $A(\mathbb{G})$. The set of completely bounded left multipliers of $A(\mathbb{G})$ will be denoted by $M_0^l(A(\mathbb{G}))$ and will be taken to have the operator space structure given by its inclusion into $CB(A(\mathbb{G}))$.

The adjoint maps $M_a^l = (m_a^l)^*$ and $M_a^r = (m_a^r)^*$ will then be completely bounded normal maps from $L_\infty(\hat{\mathbb{G}})$ to itself. The restriction $M_a^l\big|_{C_\lambda^*(\mathbb{G})}$ is a completely bounded map from $C_\lambda^*(\mathbb{G})$ into $C_\lambda^*(\mathbb{G})$. There is a contractive inclusion of $A(\mathbb{G})$ into $M_0^l(A(\mathbb{G}))$ given by the left multiplication map

$$m_{\hat{\omega}}^l(\hat{\omega}') = \hat{\omega} * \hat{\omega}',$$

for $\hat{\omega}, \hat{\omega}' \in A(\mathbb{G})$.

In the notation of Chapter 4, we have that for $\hat{\omega} \in A(\mathbb{G})$

$$\begin{aligned} M_{\hat{\omega}}^l &= (m_{\hat{\omega}}^l)^* = \hat{\Theta}^l(\hat{\omega}), \text{ and} \\ m_{\hat{\omega}}^l &= \hat{\Theta}^l(\hat{\omega})_*. \end{aligned}$$

It is shown in [23] that $M_0^l(A(\mathbb{G}))$ is a dual operator space; its predual will be denoted by $Q^l(\mathbb{G})$.

## 3.4 Approximation properties for Kac algebras

Let $G$ be a locally compact group. Let $A_c(G)$ denote the space of elements in $A(G)$ with compact support. The amenability of $G$ is equivalent to the existence of a net $\{\varphi_\alpha\}$ in $A_c(G)$ such that $\|\varphi_\alpha\| \leq 1$ and $\Theta^l(\varphi_\alpha)_* \to id_{A(G)}$ in the point-norm topology. A group $G$ is said to be *weakly amenable* if there exists a net $\{\varphi_\alpha\}$ in $A_c(G)$ such that $\left\|\Theta^l(\varphi_\alpha)_*\right\|_{cb} \leq \lambda$ (for some positive $\lambda$) and $\Theta^l(\varphi_\alpha)_* \to id_{A(G)}$ in the point-norm topology. A group $G$ has the *approximation property* if there exists a net $\{\varphi_\alpha\}$ in $A_c(G)$ such that $\Theta^l(\varphi_\alpha)_* \to id_{A(G)}$ in the $\sigma(M_0^l(A(G)), Q^l(G))$-topology.

Approximation properties for Kac algebras were introduced by Kraus and Ruan in [24] and generalize the above definitions.

**Definition 3.4.1.** Let $\mathbb{G}$ be a Kac algebra. A net $\{\hat{\omega}_i\}$ in $A(\mathbb{G})$ is said to be a *left stable weak* approximate identity* for $A(\mathbb{G})$ if $\hat{\Theta}^l(\hat{\omega}_i) \to id_{\hat{M}}$ in the stable point weak* topology of $CB(\hat{M})$. The Kac algebra $\mathbb{G}$ is said to have the *approximation property* if $A(\mathbb{G})$ has a left stable weak* approximate identity. The Kac algebra $\mathbb{G}$ has the *weak approximation property* if 1 is in the $\sigma(M_0^l(A(\mathbb{G}), Q^l(\mathbb{G}))$-closure of $A(\mathbb{G})$ in $M_0^l(A(\mathbb{G}))$. By Theorem 5.4 in [24], if $\mathbb{G}$ is a discrete Kac algebra then the approximation property and the weak approximation property are equivalent.

The Kac algebra $\mathbb{G}$ is said to be weakly amenable if $A(\mathbb{G})$ has a left approximate identity $\{\hat{\omega}_i\}$ such that

$$\sup \left\| \hat{\lambda}(\hat{\omega}_i) \right\|_{M_0^l(A(\mathbb{G}))} \leq L$$

for some positive number $L$.

If $G$ is a discrete group, then $G$ has the approximation property if and only if the group von Neumann algebra $VN(G)$ has the weak* operator approximation property, as was shown by Haagerup and Kraus in [14]. Theorems 5.13 and 5.14 of [24] extend the above result to discrete Kac algebras.

**Theorem 3.4.2.** *Let $\mathbb{G}$ be a discrete Kac algebra. Then the following conditions are equivalent:*

1. *$\mathbb{G}$ has the approximation property.*
2. *$L_\infty(\hat{\mathbb{G}})$ has the weak* operator approximation property.*
3. *$C_\lambda^*(\mathbb{G})$ has the operator approximation property.*
4. *$A(\mathbb{G})$ has the operator approximation property.*

*We also have that the following set of conditions are equivalent:*

1. *$\mathbb{G}$ is weakly amenable.*
2. *$L_\infty(\hat{\mathbb{G}})$ has the weak* completely bounded approximation property.*
3. *$C_\lambda^*(\mathbb{G})$ has the completely bounded approximation property.*
4. *$A(\mathbb{G})$ has the completely bounded approximation property.*

The approximation property and weak amenability of a discrete group $G$ are also related to approximation properties of the noncommutative $L_p$-spaces associated with the discrete group, as is shown in [18].

**Theorem 3.4.3.** *Let $1 < p < \infty$. If $G$ is a discrete group with the approximation property, the $L_p(VN(G))$ has the operator approximation property.*

*If $G$ is a weakly amenable discrete group, then $L_p(VN(G))$ has the completely bounded approximation property.*

The above result will be extended to discrete Kac algebras.

# Chapter 4

# Noncommutative $L_p$-spaces associated with Kac Algebras

Let $G$ be a locally compact group with right Haar measure $ds$. Then $L_1(G)$ acts contractively by right convolution both on $L_1(G)$ and on $L_\infty(G)$:

$$\left(\Theta_1^r(f)(g)\right)(t) = \int_G g(ts)f(s)\ ds,$$
$$\left(\Theta_\infty^r(f)(h)\right)(t) = \int_G h(ts)f(s)\ ds,$$

for $f, g \in L_1(G)$, and $h \in L_\infty(G)$. Clearly these two actions agree on $L_1(G) \cap L_\infty(G)$. Interpolating between these two cases yields a right action of $L_1(G)$ on $L_p(G)$. We shall extend this result to Kac algebras.

Let $G$ be a locally compact group. Let $A(G)$ denote the Fourier algebra of the group $G$, which can be identified with the predual of the group von Neumann algebra $VN(G)$. In [5], Daws considers a way of representing the Fourier algebra $A(G)$ on the noncommutative $L_p$-spaces $L_p(VN(G))$ associated with the group von Neumann algebra $VN(G)$. The results in this chapter generalize this representation to Kac algebras.

Let $\delta$ denote the modular function of $G$ and let $dt$ denote the left Haar measure on $G$. Let $C_c(G)$ denote the continuous functions on $G$ with compact support. This is a left Hilbert algebra with respect to the following product, involution, and inner product:

$$(\xi * \eta)(s) = \int_G \xi(t)\eta(t^{-1}s)\, dt;$$
$$\xi^{\#}(s) = \delta(s)^{-1}\overline{\xi(s^{-1})}, \quad s \in G;$$
$$(\xi \mid \eta) = \int_G \xi(t)\overline{\eta(t)}\, dt.$$

The left von Neumann algebra generated by $C_c(G)$ is isomorphic to $VN(G)$ and we can identify the maps $\pi_l(\xi)$ with $\lambda(\xi)$ for $\xi \in C_c(G)$. In the usual way, this left Hilbert algebra defines a weight (*the Plancherel weight*) on $VN(G)$ satisfying

$$\varphi(\lambda(\xi)^*\lambda(\xi)) = (\xi^\# * \xi)(e) = \|\xi\|_2^2, \qquad \xi \in C_c(\mathbb{G}).$$

The Fourier algebra $A(G)$ is the set

$$\{\omega_{\xi,\eta} \,:\, \xi, \eta \in L_2(G)\},$$

with the *-algebra structure given by the following inclusion into $C_0(G)$:

$$\omega_{\xi,\eta}(s) = (\lambda(s)\xi \mid \eta) = \int_G \xi(s^{-1}t)\overline{\eta(t)}\,dt.$$

The norm on $A(G)$ is given by

$$\|f\|_{A(G)} = \inf\{\|\xi\|_2 \,\|\eta\|_2 \,:\, f = \omega_{\xi,\eta}, \xi, \eta \in L_2(G)\}.$$

We refer the reader to Section VII.3 of [33] for a more detailed discussion of the Fourier algebra and the Plancherel weight.

Let $J : L_2(G) \to L_2(G)$ denote the modular conjugation for $\varphi$; this is given by

$$(J\xi)(s) = \delta(s)^{-1/2}\overline{\xi(s^{-1})}, \quad \xi \in L_2(G).$$

Suppose that $f, g \in C_c(G)$; we then have that $\lambda(f^\# * g) = \lambda(f)^*\lambda(g) \in VN(G)$. Using Terp's interpolation method, the corresponding element in $A(G)$ is $\varphi_{\lambda(f)^*\lambda(g)} = \omega_{Jf,Jg}$. We now find the element in $C_c(G)$ corresponding to $\varphi_{\lambda(f)^*\lambda(g)}$:

$$\begin{aligned}\varphi_{\lambda(f)^*\lambda(g)}(\lambda_s) = \langle \lambda_s, \omega_{Jf,Jg}\rangle &= \int_G \lambda_s(Jf)(t)\overline{(Jg)(t)}\,dt \\ &= \int_G (Jf)(s^{-1}t)\overline{(Jg)(t)}\,dt,\end{aligned}$$

which, by the definition of the modular conjugation $J$, is equal to

$$\begin{aligned}
&= \int_G \overline{f(t^{-1}s)} g(t^{-1}) \delta(s)^{1/2} \delta(t)^{-1}\, dt \\
&= \int_G \overline{f(t^{-1})} g(t^{-1}s) \delta(s)^{1/2} \delta(s)^{-1} \delta(t)^{-1}\, dt \\
&= \int_G f^{\#}(t) g(t^{-1}s) \delta(s)^{-1/2}\, dt, \\
&= \delta(s)^{-1/2} (f^{\#} * g)(s).
\end{aligned}$$

The Fourier algebra $A(G)$ acts on itself by multiplication defining a map $\hat{\Theta}_1(a) : A(G) \to A(G)$:

$$\hat{\Theta}(a)(b)(s) = (a \cdot b)(s) = a(s)b(s), \quad a, b \in A(G).$$

(We omit the superscript $r$ as, by commutativity of $A(G)$, this is both a left and right action.)

Dualizing this action yields an action $\hat{\Theta}_\infty$ of $A(G)$ on $VN(G)$:

$$\langle a \cdot \lambda(f), b \rangle = \langle \lambda(f), a \cdot b \rangle = \int_G a(t) f(t) b(t)\, dt = \langle \lambda(a \cdot f), b \rangle,$$

for $a, b \in A(G)$ and $f \in L_1(G)$. It follows from the above calculations that

$$\varphi_{a \cdot \lambda(f)} = a \cdot \varphi_{\lambda(f)},$$

for $a \in A(G)$ and $f \in C_0(G)^2$. Approximation and interpolation arguments now yield an action of $A(G)$ on $L_p(VN(G))$, the noncommutative $L_p$-spaces associated with the Plancherel weight $\varphi$. For further details, see [5].

By working at the level of Kac algebras, we can handle both of these situations simultaneously. If we take the Kac algebra $\mathbb{G}$ to be equal to $L_\infty(G)$, we are considering the case where $L_1(G)$ is acting by convolution on $L_p(G)$. If we take the Kac algebra $\mathbb{G}$ to be equal to $VN(G)$, we are considering the case where $A(G)$ acts by multiplication on $L_p(VN(G))$.

Throughout this chapter, $\mathbb{G}$ will denote a Kac algebra with comultiplication $\Gamma$, antipode $\kappa$, left Haar weight $\varphi$, and right Haar weight $\psi$. Using the results of [17], we see how to represent $L_1(\mathbb{G})$ on $L_\infty(\mathbb{G})$ using the map $\Theta^r(f)$. In this chapter, we show how this can be used to get a representation of $L_1(\mathbb{G})$ on the noncommutative $L_p$-space $L_p(\mathbb{G})$. We trace our way through the construction of the Haagerup non-commutative $L_p$-spaces and prove that the correct map in this context is given by

$$\Theta_p^r(f)(D^{1/2p}xD^{1/2p}) = D^{1/2p}\Theta^r(f)(x)D^{1/2p},$$

for $f \in L_1(\mathbb{G})$ and $x \in \mathfrak{M}_\psi$.

We will show that this map is well-defined and bounded in the case $p = 1$. We then interpolate between the two cases $p = 1$ and $p = \infty$ to prove the general case. Obviously, one can also consider the map $\Theta^l(f)$ that corresponds to left convolution by $f$. We consider $\Theta^r$ as it is a homomorphism, whereas $\Theta^l$ is a anti-homomorphism.

Along the way we will prove Proposition 4.2.6. This proposition will be used to consider the operator space version of the representation of $L_1(\mathbb{G})$. It will also be used to prove that if $\mathbb{G}$ is a discrete Kac algebra with the approximation property, then $L_p(\hat{\mathbb{G}})$ has the operator approximation property.

## 4.1 The map $\Theta^r(f)$

Let $\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \kappa, \varphi, \psi)$ be a Kac algebra with left Haar weight $\varphi$, right Haar weight $\psi$, and antipode $\kappa$. We shall take $L_\infty(\mathbb{G})$ to be acting in standard form on the Hilbert space $\mathcal{H} = \mathcal{H}_\psi$. We shall denote the predual $(L_\infty(\mathbb{G})))_*$ by $L_1(\mathbb{G})$. Let $V$ be the multiplicative unitary acting on $\mathcal{H} \otimes \mathcal{H}$ such that

$$\Gamma(x) = V(x \otimes 1)V^*, \qquad \text{for } x \in L_\infty(\mathbb{G}).$$

We start by working with positive linear functionals and later extend by linearity. Let $f$ be a normal state on $M$. As $M$ is in standard form, there exists $\xi \in \mathcal{H}$, with $\|\xi\| = 1$ such that

$$f(x) = (x\xi \mid \xi), \qquad \text{for } x \in L_\infty(\mathbb{G}).$$

Let $\sigma_t^\psi$ be the modular automorpsism group for $L_\infty(\mathbb{G})$ associated with $\psi$ and let $N$ denote the crossed product $N = L_\infty(\mathbb{G}) \rtimes_{\sigma^\psi} \mathbb{R}$.

We define $\Theta^r(f) : L_\infty(\mathbb{G}) \to L_\infty(\mathbb{G})$ by

$$\Theta^r(f)(x) = (\iota \otimes f)\Gamma(x) = (\iota \otimes f)V(x \otimes 1)V^*.$$

As the composition of two normal, unital, completely positive, completely contractive maps, $\Theta^r(f)$ is normal, unital, completely positive and completely contractive. It follows directly from the definition that the preadjoint $\Theta^r(f)_* = m_f^r$ is the right multiplication map on $L_1(\mathbb{G})$ given by

$$m_f^r(g) = g * f, \qquad \text{for } g \in L_1(\mathbb{G}).$$

Clearly, $m_{f*g}^r = m_g^r \circ m_f^r$ and thus $\Theta^r(f * g) = \Theta^r(f) \circ \Theta^r(g)$.

Similarly, one can define $\Theta^l(f) : L_\infty(\mathbb{G}) \to L_\infty(\mathbb{G})$ by

$$\Theta^l(f)(x) = (f \otimes \iota)\Gamma(x) = (f \otimes \iota)W^*(1 \otimes x)W.$$

In fact, as $W \in L_\infty(\mathbb{G})\bar{\otimes}B(\mathcal{H})$, this last formula can be used to define a map $\Theta^l(f) : B(\mathcal{H}) \to B(\mathcal{H})$. Proposition 3.1 from [17] shows the following.

**Proposition 4.1.1.** *Let $\mathbb{G}$ be a locally compact quantum group.*

1. *$\Theta^l$ is an injective completely contractive anti-homomorphism from $L_1(\mathbb{G})$ into $CB^{\sigma, L_\infty(\mathbb{G})}_{L_\infty(\hat{\mathbb{G}})'}(B(\mathcal{H}))$.*
2. *$\Theta^r$ is an injective completely contractive homomorphism from $L_1(\mathbb{G})$ into $CB^{\sigma, L_\infty(\mathbb{G})}_{L_\infty(\hat{\mathbb{G}})}(B(\mathcal{H}))$.*

Here $CB^{\sigma,A}_B(B(\mathcal{H}))$ denotes the normal completely bounded $B$-bimodule maps on $B(\mathcal{H})$ which map $A$ into $A$. Junge, Neufang, and Ruan then show the map $\Theta^r$ can be extended to completely isometrically and algebraically identify $M_0^r(L_1(\mathbb{G}))$ with $CB^{\sigma, L_\infty(\mathbb{G})}_{L_\infty(\hat{\mathbb{G}})}(B(\mathcal{H}))$.

The map $\Theta^r(f)$ can be extended to $B(L_2(\mathbb{R}) \otimes \mathcal{H})$ as follows

$$\begin{aligned}\Theta^r(f) :& B(L_2(\mathbb{R}) \otimes \mathcal{H}) \to B(L_2(\mathbb{R}) \otimes \mathcal{H})\\ \Theta^r(f)(x) &= (\iota \otimes \iota \otimes f)(\iota \otimes V)(x \otimes 1)(\iota \otimes V^*).\end{aligned}$$

We now check that this extension maps the crossed product $N$ to itself. For $s \in \mathbb{R}$ and $x \in L_\infty(\mathbb{G})$, by Theorem 3.2.7,

$$\sigma_{-s}((\iota \otimes f)\Gamma(x)) = (\iota \otimes f)\Gamma(\sigma_{-s}(x))$$

and thus $\Theta^r(f)(\pi(x)) = \pi(\Theta^r(f)(x))$. Writing $\lambda_t = l_t \otimes 1$, we have

$$\begin{aligned}(\iota \otimes V)(\pi(x)\lambda_t \otimes 1)(\iota \otimes V^*) =& (\iota \otimes V)(\pi(x)(l_t \otimes 1) \otimes 1)(\iota \otimes V^*)\\ =& (\iota \otimes V)(\pi(x) \otimes 1)(\iota \otimes V^*)(l_t \otimes 1 \otimes 1)\\ =& (\iota \otimes V)(\pi(x) \otimes 1)(\iota \otimes V^*)(\lambda_t \otimes 1)\\ \Theta^r(f)(\pi(x)\lambda_t) =& (\iota \otimes \iota \otimes f)\left((\iota \otimes V)(\pi(x) \otimes 1)(\iota \otimes V^*)(\lambda_t \otimes 1)\right)\\ =& \Theta^r(f)(\pi(x))\lambda_t = \pi(\Theta^r(f)(x))\lambda_t.\end{aligned}$$

As span$\{\pi(x)\lambda_t\}$ is $\sigma$-weakly dense in $N$ and $\Theta^r(f)$ is normal, it follows that $\Theta^r(f)$ is a map from $N$ to $N$. From this point on, $\pi(L_\infty(\mathbb{G}))$ will be identified with $L_\infty(\mathbb{G})$.

We note that the existence of such an extension to the crossed product has already been noted in a more general context as Theorem 4.1 in the paper [13].

## 4.2 Extending $\Theta^r(f)$ to $L_p(\mathbb{G})$

We now extend $\Theta^r(f)$ to a map from $\hat{N}_+$ to $\hat{N}_+$. Why do we extend it to $\hat{N}_+$ and not the more directly relevant $\tau$-measurable maps $\tilde{N}$? Consider the following example.

Let $G$ be the group $((0,1],+)$, where addition is considered modulo 1. The Haar measure on this group is Lebesgue measure. The function

$$f(t) = \sum_{n=1}^{\infty} 2^n \chi_{(\frac{1}{n+1},\frac{1}{n}]}(t)$$

is measurable and the function $g(t) \equiv 1$ is an element in $L_1(G)$. However $(f * g)(s) \equiv +\infty$ and is thus not measurable.

We thus should not expect to be able to extend $\Theta^r(f)$ to a map from $\tilde{N}$ to $\tilde{N}$. Instead we extend $\Theta^r(f)$ to a map from the generalized positive operators $\hat{N}_+$ to $\hat{N}_+$ (and this extension will be denoted by $\hat{\Theta}^r(f)$). We will then examine the restriction of this map to $L_p(\mathbb{G})$ and show that this restriction is a bounded map from $L_p(\mathbb{G})$ to $L_p(\mathbb{G})$; this restriction will be denoted by $\Theta^r_p(f)$. Throughout this section, it will be necessary to clearly distinguish between an operator and its closure. To this end, we recall some notation that we fixed earlier. The closure of a pre-closed operator $x$ will be denoted by $[x]$, and the domain of an operator $x$ will be denoted by $\mathcal{D}(x)$.

As $\Theta^r(f)$ is positive and normal, we can extend it to a map $\hat{\Theta}^r(f)$ on $\hat{N}_+$ by

$$\begin{aligned}\hat{\Theta}^r(f) :&\hat{N}_+ \to \hat{N}_+,\\ \langle \hat{\Theta}^r(f)(x), \omega\rangle =&\langle x, (\Theta^r(f))_*(\omega)\rangle,\end{aligned}$$

for all $x \in \hat{N}_+$, and $\omega \in N_{*,+}$.

Having defined this extension, we shall later restrict it to a map $L_p(\mathbb{G})_+ \to L_p(\mathbb{G})_+$. Our next goal is to show that for $a \in \mathfrak{N}_\psi$,

$$\hat{\Theta}^r(f)\left(D^{1/2p}a^*[aD^{1/2p}]\right) = D^{1/2p}\left(\Theta^r(f)\,(a^*a)\right)^{1/2}\left[\left(\Theta^r(f)\,(a^*a)\right)^{1/2}D^{1/2p}\right].$$

By Proposition 2.8 in [9], we know that the right hand operator is in $L_p(\mathbb{G})$ and is thus $\tau$-measurable. We shall show that the operator on the left is $\tau$-measurable and that these two measurable operators coincide on sufficiently many elements in $\mathcal{H}$ to imply that they must be equal. Our approach is to consider both sides of this equation as generalized positive operators and to show the two sides take the same value on certain vector states. We shall freely go back and forth between considering generalized positive operators as weights on $N_*$ and as operators on the Hilbert space $\mathcal{H}$ (as described in Section 2.5).

Suppose that $x \in \hat{N}_+$ has spectral decomposition $\int_0^\infty \lambda \; de_\lambda + \infty \cdot p$. We set $x_n$ equal to $\int_0^n \lambda \; de_\lambda + n \cdot p$. Then we have that $x_n \in N_+$ and

$$\langle x, \omega\rangle = \lim_n \langle x_n, \omega\rangle.$$

We now use this to describe the action of the extension $\hat{\Theta}^r(f) : \hat{N}_+ \to \hat{N}_+$ in terms of the map $\Theta^r(f) : N \to N$.

$$\begin{aligned}\langle \hat{\Theta}^r(f)(x), \omega\rangle =&\langle x, \Theta^r(f)_*(\omega)\rangle\\ =&\lim_n \langle x_n, \Theta^r(f)_*(\omega)\rangle\\ =&\lim_n \langle \Theta^r(f)(x_n), \omega\rangle.\end{aligned}$$

In particular, let us suppose that $\eta \in L^2(\mathbb{R}, \mathcal{H})$. Let $\omega_{\eta,\eta} \in L_1(\mathbb{G})$ be the map defined by $\omega_{\eta,\eta}(x) = (x\eta \mid \eta)$. We then calculate

$$\begin{aligned}\langle \hat{\Theta}^r(f)(x), \omega_{\eta,\eta}\rangle &= \lim_n \langle \Theta^r(f)(x_n), \omega_{\eta,\eta}\rangle \\ &= \lim_n \left((\iota \otimes \iota \otimes \xi^*)(\iota \otimes V)(x_n \otimes 1)(\iota \otimes V^*)(\iota \otimes \iota \otimes \xi)\eta \mid \eta\right) \\ &= \lim_n \left\|(x_n^{1/2} \otimes 1)(\iota \otimes V^*)(\eta \otimes \xi)\right\|^2 .\end{aligned}$$

Considering the spectral resolution for $x^{1/2} \otimes 1$, we see that this is equal to

$$\begin{cases} \left\|(x^{1/2} \otimes 1)(\iota \otimes V^*)(\eta \otimes \xi)\right\|^2, & \text{if } (\iota \otimes V^*)(\eta \otimes \xi) \in \mathcal{D}(x^{1/2} \otimes 1), \\ \infty, & \text{otherwise.} \end{cases}$$

We have already noted that $\lambda_t = l_t \otimes 1$, where $l_t$ is the translation operator on $L_2(\mathbb{R})$. There exists a self-adjoint operator $C$ on $L_2(\mathbb{R})$ such that $e^{iCt} = l_t$; let $D_1 = e^{iC}$. By applying Theorem 5 from [30] with $A = iC$ and $B = 0$, we have that $D = D_1 \otimes 1$ and that

$$\mathcal{D}_0 = \operatorname{span}\{g \otimes \eta : g \in \mathcal{D}(D_1), \eta \in \mathcal{H}\}.$$

is a core for $D$.

Consider the case where

$$x = D^{1/2p} a^* [a D^{1/2p}] = |[a D^{1/2p}]|^2, \qquad \text{for } a \in \mathfrak{N}_\psi$$

and $\eta = g \otimes \eta_0$ where $g \in \mathcal{D}(D_1^{1/2p}) \subset L_2(\mathbb{R})$ and $\eta_0 \in \mathcal{H}$. Then

$$\begin{aligned}\langle \hat{\Theta}^r(f)\left(D^{1/2p} a^*[a D^{1/2p}]\right), \omega_{\eta,\eta}\rangle &= \left\|[a D^{1/2p} \otimes 1](\iota \otimes V^*)(\eta \otimes \xi)\right\|^2 \\ &= \left\|(a D^{1/2p} \otimes 1)(\iota \otimes V^*)(g \otimes \eta_0 \otimes \xi)\right\|^2 \\ &= \left\|(a \otimes 1)(D_1^{1/2p} g \otimes V^*(\eta_0 \otimes \xi))\right\|^2 .\end{aligned}$$

We shall now compare this to $D^{1/2p}\Theta^r(f)(a^*a)^{1/2}[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]$ evaluated at the same vector state. Recall that by Lemma 3.2.6 we have that $\Theta^r(f)(a^*a) \in \mathfrak{M}_\psi$ and thus that $D^{1/2p}\Theta^r(f)(a^*a)^{1/2}[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]$ is a $\tau$-measurable operator in $L_p(\mathbb{G})$. Note that we have $\eta \in \mathcal{D}(D^{1/2p}) \subset$

$\mathcal{D}(|[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]|)$.

$$
\begin{aligned}
&\langle D^{1/2p}\Theta^r(f)(a^*a)^{1/2}[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}], \omega_{\eta,\eta}\rangle \\
=& \left\|[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]\eta\right\|^2 \\
=& \left\|\Theta^r(f)(a^*a)^{1/2}D^{1/2p}\eta\right\|^2 \\
=&\langle \Theta^r(f)(a^*a)^{1/2}D^{1/2p}\eta \mid \Theta^r(f)(a^*a)^{1/2}D^{1/2p}\eta\rangle \\
=&\langle \Theta^r(f)(a^*a)D^{1/2p}\eta \mid D^{1/2p}\eta\rangle \\
=&\langle (\iota\otimes\iota\otimes\xi)(\iota\otimes V^*)(a^*a\otimes 1)(\iota\otimes V)(\iota\otimes\iota\otimes\xi^*)D^{1/2p}\eta \mid D^{1/2p}\eta\rangle \\
=& \left\|(a\otimes 1)(\iota\otimes V)(D^{1/2p}\eta\otimes\xi)\right\|^2 \\
=& \left\|(a\otimes 1)(D_1^{1/2p}g\otimes V^*(\eta_0\otimes\xi))\right\|^2 .
\end{aligned}
$$

The above discussion is summarized in the lemma below.

**Lemma 4.2.1.** *Let $f \in L_1(\mathbb{G})_+$ and $a \in \mathfrak{N}_\psi$. We then have that*

$$\langle \hat{\Theta}^r(f)\left(D^{1/2p}a^*[aD^{1/2p}]\right), \omega_{\eta,\eta}\rangle = \langle D^{1/2p}\Theta^r(f)(a^*a)^{1/2}[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}], \omega_{\eta,\eta}\rangle,$$

*for all $\eta \in \operatorname{span}\{g\otimes\eta_0 : g\in\mathcal{D}(D_1^{1/2p}), \eta_0\in\mathcal{H}\}$.*

**Lemma 4.2.2.** *The set*

$$\mathcal{D}_0 = \operatorname{span}\{g\otimes\eta : g\in\mathcal{D}(D_1^{1/2p}), \eta\in\mathcal{H}\}$$

*is a core for $|[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]|$.*

*Proof.* The set $\mathcal{D}_0$ is clearly a core for $D_1^{1/2p}\otimes 1 = D^{1/2p}$. Suppose $\eta \in \mathcal{D}(D^{1/2p})$. Then there exists $(\eta_i) \in \mathcal{D}_0$ such that $\eta_i \to \eta$ and $D^{1/2p}\eta_i \to D^{1/2p}\eta$. As $\Theta^r(f)(a^*a)^{1/2}$ is bounded, it follows that

$$[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]\eta_i \to [\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]\eta,$$

and thus that

$$|[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]|\eta_i \to |[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]|\eta.$$

This shows that

$$\mathcal{D}(D^{1/2p}) \subset \mathcal{D}\left(\left[\,|[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]|\big|_{\mathcal{D}_0}\right]\right).$$

As $\mathcal{D}(D^{1/2p})$ is a core for $|[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]|$, it follows that

$$\left[|[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]|\big|_{\mathcal{D}_0}\right] = |[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}]|.$$

□

The above calculations and the following proposition will allow us to conclude that

$$\hat{\Theta}^r(f)\left(D^{1/2p}a^*[aD^{1/2p}]\right) = D^{1/2p}\Theta^r(f)(a^*a)^{1/2}[\Theta^r(f)(a^*a)^{1/2}D^{1/2p}].$$

**Proposition 4.2.3.** *Let $N \subset B(K)$ be a von Neumann algebra and $\tau$ a faithful normal semifinite trace on $N$. Suppose that $x \geq 0$ is a $\tau$-measurable operator and that $y \geq 0$ is a positive self-adjoint operator affiliated to $N$. Suppose that $D$ is a core for $x^{1/2}$ and that $D \subset \mathcal{D}(y^{1/2})$. If*

$$\left\|x^{1/2}\xi\right\| = \left\|y^{1/2}\xi\right\|, \quad \forall \xi \in D,$$

*then $x = y$.*

*Proof.* It shall first be shown that $\mathcal{D}(x^{1/2}) \subset \mathcal{D}(y^{1/2})$ and that

$$\left\|x^{1/2}\xi\right\| = \left\|y^{1/2}\xi\right\|, \quad \forall \xi \in \mathcal{D}(x^{1/2}).$$

Suppose that $\eta \in \mathcal{D}(x^{1/2})$, then there exists a sequence $(\eta_i)$ in $D$ such that $\eta_i \to \eta$ and $\left\|x^{1/2}(\eta_i) - x^{1/2}(\eta)\right\| \to 0$. This implies that $(y^{1/2}\eta_i)$ is a Cauchy sequence and thus converges to some $\zeta$. As $y^{1/2}$ is closed, $\eta \in \mathcal{D}(y^{1/2})$, $\zeta = y^{1/2}(\eta)$ and

$$\left\|y^{1/2}(\eta)\right\| = \lim_i \left\|y^{1/2}(\eta_i)\right\| = \lim_i \left\|x^{1/2}(\eta_i)\right\| = \left\|x^{1/2}(\eta)\right\|.$$

As $x^{1/2} \in \tilde{N}$, given $\delta > 0$, there exists a projection $p \in N$ such that $\tau(1-p) < \delta$, $pK \subset \mathcal{D}(x^{1/2}) \subset \mathcal{D}(y^{1/2})$ and $\left\|x^{1/2}p\right\| < \infty$. Since $\left\|y^{1/2}p\right\| = \left\|x^{1/2}p\right\| < \infty$, we see that $y^{1/2}$ is $\tau$-measurable. As the set of $\tau$-measurable operators form an algebra, this implies that $y$ is $\tau$-measurable.

Given $\delta > 0$, by Proposition 21 in [34] we have that for sufficiently large $\lambda$, $\tau(1 - \chi_{[0,\lambda)}(x)) < \delta$ and $\chi_{[0,\lambda)}(x)K \subset \mathcal{D}(x) \subset \mathcal{D}(x^{1/2})$. Also note that $\chi_{[0,\lambda)}(x)K \subset \mathcal{D}(y^{1/2})$ and $\left\|x^{1/2}\chi_{[0,\lambda)}(x)\right\| = \left\|y^{1/2}\chi_{[0,\lambda)}(x)\right\| < \infty$ and that

$$\left\|x^{1/2}\chi_{[0,\lambda)}(x)\xi\right\| = \left\|y^{1/2}\chi_{[0,\lambda)}(x)\xi\right\|, \quad \forall \xi \in K.$$

However by the polarization identity and uniqueness of square roots, if $A, B \in B(K)$, $A \geq 0, B \geq 0$, and $\|A\xi\| = \|B\xi\|$, $\forall \xi \in K$, then $A = B$. Thus

$$\begin{aligned} x^{1/2}\chi_{[0,\lambda)}(x) &= |y^{1/2}\chi_{[0,\lambda)}(x)| \\ x\chi_{[0,\lambda)}(x) &= [\chi_{[0,\lambda)}(x)y^{1/2}]y^{1/2}\chi_{[0,\lambda)}(x) \end{aligned} \tag{1}$$

and

$$\chi_{[0,\lambda)}(x)y\chi_{[0,\lambda)}(x) \subset [\chi_{[0,\lambda)}(x)y^{1/2}]y^{1/2}\chi_{[0,\lambda)}(x).$$

As $y$ and $\chi_{[0,\lambda)}(x)$ are $\tau$-measurable, $\chi_{[0,\lambda)}(x)y\chi_{[0,\lambda)}(x)$ is $\tau$-premeasurable. Thus given $\delta > 0$, there exists a projection $q \in N$ such that

$$qK \subset \mathcal{D}(\chi_{[0,\lambda)}(x)y\chi_{[0,\lambda)}(x)) \text{ and } \tau(1-q) < \delta.$$

Let $r = \chi_{[0,\lambda)}(x) \wedge q$ and then $\tau(1-r) < 2\delta$ and

$$rK \subset (pK \cap \mathcal{D}(\chi_{[0,\lambda)}(x)y\chi_{[0,\lambda)}(x))).$$

Then for all $\xi \in rK$,

$$\begin{aligned} x\xi = x\chi_{[0,\lambda)}(x)\xi &= [\chi_{[0,\lambda)}(x)y^{1/2}]y^{1/2}\chi_{[0,\lambda)}(x)\xi \\ &= \chi_{[0,\lambda)}(x)y\chi_{[0,\lambda)}(x)\xi = \chi_{[0,\lambda)}(x)y\xi. \end{aligned}$$

Suppose that $\lambda' > \lambda$ and $\xi \in rK$. Then $\xi \in \chi_{[0,\lambda)}(x)K$,

$$\chi_{[0,\lambda')}(x)\xi = \chi_{[0,\lambda)}(x)\xi = \xi,$$

and $\xi \in \mathcal{D}(\chi_{[0,\lambda)}(x)y\chi_{[0,\lambda)}(x))$ and $\chi_{[0,\lambda)}(x)y\chi_{[0,\lambda)}(x)\xi = \chi_{[0,\lambda)}(x)y\xi$ so $\xi \in \mathcal{D}(y)$. Thus $\xi \in \mathcal{D}(\chi_{[0,\lambda')}(x)y\chi_{[0,\lambda')}(x))$ and from (1),

$$\begin{aligned} \chi_{[0,\lambda')}(x)y\xi &= \chi_{[0,\lambda')}(x)y\chi_{[0,\lambda')}(x)\xi = x\chi_{[0,\lambda')}(x)\xi \\ &= x\xi = x\chi_{[0,\lambda)}(x)\xi = \chi_{[0,\lambda)}(x)y\chi_{[0,\lambda)}(x)\xi = \chi_{[0,\lambda)}(x)y\xi. \end{aligned}$$

However $\chi_{[0,\lambda)}(x) \to 1$ strongly as $\lambda \to \infty$ and thus $x\xi = y\xi$ for all $\xi \in rK$. It follows that $x\xi = y\xi$ for all $\xi$ in a $\tau$-dense subset of $K$. As $x$ and $y$ are

$\tau$-measurable, it follows from Proposition 2.3.2 that $x = y$. $\square$

**Proposition 4.2.4.** *Suppose $a \in \mathfrak{N}_\psi$. Then*

$$\hat{\Theta}^r(f)(D^{1/2p}a^*aD^{1/2p}) = D^{1/2p}\Theta^r(f)(a^*a)D^{1/2p}.$$

*Proof.* As $\hat{\Theta}^r(f)$ is a map $\hat{N}_+ \to \hat{N}_+$, $\Theta^r(f)(D^{1/2p}xD^{1/2p})$ is a positive, self-adjoint operator affiliated to $N$. By the right invariance of $\psi$, we have that $\Theta^r(f)(a^*a) \in \mathfrak{M}_\psi$; it follows that $D^{1/2p}\Theta^r(f)(a^*a)D^{1/2p}$ is in $L_p(M, \psi)$ and is $\tau$-measurable. For all $\eta \in \mathcal{D}_0$,

$$\left\| \left( \hat{\Theta}^r(f)(D^{1/2p}a^*aD^{1/2p}) \right)^{1/2} \eta \right\| = \left\| (D^{1/2p}\Theta^r(f)(a^*a)D^{1/2p})^{1/2}\eta \right\|.$$

The result follows from Proposition 4.2.3. $\square$

We now define the map $\Theta^r_p(f)$ as follows:

$$\Theta^r_p(f) = \hat{\Theta}^r(f)\Big|_{L_p(M)_+}.$$

Using the polarization identity, we extend by linearity to obtain a map $\Theta^r_p(f) : D^{1/2p}\mathfrak{M}_\psi D^{1/2p} \to D^{1/2p}\mathfrak{M}_\psi D^{1/2p}$ such that

$$\Theta^r_p(f)(D^{1/2p}xD^{1/2p}) = D^{1/2p}\Theta^r(f)(x)D^{1/2p}.$$

As $D^{1/2p}\mathfrak{M}_\psi D^{1/2p}$ is dense in $L_p(M, \psi)$, once we show that $\Theta^r_p(f)$ is bounded with respect to the $\|\cdot\|_p$-norm, we can extend it to a map $L_p(M, \psi) \to L_p(M, \psi)$. Using the Jordan decomposition, we can also extend by linearity to not necessarily positive $f \in L_1(\mathbb{G})$.

Throughout this chapter, we shall use the case $\alpha = 0$ of Izumi's interpolation method. In this situation, by the remark on pages 1036-8 of [16], Izumi's approach is identical to the approach taken by Terp in [36]. In this chapter $\psi_x$ will stand for $\psi_x^{(0)}$ as defined in Section 2.8 and $L = L_0$ will denote the intersection of $L_1(\mathbb{G}, \psi)$ and $M = L_\infty(\mathbb{G}, \psi)$.

As we are dealing with Terp's method of interpolation, elements in $L = L_1(\mathbb{G}) \cap L_\infty(\mathbb{G})$ can be approximated by elements in $\mathfrak{M}_\psi$. We have the following result which can be found as Theorem 8 and the comments following it in [36].

**Proposition 4.2.5.** *1. Suppose $x \in L_\infty(\mathbb{G})$ and that there exists a net $(x_i)$ in $\mathfrak{M}_\psi$ such that $x_i \to x$ $\sigma$-weakly and $\psi_{x_i}$ is a Cauchy sequence in $L_1(\mathbb{G})$. Then $x \in L$ and $\psi_x = \lim \psi_{x_i}$.*

*2. Suppose $x \in L$. Then there exists a net $(x_i)$ in $\mathfrak{M}_\psi$ such that $\sup_i \|x_i\|_L < \infty$, $x_i \to x$ $\sigma$-weakly, $\|\psi_{x_i} - \psi_x\|_1 \to 0$.*

Terp deals with spatial noncommutative $L_p$-spaces in her paper on interpolation; however, as noted in Section 7 of (the extended, unpublished version of) [13], this can be adapted to the Haagerup noncommutative $L_p$-spaces. Considered as spaces of operators, the Haagerup noncommutative $L_p$-spaces have trivial intersection. We can only speak of their intersection after embedding them into a larger Banach space. The space $L$ (which will be the intersection of $L_1(\mathbb{G})$ and $L_\infty(\mathbb{G})$ under these embeddings) is defined in the same way as for spatial noncommutative $L_p$-spaces (see Section 2.7). The spaces $L_1(\mathbb{G})$ and $L_\infty(\mathbb{G})$ are included into $L^*$ in the same way as before. The proofs from [36] can be adapted to this situation to show the following. The isometric identification between $(L_\infty(\mathbb{G}), L_1(\mathbb{G}))_{1/p}$ and $L_p(\mathbb{G})$ is given by

$$\mu_p(x) = D^{1/2p} x D^{1/2p}, \qquad x \in \mathfrak{M}_\psi.$$

We also recall that $D^{1/2p}\mathfrak{M}_\psi D^{1/2p}$ is norm dense in $L_p(\mathbb{G})$ for $1 \leq p < \infty$. We note in particular the identification between $L_\infty(\mathbb{G})_*$ and $L_1(\mathbb{G})$:

$$\varphi_x \mapsto \mu_1(x) = D^{1/2} x D^{1/2}, \qquad x \in \mathfrak{M}_\psi.$$

We shall now show the boundedness of $\Theta_1^r(f)$ by proving it is the preadjoint of the normal map $\Theta^r(f)$. We shall also make use of this proposition again later in this chapter.

**Proposition 4.2.6.** *For $f \in L_1(\mathbb{G})$, $\Theta_1^r(f)^* = \Theta^r(f \circ \kappa)$. Thus*

$$\|\Theta_1^r(f)\| = \|\Theta^r(f \circ \kappa)\| \leq \|f \circ \kappa\| = \|f\|.$$

*Proof.* We start by assuming that $f$ is a state; we later extend this result by linearity to all normal linear functionals $f \in L_1(\mathbb{G})$. Suppose that $b \in \mathfrak{M}_\psi$ is entire and that $c \in \mathfrak{N}_\psi$. By Proposition 4.2.4, we have that

$$\langle \Theta_1^r(f)(\mu_1(b)), c\rangle = \langle \mu_1(\Theta^r(f)(b)), c\rangle.$$

By Theorem 2.6.3 and Theorem 3.2.7, this is in turn equal to

$$\psi(\sigma_{i/2}(\Theta^r(f)(b))c) = \psi(\Theta^r(f)(\sigma_{i/2}(b))c).$$

Expanding and then using that the fact that $f$ is a state, we get

$$\psi\left((\iota \otimes f)\left(\Gamma(\sigma_{i/2}(b))\right)c\right) = \psi\left((\iota \otimes f)\left(\Gamma(\sigma_{i/2}(b))(c \otimes 1)\right)\right).$$

Using the strong right invariance of the right Haar weight (Definition 3.2.3), this is equal to

$$f\left((\psi \otimes \iota)\left(\Gamma(\sigma_{i/2}(b))(c \otimes 1)\right)\right) = (f \circ \kappa)\left((\psi \otimes \iota)\left((\sigma_{i/2}(b) \otimes 1)\Gamma(c)\right)\right).$$

As $\kappa$ is an anti-automorphism, we have that $f \circ \kappa \geq 0$ is a state. We then have that

$$\psi\left((\iota \otimes (f \circ \kappa))\left((\sigma_{i/2}(b) \otimes 1)\Gamma(c)\right)\right) = \psi\left(\sigma_{i/2}(b)(\iota \otimes (f \circ \kappa))\left(\Gamma(c)\right)\right).$$

Using Theorem 2.6.3 and Theorem 3.2.7 again, we have that

$$\psi\left(\sigma_{i/2}(b)\Theta^r(f \circ \kappa)(c)\right) = \langle \mu_1(b), \Theta^r(f \circ \kappa)(c)\rangle.$$

This shows that, for $b \in \mathfrak{M}_\infty$ and $c \in \mathfrak{N}_\psi$,

$$\langle \Theta_1^r(f)(\mu_1(b)), c\rangle = \langle \mu_1(b), \Theta^r(f \circ \kappa)(c)\rangle.$$

As the maps $\mu_1(b)$, $\Theta_1^r(f)(\mu_1(b))$ and $\Theta^r(f \circ \kappa)$ are normal on $L_\infty(\mathbb{G})$, and $\mathfrak{N}_\psi$ is $\sigma$-weakly dense in $L_\infty(\mathbb{G})$, it follows that

$$\langle \Theta_1^r(f)(\mu_1(b)), c\rangle = \langle \mu_1(b), \Theta^r(f \circ \kappa)(c)\rangle,$$

for all $b \in \mathfrak{M}_\infty$ and $c \in L_\infty(\mathbb{G})$.

As $\mu_1(\mathfrak{M}_\infty)$ is norm dense in $L_1(\mathbb{G})$, the result follows. □

*Remark* 4.2.7. We will use the above proposition several times. It provides a way of describing the convolution in $L_1(\mathbb{G})$ that is compatible with the inclusion of $L = L_1(\mathbb{G}) \cap L_\infty(\mathbb{G})$ into $L_1(\mathbb{G})$. Let $x \in L$ and let $\psi_x$ be the linear functional in $L_\infty(\mathbb{G})_*$ corresponding to the element $\mu_1(x)$ in the Haagerup noncommutative $L_1$-space. For $f \in L_1(\mathbb{G})$, $y \in L_\infty(\mathbb{G})$, and $x \in \mathfrak{M}_\psi$, we have

$$\begin{aligned}\langle \psi_x * f, y\rangle &= \langle m_f^r(\psi_x), y\rangle \\ &= \langle \psi_x, \Theta^r(f)(y)\rangle \\ &= \langle \mu_1(x), \Theta^r(f)(y)\rangle \\ &= \langle \mu_1(\Theta^r(f\circ\kappa)(x)), y\rangle \\ &= \langle \psi_{\Theta^r(f\circ\kappa)(x)}, y\rangle\end{aligned}$$

It is thus clear that for $x \in \mathfrak{M}_\varphi$

$$\psi_x * f = \psi_{\Theta^r_\infty(f\circ\kappa)(x)}.$$

However, by applying Proposition 4.2.5, we see that if $x \in L$, then we have $\Theta^r_\infty(f\circ\kappa)(x) \in L$ and the above equality holds for all $x \in L$.

It is now easy to prove the following result:

**Theorem 4.2.8.** *The map $\Theta^r_p$ is an injective contractive homomorphism from $L_1(\mathbb{G})$ into $B(L_p(\mathbb{G}))$.*

*Proof.* We interpolate between the two cases $p = 1$ and $p = \infty$. Let $f \in L_1(\mathbb{G})$. We know from Proposition 4.2.6 that both of these maps are contractive (and bounded by $\|f\|_1$). By Proposition 4.2.4, we know that $\Theta^r_1(f)$ and $\Theta^r(f)$ agree on $\mathfrak{M}_\psi \subseteq L = L_1(\mathbb{G}) \cap L_\infty(\mathbb{G})$, i.e., for $x \in \mathfrak{M}_\psi$,

$$\Theta^r_1(\mu_1(x)) = \mu_1(\Theta^r(f)(x)).$$

Let $x \in L$. Part 2 of Proposition 4.2.5 provides a bounded sequence $(x_i)$ in $\mathfrak{M}_\psi$ such that $\|\mu_1(x_i) - \mu_1(x)\|_1 \to 0$ and $x_i \to x$ $\sigma$-weakly. As $\Theta^r_1(f)$ is bounded and $\Theta^r(f)$ is normal, we can use part 1 of Proposition 4.2.5 to conclude that $\Theta^r(f)(x) \in L$ and that

$$\Theta^r_1(\mu_1(x)) = \mu_1(\Theta^r(f)(x)),$$

for all $x \in L$.

As the complex interpolation method is an exact interpolation functor (see [1]), we can interpolate to get a bounded map $\Theta^r_p(f) : L_p(\mathbb{G}) \to L_p(\mathbb{G})$

with norm less than $\|f\|_1$ satisfying

$$\Theta_p^r(f)\left(D^{1/2p}xD^{1/2p}\right) = D^{1/2p}\Theta^r(f)(x)D^{1/2p},$$

for all $x \in M_\psi$.

In a similar fashion, we can use the fact that $\Theta^r(f) \circ \Theta^r(g) = \Theta^r(f * g)$ to conclude that

$$\Theta_p^r(f) \circ \Theta_p^r(g) = \Theta_p^r(f * g),$$

for all $f, g \in L_1(\mathbb{G})$. Injectivity follows from the injectivity of the map $\Theta^r$. $\square$

Similarly, we have a version of this result corresponding to left convolution. There are two differences that should be noted. The map $\Theta_p^l$ satisfies

$$\Theta_p^l(f)\left(D_\varphi^{1/2p}xD_\varphi^{1/2p}\right) = D_\varphi^{1/2p}\Theta^l(f)(x)D_\varphi^{1/2p},$$

for $x \in \mathfrak{M}_\varphi$, where in general $D_\varphi \neq D = D_\psi$. Also, as $\Theta^l$ is an antihomomorphism, $\Theta_p^l$ will also be an antihomomorphism.

**Theorem 4.2.9.** *The map $\Theta_p^l$ is an injective contractive antihomomorphism from $L_1(\mathbb{G})$ into $B(L_p(\mathbb{G}))$.*

*Remark* 4.2.10. Let $M$ and $N$ be von Neumann algebras with normal faithful semifinite weights $\varphi$ and $\psi$ respectively. Consider a positive map $T : M \to N$ such that for some positive constant $C$,

$$\psi(T(x)) \leq C\varphi(x),$$

for all $x \in \mathfrak{M}_{\varphi,+}$. For $1 \leq p < \infty$, define

$$\begin{aligned} T_p : D_\varphi^{1/2p}\mathfrak{M}_\varphi D_\varphi^{1/2p} &\to D_\psi^{1/2p}\mathfrak{M}_\psi D_\psi^{1/2p}, \\ D_\varphi^{1/2p}xD_\varphi^{1/2p} &\mapsto D_\psi^{1/2p}T(x)D_\psi^{1/2p}. \end{aligned}$$

By Theorem 5.1 in [13], the map $T_p$ extends to a positive bounded map from $L_p(M,\varphi)$ into $L_p(N,\psi)$ for all $1 \leq p < \infty$.

Returning to the maps considered in this chapter, by the right invariance of $\psi$, we have

$$\psi(\Theta^r(f)(x)) = \psi((\iota \otimes f)\Gamma(x)) = \psi(x)f(1),$$

for $f \in L_1(\mathbb{G})_+$ and $x \in \mathfrak{M}_{\psi,+}$. The more general results of [13] thus apply to show that $\Theta^r_p(f)$ is bounded.

However, in the specific case considered in this chapter, we can make a stronger statement. Instead of assuming that $\Theta^r_p(f)$ should be of the form given above, we prove that $\Theta^r_p(f)$ is the correct extension of $\Theta^r(f)$ to $L_p(\mathbb{G})$ in this context. Most of the preceding work is not devoted to showing that $\Theta^r_p(f)$ is bounded; instead it is devoted to proving a result that must be taken as a hypothesis in the more general setting of [13].

We also take a different path to showing that $\Theta^r_1(f)$ is bounded on $L_1(\mathbb{G})$. The approach taken above leads to Proposition 4.2.6, which will be useful later.

*Remark* 4.2.11. We now identify the map $\Theta^r_2(f)$ acting on $L_2(\mathbb{G})$ with the right regular representation $\rho(f)$ acting on $\mathcal{H}$. This follows from Theorem 3.2.7 and Theorem 2.6.3 and arguments similar to those in Section 2.4 in [7].

First we recall that we can isometrically identify $L_2(M,\psi)$ with $\mathcal{H}$ using the map

$$T : \Lambda(x) \mapsto xD^{1/2},$$

for $x \in \mathfrak{N}_\psi$. We can then write

$$\begin{aligned}\Theta^r_2(f)T\Lambda(x) &= \Theta^r_2(f)xD^{1/2}\\ &= \Theta^r_2(f)D^{1/4}\sigma_{i/4}(x)D^{1/4} = D^{1/4}\Theta^r(f)(\sigma_{i/4}(x))D^{1/4}\\ &= D^{1/4}\sigma_{i/4}(\Theta^r(f)(x))D^{1/4} = \Theta^r(f)(x)D^{1/2}\\ &= T\Lambda(\Theta^r(f)(x)).\end{aligned}$$

We will temporarily assume that $f$ is of the form $\omega_{\Lambda(z)}$ for $z \in \mathfrak{N}_\psi$. For $x, y \in \mathfrak{N}_\psi$, we have that

$$\begin{aligned}\omega_{\Lambda(z)}\left((\psi\otimes\iota)\left(\Gamma(y^*)(x\otimes 1)\right)\right) &= (\psi\otimes\psi)\left((1\otimes z^*)\Gamma(y^*)(x\otimes 1)(1\otimes z)\right)\\ &= (\Lambda(x)\otimes\Lambda(z)\mid(\Lambda\otimes\Lambda)\left(\Gamma(y)(1\otimes z)\right))\\ &= (\Lambda(x)\otimes\Lambda(z)\mid V\left(\Lambda(y)\otimes\Lambda(z)\right))\\ &= \big(\Lambda(x)\mid(\iota\otimes\omega_{\Lambda(z)})V\Lambda(y)\big).\end{aligned}$$

Extending by linearity and continuity, we have that for all $f \in L_1(\mathbb{G})$

$$f\left((\psi\otimes\iota)\left(\Gamma(y^*)(x\otimes 1)\right)\right) = (\Lambda(x)\mid(\iota\otimes f)V\Lambda(y)).$$

Using this, we then have that for $x, y \in \mathfrak{N}_\psi$

$$\begin{aligned}(\Lambda(x) \mid \Lambda((\iota \otimes f)\Gamma(y))) &= \psi(((\iota \otimes \omega)\Gamma(y))\, x) \\ &= f((\psi \otimes \iota)(\Gamma(y^*)(x \otimes 1))) \\ &= (\Lambda(x) \mid (\iota \otimes f)V\Lambda(y)).\end{aligned}$$

Combining the above results leads to the following identication:

$$T^{-1}\Theta_2^r(f)T = \rho(f).$$

## 4.3 Decomposable maps on noncommutative $L_p$-spaces

We will now show that the maps $\Theta_p^r(f) : L_p(\mathbb{G}) \to L_p(\mathbb{G})$ are decomposable for all $f \in L_1(\mathbb{G})$. We begin by recalling some definitions from [19].

**Definition 4.3.1.** Given $C^*$-algebras $A$ and $B$, a linear map $u : A \to B$ is *decomposable* if there exist completely positive maps $v_i : A \to B$ such that the induced map

$$\Phi = \begin{bmatrix} v_1 & u \\ u^* & v_2 \end{bmatrix} : \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \in M_2(A) \mapsto \begin{bmatrix} v_1(x_1) & u(x_2) \\ u(x_3^*)^* & v_2(x_4) \end{bmatrix} \in M_2(B)$$

is completely positive.

Similarly, a bounded linear map $U : L_p(N) \to L_q(M)$ is said to be *decomposable* if there exist completely positive maps $v_i : L_p(N) \to L_q(M)$ such that the induced map

$$\Phi = \begin{bmatrix} v_1 & u \\ u^* & v_2 \end{bmatrix} : L_p(M_2 \bar{\otimes} N) \to L_q(M_2 \bar{\otimes} M)$$

is completely positive. Here the matrix order is given by the positive cones $L_p(M_n(M))_+$.

The decomposable norm is defined to be

$$\|u\|_{dec} = \inf\{\max\{\|v_1\|, \|v_2\|\}\},$$

where the infimum is taken over all decompositions of the above form.

By the polar decomposition for normal linear functionals, we can write $f = ug$, where $g \in L_1(\mathbb{G})_+$ and $u$ is a partial isometry. As $L_\infty(\mathbb{G})$ is in standard form, we can write $g = \omega_{\xi,\xi}$ for some $\xi \in \mathcal{H}$. It follows that $f = \omega_{u\xi,\xi}$. We then consider the maps

$$v_1 = \Theta_p^r(\omega_{\xi,\xi}) \qquad \text{and} \qquad v_2 = \Theta_p^r(\omega_{u\xi,u\xi}).$$

As $\Theta^r(f) = (\iota \otimes \xi^*)V(x \otimes 1)V^*(\iota \otimes u\xi)$, it follows that $\Theta_p^r(\omega_{u\xi,\xi})(x^*)^* = \Theta_p^r(\omega_{\xi,u\xi})$, (i.e., $\Theta_p^r(\omega_{\xi,u\xi}) = \Theta_p^r(f)^*$ in the sense of Definition 4.3.1).

We recall that the map $\Theta_p^r(g)$ is defined to be a restriction of the map sending $x$ to $(\iota \otimes \iota \otimes \xi^*)(\iota \otimes V)(x \otimes 1)(\iota \otimes V^*)(\iota \otimes \iota \otimes \xi)$ and thus clearly preserves the matrix order on $L_p(\mathbb{G})$. As a result, the maps $v_1$ and $v_2$ are completely positive. As commented in Section 3 of [19], every completely positive contraction $T : L_p(M) \to L_q(N)$ between noncommutative $L_p$-spaces is in fact completely contractive. Thus the maps $v_1$ and $v_2$ are completely contractive.

The map $\Phi : M_2(L_p(\mathbb{G})) \to M_2(L_p(\mathbb{G}))$ given by

$$\Phi\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right) = \begin{bmatrix} \Theta_p^r(\omega_{\xi,\xi})(x_1) & \Theta_p^r(f)(x_2) \\ \Theta_p^r(\omega_{\xi,u\xi})(x_3) & \Theta_p^r(\omega_{u\xi,u\xi})(x_4) \end{bmatrix},$$

is completely positive, as it can be written in the form

$$\begin{bmatrix} (\iota \otimes \iota \otimes \xi^*)(\iota \otimes V) \\ (\iota \otimes \iota(v\xi)^*)(\iota \otimes V) \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} (\iota \otimes V^*)(\iota \otimes \iota \otimes \xi) \\ (\iota \otimes V^*)(\iota \otimes \iota \otimes (v\xi)) \end{bmatrix}.$$

It is also then immediate that $\left\|\Theta_p^r(f)\right\|_{dec} \leq \|f\|_1$.

## 4.4 A completely contractive representation of $L_1(\mathbb{G})$ on $L_p(\mathbb{G})$

Suppose that $G$ is a locally compact group with left Haar measure $ds$. We have a left action of $L_1(G)$ on $L_1(G)$ given by left multiplication. For $f, g \in L_1(G)$, we have

$$(\lambda(f)g)(t) = (f * g)(t) = \int f(s)g(s^{-1}t)\, ds.$$

This can be dualized to give a right action of $L_1(G)$ on $L_\infty(G) \simeq L_\infty(G)^{op}$, which yields for $f \in L_1(G)$ and $h \in L_\infty(G)$

$$(h \cdot f)(t) = \int f(s)h(st)\, ds.$$

However these two actions do not coincide on $L_1(G) \cap L_\infty(G)$. Hence we need to consider

$$(h \cdot (\kappa_*(f)))(t) = \int f(s)h(s^{-1}t)\, ds.$$

Bearing this in mind, we see that Proposition 4.2.6 will be quite useful in this situation. We can now interpolate between these two actions to get an action of $L_1(G)$ on $L_p(G)$. The approach taken below is inspired by the approach used in the group case by Daws in [5].

As explained in Section 2.9, it is necessary to distinguish between the predual $L_\infty(\mathbb{G})_*$ and the noncommutative $L_p$-space $L_1(\mathbb{G}) = (L_\infty(\mathbb{G})_*)^{op}$. The antipode $\kappa$ allows us to identify $L_\infty(\mathbb{G})$ with $L_\infty(\mathbb{G})^{op}$ (as operator spaces). Thus the map $\kappa_*$ provides us the identification

$$\begin{aligned} \kappa_* : L_\infty(G)_* &\to L_1(\mathbb{G}) \\ \omega &\mapsto \omega \circ \kappa. \end{aligned}$$

We also have the following identification of $L_\infty(\mathbb{G})^{op}$ with $L_\infty(\mathbb{G})'$:

$$\begin{aligned} \Phi : L_\infty(G)^{op} &\to L_\infty(G)' \\ x &\mapsto Jx^*J. \end{aligned}$$

Using these, the following elements in $L_\infty(\mathbb{G})_*$ and $L_\infty(\mathbb{G})^{op}$ correspond. Let $x$ be an element in $L = L_\infty(\mathbb{G}) \cap L_\infty(\mathbb{G})_*$. Then $x$ corresponds to the element $\varphi_x \in L_\infty(\mathbb{G})_*$ (where $\varphi_x$ is as in Terp's interpolation method or $\varphi_x^{(0)}$ in Izumi's interpolation method). Using the map $\Phi$, we can consider the pair $(L_\infty(\mathbb{G})', L_\infty(\mathbb{G})_*)$ to be a compatible couple in the sense of interpolation theory. We are then identifying $\Phi(x) \in L_\infty(\mathbb{G})'$ with $\varphi_x \in L_\infty(\mathbb{G})_*$, for each $x \in L$.

The multiplication in $L_\infty(\mathbb{G})_*$ gives us a completely contractive left action of $L_\infty(\mathbb{G})_*$ on $L_\infty(\mathbb{G})_*$ (by convolution on the left). For $x \in \mathfrak{M}_\varphi$, this action sends $\varphi_x$ to $\omega * \varphi_x$.

If we dualize the completely contractive left action of $L_\infty(\mathbb{G})_*$ on $L_\infty(\mathbb{G})_*$, we get a completely contractive right action of $L_\infty(\mathbb{G})_*$ on $L_\infty(\mathbb{G})$. For $\omega, \omega' \in L_\infty(\mathbb{G})_*$ and $x \in L_\infty(\mathbb{G})$, we have

$$\langle \omega', x \cdot \omega \rangle = \langle \omega * \omega', x \rangle.$$

We use this right action to define a left action of $L_\infty(\mathbb{G})_*$ on $L_\infty(\mathbb{G})' \simeq L_\infty(\mathbb{G})^{op}$. Let $\omega \in L_\infty(\mathbb{G})_*$ and let $x \in L_\infty(\mathbb{G})$. We then define

$$\omega \cdot \Phi(x) = \Phi(x \cdot \kappa_*(\omega)).$$

This will be a left action as

$$\kappa_*(\omega_1) * \kappa_*(\omega_2) = \kappa_*(\omega_2 * \omega_1).$$

We now verify that this left action is completely contractive. Suppose that $[\omega_{ij}] \in M_n(L_\infty(\mathbb{G})_*)$ and $[x_{kl}] \in M_m(L_\infty(\mathbb{G}))$. We then have that

$$\|[\Phi(x_{kl})]\| = \|[x_{lk}]\|,$$

and also that

$$\begin{aligned}
\|[\omega_{ij} \cdot \Phi(x_{kl})]\| &= \|[\Phi(x_{kl} \cdot (\omega_{ij} \circ \kappa))]\| \\
&= \|[x_{lk} \cdot (\omega_{ji} \circ \kappa)]\| \\
&\leq \|[x_{lk}]\| \, \|[\omega_{ji} \circ \kappa]\| \\
&= \|[\Phi(x_{kl})]\| \, \|[\omega_{ij}]\|.
\end{aligned}$$

It remains to show that these two actions coincide on $L_\infty(\mathbb{G})_* \cap L_\infty(\mathbb{G})'$.

For $x \in \mathfrak{M}_\varphi$, the action of $\omega$ on $\varphi_x$ is as follows. Let $y \in L_\infty(\mathbb{G})$.

$$\begin{aligned}
\langle \omega * \varphi_x, y \rangle &= \langle \omega \otimes \varphi_x, \Gamma(y) \rangle \\
&= \langle \varphi_x, (\omega \otimes \iota)\, \Gamma(y) \rangle \\
&= \langle \varphi_x, \Theta^l(\omega)(y) \rangle \\
&= \langle \Theta^l(\omega)_*(\varphi_x), y \rangle \\
&= \langle \varphi_{\Theta^l(\omega \circ \kappa)(x)}, y \rangle
\end{aligned}$$

Here $\Theta^l(\omega\circ\kappa)(x)$ is in $\mathfrak{M}_\varphi$ by Lemma 3.2.6. It is thus valid to write $\varphi_{\Theta^l(\omega\circ\kappa)(x)}$ and it then follows from Proposition 4.2.6 that

$$\Theta^l(\omega)_*(\varphi_x) = \varphi_{\Theta^l(\omega\circ\kappa)}.$$

We also have that for $\omega, \omega_0 \in L_\infty(\mathbb{G})_*$ and $x \in L_\infty(\mathbb{G})$,

$$\begin{aligned} \langle \omega_0, x \cdot (\omega \circ \kappa) \rangle &= \langle (\omega \circ \kappa) * \omega_0, x \rangle \\ &= \langle (\omega \circ \kappa) \otimes \omega_0, \Gamma(x) \rangle \\ &= \langle \omega_0, ((\omega \circ \kappa) \otimes \iota)\Gamma(x) \rangle \\ &= \langle \omega_0, \Theta^l(\omega \circ \kappa)(x) \rangle \end{aligned}$$

We then have the following commutative diagram for $x \in \mathfrak{M}_\varphi$:

$$\begin{array}{ccc} & \Phi(x) \longrightarrow & \omega \cdot \Phi(x) = \Phi\left(x \cdot (\omega \circ \kappa)\right) = \Phi\left(\Theta^l(\omega \circ \kappa)(x)\right) \\ \nearrow & & \updownarrow \\ x & & \\ \searrow & & \\ & \varphi_x \longrightarrow & \omega * \varphi_x = \varphi_{\Theta^l(\omega\circ\kappa)(x)} \end{array}$$

and the left actions on $L_\infty(\mathbb{G})'$ and $L_\infty(\mathbb{G})_*$ are completely contractive.

Given $x \in L$, it can be approximated by a net $(x_i)$ in $\mathfrak{M}_\varphi$ satisfying the conditions in Proposition 4.2.5. It follows that the above diagram commutes for all $x \in L = L_\infty(\mathbb{G})_* \cap L_\infty(\mathbb{G})^{op}$. We also have that $(L_\infty(\mathbb{G})^{op}, L_\infty(\mathbb{G})_*)_{1/p} = L_p(\mathbb{G})^{op}$. Interpolating yields the following result.

**Proposition 4.4.1.** *There is a completely contractive representation of $L_\infty(\mathbb{G})_*$ on $L_p(\mathbb{G})^{op}$ obtained by interpolating between the left action of $L_\infty(\mathbb{G})_*$ on $L_\infty(\mathbb{G})_*$*

$$\omega \cdot \omega_0 = \omega * \omega_0, \qquad \omega, \omega_0 \in L_\infty(\mathbb{G})_*,$$

*and the left action of $L_\infty(\mathbb{G})_*$ on $L_\infty(\mathbb{G})' \simeq L_\infty(\mathbb{G})^{op}$*

$$\omega \cdot \Phi(x) = \Phi(x \cdot (\kappa_*(\omega))) = \Phi\left(\Theta^l(\omega \circ \kappa)(x)\right), \qquad \omega \in L_\infty(\mathbb{G})_*, x \in L_\infty(\mathbb{G}).$$

The same approach gives the following representation of $L_1(\mathbb{G})$ by switching which noncommutative $L_p$-spaces are given their opposite operator space structure. Let $\varphi'(x) = \varphi(JxJ)$ for $x \in L_\infty(\mathbb{G})$. We note that $(L_\infty(\mathbb{G})')_* \simeq L_1(\mathbb{G})$ and view $(L_\infty(\mathbb{G}), L_1(\mathbb{G}))$ as a compatible pair by identifying $\Phi^{-1}(x') \in$

$L_\infty(\mathbb{G})$ with $\varphi'_{x'}$ for each $x' \in L' = L_\infty(\mathbb{G})' \cap (L_\infty(\mathbb{G})')_*$. We then have the following commutative diagram of completely contractive actions for $x' \in L'$ and $\omega' \in (L_\infty(\mathbb{G})')_* \simeq L_1(\mathbb{G})$:

$$\begin{array}{ccccc} & \nearrow & \Phi^{-1}(x') & \longrightarrow & \omega' \cdot \Phi^{-1}(x') = \Phi^{-1}\left(x' \cdot (\omega' \circ \kappa')\right) = \Phi^{-1}\left(\Theta^{l\prime}(\omega' \circ \kappa')(x')\right) \\ x' & & & & \updownarrow \\ & \searrow & \varphi'_{x'} & \longrightarrow & \omega' * \varphi'_{x'} = \varphi'_{\Theta^{l\prime}(\omega' \circ \kappa')(x')} \end{array}$$

**Proposition 4.4.2.** *There is a completely contractive representation of $L_1(\mathbb{G})$ on $L_p(\mathbb{G})$ obtained by interpolating between the left action of $L_1(\mathbb{G})$ on $L_1(\mathbb{G})$*

$$\omega' \cdot \omega'_0 = \omega' * \omega'_0, \qquad \omega', \omega'_0 \in L_1(\mathbb{G}) \simeq (L_\infty(\mathbb{G})')_*,$$

*and the left action of $L_\infty(\mathbb{G})_*$ on $L_\infty(\mathbb{G})' \simeq L_\infty(\mathbb{G})^{op}$*

$$\omega' \cdot \Phi^{-1}(x') = \Phi^{-1}\left(x' \cdot (\omega' \circ \kappa')\right) = \Phi^{-1}\left(\Theta^{l\prime}(\omega' \circ \kappa')(x')\right),$$

*for $\omega' \in L_1(\mathbb{G})$, and $x' \in L_\infty(\mathbb{G})'$.*

*Remark* 4.4.3. Let $G$ be a locally compact group and let $A(G) = VN(G)_*$ be its Fourier algebra. Let $L_p(\hat{G})$ be the noncommutative $L_p$-space $L_p(VN(G))$. In [5], Daws extends the action of $A(G)$ to an action of the completely bounded multiplier algebra $M_{cb}(A(G))$ on $L_p(\hat{G})$. This is then used to find an completely isometric representation of $M_{cb}(A(G))$ on a direct sum of noncommutative $L_p$-spaces.

We seek to extend this result to Kac algebras but have not yet done so. We briefly indicate the main difficulty with extending Daws' argument to Kac algebras.

Given $\omega \in L_\infty(\mathbb{G})_*$, its left action on $L_\infty(\mathbb{G})_*$ is given by

$$\omega \cdot \omega_0 = (\Theta^l_\infty(\omega))_*(\omega_0),$$

and its left action on $L_\infty(\mathbb{G})'$ is given by

$$\omega \cdot \Phi(x) = \Phi(\Theta^l_\infty(\omega \circ \kappa)(x)).$$

The left action of $\omega$ is defined in terms of the left action of $\omega \circ \kappa$. However, if we are considering more general left multipliers, then what is the analogue of the left action of $\omega \circ \kappa$? In defining this left action of $\omega$, we have in fact used the natural right action of $\omega$ on $L_\infty(\mathbb{G})$. More precisely,

$$\begin{aligned}\Theta^l(\omega \circ \kappa)(x) &= ((\omega \circ \kappa) \otimes \iota)\Gamma(x) \\ &= (\iota \otimes (\omega \circ \kappa))(\kappa \circ \kappa)\Gamma(\kappa(x)) \\ &= \kappa((\iota \otimes \omega)\Gamma(\kappa(x))) \\ &= \kappa(\Theta^r(\omega)(\kappa(x)))\end{aligned}$$

and thus we have that $\Theta^l(\omega \circ \kappa) = \kappa\Theta^r(\omega)\kappa$. Our definition for the left action of $\omega$ is to conjugate a right action of $\omega$ by the antipode.

If we are working with a group $G$, then the left and right multipliers of $A(G)$ coincide and thus there is again a natural right action of any multiplier which can be used to define the left action of that multiplier. It thus appears that one way of generalizing these multiplier results from groups to Kac algebras would be to associate a right multiplier to each left multiplier in such a way that the calculations above can be extended.

## 4.5 Approximation properties for discrete Kac algebras

Let $\mathbb{G}$ be a discrete Kac algebra. We will use the notation $A(\mathbb{G})$ for $L_\infty(\hat{\mathbb{G}})_*$ to suggest its role as the Fourier algebra of the Kac algebra $\mathbb{G}$. We then have the identification $\kappa_*(A(\mathbb{G})) = L_1(\hat{\mathbb{G}})$. We note that $\hat{\mathbb{G}}$ will be a compact Kac algebra whose Haar weight $\tau$ is a tracial state. We can thus apply Theorem 2.8.6 when convenient.

We refer to Sections 3.3 and 3.4 for the definitions and notation that will be used in this section. We will extend Theorem 3.4.3 to discrete Kac algebras. The approximation property will provide a pair of nets of maps on $A(\mathbb{G})$ and $L_\infty(\hat{\mathbb{G}})$ satisfying certain properties. We will then interpolate between these pairs of maps to reach our conclusion. The results in this section rely heavily on the approaches taken in [18] and [24]. Some proofs will be repeated so as to indicate why the results hold but the reader will be directed to the aforementioned papers for other proofs. For readers already familiar with [18], we note the two differences from the approach taken there.

Firstly, the nets of maps are no longer directly provided by the definition of the approximation property; we instead taken a pair of nets of maps provided by Theorem 4.5.4. Secondly, we use Proposition 4.2.6 to show that these pairs of maps are compatible in the sense of interpolation theory.

Key to the proof is a characterization of the elements of $Q^l(\mathbb{G})$, the predual of $M_0^l(A(\mathbb{G}))$. We fix notation for the following linear functionals on $M_0^l(A(\mathbb{G}))$. We define

$$\langle a, \omega_{x,\Omega}\rangle = \langle \left(M_a^l \otimes id_{M_\infty}\right)(x), \Omega\rangle = \langle x, \left(m_a^l \otimes id_{T_\infty}\right)\Omega\rangle,$$

where $a \in M_0^l(A(\mathbb{G}))$ and the pair $(x, \Omega)$ is of one of the following forms:

$$\begin{aligned} x &\in C_\lambda^*(\hat{\mathbb{G}})\check{\otimes} K_\infty \text{ and } \Omega \in (L_\infty(\hat{\mathbb{G}})\bar{\otimes} M_\infty)_*, \\ x &\in (A(\mathbb{G})\check{\otimes} K_\infty)^* \text{ and } \Omega \in A(\mathbb{G})\check{\otimes} K_\infty, \\ x &\in C_\lambda^*(\hat{\mathbb{G}})\check{\otimes} K_\infty \text{ and } \Omega \in (C_\lambda^*(\hat{\mathbb{G}})\check{\otimes} K_\infty)^*. \end{aligned}$$

Theorem 3.3 from [24] provides the first description below of the predual $Q^l(\mathbb{G})$ of $M_0^l A(\mathbb{G})$. The second equality follows from this description and the proof of Proposition 2.1 from [18] (where it is stated for the group case).

**Theorem 4.5.1.** *Let $\mathbb{G}$ be a Kac algebra. Then the predual $Q^l(\mathbb{G})$ of $M_0^l(A(\mathbb{G}))$ is equal to*

$$\begin{aligned} Q^l(\mathbb{G}) &= \left\{\omega_{x,\Omega} \,:\, x \in C_\lambda^*(\hat{\mathbb{G}})\check{\otimes} K_\infty \textit{ and } \Omega \in (L_\infty(\hat{\mathbb{G}})\bar{\otimes} M_\infty)_*\right\} \\ &= \left\{\omega_{x,\Omega} \,:\, x \in (A(\mathbb{G})\check{\otimes} K_\infty)^* \textit{ and } \Omega \in A(\mathbb{G})\check{\otimes} K_\infty\right\}. \end{aligned}$$

Suppose now that $\mathbb{G}$ is a discrete Kac algebra. Proposition 5.2 in [24] shows that $\omega_{x,\Omega} \in Q^l(\mathbb{G})$ whenever $x \in C_\lambda^*(\hat{\mathbb{G}})\check{\otimes} B(H)$ and $\Omega \in (C_\lambda^*(\hat{\mathbb{G}})\check{\otimes} B(H))^*$. The same argument proves the following result:

**Theorem 4.5.2.** *Suppose that $\mathbb{G}$ is a discrete Kac algebra. Then $\omega_{x,\Omega} \in Q^l(\mathbb{G})$ whenever $x \in C_\lambda^*(\hat{\mathbb{G}})\check{\otimes} K_\infty$ and $\Omega \in (C_\lambda^*(\hat{\mathbb{G}})\check{\otimes} K_\infty)^*$.*

*Remark* 4.5.3. In [18], the set of finitely supported functions in $A(G)$ is denoted by $A_c(G)$, where $G$ is a discrete group. In the discrete Kac algebra case, its analogue will be denoted by $A_c(\mathbb{G})$. For a discussion of the structure of discrete Kac algebras and a full description of $A_c(\mathbb{G})$, see Section 4 of [24].

We will use Proposition 5.11 in [24] to characterize the elements in $A_c(\mathbb{G})$.

$$\begin{aligned} A_c(\mathbb{G}) &= \{\hat{\omega} \in A(\mathbb{G}) \,:\, \Theta^l_\infty(\hat{\omega}) \in F(L_\infty(\hat{\mathbb{G}}))\} \\ &= \{\hat{\omega} \in A(\mathbb{G}) \,:\, \Theta^l_\infty(\hat{\omega}) \in F(C^*_\lambda(\hat{\mathbb{G}}))\}, \end{aligned}$$

where $F(V)$ denotes the finite rank maps on a Banach space $V$. We will also use that $A_c(\mathbb{G})$ is norm dense in $A(\mathbb{G})$.

The definition of the approximation property for a Kac algebra provides a left stable weak* approximate identity for $A(\mathbb{G})$. The following result allows us to choose an approximate identity $\{\hat{\omega}_\alpha\}$ that is in the centre of $A(\mathbb{G})$, thus easing the transition from the commutative case to the noncommutative case.

**Lemma 4.5.4.** *If $\mathbb{G}$ is a discrete Kac algebra with the AP, then there exists a net $\hat{\omega}_\alpha$ that lies in the intersection of $A_c(\mathbb{G})$ and the centre of $A(\mathbb{G})$ such that $\hat{\Theta}^l_1(\hat{\omega}_\alpha) \to id_{A(\mathbb{G})}$ in the stable point-norm topology.*

*Proof.* Theorem 5.15 in [24] provides a left stable weak* approximate identity $\{\hat{\omega}_\alpha\}$ for $A(\mathbb{G})$ that lies in the intersection of $A_c(\mathbb{G})$ and the centre of $A(\mathbb{G})$. By Corollary 3.4 in [24], we then have that $\hat{\Theta}^l_1(\hat{\omega}_\alpha) \to id_{A(\mathbb{G})}$ in the $\sigma(M^l_0 A(\mathbb{G}), Q^l(\mathbb{G}))$-topology. By Theorem 4.5.1, the net $\hat{\Theta}^l_1(\hat{\omega}_\alpha) \to id_{A(\mathbb{G})}$ in the stable point-weak topology. By a standard convexity argument, we can then find a net $\hat{\omega}'_\alpha$ such that $\hat{\Theta}^l_1(\hat{\omega}_\alpha) \to id_{A(\mathbb{G})}$ converges to $1_{A(\mathbb{G})}$ in the stable point-norm topology. □

We will also need the following modification of the above lemma.

**Lemma 4.5.5.** *Suppose that $\mathbb{G}$ is a weakly amenable discrete Kac algebra. Then there is an $L > 0$ and a net $\{\hat{\omega}_i\}$ in the intersection of $A_c(\mathbb{G})$ and the centre of $A(\mathbb{G})$ such that*

$$\sup_i \left\| \hat{\Theta}^l_1(\hat{\omega}_i) \right\|_{M^l_0 A(\mathbb{G})} \leq L,$$

*and $\hat{\Theta}^l_1(\hat{\omega}_i) \to id_{A(\mathbb{G})}$ in the stable point-norm topology.*

*Proof.* Suppose that $\mathbb{G}$ is a weakly amenable discrete Kac algebra. Then $A(\mathbb{G})$ has a left approximate identity $\{\hat{\omega}_i\}$ such that, for some $L > 0$,

$$\sup \left\| \hat{\Theta}^l_1(\hat{\omega}_i) \right\|_{M^l_0 A(\mathbb{G})} \leq L.$$

As the net is bounded, $\hat{\Theta}_1^l(\hat{\omega}_i)\otimes id_\infty \to id_{A(\mathbb{G})\check{\otimes}K_\infty}$ in the point-norm topology on $A(\mathbb{G})\check{\otimes}K_\infty$. Let

$$C = \left\{\hat{\Theta}_1^l(\hat{\omega}) \,:\, \hat{\omega} \in A(\mathbb{G}) \text{ and } \left\|\hat{\Theta}_1^l(\hat{\omega})\right\|_{M_0^l A(\mathbb{G})} \leq L\right\}.$$

By Theorem 4.5.1, we see that $id_{A(\mathbb{G})}$ is in the $\sigma(M_0^l A(\mathbb{G}), Q^l(\mathbb{G}))$-closure of $C$. As $A_c(\mathbb{G})$ is $\|\cdot\|_{A(\mathbb{G})}$-dense in $A(\mathbb{G})$ and $\left\|\hat{\Theta}_1^l(\hat{\omega})\right\|_{M_0^l A(\mathbb{G})} \leq \|\hat{\omega}\|_{A(\mathbb{G})}$, it follows that $\{\hat{\Theta}_1^l(\hat{\omega}) \,:\, \hat{\omega} \in A(\mathbb{G})\}$ is in the $\|\cdot\|_{M_0^l A(\mathbb{G})}$-closure of $\{\hat{\Theta}_1^l(\hat{\omega}) \,:\, \hat{\omega} \in A_c(\mathbb{G})\}$. A diagonalization argument yields that $id_{A(\mathbb{G})}$ is in the $\sigma(M_0^l A(\mathbb{G}), Q^l(\mathbb{G}))$-closure of $C \cap \{\hat{\Theta}_1^l(\hat{\omega}) \,:\, \hat{\omega} \in A_c(\mathbb{G})\}$.

Let $\{\hat{\omega}_i'\}$ be a net in $C \cap A_c(\mathbb{G})$ such that $\hat{\Theta}_1^l(\omega_i') \to id_{A(\mathbb{G})}$ in the $\sigma(M_0^l A(\mathbb{G}), Q^l(\mathbb{G}))$-topology. For each $i$, set $T_i = \Theta_\infty^l(\hat{\omega}_i')$. Let $E$ denote the unique faithful normal $\tau\otimes\tau$-invariant conditional expectation from $L_\infty(\hat{\mathbb{G}})\bar{\otimes}L_\infty(\hat{\mathbb{G}})$ onto $\hat{\Gamma}(L_\infty(\hat{\mathbb{G}}))$. Then by Theorem 5.5 in [24], there exists a unique $\hat{\omega}_{T_i} \in A(\mathbb{G})$ such that

$$\hat{\Theta}_\infty^l(\hat{\omega}_{T_i}) = \hat{\kappa} \circ \hat{\Gamma}^{-1} \circ E \circ (T_i \otimes id_{L_\infty(\hat{\mathbb{G}})}) \circ \hat{\Gamma} \circ \hat{\kappa}. \tag{4.1}$$

The proof of Theorem 5.15 in [24] shows that $\hat{\omega}_{T_i}$ is in the intersection of $A_c(\mathbb{G})$ and the centre of $A(\mathbb{G})$. As $\hat{\kappa}, \hat{\Gamma}^{-1}, E$, and $\hat{\Gamma}$ are complete contractions, (4.1) shows that

$$\hat{\Theta}_\infty^l(\hat{\omega}_{T_i}) \leq \|T_i\|_{cb} \leq \left\|\hat{\Theta}_\infty^l(\hat{\omega}_i')\right\|_{M_0^l A(\mathbb{G})} \leq L.$$

It follows from the proof of Theorem 5.13 of [24] that $\{\hat{\omega}_{T_i}\}$ is a left stable weak* approximate identity for $A(\mathbb{G})$. By Proposition 5.3 in [24], we have that $\hat{\Theta}_1^l(\hat{\omega}_{T_i}) \to id_{A(\mathbb{G})}$ in the $\sigma(M_0^l A(\mathbb{G}), Q^l(\mathbb{G}))$-topology and thus we can find a net in the convex hull of $\{\hat{\Theta}_1^l(\hat{\omega}_{T_i})\}$ that converges to $id_{A(\mathbb{G})}$ in the stable point-norm topology. □

The proof below is basically that of Proposition 2.2 in [18], where discrete groups are discussed. We do however choose the net $\hat{\omega}_\alpha$ to be in the centre of the Fourier algebra by using Lemma 4.5.4.

**Proposition 4.5.6.** *Let $\mathbb{G}$ be a discrete Kac algebra with the AP. Then there exists a net $\{\hat{\omega}_\alpha\}$in $A_c(\mathbb{G}) \cap Z(A(\mathbb{G}))$ such that $\hat{\Theta}_1^l(\hat{\omega}_\alpha) \to id_{A(\mathbb{G})}$ in the stable point-norm topology on $A(\mathbb{G})$ and $\hat{\Theta}_\infty^l(\hat{\omega}_\alpha) \to id_{C_\lambda^*(\mathbb{G})}$ in the stable point-norm* *topology on* $C^*_\lambda(\mathbb{G})$.

*Proof.* Suppose that $f \in A(\mathbb{G})\check{\otimes}K_\infty$, $a \in C^*_\lambda(\mathbb{G})\check{\otimes}K_\infty$, and $\varepsilon > 0$. (It suffices to show we can find an approximation for a single pair $(f, a)$ by the remarks at the beginning of Section 11.2 in [6].)

Let $F = \{f_1, \ldots, f_n\}$ be a nonempty finite subset of $(C^*_\lambda(\hat{\mathbb{G}})\check{\otimes}K_\infty)^*$. By Theorem 4.5.2, we have that $\omega_{a,f_i} \in Q^l(\mathbb{G})$. By Theorem 4.5.1, there exist $\tilde{a}_i \in L_\infty(\hat{\mathbb{G}})\widehat{\otimes}T_\infty$ and $\tilde{f}_i \in A(\mathbb{G})\check{\otimes}K_\infty$ such that $\omega_{a,f_i} = \omega_{\tilde{a}_i,\tilde{f}_i}$.

As $\mathbb{G}$ has the approximation property, by Lemma 4.5.4 there exists $\hat{\omega}_F \in A_C(\mathbb{G}) \cap Z(A(\mathbb{G}))$ such that

$$\begin{aligned}\left\|(\hat{\Theta}^l_1(\hat{\omega}_F) \otimes id_\infty)(f) - f\right\| &< \varepsilon\\ \left\|(\hat{\Theta}^l_1(\hat{\omega}_F) \otimes id_\infty)(\tilde{f}_i) - \tilde{f}_i\right\| &< \frac{1}{n(K+1)},\end{aligned}$$

where $K = \max \|\tilde{a}_i\|$. We then have that

$$\begin{aligned}\left|\langle(\hat{\Theta}^l_1(\hat{\omega}_F) \otimes id_\infty)(a) - a, f_i\rangle\right| &= |\omega_{a,f_i}(\hat{\omega}_F - 1)| = \left|\omega_{\tilde{a}_i,\tilde{f}_i}(\hat{\omega}_F - 1)\right|\\ &= \left|\langle\tilde{a}_i, (\hat{\Theta}^l_1(\hat{\omega}_F) \otimes id_\infty)(\tilde{f}_i) - \tilde{f}_i\rangle\right| < \frac{1}{n}.\end{aligned}$$

We then have a net $\{\hat{\omega}_F\}$ of elements in $A_c(\mathbb{G})$ indexed by finite subsets of $(C^*_\lambda(\mathbb{G})\check{\otimes}K_\infty)^*$, such that

$$\left\|(\hat{\Theta}^l_1(\hat{\omega}_F) \otimes id_\infty)(f) - f\right\| < \varepsilon,$$

for all $\hat{\omega}_F$ and $\hat{\Theta}^l_\infty(\hat{\omega}_F) \to id_{C^*_\lambda(\hat{\mathbb{G}})}$ in the stable point-weak topology. By a standard convexity argument, we may find an element $\hat{\omega} \in A_c(\mathbb{G})$ such that

$$\left\|(\hat{\Theta}^l_1(\hat{\omega}) \otimes id_\infty)(f) - f\right\| < \varepsilon \qquad \text{and} \qquad \left\|(\hat{\Theta}^l_\infty(\hat{\omega}) \otimes id_\infty)(a) - a\right\| < \varepsilon.$$

□

In the weakly amenable case we have the following result, which corresponds to Proposition 2.3 in [18].

**Proposition 4.5.7.** *Suppose that* $\mathbb{G}$ *is a weakly amenable discrete Kac algebra. Then there exists a net* $\{\hat{\omega}_\alpha\}$ *in* $A_c(\mathbb{G})$ *such that*

$$\left\|\hat{\Theta}^l_1(\hat{\omega}_\alpha)\right\|_{cb} = \left\|\hat{\Theta}^l_\infty(\hat{\omega}_\alpha)\right\|_{cb} \leq \Lambda(\mathbb{G}),$$

*and* $\hat{\Theta}_1^l(\hat{\omega}_\alpha) \to id_{A(\mathbb{G})}$ *and* $\hat{\Theta}_\infty^l(\hat{\omega}_\alpha) \to id_{C_\lambda^*(\hat{\mathbb{G}})}$ *in the point-norm topologies on* $A(\mathbb{G})$ *and* $C_\lambda^*(\hat{\mathbb{G}})$ *respectively.*

*Proof.* Fix arbitrary $\{f^1, \dots, f^k\} \in A(\mathbb{G})$ and $\{a^1, \dots, a^l\} \in C_\lambda^*(\mathbb{G})$. If $F = \{f_1, \dots, f_n\}$ is a finite subset of $C_\lambda^*(\mathbb{G})^*$, then $\omega_{a^j, f_i} \in Q^l(\mathbb{G})$ by Theorem 4.5.2. By Theorem 4.5.1, there exists $\tilde{a}_i^j \in \hat{L}_\infty(\mathbb{G})\widehat{\otimes}T_\infty$ and $\tilde{f}_i^j \in A(\mathbb{G})\check{\otimes}K_\infty$ such that $\omega_{a^j, f_i} = \omega_{\tilde{a}_i^j, \tilde{f}_i^j}$.

By Lemma 4.5.5, for any $\varepsilon > 0$, there exists a net (indexed by $F$) of elements $\hat{\omega}_F$ in $A_c(\mathbb{G})$ such that $\left\|\hat{\Theta}_1^l(\hat{\omega}_F)\right\|_{cb} \leq \Lambda(\mathbb{G}) < \infty$, and

$$\begin{aligned} \left\|\Theta_1^l(\hat{\omega}_F)(f^j) - f^j\right\| &< \varepsilon \\ \left\|(\Theta_1^l(\hat{\omega}_F) \otimes id_\infty)(\tilde{f}_i^j) - \tilde{f}_i^j\right\| &< \frac{1}{n(K+1)}, \end{aligned}$$

where $K = \max \left\|\tilde{a}_i^j\right\|$.

Then

$$\begin{aligned} \left|\langle(\hat{\Theta}_\infty^l(\hat{\omega}_F) \otimes id_\infty)(a^j) - a^j, f_i\rangle\right| &= \left|\omega_{a^j, f_i}(\hat{\omega}_F - 1)\right| = \left|\omega_{\tilde{a}_i^j, \tilde{f}_i^j}(\hat{\omega}_F - 1)\right| \\ &= \left|\langle \tilde{a}_i^j, (\Theta_1^l(\hat{\omega}_F) \otimes id_\infty)(\tilde{f}_i^j) - \tilde{f}_i^j\rangle\right| < \frac{1}{n}. \end{aligned}$$

There then exists $\hat{\omega}$ in the convex hull of $\{\hat{\omega}_F\}$ for which

$$\begin{aligned} \left\|\Theta_1^l(\hat{\omega})\right\|_{cb} = \left\|\hat{\Theta}_\infty^l(\hat{\omega})\right\|_{cb} &\leq \Lambda(\mathbb{G}) \\ \left\|\Theta_1^l(\hat{\omega})(f) - f\right\| < \varepsilon \text{ and } \left\|\hat{\Theta}_\infty^l(\hat{\omega})(a) - a\right\| &< \varepsilon. \end{aligned}$$

□

We now modify these maps to obtain maps that are compatible with the inclusions of our interpolation method.

**Proposition 4.5.8.** *If* $\mathbb{G}$ *is a discrete Kac algebra with the AP, then there exists a net* $\{\hat{\omega}_\alpha\}$ *that lies in the intersection of* $A_c(\mathbb{G})$ *and the centre of* $A(\mathbb{G})$ *such that* $\hat{\Theta}_1^l(\hat{\omega}_\alpha \circ \hat{\kappa}) \to id_{L_1(\hat{\mathbb{G}})}$ *and* $\hat{\Theta}_\infty^l(\hat{\omega}_\alpha) \to id_{C_\lambda^*(\mathbb{G})}$ *in their stable point-norm topologies.*

*If* $\mathbb{G}$ *is a weakly amenable discrete Kac algebra, then there exists a net* $\{\hat{\omega}_\alpha\}$ *that lies in the intersection of* $A_c(\mathbb{G})$ *and the centre of* $A(\mathbb{G})$ *such that* $\hat{\Theta}_1^l(\hat{\omega}_\alpha \circ \hat{\kappa}) \to id_{L_1(\mathbb{G})}$ *and* $\hat{\Theta}_\infty^l(\hat{\omega}_\alpha) \to id_{C_\lambda^*(\hat{\mathbb{G}})}$ *in their point-norm topologies.*

*These maps satisfy*

$$\left\|\hat{\Theta}_1^l(\hat{\omega}_\alpha \circ \hat{\kappa})\right\|_{cb} = \left\|\hat{\Theta}_\infty^l(\hat{\omega}_\alpha)\right\|_{cb} \leq \Lambda(\mathbb{G}).$$

*Given $\omega \in Z(A(\mathbb{G}))$, the pair $(\hat{\Theta}_\infty^l(\hat{\omega}), \hat{\Theta}_1^l(\hat{\omega} \circ \hat{\kappa}))$ is a compatible pair of completely bounded maps on the compatible couple $(C_\lambda^*(\mathbb{G}), L_1(\hat{\mathbb{G}}))$.*

*Proof.* As $\hat{\omega}_\alpha$ is in the centre of $A(\mathbb{G})$, it follows that

$$\hat{\Theta}_1^l(\hat{\omega}_\alpha \circ \kappa) = \kappa_* \hat{\Theta}_1^r(\hat{\omega}_\alpha)\kappa_* = \kappa_* \hat{\Theta}_1^l(\hat{\omega}_\alpha)\kappa_*.$$

As $\kappa_*(A((G)) = L_1(\hat{\mathbb{G}})$, the first two statements then follow from Propositions 4.5.6 and 4.5.7.

The element $x \in C_\lambda^*(\mathbb{G})$ corresponds to the element $\hat{\varphi}_x$ and the element $\hat{\Theta}_\infty^l(\hat{\omega})(x)$ corresponds to $\hat{\varphi}_{\hat{\Theta}_\infty^l(\hat{\omega})(x)}$. Then by Proposition 4.2.6,

$$\begin{aligned}\hat{\Theta}_1^l(\hat{\omega} \circ \kappa)(\hat{\varphi}_x) &= (\omega \circ \hat{\kappa}) * (\hat{\varphi}_x) \\ &= \hat{\varphi}_{\hat{\Theta}_\infty^l(\hat{\omega})(x)}.\end{aligned}$$

This shows the above pairs of maps are compatible. □

We are now ready to prove the following theorem. The proof is the same as the proof of Theorem 1.1 from [18].

**Theorem 4.5.9.** *Let $1 < p < \infty$. If $\mathbb{G}$ is a discrete Kac algebra with the approximation property, then $L_p(\hat{\mathbb{G}})$ has the operator space approximation property.*

*Proof.* We will show that for each $a \in L_p(\mathbb{G})\check{\otimes}K_\infty$ and $\varepsilon > 0$, there exists a finite rank map $T$ on $L_p(\mathbb{G})$ such that

$$\|(T \otimes id_{K\infty})(a) - a\| < \varepsilon.$$

It follows from Theorem 2.8.6 (and Remark 3.4 in [18]) that

$$L_p(\hat{\mathbb{G}})\check{\otimes}K_\infty = (C_\lambda^*(\mathbb{G})\check{\otimes}K_\infty, L_1(\mathbb{G})\check{\otimes}K_\infty)_{1/p}.$$

By the complex interpolation method, there is a continuous and bounded map $f : S \to C_\lambda^*(\mathbb{G})\check{\otimes}K_\infty + L_1(\mathbb{G})\check{\otimes}K_\infty$ in $\mathcal{F}$ such that $a = f(1/p)$. Since

$f(it) \in C_\lambda^*(\mathbb{G})\check{\otimes}K_\infty$ and $f(it) \to 0$ as $t \to \pm\infty$, the set $\{f(it)\}_{t\in\mathbb{R}}$ is contained in a compact subset of $C_\lambda^*(\hat{\mathbb{G}})\check{\otimes}K_\infty$. However a subset of a Banach space is compact if and only if it is contained in the closed convex hull of null sequence (see Lemma 11.1.1 in [6] for example). Thus there exists an element $(x_n) \in C_\lambda^*(\mathbb{G})\check{\otimes}K_\infty \otimes c_0 \subseteq C_\lambda^*(\mathbb{G})\check{\otimes}K_\infty$ such that $f(it)$ is contained in the norm closure of the convex hull of $\{x_n\}$ in $C_\lambda^*(\mathbb{G})\check{\otimes}K_\infty$. Similarly, there exists $(y_n) \in L_1(\mathbb{G})\check{\otimes}K_\infty$ such that the set $\{f(1+it)\}_{t\in\mathbb{R}}$ is contained in the norm closure of the convex hull of $\{y_n\}$.

As $\mathbb{G}$ has the approximation property, by Proposition 4.5.8 there exists $\hat{\omega} \in A_c(\mathbb{G})$ such that

$$\left\|(\hat{\Theta}^l_\infty(\hat{\omega}) \otimes id_\infty)(f(it)) - f(it)\right\| < \varepsilon$$

and

$$\left\|(\hat{\Theta}^l_1(\hat{\omega} \circ \hat{\kappa}) \otimes id_\infty)(f(1+it)) - f(1+it)\right\| < \varepsilon,$$

for all $t \in \mathbb{R}$.

As $\mathbb{G}$ is discrete and $\hat{\omega} \in A_c(\mathbb{G})$, $\hat{\Theta}^l_\infty(\hat{\omega})$ and $\hat{\Theta}^l_1(\hat{\omega} \circ \hat{\kappa})$ are finite rank maps. Since $(\hat{\Theta}^l_\infty(\hat{\omega}), \hat{\Theta}^l_1(\hat{\omega} \circ \hat{\kappa}))$ is a compatible pair of finite rank maps, they induce a finite-rank map $T : C_\lambda^*(\mathbb{G}) + L_1(\hat{\mathbb{G}})) \to C_\lambda^*(\mathbb{G}) + L_1(\hat{\mathbb{G}}))$ such that $(T \otimes id_\infty) \circ f \in \mathcal{F}$ and $(T \otimes id_\infty)(a) = (T \otimes id_\infty)(f(1/p))$. Then

$$\begin{aligned} &\|(T \otimes id_\infty)(a) - a\|_{L_p(\hat{\mathbb{G}})\check{\otimes}K_\infty} \leq \|(T \otimes id_\infty) \circ f - f\|_{\mathcal{F}} \\ = \max\{ &\sup_{t\in\mathbb{R}} \left\|(\hat{\Theta}^l_\infty(\hat{\omega} \circ \hat{\kappa}) \otimes id_\infty)(f(it)) - f(it)\right\|_{C_\lambda^*(\mathbb{G})\check{\otimes}K_\infty}, \\ &\sup_{t\in\mathbb{R}} \left\|(\hat{\Theta}^l_\infty(\hat{\omega}) \otimes id_\infty)(f(1+it)) - f(1+it)\right\|_{L_1(\hat{\mathbb{G}})\check{\otimes}K_\infty} \} < \varepsilon. \end{aligned}$$

□

There is also the weakly amenable version of this result, which is proved in the same fashion as Proposition 3.5 of [18].

**Theorem 4.5.10.** *Let $\mathbb{G}$ be a weakly amenable discrete group and let $1 < p < \infty$. Then $L_p(\hat{\mathbb{G}})$ has the completely bounded approximation property with*

$$\Lambda(L_p(\hat{\mathbb{G}})) \leq \Lambda(\mathbb{G}).$$

*Proof.* By Proposition 4.5.8, we choose a net $\hat{\omega}_\alpha$ in $A_c(\mathbb{G})$ such that

$$\left\|\hat{\Theta}_1^l(\hat{\omega}_\alpha \circ \hat{\kappa})\right\|_{cb} = \left\|\hat{\Theta}_\infty^l(\hat{\omega}_\alpha)\right\|_{cb} \leq \Lambda(\mathbb{G}),$$

$\hat{\Theta}_1^l(\hat{\omega}_\alpha \circ \hat{\kappa}) \to id_{L_1(\hat{\mathbb{G}})}$ and $\hat{\Theta}_\infty^l(\hat{\omega}_\alpha) \to id_{C_\lambda^*(\mathbb{G})}$ in the point-norm topology. As these maps are compatible in the sense of complex interpolation theory, they induce completely bounded finite ranks maps $T_\alpha^p$ on $L_p(\hat{\mathbb{G}})$ such that

$$\|T_\alpha^p\|_{cb} \leq \left\|\hat{\Theta}_1^l(\hat{\omega}_\alpha \circ \hat{\kappa})\right\|_{cb}^{1-1/p} \left\|\hat{\Theta}_\infty^l(\hat{\omega}_\alpha)\right\|_{cb}^{1/p} \leq \Lambda(\mathbb{G}).$$

The net $\hat{\omega}_\alpha$ can be chosen so that $T_\alpha^p \to id_{L_p(\hat{\mathbb{G}})}$ in the point-norm topology by following an approach similar to the proof of the previous theorem. □

# Chapter 5

# The Hausdorff–Young Inequality

Let $G$ be a locally compact abelian group with Haar measure $\mu$ and dual group $\hat{G}$. The Fourier transform of a function $f \in L_1(G,\mu)$ is defined by

$$\mathcal{F}_1(f)(\xi) = \hat{f}(\xi) = \int_G f(s)\overline{\xi(s)}d\mu(s), \qquad \xi \in \hat{G}.$$

Clearly, $\|\mathcal{F}_1(f)\|_\infty \leq \|f\|_1$. For a suitably normalized Haar measure $\hat{\mu}$ on $\hat{G}$, if $f \in L_1(G,\mu) \cap L_2(G,\mu)$, then we have $\|\mathcal{F}_1(f)\|_2 = \|f\|_2$ and the Fourier transform extends to a unitary, $\mathcal{F}_2$, from $L_2(G,\mu)$ onto $L_2(\hat{G},\hat{\mu})$. Interpolating between these cases yields the Hausdorff–Young inequality: for $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mathcal{F}_p : L_p(G,\mu) \to L_q(\hat{G},\hat{\mu})$ is a contraction, i.e., $\|\mathcal{F}_p(f)\|_q \leq \|f\|_p$ (see, for example, [8]).

For $f \in L_1(G,\mu)$, let $\lambda(f)$ denote the operator on $L_2(G,\mu)$ given by $\lambda(f)(g) = f * g$. As $\widehat{f * g} = \hat{f}\hat{g}$, the operator $\lambda(f)$ is unitarily equivalent to the operator $\hat{f} \in L_\infty(\hat{G},\hat{\mu})$ acting by multiplication on $L_2(\hat{G},\hat{\mu})$. Inspired by this, Kunze dealt with the unimodular group case by making the definition $\mathcal{F}_1(f) = \lambda(f)$. The group von Neumann algebra $L(G)$ is generated by $\{\lambda(f) \mid f \in L_1(G)\}$ and plays the role of $L_\infty(\hat{G})$. Using the noncommutative $L_p$ spaces associated with a trace on $L(G)$, Kunze showed in [25] that the Hausdorff–Young inequality holds for unimodular groups.

Terp extended the Hausdorff–Young inequality to locally compact groups in [35] by using the following definition for the Fourier transform:

**Definition 5.0.1.** Let $G$ be a locally compact group with modular function $\Delta$. Let $f \in L_p(G)$, $1 \leq p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$. The *$L_p$-Fourier transform of $f$* is the operator $\mathcal{F}_p(f)$ on $L_2(G)$ given by

$$\mathcal{F}_p(f)\xi = f * \Delta^{1/q}\xi, \quad \xi \in \mathcal{D}(\mathcal{F}_p(f)),$$

where $\mathcal{D}(\mathcal{F}_p(f)) = \{\xi \in L_2(G) \mid f * \Delta^{1/q}\xi \in L_2(G)\}$.

Terp showed that $\mathcal{F}_p(f) \in L_q(VN(G))$, where $L_q(VN(G))$ is a non-commutative $L_q$ space constructed using the Plancherel weight on the group von Neumann algebra $L(G)$, and showed that the Hausdorff–Young inequality holds. This is the definition that shall be extended to the locally compact quantum group case. We also note that a version of this chapter will be published as [4].

## 5.1 The Hausdorff-Young inequality for locally compact quantum groups

Let $\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \varphi, \varphi_r)$ be a locally compact quantum group with left Haar weight $\varphi$ and right Haar weight $\varphi_r$. Let $\hat{\mathbb{G}} = (L_\infty(\hat{\mathbb{G}}), \hat{\Gamma}, \hat{\varphi}, \hat{\varphi}_r)$ be its dual locally compact quantum group.

*Remark* 5.1.1. Throughout this chapter, we shall use the case $\alpha = -1/2$ of Izumi's method of constructing the noncommutative $L_p$-spaces by complex interpolation. As a result, $\varphi_x$ shall stand for $\varphi_x^{(-1/2)}$ as defined in Section 2.8. Especially important throughout this chapter are $L = L_{-1/2}$, the intersection of $L_1(M)$ and $L_\infty(M)$, and $j^* = j^*_{1/2}$, the inclusion of $L_\infty(\mathbb{G})$ into $L^*_{1/2}$, the space containing all of the interpolation spaces that arise. The notation $L_p(M, \varphi)_\alpha$ indicates the $L_p$ space constructed using interpolation, whereas $L_p(M, \varphi)$ denotes a spatial $L_p$ space in the sense of Hilsum. Here $M$ denotes either of the von Neumann algebras $L_\infty(\mathbb{G})$ or $L_\infty(\hat{\mathbb{G}})$.

Let $\psi$ (respectively $\hat{\psi}$) be a nfs weight on the commutant $L_\infty(\mathbb{G})'$ (respectively $L_\infty(\hat{\mathbb{G}})'$). For example, one could take $\psi = \varphi(J \cdot J)$ (respectively $\hat{\psi} = \hat{\varphi}(\hat{J} \cdot \hat{J})$). Let $d = \frac{d\varphi}{d\psi}$ and $\hat{d} = \frac{d\hat{\varphi}}{d\hat{\psi}}$. Let $L_p(\mathbb{G})$ (respectively $L_q(\hat{\mathbb{G}})$) be the spatial non-commutative $L_p$ space associated to the pair $(L_\infty(\mathbb{G}), \psi)$ (respectively $(L_\infty(\hat{\mathbb{G}}), \hat{\psi})$). As noted above, $L_p(\mathbb{G})_{-1/2}$ denotes an $L_p$ space constructed by interpolation. Let $\mathfrak{A}_0$ be the Tomita algebra associated with the pair $(L_\infty(\mathbb{G}), \varphi)$ and let $\pi_l(\mathfrak{A}_0^2) = \text{span}\{y^* z \,:\, y, z \in \pi_l(\mathfrak{A}_0)\}$.

*Remark* 5.1.2. We will be interested in the following special case of Theorem 2.8.5. We have

$$U_{p,0,-1/2}(j^*(a)) = j_0^*(\sigma^{\varphi}_{\frac{i}{2p}}(a)).$$

Using Theorem 27 of [36] (which provides a map identifying Terp's interpolation $L_p$-spaces with the spatial non-commutative $L_p$-spaces), we can compose to get an isometric isomorphism $U_p : L_p(M,\varphi)_{-1/2} \to L_p(M,\varphi)$ such that for $a \in \pi_l(\mathfrak{A}_0^2)$, we have

$$U_p(j^*(a)) = d^{1/2p}\sigma^{\varphi}_{\frac{i}{2p}}(a)d^{1/2p}.$$

For $p \geq 2$, $a \in \pi_l(\mathfrak{A}_0^2)$, Theorem 2.7.2 allows this to be written as

$$U_p(j^*(a)) = ad^{1/p}.$$

*Remark* 5.1.3. Combining Remark 5.1.2 with Theorem 2.7.1 yields an isometry from $L_2(M,\varphi)_{-1/2}$ onto $\mathcal{H}$, which maps $U_2^{-1}(xd^{1/2})$ onto $\Lambda(x)$, for $x \in \pi_l(\mathfrak{A}_0^2)$.

The results from Section 2.8 and the remarks above describe the image of $x \in \pi_l(\mathfrak{A}_0^2)$ under the inclusions into $L_p(\mathbb{G})_{-1/2}$ and $L_p(\mathbb{G})$. To prove the Hausdorff–Young inequality, it is necessary to identify the intersection $L_1(\mathbb{G})_{-1/2} \cap L_2(\mathbb{G})_{-1/2}$ and approximate its elements by elements in $\pi_l(\mathfrak{A}_0^2)$. With this in mind, the following approximation theorem will prove most useful.

**Theorem 5.1.4.** *Suppose that $x \in \mathfrak{N}_\varphi$. There exists a sequence $(\Lambda(x_n)) \subset \mathfrak{A}_0^2$ such that*

1. $\lim_{n\to\infty} \|\Lambda(x_n) - \Lambda(x)\|_2 = 0,$
2. *$x_n$ converges to $x$ in the $\sigma$-strong operator topology, and*
3. $\|x_n\| \leq \|x\|$.

*Proof.* We apply an approximation result due to Haagerup to the Tomita algebra. See [10] or Theorem VI.1.26 in [33]. These sources state that the sequence lies in $\mathfrak{A}_0$ but, examining the proof, we see that the stronger statement that $\Lambda(x_n) \in \mathfrak{A}_0^2$ holds. □

**Theorem 5.1.5.** *For all $x \in \mathfrak{N}_\varphi$, $2 \leq p < \infty$,*

$$U_p(j^*(x)) = xd^{1/p}.$$

*Proof.* First we consider the case $p = 2$. Let $x \in \mathfrak{N}_\varphi$. By Theorem 2.7.1 and Remark 5.1.2, we have that $xd^{1/2} \in L_2(\mathbb{G})$ and $U_2^{-1}(xd^{1/2}) \in L_2(\mathbb{G})_{-1/2}$.

By Theorem 5.1.4, there exists a sequence $\{x_n\} \subset \pi_l(\mathfrak{A}_0^2)$ such that $x_n \to x$ strongly, $\|x_n\| \leq \|x\|$, and $\|\Lambda(x_n) - \Lambda(x)\|_2 \to 0$. By Remark 5.1.3, convergence in $\mathcal{H}$ implies convergence in $L_2(M,\varphi)_{-1/2}$. As the inclusion of $L_2(M,\varphi)_{-1/2}$ into $L_{1/2}^*$ is continuous, convergence in $L_2(\mathbb{G})_{-1/2}$ in turn implies that $U_2^{-1}(x_n d^{1/2}) \to U_2^{-1}(xd^{1/2})$ in the $\sigma(L_{1/2}^*, L_{1/2})$-topology.

The inclusion $j^*$ of $L_\infty(\mathbb{G})$ into $L_{1/2}^*$ is the adjoint of the map $j : L_{1/2} \to L_\infty(\mathbb{G})_*$. Thus the map $j^*$ is $\sigma(L_\infty(\mathbb{G}), L_\infty(\mathbb{G})_*) - \sigma(L_{1/2}^*, L_{1/2})$ continuous. As $\{x_n\}$ is a strongly convergent, $\|\cdot\|_\infty$-bounded sequence, this sequence also converges in the $\sigma(L_\infty(\mathbb{G}), L_\infty(\mathbb{G})_*)$-topology. Thus $j^*(x_n) \to j^*(x)$ in the $\sigma(L_{1/2}^*, L_{1/2})$-topology.

As $x_n \in \mathfrak{A}_0^2$, applying Remark 5.1.2 yields that $j^*(x_n) = U_2^{-1}(x_n d^{1/2})$. Since the $\sigma(L_{1/2}^*, L_{1/2})$-topology is a Hausdorff topology, it follows that $j^*(x) = U_2^{-1}(xd^{1/2})$, i.e., $U_2(j^*(x)) = xd^{1/2}$ for $x \in \mathfrak{N}_\varphi$.

Now we consider the case $p > 2$. Let $x \in \mathfrak{N}_\varphi$ and again let $\{x_n\}$ be a sequence $\{x_n\} \subset \pi_l(\mathfrak{A}_0^2)$ such that $x_n \to x$ strongly, $\|x_n\| \leq \|x\|$, and $\|\Lambda(x_n) - \Lambda(x)\|_2 \to 0$. By Theorem 2.7.1, we have that $xd^{1/p} \in L_p(\mathbb{G})$ and thus $(x_n - x)d^{1/p} \in L_p(\mathbb{G})$.

The three line theorem and Proposition 25 from [36] are now used to estimate $\left\|(x_n - x)d^{1/p}\right\|_p$. Let $a_n \in \mathbb{R}$ be such that $e^{a_n} = \|\Lambda(x_n - x)\|_2^{-2} \|x_n - x\|^2$. We now repeat the proof of Theorem 26 in [36] but with the function $F(z)$ replaced by $G_n(z) = e^{a_n(z-1/p)}F(z)$. This yields

$$\left\|(x_n - x)d^{1/p}\right\|_p \leq \|\Lambda(x_n - x)\|_2^{2/p} \|x_n - x\|^{1-2/p}.$$

As $\|x_n - x\| \leq 2$, the convergence of $x_n d^{1/2} \to xd^{1/2}$ in $\|\cdot\|_2$-norm implies that $x_n d^{1/p} \to xd^{1/p}$ in $\|\cdot\|_p$-norm. Proceeding as in the $p = 2$ case, we obtain the stated result. □

*Remark* 5.1.6. This theorem will be used in the following two ways. If $x \in \mathfrak{N}_\varphi \subset L_\infty(\mathbb{G})$, then the corresponding element of $L_2(\mathbb{G})$ is $xd^{1/2}$ and this inclusion agrees with the interpolation method being used. If $\omega \in \mathcal{I}$ and $q \geq 2$, then $\lambda(\omega)$ lies in $\mathfrak{N}_{\hat{\varphi}} \subset L_\infty(\hat{\mathbb{G}})$, the corresponding element of $L_q(\mathbb{G})$ is $\lambda(\omega)\hat{d}^{1/q}$ and this inclusion agrees with the interpolation method being used.

**Lemma 5.1.7.** *The intersection $L$ is a subspace of $\mathfrak{N}_\varphi$.*

*Proof.* **Claim:** Given $x \in L$, there exists a net $(x_k)$ in $L \cap \mathfrak{N}_\varphi$, such that

1. $\sup_k \|x_k\| < \infty$
2. $x_k \to x$ $\sigma$-strongly
3. $\varphi_{x_k} \to \varphi_x$ in $\|\cdot\|_1$-norm

This will be shown in a manner similar to Lemma 9 in [36] and the proof of Theorem 2.5 in [16].

By the Kaplansky Density Theorem, there is a net $\{f_k\}_k$ of self-adjoint elements in the unit ball of $\mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$ such that $f_k \to 1$ in the $\sigma$-strong*-topology. Define the net of elements $\{e_k\}_k$ by

$$e_k = \sqrt{\frac{1}{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sigma_t^\varphi(f_k)\ \mathrm{d}t.$$

Proposition 2.16 in [31] shows that $\{e_k\}_k \subset \pi_l(\mathfrak{A}_0)$, and that for every $\alpha \in \mathbb{C}$

$$\sigma_\alpha^\varphi(e_k) \to 1 \text{ in the } \sigma\text{-strong}^* \text{ topology},$$
$$\|\sigma_\alpha^\varphi(e_k)\| \leq e^{(Im\ \alpha)^2}.$$

Set $x_k = e_k x \sigma_{-i}^\varphi(e_k)$. By Proposition 2.8.3, $L$ is an $\pi_l(\mathfrak{A}_0)$-bimodule. Thus $x_k \in L$ and $\varphi_{x_k} = e_k \varphi_x e_k$. As $\sigma_{-i}^\varphi(e_k) \in \pi_l(\mathfrak{A}_0) \subset \mathfrak{N}_\varphi$, it follows that $x_k \in L \cap \mathfrak{N}_\varphi$. As $\{e_k\}$ and $\{\sigma_{-i}^\varphi(e_k)\}$ are bounded nets converging strongly to 1, $\{x_k\}$ is a bounded net converging strongly to $x$.

As $L_\infty(\mathbb{G})$ is a von Neumann algebra in standard form, there exist $\xi, \eta \in \mathcal{H}$ such that $\varphi_x = (\cdot\ \xi \mid \eta)$. Fix $y \in L_\infty(\mathbb{G})$.

$$\begin{aligned}|\varphi_{x_k}(y) - \varphi_x(y)| =& (e_k y e_k \xi \mid \eta) - (y\xi \mid \eta)| \\ =& |(y(e_k\xi) \mid \eta) - (y\xi \mid e_k\eta) + (y\xi \mid e_k\eta) - (y\xi \mid \eta)| \\ =& |(y(e_k\xi - \xi) \mid e_k\eta) + (y\xi \mid (e_k\eta - \eta))| \\ \leq& \|y\| \left(\|e_k\xi - \xi\| \|\eta\| + \|\xi\| \|e_k\eta - \eta\|\right)\end{aligned}$$

and thus $\|\varphi_{x_k} - \varphi_x\|_1 \to 0$. This completes the proof of the claim.

It remains to show that that $x \in \mathfrak{N}_\varphi$. As $x_k - x \in L$, the intersection of $L_\infty(\mathbb{G})$ and $L_\infty(\mathbb{G})_*$ in $L^*$, the norm $\|j^*(x_k - x)\|_2$ can be estimated using the norms $\|\varphi_{x_k} - \varphi_x\|_1$ and $\|x_k - x\|_\infty$. Consider the function $f_k(z) = e^{a_k(z^2 - (1/2)^2)} j^*(x_k - x)$, where $e^{a_k} = \|x_k - x\|_1 \|x_k - x\|_\infty^{-1}$. The definition of the norm on the interpolation space $L_p(\mathbb{G})_{-1/2}$ yields that

$$\|j^*(x_k - x)\|_2 \leq \|j^*(x_k - x)\|_1^{3/4} \|j^*(x_k - x)\|_\infty^{1/4} = \|\varphi_{x_k} - \varphi_x\|_1^{3/4} \|x_k - x\|_\infty^{1/4}.$$

As $x_k - x$ is $\|\cdot\|_\infty$-bounded and $\varphi_{x_k} \to \varphi_x$ in $\|\cdot\|_1$, it follows that

$$\|j^*(x_k) - j^*(x)\|_2 \to 0.$$

Let $\xi \in \mathcal{H}$ be the image of $j^*(x)$ under the isometry from $L_2(\mathbb{G})_{-1/2}$ onto $\mathcal{H}$, as given by Remark 5.1.3. For any $\eta \in \mathfrak{A}'$,

$$\pi_r(\eta)\xi = \lim \pi_r(\eta)\Lambda(x_k) = \lim x_k\eta = x\eta.$$

Thus $\xi$ is left bounded and $x = \pi_l(\xi)$. By Theorem 2.5 of [33], $x \in \mathfrak{N}_\varphi$ and $\xi = \Lambda(x)$. □

**Lemma 5.1.8.** *Let $\mathbb{G}$ be a locally compact quantum group and let $\mathcal{I}$ be as in Section 3.1. If $x \in L$, then $\varphi_x \in \mathcal{I}$ and*

$$\hat{\Lambda}(\lambda(\varphi_x)) = \xi(\varphi_x) = \Lambda(x).$$

*Proof.* Let $x \in L$ which implies by Lemma 5.1.7 that $x \in \mathfrak{N}_\varphi$. The definition of $L$ provides a normal linear functional $\varphi_x \in L_\infty(\mathbb{G})_*$ such that for all $y, z \in \pi_l(\mathfrak{A}_0^2)$

$$\begin{aligned}
\varphi_x(y^*z) &= \big(xJ\Delta^{-1/2}\Lambda(y)\big|J\Delta^{1/2}\Lambda(z)\big) \\
&= (x\Lambda(\sigma_{-i}(y^*))|\Lambda(z^*)) \\
&= \varphi\left(zx\sigma_{-i}(y^*)\right)) \\
&= \varphi\left(y^*zx\right) \\
&= (\Lambda(x)|\Lambda((y^*z)^*))
\end{aligned}$$

Since $zx \in \mathfrak{M}_\varphi$, Proposition 2.2.2 justifies the second last equality.

The above is now used to show that $\varphi_x(a^*) = (\Lambda(x)|\Lambda(a))$ for all $a \in \mathfrak{N}_\varphi$. By Theorem 5.1.4, there exists a sequence $\{a_n\} \subset \pi_l(\mathfrak{A}_0^2)$ such that $\|\Lambda(a_n) - \Lambda(a)\|_2 \to 0$ and $a_n^* \to a^*$ $\sigma$-weakly. As $\varphi_x$ is a normal functional and $a_n^* \to a^*$ $\sigma$-weakly, $\varphi_x(a_n^*) \to \varphi_x(a^*)$, and then

$$\varphi_x(a^*) = \lim \varphi_x(a_n^*) = \lim (\Lambda(x)|\Lambda(a_n)) = (\Lambda(x)|\Lambda(a)).$$

The result now follows from the construction of the map $\hat{\Lambda}$ (see Section 3.1). □

The $L_p$-Fourier transform can now be defined and shown to satisfy the Hausdorff–Young inequality.

**Theorem 5.1.9.** *Let* $1 \leq p \leq 2$ *with* $\frac{1}{p} + \frac{1}{q} = 1$*. The* $L_p$-Fourier transform of $\mathbb{G}$ *is the extension of the map*

$$\mathcal{F}_p\left(U_p\left(j^*(a)\right)\right) = \lambda\left(\varphi_a\right)\hat{d}^{1/q}, \qquad a \in L,$$

*to a contraction* $\mathcal{F}_p : L_p(\mathbb{G}, \varphi) \to L_q(\hat{\mathbb{G}}, \hat{\varphi})$ *and thus*

$$\left\|\mathcal{F}_p\left(U_p\left(j^*(a)\right)\right)\right\|_q \leq \left\|ad^{1/p}\right\|_p.$$

*Remark* 5.1.10. For $a \in \pi_l(\mathfrak{A}_0^2)$, there is a more explicit description of $\mathcal{F}_p$:

$$\mathcal{F}_p\left(d^{1/2p}\sigma^{\varphi}_{i/2p}(a)d^{1/2p}\right) = \lambda(\varphi_a)\hat{d}^{1/q}, \quad a \in \pi_l(\mathfrak{A}_0^2).$$

If $\varphi$ is a state, we have the simpler expression

$$\mathcal{F}_p(ad^{1/p}) = \lambda(\varphi_a)\hat{d}^{1/q}, \qquad a \in L_\infty(\mathbb{G}).$$

*Proof.* The case $p = 1$ is obvious. For $\omega \in L_\infty(\mathbb{G})_*$, $\mathcal{F}_1$ is the map $\omega \mapsto \lambda(\omega) = (\omega \otimes \iota)W$, which is a contraction.

We now consider the case $p = 2$. By Theorem 2.7.1, we can identify the Hilbert spaces $\mathcal{H}_\varphi$ and $L_2(\mathbb{G})$ by using the isometry $\alpha : \Lambda(x) \mapsto xd^{1/2}$, for $x \in \mathfrak{N}_\varphi$. Similarly, we can identify $L_2(\hat{\mathbb{G}})$ and $\mathcal{H}_{\hat{\varphi}}$ by using the isometry $\beta : \lambda(\omega) \mapsto \hat{\Lambda}(\lambda(\omega))$, for $\lambda(\omega) \in \mathfrak{N}_{\hat{\varphi}}$. Furthermore, from Lemma 5.1.8 and the construction of the dual Haar weight, the map $\beta \circ \mathcal{F}_2 \circ \alpha : \mathcal{H}_\varphi \to \mathcal{H}_{\hat{\varphi}}$ is the identity map. This implies that $\mathcal{F}_2$ is an isometry from $L_2(\mathbb{G})$ onto $L_2(\hat{\mathbb{G}})$.

The next step is to show that $\mathcal{F}_1$ and $\mathcal{F}_2$ agree on $L_1(\mathbb{G})_{-1/2} \cap L_2(\mathbb{G})_{-1/2} \subset L^*_{1/2}$.

Suppose that $x \in L_1(\mathbb{G})_{-1/2} \cap L_2(\mathbb{G})_{-1/2}$, with $U_1(x) = \psi \in L_\infty(\mathbb{G})_*$ and $U_2(x) = \alpha(\xi) \in L_2(\mathbb{G})$ for $\xi \in \mathcal{H}_\varphi$. Let $z_i$ be a sequence in $\mathfrak{A}_0^2$ such that $\|\Lambda(z_i) - \xi\|_2 \to 0$. As the inclusions into $L_{1/2}^*$ are continuous, this implies that $j^*(z_i) \to x$ in the $\sigma(L_{1/2}^*, L_{1/2})$-topology. Therefore for $y \in \pi_l(\mathfrak{A}_0^2)$,

$$\langle y, \psi \rangle_{L_\infty(\mathbb{G}), L_\infty(\mathbb{G})_*} = (\xi \mid \Lambda(y^*)),$$

since

$$\begin{aligned} & \langle y, \psi \rangle_{L_\infty(\mathbb{G}), L_\infty(\mathbb{G})_*} && = \langle y, x \rangle_{L_{1/2}, L_{1/2}^*} \\ = \ & \lim \langle y, j^*(z_i) \rangle_{L_{1/2}, L_{1/2}^*} && = \lim \langle y, \varphi_{z_i} \rangle \\ = \ & \lim(\Lambda(z_i) \mid \Lambda(y^*)) && = (\xi \mid \Lambda(y^*)). \end{aligned}$$

Suppose that $y \in \mathfrak{N}_\varphi^*$. By Theorem 5.1.4, there exists a sequence $\{y_n^*\}$ in $\pi_l(\mathfrak{A}_0^2)$ such that $\|\Lambda(y_n^*) - \Lambda(y^*)\|_2 \to 0$ and $y_n \to y$ $\sigma$-weakly. Thus $\psi(y_n) \to \psi(y)$ and it follows that

$$\langle y, \psi \rangle = \lim \langle y_n, \psi \rangle = \lim(\xi \mid \Lambda(y_n^*)) = (\xi \mid \Lambda(y^*)), \qquad \forall y \in \mathfrak{N}_\varphi^*.$$

Therefore $\psi \in \mathcal{I}$ and $\xi(\psi) = \xi$. However by the construction of the Haar weight on $L_\infty(\hat{\mathbb{G}})$, this implies that $\lambda(\psi) \in \mathfrak{N}_{\hat{\varphi}}$ and, by Theorem 2.7.1, $\lambda(\psi)\hat{d}^{1/2} \in L_2(\hat{\mathbb{G}})$. Then $\beta(\lambda(\psi)\hat{d}^{1/2}) = \hat{\Lambda}(\lambda(\psi)) = \xi(\psi) = \xi$. As $\beta \circ \mathcal{F}_2 \circ \alpha = id_{\mathcal{H}_\varphi}$, this implies $\beta^{-1}(\xi) = \mathcal{F}_2(\alpha(\xi))$ or $\mathcal{F}_2(U_2(x)) = \lambda(\psi)\hat{d}^{1/2}$.

Thus $\mathcal{F}_1(U_1(x)) = \lambda(\psi)$ and $\mathcal{F}_2(U_2(x)) = \lambda(\psi)\hat{d}^{1/2}$. By Remark 5.1.6, these two elements are the same when considered as elements of $\hat{L}_{1/2}^*$, and the Fourier transform is well-defined as a map

$$L_1(\mathbb{G})_{-1/2} \cap L_2(\mathbb{G})_{-1/2} \to L_2(\hat{\mathbb{G}})_{-1/2} \cap L_\infty(\hat{\mathbb{G}})_{-1/2}.$$

The Hausdorff–Young inequality now follows after interpolating between the cases $p = 1$ and $p = 2$ as the complex interpolation method is an exact interpolation functor. □

We note that the non-commutative $L_p$ spaces are (up to isometric isomorphism) independent of the choice of the weights on the commutants and thus so are the results in this chapter. However the connection with the group case is clearer when we work with the following choice of weights on the commutants, $\varphi' = \varphi(J \cdot J)$ and $\hat{\varphi}' = \hat{\varphi}(\hat{J} \cdot \hat{J})$. Suppose $G$ is a locally compact group with modular function $\Delta$. This defines an unbounded operator acting by multiplication on $L_2(G)$, $(\Delta f)(s) = \Delta(s)f(s)$, for $s \in G$. The left Haar weight $\varphi$ is given by integration against the left Haar measure. If $\varphi' = \varphi(J \cdot J)$, then $d = \frac{d\varphi}{d\varphi'} = 1$. Let $\hat{\varphi}$ be the Plancherel weight on the group von Neumann algebra $L(G)$ (for example, see Section VII.3 of [33]). If we let $\hat{\varphi}' = \hat{\varphi}(\hat{J} \cdot \hat{J})$, then $\hat{d} = \frac{d\hat{\varphi}}{d\hat{\varphi}'} = \Delta$ (see [34] or [35]). Terp's Fourier transform can then be written as $\mathcal{F}_p(f) = \lambda(f)\hat{d}^{1/q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. This shows the locally compact quantum group definition generalizes the locally compact group case. It is to preserve this connection with the group case that Hilsum's construction of non-commutative $L_p$-spaces will be used. The $L_p$-Fourier transform is also compatible with the Fourier transform from a suitable subset of $L_\infty(\mathbb{G})$ into $L_\infty(\hat{\mathbb{G}})$ considered in [21] and [38].

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