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DISPERSE ESTIMATES FOR THE SCHRÖDINGER EQUATION

BY

WILLIAM ROBERT GREEN

DISSERATION

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Doctoral Committee:

Associate Professor Richard Laugesen, Chair
Associate Professor M. Burak Erdoğan, Director of Research
Associate Professor Jared Bronski
Associate Professor Dirk Hundertmark
Abstract

In this document we explore the issue of $L^1 \to L^\infty$ estimates for the solution operator of the linear Schrödinger equation,

$$iu_t - \Delta u + Vu = 0 \quad u(x,0) = f(x) \in S(\mathbb{R}^n).$$

We focus particularly on the five and seven dimensional cases. We prove that the solution operator pre-composed with projection onto the absolutely continuous spectrum of $H = -\Delta + V$ satisfies the following estimate $\|e^{itH}P_{ac}(H)\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{n}{2}}$ under certain conditions on the potential $V$. Specifically, in Chapter 1 we prove the dispersive estimate is satisfied with optimal assumptions on smoothness, that is $V \in C^{\frac{n+3}{2}}(\mathbb{R}^n)$ for $n = 5, 7$ assuming that zero is regular, $|V(x)| \lesssim \langle x \rangle^{-\beta}$ and $|\nabla^j V(x)| \lesssim \langle x \rangle^{-\alpha}$, $1 \leq j \leq \frac{n-3}{2}$ for some $\beta > \frac{3n+5}{2}$ and $\alpha > 3, 8$ in dimensions five and seven respectively.

In Chapter 2 we show that for the five dimensional result one only needs that $|V(x)| \lesssim \langle x \rangle^{-4}$ in addition to the assumptions on the derivative and regularity of the potential. This more than cuts in half the required decay rate in the first chapter.

Finally in Chapter 3 we consider a problem involving the non-linear Schrödinger equation. In particular, we consider the following equation that arises in fiber optic communication systems,

$$iu_t + d(t)u_{xx} + |u|^2u = 0.$$ 

We can reduce this to a non-linear, non-local eigenvalue equation that describes the so-called dispersion management solitons. We prove that the dispersion management solitons decay exponentially in $x$ and in the Fourier transform of $x$. 


To my family.
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Chapter 1

Five and seven dimensional cases

1.1 Introduction

Consider the linear Schrödinger equation with time-independent potential $V$,

\[ iu_t + (-\Delta + V)u = 0 \quad u(x,0) = f(x) \in S(\mathbb{R}^n). \quad (1.1) \]

This equation is the (non-dimensionalized) fundamental law of motion for small particles, used to describe the evolution of the quantum state of a system. The solution $u(x,t)$ of (1.1) describes a wave function or probability amplitude for the particle. That is, physically $u(x,t)$ represents a probability density of finding the system in a given state. The probability of finding a particle described by (1.1) in a region $\Omega \subset \mathbb{R}^n$ at time $t$ is given by

\[ Pr(\Omega,t) = \int_{\Omega} |u(x,t)|^2 \, dx. \quad (1.2) \]

Physically, this probability leads us to search for solutions $u \in L^2(\mathbb{R}^n)$ and the assumption on the initial condition $f(x) \in L^2(\mathbb{R}^n)$, which is slightly stricter than that in (1.1).

The time-independent version of (1.1) was introduced by Schrödinger in 1926, [69], as a mathematical model for quantum mechanics.

\[ h^2 \Delta \psi + 8\pi m (E - V) \psi = 0 \quad (1.3) \]

Schrödinger wanted to use an “equation of wave propagation” to elucidate the inherent wave-particle duality of sub-atomic particles in quantum mechanics.

When $V = 0$, the solution in $\mathbb{R}^n$ can be described as a convolution integral

\[ u(x,t) = e^{-it\Delta} f(x) = (4\pi it)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{i(x-y)^2}{4t}} f(y) \, dy. \quad (1.4) \]
By the triangle inequality, it is clear that the solution operator $e^{-it\Delta}$ has the mapping property

$$\|e^{-it\Delta}\|_{L^1 \to L^\infty} \leq C_n |t|^{-\frac{n}{2}}.$$  \hspace{1cm} (1.5)

This estimate illustrates the dispersive nature of the free Schrödinger equation. That is the solution to (1.1) when $V = 0$ is made up of components with varying frequencies and hence velocities. Thus, as time increases the solution disperses by flattening out in space. Physically this corresponds to the probability density of finding a particle in any compact set becoming arbitrarily small as $t \to \infty$, see (1.2).

Formally, defining the operator $H = -\Delta + V$ one can see that (1.1) is solved by

$$u(x,t) = e^{itH}f(x).$$  \hspace{1cm} (1.6)

It is of interest to see if the solution to (1.1) given formally in (1.6) satisfies a mapping property similar to (1.5). In particular, we consider the Schrödinger operator $H = -\Delta + V$, where $V$ is a real-valued potential. In general the evolution $e^{itH}$ cannot satisfy the dispersive estimate as in (1.5) due to the possibility of bound states. In recent years there has been interest in the following question. Under what conditions on $V$, does $e^{itH}P_{ac}$ satisfy the $L^1 \to L^\infty$ dispersive estimates,

$$\|e^{itH}P_{ac}f\|_{\infty} \lesssim |t|^{-n/2}\|f\|_1, \quad f \in \mathcal{S}(\mathbb{R}^n),$$  \hspace{1cm} (1.7)

where $P_{ac}$ is the projection onto the absolutely continuous spectrum of $H$?

One wishes to in some way mimic the solution in the free case (1.4) which is obtained by simply applying the Fourier transform in the $x$ variable. Upon application of the Fourier transform to the partial differential equation (1.1) becomes an ordinary differential equation in frequency space. This is easily solved and inverse Fourier transformed to yield (1.4). Unfortunately a non-zero potential makes the resulting frequency space ordinary differential equation much more difficult to work with. This is due to the non-local interaction of the resulting $\hat{V} \ast \hat{u}$ term.

We show in Section 1.2 that one can reduce the estimate (1.7) to an oscillatory integral bound.

$$\left| \int_{\mathbb{R}} e^{it\phi(\lambda)}a(\lambda) \, d\lambda \right| \lesssim |t|^{-\frac{n}{2}}.$$  \hspace{1cm} (1.8)

Intuitively, this estimate should hold due to its similarity to a stationary phase integral. However, $a(\lambda)$ is an iterated integral operator which is highly singular, with singularities of order $n - 2$ in $\mathbb{R}^n$, and the phase $\phi$
is not even $C^1(\mathbb{R}^n)$. As one can see in [83], classical stationary phase estimates require both $\phi, a \in C^\infty(\mathbb{R})$. Despite the shortcoming of the stationary phase methods, we develop an ad hoc method for establishing that (1.7) and (1.8) hold.

The spectral theoretic background necessary to make sense of the operator $e^{itH}$ is laid out in Section 1.2. In particular, one can use the spectral theorem to express the evolution of $e^{itH}$ as an integral provided $H$ is self-adjoint.

1.1.1 Problem History

Dispersive estimates for the perturbed, i.e. when $V \neq 0$, Schrödinger equation (1.1) have been studied in the weighted $L^2$ sense for many years. In the context of exponential weights, with a function $\rho \in C^\infty_c(\mathbb{R}^3)$ such that $\rho(x) = |x|$ for $|x|$ large, Rauch proved the following.

**Theorem** (Rauch 1978). If zero is regular, and for some $\epsilon > 0$ and $p > 2$, $e^{\epsilon|V|} \in L^p(\mathbb{R}^3)$, $\exists V \geq 0$ and $e^{\epsilon\rho(\cdot)}u(\cdot, 0) \in L^2(\mathbb{R}^n)$, then

$$\|e^{itH}P_{ac}(H)\|_{L^2(e^{\rho(x)}dx)\rightarrow L^2(e^{-\rho(x)}dx)} \lesssim |t|^{-3/2}.$$  

At the time the concept of regularity at zero was not fully developed. Instead Rauch used the assumption that

$$\limsup_{\lambda \rightarrow 0} \|e^{\epsilon\rho}(H - (\lambda \pm i - 0))^{-1}e^{\epsilon\rho}\|_{L^2\rightarrow L^2} < \infty.$$  

In fact, this paper proved the bound of

$$\|e^{itH}P_{ac}(H)\|_{L^2(e^{\rho(x)}dx)\rightarrow L^2(e^{-\rho(x)}dx)} \lesssim (1 + t)^{-1/2} \quad t \geq 0$$

without any assumptions on zero energy. Jensen and Kato improved this result from exponential to polynomial weights in [38]. In particular, they proved the following.

**Theorem** (Jensen and Kato 1979). If zero is regular and $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 3$, then for $s, s' > \frac{3}{2}$, $s < \frac{3}{2}$,

$$\|e^{itH}P_{ac}(H)\|_{L^2(\mathbb{R}^3)\rightarrow L^2(\mathbb{R}^3)} \lesssim |t|^{-3/2}$$

Their approach relied on asymptotic expansions of the resolvent around zero energy. Jensen continued to use the asymptotic expansions to obtain results in higher dimensions, [37].
Theorem (Jensen 1980). In odd dimension $n \geq 5$, if zero is regular and $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > n$, then for $s, s' > \frac{n}{2} + 1$,

$$\|e^{it\partial_x^2}P_{ac}(H)\|_{L^2 \to L^2} \lesssim |t|^{-n/2}$$

A similar result holds in even dimensions, but requires more decay from the potential and larger weights $s, s'$.

The first work to consider dispersive estimates in the context of $L^1 \to L^\infty$ estimates was the paper of Journé, Soffer and Sogge, [41]. In particular they proved the following.

Theorem (Journé, Soffer and Sogge 1991). In dimension $n \geq 3$ if $V(x)$ satisfies $\hat{V} \in L^1(\mathbb{R}^n)$ and $\langle x \rangle^\alpha V(x) : H^\nu \mapsto H^\nu$ for some $\alpha > n + 4$ and $\nu > 0$, and zero is regular,

$$\|e^{it\partial_x^2}P_{ac}(H)\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{n}{2}}.$$ 

The constraint of $n \geq 3$ arises from the proof depending on the integrability of $|t|^{-n/2}$ at infinity. The constraint on the Fourier transform of $V$ arises from the use of Duhamel’s formula and the bound

$$\|e^{it\partial_x^2}V e^{-it\partial_x^2}\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \leq \|\hat{V}\|_{L^1(\mathbb{R}^n)}.$$ 

In this paper, they always conjectured that $|V(x)| \lesssim \langle x \rangle^{-2}$ and regularity of zero should be enough to assure $L^1 \to L^\infty$ dispersive estimates hold. This conjecture has proven to be false, as seen in [29]. We note that the $L^1 \to L^\infty$ estimates imply weighted $L^2$ estimates by the embeddings $L^{2, -\sigma}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n) \hookrightarrow L^{2, \sigma}(\mathbb{R}^n)$ for $\sigma > \frac{n}{2}$. Here, the weighted $L^2$ space is defined by $L^{2, -\sigma} = \langle x \rangle^\sigma L^2$.

Yajima proved $W^{k,p}(\mathbb{R}^n)$-continuity of the wave operators with zero regularity and sufficient decay of the potential in [84], [85] and [87]. The wave operator is defined by

$$W := s - \lim_{t \to \infty} e^{-it\partial_x^2} e^{-it\partial_x^2}.$$ 

Using that $WW^* = P_{ac}$, Yajima was able to conclude

$$\|e^{it\partial_x^2}P_{ac}(H)\|_{L^1 \to L^\infty} = \|W e^{-it\partial_x^2} W^*\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{n}{2}}$$

for $n \geq 3$.

The results of Journé, Soffer and Sogge were improved by Rodnianski and Schlag in [62]. They were able
to extend the dispersive estimates to a class of potentials that are small in the Rollnik norm

\[ \|V\|^2_R := \int_{\mathbb{R}^6} \frac{|V(x)||V(y)|}{|x-y|^2} \, dx \, dy, \]

and Global Kato norm

\[ \|V\|_K := \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|} \, dy. \]

**Theorem** (Rodnianski and Schlag 2004). *If \( V \) obeys

\[ \|V\|^2_R < (4\pi)^2, \quad \|V\|_K < 4\pi, \]

then for all \( t > 0 \),

\[ \|e^{itH}\|_{L^1 \to L^\infty} \lesssim |t|^{-3/2} \]

This smallness assumption was necessitated by the authors expanding into an infinite series. In this result, the assumption that zero is regular is not needed. A classical result of Kato, [42], was used which states that for a potential satisfying the assumption on smallness of the potential in the Rollnik norm \(-\Delta + V\) is self-adjoint and unitarily equivalent to \(-\Delta\). Under these assumptions on \( V \), the spectrum is purely absolutely continuous, thus accounting for the lack of \( P_{ac}(H) \) in the stated \( L^1 \to L^\infty \) bound. Rodnianski and Schlag’s paper delved into time-dependent potentials and various other Strichartz estimates.

This result was shortly thereafter improved in some respects by Goldberg and Schlag. In [27] they removed the smallness condition on the potential, proving the following.

**Theorem** (Goldberg and Schlag 2004). *In dimension three, if zero is regular and \(|V(x)| \lesssim \langle x \rangle^{-\beta} \) for some \( \beta > 3 \), then

\[ \|e^{itH}P_{ac}(H)\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{3}{2}}. \]

The result of [27] does not necessitate smallness of \( V \) in Rollnik or Global Kato norms as in [62]. However, one does require a faster decay rate at infinity, \( \langle x \rangle^{-3-} \) versus \( \langle x \rangle^{-2-} \), and now requires that the potential be bounded.

The proof used the limiting absorption principle of Agmon, [3], to improve the previously known results. This paper also proved dispersive estimates for the one-dimensional Schrödinger operator if \( V \) is in a weighted
The methods for dimensions $n = 1, 2$ differ greatly from those used in dimensions $n \geq 3$. As such, we make only passing mention to the results. In addition to Goldberg and Schlag, the one-dimensional problem was studied by Artzabar and Yajima [4] and Weder [81, 82]. The two dimensional result was most notably studied by Schlag [65] and Yajima [87].

Here we wish to note that in dimensions $n \leq 3$ pointwise decay and/or integrability of the potential $V$ suffice to establish the $L^1 \to L^\infty$ estimates. In dimensions $n > 3$, one must have sufficient control over the derivatives of the potential for the $L^1 \to L^\infty$ dispersive estimates to hold. In [29] the following negative result was established.

**Theorem** (Goldberg and Visan 2005). In dimension $n > 3$, for any $\alpha < \frac{n-3}{2}$, there exists a compactly supported potential $V \in C^\alpha(\mathbb{R}^n)$ for which there cannot exist a bound of the form

$$\|e^{itH}P_{ac}(H)\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{n}{2}}.$$

Heuristically, this result was obtained by constructing a potential $V$ whose “non-smooth” part traps the high frequency components, causing too large a singularity at small times. Their proof relied on, among other things, the Uniform Boundedness Principle and estimates for the tail of Born series.

The result of Journé, Soffer and Sogge, [41], applies in dimensions $n > 3$ despite having no explicit smoothness requirement that the result necessitated in [29]. Their result required the assumption $\hat{V} \in L^1(\mathbb{R}^n)$, which requires some control of the derivatives of $V$. We note that, for instance, the assumption that $V \in H^{\frac{n}{2}+}(\mathbb{R}^n)$ implies $\hat{V} \in L^1(\mathbb{R}^n)$. This follows from the identification $\mathcal{F}[H^\sigma(\mathbb{R}^n)] = L^{2,\sigma}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$.

In the positive direction, Yajima improved the wave-operator theory in [86] to include potentials who obeyed an integrability condition.

**Theorem** (Yajima 2006). In odd dimensions $n \geq 3$ if zero is regular, and we take $n_\star = \frac{n-1}{n-2}$, letting $\mathcal{F}$ denote the spatial Fourier transform, if $\mathcal{F}(\langle x \rangle^{2\sigma}V) \in L^{n_\star}(\mathbb{R}^n)$ for some $\sigma > \frac{1}{n_\star}$, and further $|V(x)| \lesssim \langle x \rangle^{-n-2-}$, then

$$\|e^{itH}P_{ac}(H)\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{n}{2}}.$$

This was extended to even dimensions.

**Theorem** (Finco and Yajima 2006). In even dimensions $n \geq 6$ if zero is regular, and we take $n_\star = \frac{n-1}{n-2}$, letting $\mathcal{F}$ denote the spatial Fourier transform, if $\mathcal{F}(\langle x \rangle^{2\sigma}V) \in L^{n_\star}(\mathbb{R}^n)$ for some $\sigma > \frac{1}{n_\star}$, and further
\[ |V(x)| \lesssim \langle x \rangle^{-n-2}, \text{ then} \]

\[ \|e^{itH}P_{ac}(H)\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{n}{2}}. \]

Unraveling these statements, one can see that the hypothesis on the Fourier transform of the potential is satisfied if more than \( \frac{n-1}{2} - \frac{1}{n-2} \) derivatives of \( V \) are in \( L^2(\mathbb{R}^n) \).

A string of papers using techniques of semi-classical analysis have recently been written, see [8, 10, 9, 58, 79, 80]. We focus on the five dimensional result of Cardoso, Cuevas and Vodev, [8].

**Theorem** (Cardoso, Cuevas and Vodev 2008). In dimension five, if \( V \in C^\alpha(\mathbb{R}^5) \) for some \( \alpha > 1 \) and \( |\nabla^j V(x)| \lesssim \langle x \rangle^{-5-j} \), then

\[ \|e^{itH}P_{ac}(H)\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{5}{2}}. \]

It was also conjectured in [8] that \( V \in C^{n-5/2}(\mathbb{R}^n) \) and \( |D^k V| \lesssim \langle x \rangle^{-2-k} \) for \( 0 \leq k \leq (n-3)/2 \) should imply (1.7).

Optimal smoothness results in dimensions five and seven were established by Erdo˘gan and Green, [16]. These results are the subject of the bulk of this chapter, and are stated in Theorem 1.1. We prove (1.7) under the optimal smoothness requirement in dimensions five and seven. We present a proof in the five dimensional case which is somewhat simpler than the seven dimensional case. The five dimensional problem is not easy by any measure, but the problem gets very complicated in dimensions seven and higher.

**Theorem 1.1** (Erdo˘gan and Green 2009). Assume that zero is not an eigenvalue\(^1\) of \( H = -\Delta + V \), \( V \in C^{(n-3)/2}(\mathbb{R}^n)^2 \) for \( n = 5, 7 \) with \( |V(x)| \lesssim \langle x \rangle^{-\beta} \) for some \( \beta > \frac{3n+5}{2} \) and for \( 1 \leq j \leq \frac{n-3}{2} \), \( |\nabla^j V(x)| \lesssim \langle x \rangle^{-\alpha} \) for some \( \alpha > 3 \) for \( n = 5 \) and \( \alpha > 8 \) for \( n = 7 \). Then

\[ \|e^{itH}P_{ac}\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{n}{2}}. \]

Many other questions in the field are open. \( L^1 \to L^\infty \) estimates are not the only kind of estimates that elucidate the dispersion nature of the Schrödinger equation. For Strichartz and Kato smoothing estimates for the magnetic Schrödinger operators, see for example [15] and [55]. When zero is not regular, it is known that a class of dispersive estimates hold, but often with a slower decay rate. See [18, 19, 24, 38, 60] for example.

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\(^1\)There cannot be a resonance at zero energy since \((-\Delta)^{-1}(x)^{-2}\) is bounded in \( L^2(\mathbb{R}^n) \) for \( n \geq 5 \).

\(^2\)In fact, we don’t need continuity of \( \nabla^{\frac{n-3}{2}} V \). It is easy to check from the proof that \( V \in C^{n-3} \) and the decay assumptions on \( |\nabla^j V(x)| \) for \( 0 \leq j \leq \frac{n-1}{2} \) suffice for the result.
Dispersive and Strichartz estimates for time-dependent potentials are also an interesting field of study. There is some work done in [5, 6, 26, 62] for example.

1.2 Spectral Theory Background

When the potential $V \neq 0$, one cannot express the solution to (1.1) as cleanly as in the free case, (1.4). One must also take care as $e^{itH}$ has not been established as an operator on any spaces. The results mentioned in Section 1.1 all have potentials $V$ that obey sufficient point-wise decay, $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 0$, or integrability conditions $V \in L^{p,\alpha}(\mathbb{R}^n)$ so that $H$ is asymptotically complete. That is the span of the eigenfunctions of $H$ and the absolutely continuous spectral subspace of $H$ are all of $L^2(\mathbb{R}^n)$. This is written

$$L^2(\mathbb{R}^n) = L^2_H(\mathbb{R}^n) \oplus L^2_{ac}(\mathbb{R}^n).$$

For these potentials, it is known by the Kato-Rellich Theorem that $H$ is self-adjoint. We note that it follows that $e^{itH}$ is an isometry on $L^2(\mathbb{R}^n)$. We thus have

$$\|e^{itH}P_{ac}(H)\|_{L^2 \to L^2} \lesssim 1.$$

Upon proving the $L^1 \to L^\infty$ dispersive bound

$$\|e^{itH}P_{ac}(H)\|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{n}{2}},$$

we can interpolate between the two to obtain a class of dispersive estimates. Specifically, for initial data $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ we have

$$\|e^{itH}P_{ac}\|_{L^p \to L^{p'}} \lesssim |t|^{-n\left(\frac{1}{2} - \frac{1}{p}\right)}, \quad t \neq 0. \quad (1.9)$$

Here $\frac{1}{p} + \frac{1}{p'} = 1$ and $p \in [1, 2]$. It is a well-known fact that via a $TT^*$ argument, the class of dispersive estimates in (1.9) yield a class of Strichartz estimates,

$$\|e^{itH}P_{ac}\|_{L^q_t L^p_x(\mathbb{R}^n)} \lesssim \|f\|_2, \quad \text{for } \frac{2}{q} + \frac{n}{p} = \frac{n}{2} \text{ and } q > 2.$$

The case of $q = 2$ was handled by different methods by Keel and Tao [44].

For a potential $V$ satisfying $|V(x)| \lesssim \langle x \rangle^{-2-}$, several important spectral theoretic results apply. The
Birman-Schwinger Theorem states that $H = -\Delta + V$ has only finitely many non-positive eigenvalues. Kato’s Theorem states that $H$ has no positive eigenvalues. The Kato-Rellich Theorem guarantees that $H$ is self-adjoint on $H^2(\mathbb{R}^n)$ and $\sigma_{ess}(H) = [0, \infty)$. Agmon-Kato-Kuroda Theorem yields $\sigma_{ac}(H) = \emptyset$, that is $\sigma_c(H) = \sigma_{ac}(H)$. These statements can be found in [61] for instance.

These spectral theoretic results characterize the spectrum of $H = -\Delta + V$ as

$$\sigma(H) = \sigma_p(H) \cup \sigma_{ac}(H) = \{\lambda_j \leq 0 : H\psi_j = \lambda_j \text{ for some } \psi_j \in H^2(\mathbb{R}^n)\}_{j=1}^N \cup [0, \infty). \quad (1.10)$$

Where $N < \infty$. Thus, the spectral theorem allows us to represent the evolution of (1.1) as an integral over the spectrum of $H$,

$$e^{itH} = \int_{\sigma(H)} e^{it\lambda} dE_H(\lambda). \quad (1.11)$$

Here $E_H$ is the unique spectral measure associated with the operator $H$. We note that (1.10) and (1.11) allow us to represent the evolution operator as

$$e^{itH} = \sum_{\lambda_j \in \sigma_p(H)} e^{it\lambda_j} P_j(H) + \int_{\sigma_{ac}(H)} e^{it\lambda} dE(\lambda). \quad (1.12)$$

$P_j$ are orthogonal projections onto the eigenspaces of $H$. If the initial data $f$ has non-zero projection onto an eigenspace, it is clear from (1.12), that $e^{itH}f(x)$ will not become arbitrarily small as $t \to \infty$. We now concern ourselves with projection away from the eigenvalues. It is known, in dimension three, that zero energy being an eigenvalue or a resonance will result in slower decaying $L^1 \to L^\infty$ estimates. For example, in [18, 19] it is shown that the three-dimensional solution operator maps with a bound of size $|t|^{-\frac{1}{2}}$ if zero is not regular.

As in [25, 27, 62], the starting point of our proof is to use Stone’s theorem on (1.12). This allows us to represent the evolution of the projection onto the absolutely continuous spectrum via the spectral representation (with $f, g \in S(\mathbb{R}^n)$)

$$\langle e^{itH} P_{ac} f, g \rangle = \int_0^\infty e^{it\lambda} \langle E'_{ac}(\lambda) f, g \rangle d\lambda = \frac{1}{2\pi i} \int_0^\infty e^{it\lambda} \langle [R_V^+(\lambda) - R_V^-(\lambda)] f, g \rangle d\lambda.$$

Here $E'_{ac}(\lambda)$ is the density of the absolutely continuous part of the spectral measure associated to $H$, and $R_V^\pm(\lambda) = (H - \lambda \pm i0)^{-1}$ is the resolvent of the perturbed Schrödinger equation.

It is known that $R_V^\pm(z)$ is a function of $\sqrt{z}$ and the $\pm i0$ determines which branch of the square root
The function is taken in the complex plane. The two branches stemming from the different choices do not agree with each other. The resolvent $R_V^\pm(z)$ are the limiting values as you approach the real line from above or below in the complex plane. The limit exists in the operator norm between weighted $L^2$ spaces. This process is explained in more detail in [3] and [15] in the proofs of the limiting absorption principle.

In light of these formulae, and a change of variable, (1.7) follows from

$$\sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi_L(\lambda) \langle [R_V^+(\lambda^2) - R_V^-(\lambda^2)] f, g \rangle d\lambda \right| \lesssim |t|^{-\frac{3}{2}} \|f\|_1 \|g\|_1, \quad (1.13)$$

where $\chi \in C_0^\infty(\mathbb{R})$ with $\chi = 1$ for $|\lambda| \leq 1$ and $\chi = 0$ for $|\lambda| > 2$, and $\chi_L(\lambda) = \chi(\frac{\lambda}{L})$. We note the resolvent identity for two closed operators $A$ and $B$, if $z \in \rho(A) \cap \rho(B)$,

$$R_A(z) - R_B(z) = R_A(z)[A - B]R_B(z).$$

Here $R_A$ and $R_B$ are the resolvents of $A$ and $B$ respectively.

This identity show that $R_V(z)$ can be expressed in terms of the free resolvent $R_0(z)$ as

$$R_V(z) = R_0(z) - R_0(z)VR_V(z).$$

This follows the above resolvent identity as $(-\Delta + V) - (-\Delta) = V$, see [32] for example. Upon iterating this identity $2m + 1$ times for some positive integer $m$ and using $R_0VVR_V = -VR_VR_0$, one obtains the symmetric finite Born series expansion

$$R_V(z) = \sum_{\kappa = 0}^{2m+1} (-1)^\kappa [V R_0(z)]^{\kappa} + [R_0(z)V]^{m+1}R_V(z)[VR_0(z)]^{m+1}. \quad (1.14)$$

The anti-symmetric identity $R_0VR_V = -VR_VR_0$ follows from the second resolvent identity $R_A[A - B]R_B = R_B[B - A]R_A$. We choose an odd number $2m + 1$ so that we can symmetrically have $m + 1$ iterations of $R_0V$ or $VR_0$ on either side of the $R_V$. We use the smoothing properties of the iterated free resolvents to map to a “nice” enough space on which $R_V$ acts.

In [29] (Theorem 4.1), Goldberg and Visan proved that under the assumptions of our Theorem 1.1 (in fact only the decay assumption for $V$ and regularity of zero are needed), if $m$ is sufficiently large, then (1.13) is satisfied for the contribution of the remainder term in (1.14). Therefore, Theorem 1.1 follows from the following

**Theorem 1.2.** If $V \in C^{(n-3)/2}(\mathbb{R}^n)$ for $n = 5, 7$ with $|\nabla^j V(x)| \lesssim \langle z \rangle^{-\beta}$, for some $\beta > 3$ when $n = 5$ and
\( \beta > 8 \) when \( n = 7, \) \( 0 \leq j \leq \frac{n - 3}{2} \) then for each \( \kappa \in \mathbb{N}, \) (1.13) is satisfied for the contribution of the \( \kappa \)th term of the Born series in (1.14).

Although we didn’t try to obtain sharp decay conditions on the potential and its derivatives in [16], it should be possible to obtain Theorem 1.2 under the condition \(|D^kV| \lesssim \langle x \rangle^{-2-k-} \) for \( 0 \leq k \leq (n-3)/2 \) by improving our integral estimates. However, this would add many more subcases to the proof, and with a lack of optimal estimates on the tail of the Born series, this is not a matter of primary concern.

**Remark 1.3.** In fact, in light of the results in Chapter 2, we can relax the assumptions on the potential in Theorem 1.1 in dimension five to only requiring \( \beta > 4, \) down from \( \beta > 10 \) in [16].

### 1.3 Contribution of the \( \kappa \)th term of the Born series

In this section we describe the basic idea behind the proof of Theorem 1.2. The technical details are in the later sections. We start with the properties of the free resolvent.

It is known that in dimensions \( n \geq 3, \) \( R_0(z) \) is an integral operator with kernel given by

\[
R_0(z)(x,y) = i \frac{z^{\frac{1}{2}}}{2\pi|x-y|} \frac{z^{\frac{n-3}{2}}}{2\pi|x-y|} H_{\frac{n-2}{2}}^{(1)}(z^{\frac{1}{2}}|x-y|). \tag{1.15}
\]

Here \( H_{\nu}^{(1)}(\cdot) \) is a Hankel function of the first kind of order \( \nu, \) a complex superposition of Bessel functions. The proof of this fact relies on the Fourier transform.

We use the following explicit representation for the kernel of the limiting resolvent operator \( R_0^\pm(\lambda^2) \) (see, e.g., [37])

\[
R_0^\pm(\lambda^2)(x,y) = G_n(\pm \lambda, |x-y|),
\]

where

\[
G_n(\lambda, r) = C_n e^{i\lambda r} \sum_{\ell=0}^{\frac{n-3}{2}} \frac{(n-3-\ell)!}{\ell!(\frac{n-3}{2}-\ell)!} (-2ir\lambda)^\ell. \tag{1.16}
\]

This expansion only holds in odd dimensions \( n \geq 3. \) The expansion in [37] relied on a series expansion for the Hankel functions of order \( \frac{n-3}{2}. \) We also define

\[
G_1(\lambda, r) = C_1 e^{i\lambda r}. \]

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Lemma 1.4. For $n \geq 3$ and odd, the following recurrence relation holds.

$$\left( \frac{1}{\lambda} \frac{d}{d\lambda} \right) G_n(\lambda, r) = \frac{1}{2\pi} G_{n-2}(\lambda, r).$$

Proof. We note the following recurrence relation for Hankel functions of order $\nu$ from [2],

$$\frac{d}{dz} H_\nu(z) = H_{\nu-1}(z) - \frac{\nu}{z} H_\nu(z).$$

Using the expansion for the resolvent (1.15), and writing $z = \lambda^2$, we have

$$\frac{1}{\lambda} \frac{d}{d\lambda} G_n(\lambda, |x - y|) = \frac{i}{4} \left[ \frac{n - 2}{2} \frac{1}{2\pi |x - y|} \left( \frac{\lambda}{2\pi |x - y|} \right)^{\frac{n-2}{2}} H_{\frac{n-2}{2}}^{(1)}(\lambda |x - y|) \right]$$

$$+ \left( \frac{\lambda}{2\pi |x - y|} \right)^{\frac{n-2}{2}} |x - y| \left\{ H_{\frac{n-2}{2}-1}^{(1)}(\lambda |x - y|) - \frac{1}{\lambda} H_{\frac{n-2}{2}}^{(1)}(\lambda |x - y|) \right\}$$

$$= \frac{1}{2\pi} \left[ \frac{\lambda}{2\pi |x - y|} \right]^{\frac{n-4}{2}} H_{\frac{n-4}{2}}^{(1)}(\lambda |x - y|)$$

$$= \frac{1}{2\pi} G_{n-2}(\lambda, |x - y|).$$

Since, with a slight abuse of notation which is standard in the literature, $R^-_0(\lambda^2) = R^+_0((-\lambda)^2)$, the contribution of the $\kappa^{th}$ term of the Born series, given as a summand in (1.14), to the integral given in (1.13) can be written as

$$\int_{-\infty}^{\infty} e^{it\lambda^2} \lambda \chi_L(\lambda) \langle R^+_0(\lambda^2) | V R^+_0(\lambda^2) \rangle^\kappa f, g \rangle d\lambda$$

$$= \int_{\mathbb{R}^{n(\kappa+2)+1}} e^{it\lambda^2} \lambda \chi_L(\lambda) \prod_{j=0}^{\kappa} G_n(\lambda, r_j) \prod_{l=1}^{\kappa} V(z_l) f(z_0) g(z_{\kappa+1}) d\lambda d\tilde{\lambda} dz_0 d\tilde{z} dz_{\kappa+1},$$

where for notational convenience we write $r_j = |z_j - z_{j+1}|$, and $d\tilde{z} = dz_1 \ldots dz_{\kappa}$. Thus, we need to prove that

$$\sup_{\lambda, z_0, z_{\kappa+1}} \left| \int_{\mathbb{R}^{n+1}} e^{it\lambda^2} \lambda \chi_L(\lambda) \prod_{j=0}^{\kappa} G_n(\lambda, r_j) \prod_{l=1}^{\kappa} V(z_l) d\tilde{\lambda} d\lambda \right| \lesssim |t|^{-n/2}. \quad (1.17)$$

Following the approach to the Born series taken in [62], we note that by $\frac{n-1}{2}$ successive integration by parts
in $\lambda$, one obtains

$$
\int_{\mathbb{R}} e^{it\lambda^2} \lambda f(\lambda) d\lambda = \left( \frac{1}{2it} \right)^{n-1} \int_{\mathbb{R}} e^{it\lambda^2} \lambda \left[ \frac{d}{d\lambda} \right]^{n-1} f(\lambda) d\lambda.
$$

(1.18)

for any $f \in C_0^{(n-1)/2}(\mathbb{R})$. We note that the cut-off function $\chi_L$ keeps the boundary terms from appearing.

In our case, $f(\lambda) = \chi_L(\lambda) \prod_{j=0}^{\kappa} G_n(\lambda, r_j)$. By Leibniz’s rule, and Lemma 1.4, we can write the integrand of the right hand side of (1.18) as a linear combination of the terms of the form:

$$
\lambda \left[ \left( \frac{1}{\lambda} \right)^{\alpha-1} \chi_L(\lambda) \right] \prod_{j=0}^{\kappa} G_{n-2\alpha_j}(\lambda, r_j),
$$

(1.19)

where $\alpha_1, \alpha_0, ..., \alpha_\kappa \in \mathbb{N}_0$ satisfy $\sum_{j=1}^{\kappa} \alpha_j = \frac{n-1}{2}$.

We first consider the case when no derivatives act on the cutoff function $\chi_L$, i.e. $\alpha_1 = 0$. Using (1.19), the contribution of this case to the integral in (1.17) can be written as a sum of terms of the form

$$
t^{(1-n)/2} \int_{\mathbb{R}^{n+1}} e^{it\lambda^2} \lambda^{\chi_L(\lambda)} \prod_{j=0}^{\kappa} G_{n-2\alpha_j}(\lambda, r_j) \prod_{k=1}^{\kappa} V(z_k) d\lambda.
$$

(1.20)

Note that by (1.16),

$$
\lambda \prod_{j=0}^{\kappa} G_{n-2\alpha_j}(\lambda, r_j) = e^{i\lambda \varphi_\kappa} P_{n,\kappa}(\lambda, r_0, ..., r_\kappa),
$$

(1.21)

where $\varphi_\kappa = \sum_{j=0}^{\kappa} r_j$ and $P_{n,\kappa}$ is a polynomial in $\lambda$ of degree $\kappa \frac{n-3}{2}$ with coefficients depending on $r_j$'s. For the $\lambda^N$ term in $P_{n,\kappa}$, we apply $N$ successive integration by parts in the variables $z_1, ..., z_\kappa$. That is, we integrate by parts up to $\frac{n-3}{2}$ times in each of the variables $z_1, ..., z_\kappa$. This requires that $V \in C^{\frac{n-3}{2}}(\mathbb{R}^n)$. To apply integration by parts, we use the identity

$$
e^{i\lambda \varphi_\kappa} = (\nabla_{z_j} e^{i\lambda \varphi_\kappa}) \frac{i \nabla_{z_j} \varphi_\kappa}{\lambda |\nabla_{z_j} \varphi_\kappa|^2}.
$$

(1.22)

We note that for $1 \leq j \leq \kappa$,

$$
\nabla_{z_j} \varphi_\kappa = \nabla_{z_j} (|z_{j-1} - z_j| - |z_j - z_{j+1}|) = -\left( \frac{z_{j-1} - z_j}{|z_{j-1} - z_j|} - \frac{z_j - z_{j+1}}{|z_j - z_{j+1}|} \right).
$$

For notational convience we denote $E_j := -\nabla_{z_j} \varphi_\kappa = \frac{z_{j-1} - z_j}{|z_{j-1} - z_j|} - \frac{z_j - z_{j+1}}{|z_j - z_{j+1}|}$. Since we gain a negative power
of \( \lambda \) from each application, we can rewrite

\[
(1.20) = t^{(1-n)/2} \int_{\mathbb{R}^{n+1}} e^{it\lambda^2} \chi_L(\lambda) e^{it\lambda^2} Z_{n,\kappa}(z_0, \vec{z}, z_{\kappa+1}) d\vec{z} d\lambda,
\]

with \( Z_{n,\kappa} \) independent of \( \lambda \). \( Z_{n,\kappa} \) is the result of repeated application of (1.22) in the \( z_1, z_2, \ldots, z_\kappa \) variables as necessary. This process is described in detail in Sections 1.3.1 and 1.4 for dimensions five and seven respectively.

It is well-known that the \( n \)-dimensional imaginary Gaussian has Fourier transform in the sense of distributions given by

\[
\hat{e^{it\lambda^2}}(\xi) = Ct^{-\frac{n}{2}} e^{i\xi^2/4t}.
\]

(1.24)

See [83] for example.

We note that by scaling of the Fourier transform and the fact that \( \chi \in \mathcal{S}(\mathbb{R}) \), we have

\[
\|\hat{\chi_L}\|_1 = \|\hat{\chi} \circ d_{L^{-1}}\|_1 = \|L^{-1}\hat{\chi} \circ d_L\|_1 = \|\hat{\chi}\|_1.
\]

Where the dilation operator \( d_a \) is defined by \( f \circ d_a(x) = f(ax) \).

Next, we use Parseval’s formula,

\[
\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is the standard \( L^2 \) pairing given by \( \langle f, g \rangle = \int f(x) \overline{g(x)} \, dx \). This fact together with the identity given in (1.24) above yields

\[
\sup_{z_0, z_{\kappa+1}} \sup_{L \geq 1} \| (1.20) \| \lesssim |t|^{-\frac{n}{2}} \sup_{L \geq 1} \|\hat{\chi_L}\|_1 \sup_{z_0, z_{\kappa+1}} \| Z_{n,\kappa}(z_0, \cdots, z_{\kappa+1})\|_1
\]

\[
\lesssim |t|^{-\frac{n}{2}} \sup_{z_0, z_{\kappa+1}} \| Z_{n,\kappa}(z_0, \cdots, z_{\kappa+1})\|_1.
\]

Where the \( L^1 \) norm is taken in each of the variables \( z_1, \ldots, z_\kappa \). This yields the statement of Theorem 1.2 (for the contribution of the terms with \( \alpha_{-1} = 0 \)) if we can prove that

\[
\sup_{z_0, z_{\kappa+1}} \| Z_{n,\kappa}(z_0, \cdots, z_{\kappa+1})\|_1 < \infty.
\]

(1.25)

Unfortunately, (1.25) holds only for \( n = 5 \) or \( \kappa = 1 \) (if \( V \) and \( \nabla V \) decay sufficiently rapidly at infinity). For
the higher values of $n$ and $\kappa$ one needs to setup the integration by parts more carefully. For this reason we first discuss the five dimensional case using a slight variation of the method above which is more suitable for generalization to higher dimensions.

### 1.3.1 Combined variable calculus

We note that the vector field in (1.22) that arises from performing integration by parts in the $z_j$ variable depends on the variables $z_{j-1}, z_j, z_{j+1}$. In particular

$$\nabla z_l E_{j-1,j,j+1} \neq 0 \text{ when } l \notin \{j-1, j, j+1\},$$

In dimension five, the powers of the line singularities remain below the critical threshold of $n-1 = 4$. However, in higher dimensions the resulting line singularities can quickly become non-integrable if care is not taken. We propose the following calculus to regulate the powers of resulting line singularities. This method, though not strictly necessary in dimension five, does simplify the analysis.

We define the corresponding combined variable as $A_I = (z_{i_1}, z_{i_2}, \ldots, z_{i_J}) \in \mathbb{R}^{nJ}$ with $z_i \in \mathbb{R}^n$. For $f : \mathbb{R}^{nk} \to \mathbb{R}$ and $F = (F_1, F_2, \ldots, F_J) : \mathbb{R}^{nk} \to \mathbb{R}^n$, we define

$$\nabla_{A_I} f := (\nabla_{i_1} f, \nabla_{i_2} f, \ldots, \nabla_{i_J} f),$$

$$\nabla_{A_I} \cdot F := \sum_{j=1}^J \nabla_{i_j} \cdot F_j,$$

where $\nabla_i = \nabla z_i$.

We perform integration by parts in the variable $A_I$ by using the identity below. We ignore the boundary terms in the integration by parts coming from the singularities. One can use smooth cut-off functions as explained in Section 1.3.4.

$$e^{i\lambda \bar{\varphi}_n} = (\nabla_{A_I} e^{i\lambda \varphi_n}) \cdot \frac{iF_I}{\lambda |F_I|^2}.$$
where \( F_I = (E_{i_1}, \ldots, E_{i_J}) \), and \( E_i = \frac{z_{i+1} - z_i}{|z_{i-1} - z_i|} - \frac{z_{i-1} - z_i}{|z_{i+1} - z_i|} \), as follows

\[
\int_{\mathbb{R}^n} e^{i\lambda \varphi_n} f(z_1, z_2, \ldots, z_{\kappa}) d\vec{z} = -\frac{i}{\lambda} \int_{\mathbb{R}^n} e^{i\lambda \varphi_n} \nabla_{A_I} \cdot \left( f(\vec{z}) \frac{F_I}{|F_I|^2} \right) d\vec{z}
\]

\[
= -\frac{i}{\lambda} \sum_{j=1}^J \int_{\mathbb{R}^n} e^{i\lambda \varphi_n} \nabla_{i_j} \cdot \left( f(\vec{z}) E_{i_j} \right) d\vec{z}
\]

\[
= -\frac{i}{\lambda} \sum_{j=1}^J \int_{\mathbb{R}^n} e^{i\lambda \varphi_n} \Phi_{I_i, i_j} f(\vec{z}) d\vec{z}. \tag{1.29}
\]

Here, for any index set \( I \subseteq \mathbb{N} \) and \( i \in I \),

\[
\Phi_{I_i, i_j} := \nabla_{i_j} \cdot \left( f \frac{E_i}{|F_I|^2} \right).
\]

For the \( \kappa^{th} \) term in (1.20), we focus on the highest \( \lambda \) power term. In Section 1.3.2, we outline how to deduce the lower power terms from the highest power term. First we apply the process in (1.29) with the index set \( I = \{1, 2, \ldots, \kappa\} \). Then, for each summand \( j \) in (1.29), we apply the same operation with the index set \( I \setminus \{i_j\} \). We continue in this manner by removing the used index from the index set in each step. After \( \kappa \) steps, we obtain a tree of height \( \kappa \), and we write \( \int_{\mathbb{R}^n} e^{i\lambda \varphi_n} f(z_1, z_2, \ldots, z_{\kappa}) d\vec{z} \) as a finite sum (with each summand corresponding to a length \( \kappa \) branch in the tree) of integrals of the form

\[
\left( -\frac{i}{\lambda} \right)^\kappa \int_{\mathbb{R}^n} e^{i\lambda \varphi_n} \Phi_{I_{i_1}, i_2} \cdots \Phi_{I_{i_{\kappa}}, i_1} f(\vec{z}) d\vec{z},
\]

where \( I_1 = \{1, \ldots, \kappa\} \), \( i_j \in I_j \) for each \( j \), and \( I_j \setminus i_j = I_{j+1} \) for each \( j = 1, 2, \ldots, \kappa - 1 \). Using this with

\[
f(\vec{z}) = \lambda^n \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2 - \alpha_j}} \prod_{k=1}^{\kappa} V(z_k),
\]

the leading \( \lambda \) power term of (1.21) multiplied by the potentials (for \( n = 5 \)), now viewed as a function of \( z_1, z_2, \ldots, z_{\kappa} \) with coefficient depending on \( \lambda \). We see that the contribution of this term to \( Z_{5,\kappa} \) in (1.23) is of size

\[
\left| \Phi_{I_{i_1}} \cdots \Phi_{I_{i_{\kappa}}} \left( \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2 - \alpha_j}} \prod_{k=1}^{\kappa} V(z_k) \right) \right|.
\]

Therefore, in light of the discussion following (1.23), the proof (for \( \alpha_{-1} = 0 \) and for the leading term in \( P_{5,\kappa} \)) follows from the following

**Proposition 1.5.** Under the hypothesis of Theorem 1.2 in dimension five, for each \( \kappa \in \mathbb{N} \), for each
\(\alpha_0, \ldots \alpha_\kappa \in \mathbb{N}_0, \sum_j \alpha_j = 2, \text{ and for each sequence } \{I_j, i_j\} \text{ as defined above, we have} \)

\[
\sup_{z_0, \ldots, z_{\kappa + 1}} \left\| \Phi_{I_0, i_0} \ldots \Phi_{I_\kappa, i_\kappa} \left( \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2-\alpha_j}} \prod_{k=1}^{\kappa} V(z_k) \right) \right\|_{L^1(\mathbb{C})} < \infty.
\]

The only difference in higher dimensions is that one should be more careful about the choice of the variables in \(A_I\). Instead of working with \(z_1, z_2, \ldots, z_\kappa\), we apply integration by parts in more suitable variables.

The first step in the proof of Proposition 1.5 is the following

**Lemma 1.6.** For any sequence \(\{I_j, i_j\}\) as defined above, we have

\[
\left| \Phi_{I_0, i_0} \ldots \Phi_{I_\kappa, i_\kappa} \left( \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2-\alpha_j}} \prod_{k=1}^{\kappa} V(z_k) \right) \right| \leq \prod_{l=1}^{\kappa} \left( \frac{|\nabla V(z_l)|}{|E_l|} + \frac{|V(z_l)|}{|E_l|^2} \left( \frac{1}{|z_{l-1} - z_l|} + \frac{1}{|z_l - z_{l+1}|} \right) \right) \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2-\alpha_j}},
\]

\[
\leq \prod_{l=1}^{\kappa} \frac{(z_l)^{-3}}{|E_l|^2} \left( 1 + \frac{1}{|z_{l-1} - z_l|} + \frac{1}{|z_l - z_{l+1}|} \right) \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2}} \sum_{i=0}^{\kappa} |z_i - z_{i+1}|^{2} \quad (1.31)
\]

**Proof.** The first inequality follows from the following simple observations. We leave the proof to the reader.

\[
|\nabla_j \cdot E_i| \lesssim \left( \frac{1}{|z_{j-1} - z_j|} + \frac{1}{|z_j - z_{j+1}|} \right), \text{ for } i = j-1, j, j+1
\]

\[
|\nabla_j |F_l|^{-1}| \lesssim |F_l|^{-2} \left( \frac{1}{|z_{j-1} - z_j|} + \frac{1}{|z_j - z_{j+1}|} \right)
\]

\[
|\nabla_j |F_l|^{-1}| = 0, \quad \text{if } I \text{ does not contain } j-1, j, j+1.
\]

For the first inequality, we note

\[
\nabla_z \cdot \frac{z-x}{|z-x|} = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \left[ \frac{z_i-x_i}{|z-x|} \right] = \sum_{i=1}^{n} \frac{1}{|z-x|} - \frac{(z_i-x_i)(z_i-x_i)}{|z-x|^3} = \frac{n-1}{|x-z|} - \frac{(x-z) \cdot (x-z)}{|x-z|^3} = \frac{n-1}{|x-z|}
\]

The other necessary derivative calculations are confined to Section 1.7 for the reader’s convenience.

Moreover, these inequalities remain valid if one applies the same \(\Phi_{I, i}\) operator to both sides of the inequality. When we apply \(\Phi_{I, j}\) in (1.30), depending on where \(\nabla_{z_j}\) acts, one gets an additional contribution.
of either \( \frac{|\nabla V(z_j)|}{|F_I|} \) (since \( |E_j| \leq |F_I| \)), or for some \( J \supseteq I \),

\[
\frac{|V(z_j)|}{|F_I||F_J|} \left( \frac{1}{|z_{j-1} - z_j|} + \frac{1}{|z_j - z_{j+1}|} \right) \leq \frac{|V(z_j)|}{|F_I|^2} \left( \frac{1}{|z_{j-1} - z_j|} + \frac{1}{|z_j - z_{j+1}|} \right),
\]

(1.32)

The derivatives may also act on \( |z_j - z_{j+1}| \) terms whose effect can also be bounded by the right hand side of (1.32). This proves (1.30) with \( E_I \) on the right hand side replaced with \( F_I \) for some \( I \) containing \( l \).

The second inequality follows immediately by the decay assumptions on \( V \) and the inequalities 

\[
|F_I| \geq |E_l| \quad \text{and} \quad |E_l| \leq 2,
\]

and the elementary fact that for \( a_j > 0 \) and \( \alpha_j \in \mathbb{N}_0 \) with \( \sum_j \alpha_j = A \),

\[
\prod_{j=0}^m \left( \frac{1}{a_j} \right)^{\alpha_j} \lesssim \sum_{j=0}^m \left( \frac{1}{a_j} \right)^A.
\]

(1.33)

First we introduce some notation.

**Lemma 1.7.** For \( x, z, w, y \in \mathbb{R}^n \), \( x \neq z \), \( w \neq y \), let \( E_{xzw} \) denote the vector field \( \frac{x - z}{|x - z|} - \frac{w - y}{|w - y|} \), then

\[
|E_{xzw}| \approx \angle(x \bar{z}, w \bar{y}), \quad |E_{xzwy}| \approx \max(\angle(x \bar{z}, x \bar{w}), \angle(z \bar{w}, x \bar{w})).
\]

(1.34)

**Proof.** For notational convenience, let \( u = \frac{x - z}{|x - z|} \), \( v = \frac{w - y}{|w - y|} \) be unit vectors. We note that if \( \theta = \angle(x \bar{z}, w \bar{y}) \),

\[
|u - v|^2 = (u - v) \cdot (u - v) = 2(1 - u \cdot v) = 2(1 - |u| |v| \cos \theta)
\]

\[
= 2(1 - \cos \theta) \approx \theta^2.
\]

This approximation is valid when \( \theta \) is small, however when \( \theta \gtrsim 1 \) the statement is trivial.

The second statement follows from the first and a simple geometric observation, assume without loss of generality that \( |z - x| < |z - w| \), denote the angle between the two unit vectors again by \( \theta \), let \( \angle(x \bar{z}, x \bar{w}) = \alpha \), and \( \angle(z \bar{w}, x \bar{w}) = \beta \). Since \( \theta = \alpha + \beta \), \( \theta \approx \max(\alpha, \beta) \).

**Figure 1.1:** Figure of the angles in Lemma 1.7.

With this notation, we have \( E_j = E_{z_{j-1}z_jz_{j+1}} \). In five dimensions we need estimates of the following
kind. Fix three distinct points \(x, w, y \in \mathbb{R}^n\). Assume that \(w\) is not on the line segment connecting \(x\) to \(y\), or equivalently, \(E_{xwwy} \neq 0\). As this is a measure zero set for \(w\), and we integrate in \(w\) after integration in \(z\), we can safely make this assumption. Consider integrals of the form

\[
\int_{\mathbb{R}^n} \frac{(z)^{-3}dz}{|x - z|^k|z - w|^\ell |E_{xzzw}|^{n-3}|E_{zwwy}|^{n-3}},
\]

(1.35)

with \(0 \leq k, \ell \leq n - 1\) and \(n - 3 \leq k + \ell\). For the first term in the Born series, only the first line singularity occurs. Note that this integral has two point singularities and two line singularities. The assumption \(E_{xwwy} \neq 0\) implies that the line singularities are separated from each other by some angle. It also implies that the point singularity at \(x\) is away from the line singularity \(E_{zwwy}\). Accordingly, our estimates depend on the angle \(|E_{xwwy}|\), and also on the length \(|x - w|\).

The proof of the following theorems are technical and are given in Section 1.5.

**Theorem 1.8.** Fix \(0 \leq k, \ell \leq n - 1, n - 3 \leq k + \ell, k + \ell \neq n, \) and \(x, w \in \mathbb{R}^n\). Then

\[
\int_{\mathbb{R}^n} \frac{(z)^{-3}dz}{|x - z|^k|z - w|^\ell |E_{xzzw}|^{n-3}|E_{zwwy}|^{n-3}} \lesssim \begin{cases} 
\left(\frac{1}{|x - w|}\right)^{\max(0,k+\ell-n)} & |x - w| \leq 1 \\
\left(\frac{1}{|x - w|}\right)^{\min(k,\ell,k+\ell+3-n)} & |x - w| > 1
\end{cases}
\]

The preceding Theorem, along with its obvious generalization given in the following Corollary to the cases in which the power of the line singularity is less than \(n - 3\), suffice for the first term of the Born series in any odd dimension.

**Corollary 1.9.** Fix \(0 \leq k, \ell \leq n - 1, m \leq n - 2, m \leq k + \ell, k + \ell \neq n, \beta > 0\) such that \(k + \ell + \beta \geq n\) and
Remark 1.10. We note that the line singularities other than $E_{zww}$ involving $z$ are determined by a base-point, either $x$ or $w$, and a direction vector $\vec{v}$. We define $E_{x,\vec{v}}(z) = \angle(\vec{x}z, \vec{v})$. For instance, $|E_{zwuy}|^{-1}$ is singular along the line emanating from $w$ with direction vector $\vec{y}w$. Thus

$$E_{zwuy} = E_{w,\vec{y}w}(z).$$

Similarly, note that $E_{zxwy} = E_{x,\vec{y}w}(z)$ and $E_{zwyu} = E_{w,\vec{y}w}(z)$.

The following theorem will suffice for nearly all cases that arise in this paper in dimensions five and seven.

**Theorem 1.11.** Fix $0 \leq k, \ell \leq n - 1$, $n - 3 \leq k + \ell$, $x, w \in \mathbb{R}^n$ and a vector $\vec{v} \in \mathbb{R}^n$. Assume $\alpha := \angle(\vec{v}, \vec{w}x) > 0$, then for any $F, G \in \{E_{zww}, E_{w,\vec{v}}(z), E_{x,\vec{v}}(z)\}$, $F \neq G$, we have: if $k + \ell \neq n$, then

$$\int_{\mathbb{R}^n} \frac{\langle z \rangle^{-3} dz}{|x - z|^{k} |z - w|^{\ell} |F|^{n-3} |G|^{n-3}} \lesssim \alpha^{-(n-3)} \left\{ \begin{array}{ll}
\left( \frac{1}{|x-w|} \right)^{\max(0,k+\ell-n)} & |x-w| \leq 1 \\
\left( \frac{1}{|x-w|} \right)^{\min(k,\ell,k+\ell+3-n)} & |x-w| > 1
\end{array} \right..$$

If $k + \ell = n$, then

$$\int_{\mathbb{R}^n} \frac{\langle z \rangle^{-3} dz}{|x - z|^{k} |z - w|^{\ell} |F|^{n-3} |G|^{n-3}} \lesssim \alpha^{-(n-3)} \left\{ \begin{array}{ll}
\left( \frac{1}{|x-w|} \right)^{0+} & |x-w| \leq 1 \\
\left( \frac{1}{|x-w|} \right)^{-\min(k,\ell,3)} & |x-w| > 1
\end{array} \right..$$

Remark 1.12. Note that this theorem applies to (1.35) with $\alpha \approx |E_{xwuy}|$. In fact, for every line singularity except $E_{zwyu}$ in Remark 1.10, we have $\alpha \approx |E_{xwuy}|$. When the singularity $|E_{zwyu}|$ appears, then $\alpha \approx |E_{xwuy}|$.

The following weaker version of this theorem will be used often:

**Corollary 1.13.** Under the assumptions of Theorem 1.11, we have

$$\int_{\mathbb{R}^n} \frac{\langle z \rangle^{-3} dz}{|x - z|^{k} |z - w|^{\ell} |F|^{n-3} |G|^{n-3}} \lesssim \alpha^{-(n-3)} \left( \frac{1}{|x-w|} \right)^{\min(k,\ell,k+\ell+3-n)}.$$

**Proof.** This follows immediately from Theorem 1.11 if $k + \ell \neq n$ since $\min(k, \ell, k+\ell+3-n) \geq \max(0, k+\ell-n)$. 

$$x, w \in \mathbb{R}^n. \text{ Then}$$

$$\int_{\mathbb{R}^n} \frac{\langle z \rangle^{-3} dz}{|x - z|^{k} |z - w|^{\ell} |E_{zww}|^{n}} \lesssim \left\{ \begin{array}{ll}
\left( \frac{1}{|x-w|} \right)^{\max(0,k+\ell-n)} & |x-w| \leq 1 \\
\left( \frac{1}{|x-w|} \right)^{\min(k,\ell,k+\ell+3-n)} & |x-w| > 1
\end{array} \right..$$
If \( k + \ell = n \), first use the inequality

\[
\frac{1}{|x - z|^k|z - w|} \lesssim \frac{1}{|x - w|^{\min(k,\ell)}} \left[ \frac{1}{|x - z|^{\max(k,\ell)}} + \frac{1}{|z - w|^{\max(k,\ell)}} \right],
\]

then apply the first part of Theorem 1.11 with \( k, \ell \) replaced by 0, \( \max(k, \ell) \) and vice versa. 

Now, we prove Proposition 1.5 using these estimates.

Proof of Proposition 1.5. First we consider the case \( \kappa = 1 \). Using (1.31), we need only show

\[
\sup_{z_0, z_2} \int_{\mathbb{R}^5} \left\langle z_1 \right\rangle^{-3} \left| z_0 - z_1 \right|^{m_0} \left| z_1 - z_2 \right|^{m_1} |E_1|^2 d\tilde{z}_1 < \infty,
\]

where, by (1.31) (for each fixed value of \( i \) in the inner sum), we have the following restrictions on \( m_0 \) and \( m_1 \):

\[
m_0, m_1 \geq 0, \text{ and } 2 \leq m_0 + m_1 \leq 3.
\]

This immediately follows from Theorem 1.8.

Now we consider the case \( \kappa > 1 \). Similarly using (1.31), it suffices to prove that

\[
\sup_{z_0, z_{\kappa+1}} \int_{\mathbb{R}^5} \frac{1}{|z_0 - z_1|^{m_0} |z_1 - z_2|^{m_1}} \prod_{\ell=1}^{\kappa} \left[ \left\langle z_\ell \right\rangle^{-3} \left| z_\ell - z_{\ell+1} \right|^{m_\ell} |E_{\ell}|^2 \right] d\tilde{z} < \infty. \tag{1.36}
\]

Where, by (1.31) (for each fixed value of \( i \) in the inner sum), we have the following restrictions on \( m_0 \) and \( m_1 \):

\[
m_0, m_1 \geq 0, \text{ and } 2 \leq m_0 + m_1 \leq 3.\]

Moreover, following Lemma 1.6 we have the following two possible cases:

i) \( m_\ell \geq 2 \) for each \( \ell \),

ii) \( m_j \in \{0, 1\} \) for some \( j \), and \( m_\ell \geq 2 \) for all \( \ell \neq j \).

Case i) By Corollary 1.13, noting that \( \alpha \approx |E_{0223}| \), we estimate the \( z_1 \) integral in (1.36) as follows

\[
\int_{\mathbb{R}^5} \left\langle z_1 \right\rangle^{-3} \left| z_0 - z_1 \right|^{m_0} \left| z_1 - z_2 \right|^{m_1} |E_1|^2 |E_2|^2 d\tilde{z}_1 \lesssim |E_{0223}|^{-2} \left( \frac{1}{|z_0 - z_2|} \right)^{m_1},
\]

where \( m_1' = \min(m_0, m_1, m_0 + m_1 - 2) \). Since \( m_0 \leq 3 \) and \( m_0, m_1 \geq 2 \), we have \( 2 \leq m_1' \leq 3 \).

By repeatedly applying Corollary 1.13 as above, we estimate the \( z_2, ..., z_{\kappa-2} \) integrals by

\[
|E_{0,\kappa-1,\kappa-1,\kappa}|^{-2} \left( \frac{1}{|z_0 - z_{\kappa-1}|} \right)^{m_{\kappa-2}'},
\]

where \( 2 \leq m_{\kappa-2}' \leq 3 \). We use \( m_j' \) to denote the leftover power of \( 1/|z_0 - z_{j+1}| \) after we estimate the \( z_j \) integrals.
integral. For $z_{\kappa-1}$ integral we use the other bound in Theorem 1.11 to estimate (1.36) by

$$
\int_{R^{10}} \frac{(z_{\kappa})^{-3}}{|z_0 - z_{\kappa}|^{m_{\kappa-1}}|z_{\kappa} - z_{\kappa+1}|^{m_{\kappa}}|E_{0,\kappa,\kappa+1}|^2} \, dz
$$

where $m'_{\kappa-1} = \max(0 + m'_{\kappa-2} + m_{\kappa-1} - 5) \in (0, 2]$. This integral is $\lesssim 1$ by Theorem 1.8 since

$$
2 \leq m'_{\kappa-1} + m_{\kappa} \leq m'_{\kappa-2} + m_{\kappa-1} + m_{\kappa} - 5 \leq 3 + 6 - 5 = 4.
$$

**Case ii)** $m_j \in \{0, 1\}$ for some $j$, and $m_{\ell} \geq 2$ for all $\ell \neq j$. Without loss of generality, we can assume that $j < \kappa$ (if $j = \kappa$, reverse the ordering of $z_1, \ldots, z_\kappa$). For $\ell < j - 1$ we estimate the $z_\ell$ integrals as in the first case, which gives $2 \leq m'_{j-1} \leq 3$. Since $m_j \in \{0, 1\}$, Corollary 1.13 implies that $m'_j = m_j$. We continue to apply Corollary 1.13 for $\ell = j + 1, \ldots, \kappa - 1$. Noting that $m'_\ell = m_j$ for $\ell = j, \ldots, \kappa - 1$ we estimate

$$
(1.36) \lesssim \sup_{z_0, z_{\kappa+1}} \int_{R^5} \frac{(z_{\kappa})^{-3}}{|z_0 - z_{\kappa}|^{m_j}|z_{\kappa} - z_{\kappa+1}|^{m_{\kappa}}|E_{0,\kappa,\kappa+1}|^2} < \infty.
$$

The last inequality follows from Theorem 1.8 since $m_j \in \{0, 1\}$, and $2 \leq m_{\kappa} \leq 3$.

### 1.3.2 Contribution of the lower order terms of $P_{5,\kappa}$ for $\alpha_{-1} = 0$

We reduce the lower $\lambda$ power terms to the highest $\lambda$ power case which we established in Proposition 1.5.

Fix $\alpha_0, \ldots, \alpha_\kappa$ as above. We consider the contribution of $\lambda^{\kappa-1}$ term, in $P_{5,\kappa}$, the others are similar. By (1.16) and the definition of $P_{5,\kappa}$, see (1.21), this term can be written as a linear combination of

$$
\lambda^{\kappa-1} \frac{1}{|z_\ell - z_{\ell+1}|} \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2-\alpha_j}} \prod_{k=1}^{\kappa} V(z_k), \quad \ell = 0, 1, \ldots, \kappa. \tag{1.37}
$$

Note that after applying the first integration by parts, see (1.29), to the leading term of $P_{5,\kappa}$, we obtain a monomial of degree $\kappa - 1$ in $\lambda$ which can be written as a sum of

$$
\Phi_{l,\ell} \left( \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{2-\alpha_j}} \prod_{k=1}^{\kappa} V(z_k) \right), \quad l = 1, \ldots, \kappa.
$$

The singularities of this term for $l = \ell$ or $l = \ell + 1$ are worse then the singularities of (1.37) since $|E_{\ell}| \lesssim 1$, see (1.31). Therefore, the rest of the procedure described before Proposition 1.5 finishes the proof for this term.
Similarly, the proof for the contribution of \( \lambda^{\kappa-K} \) term is done by comparing the coefficient with \( \Phi_{I_1,i_1} \cdots \Phi_{I_K,i_K} \left( \prod_{j=0}^{\kappa} \frac{1}{|z_j-z_{j+1}|^{2-\alpha_j}} \prod_{k=1}^{\kappa} V(z_k) \right), \quad l = 1, \ldots, \kappa. \)

for a suitable sequence \( \{I_1,i_1\}, \{I_2,i_2\}, \ldots, \{I_K,i_K\} \).

### 1.3.3 The case \( \alpha_{-1} \in \{1, 2\} \)

This will also follow from our previous discussion. First note that for any \( \alpha_{-1} \geq 1 \)

\[
\left( \frac{1}{\lambda} \frac{d}{d\lambda} \right)^{\alpha_{-1}} \chi_{L}(\lambda) = \frac{1}{\lambda^{2\alpha_{-1}}} \sum_{j=1}^{\alpha_{-1}} C_{\alpha_{-1},j} \left( \frac{\lambda}{L} \right)^{j} \chi^{(j)}(\lambda/L) =: \frac{1}{\lambda^{2\alpha_{-1}}} \bar{\chi}_{L}(\lambda).
\]

This follows from the product and chain rules and noting that each time a derivative acts, we either gain a power of \( L^{-1} \) if the derivative acts on the cut-off, or a power of \( \lambda^{-1} \) if the derivative acts on a \( \lambda^{-j} \) term.

**Lemma 1.14.** For \( \chi \in \mathcal{S}(\mathbb{R}) \) and any \( \alpha_{-1} \in \mathbb{N}_0 \),

\[
\mathcal{F} \left[ \left( \frac{1}{\lambda} \frac{d}{d\lambda} \right)^{\alpha_{-1}} \chi_{L}(\lambda) \right] \in L^1(\mathbb{R}).
\]

**Proof.** Since, for \( j \geq 1 \), \( \chi^{(j)} \) is a Schwarz function supported in the set \( |\lambda| \approx 1 \), and \( L > 1 \), we note the following

\[
\left( \frac{1}{\lambda} \frac{d}{d\lambda} \right)^{\alpha_{-1}} \chi_{L}(\lambda) = L^{-2\alpha_{-1}} \left( \frac{1}{\lambda} \frac{d}{d\lambda} \right)^{\alpha_{-1}} \chi \circ d_{L^{-1}}(\lambda).
\]

Thus, we have

\[
\left\| \mathcal{F} \left[ \left( \frac{1}{\lambda} \frac{d}{d\lambda} \right)^{\alpha_{-1}} \chi_{L}(\lambda) \right] \right\|_1 = L^{-2\alpha_{-1}} \left\| \mathcal{F} \left[ \left( \frac{1}{\lambda} \frac{d}{d\lambda} \right)^{\alpha_{-1}} \chi \circ d_{L^{-1}} \right] \right\|_1
\]

\[
= L^{-2\alpha_{-1}} \|\mathcal{F}(\bar{\chi})\|_1.
\]

Where \( \bar{\chi} \in \mathcal{S}(\mathbb{R}) \), and thus has \( L^1 \) Fourier transform. Thus, taking the supremum over \( L \geq 1 \), one obtains

\[
\sup_{L \geq 1} \left\| \mathcal{F} \left[ \left( \frac{1}{\lambda} \frac{d}{d\lambda} \right)^{\alpha_{-1}} \chi_{L}(\lambda) \right] \right\|_1 = \sup_{L \geq 1} L^{-2\alpha_{-1}} \|\mathcal{F}(\bar{\chi})\|_1
\]

\[
= \|\mathcal{F}(\bar{\chi})\|_1 < \infty.
\]
We present the case $\alpha_1 = 2$, the case $\alpha_1 = 1$ is essentially the same. In this case, the integral in (1.20) takes the form, with $r_j = |z_j - z_{j+1}|$,

$$
\int_{\mathbb{R}^{n+1}} e^{it\lambda^2} \tilde{\chi}_L(\lambda) \frac{1}{\lambda^3} \prod_{j=0}^{\kappa} G_n(\lambda, r_j) \prod_{k=1}^{\kappa} V(z_k) d\vec{z} d\lambda.
$$

(1.38)

Thus, (1.21) is replaced with

$$
\lambda^{-3} \prod_{j=0}^{\kappa} G_5(\lambda, r_j) = e^{i\lambda \varphi_n} \tilde{P}_{5,\kappa}(\lambda, r_0, \ldots, r_\kappa).
$$

(1.39)

The main difference from the case $\alpha_1 = 0$ is that $\tilde{P}_{5,\kappa}(\lambda, r_0, \ldots, r_\kappa)$ has two components, a polynomial of degree $\kappa - 2$, and a non-polynomial part containing the $\lambda$ singular terms, factors of $\lambda^{-1}$ and $\lambda^{-2}$. However, these terms do not create additional problems since $\lambda^{-N} \tilde{\chi}_L(\lambda)$ has $L^1$ Fourier transform by Lemma 1.14.

The leading term of $\tilde{P}_{5,\kappa}(\lambda, r_0, \ldots, r_\kappa)$ is given by

$$
\lambda^{\kappa-2} \prod_{j=0}^{\kappa} \frac{1}{r_j^2}.
$$

We perform $\kappa - 2$ integration by parts as described before Proposition 1.5. The resulting $\vec{z}$ integrals can be estimated in exactly the same way as in the case i) of the proof of Proposition 1.5. The proof for the lower order terms are done as in the previous section.

### 1.3.4 Justification of integration by parts with smooth cut-offs

In integration by parts, we use smooth cut-off functions around the singularities to eliminate the boundary terms. Let $\rho(x)$ be a smooth cut-off around zero, $\rho(x) = 1$ when $|x| > 2$ and $\rho(x) = 0$ when $|x| < 1$. Note the following.

**Lemma 1.15.** For $\rho \in C^\infty$ as above,

$$
\sup_{\epsilon} \left| \frac{d}{dx} \rho(x/\epsilon) \right| \lesssim \frac{1}{|x|}.
$$

**Proof.** We note that

$$
\left| \frac{d}{dx} \rho(x) \right| \begin{cases} 
0 & |x| < 1 \\
1 & 1 < |x| < 2 \\
0 & |x| > 2
\end{cases}
$$
Adding in the $\epsilon$ term, we have

$$\left| \frac{d}{dx} \rho \left( \frac{x}{\epsilon} \right) \right| \lesssim \begin{cases} 0 & |x| < \epsilon \\ \frac{1}{\epsilon} & \epsilon < |x| < 2\epsilon \lesssim \frac{1}{|x|} \\ 0 & |x| > 2\epsilon \end{cases}$$

Therefore applying Lemma 1.15, for any line singularity $F$,

$$\sup_{\epsilon} |\nabla_z \rho(|F|^2/\epsilon)| \lesssim \frac{1}{|F|^2} |\nabla_z |F|^2|$$

$$\lesssim \frac{1}{|F|} |\nabla_z |F||.$$

Which has the same size as if the derivative had acted on the line singularity $\frac{1}{|F|}$ itself. Higher order derivatives behave similarly. We also use the cut-off $\rho(| \cdot - z |^2/\epsilon)$ for point singularities, as

$$\sup_{\epsilon} |\nabla_z \rho(| \cdot - z |^2/\epsilon)| \lesssim \frac{1}{| \cdot - z |^2} |\nabla_z | \cdot - z |^2|$$

$$\lesssim \frac{1}{| \cdot - z |} |\nabla_z | \cdot - z ||.$$

Using these cut-offs, we have justified the lack of boundary terms from the integration by parts scheme in Section 1.3.

### 1.4 Seven Dimensional Case

As in the five dimensional case, we set up an integration by parts scheme. We ignore the issues of smooth cut-offs, derivatives acting on $\chi_L$ and lower order $\lambda$ terms, they are handled as in the five dimensional case. Consider the leading, $\lambda^{2k}$, term in the polynomial $P_{7,\kappa}$ of (1.21). Here we must perform $2\kappa$ integration by parts, the assumption $V \in C^2$ necessitates that we perform two integration by parts in each $z_j$ variable.

This introduces a new difficulty since differentiating $|z_j - z_{j+1}|^{-3}$ twice in both $z_j$ and $z_{j+1}$ leads to a non-integrable singularity.

To overcome this difficulty, we integrate by parts with respect to the new variable $b_j = z_j + z_{j+1}$ as needed by using the formula

$$e^{i\lambda \varphi_{\kappa}} = \frac{2}{i\lambda} \left( \nabla_{b_j} e^{i\lambda \varphi_{\kappa}} \right) \cdot \left( \frac{E_{j-1,j,j+1,j+2}}{E_{j-1,j,j+1,j+2}^2} \right), \quad (1.40)$$
where \( E_{ijkl} = \frac{z_i - z_j}{|z_i - z_j|} - \frac{z_k - z_l}{|z_k - z_l|} \) and \( \varphi_\kappa = \sum_{j=0}^\kappa |z_j - z_{j+1}| \). We can now perform integration by parts in the \( b_j \) variable without affecting the singularity involving \(|z_j - z_{j+1}|\) as \( \nabla_{b_j} |z_j - z_{j+1}| = 0 \). Integration by parts in this variable will allow us to avoid non-integrable point singularities.

We first discuss higher Born series terms, when \( \kappa > 2 \). The terms \( \kappa = 0, 1 \) are handled by known results about the free Schrödinger evolution and the discussion on the \( \kappa = 1 \) term following Theorem 1.11. In odd dimensions \( n \geq 7 \), one must take care with the second, \( \kappa = 2 \), Born series term. This is discussed in Section 1.4.2

1.4.1 Higher Born series terms

We first discuss how to handle the terms of the Born series with \( \kappa > 2 \). The highest \( \lambda \) power term has power \( 2\kappa \) and we wish to perform \( 2\kappa \) integration by parts twice in each of the \( z_1, z_2, \ldots, z_\kappa \) variables.

As in the five dimensional case, when we integrate by parts in the combined variables we obtain a sum of terms, in this case we get a tree of height \( 2\kappa \). Again, each branch of the tree represents a sequence of \( 2\kappa \) integration by parts. However, we note several ways in which the seven dimensional case differs from the five dimensional case. We start by integrating by parts in the combined variable \((z_1, z_2, \ldots, z_\kappa)\) on the function

\[
\varphi_\kappa = \sum_{j=1}^\kappa \prod_{\ell=1}^\kappa \frac{1}{|z_j - z_{j+1}|^{3-\alpha_j}} \prod_{\ell=1}^\kappa V(z_\ell).
\]

Recall that \( \alpha_j \in \mathbb{N}_0 \) are such that \( \sum \alpha_j = 3 \) in dimension seven. It is clear that our choice of index sets in the five dimensional case will not generalize easily to a procedure for integrating by parts in the seven dimensional case. In seven dimensions, we cannot remove a variable \( z_j \) from the combined variable after it is used as we need to use each \( z_j \) twice.

In the seven dimensional case, the use of combined variables is strictly necessary to avoid the line singularity terms becoming too large. One obtains line singularities of order \( n-1 = 6 \), which are non-integrable, if one does not use the combined variable approach. We note that the combined variables results in a sum of terms for the combined variable \( \mathcal{A} = (a_1, a_2, \ldots, a_I) \) with associated combined line singularity \( F = (F_1, F_2, \ldots, F_I) \),

\[
\int_{\mathbb{R}^7} e^{i\lambda \varphi_\kappa} f(\bar{z}) \, d\bar{z} = -i \int_{\mathbb{R}^7} e^{i\lambda \varphi_\kappa} \nabla \mathcal{A} \cdot \left( f(\bar{z}) \frac{F}{|F|^2} \right) \, d\bar{z}
\]

\[
= -i \sum_{j=1}^I \int_{\mathbb{R}^7} e^{i\lambda \varphi_\kappa} \nabla a_j \cdot \left( f(\bar{z}) \frac{F_j}{|F|^2} \right) \, d\bar{z}.
\]

Unlike the five dimensional case, we must keep track of the different summands that arise in each \( \nabla_j f(\bar{z}) \)

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term, as each derivative can act by increasing the power one of two point singularities or act on a potential. The choice of the next combined variable to use depends on the summand chosen. To denote this, we write

$$\nabla_{a_j} \cdot \left( f(\bar{z}) \frac{F_j}{|F|^2} \right) = f_{a_{j,1}} + f_{a_{j,2}} + f_{a_{j,3}}. \quad (1.41)$$

Where in $f_{a_{j,1}}$ the derivative increased the power on a point singularity, in $f_{a_{j,2}}$ the derivative increased the power on a different point singularity and in $f_{a_{j,3}}$ the derivative acted on a potential function. In dimension five, the derivative used determined a branch in the tree, but in dimension seven our scheme depends on the derivative used and the summand in (1.41).

We discuss our scheme for following a branch of the resulting tree, i.e. we select a summand in (1.41) after each integration by parts. We integrate by parts in a combined variable, starting with $(z_1, z_2, \ldots, z_\kappa)$ first, until one of the following occurs,

i. For some $j$, we integrate by parts in $z_j$ twice.

ii. We reach $|z_j - z_{j+1}|^{-6}$ for some $j$ and we have integrated by parts in $z_j$ or $z_{j+1}$ only once.

Note that these two criteria can occur simultaneously. If i. occurs and ii. does not, we simply remove $z_j$ from the combined variable. From here we restart the process with the combined variable that results from removing $z_j$ until we reach the above criteria again.

If ii. occurs and i. does not, we note that we must have that we are working with a summand in which, up to switching the roles of $z_j$ and $z_{j+1}$, the $z_j$ derivative has just acted on the point singularity and two $z_{j+1}$ derivatives acted on the same point singularity. Hence $z_{j+1}$ was removed from the combined variable by i. Here we remove $z_j$ from the combined variable and replace it with $b_j$.

If both i. and ii. occur simultaneously, we note that, again up to switching the roles of $z_j$ and $z_{j+1}$, a $z_j$ derivative has just acted on the point singularity for the second time and one $z_{j+1}$ derivative previously acted on the same point singularity. Here we remove both $z_j$ and $z_{j+1}$ from the combined variable and replace them with one $b_j$. Derivatives in $b_j$ can act on both $V(z_j)$ and $V(z_{j+1})$, however this requires no more differentiability on $V$ as neither of these potentials have been differentiated at this point. All the $z_j$ and $z_{j+1}$ derivatives have acted on a point singularity.

In each of these cases, we restart the process with the resulting modified combined variable. At this point we have added the $b_j$ variables to the process, adding another condition for which the combined variable changes.
iii. We integrate by parts in \( b_j \) once.

In the third case we simply remove \( b_j \) from the combined variable and restart the process. These three rules completely characterize the choice of combined variables in each branch.

We note that for the use of \( b_j \) variables to occur, three derivatives out of the four \( z_j \) and \( z_{j+1} \) derivatives must have acted on a single point singularity, in particular, we will never use both \( b_j \) and \( b_{j+1} \). To use a \( b_j \), three of the four available \( z_j \) and \( z_{j+1} \) derivatives have been used on \( |z_j - z_{j+1}| \) with a fourth derivative to be used as \( b_j \). In particular one \( z_{j+1} \) derivative could not have acted on \( |z_{j+1} - z_{j+2}| \). Thus the \( |z_j - z_{j+1}| \) singularity can only be acted on by three derivatives. Thus if we use both \( b_j \) and \( b_\ell \), it must be true that \(|j - \ell| \geq 2\).

For \( \ell \in \{1, 2, 3\} \), we define \( \Psi_{F,a,\ell} \) so that

\[
\Psi_{F,a,1}(f) + \Psi_{F,a,2}(f) + \Psi_{F,a,3}(f) = \nabla_a \cdot \left( f \frac{F}{|F|^2} \right),
\]

where the \( \ell \) selects the summand, as in (1.41), of the above operator on which we continue. Then, there is a sequence of combined line singularities \( J_1, J_2, \ldots, J_{2\kappa} \) determined by the choice of variables \( a_1, a_2, \ldots, a_{2\kappa} \) and a sequence in \( \{\ell_i\}_{i} \in \{1, 2, 3\}^{2\kappa} \) so that

\[
\Psi_{J_{2\kappa},a_{2\kappa},\ell_{2\kappa}} \cdots \Psi_{J_{1},a_{1},\ell_{1}} \left( \prod_{j=0}^{\kappa} \frac{1}{|z_j - z_{j+1}|^{3-a_j}} \prod_{i=1}^{\kappa} V(z_i) \right)
\]  

(1.42)

corresponds to a branch of the tree. Every branch can be represented as such.

A similar argument as in Lemma 1.6 along with the fact that \(|E_j|^2 |E_{j-1,j,j+1,j+2}|^{-2} \leq |E_j|^{-4} + |E_{j-1,j,j+1,j+2}|^{-4}\) (this is another application of (1.33)) implies that we can bound the contribution to the \( \vec{z} \) integral of the highest \( \lambda \) power of \( P_{7,\kappa} \) by a sum of integrals of the form

\[
\int_{R^{7\kappa}} \frac{1}{|z_0 - z_1|^{m_0} \prod_{j=1}^{\kappa} |z_j - z_{j+1}|^{m_j} |E_j|^4} d\vec{z}.
\]  

(1.43)

Here \( E_j \in \{E_j, E_{j-2,j-1,j,j+1}, E_{j-1,j,j+1,j+2}\} \), the line singularities corresponding to \( z_j, b_j \) and \( b_{j+1} \) respectively as the \( z_j \) can be replaced by \( b_{j-1} \) or \( b_j \) in this scheme. We also have the restriction on \( E_j \) that \( E_1 \neq E_{-1012} \) nor \( E_{\kappa} \neq E_{-1,\kappa,\kappa+1,\kappa+2} \), as we do not use \( b_0 \) or \( b_{\kappa} \). This is easy to see from the rules for selecting combined variables, and the fact that we don’t use the variables \( z_0 \) or \( z_{\kappa+1} \). We also have the restriction that arises from the \( b_j \) separation condition described above, namely that any sequence of line singularities cannot contain both \( E_{j-1,j,j+1,j+2} \) and \( E_{j,j+1,j+2,j+3} \), the line singularities for \( b_j \) and \( b_{j+1} \) respectively, for
any $1 \leq j \leq \kappa - 2$.

For instance, for $\kappa = 5$ we have a branch with the sequence of line singularities $(E_1, E_2, E_3, E_4, E_5) = (E_1, E_{0123}, E_{2345}, E_4, E_5)$ where in addition to using $z_1, z_2, \ldots, z_5$ we use $b_1$ once in place of $z_2$ and $b_3$ once in place of $z_3$.

Moreover, we have $3 \leq m_j \leq 6$ for all $j$ except possibly one $0 \leq m_{j_0} \leq 4$, and the constrictions that for $\ell \geq 1$,

$$3\ell \leq m_j + m_{j+1} + \cdots + m_{j+\ell} \leq 5\ell + 7,$$

for any $j + \ell \leq \kappa$, with the upper bound being $5\ell + 5$ if $j = 0$ with $\ell < \kappa$ and the upper bound is $5\kappa$ if we sum over all the $m_j$'s.

**Remark 1.16.** Since we are using the same estimate on $V$, $\nabla V$, and $\nabla^2 V$, we make no distinction on whether the derivatives act on the potential, the point singularities or the line singularities. In fact, one must be more careful when $\kappa = 2$, as we see in Section 1.4.2. A more thorough case-analysis would likely allow for weaker assumptions on the decay of the potential and its derivatives.

We have two types of line singularities, $z$ type, which arise from integration by parts in a $z_j$ and $b$ type, which arise from integration by parts in a $b_j$. We will call a line singularity $b_+$ type if $b_j$ acts in place of $z_j$ and $b_-$ type if $b_{j-1}$ acts in place of $z_j$. We summarize in the following table the types of line singularities that can occur from the $z_j$ variable according to the rules for selecting combined variables laid out after (1.41).

<table>
<thead>
<tr>
<th>Variable used for $z_j$</th>
<th>Line singularity produced</th>
<th>Line singularity type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_j$</td>
<td>$E_{j-1,j,j,j+1}$</td>
<td>$z$ type</td>
</tr>
<tr>
<td>$b_j = z_j + z_{j+1}$</td>
<td>$E_{j-1,j,j+1,j+2}$</td>
<td>$b_+$ type</td>
</tr>
<tr>
<td>$b_{j-1} = z_{j-1} + z_j$</td>
<td>$E_{j-2,j-1,j,j+1}$</td>
<td>$b_-$ type</td>
</tr>
</tbody>
</table>

We can view the line singularities as a sequence, with the $j^{th}$ entry corresponding to the type of line singularity that arises from the $z_j$ integration by parts. For the $\kappa^{th}$ term of the Born series, we have a sequence in $\{z, b_-, b_+\}^\kappa$. We note that the restriction on the use of the $b_j$ variables yields that the first entry in the sequence cannot be $b_-$ and the last entry cannot be $b_+$. The restrictions also imply that two $b_+$'s or two $b_-$'s must have at least one $z$ between them, a $b_+$ must have two $z$'s after it before a $b_-$ can occur. Integration takes a sequence of length $\kappa$ to a sequence of length $\kappa - 1$. In this notation, denoting integration
in $z_1$ by $\mapsto$, Theorem 1.11 (see the remark following the theorem) can be phrased as
\begin{align}
(z, z, Z)_k, (b_+, z, Z)_k, (z, b_-, Z)_k & \mapsto (z, Z)_{k-1}, \quad (1.45) \\
(z, b_+, Z)_k & \mapsto (b_+, Z)_{k-1}. \quad (1.46)
\end{align}

Where $Z$ is a sequence, the subscript is a placekeeper for the length of the sequence, and in $(z, z, Z)_k$ the first entry of $Z$ is not $b_-$. In a slight abuse of notation, if we use $\mapsto$ to denote integration in $z_1$ followed by integration in $z_2$, we can rephrase Theorem 1.18, which is stated below to estimate integrals with three line singularities involving $z_1$, as
\begin{equation}
(z, z, b_-, Z)_k \mapsto (z, Z)_{k-2}. \quad (1.47)
\end{equation}

We note that if we approach integration from $z_\kappa$ first instead of $z_1$, the sequence reverses order with $b_-$ and $b_+$ switching places. This is because from the point of view of $z_j$ a $b_-$ type singularity if of the form $E_{j-2,j-1,j,j+1}$. If we reverse the order of integration, we make the change $j \rightarrow \kappa - j + 1$ from the point of view of $z_j$, which is now $z_l = z_{\kappa-j+1}$ the singularity is of the form $E_{l+2,l+1,l,l-1} = -E_{l-1,l,l+1,l+2}$, which is $b_+$ type.

**Lemma 1.17.** For any integer $\kappa > 2$ and any sequence in $\{z, b_-, b_+\}^\kappa$ that arises in the integration by parts scheme for dimension seven, there exists a sequence of integrations such that the sequence can be reduced to $(z, z)$.

**Proof.** We establish this inductively. We take base cases $\kappa = 3$ and $\kappa = 4$. For $\kappa = 3$, by reversing the sequences, we need only consider the cases $(z, z, z), (z, b_-, z),$ and $(b_+, z, z)$. They are all handled by integrating first in $z_1$, by (1.45) the resulting sequence is $(z, z)$.

For $\kappa = 4$, we have the cases $(z, z, z, z), (z, b_-, z, z), (z, b_+, z, z), (b_+, z, z, z), (z, b_-, z, b_-), (z, b_-, b_+, z),$ and $(b_+, z, z, b_-)$. Using (1.45) and (1.46), the first four sequences are handled by successive integrations in $z_1$ and $z_2$, the last three are handled by integrations in $z_1$ and $z_4$. The case which would require (1.47) is avoided by reversing the order of integration. This can only be done because $\kappa = 4$ is too small to have a sequence at the start and end which would require this rule.

Now, we assume that every sequence of length $k \leq K_0$ can be reduced to $(z, z)$, we call such a sequence admissible. Now we take an arbitrary sequence that arises in the integration by parts scheme of length $K_0 + 1$. Call this sequence $(a, X)$ where $X \in \{z, b_-, b_+\}^{K_0}$. We note that (1.45), (1.46) and (1.47) all map a sequence to a shorter sequence that is better, in the sense that $b$ type singularities are converted to $z$ type,
or stay the same. Further, the new sequence follows the rules on the separation of \( b \) type singularities. If \( a = b_i \) then the first term in \( X \) must be \( z \) and we integrate in \( z_i \) to obtain an admissible sequence of length \( K_0 \). If \( a = z \), we can apply (1.45), (1.46) to obtain admissible sequences of length \( K_0 \) or apply (1.47) to obtain an admissible sequence of length \( K_0 - 1 \). 

Recall that Theorem 1.11 and Corollary 1.13 contain estimates for integrals involving the line singularities that \( b_j \) variables produce. We also need the following estimate, which handles the case when have three fourth power line singularities containing \( z_j \). This arises when one uses \( z_j, z_{j+1} \) and \( b_{j+1} \) in place of \( z_{j+2} \).

**Theorem 1.18.** Fix \( 0 \leq k, \ell, m \leq n - 1 \) satisfying \( k + m \geq n - 3 \), \( \ell + p \geq n - 3 \) where \( p = \max(0, k + m - n) \) or \( p = \min(k, m, k + m + 3 - n) \). Fix \( x, y, u \in \mathbb{R}^n \). Assume that \( \alpha := |E_{xuy}| > 0 \), then if \( k + m \neq n \) and \( \ell + p \neq n \),

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\langle z \rangle^{-3} \langle w \rangle^{-3}}{|z - w|^m |E_{xzzw}|^{n-3} |E_{zwuy}|^{n-3} |E_{zwuy}|^{n-3}} \, dz \, dw \lesssim \alpha^{-(n-3)} \left\{ \begin{array}{ll} \frac{1}{|x-y|} & |x-y| \leq 1 \\
\min(\ell, p, \ell+p+3-n) & |x-y| > 1 \end{array} \right.
\]

If \( k + m = n \) or \( \ell + p = n \),

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\langle z \rangle^{-3} \langle w \rangle^{-3}}{|z - w|^m |E_{xzzw}|^{n-3} |E_{zwuy}|^{n-3} |E_{zwuy}|^{n-3}} \, dz \, dw \lesssim \alpha^{-(n-3)} \left\{ \begin{array}{ll} \frac{1}{|x-y|^{\max(0, \ell+p-n)+}} & |x-y| \leq 1 \\
\min(\ell, p, \ell+p+3-n) & |x-y| > 1 \end{array} \right.
\]

The end estimate here is of the same form we would expect for estimating the \( z \) integral and then then \( w \) integral if each had two line singularities. With the two choices for \( p \), we can bound the point singularity with order \( \min(k, \ell, m, k + m - n, k + \ell + 3 - n, k + m + 3 - n, k + \ell + m + 6 - 2n) \) or \( \max(0, k + \ell + m - 2n) \) as needed.

To finish the proof in dimension seven, we further divide into subcases based on the size of \( m_0 + m_n \) to show that for any sequence \( \{\ell_i\} \in \{1, 2, 3\}^\kappa \), with \( \kappa > 2 \), and any sequence of variables \( a_i \) in the combined variables chosen by the integration by parts scheme,

\[
\sup_{z_0, z_{n+1}} |(1.43)| < \infty.
\]
Case i: First we consider the case when \( m_0 + m_\kappa \geq 4 \). We note that

\[
\langle x \rangle^{-1} \langle w \rangle^{-1} \lesssim \langle x - w \rangle^{-1} \lesssim |x - w|^{-1},
\]

which allows us to create point singularity decay. We divide into cases based on the size of \( m_0 + m_\kappa \). If \( m_0 + m_\kappa \leq 6 \), we use (1.48) so that with the constraints on the \( m_j \)’s in (1.44), we have the following bound.

\[
\left| (1.43) \right| \lesssim \int_{\mathbb{R}^\kappa} \frac{1}{|z_0 - z_1|^{m_0} \prod_{j=1}^{\kappa-1} |z_j - z_{j+1}|^{\tilde{m}_j} |E_j|^4 |z_\kappa - z_{\kappa+1}|^{m_\kappa} |E_\kappa|^4} \, dz,
\]

(1.49)

where \( \tilde{m}_j = \max(m_j, 4) \). This process of using (1.48) to create point singularities is why we need the potential and its derivatives to decay faster than eighth power at infinity. Consider when there is a \( j_0 \) such that \( m_{j_0} = 0 \) and \( m_{j_0+1} = 3 \). The potential \( V(z_{j_0}) \), or one of its derivatives, must contribute \( \langle z_{j_0} \rangle^{-4} \) to bring \( m_{j_0} \) up to fourth order, similarly it must contribute \( \langle z_{j_0} \rangle^{-1} \) to bring \( m_{j_0+1} \) up to fourth order. Thus, there is only \( \langle z_{j_0} \rangle^{5-\beta} \) decay left for the integration as in Theorems 1.8, 1.11, and 1.18. Since we need \( \beta - 5 > 3 \) for these results to apply, we see that we must take \( \beta > 8 \).

Now, as \( m_0 \leq 4 \) by assumption, \( \min(m_0, \tilde{m}_j, m_0 + \tilde{m}_j - 4) = m_0 \) for all \( 1 \leq j \leq \kappa - 1 \). We can estimate the \( z_1 \) integral by

\[
\left( \frac{1}{|z_0 - z_1|^{m_0}} \right)^{\tilde{m}_1} |\tilde{E}_2|^{-4}.
\]

With \( \tilde{E}_2 = E_{0223} \) or \( E_{0234} \) using Corollary 1.13 depending on which branch of the integration by parts tree we are considering. If necessary, we use Theorem 1.18 to estimate the \( z_1 \) and \( z_2 \) integrals by

\[
\left( \frac{1}{|z_0 - z_2|^{m_0}} \right) |E_{0334}|^{-4}.
\]

Similar calculations apply if we integrate in \( z_\kappa \) and \( z_{\kappa-1} \), where we use that \( \min(m_\kappa, \tilde{m}_j, m_\kappa + \tilde{m}_j - 4) = m_\kappa \). Repeatedly applying Corollary 1.13 and Theorem 1.18, for some \( j \), the final integral is bounded by

\[
\int_{\mathbb{R}^7} \langle z_j \rangle^{-3} \frac{1}{|z_0 - z_j|^{m_0} |z_j - z_{\kappa+1}|^{m_{\kappa+1}} |E_{0,j,j+1,\kappa+1}|^4} \, dz_j.
\]

This integral is \( \lesssim 1 \) by Theorem 1.8 since \( 4 \leq m_0 + m_\kappa \leq 6 \).

When \( m_0 + m_\kappa \geq 7 \), we must be more careful, in the approach above the final integral estimate would be unbounded as \( |z_0 - z_{\kappa+1}| \to 0 \). We have thus far used Corollary 1.13 as a labor-saving device, simplifying the iterated integration process with a less precise estimate. In dimension five, we had to take care in the final
integral using Theorem 1.11 directly instead of Corollary 1.13 in the second to last integral. In dimension seven, we need to take care with at most two integrations.

As \( m_0, m_\kappa \leq 5 \), since each point singularity from (1.19) is of order at most three and at most two derivatives act on \( r_0 \) and \( r_\kappa \), we need to consider when \( 7 \leq m_0 + m_\kappa \leq 10 \). By symmetry and (1.33) if necessary, we can assume that \( m_0 = 5 \), \( 2 \leq m_\kappa \leq 5 \). Using the constraints in (1.44) we deduce that either there exists \( 1 \leq j_1 < j_2 \leq \kappa - 1 \) with \( 0 \leq m_{j_1}, m_{j_2} \leq 4 \) or there exists \( 1 \leq j_0 \leq \kappa - 1 \) with \( 0 \leq m_{j_0} \leq 5 - m_\kappa \). Symmetrizing by (1.33) as above, we can assume in both cases that there is a \( j_0 \) with \( m_{j_0} \leq 3 \). Then using (1.48), we can guarantee all \( m_\ell \geq 4 \) for \( \ell \not\in \{j_0, \kappa\} \) and \( m_{j_0} = 3 \).

Now, we use Corollary 1.13 and Theorem 1.18, until we reach the \( z_{j_0-1} \) integral from the left or the \( z_{j_0+1} \) integral from the right. Note that if we approach from the left \( m'_{j_0-1} \in \{4, 5\} \), and if we approach from the right, \( m'_{j_0+1} \in \{2, 3, 4, 5\} \). In each of these cases, we do not need to pass forward the point decay to the final integral, we instead wish to control the size of the singularity. As such, we modify the estimates of Theorems 1.11, and 1.18 to bound by \( \alpha^{-4}|x-w|^{-\max(0+k+\ell-7)} \) since \( \min(k, \ell, k+\ell-4) \geq \max(0, k+\ell-7) \), we insert a “+” sign into the inequality for the case \( k+\ell = 7 \).

Assuming that we approached from left, we use \( m'_{j_0} = \max(0+, m'_{j_0-1} + m_{j_0} - 7) \in (0, 1] \) if \( m_\kappa \in \{4, 5\} \) and we use \( m'_{j_0} = \min(m'_{j_0-1}, m_{j_0}, m'_{j_0-1} + m_{j_0} - 4) = 3 \) if \( m_\kappa \in \{2, 3\} \). In both cases, we have reduced to the previous case in which \( 4 \leq m'_{j_0} + m_\kappa \leq 6 \). Continuing as in the previous case, the final integral is bounded by

\[
\int_{R^n} \frac{(z_j)^{-3}}{|z_0 - z_j|^{m'_{j-1}}|z_j - z_{\kappa+1}|^{m'_j}|E_{0,j,j,\kappa+1}|^4} dz_j \lesssim 1,
\]

by Theorem 1.8 since \( 4 \leq m'_{j-1} + m'_j \leq 6 \).

If we approach from the right, we use the same idea. Now, \( m'_{j_0} = \max(O+, m_{j_0+1} + m_{j_0} - 7) \in (0, 1] \) as \( m_{j_0} = 3 \) and \( m_{j_0+1} \leq 5 \). The second inequality follows from \( m_\ell \leq 6 \) and \( m_{j_0+1} \leq \max(0+, m_{j_0+1} + m'_{j_0+2} - 7) \leq 6 + 6 - 7 = 5 \). Now, \( m_0 + m_{j'_0} \in \{5, 6\} \), and we achieve boundedness as in the approach from the left.

**Case ii:** For the case when \( m_0 + m_\kappa = 3 \), by symmetry, i.e. (1.33), we can assume \( m_0 = 3 \), \( m_\kappa = 0 \). We can not use (1.48) here as we do not have a potential in \( z_0 \) or \( z_{\kappa+1} \). We are also guaranteed that \( m_\ell \geq 3 \) for all \( 1 \leq \ell \leq \kappa - 1 \). Lemma 1.17 tells us that there is a \( 1 \leq j \leq \kappa \) such that we can iterate Corollary 1.13 and Theorem 1.18, to bound (1.43) by (to do this we use (1.48) for \( \ell \neq j \) to ensure \( n_\ell \geq 4 \))

\[
\int_{R^n} \int_{R^n} \frac{(z_j)^{-7}(z_{j+1})^{-7} dz_j dz_{j+1}}{|z_0 - z_j|^3|z_j - z_{j+1}|^{m'_j}|E_{0,j,j,j+1}|^4|E_{j,j+1,j+1,\kappa+1}|^7}.
\]

Since the \( z_j \) potential is only used at most once in (1.48) and the \( z_{j+1} \) potential is likewise only used once.
The following establishes boundedness.

**Proposition 1.19.** Fix \(3 \leq \ell \leq 6\). Then

\[
\int_{\mathbb{R}^7} \int_{\mathbb{R}^7} \frac{\langle z \rangle^{-7-} \langle w \rangle^{-7-} \, dz \, dw}{|x - z|^3 |z - w|^\ell |E_{xzzw}|^4 |E_{zwzy}|^4} < \infty.
\]

### 1.4.2 The second term of the Born series

The second term of the Born series expansion, (1.14), can be handled in exactly the same manner as for \(\kappa > 2\) provided \(m_0 + m_2 \geq 4\). The only difference is that the last two singularities are not necessarily \(z\) type.

When \(m_0 + m_2 = 3\), which by (1.33) can be assumed to be \(m_0 = 3\) and \(m_2 = 0\), if both line singularities are of \(z\) type, that is if \(b_1\) was not used, we can apply Proposition 1.19.

When we use \(b_1\), since \(m_0 = 3\) and \(m_2 = 0\), \(b_1\) derivative must have acted on a potential. Therefore the line singularity from \(b_1\) has only power one. We also have \(m_1 = 6\), since \(b_1\) is used. The combined variables that were used are now either

i. \((z_1, z_2)\) three times followed by \(b_1\) once, or

ii. \((z_1, z_2)\) two times followed by \(z_1\) once and \(b_1\) once, or

iii. the same as ii. with \(z_2\) instead of \(z_1\).

The line singularities possible for case ii. are of the form

\[
\frac{1}{(|E_1|^2 + |E_2|^2)^2 |E_1|^2 |E_{0123}|}, \quad \frac{1}{(|E_1|^2 + |E_2|^2)^{5/2} |E_1||E_{0123}|} \lesssim \frac{1}{|E_1|^3 |E_2|^3 |E_{0123}|}.
\]

Case iii. is identical with \(E_1\) and \(E_2\) switching places. Case i. has \(|E_1|^2 + |E_2|^2 |E_{0123}|\) and is bounded in the same way. We need the following

**Proposition 1.20.** Fix \(x, w, y \in \mathbb{R}^7\). Assume \(\alpha := |E_{xwwy}| > 0\), then

\[
\int_{\mathbb{R}^7} \frac{\langle z \rangle^{-4-} \, dz}{|x - z|^3 |z - w|^6 |E_{xzzw}|^3 |E_{zwzy}|^3 |E_{xzwy}|} \lesssim \alpha^{-3} |x - w|^{-3}.
\]

Now there is enough point-wise decay in the resulting \(z_2\) integral to apply Corollary 1.9, to ensure boundedness in \(z_0\) and \(z_3\).

This yields Theorem 1.2 for \(n = 7\).
1.5 Proofs of integral estimates

In this section, we present proofs of theorems on the estimates for integrals involving point and line singularities. We start with estimates on the size of line singularities. For $0 < \alpha < 1$, define $T_\alpha(x, w)$ to be the intersection of solid cones of opening angle $\alpha$ from $x$ towards $w$ and from $w$ towards $x$. Define $E_\alpha(w, \vec{v})$ to be the solid cone of opening angle $\alpha$ from $w$ in direction $\vec{v}$. It is easy to see that outside $T_1(x, w)$, $|E_{xzzw}| \gtrsim 1$. Similarly, outside $E_1(w, \vec{v})$, $|E_{w,\vec{v}}(z)| \gtrsim 1$. The following lemmas are immediate from the definition of line singularities.

**Lemma 1.21.** Fix $x, w \in \mathbb{R}^n$. Let $r$ be the distance between a point $z \in \mathbb{R}^n$ and the line segment $\overline{xw}$.

i) For $z \in T_1(x, w)$, we have $|E_{xzzw}| \approx \frac{r}{\min(|x-z|, |w-z|)}$.

ii) For $0 < \alpha \leq 1$ and $z \notin T_\alpha(x, w)$, we have $|E_{xzzw}| \gtrsim \alpha$.

**Proof.** Part ii) follows immediately from Lemma 1.7. Part i) follows from the previousLemma and a simple geometric observation. Assume that $|x-z| < |z-w|$, then $\theta = \angle(x\vec{z}, x\vec{w})$ is the larger of the angles. For $0 < \theta < 1$, we have

$$\theta \approx \sin \theta = \frac{r}{|x-z|} = \frac{r}{\min(|x-z|, |z-w|)}.$$

**Lemma 1.22.** Fix $w \in \mathbb{R}^n$. Let $r$ be the distance between the point $z$ and the ray $\{w+s\vec{v}: s \geq 0\}$.

i) For $z \in E_1(w, \vec{v})$, we have $|E_{w,\vec{v}}(z)| \approx \frac{r}{|w-z|}$.

ii) For $0 < \alpha \leq 1$ and $z \notin E_\alpha(w, \vec{v})$, we have $|E_{w,\vec{v}}(z)| \gtrsim \alpha$.

**Proof.** As in Lemma 1.7 and 1.21, we have a superposition of unit vectors. It is clear that for $\theta = \angle(w\vec{z}, \vec{v})$,

$$|E_{w,\vec{v}}(z)|^2 = 2(1 - \cos \theta) \approx \theta^2.$$
The proof is now analogous to the previous Lemma.

The following lemma is used repeatedly in the rest of this section.

**Lemma 1.23.** I) Fix $u_1, u_2 \in \mathbb{R}^n$, and let $0 \leq k, \ell, k + \ell < n, \ h > 0$. We have

$$\int_{B(0, h) \subset \mathbb{R}^n} \frac{dz}{|z - u_1|^k |z - u_2|^\ell} \lesssim h^{n-k-\ell}$$

II) Fix $u_1, u_2 \in \mathbb{R}^n$, and let $0 \leq k, \ell < n, \beta > 0, \ k + \ell + \beta \geq n, \ k + \ell \neq n$. We have

$$\int_{\mathbb{R}^n} \frac{\langle z \rangle^{-\beta}dz}{|z - u_1|^k |z - u_2|^\ell} \lesssim \begin{cases} \frac{1}{|u_1 - u_2|} & |u_1 - u_2| \leq 1 \\ \left( \frac{1}{|u_1 - u_2|} \right)^{\max(0, k + \ell - n)} & |u_1 - u_2| > 1. \end{cases}$$

**Proof.** Proof of I) immediately follows from the inequalities

$$\frac{1}{|z - u_1|^k |z - u_2|^\ell} \lesssim \frac{1}{|z - u_1|^{k + \ell}} + \frac{1}{|z - u_2|^{k + \ell}},$$

and

$$\int_{B(0, h) \subset \mathbb{R}^n} \frac{dz}{|z - u_1|^{k + \ell}} \lesssim \int_{B(0, h)} \frac{dz}{|z|^{k + \ell}} \lesssim h^{n-k-\ell}.$$

Now, we consider part II. For $|u_1 - u_2| < 1$ and $k + \ell < n$, the inequality can be proved as in part I. For $|u_1 - u_2| < 1$ and $k + \ell > n$, ignore the $\langle z \rangle^{-\beta}$ term. We proceed by a standard argument of dividing $\mathbb{R}^n$ into four regions. First on $|z - u_1| < \frac{1}{2}|u_1 - u_2|$, which we denote $R_1$, we note that by the triangle inequality

$$\frac{1}{2}|u_1 - u_2| \leq |u_1 - u_2| - |z - u_2| \leq |z - u_1| + |u_1 - u_2| \leq \frac{3}{2}|u_1 - u_2|.$$ 

$$\int_{R_1} \frac{\langle z \rangle^{-\beta}dz}{|z - u_1|^k |z - u_2|^\ell} \lesssim |u_1 - u_2|^{-\ell} \int_{R_1} \frac{dz}{|z - u_1|^k} \lesssim |u_1 - u_2|^{-\ell} \int_0^{\frac{1}{2}|u_1 - u_2|} r^{n-1-k}dr \lesssim |u_1 - u_2|^{n-k-\ell}.$$

Where we switched to polar coordinates in the last integral. The region, $R_2$ where $|z - u_2| < \frac{1}{2}|u_1 - u_2|$ is identical in form with $u_1, k$ switching roles with $u_2, \ell$.

The third region $R_3$ is the complement of the first two regions subject to $|z - u_1| \leq 2|u_1 - u_2|$. Again,
by the triangle inequality $|z-u_1| \approx |z-u_2| \approx |u_1-u_2|$. 

$$\int_{R_3} \frac{(z)^{-\beta}}{|z-u_1|^k|z-u_2|^\ell} \, dz \lesssim |u_1-u_2|^{-k-\ell} \text{Vol}(R_3) \lesssim |u_1-u_2|^{n-k-\ell}.$$ 

Finally, on $R_4$, where $|z-u_1| \geq 2|u_1-u_2|$, we have $|z-u_1| \approx |z-u_2|$, and we use 

$$\int_{R_4} \frac{(z)^{-\beta}}{|z-u_1|^k|z-u_2|^\ell} \, dz \lesssim \int_{R_4} \frac{(z)^{-\beta}}{|z-u_1|^{k+\ell}} \, dz \lesssim ||\cdot||^{-k-\ell} \chi_{\{|\cdot| > |u_1-u_2|\}} \|\cdot\|^{-\beta} \|\cdot\|_{\infty} \lesssim |u_1-u_2|^{n-k-\ell}.$$ 

For $|u_1-u_2| > 1$, let $m := \min(k,\ell, k+\ell + \beta - n)$. Note that $m \geq 0$, $0 \leq k+\ell-m < n$, and $\beta + k+\ell - m \geq n$. The statement follows from the following inequality 

$$\frac{1}{|z-u_1|^k|z-u_2|^\ell} \lesssim \frac{1}{|u_1-u_2|^m} \left( \frac{1}{|z-u_1|^{k+\ell-m}} + \frac{1}{|z-u_2|^{k+\ell-m}} \right),$$ 

and Hölder's inequality. On $B(u_1,1)$, we ignore the $(z)^{-\beta}$ and bound by 

$$|u_1-u_2|^{-m} \int_{B(u_1,1)} \frac{1}{|z-u_1|} \, dz \lesssim |u_1-u_2|^{-m}.$$ 

Away from $u_1$, we have 

$$\int_{R_2} \frac{(z)^{-\beta}}{|z-u_1|^k|z-u_2|^\ell} \, dz \lesssim |u_1-u_2|^{-m} \int_{R_2} \frac{(z)^{-\beta}}{|z-u_1|^{k+\ell-m}} \, dz \lesssim |u_1-u_2|^{-m} \|\cdot\|^{-\beta} \|\cdot\|_p \|\cdot\|^{k+\ell-m} \chi_{\{|\cdot| > 1\}} \|\cdot\|_{p'} \lesssim |u_1-u_2|^{-m}.$$ 

Here, $k+\ell-m = \max(k,\ell, n-\beta)$. Taking $p = \frac{n}{\beta} +$ and $p' = \frac{n}{n-\beta}$. 

Now we are ready to prove Theorems 1.8 and 1.11. For the reader’s convenience, we restate the Theorems immediately before their respective proofs.
**Theorem 1.8** (1.8). Fix $0 \leq k, \ell \leq n - 1$, $n - 3 \leq k + \ell$, $k + \ell \neq n$, and $x, w \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} \frac{(z)^{-3} dz}{|x - z|^k |z - w|^\ell |E_{zzww}|^{n-3}} \lesssim \begin{cases} \left(\frac{1}{|x-w|}\right)^{\max(0, k + \ell - n)} & |x - w| \leq 1 \\ \left(\frac{1}{|x-w|}\right)^{\min(k, \ell, k + \ell + 3 - n)} & |x - w| > 1 \end{cases}. $$

*Proof of Theorem 1.8.* Outside of $T_1$, $|E_{zzww}| \approx 1$, and we can apply part II of Lemma 1.23 with $\beta = 3$. Divide $T_1$ into $T_{11}$ on which $|x - z| < |w - z|$ and $T_{12}$ on which $|x - z| > |w - z|$. 

![Figure 1.4: The region $T_{11}$ in the proof of Theorem 1.8.](image)

By symmetry, i.e. switching the roles of $|x - z|, k$ and $|z - w|, \ell$, it suffices to consider the integral on $T_{11}$. Let $h$ denote the distance between $x$ and the orthogonal projection of $z$ on to the line $\text{span}(x, w)$. We use the coordinates $z = (h, z^\perp) \in \mathbb{R} \times \mathbb{R}^{n-1}$, with $z^\perp$ the coordinate on the $n - 1$ dimensional plane perpendicular to $\text{span}(x, w)$. In this cone, we note that $0 \leq h \leq \frac{1}{2} |x - w|$ and $|z^\perp| \lesssim h$.

We note that $(h, 0)$ is the line $\text{span}(x, w)$. Note that by the triangle inequality $|z - w| \approx |x - w|$, and $|z - x| \approx h$ as $|z - x|^2 = h^2 + |z^\perp|^2 \lesssim h^2$ since $z \in T_1(x, w)$ yields $|z^\perp| \lesssim h$. Further,

$$|E_{zzww}| \approx \frac{|z^\perp|}{\min(|x - z|, |w - z|)} \approx \frac{|z^\perp|}{\min(h, |x - w|)} \approx |z^\perp|/h.$$

We also have $\langle z \rangle \approx (z^\perp - z^\perp_0) + (h - h_0)$, where $(h_0, z^\perp_0)$ is the origin in this coordinates. We have

$$\int_{T_{11}} \frac{(z)^{-3} dz}{|x - z|^k |z - w|^\ell |E_{zzww}|^{n-3}} \lesssim \int_0^{\min(h, |x - w|)} \int_{|z^\perp| \leq h} \frac{h^{n-3-k}(z^\perp - z^\perp_0)^{-2} (h - h_0)^{-1}}{|x - w|^\ell |z^\perp|^{n-3}} dz^\perp dh \lesssim |x - w|^{-\ell} \int_0^{\min(h^2, 1)} h^{n-3-k} dh.$$  

Where the minimum term in the last inequality arises from considering the cases of $|x - w| < 1$ and $|x - w| \geq 1$. 

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When $|x - w| < 1$, we have that $h < 1$ as well. Here we ignore the $(z^\perp - z_0^\perp)^{-2}$ term and bound
\[
\int_{B(0,h)\subset \mathbb{R}^{n-1}} \frac{1}{|z^\perp|_{n-3}} dz^\perp \lesssim h^{n-1-(n-3)} = h^2.
\]
We have used part I of Lemma 1.23 to bound the integral.

When $|x - w| > 1$, instead of integrating over a ball of radius $r \lesssim h$, we expand to all of $\mathbb{R}^{n-1}$.
\[
\int_{|z^\perp| \lesssim h} \frac{(z^\perp - z_0^\perp)^{-2}}{|z^\perp|_{n-3}} dz^\perp \lesssim \int_{R^{n-1}} \frac{(z')^{-2}}{|z' - (-z_0^\perp)|_{n-3}} dz'
\]
\[
\lesssim \begin{cases} 
1 & \text{when } |z_0^\perp| < 1 \\
\left(\frac{1}{|z_0^\perp|}\right)^{\min(0,n-3,n-3+2-(n-1))} & \text{when } |z_0^\perp| > 1
\end{cases} \lesssim 1.
\]
Here we made the substitution $z' = z^\perp + z_0^\perp$ in the first inequality, and bounded by part II of Lemma 1.23.

For $|x - w| < 1$, this immediately implies the required bound (by ignoring the term $(h - h_0)^{-1}$).
\[
|x - w|^{-\ell} \int_0^{|x - w|} h^{n-3-k} \min(h^2,1)(h - h_0)^{-1-} dh \lesssim |x - w|^{-\ell} \int_0^{|x - w|} h^{n-1-k} dh \lesssim |x - w|^{n-k-\ell}.
\]

For $|x - w| > 1$, note that
\[
|x - w|^{-\ell} \int_0^{|x - w|} h^{n-3-k} \min(h^2,1)(h - h_0)^{-1-} dh \lesssim |x - w|^{-\ell} \left(1 + \int_1^{|x - w|} h^{n-3-k}(h - h_0)^{-1-} dh\right) \lesssim |x - w|^{-\ell}(1 + |x - w|^{3-k-n}),
\]
which implies the required bound.

\[\square\]

**Theorem 1.11.** Fix $0 \leq k, \ell \leq n - 1$, $n - 3 \leq k + \ell$, $x, w \in \mathbb{R}^n$ and a vector $\overset{\rightharpoonup}{v} \in \mathbb{R}^n$. Assume $\alpha := \angle(\overset{\rightharpoonup}{v}, \overset{\rightharpoonup}{w}x) > 0$, then for any $F, G \in \{E_{xzzw}, E_{ww, v}(z), E_{z, -v}(z)\}$, $F \neq G$, we have: if $k + \ell \neq n$, then
\[
\int_{\mathbb{R}^n} \frac{\langle z \rangle^{-3} dz}{\langle x - z \rangle^k \langle z - w \rangle^\ell |F|^{n-3}|G|^{n-3}} \lesssim \alpha^{-(n-3)} \begin{cases} 
1 & |x - w| \leq 1 \\
\left(\frac{1}{|x - w|}\right)^{\max(0,k+\ell-n)} & |x - w| > 1
\end{cases}
\]

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If $k + \ell = n$, then
\[
\int_{\mathbb{R}^n} \frac{(z)^{-3-} dz}{|x - z|^{k} |z - w|^\ell |F|^n |G|^{n-3}} \lesssim \alpha^{-(n-3)} \left\{ \begin{array}{ll}
\left( \frac{1}{|x - w|} \right)^{0+} & |x - w| \leq 1 \\
\left( \frac{1}{|x - w|} \right)^{\min(k, \ell, 3)-} & |x - w| > 1
\end{array} \right.
\]

Proof of Theorem 1.11. For each choice of $F$ and $G$ the integral involves two point singularities and two line singularities. The condition on the angle between $\vec{v}$ and the line $\vec{w}$ separates the line singularities from each other and also separates line singularities from the point singularities. Therefore, we prove the statement for $F = E_{zzzzw}$, $G = E_{w,z}(z)$. We note that the proof for $G = E_{x,z}(z)$ is identical in form with $x$ and $w$ switching roles.

Fix $x, w$ with $\alpha > 0$. Recall that $|E_{zzzzw}|^{3-n}$ and $|E_{w,z}(z)|^{3-n}$ are singular along the line between $x$ and $w$ and on the ray with direction $\vec{v}$ from $w$, respectively. The case $\alpha \gtrsim 1$ is easier since the line singularities are separated by an angle $\gtrsim 1$. In this case, $E_{\alpha/2}(w, \vec{v}) \cap T_{\alpha/2}(w, \vec{v}) = \emptyset$. So, we apply Theorem 1.8 to obtain our desired result.

We note that $\text{dist}(x, E_{\alpha/2}(w, \vec{v})) \approx \alpha |x - w| \approx |x - w|$. We divide $E_{\alpha/2}(w, \vec{v})$ into two distinct regions. First when $|z - w| \leq 2|x - w|$, call this region $P_1$ we assign coordinates as in the proof of Theorem 1.8. $h$ is the projection along the line singularity $G = E_{w,z}(z)$ and $z^\perp$ is the $n - 1$ dimensional coordinate along planes perpendicular to $h$. Again $|z^\perp| \lesssim h$, and by Lemma 1.22
\[
|E_{w,z}(z)| \approx \left| \frac{z^\perp}{z - w} \right|.
\]

With $(h_0, z_0^\perp)$ being the origin in these coordinates, the integral can be bounded as follows.

\[
\int_{P_1} \frac{(z)^{-3-}}{|x - z|^k |z - w|^\ell |E_{w,z}(z)|^{n-3}} dz \lesssim |x - w|^{-k} \int_0^{1}|x - w|^{-k} \int_{|z^\perp| \leq h} \frac{(z^\perp - z_0^\perp)^{-2} (h - h_0)^{-1}}{h^\ell (|z^\perp|/h)^{n-3}} dz^\perp dh \lesssim |x - w|^{-k} \int_0^{1}|x - w|^{-k} (h - h_0)^{-1} h^{n-2} \int_{|z^\perp| \leq h} \frac{(z^\perp - z_0^\perp)^{-2}}{|z^\perp|^n} dz^\perp dh.
\]

We have reduced it to the integral estimated in Theorem 1.8 in (1.50). Thus, we have the bound

\[
\int_{P_1} \frac{(z)^{-3-}}{|x - z|^k |z - w|^\ell |E_{w,z}(z)|^{n-3}} dz \lesssim \left\{ \begin{array}{ll}
\left( \frac{1}{|x - w|} \right)^{\max(0, k + \ell - n)} & |x - w| < 1 \\
|x - w|^{-k} (1 + |x - w|^{n-\ell}) & |x - w| > 1
\end{array} \right.
\]

On the complement of $P_1$, which we denote $P_2$, we have $|z - w| > 2|x - w|$. Further by the triangle inequality,
we have $|z - w| \approx |x - z|$. Assuming $|x - w| > 1$, we have

$$
\int_{P_2} \frac{(z)^{-3}}{|z - z|^{\alpha}} \langle z \rangle \left| z - w \right|^{n-k} d\mathcal{E}_{\alpha,\nu}(z) \leq \int_{|z - w|}^{\infty} \int_{|z| \leq h} \frac{(z)^{-2}}{|z|^{n-3}} d\mathcal{E}_{\alpha,\nu}(z) dh.
$$

where we used Hölder in the last step. If $|x - w| < 1$, we use the above calculation with lower bound 1 instead of $|x - w|$, and

$$
\int_{1}^{\infty} h^{n-1-k-\ell} dh \lesssim \int_{1}^{\infty} dh \lesssim 1.
$$

Where we used that $k + \ell \geq n - 3$. On $(|x - w|, 1)$ we bound with

$$
\int_{|x - w|}^{1} h^{n-1-k-\ell} dh \lesssim 1 + |x - w|^{n-1-k-\ell}.
$$

We now consider the case when $\alpha \ll 1$. Define $C_1$ to be $E_1(w, \vec{v})$, the cone opening opening around the line singularity $G$. Note that $T_{1/2}(x, w) \subset C_1$.

Figure 1.5: The region $C_1$ in the proof of Theorem 1.11.
Therefore, outside $C_1$, we have $|E_{zzzw}|, |E_{w,v}(z)| \gtrsim 1$, and hence the statement for the contribution of the integral outside $C_1$ follows from Lemma 1.23.

We divide $C_1$ into several regions. Let $C_2$ be the intersection of $C_1$ and the cone from $x$ opening with angle one towards $w$.

Consider the first region, denoted $R_1$, which is $C_2 \cap \{|w-z| \leq \frac{1}{2}|x-w|\}$. The triangle inequality implies that here $|x-z| \approx |x-w|$. We define new coordinates on this region. Let $h$ be the coordinate along the continuation of $\bar{v}$ from $w$, $0 \leq h \leq \frac{1}{2}|x-w|$ and $z^\perp$ the $n-1$ dimensional coordinate on planes perpendicular to the line defining $h$. In $C_1$, it follows that $|z^\perp| \lesssim h$ as we are in a cone starting at $h = 0$. It follows by the triangle inequality that $|z-w| \approx h$. The line $\overline{wx}$ in these coordinates can be written as $(h,z^\perp)$ with $|z^\perp| \approx \alpha h$, as $\text{dist}((h,0), \overline{wx}) \approx \alpha |z-w|$. Note that the distance of a point $z = (h,z^\perp)$ to the line $\overline{wx}$ is $\approx |z^\perp - z_h|$. As $\alpha \ll 1$ the component of the distance along $\bar{v}$ is small compared to $|z^\perp - z_h|$. Therefore for $z = (h,z^\perp)$, we have

$$|E_{zzzw}| \approx \frac{\text{dist}(z, \overline{wx})}{\min(|x-z|, |z-w|)} \approx |z^\perp - z_h|/h,$$

$$|E_{w,v}(z)| \approx \frac{\text{dist}(z, w+\bar{v})}{|z-w|} \approx |z^\perp|/h.$$
Also, let \((h_0, z_0)\) be the coordinates of the origin. We have

\[
\int_{R_1} \frac{\langle z \rangle^{-3} dz}{|x-z|^{|z-w|^\ell |E_{zzzw}|^{n-3} |E_w,\tilde{v}(z)|^{n-3}}}
\]

Using the inequality

\[
\leq \int_0 \int_{|z^\perp| \leq h} \frac{h^n \min(h^2, 1)}{|x-w|^{n-3} h^{n-3}} dh
\]

and Lemma 1.23, we have

\[
(1.51) \lesssim \alpha^{-n-3} \left\{ \begin{array}{ll}
|x-w|^{n-k-\ell} & |x-w| < 1 \\
|x-w|^{-k} + |x-w|^{n-3-k-\ell} & |x-w| > 1
\end{array} \right.
\]

where the last inequality follows as in the proof of Theorem 1.8.

Now consider the second region, denoted \(R_2, C_2 \cap \{ |x-z| \leq \frac{1}{2} |x-w| \} \). See Figure 1.5. Here the triangle inequality implies that \(|z - w| \approx |x - w|\). Define new coordinates on this region. Let \(h\) be the coordinate along the line \(\overline{xw}\), \(0 \leq h \leq \frac{|x-w|}{2}\) and \(z^\perp\) the coordinate on planes perpendicular to \(\overline{xw}\). Again \(|z^\perp| \lesssim h\), and here \(|x-z| \approx h\). The continuation of \(\tilde{v}\) from \(w\) has coordinates \((h, z_h)\) where \(|z_h| \approx \alpha |x-w|\). As in the previous case, we have

\[
\int_{R_2} \frac{\langle z \rangle^{-3} dz}{|x-z|^{|z-w|^\ell |E_{zzzw}|^{n-3} |E_w,\tilde{v}(z)|^{n-3}}}
\]

As in the previous case, we have the bound

\[
\frac{1}{|z^\perp - z_h|^{n-3} |z^\perp|^{n-3}} \lesssim \frac{1}{z_h^{n-3}} \left[ \frac{1}{|z^\perp - z_h|^{n-3}} + \frac{1}{|z^\perp|^{n-3}} \right]
\]

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which implies

$$
(1.52) \lesssim \alpha^{-(n-3)}|x-w|^{-\ell} \int_0^{[x-w]} h^{n-3-k}(h-h_0)^{-1} \min(h^2,1) \, dh
$$

$$
\lesssim \alpha^{-(n-3)} \begin{cases} 
|x-w|^{n-k-l} & |x-w| < 1 \\
|x-w|^{-\ell}(1+|x-w|^{n-3-k}) & |x-w| > 1 
\end{cases}.
$$

The final region $R_3 = C_1 \setminus C_2$, here $|E_{zzw}| \gtrsim 1$. This region is composed of the infinite cone in $C_1$ bounded away from where $|E_{zzw}|$ is small. In particular, notice that $R_3 \subseteq C_1 \setminus B(w, |x-w|/2)$. We define new coordinates on this region. Let $h$ be the coordinate along the continuation of $\vec{v}$ from $w$ and $z^\perp$ the coordinate on planes perpendicular to the line defining $h$, $|x-w| \lesssim h < \infty$. Again $|z^\perp| \lesssim h$ and $h \approx |z-w|$. The point $x$ has coordinates $(h_x, z_x)$ where $h_x \approx |x-w|$ and $|z_x| \approx \alpha|x-w|$. This differs from the previous two regions as the point singularity is not at the tip of the cone, but somewhere in the interior.

$$
\int_{R_3} \frac{\langle z \rangle^{-3} \, dz}{|x-z|^k|z-w|^\ell(E_{zzw}|^{n-3}E_{w,\vec{v}}(\vec{z}))^{n-3}}
$$

$$
\lesssim \int_{h \geq |x-w|} \int_{|z^\perp| \lesssim h} \frac{h^{n-3-\ell}(h-h_0)^{-1-\langle z^\perp-z_0^\perp \rangle^{-2}-\langle z^\perp \rangle^{-n-3}} \, dz^\perp \, dh}
$$

$$
\lesssim \int_{h \gg |x-w|} \frac{h^{n-3-k-\ell}(h-h_0)^{-1} \min(h^2,1) \, dh}
$$

$$
\lesssim \begin{cases} 
1 + |x-w|^{n-k-l} & |x-w| < 1 \\
|x-w|^{n-3-k-\ell} & |x-w| > 1 
\end{cases}.
$$

We divide the $h$ integral into the regions i) $h \gg |x-w|$, and ii) $h \approx |x-w|$. For $h \gg |x-w|$, we have $|h-h_x| \gtrsim h$ by the triangle inequality, which implies

$$
\int_{h \gg |x-w|} \int_{|z^\perp| \lesssim h} \frac{h^{n-3-\ell}(h-h_0)^{-1-\langle z^\perp-z_0^\perp \rangle^{-2}-\langle z^\perp \rangle^{-n-3}} \, dz^\perp \, dh}
$$

$$
\lesssim \int_{h \gg |x-w|} \frac{h^{n-3-k-\ell}(h-h_0)^{-1} \min(h^2,1) \, dh}
$$

$$
\lesssim \begin{cases} 
1 + |x-w|^{n-k-l} & |x-w| < 1 \\
|x-w|^{n-3-k-\ell} & |x-w| > 1 
\end{cases}.
$$

The calculations here are essentially identical to those in $R_1$. The two bounds for $|x-w| < 1$ come from considering the regions $(|x-w|,1)$ and $(1,\infty)$ respectively. The bound for $|x-w| > 1$ comes from Hölder and the integrability of $(h-h_0)^{-1-\ell}$ on $\mathbb{R}$.

For $h \approx |x-w|$, we have

$$
\int_{h \approx |x-w|} \int_{|z^\perp| \lesssim h} \frac{h^{n-3-\ell}(h-h_0)^{-1-\langle z^\perp-z_0^\perp \rangle^{-2}-\langle z^\perp \rangle^{-n-3}} \, dz^\perp \, dh}
$$

$$
\lesssim |x-w|^{n-3-\ell} \int_{|z^\perp| \lesssim |x-w|} \int_{h \approx |x-w|} \frac{\langle h-h_0 \rangle^{-1-\langle z^\perp-z_0^\perp \rangle^{-2}} \, dh \, dz^\perp}{(|h-h_x| + |z^\perp - z_x|)^k |z^\perp|^{n-3}}.
$$

(1.54)
Note that in the second integral we have used that \( h \approx |x - w| \) to remove the \( h \) dependence from the \( z^\perp \) integral. First assume that \( k < n - 1 \). Using

\[
\frac{1}{|z^\perp - z_x|^k |z^\perp|^n-3} \lesssim \frac{1}{|z_x|^{\min(n-3,k)}} \left[ \frac{1}{|z^\perp - z_x|^{|\max(n-3,k)|}} + \frac{1}{|z^\perp|^{|\max(n-3,k)|}} \right] \\
\approx \frac{1}{(\alpha|x - w|)^{\min(n-3,k)}} \left[ \frac{1}{|z^\perp - z_x|^{|\max(n-3,k)|}} + \frac{1}{|z^\perp|^{|\max(n-3,k)|}} \right],
\]

we have

\[
(1.54) \lesssim \frac{|x - w|^{n-3-k}}{(\alpha|x - w|)^{\min(n-3,k)}} \int_{h \geq |x - w|} (h - h_0)^{-1} \int_{|z^\perp| \leq |x - w|} \frac{(z^\perp - z^\perp_0)^{-2} \, dh \, dz^\perp}{|z^\perp|^{|\max(n-3,k)|}}
\]

\[
\lesssim |x - w|^{n-3-k} \left\{ \begin{array}{ll}
|x - w| & |x - w| < 1 \\
|x - w|^{n-3-k-\ell} + |x - w|^{-\ell} & |x - w| > 1.
\end{array} \right.
\]

Here the first minimum term comes from the \( h \) integral, and the second minimum term comes from the \( z^\perp \) integral as in (1.50). The final bound comes from considering the two cases of \( |x - w| < 1 \) and \( |x - w| > 1 \).

When the \( |z^\perp - z_x| \) term arises, we note that if \( |z^\perp| \lesssim |x - w| \) and \( |z_x| \approx \alpha|x - w| \) that \( |z^\perp - z_x| \lesssim |x - w| \) as well by the triangle inequality. Changing variables, the calculations are identical. We need only view this as a shift of the origin, in the \( z^\perp \) plane.

For \( k = n - 1 \), one needs to proceed slightly differently. Note that (for \( n - 2 \leq k \leq n - 1 \)),

\[
(1.54) \lesssim |x - w|^{n-3-k} \int_{h \geq |x - w|} \frac{1}{|h - h_x|} \int_{|z^\perp| \leq |x - w|} \frac{(z^\perp - z^\perp_0)^{-2} \, dz^\perp}{|z^\perp|^{|k-1| |z^\perp| n-3}}
\]

\[
\lesssim |x - w|^{n-3-k} \frac{1}{|z_x|^{n-3}} \min(1, |x - w|^{n-1-(k-1)})
\]

\[
\lesssim \alpha^{-(n-3)} |x - w|^{-\ell} \min(1, |x - w|^{n-k}).
\]

Where we used \( |h - h_x| \geq |h| - |h_x| \approx (1 - \alpha)|x - w| \) and \( \alpha \ll 1 \). The integral over the annulus \( h \approx |x - w| \) is of order one. For the case of \( k + \ell = n \), use that

\[
\frac{1}{|x - z|^k |z - w|} \lesssim \frac{1}{|x - z|^k |z - w|^\ell} + \frac{1}{|x - z|^{k-\ell} |z - w|^\ell},
\]

and bound with the dominant terms. For \( |x - w| < 1 \) the \( k^+ \) terms dominate, and for \( |x - w| > 1 \) the \( k^- \) terms dominate.

We note that when \( F = E_{w,\varphi}(z) \) and \( G = E_{x,-\varphi}(z) \), the situation is slightly different (though morally the
same), in that we have two line singularities that extend off to infinity regardless of the size of $|x - w|$. This actually does not have a damaging effect, as we gain the property that the two parallel line singularities are separated by a distance comparable to $\approx \alpha |x - w|$.

Figure 1.7: Regions of interest in the second case in the proof of Theorem 1.11.

We proceed in an analogous way to the case when one of the line singularities was $E_{xzzw}$. We first define $C_1$ to be the cone with angle of opening one from $x$ about the vector $-\vec{v}$. Similarly, $C_2$ is the cone with angle of opening one from $x$ about the vector $\vec{v}$.

The first region we consider is $|x - z| \leq \frac{1}{2} |x - w|$ inside $C_1 \cap C_2$. Again, we put coordinates on this region, we let $h$ be the distance from $x$ along $-\vec{v}$, and $z^\perp$ the $n - 1$ dimensional coordinate on planes perpendicular to $\vec{v}$. Again, we have $|z^\perp| \lesssim h$, $|x - z| \approx h$. The line singularity $E_{x,-\vec{v}}(z)$ is along the $h$-axis, while $E_{w,\vec{v}}(x)$ is along the line with coordinates $(h, z_h)$ with $|z_h| \approx \alpha |x - w|$. This again emphasizes that the line singularities are separated by a distance comparable to $\alpha |x - w|$.

The second region we consider is $|z - w| \leq \frac{1}{2} |x - w|$ inside $C_1 \cap C_2$. This region is, up to reflection and rotation, identical to the first region considered. On both regions, we bound as in the case when one of the line singularities is $E_{xzzw}$.

On $C_1 \setminus C_2$ we have $|E_{w,\vec{v}}(z)| \gtrsim 1$. We note that $C_1 \setminus C_2 \subset \mathbb{R}^n \setminus B(x, \frac{1}{2} |x - w|)$. We have the same issues here as in the analogous region in the case when $E_{xzzw}$ appears. We again put coordinates on this region, with $h$ being the projection onto the line emanating from $x$ in direction $-\vec{v}$, and $z^\perp$ the $n - 1$ dimensional coordinates on planes perpendicular to $h$. We note now that $w$ is inside this region with coordinates $(h_w, z_w)$, with $h_w \approx |x - w|$ and $|z_w| \approx \alpha |x - w|$. The calculations now proceed identically to the previous case, we
need to take care again when one of the point singularities is of order \( n - 1 \), though this time it is \( \ell \) instead of \( k \).

\[ \text{Theorem 1.18.} \text{ Fix } 0 \leq k, \ell, m \leq n - 1 \text{ satisfying } k + m \geq n - 3, \ell + p \geq n - 3 \text{ where } p = \max(0, k + m - n) \text{ or } p = \min(k, m, k + m + 3 - n). \text{ Fix } x, y, u \in \mathbb{R}^n. \text{ Assume that } \alpha := |E_{xyu}| > 0, \text{ then if } k + m \neq n \text{ and } \ell + p \neq n, \]

\[
\begin{align*}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} & \frac{\langle z \rangle^{-3} - \langle w \rangle^{-3}}{|z - w|^m |E_{xzzw}|^{n-3} |E_{zwuy}|^{n-3} |E_{zwyu}|^{n-3}} dz dw \\
& \lesssim \alpha^{-(n-3)} \begin{cases} 
(\frac{1}{|x-y|})^{\max(0,\ell+p-n)} & |x - y| \leq 1 \\
(\frac{1}{|x-y|})^{\min(\ell,p,\ell+p+3-n)} & |x - y| > 1 
\end{cases}.
\end{align*}
\]

If \( k + m = n \) or \( \ell + p = n \),

\[
\begin{align*}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} & \frac{\langle z \rangle^{-3} - \langle w \rangle^{-3}}{|z - w|^m |E_{xzzw}|^{n-3} |E_{zwuy}|^{n-3} |E_{zwyu}|^{n-3}} dz dw \\
& \lesssim \alpha^{-(n-3)} \begin{cases} 
(\frac{1}{|x-y|})^{\max(0,\ell+p-n)+} & |x - y| \leq 1 \\
(\frac{1}{|x-y|})^{\min(\ell,p,\ell+p+3-n)-} & |x - y| > 1 
\end{cases}.
\end{align*}
\]

\[ \text{Proof of Theorem 1.18.} \text{ Morally speaking, this proof is an iteration of the previous two proofs after using a suitable change of variables.} \]
Let us define \( a = \frac{1}{2}(z + w) \) and \( b = \frac{1}{2}(z - w) \). We note the Jacobian of the change of variables \((z, w) \mapsto (a, b)\) is constant. Further define \( c = x - b \) and \( d = b + y \), then

\[
E_{zww} = \frac{x - z}{|x - z|} - \frac{z - w}{|z - w|} = \frac{(x - b) - a}{|(x - b) - a|} - \frac{b - 0}{|b - 0|} = E_{c|a} = E_{c,-b}(a).
\]

Noting that \( w = a - b, y = d - b \), and \( \frac{z - w}{|z - w|} = \frac{b - 0}{|b - 0|} \), we have

\[
E_{zw} = \frac{z - w}{|z - w|} - \frac{w - y}{|w - y|} = \frac{b - 0}{|b - 0|} - \frac{a - b - (d - b)}{|a - b - (d - b)|} = E_{db}
\]

Noting that \( |E_{db}| = |E_{ad}| \) we have

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\langle z \rangle^{-3} \langle w \rangle^{-3} dz dw}{|z - w|^k|w - y|^l |E_{zww}|^{n-3}|E_{zw}w|^{n-3}|E_{zw}|^{n-3}} \lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\langle a + b \rangle^{-3} \langle a - b \rangle^{-3} da db}{|c - a|^k|a - d|^m |E_{c|a}|^{n-3}|E_{c|b}|^{n-3}|E_{db}|^{n-3}}. \tag{1.55}
\]

We now consider two regions, first when \( \langle a + b \rangle^{-3} \langle a - b \rangle^{-3} \lesssim \langle b \rangle^{-3} \langle a - b \rangle^{-3} \) and secondly when \( \langle a + b \rangle^{-3} \langle a - b \rangle^{-3} \lesssim \langle b \rangle^{-3} \langle a + b \rangle^{-3} \). In either case, we obtain decay for the \( b \) integral. If we consider only the \( a \) integral in (1.55), we have

\[
\int_{\mathbb{R}^n} \frac{\langle a \pm b \rangle^{-3} da}{|c - a|^k|a - d|^m |E_{c|a}|^{n-3}|E_{d|b}(a)|^{n-3}}. \tag{1.56}
\]

This integral is now in the correct form for applying Theorem 1.11. Viewing the \( \langle a \pm b \rangle^{-3} \) term as a shift of the origin, by Theorem 1.11 we have

\[
(1.56) \lesssim \gamma^{-(n-3)} \begin{cases} 
\left( \frac{1}{|c - d|} \right)^{\max(0,k+m-n)} & |c - d| \leq 1 \\
\left( \frac{1}{|c - d|} \right)^{\min\{k,m,k+m+3-n\}} & |c - d| > 1
\end{cases},
\]

with \( \gamma = |E_{c|a}| \). We now can evaluate the \( b \) integral, as it now has only two line singularities.

\[
(1.55) \lesssim \int_{\mathbb{R}^n} \frac{\langle b \rangle^{-3} db}{|c - d|^p|b|^l |E_{c|a}|^{n-3}|E_{db}|^{n-3}},
\]

where \( p \) can be \( \max(0, k + m - n) \) or \( \min(k, m, k + m + 3 - n) \). We can use either choice for \( p \) as \( \max(0, k + m - n) \geq \min(k, m, k + m + 3 - n) \). This flexibility in \( p \) is necessary in application of this theorem. As both
c and d depend on b, we define $e = \frac{1}{2}(y - x)$ and $f = \frac{1}{2}(u - y)$. Then we have

$$(1.55) \lesssim \int_{\mathbb{R}^n} \frac{\langle b \rangle^{-3} - db}{|b - e|^p|b|^\ell|E_{xzzw}|^{n-3}|E_{zwwy}|^{n-3}}.$$  

This integral is in the correct form for application of Theorem 1.11. The result now follows from Theorem 1.11, and we have

$$(1.55) \lesssim \alpha^{-<(n-3)} \left\{ \begin{array}{ll}
\left( \frac{1}{|z-y|} \right)^{\max(0,\ell+p-n)} & |e| \leq 1 \\
\left( \frac{1}{|z-y|} \right)^{\min(p,\ell+p+3-n)} & |e| > 1
\end{array} \right.,$$

$$\lesssim |E_{xyyu}|^{-<n-3} \left\{ \begin{array}{ll}
\left( \frac{1}{|x-y|} \right)^{\max(0,\ell+p-n)} & |x - y| \leq 1 \\
\left( \frac{1}{|x-y|} \right)^{\min(\ell,p,\ell+p+3-n)} & |x - y| > 1
\end{array} \right..$$

Where we used the definitions of $e$ and $f$ in the last step.

\[\square\]

**Proposition 1.19.** Fix $3 \leq \ell \leq 6$. Then

$$\int_{\mathbb{R}^7} \int_{\mathbb{R}^7} \frac{\langle z \rangle^{-7} - \langle w \rangle^{-7} - dz dw}{|x - z|^3 |z - w|^\ell |E_{xzzw}|^4 |E_{zwwy}|^4} < \infty.$$  

**Proof of Proposition 1.19.** We use coordinates $z = (z_1, \tilde{z})$ with $z_1 \in \mathbb{R}$ the projection of $z$ onto the line $\overline{xy}$ and $\tilde{z} \in \mathbb{R}^6$ the coordinate on planes perpendicular to $\overline{xy}$. Similarly $w = (w_1, \tilde{w})$, and $x = (x_1, 0), y = (y_1, 0)$.

![Coordinate system for the proof of Proposition 1.19.](image)

Figure 1.9: Coordinate system for the proof of Proposition 1.19.

We can further assume that $|E_{xzzw}|, |E_{zwwy}| \ll 1$, as we know how to handle the other cases using Theorem 1.8 and Lemma 1.23. Note that in this case we have $x_1 < z_1 < w_1 < y_1$ or $x_1 > z_1 > w_1 > y_1$, that is, we must have the case of either the figure above, or its mirror image. Using the figure above as a guide,
one can see that if \( z_1 \) is to the right of \( w_1, z \) will be outside of the regions in which the line singularities are large. Further, if either is outside of \([x_1, y_1]\), the line singularities will be separated by some angle \( \theta \gtrsim 1 \).

Define \( \tilde{a} \) by the six dimensional coordinate so that \((z_1, \tilde{a})\) is on the line \( \overline{xw} \). Similarly for \((z_1, \tilde{b})\) and the continuation of \( \overline{yw} \). A similar triangles argument shows that

\[
\frac{\tilde{a}}{\tilde{w}} = \frac{|x_1 - z_1|}{|x_1 - w_1|}, \quad \frac{\tilde{b}}{\tilde{w}} = \frac{|y_1 - z_1|}{|y_1 - w_1|}.
\]

Hence,

\[
\tilde{a} = \tilde{w} \frac{|x_1 - z_1|}{|x_1 - w_1|}, \quad \tilde{b} = \tilde{w} \frac{|y_1 - z_1|}{|y_1 - w_1|}.
\]

We use the ordering \( x_1 < z_1 < w_1 < y_1 \) to establish

\[
|y_1 - z_1| = y_1 - z_1 = y_1 - w_1 + w_1 - z_1
\]
\[
|x_1 - z_1| = z_1 - x_1 = z_1 - w_1 + w_1 - x_1
\]

Thus, as a consequence,

\[
|\tilde{a} - \tilde{b}| = |\tilde{w}| \left| \frac{|y_1 - z_1|}{|y_1 - w_1|} - \frac{|x_1 - z_1|}{|x_1 - w_1|} \right| \approx |\tilde{w}| \frac{1 + \frac{w_1 - z_1}{|y_1 - w_1|} - 1 - \frac{w_1 - z_1}{|x_1 - w_1|}}{\min(|y_1 - w_1|, |x_1 - w_1|)}.
\]

We also have the following estimates for the singularities in these coordinates.

\[
|E_{xzzw}| \approx \frac{|\tilde{z} - \tilde{a}|}{\min(|x_1 - z_1|, |z_1 - w_1|)}, \quad |E_{zwyy}| \gtrsim \frac{|\tilde{z} - \tilde{b}|}{|z_1 - w_1|},
\]
\[
|x - z| \approx |x_1 - z_1|, \quad |z - w| \gtrsim |z_1 - w_1|.
\]

The integral is now bounded by

\[
\int_{\mathbb{R}^{14}} (z_1)^{-7} (w)^{-7} |z_1 - w_1|^4 \min(|z_1 - w_1|, |x - z_1|)^4 d\tilde{z} d\tilde{w} dw_1.
\]

We apply Lemma 1.23 part II to the \( \tilde{z} \) integral, along with the estimate on \( |\tilde{a} - \tilde{b}| \),

\[
\int_{\mathbb{R}^6} \frac{d\tilde{z}}{|\tilde{z} - \tilde{a}|^4 |\tilde{z} - \tilde{b}|^4} \lesssim \frac{1}{|\tilde{a} - \tilde{b}|^2} \approx \left( \frac{\min(|y_1 - w_1|, |x_1 - w_1|)}{|\tilde{w}| |w_1 - z_1|} \right)^2,
\]
to obtain

\[(1.57) \lesssim \int_{\mathbb{R}^8} \frac{(z_1)^{-7} (w)^{-7} \min(|z_1 - w_1|, |x_1 - z_1|)^4 \min(|y_1 - w_1|, |x_1 - w_1|)^2}{|x_1 - z_1|^3 |z_1 - w_1|^\ell - 2} \, dz_1 \, dw_1 \, d\tilde{w}.\]

Using \((w)^{-7} \lesssim \langle w_1 \rangle^{-3} \langle \tilde{w} \rangle^{-4}\), the \(\tilde{w}\) integral is bounded by

\[
\int_{\mathbb{R}^6} \frac{\langle \tilde{w} \rangle^{-4}}{|\tilde{w}|^2} \, d\tilde{w} \lesssim \int_{\{|\tilde{w}| < 1\} \subset \mathbb{R}^6} \frac{d\tilde{w}}{|\tilde{w}|^2} + \int_{\{|\tilde{w}| > 1\} \subset \mathbb{R}^6} |\tilde{w}|^{-6} \lesssim 1.
\]

Where we switch to polar coordinates to establish boundedness of these integrals. (Or one could apply Lemma 1.23 part II as \(|o| < 1\) trivially.) We now have

\[(1.57) \lesssim \int_{\mathbb{R}^2} \frac{(z_1)^{-7} \langle w_1 \rangle^{-3}}{|x_1 - z_1|^3 |z_1 - w_1|^\ell - 2} \, dz_1 \, dw_1.
\]

As \(x_1 < z_1 < w_1 < y_1\) all lie along a line, we have that either \(|x_1 - z_1| \geq \frac{1}{2} |x_1 - w_1|\) or \(|z_1 - w_1| \geq \frac{1}{2} |x_1 - w_1|\). We note the following

\[
\min(|z_1 - w_1|, |x_1 - z_1|)^4 \min(|y_1 - w_1|, |x_1 - w_1|)^2 \leq \min(|z_1 - w_1|, |x_1 - z_1|)^4 |x_1 - w_1|^2
\]

\[
\lesssim \min(|z_1 - w_1|, |x_1 - z_1|)^4 \max(|x_1 - z_1|, |z_1 - w_1|)^2 \lesssim |x_1 - z_1|^3 - |z_1 - w_1|^3 + |z_1 - w_1|^3.
\]

Here we bound \(1+\) or \(1-\) powers of the minimum by the maximum, depending on whether \(|x_1 - z_1| < |z_1 - w_1|\).

Therefore,

\[(1.57) \lesssim \int_{\mathbb{R}^2} \frac{(z_1)^{-7} \langle w_1 \rangle^{-3}}{|x_1 - z_1|^0 |z_1 - w_1|^\ell - 5} \, dz_1 \, dw_1.
\]

To see that this integral is bounded in \(x_1\) we note that if \(3 \leq \ell \leq 5\), we have a term growing in \(|z_1 - w_1|\). In this case \(5 - \ell := j \in [0, 2]\). We now bound as

\[
\int_{\mathbb{R}^2} \frac{(z_1)^{-7} \langle w_1 \rangle^{-3} |z_1 - w_1|^j}{|x_1 - z_1|^0} \, dz_1 \, dw_1 \lesssim \int_{\mathbb{R}^2} \frac{(z_1)^{j-7} \langle w_1 \rangle^{j-3}}{|x_1 - z_1|^0} \, dz_1 \, dw_1
\]

where we used (1.48) to nullify the \(|z_1 - w_1|\) growth. Two applications of Lemma 1.23, one in \(z_1\) followed by \(w_1\) finishes the bound. The bound is immediate from Lemma 1.23 if \(\ell = 6\) as \(6 - 5- < 1\).
Proposition 1.20. Fix $x, w, y \in \mathbb{R}^7$. Assume $\alpha := |E_{xwwy}| > 0$, then

$$\int_{\mathbb{R}^7} \frac{(z)^{-4-} dz}{|x-z|^3|z-w|^6|E_{zwwy}|^3|E_{xwwy}|^3} \lesssim \alpha^{-3}|x-w|^{-3}.$$  

Proof of Proposition 1.20. We note that outside of $T_1(x, w)$, we can bound the integral by

$$\int_{\mathbb{R}^7} \frac{(z)^{-4-} dz}{|x-z|^3|z-w|^6|E_{zwwy}|^3|E_{xwwy}|^3}.$$  

This is bounded by Corollary 1.13 with obvious modifications to get $\alpha^{-3}|x-w|^{-3}$.

Inside $T_1$, we break into the regions $T_{11}$, on which $|x-z| < |z-w|$ and $T_{12}$ on which $|z-w| < |x-z|$. We only consider $T_{11}$ as, by symmetry, the calculations on $T_{12}$ will be identical in form. We define variables $(h, z^\perp)$ where $h$ is distance along the line $xw$ and $z^\perp$ is the six dimensional variable on planes perpendicular to $h$. Here $0 \leq h \leq \frac{1}{2} |x-w|$, $|z^\perp| \lesssim h$ and $|x-z| \approx h$.

The singular lines for $E_{zwwy}$ and $E_{xwwy}$ have coordinates $(h, z_h)$ and $(h, \tilde{z})$ with $|z_h| \approx \alpha|x-w|$ and $|\tilde{z}| \approx \alpha h$ respectively. We have

$$|E_{zwwy}| \gtrsim |z^\perp - z_h|/|x-w|, \quad |E_{xwwy}| \gtrsim |z^\perp - \tilde{z}|/h.$$  

The integral is now bounded by

$$|x-w|^{-3} \int_0^{[x-w]} \int_{|z^\perp| \leq h} \frac{(h-h_0)^{-4-h}}{|z^\perp|^3|z^\perp - z_h|^3|z^\perp - \tilde{z}|} dz^\perp dh \lesssim |x-w|^{-3} \int_0^{[x-w]} \int_{\mathbb{R}^6} \frac{(h-h_0)^{-4-h}}{|z^\perp|^3} \left( \frac{1}{|z^\perp - z_h|^4} + \frac{1}{|z^\perp - \tilde{z}|^4} \right) dz^\perp dh.$$  

We can now apply Lemma 1.23 part II to the integral in $\mathbb{R}^6$. The size estimates on $z_h$ and $\tilde{z}$ bound the integral by

$$\alpha^{-1}|x-w|^{-3} \int_0^{[x-w]} \langle h-h_0 \rangle^{-4-h} dh \lesssim \alpha^{-1}|x-w|^{-3}.$$  

\[ \square \]

## 1.6 Higher Odd Dimensions

The following is to be considered as an extended remark. No proof is offered for dimensions $n > 7$, this is merely a heuristic for how one might approach this situation.
The integration by parts scheme we develop for dimensions five and seven in Sections 1.3 and 1.4 can be generalized to higher odd dimensions. There will, of course, be more complications which we will not tackle in this paper.

We note that in dimension three, see [27], one need not perform integration by parts in the \( z_j \) variables at all. In dimension five, one must integrate by parts once in each variable \( z_j \). In dimension seven, one must integrate by parts in variables \( z_j \) and \( b_j = z_j + z_{j+1} \), twice for each \( j \in \{1, \ldots, \kappa\} \). To avoid non-integrable singularities, in higher odd dimension \( n \), one must employ \( \frac{n-3}{2} \) variables of the form \( z_j + z_{j+1} + \cdots + z_{j+\ell} \) with \( 0 \leq \ell \leq \frac{n-5}{2} \). Use of such variables will complicate the scheme needed to integrate by parts and produce a larger class of line singularities.

The necessary integration by parts scheme mirrors that of seven dimensions. Denoting \(|z_j - z_{j+1}|^{-1}\) by \( r_j \), we use the variable \( b^k_j := z_j + z_{j+1} + \cdots + z_{j+k} \) of length \( k+1 \) when there are \( k \) consecutive point singularities, \( r_j, r_{j+1}, \ldots, r_{j+k-1} \), all have power \( n - 1 \) but \( r_{j-1} \) and \( r_{j+k} \) have smaller powers. Note that \( \nabla b^k_j \) leaves \( r_j, r_{j+1}, \ldots, r_{j+k-1} \) alone and acts on the neighboring \( r_{j-1} \) and \( r_{j+k} \). Since the total number of point singularities is at most \( \kappa \frac{n-1}{2} \) and we perform \( \kappa \frac{n-3}{2} \) integration by parts, the total number of point singularities at the end is at most \( \kappa(n-2) \), which can be safely distributed over \( \kappa + 1 \) different \( r_j \)’s using this scheme.

It is, of course, necessary to use estimates for integrals which involves many different line singularities, which differs from our estimates presented previously.

1.7 Derivative Calculations

In this section, we provided the derivative calculations necessitated by our integration by parts scheme. We first note that the use of cut-off functions as described in Section 1.3.4 allows us to assume smoothness and commutativity of derivatives.

Note the following formula.

\[
\nabla_z (f(z) \cdot g(z)) = \nabla_z f(z)^T g(z) + \nabla_z g(z)^T f(z) \tag{1.58}
\]

Where \( \nabla_z f(z) \in \mathbb{R}^{n \times n} \) is defined by \( \nabla_z f(z)_{ij} = \frac{\partial}{\partial z_j} f_i(z) \). It follows from this definition that

\[
\nabla_{\alpha}|F| = \frac{1}{|F|}(\nabla_{\alpha} F)^T F,
\]

where \( \alpha \) is any variable and \( F \) is any vector field.
Note the following calculations. If we define \( G(x, z) \in \mathbb{R}^{n \times n} \) by \( G(x, z)_{ij} = e_x(z)_i e_x(z)_j \), then

\[
|G(x, z)| \leq 1.
\]

This is in the sense that any entry of the matrix is of size less than one. This implies that

\[
|G(x, z)v| \leq |v|
\]

for any vector \( v \). Let \( E := E_{xzw} \) and \( e_x(z) = \frac{x - z}{|x - z|} \), then

\[
\nabla_z E = \frac{-1}{|x - z|}(I - G(x, z)) + \frac{-1}{|w - z|}(I - G(z, w)),
\]
\[
\nabla_z \cdot E = C \left( \frac{1}{|x - z|} + \frac{1}{|z - w|} \right),
\]
\[
\nabla_z |x - z|^{-k} = \frac{C e_x(z)}{|x - z|^{k+1}},
\]
\[
\nabla_z \cdot G(z, w) = \frac{1}{|w - z|} e_w(z),
\]
\[
e_x(z) \cdot E = e_w(z) \cdot E = \frac{1}{2} |E|^2,
\]
\[
\nabla_z |E| = -\frac{1}{|E|} \left[ \frac{1}{|x - z|}(I - G(x, z)) + \frac{1}{|w - z|}(I - G(z, w)) \right] E,
\]
\[
\nabla_w E := W = (I - G(x, z))E.
\]

Repeated use of the above calculations and the product rule

\[
\nabla_z(Av) = (\nabla_z \cdot A)^T v + A(\nabla_z v)^T,
\]

where \( A \in \mathbb{R}^{n \times n} \) and \( v \in \mathbb{R}^n \), yield the following result.

\[
|\nabla^\ell |E|^{-k}| \lesssim \left( \frac{1}{|x - z|} + \frac{1}{|w - z|} \right)^\ell |E|^{-(k+\ell)}.
\]

Noting

\[
\nabla_w |W| = \frac{1}{|w - z||W|} (I - G(x, z))(I - G(z, w))(I - G(z, w))E,
\]
and the previous calculations, we have the following result.

\[ |\nabla_w^\ell |W|^{-k}| \lesssim |w - z|^{-\ell} |W|^{-(k + \ell)} \]  \hspace{1cm} (1.62)

Similarly, we have

\[ \nabla_x E = \frac{1}{|x - z|} (I - G(z, w)). \]  \hspace{1cm} (1.63)

Call \((\nabla_x E) = \tilde{W} \). Then,

\[ \nabla_x |\tilde{W}| = \frac{1}{|x - z||W|} (I - G(x, z))^3 E \]  \hspace{1cm} (1.64)

Combining (1.60), (1.62) and (1.64) we have the following.

\[ |\nabla_w^\ell |\nabla^m_z |\nabla^p_x |E|^{-k}| \lesssim \frac{|x - z|^{-p}|w - z|^{-\ell}}{|E|^{k + \ell + m + p}} \left( \frac{1}{|x - z|} + \frac{1}{|w - z|} \right)^m \]  \hspace{1cm} (1.65)

Define \( \Gamma_z(f) = \nabla_z \cdot (f \frac{F}{|F|^2}) \). Then, we have

\[ \left| \Gamma^k_z (f) \right| \lesssim \sum_{\ell=0}^{k} \frac{|\nabla^\ell_z f|}{|F|^{2k-\ell}} \left( \frac{1}{r_{j-1}} + \frac{1}{r_j} \right)^{k-\ell}. \]  \hspace{1cm} (1.66)

Noting commutativity of derivatives when we are away from the singularities. This is justified by use of suitable cut-offs as described earlier, which make our functions smooth. It does not matter the order in which these \( \Gamma \)'s act, only the total number. So, (1.60), (1.62) and (1.64) give us

\[ \left| \Gamma^{m_j}_{z_{j-1}} \Gamma^{m_j}_{z_j} \Gamma^{m_{j+1}}_{z_{j+1}} (f) \right| \lesssim \frac{1}{|F|^{2M}} \sum |\nabla^{\alpha_{j-1}} \nabla^\alpha_j \nabla^\alpha_{j+1} f| |F|^A \prod_{\ell=1}^{j+1} \left( \frac{1}{r_{\ell-1}} + \frac{1}{r_\ell} \right)^{m_{\ell} - \alpha_\ell}. \]

Where \( M = m_{j-1} + m_j + m_{j+1} \), \( A = \alpha_{j-1} + \alpha_j + \alpha_{j+1} \) and the sum is taken over all combinations of \( \alpha_i \) with \( 0 \leq \alpha_i \leq m_i \). We can clearly extend this to

\[ \left| \Gamma^{n_k}_{z_k} \cdots \Gamma^{n_1}_{z_1} (f) \right| \lesssim \frac{1}{|F|^{2M}} \sum |\nabla^{\alpha_k} \cdots \nabla^\alpha_1 f| |F|^A \prod_{\ell=0}^{k} \left( \frac{1}{r_{\ell-1}} + \frac{1}{r_\ell} \right)^{m_{\ell} - \alpha_\ell}. \]

Here \( M = \sum m_i \) and \( A = \sum \alpha_i \) with the sum taken over all combinations of the \( \alpha_i \)'s with \( \alpha_i \in \mathbb{N}_0 \). The
addition of $b_j$ variables only changes the estimate in that the point singularities are affected as follows.

$$\nabla b_j \left( \frac{1}{r_j-1r_jr_{j+1}} \right) = C \left( \frac{1}{r_j-1r_jr_{j+1}} \right) \left( \frac{e_{z_{j-1}}(z_j)}{r_{j-1}} + \frac{e_{z_{j+2}}(z_{j+1})}{r_{j+1}} \right)$$  (1.67)
Chapter 2

Improved five dimensional tail estimates

In this chapter we present a result that improves the decay requirements we use in [16], the main result of Chapter 1. The history and background of this problem are recounted in Chapter 1, Section 1.1.

In [16], we used the tail estimate of Goldberg and Visan for dimension \( n > 3 \), [29], which required that the potential be in \( L^\infty \) and decay faster than \( \langle x \rangle^{-\frac{3n+5}{2}} \) at infinity. We present a proof for dimension five that weakens the assumptions needed on the potential from \( |V(x)| \lesssim \langle x \rangle^{-\beta} \) for some \( \beta > 10 \) to \( \beta > 4 \). In particular, we prove a dispersive bound on the tail of the Born series (1.14).

Theorem 2.1. In dimension five, if zero is regular and \( |V(x)| \lesssim \langle x \rangle^{-\beta} \) for some \( \beta > 4 \) and \( m \geq 6 \),

\[
\sup_{L \geq 1} \left| \int_{\mathbb{R}} e^{it\lambda^2} \lambda \chi_L(\lambda) \int_{\mathbb{R}^{2m+3}} \left( R_0^+ V \right)^{m+1} \left( V R_0^+ \right)^{m+1} d\vec{z} d\lambda \right| \lesssim |t|^{-\frac{5}{2}}.
\]

Theorem 2.1 immediately improves the result of [16], see (1.14) and Theorem 1.2. As such, we state an improved version of Theorem 1.1 from Chapter 1.

Theorem 2.2. In dimension five, if zero is regular, and \( V \in C^1(\mathbb{R}^5) \) satisfying \( |V(x)| \lesssim \langle x \rangle^{-4} \), \( |\nabla V(x)| \lesssim \langle x \rangle^{-3} \), then

\[
\| e^{itH} P_{ac}(H) \|_{L^1 \rightarrow L^\infty} \lesssim |t|^{-\frac{5}{2}}.
\]

It appears that this method can be extended to higher odd dimensions \( n \geq 3 \), but such extensions will not be presented in this thesis.

The method of this proof presented in this chapter was inspired by method of Goldberg and Schlag in [27]. In particular in Section 2.2 we extend their small energy strategy of expanding a perturbation of the zero energy resolvent in a Neumann series in certain Hilbert-Schmidt norms to higher dimensions. Morally speaking, this is the difficult part of the argument. In Section 2.1, we estimate the high energy portion by direct computation and applications of the limiting absorption principle.

We break up our analysis into “high” energy, when \( \lambda > \lambda_0 \), which is handled in Section 2.1 and “low”
energy, when \( \lambda < \lambda_0 \) which is explored in Section 2.2. We need to separate the behavior near zero from the behavior as \( \lambda \to \infty \). The exact separation value \( \lambda_0 \) is chosen in the low energy section. We first present the argument for the high energy.

### 2.1 Five dimensional high energy

We wish to control the quantity from Theorem 2.1, in this section we focus on the high energy portion of the integral given by

\[
\sup_{L \geq 1} \left| \int e^{it\lambda^2} \lambda \chi_L(\lambda) \left( 1 - \chi(\lambda/\lambda_0) \right) \int_{\mathbb{R}^{(2m+3)5}} (R_0^+ V)^m R_0^{+1} (VR^{+1}_0)^m d\bar{z} d\lambda \right|.
\]

Here \( \lambda_0 > 0 \) is a small number we choose based on the results in Section 2.2 to cut-off from zero energy.

We define the following kernels

\[
G_{\mp,x}(\lambda^2)(\cdot) := e^{\mp i\lambda|\cdot|} R_0(\lambda^2 \pm i0)(\cdot, x).
\] (2.1)

Such kernels have been used first by Yajima, see [87], and in the three-dimensional case by Goldberg and Schlag, [27]. These kernels used in place of \( R_0^+ \) guarantee that the \( \lambda \) derivatives do not lead to growth in \( x \) or \( y \), as we see in Lemma 2.5. Further, these kernels merely shift the phase in the \( \lambda \) integral, which does not effect the use of stationary phase like methods. Specifically, we move the critical point of the phase away from zero, to a value \( \lambda_1 = \mp \frac{|x| + |y|}{2t} \). As such, we examine

\[
\int_0^\infty e^{it\lambda^2} e^{\pm i\lambda(|x|+|y|)} \chi(\lambda/L) \left[ 1 - \chi(\lambda/\lambda_0) \right] \lambda(VR^{+1}_0(\lambda^2)VR^{+1}_0(\lambda^2))^m G_{\mp,x}(\lambda^2), (R_0^{+1}(\lambda^2)V)^m G_{\mp,y}(\lambda^2)) d\lambda
\] (2.2)

Our analysis depends on estimates on the limiting absorption principle of Agmon, [3], and some estimates we establish for certain functions.

**Lemma 2.3 (The Limiting Absorption Principle).** In dimension \( n \), for all \( \lambda > \lambda_0 \),

\[
\| R_0^+ (\lambda^2) \|_{L^2, \frac{1}{2}+L^2,-\frac{1}{2}} \lesssim \lambda^{-1},
\]

\[
\left\| \left( \frac{d}{d\lambda} \right)^j R_0^+ (\lambda^2) \right\|_{L^2, \frac{1}{2}+j+L^2,-\frac{1}{2}-j} \lesssim 1,
\]

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where $0 \leq j \leq \frac{n+1}{2}$. Further, if $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > \frac{n+1}{2}$,

$$
\|R_V^\pm(\lambda^2)\|_{L^2, 1/2} \lesssim \lambda^{-1},
$$

$$
\left\| \left( \frac{d}{d\lambda} \right)^j R_V^\pm(\lambda^2) \right\|_{L^2, 1/2 - j} \lesssim 1,
$$

This result is due to Agmon, [3]. For another proof, one can see [15]. One transfers the results on the free resolvent to the perturbed resolvent by the following relations as in [27, 29].

$$
R_V^\pm(\lambda^2) = [S^\pm(\lambda)]^{-1} R_0^\pm(\lambda^2),
$$

$$
\frac{d}{d\lambda} [S^\pm(\lambda)]^{-1} = -[S^\pm(\lambda)]^{-1} \frac{d}{d\lambda} R_0^\pm(\lambda^2) V[S^\pm(\lambda)]^{-1}.
$$

Here $S^\pm(\lambda)$ is defined in (2.22) and its properties are explored in Proposition 2.9.

We state the following lemma bounding a certain class of integrals. This lemma and its proof are presented in Lemma 3.8 of [29]. Note the similarity in form to Lemma 1.23, which handled integrals of similar form with two distinct point singularities.

**Lemma 2.4.** Let $\mu$ and $\sigma$ be such that $\mu < n$ and $n < \sigma + \mu$. Then

$$
\int_{\mathbb{R}^n} \frac{dy}{\langle y \rangle^\sigma |x - y|^\mu} \lesssim \begin{cases} 
\langle x \rangle^{n-\sigma-\mu} & \sigma < n \\
\langle x \rangle^{-\mu} & \sigma > n
\end{cases}.
$$

**Proof.** This proof follows by dividing $\mathbb{R}^n$ into three disjoint domains. First, consider the domain on which $|y| \leq \frac{1}{2}|x|$. By the triangle inequality, we have $|x - y| \approx |x|$. The contribution of this region is given by

$$
|x|^{-\mu} \int_{|y| \leq \frac{1}{2}|x|} \langle y \rangle^{-\sigma} dy \lesssim |x|^{-\mu} \int_0^{|x|} r^{n-1} (1 + r)^{-\sigma} dr. 
$$

(2.3)

Now if $|x| < 1$, we bound with

$$
(2.3) \lesssim |x|^{-\mu} \int_0^{|x|} r^{n-1} dr \lesssim |x|^{n-\mu} \lesssim 1.
$$

If $|x| > 1$, we bound with

$$
(2.3) \lesssim |x|^{-\mu} \left[ \int_0^1 r^{n-1} dr + \int_{|x|}^{|x|} r^{n-1-\sigma} dr \right] \lesssim |x|^{-\mu} (1 + |x|^{n-\sigma})
$$

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When $\sigma < n$, the dominant term is $|x|^{n-\sigma-\mu}$. When $\sigma > n$, the dominant term is $|x|^{-\mu}$.

The second domain is when $|x - y| \leq \frac{1}{2}|x|$. On this domain, the triangle inequality gives us that $|x| \approx |y|$.

The contribution of this region is bounded by

$$\int_{|x - y| \leq \frac{1}{2}|x|} \frac{dy}{\langle x \rangle^{\sigma}|x - y|^\mu} \lesssim \langle x \rangle^{-\sigma} \int_0^{|x|} r^{n-1-\mu} dr \lesssim \langle x \rangle^{-\sigma}|x|^{n-\mu} dr \lesssim \langle x \rangle^{n-\sigma-\mu}. \quad (2.4)$$

Where the integral is finite by the assumption $\mu < n$.

The third domain is the complement of the first two. Namely, when $|y|, |x - y| > \frac{1}{2}|x|$. Noting that $n - \sigma - \mu < 0$, we can bound with

$$\int_{|y| > \frac{1}{2}|x|} \frac{dy}{\langle y \rangle^{\sigma}|x - y|^\mu} \lesssim \int_{|y| > \frac{1}{2}|x|} \langle y \rangle^{-\sigma}|y|^{-\mu} dy. \quad (2.5)$$

First if $|x| > 1$ we can bound the above integral with

$$(2.5) \lesssim \int_{|x|}^\infty r^{n-1-\sigma-\mu} dr \lesssim |x|^{n-\sigma-\mu}.$$  

When $|x| < 1$ we have

$$(2.5) \lesssim \int_{|x|}^1 r^{n-1-\mu} dy + \int_1^\infty r^{n-1-\sigma-\mu} dr \lesssim |x|^{n-\mu} + 1 \lesssim 1.$$  

The bounds on (2.3), (2.4), and (2.5) establish the desired bound.

We now define the following.

$$J_y^\pm(\lambda, \cdot) := \int_{\mathbb{R}^5} R_0^\pm(\lambda^2)(\cdot, z)V(z)G_{\pm, y}(\lambda^2)(z) dz \quad (2.6)$$

We establish estimates on three derivatives of $J_y^\pm$, first pointwise bounds which then imply weighted $L^2$ bounds. $J_y^\pm$ is a kernel that is obtained by iterating a modulation of $R_0^\pm V R_0^\pm$ and integrating in the inner variable. In dimension five, it has a polynomial behavior in $\lambda$, the $\lambda^0$ term constrains the local $L^2$ behavior, i.e. dictates how many iterations of the resolvent and potential are necessary, and the $\lambda^2$ term dictates the weight needed to achieve $L^2$ decay at infinity.

**Lemma 2.5.** If $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 4$, the following estimates hold for $\lambda$ derivatives of $J_y^\pm$. If
0 \leq j \leq 2,

\left| \left( \frac{d}{d\lambda} \right)^j J_y^\pm (\lambda, x) \right| \lesssim \langle \lambda \rangle^2 \begin{cases} 
|x-y|^{-1}, & |x-y| < 1 \\
|x-y|^{-2}, & |x-y| > 1
\end{cases}

and

\left| \frac{d^3}{d\lambda^3} J_y^\pm (\lambda, x) \right| \lesssim \langle \lambda \rangle^2 \begin{cases} 
|x-y|^{-1}, & |x-y| < 1 \\
\lambda(x), & |x-y| > 1
\end{cases}

Proof. Recall that in dimension five,

\[ R_0^\pm (\lambda) = C_5 e^{\pm i\lambda |x-y|} \left( \frac{-i\lambda |x-y| + 1}{|x-y|^3} \right) \]

We note the following inequality.

\[ \left| \left( \frac{d}{d\lambda} \right)^j \left[ e^{i\lambda \phi} (a\lambda^2 + b\lambda + c) \right] \right| \lesssim \lambda^2 [a\phi^j] + \lambda [a\phi^{j-1} + b\phi^j] + [a\phi^{j-2} + b\phi^{j-1} + c\phi^j], \quad (2.7) \]

where we take \( \phi^\ell = 0 \) if \( \ell < 0 \). Taking \( \phi = |x-z| + |z-y| - |y| \) and \( a, b, c \) the coefficients of the \( \lambda \) powers that arise in \( J_y^\pm \). Specifically, \( a = |x-z|^{-2} |z-y|^{-2} \), \( b = a(|x-z|^{-1} + |z-y|^{-1}) \), and \( c = |x-z|^{-3} |z-y|^{-3} \).

The proof now follows Lemma 1.23, the fact that \( |z-y| - |y| \lesssim |z| \).

We use that \(|\phi| \lesssim |x-z| + \langle z \rangle\), so that

\[ |a\phi^\ell| \lesssim \frac{1}{|z-y|^3} \sum_{i=0}^\ell \frac{\langle z \rangle^{\ell-i}}{|x-z|^{2-i}}, \quad (2.8) \]

\[ |b\phi^\ell| \lesssim \left( \frac{1}{|z-y|} + \frac{1}{|x-z|} \right) \left( \frac{1}{|z-y|^3} \sum_{i=0}^\ell \frac{\langle z \rangle^{\ell-i}}{|x-z|^{2-i}} \right), \quad (2.9) \]

\[ |c\phi^\ell| \lesssim \frac{1}{|z-y|^3} \sum_{i=0}^\ell \frac{\langle z \rangle^{\ell-i}}{|x-z|^{3-i}}. \quad (2.10) \]

In essence, either \( \beta \) goes down by one, or \( k \) goes down by one in the applications of Lemma 1.23 used to bound the derivatives of \( J_y^\pm \).

We also note that

\[ |b\phi^\ell| \lesssim |a\phi^\ell| + |c\phi^\ell|. \]

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Accordingly, we only bound the $\lambda^2$ and $\lambda^0$ terms.

For each case, we find the terms that correspond to each power of $\lambda$ and bound with the dominant behavior. For $j = 0$, we have

$$|J_y^\pm(\lambda, x)| \lesssim \int_{\mathbb{R}^5} \langle z \rangle^{-4-} \left( \frac{\lambda^2}{|x-z|^2 |z-y|^2} + \frac{1}{|x-z|^3 |z-y|^3} \right) dz$$

$$\lesssim \langle \lambda \rangle^2 \begin{cases} \frac{1}{|x-y|} & |x-y| < 1 \\ \left( \frac{1}{|x-y|} \right)^2 & |x-y| > 1 \end{cases}.$$ 

The last inequality follows from Lemma 1.23. The $\lambda^2$ term determines the decay rate and the $\lambda^0$ term determines the singularity.

For $j = 1$, we use (2.7) along with the estimates (2.8), (2.9), (2.10) to see

$$\left| \frac{d}{d\lambda} J_y^\pm(\lambda, x) \right| \lesssim \int_{\mathbb{R}^5} \langle z \rangle^{-4-} \left( \lambda^2 \frac{1}{|x-z|^2 |z-y|^2} + \frac{\langle z \rangle}{|x-z|^3 |z-y|^3} \right) + \frac{1}{|x-z|^3 \left( \frac{1}{|z-y|^2} + \frac{1}{|z-y|^3} \right) + \frac{1}{|x-z|^3 \left( \frac{1}{|z-y|^2} + \frac{1}{|z-y|^3} \right) + \frac{1}{|x-z|^3 \left( \frac{1}{|z-y|^2} + \frac{1}{|z-y|^3} \right)}} dz.$$

We now apply Lemma 1.23. For the $\lambda^2$ term we have either $(k, \ell, \beta) = (1, 2, 4)$ or $(2, 2, 3)$. Each pair can be bounded by 1 when $|x-y| < 1$ as $k+\ell \leq 4 < 5$ and $|x-y|^{-1}$ for $|x-y| > 1$ as $\min(k, \ell, k+\ell+\beta-n) \geq 1$ for these values of $(k, \ell, \beta)$.

For the $\lambda^0$ term we have $(k, \ell, \beta) = (3, 2, 4), (3, 3, 3)$, or $(2, 3, 4)$. Clearly, $\max(k+\ell-5, 0) = 1$ and $\min(k, \ell, k+\ell+\beta-5) = 2$ for these choices. Thus, we have

$$\left| \frac{d}{d\lambda} J_y^\pm(\lambda, x) \right| \lesssim \langle \lambda \rangle^2 \frac{1}{|x-y|}.$$ 

For the case when $j = 2$, we use (2.7) and (2.8), (2.9), (2.10) to bound as

$$\left| \frac{d^2}{d\lambda^2} J_y^\pm(\lambda, x) \right| \lesssim \int_{\mathbb{R}^5} \langle z \rangle^{-4-} \left\{ \lambda^2 \left[ \frac{\langle z \rangle^2}{|z-y|^2} + \frac{\langle z \rangle}{|z-y|^3} + 1 \right] \right\} + \left[ \frac{1}{|x-z|^2 |z-y|^2} + \frac{1}{|z-y|^2} + \frac{1}{|x-z|^2} \right] \left[ \frac{1}{|z-y|^3} + \frac{1}{|x-z|^3} \right] dz.$$ 

Applying Lemma 1.23, we see that for the $\lambda^2$ term, we have $(k, \ell, \beta) = (2, 2, 2), (2, 1, 3)$ or $(2, 0, 4)$. For these choices, we have $\max(0, k+\ell-5) = 0$ and $\min(k, \ell, k+\ell+\beta-5) = 0$.

For the $\lambda^0$ term, we have $(k, \ell, \beta) = (2, 2, 4), (3, 1, 4), (3, 2, 3), (2, 3, 3)$, or $(3, 3, 2)$. For these choices, we
have \( \max(0, k + \ell - 5) = 1 \) and \( \min(k, \ell, k + \ell + \beta - 5) = 1 \). We have shown

\[
\left| \frac{d^2}{d\lambda^2} J_\pm^\lambda(\lambda, x) \right| \lesssim \langle \lambda \rangle^2 \begin{cases} \frac{1}{|x-y|} & |x-y| < 1 \\ 1 & |x-y| > 1 \end{cases}.
\]

For the case when \( j = 3 \), we first note that \( |x-z| \leq |x| + |z| \leq \langle x \rangle + \langle z \rangle \). We use this when the derivatives cause a positive power of \( |x-z| \), which we have not yet dealt with. Again, we use (2.7) and (2.8), (2.9), (2.10) to bound with

\[
\int_{\mathbb{R}^n} \langle z \rangle^{-4} \left\{ \lambda^2 \left[ \frac{1}{|z-y|^2} \left( \frac{\langle z \rangle^3}{|x-z|^2} + \frac{\langle z \rangle^2}{|x-z|} + \langle z \rangle + \langle x \rangle \right) \right] 
+ \left[ \frac{1}{|z-y|^2 \langle x-z \rangle^2} + \frac{1}{|x-z|^2 \langle x-y \rangle^2} \left( \frac{1}{|x-z|^2} + \frac{1}{|x-y|} \right) \left( \frac{\langle z \rangle^2}{|x-z|^2} + \frac{\langle z \rangle}{|x-z|} + 1 \right) 
+ \frac{1}{|z-y|^3 \langle x-z \rangle^3} \left( \frac{\langle z \rangle^3}{|x-z|^3} + \frac{\langle z \rangle^2}{|x-z|^2} + \frac{\langle z \rangle}{|x-z|^2} + 1 \right) \right] \right\} dz.
\]

We again apply Lemma 1.23. For the \( \lambda^2 \) term, we have \((k, \ell, \beta) = (2, 2, 1), (2, 1, 2), (2, 0, 3) \) or \((2, 0, 4) \) with a factor of \( \langle x \rangle \). The first three choices have \( \max(0, k + \ell - 5) = 0 \) and \( \min(k, \ell, k + \ell + \beta - 5) = 0 \). So these terms can be bounded by 1. The fourth term is bounded the same way, and we finally use that \( \langle x \rangle \gtrsim 1 \) to obtain the desired bound.

For the \( \lambda^0 \) term, we have \((k, \ell, \beta) = (2, 2, 3), (2, 1, 4), (3, 0, 4), (2, 3, 2), (3, 3, 1), (3, 2, 2), \) or \((3, 1, 3) \). Here we have \( \max(0, k + \ell - 5) = 1 \) and \( \min(k, \ell, k + \ell + \beta - 5) = 0 \). This establishes the desired bound.

\[\square\]

**Corollary 2.6.** If \( |V(x)| \lesssim \langle x \rangle^{-\beta} \) for some \( \beta > 4 \), and \( 0 \leq j \leq 3 \),

\[
\left\| \left( \frac{d}{d\lambda} \right)^j J_\gamma(\lambda, z) \right\|_{L^2_{\check{x}}} \lesssim \langle \lambda \rangle^2
\]

for \( \sigma > j + \frac{1}{2} \).

**Proof.** The first statement follows immediately from the bounds in Lemma 2.5. The estimates clearly establish local \( L^2 \) behavior, since the highest order singularity is of first order. The weight needed takes the decay at infinity to \(-\frac{5}{2}-\). The decay at infinity from the estimates is of order \(2 - j\).

\[\square\]

We can actually improve these estimates to push forward decay in \( y \).
Proposition 2.7. If $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 4$, then

$$
\|J_y^\pm(\lambda, z)\|_{L^2_x, -\frac{\beta}{2}} \lesssim \frac{\langle \lambda \rangle^2}{\langle y \rangle^2},
$$

$$
\left\| \frac{d}{d\lambda} J_y^\pm(\lambda, z) \right\|_{L^2_x, -\frac{\beta}{2}} \lesssim \frac{\langle \lambda \rangle^2}{\langle y \rangle^2}.
$$

Proof. The statement for $j = 0$ arises from the following calculations. First consider the contribution of the $\lambda^2$ term of $J_y^\pm$, (2.6), to the weighted $L^2$ norm.

$$
\left[ \int_{\mathbb{R}^5} \int_{\mathbb{R}^5} \frac{(z)^{-4} - 1}{|x - z|^2 |z - y|^2} dz \right] \langle x \rangle^{-3} \langle y \rangle^{-3} dx \lesssim \frac{1}{\langle x \rangle^{-3} |x - y|^4} \left[ \frac{1}{|x - z|^2} + \frac{1}{|z - y|^2} \right].
$$

We note that as in the proof of Theorem 1.11,

$$
\frac{1}{|x - z|^2 |z - y|^2} \lesssim \frac{1}{|x - y|^2} \left[ \frac{1}{|x - z|^2} + \frac{1}{|z - y|^2} \right] .
$$

Using this fact and Lemma 2.4, we bound with

$$
(2.11) \lesssim \left[ \int_{\mathbb{R}^5} \int_{\mathbb{R}^5} \frac{(z)^{-4} - 1}{|x - z|^2 |z - y|^2} dz \right] \langle x \rangle^{-3} \langle y \rangle^{-3} dx \lesssim \frac{1}{\langle y \rangle^2}.
$$

Applying (2.13), we have

$$
(2.14) \lesssim \left[ \int_{\mathbb{R}^5} \int_{\mathbb{R}^5} \frac{(z)^{-4} - 1}{|x - z|^2 |z - y|^2} dz \right] \langle x \rangle^{-3} \langle y \rangle^{-3} dx \lesssim \frac{1}{\langle y \rangle^2}.
$$

We handle the $\lambda^0$ term of $J_y^\pm$, (2.6), similarly, we note

$$
\frac{1}{|x - z|^3 |z - y|^3} \lesssim \frac{1}{|x - y|^2} \left[ \frac{1}{|x - z|^2} + \frac{1}{|z - y|^2} \right] .
$$

Thus, the $\lambda^0$ term of (2.6) contributes the following to the weighted $L^2$ norm.

$$
\left[ \int_{\mathbb{R}^5} \int_{\mathbb{R}^5} \frac{(z)^{-4} - 1}{|x - z|^3 |z - y|^3} dz \right] \langle x \rangle^{-3} \langle y \rangle^{-3} dx \lesssim \frac{1}{\langle y \rangle^2}.
$$

Applying (2.13), we have

$$
(2.14) \lesssim \left[ \int_{\mathbb{R}^5} \int_{\mathbb{R}^5} \frac{(z)^{-4} - 1}{|x - z|^3 |z - y|^3} dz \right] \langle x \rangle^{-3} \langle y \rangle^{-3} dx \lesssim \frac{1}{\langle y \rangle^2}.
$$
For the term with a \(\lambda\) derivative, we take a bit more care. Expanding out (2.6), we have that

\[
J_y^\pm (\lambda, z) = \int_{\mathbb{R}^3} V(x) e^{\pm i\lambda (|z-x|+|x-y|-|y|)} \left( \frac{(-i\lambda |z-x|+1)(-i\lambda |x-y|+1)}{|z-x|^3|x-y|^3} \right) dx.
\] (2.15)

We write the \(\lambda\) independent portion of the phase as \(\phi = |z-x| + |x-y| - |y|\). We note that in Lemma 2.5, we bound \(|\phi| \lesssim |x-z| + \langle z \rangle\), for this estimate, we wish to retain the \(|x-z|\) decay, so instead we use the bound \(|\phi| \lesssim \langle x \rangle + \langle z \rangle\). Again, we need only concern ourselves with the \(\lambda^2\) and \(\lambda^0\) terms of the integrand of (2.15). We bound these terms as follows,

\[
\begin{align*}
\frac{\lambda^2}{|x-z|^2|x-y|^2} (\langle x \rangle + \langle z \rangle), \\
\frac{1}{|x-z|^3|x-y|^3} (\langle x \rangle + \langle z \rangle).
\end{align*}
\] (2.16) (2.17)

We note that each bound is a sum of two terms. However, if we take the term with \(\langle x \rangle\), it reduces down to the case with no derivatives since this term merely cancels out the extra weight \(\sigma > \frac{5}{2}\). That is, (2.16) reduces down to (2.11) and (2.14) reduces to (2.17). Let us first consider the \(\lambda^2\) term. We need to bound the weighted \(L^2\) norm, that is for the \(\langle z \rangle\) term of (2.16), we use

\[
\left[ \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \frac{\langle z \rangle^{-3}}{|x-z|^2|x-y|^2} \, dz \right|^2 (\langle x \rangle^{-5}) \, dx \right]^{\frac{1}{2}} \lesssim \left[ \int_{\mathbb{R}^3} \frac{\langle x \rangle^{-5}}{|x-y|^4} \, dx \right]^{\frac{1}{2}} \lesssim \frac{1}{\langle y \rangle^2}.
\]

Where we used Lemma 1.23 in the second to last line with \(\max(0, k+\ell-n) \leq \min(k, \ell, k+\ell+\beta-n) = 2\) as \(k = \ell = 2\) and \(\beta = 3\), and we used Lemma 2.4 in the last line.

Turning our attention to (2.17), the \(\lambda^0\) term, as in the case of no derivatives and using (2.13), we need to bound

\[
\left[ \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \left( \frac{1}{|z-y|^3} + \frac{1}{|x-z|^3} \right) \, dz \right|^2 (\langle x \rangle^{-5}) \, dx \right]^{\frac{1}{2}} \lesssim \left[ \int_{\mathbb{R}^3} \left( \langle y \rangle^{-2} + \langle z \rangle^{-2} \right) \frac{\langle x \rangle^{-5}}{|x-y|^4} \right]^{\frac{1}{2}} \lesssim \frac{1}{\langle y \rangle^2}.
\]

Where we used Lemma 2.4 throughout this calculation.
We rewrite the high energy evolution, (2.2), as

\[
I_{x,y}^{\pm}(t) = \int_{0}^{\infty} e^{it\lambda^2 \pm i\lambda(|x|+|y|)} \chi L(\lambda)(1 - \chi_0(\lambda))\lambda VR_0^{\pm} VR_0^{\pm} \langle VR_0^{\pm} \rangle^{m-1} J_y^{\pm}(\lambda,\cdot), (R_0^{\pm})^{m-1} J_x^{\pm}(\lambda,\cdot) \rangle d\lambda
\]

\[
= \int_{0}^{\infty} e^{it\lambda(\lambda - 2\lambda_1)} a_{x,y}^{\pm}(\lambda) d\lambda. \tag{2.18}
\]

Here \( \lambda_1 = \pm \frac{|x|+|y|}{2t} \) is the unique critical point of the phase.

For \( 0 < t < 1 \), we note that by Corollary 2.6 and Lemma 2.3,

\[
|a_{x,y}^{\pm}(\lambda)| \lesssim \lambda \|VR_0^{\pm} VR_0^{\pm} \langle VR_0^{\pm} \rangle^{m-1} J_y^{\pm}(\lambda,\cdot)\|_{L^2,-\frac{1}{2}} + \|VR_0^{\pm} VR_0^{\pm} \langle VR_0^{\pm} \rangle^{m-1} J_y^{\pm}(\lambda,\cdot)\|_{L^2,-\frac{1}{2}}
\]

\[
\lesssim \lambda \|J_x\|_{L^2,-\frac{1}{2}} \|R_0^{\pm} V\|^{2m-2}_{L^2,-\frac{1}{2} - L^2,-\frac{1}{2}} \|VR_0 V\|_{L^2,-\frac{1}{2} - L^2,-\frac{1}{2}}
\]

\[
\times \|V\|_{L^2,-\frac{1}{2} - L^2,\frac{1}{2}} + \|J_y^{\pm}\|_{L^2,-\frac{1}{2}}
\]

\[
\lesssim \lambda(\lambda)^4(\lambda^{-1})^{2m-2}\lambda^{-1} \lesssim \lambda^{6-2m}
\]

We see taking \( m = 4 \) will be sufficient and that for \( 0 < t < 1 \),

\[
|I_{x,y}^{\pm}(t)| \lesssim \int_{0}^{\infty} (\lambda)^{-2} d\lambda \lesssim 1.
\]

The implicit constant depends on \( \lambda_0 > 0 \), the small constant we choose in Section 2.2 when treating the small energies.

When \( t > 1 \), we note that the phase, a constant (in \( \lambda \)) multiple of \( \varphi(\lambda) = \lambda^2 - 2\lambda_1\lambda \), has critical point \( \lambda_1 = \pm \frac{|x|+|y|}{2t} \), and is stationary since \( \frac{d^2}{d\lambda^2}\varphi(\lambda) = 2 \neq 0 \). We have that \( a_{x,y}^{\pm} \) has three derivatives in \( \lambda \) that satisfy the following bounds

\[
\left| \frac{d}{d\lambda} \right|^j a_{x,y}^{\pm}(\lambda) \lesssim (\lambda)^{-2} \langle x \rangle^{-2} \langle y \rangle^{-2} \text{ for } j = 0, 1 \text{ for all } \lambda > 1,
\]

\[
\left| \frac{d}{d\lambda} \right|^j a_{x,y}^{\pm}(\lambda) \lesssim (\lambda)^{-2} \text{ for } j = 2, 3 \text{ for all } \lambda > 1.
\]

In fact, these bounds hold for all \( \lambda > \lambda_0 \) with an implicit constant that depends on \( \lambda_0 \).

The above bounds follow from an analysis as in the small time \( 0 < t < 1 \) case handled above. \( \lambda \) derivatives can affect either the small-energy cut-off, the stray power of \( \lambda \), one of the free or perturbed resolvents or finally one of the \( J^{\pm} \)'s. When derivatives act on the resolvents, free or perturbed, the effect is the same. This
can be seen by Lemma 2.3. In effect, we have to bound a sum of terms with \( j_\ell \in \mathbb{N}_0 \) such that \( j_1 + \cdots + j_7 = j \)

\[
\langle V \left[ \left( \frac{d}{d\lambda} \right)^{j_3} R^*_1 V \right] \left[ \left( \frac{d}{d\lambda} \right)^{j_4} (R^*_0 V)^{m-1} \right] \left[ \left( \frac{d}{d\lambda} \right)^{j_5} J^\pm_y (\lambda, \cdot) \right] \left[ \left( \frac{d}{d\lambda} \right)^{j_6} (R^*_0 V)^{m-1} \right] \left[ \left( \frac{d}{d\lambda} \right)^{j_7} J^\pm_x (\lambda, \cdot) \right] \rangle
\]

\[
\times \left[ \left( \frac{d}{d\lambda} \right)^{j_1} (1 - \chi(\lambda)) \right] \left[ \left( \frac{d}{d\lambda} \right)^{j_2} \chi_j \right] \left\| \frac{d^j}{d\lambda^j} J_y \right\|_{L^2, -\frac{1}{2} - j_5} \left\| R_0 V \right\|_{L^2, -\frac{1}{2} - j_5} \left\| R_0 V \right\|_{L^2, -\frac{1}{2} - j_5} \left\| J_x \right\|_{L^2, -\frac{1}{2} - j_5} \left\| \frac{d^j}{d\lambda^j} J_y \right\|_{L^2, -\frac{1}{2} - j_5} \left\| R_0 V \right\|_{L^2, -\frac{1}{2} - j_5} \left\| R_0 V \right\|_{L^2, -\frac{1}{2} - j_5} \left( \lambda^{-1} \right)^{2m - 2 - j_5 - j_6} \langle \lambda \rangle^4
\]

\[
\lesssim \chi_j \left( \lambda = \lambda_0 \right) \lambda^{j+2 - 2m} \langle \lambda \rangle^4
\]

We note that if \( j_1 \neq 0 \), we are confined to the annulus \( \lambda \approx \lambda_0 \), and to have a non-zero contribution, we must have that \( j_2 \leq 1 \). To attain the desired estimate, we see that we need \( m \geq 4 + \frac{j}{2} \). In particular, as these functions decay sufficiently at infinity, this justifies taking \( L = \infty \) in (2.2).

The requirement that \( |V(x)| \lesssim \langle x \rangle^{-4-} \) is needed when \( j_3 = 3 \), \( V \) must map \( L^{2, -\frac{1}{2}} \) to \( L^{2, \frac{1}{2}+} \). This requirement is also realized when all three derivatives act on either \( J^\pm_y \) or \( J^\pm_x \).

We note that for \( I^+_x \), the critical point of the phase is outside of the support of \( a^+_x \). Three integration by parts in \( \lambda \) yield the bound \( |I^+_x(t)| \lesssim |t|^{-3} \). In particular, we note that

\[
\int_0^\infty e^{it\lambda (\lambda - 2\lambda_1)} a^+_x (\lambda) d\lambda = \left( \frac{1}{2it} \right)^3 \int_0^\infty e^{it\lambda (\lambda - 2\lambda_1)} \left( \frac{d}{d\lambda} \right)^3 a^+_x (\lambda) d\lambda
\]

For \( I^+_x \), \( \lambda - \lambda_1 \gtrsim \lambda \). Thus, we have

\[
\int_0^\infty e^{it\lambda (\lambda - 2\lambda_1)} a^+_x (\lambda) d\lambda \lesssim |t|^{-3} \int_0^\infty \left| \frac{d}{d\lambda} \frac{1}{\lambda - \lambda_1} \right|^3 a^+_x (\lambda) d\lambda
\]

\[
\lesssim |t|^{-3} \int_0^\infty (\lambda)^{-2} d\lambda \lesssim |t|^{-3}.
\]

We can similarly integrate by parts and bound \( |I^+_x(t)| \lesssim |t|^{-3} \) away from the critical point of the phase. Further if \( \lambda_1 \ll \lambda_0 \), we can again integrate by parts three times. Finally, if \( \lambda_1 \gtrsim \lambda_0 \) we can also have the

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is not a problem as for some $c$ proved bounds for $a_{x,y}$. This completes the proof of the desired bound.

**Taylor’s Theorem,**

Writing $1 = \eta_2 + (1 - \eta_2)$, we rewrite the integral as follows

$$
\left| \int_{-\infty}^{\infty} e^{it\phi(\lambda)} a(\lambda) d\lambda \right| \lesssim \left| \int_{-\infty}^{\infty} e^{it\phi(\lambda)} a(\lambda) \eta_2(\lambda) d\lambda \right| + \left| \int_{-\infty}^{\infty} e^{it\phi(\lambda)} a(\lambda)(1 - \eta_2(\lambda)) d\lambda \right|
$$

The first term is bounded as in the statement since $\text{supp}(\eta_2) = [-|t|^{-\frac{1}{2}}, |t|^{-\frac{1}{2}}]$. For the second term, we integrate by parts once in $\lambda$ to bound with

$$
|t|^{-1} \left| \int_{-\infty}^{\infty} e^{it\phi(\lambda)} \left( \frac{a(\lambda)(1 - \eta_2(\lambda))}{\phi'(\lambda)} \right)' \lambda \phi''(0) + \lambda \phi''(c) \right| \approx |\phi'(\lambda)| \approx |\lambda|.
$$

This completes the proof of the desired bound.

**Lemma 2.8.** Let $\phi'(0) = 0$ and $1 \leq \phi'' \leq C$. Then,

$$
\left| \int_{-\infty}^{\infty} e^{it\phi(\lambda)} a(\lambda) d\lambda \right| \lesssim \int_{|\lambda| < |t|^{-\frac{1}{2}}} |a(\lambda)| d\lambda + |t|^{-1} \int_{|\lambda| > |t|^{-\frac{1}{2}}} \left( \frac{|a(\lambda)|}{|\lambda|^2} + \frac{|a'(\lambda)|}{|\lambda|} \right) d\lambda.
$$

**Proof.** Let $\eta \in C^\infty_c$ be such that $\eta(x) = 1$ if $|x| < 1$ and $\eta(x) = 0$ if $|x| > 2$. Let $\eta_2(x) = \eta(x/2|t|^{1/2})$.

Following line of reasoning after (2.19) using Proposition 2.7 instead of Corollary 2.6 we are led to im-

\begin{align*}
|\phi'(\lambda)| &= \phi'(0) + \lambda \phi''(c) = \lambda \phi''(c)
\end{align*}

for some $c$ between 0 and $\lambda$. By assumptions, we have that $\phi''$ is bounded above and below, we have

|\phi'(\lambda)| \approx |\lambda|.

Following line of reasoning after (2.19) using Proposition 2.7 instead of Corollary 2.6 we are led to im-

\begin{align*}
a_{x,y}^{\pm}(\lambda + \lambda_1) \text{ and } \phi(\lambda) = \lambda^2 - \lambda_1^2, \text{ that is taking the integrand of } I_{x,y}^{\pm}(t) \text{ and translating}
\end{align*}

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by $\lambda_1$, we see that Lemma 2.8 applies, and we have the bounds

$$|a(\lambda)| \lesssim \frac{(\lambda + \lambda_1)^{-2}}{\langle x \rangle^2 \langle y \rangle^2},$$

$$\left| \frac{d}{d\lambda} a(\lambda) \right| \lesssim \frac{(\lambda + \lambda_1)^{-2}}{\langle x \rangle^2 \langle y \rangle^2}.$$ 

Where the implicit constant depends on $\lambda_0$, a constant chosen in Section 2.2.

Thus, we have if $\lambda_1 \gtrsim \lambda_0$, the contribution to $I_{x,y}(t)$ is bounded by

$$|I_{x,y}(t)| \lesssim |t|^{-\frac{5}{2}} + \left[ \int_{|\lambda| < |t|^\gamma} |a(\lambda)| \, d\lambda + \int_{|\lambda| > |t|^\gamma} \left( \frac{|a(\lambda)|}{|\lambda|^2} + \frac{|a'(\lambda)|}{|\lambda|} \right) \, d\lambda \right].$$

This suffices to show that

$$|I_{x,y}(t)| \lesssim |t|^{-\frac{5}{2}},$$

which proves Theorem 2.1 when restricted to $\lambda > \lambda_0$.

### 2.2 Five dimensional low energy

In this section we wish to control the low energy portion of the integral in Theorem 2.1,

$$\left| \int e^{it\lambda^2} \chi(\lambda/\lambda_0) \int_{\mathbb{R}^{2m+3s}} (R_0^+ V)^{m+1} R_V^\pm (VR_0^\pm)^{m+1} d\zeta d\lambda \right|. \quad (2.20)$$

We omit the cut-off $\chi_L$ as $\chi_L \chi_0 = \chi_0$ where $\chi_0(\lambda) = \chi(\lambda/\lambda_0)$.

We use the resolvent identity

$$R_V^\pm (\lambda^2) = R_0^\pm (\lambda^2) - R_0^\pm (\lambda^2)VR_0^\pm (\lambda^2), \quad (2.21)$$

and define the operator $S^\pm(\lambda)$ by

$$S^\pm(\lambda) = I + R_0^\pm (\lambda^2) V, \quad (2.22)$$
so that formally,

\[ R^\pm_V(\lambda^2) = [S^\pm(\lambda)]^{-1} R^\pm_0(\lambda^2). \]

We further define \( S_0 = S^\pm(0) \), and note that (2.20) and (2.21) lead us to bounding

\[
\sup_{x,y \in \mathbb{R}^5} \left| \int_0^\infty \lambda \chi_0(\lambda) \left[ (R^+_0(\lambda^2) VR^+_0(\lambda^2) S_0^{-1} B^+_5(\lambda) R^+_0(\lambda^2) VR^+_0(\lambda^2)) - R^+_0(\lambda^2) VR^+_0(\lambda^2) S_0^{-1} B^+_5(\lambda) R^+_0(\lambda^2) VR^+_0(\lambda^2)) \right] (x, y) d\lambda \right| \tag{2.23}
\]

Where, \( B^+_5(\lambda) \) is defined in (2.28), later in this section. We iterate enough times so that the iterated resolvents on either side of the \( B^+_5 \) are in weighted \( L^2 \) spaces.

The free resolvents in dimension five have explicit expansion

\[
R^+_0(\lambda^2)(x, y) = C_5 e^{\pm i\lambda |x-y|} \left( 1 \mp i\lambda |x-y| |x-y|^3 \right). \tag{2.24}
\]

We note that in particular \( R^+_0(\lambda^2)(x, y) \) is not locally in \( L^2(\mathbb{R}^5) \), so it cannot be Hilbert-Schmidt as in the three-dimensional case.

We follow the approach of Goldberg and Schlag in [27], in particular we will establish invertibility and control the size of \( S^\pm(\lambda) = (I + R^\pm_0(\lambda^2) V) \) as a perturbation from zero energy. We then expand the resulting perturbation in a Neumann series in certain Hilbert-Schmidt norms.

In this section we establish estimates on the Hilbert-Schmidt norm of \( \chi_0 B^+_5 \) and its derivatives as a linear mapping from \( L^{2,\sigma}(\mathbb{R}^5) \) to \( L^{2,-\alpha}(\mathbb{R}^5) \). We recall that the norm is defined by

\[
\| F \|_{HS(\sigma,-\alpha)}^2 = \int \int_{\mathbb{R}^{10}} \langle x \rangle^{-2\sigma} |F(x, y)|^2 \langle y \rangle^{-2\alpha} dx dy. \tag{2.25}
\]

We further define the following notation. For an integral operator with kernel \( F(\lambda)(x, y) \), we define the norm

\[
\| F \|_{L^1(HS(\sigma,-\alpha))} = \left( \int \int_{\mathbb{R}^{10}} \langle x \rangle^{-2\sigma} |F(\lambda)(x, y)|^2 \langle y \rangle^{-2\alpha} d\lambda \right)^{\frac{1}{2}}
\]

We note Proposition 4.3 from [29].

**Proposition 2.9.** Suppose \( |V(x)| \lesssim \langle x \rangle^{-\beta} \) for some \( \beta > \frac{n+1}{2} \) and also that zero energy is neither an
eigenvalue nor a resonance of $H = -\Delta + V$. Then

$$\sup_{\lambda \geq 0} \|[S^\pm(\lambda^2)]^{-1}\|_{L^2(-\sigma) \to L^2(-\sigma)} < \infty$$

for all $\sigma \in \left(\frac{1}{2}, \beta - \frac{1}{2}\right)$.

This proposition is vital to our approach, and relies on using the Fredholm alternative and the assumption that zero is regular.

Noting that

$$R^\pm_0(0)(x,y) = C_5 \frac{1}{|x-y|^3},$$

we rewrite $R^\pm_0(\lambda^2) = R^\pm_0(0) + B^\pm_5(\lambda)$. Then, we can write

$$[I + R^\pm_0(\lambda^2)V]^{-1} = [I + R^\pm_0(0)V + B^\pm_5(\lambda)V]^{-1}$$

$$= [S_0 + B^\pm_5(\lambda)V]^{-1}$$

$$= [(I + B^\pm_5(\lambda)V S_0^{-1})S_0]^{-1}$$

$$= S_0^{-1}[I + B^\pm_5(\lambda)V S_0^{-1}]^{-1}.$$

where $S_0 = S^\pm(0) = I + R^\pm_0(0)V$. Proposition 2.9 establishes the existence of $S_0^{-1}$ as an operator on certain weighted $L^2$ spaces. The integral kernel has form

$$B^\pm_5(\lambda) = C_5 \left( e^{\pm i\lambda|x-y|} \frac{1 \mp i\lambda|x-y|}{|x-y|^3} - \frac{1}{|x-y|^3} \right),$$

which satisfies the size estimate

$$|B^\pm_5(\lambda^2)| \lesssim \frac{\lambda}{|x-y|^2};$$

(2.26)

This follows since,

$$|B^\pm_5(\lambda^2)| \lesssim \left| \frac{e^{\pm i\lambda|x-y|} - 1}{|x-y|^3} \mp \frac{i\lambda e^{\pm i\lambda|x-y|}}{|x-y|^2} \right|,$$

$$\lesssim \left| \frac{e^{\pm i\lambda|x-y|} - 1}{|x-y|^3} \right| + \frac{\lambda}{|x-y|^2},$$

and $|e^{i\theta} - 1| \lesssim |\theta|$. 

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Proposition 2.10. If \( \sigma, \alpha > \frac{1}{2} \), and \( \sigma + \alpha > 3 \), then

\[
\sup_{\lambda \geq 0} \lambda^{-1} \| B_5^\pm (\lambda) \|_{HS(\sigma, -\alpha)} \leq C_{\sigma, \alpha}.
\]

Proof. Consider

\[
\| B_5^\pm (\lambda) \|^2_{HS(\sigma, -\alpha)} = \int_{\mathbb{R}^10} (x)^{-2\sigma} |B_5^\pm (\lambda)|^2 (y)^{-2\alpha} \, dx \, dy
\]

\[
\lesssim \int_{\mathbb{R}^10} (x)^{-2\sigma} \frac{\lambda^2}{|x-y|^4} (y)^{-2\alpha} \, dx \, dy
\]

By symmetry between \( x \) and \( y \), let us assume that \( \sigma < \alpha \), using Lemma 2.4, we can first integrate in \( x \) to obtain

\[
\| B_5^\pm (\lambda) \|^2_{HS(\sigma, -\alpha)} \lesssim \lambda^2 \int_{\mathbb{R}^5} (x)^{1-2\alpha-2\sigma} \, dx \lesssim \lambda^2.
\]

The last inequality follows as \( 1 - 2\alpha - 2\sigma < -5 \). Here we assumed \( 2\sigma < 5 \), if \( 2\sigma > 5 \), the \( x \) integral passes forth \(-4\) decay, which is bounded similarly as \( 2\alpha + 4 > 5 \). \( \square \)

Corollary 2.11. If \( \sigma, \alpha > \frac{1}{2} \), and \( \sigma + \alpha > 3 \), then \( \lim_{\lambda \to 0} \| B_5^\pm (\lambda) \|_{HS(\sigma, -\alpha)} = 0 \).

Corollary 2.12. If \( |V(x)| \lesssim \langle x \rangle^{-\beta} \) for some \( \beta > 4 \), then

\[
\lim_{\lambda \to 0} \| B_5^\pm (\lambda) V S_0^{-1} \|_{HS(\sigma, \sigma)} = 0
\]

for all \( \sigma \in \left[ -\frac{7}{2}, -\frac{1}{2} \right) \).

Proof. We know \( V S_0^{-1} : L^{2,\sigma} \to L^{2,\sigma+4+} \) for \( -\frac{7}{2} < \sigma < -\frac{1}{2} \) by Propostion 2.9. Proposition 2.11 implies that

\[
\| B_5^\pm \|_{HS(\sigma+4+, \sigma)} \to 0 \text{ as } \lambda \to 0.
\]

\( \square \)

For derivatives of \( B_5^\pm (\lambda)(x, y) \), we note that by Lemma 1.4 \( (B_5^\pm)'(\lambda)(x, y) = C e^{\pm i\lambda |x-y|} \frac{\lambda}{|x-y|^4} \). So that \( \frac{1}{\lambda} \frac{d}{d\lambda} B_5^\pm (\lambda)(x, y) = CR_3(\lambda^2)(x, y) \) where \( R_3(\lambda^2)(x, y) \) is the three-dimensional free resolvent. Further, we have that \( \frac{d}{d\lambda} \frac{d}{d\lambda} B_5^\pm (\lambda^2)(x, y) = C e^{\pm i\lambda |x-y|} \).

Lemma 2.13. \( \| \frac{1}{\lambda} \frac{d}{d\lambda} B_5^\pm (\lambda) \|_{HS(\sigma, -\alpha)} \leq C \) if \( \sigma, \alpha > \frac{3}{2} \) and \( \sigma + \alpha > 4 \).
Proof.

\[
\left\| \frac{1}{\lambda} \frac{d}{d\lambda} B_5^\pm (\lambda) \right\|_{HS(\sigma, -\alpha)}^2 = \int_{\mathbb{R}^10} \langle x \rangle^{-2\alpha} \left| \frac{1}{\lambda} \frac{d}{d\lambda} B_5^\pm (\lambda) \right|^2 \langle y \rangle^{-2\alpha} \, dx \, dy \\
\lesssim \int_{\mathbb{R}^10} \langle x \rangle^{-2\alpha} \frac{1}{|x-y|^2} \langle y \rangle^{-2\alpha} \, dx \, dy
\]

By symmetry between \( x \) and \( y \), let us assume that \( \sigma < \alpha \), using Lemma 2.4, we can first integrate in \( x \) to obtain

\[
\| B_5^\pm (\lambda) \|_{HS(\sigma, -\alpha)}^2 \lesssim \int_{\mathbb{R}^10} \langle x \rangle^{3-2\alpha-2\sigma} \, dx \lesssim 1.
\]

The last bound follows as \( 3 - 2\sigma - 2\alpha < -5 \). Here we assumed \( 2\sigma < 5 \), if \( 2\sigma > 5 \), the \( x \) integral passes forth \( -2 \) decay, which is bounded similarly as \( 2\alpha + 2 > 5 \).

\[
\square
\]

**Lemma 2.14.** \( \| \frac{d}{d\lambda} \frac{d}{d\lambda} B_5^\pm (\lambda) \|_{HS(\sigma, -\alpha)} \leq C \text{ if } \sigma, \alpha > \frac{5}{2} \).

Proof. This is trivial because

\[
\langle x \rangle^{-2\alpha} \langle y \rangle^{-2\sigma}
\]

is integrable over \( \mathbb{R}^{10} \) when \( \sigma, \alpha > \frac{5}{2} \).

\[
\square
\]

We prove the existence of a small constant \( \lambda_0 > 0 \) such that for \( \lambda < \lambda_0 \) we can expand

\[
\widetilde{B_5^\pm (\lambda)} = [I + B_5^\pm (\lambda) V S_0^{-1}]^{-1}
\]

as a Neumann series in the norms \( \| \cdot \|_{HS(\sigma, \alpha)} \) for some appropriate \( \sigma \) and \( \alpha \).

We now establish some integrability properties of iterated resolvents.
Lemma 2.15. If $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 4$ in dimension five, for each $\lambda \in \mathbb{R}$, for any $\sigma > \frac{1}{2}$,

$$
\left\| \int_{\mathbb{R}^5} R_0^\pm (\lambda^2)(x, z)V(z)R_0^\pm (\lambda^2)(z, y)\, dz \right\|_{L^2_{x, y, \sigma}} \lesssim 1,
$$

$$
\left\| \int_{\mathbb{R}^5} \left( 1 + \frac{1}{\lambda} \right) \frac{d}{d\lambda} \left[ R_0^\pm (\lambda^2)(x, z)V(z)R_0^\pm (\lambda^2)(z, y) \right]\, dz \right\|_{L^2_{x, y, \sigma}} \lesssim 1,
$$

$$
\left\| \int_{\mathbb{R}^5} \frac{d}{d\lambda} \frac{d}{d\lambda} \left[ R_0^\pm (\lambda^2)(x, z)V(z)R_0^\pm (\lambda^2)(z, y) \right]\, dz \right\|_{L^2_{x, y, \sigma}} \lesssim 1,
$$

uniformly for $y \in \mathbb{R}^5$.

Proof. Using the explicit expansion for the kernel of $R_0^\pm$ from (1.16) and apply Lemma 1.23. We can see that one integration is enough to establish local $L^2$ behavior. Considering the slowest decaying terms that result from the integration, one can establish the weighted $L^2$ behavior.

Consider when no derivatives act, we have to bound the following

$$
\left\| \int_{\mathbb{R}^5} V(z) \left( \frac{\lambda}{|x - z|^2} + \frac{1}{|x - z|^3} \right) \left( \frac{\lambda}{|z - y|^2} + \frac{1}{|z - y|^3} \right)\, dz \right\|,
$$

where $\beta$ is the weight. The weight is now constrained by the decay, $\min(k, \ell, k + \ell - 2) = 2$ for all the possible pairs $(k, \ell)$ that arise in the iterated resolvent. Thus, we need an extra weight of $-\frac{1}{2}$ to be integrable at infinity.

That is, $k, \ell \in \{2, 3\}$ and $4 \leq k + \ell \leq 6$. Applying Lemma 1.23, we have

$$
(2.29) \lesssim \langle \lambda \rangle^2 \left\{ \begin{array}{ll}
\left( \frac{1}{|x - y|} \right)^{\max(0, k + \ell - 5)} & |x - y| < 1 \\
\left( \frac{1}{|x - y|} \right)^{\min(k, \ell, k + \ell - 2)} & |x - y| > 1
\end{array} \right.
$$

Noting that $k + \ell - 5 \leq 1 < \frac{5}{2}$, we have established local $L^2$ behavior. The weight is now constrained by the decay, $\min(k, \ell, k + \ell - 2) = 2$ for all the possible pairs $(k, \ell)$ that arise in the iterated resolvent. Thus, we need an extra weight of $-\frac{1}{2}$ to be integrable at infinity.

For the second case where one $\lambda$ derivative acts, we note that by Lemma 1.4, we have

$$
\left\| \left( 1 + \frac{1}{\lambda} \right) \frac{d}{d\lambda} \left[ R_0^\pm (\lambda^2)(x, z)V(z)R_0^\pm (\lambda^2)(z, y) \right] \right\| \lesssim \langle \lambda \rangle^2 \int_{\mathbb{R}^5} \langle z \rangle^{-4-} \left( \frac{1}{|x - z||z - y|^2} \right) \left( 1 + \frac{1}{|z - y|} \right)\, dz,
$$

(2.30)
up to switching the roles of $x$ and $y$. The powers depend on which resolvent is affected by the $\lambda$ derivative.

We again apply Lemma 1.23 and bound with

$$(2.30) \lesssim \langle \lambda \rangle^2 \begin{cases} \left( \frac{1}{|x-y|} \right)^{\max(0,k+\ell-5)} & |x-y| < 1 \\ \left( \frac{1}{|x-y|} \right)^{\min(k,\ell,k+\ell-2)} & |x-y| > 1 \end{cases}$$

$$\lesssim \langle \lambda \rangle^2 \left( \frac{1}{|x-y|} \right).$$

Where here we have $k = 1$ and $\ell \in \{2, 3\}$. In both of these cases, the local singularity is integrated out. The limiting factor on the decay rate is when $\ell = 2$, then $\min(k,\ell,k+\ell-2) = 1$, and an extra factor of $-\frac{3}{2}$ decay is needed for $L^2$ integrability at infinity.

For the final case, when two $\lambda$ act on the combined resolvents, using Lemma 1.4, we have an integral with kernels determined by either $\lambda R_5 R_1$, $\lambda R_3 R_3$ or $\lambda R_1 R_5$. By symmetry between the $x$ and $y$ variables, the first and last case are essentially the same. In these cases we bound with

$$\int_{\mathbb{R}^5} R_0^\pm(\lambda^2)(x,z)V(z) \frac{d}{d\lambda} \frac{d}{d\lambda} R_0^\pm(\lambda^2)(z,y) \, dz \lesssim \langle \lambda \rangle \int_{\mathbb{R}^5} \frac{\langle z \rangle^{-4}}{|x-z|^k} \, dz$$

where $k = 2, 3$. Applying Lemma 1.23, we quickly see that we can bound with 1 for $|x-y|$ both small and large. This follows as $\ell = 0$, $\max(0, k+\ell-5) = 0$ and $\min(k,\ell,k+\ell-2) = 0$. This is clearly in $L^{2,-\frac{3}{2}}(\mathbb{R}^5)$.

When we have $\lambda R_3 R_3$, we can bound with

$$\int_{\mathbb{R}^5} \frac{d}{d\lambda} R_0^\pm(\lambda^2)(x,z)V(z) \lambda \frac{d}{d\lambda} R_0^\pm(\lambda^2)(z,y) \, dz \lesssim \langle \lambda \rangle \int_{\mathbb{R}^5} \frac{\langle z \rangle^{-4}}{|x-z||z-y|} \, dz$$

Applying Lemma 1.23, we can bound by 1 as in the last case. Again, this is because $\max(0, k+\ell-5) = 0$ and $\min(k,\ell,k+\ell-2) = 0$. This is also clearly in $L^{2,-\frac{3}{2}}(\mathbb{R}^5)$.

We bound the tail of the Born series as follows.

$$(2.23) \lesssim \sup_{x,y \in \mathbb{R}^5} \left| \int_{-\infty}^{\infty} \int_{\mathbb{R}^5} A(\lambda,|x-x_1|)(V S_0^{-1}(\chi_0 B_0^+)\lambda)(x_1,x_2)A(\lambda,|x_2-y|) \, dx_1 \, dx_2 \, d\lambda \right|$$

Where $A := \int R_0^\pm V R_0^\pm$, where the integral is taken in the appropriate variable. $A(\lambda,\cdot)$ is seen to be in
certain weighted $L^2$ spaces by Lemma 2.15. The term for $\tilde{B}_5$ is bounded identically, we use the size of these kernels and not their oscillatory behavior. Following the standard approach first laid out in [62] by Rodnianski and Schlag, we integrate by parts in $\lambda$ twice and bound.

$$\sup_{x, y \in \mathbb{R}^5} \left| \frac{1}{t^2} \int_{-\infty}^{\infty} e^{it\lambda^2} \int_{\mathbb{R}^10} \frac{d}{d\lambda} \frac{d}{d\lambda} \left[ A(\lambda, |x - x_1|)(VS_0^{-1}(\chi_0B_5^+)(\lambda)(x_1, x_2)A(\lambda, |x_2 - y|) dx_1 dx_2 d\lambda \right] \right|$$

(2.31)

There are several different cases to consider, depending on where the $\lambda$ derivatives act. We first consider when derivatives don’t act on the cut-off function $\chi_0(\lambda)$. One should note that in this section, the constant $\lambda_0$ can be assumed to be small. That is, we make the standing assumption that $0 < \lambda_0 < 1$.

**Lemma 2.16.** The inverse Fourier transform of $\chi_0B_5^+$ in $\lambda$ satisfies

$$\int_{-\infty}^{\infty} \| [\chi_0B_5^+]^\vee(u) \|_{HS(\sigma, -\alpha)} du < C\lambda_0$$

if $\sigma, \alpha > 1$ and $\sigma + \alpha > 4$.

**Proof.** We first note that by the definition of $B_5^+$, (2.26), up to a common constant $C_5$, distributionally,

$$[\chi_0B_5]^\vee(u)(x, y) = \frac{d}{du} \chi_0^\vee(u + |x - y|) \frac{\chi_0^\vee(u + |x - y|) - \chi_0^\vee(u)}{|x - y|^3}.$$

We first consider the case when $|u| \leq \frac{2}{\lambda_0}$. As $\chi \in S(\mathbb{R})$, we can take it to be $C^\infty(\mathbb{R})$ and the Mean Value Theorem applies to the second term,

$$\frac{\chi_0^\vee(u + |x - y|) - \chi_0^\vee(u)}{|x - y|^3} = \frac{d}{du} \chi_0^\vee(c) \frac{\chi_0^\vee(u + |x - y|) - \chi_0^\vee(u)}{|x - y|^3}$$

for some $c$ between $u$ and $u + |x - y|$. So that,

$$|[\chi_0B_5]^\vee(u)(x, y)| \lesssim \frac{\lambda_0^2}{|x - y|^2}.$$

The $\lambda_0^2$ comes from the scaling of the Fourier transform and the chain rule. Notice that

$$\left| \frac{d}{du} \chi_0^\vee(u) \right| \left| \frac{d}{du} \chi_0^\vee(u) \right| = \left| \int \lambda \chi(\lambda/\lambda_0) e^{iu\lambda} d\lambda \right| = \lambda_0^2 \left| \int s \chi(s) e^{isu} ds \right|.$$

As in the proof of Proposition 2.10, applying Lemma 2.4 repeatedly, we have

$$\| [\chi_0B_5]^\vee(u) \|_{HS(\sigma, -\alpha)} \lesssim \lambda_0^2.$$

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In fact, one only needs $\sigma, \alpha > \frac{1}{2}$ and $\sigma + \alpha > 3$ here. Thus, on this region we have

$$\int_{|u| \leq \frac{2}{\lambda_0}} \| [\chi_0 B_5]^\gamma (u) \|_{H^s (\sigma, -\alpha)} du \lesssim \int_{|u| \leq \frac{2}{\lambda_0}} \lambda_0^2 du \lesssim \lambda_0.$$ 

As $\lambda_0$ is small, this satisfies the desired bound.

In the case that $|u| \geq \frac{2}{\lambda_0}$, we need to attain some decay in the $u$ variable to ensure that the integral converges. We note that since $\chi$ and its derivatives are Schwartz functions, we have

$$\left\| \left( \frac{d}{du} \right)^j \chi^\gamma (u) \right\| \lesssim (u)^{-10}.$$ 

We divide the bound up into several regions. First, consider the region on which $2 |u| < |x - y|$. Here, $|u + |x - y|| \geq |u|$ by the triangle inequality, and $|x - y|^{-1} \lesssim \lambda_0$ yielding

$$[\| [\chi_0 B_5]^\gamma (u) \|_{H^s (\sigma, -\alpha)}} \lesssim \frac{\lambda_0^2 (\chi_0^\gamma)'(u + |x - y|) + \chi_0^\gamma (u + |x - y|) + \chi_0^\gamma (u)}{|x - y|^2}$$

$$\lesssim \frac{\lambda_0^2}{|x - y|^2} (\lambda_0 u)^{-10}$$

$$\lesssim \frac{1}{\lambda_0^6 |u|^{10} |x - y|^2}$$

Again, similar to the proof of Proposition 2.10, the Hilbert-Schmidt norm is bounded as

$$\| [\chi_0 B_5]^\gamma (u) \|_{H^s (\sigma, -\alpha)} \lesssim \frac{1}{\lambda_0^6 |u|^{10}}.$$ 

So that on this region

$$\int_{|u| \geq \frac{2}{\lambda_0}} \| [\chi_0 B_5]^\gamma (u) \|_{H^s (\sigma, -\alpha)} du \lesssim \lambda_0^{-8} \int_{|u| \geq \frac{2}{\lambda_0}} |u|^{-10} du \lesssim \lambda_0.$$ 

Secondly, we consider the region on which $|x - y| < \frac{1}{2} |u|$. By the triangle inequality, we have that $|u + |x - y|| \approx |u|$ here. We again use the Mean Value Theorem on the second term of the inverse Fourier transform. On this region, the value $c$ obeys $|c| \approx |u|$, so that

$$[\| [\chi_0 B_5]^\gamma (u) \|_{H^s (\sigma, -\alpha)}} \lesssim \frac{\lambda_0^2 (\chi_0^\gamma)'(u + |x - y|) + \chi_0^\gamma (u + |x - y|) + \chi_0^\gamma (u)}{|x - y|^2}$$

$$\lesssim \frac{\lambda_0^2}{|x - y|^2} (\lambda_0 u)^{-10}$$

$$\lesssim \frac{1}{\lambda_0^6 |u|^{10} |x - y|^2}$$

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Similar to the last case, on this region
\[
\int_{|u| \geq \frac{\lambda_0}{4}} \|\chi_0 B_5\|^y H_{\sigma, -\alpha} \, du \lesssim \lambda_0^{-8} \int_{|u| \geq \frac{\lambda_0}{4}} |u|^{-10} \, du \lesssim \lambda_0.
\]

The final region to consider is when \( \frac{1}{2} |u| < |x - y| < 2 |u| \). As in the previous cases, we can estimate as follows.
\[
\|\chi_0 B_5\|^y (u) \lesssim \frac{\lambda_0^2}{|x - y|^2} (\lambda_0 (u + |x - y|))^{-10} + (\lambda_0 u)^{-10}).
\]
The second term in the sum can be handled as in the previous cases. The first term is different, as on this region, we can have \( u + |x - y| = 0 \) here. Accordingly, we must take some care in our treatment. We wish to bound
\[
\lambda_0^2 \int_{|u| \geq \frac{\lambda_0}{4}} \left[ \int_{|x - y| \leq |u|} \frac{\langle x \rangle^{-2\sigma} \langle u \rangle^{-20} \langle x + \tilde{r} \theta \rangle^{-2\alpha}}{|x - y|^4} \, dx \, dy \right]^{\frac{1}{2}} \, du \tag{2.32}
\]
We make the change to polar coordinates to estimate (2.32). Let \( y = x + r \theta \) where \( \theta \in S^4 \), the four-dimensional unit sphere and \( r \geq 0 \). Then, \( |x - y| = |x - (x + r \theta)| = |r \theta| = r \). We now have
\[
\left(2.32\right) \lesssim \lambda_0^2 \int_{|u| \geq \frac{\lambda_0}{4}} \left[ \int_{S^4} \int_{r \leq |u|} \int_{\mathbb{R}^5} \langle x \rangle^{-2\sigma} \langle u \rangle^{-20} \langle x + r \theta \rangle^{-2\alpha} \, dx \, dr \, d\theta \right]^{\frac{1}{2}} \, du.
\]
We note that that \( |x - y|^{-4} \) is cancelled out by the \( r^4 \) that arises in the change to polar coordinates. We now make the following change of variables to scale out the \( \lambda_0 \) from the integrals. We let \( s = u \lambda_0 \) and \( \tilde{r} = r \lambda_0 \). Now,
\[
\left(2.32\right) \lesssim \lambda_0 \int_{|s| \geq 1} \left[ \int_{S^4} \int_{\tilde{r} \leq |s|} \int_{\mathbb{R}^5} \langle x \rangle^{-2\sigma} \langle s + \tilde{r} \rangle^{-20} \left( x + \frac{\tilde{r} \theta}{\lambda_0} \right)^{-2\alpha} \, dx \, d\tilde{r} \, d\theta \right]^{\frac{1}{2}} \, ds. \tag{2.33}
\]
We note that
\[
\int_{\mathbb{R}^5} \langle x \rangle^{-2\sigma} \left( x + \frac{\tilde{r} \theta}{\lambda_0} \right)^{-2\alpha} \, dx \lesssim \frac{(\tilde{r} \theta)^{-q}}{\lambda_0^q}, \tag{2.34}
\]
where \( q = \min(2\alpha, 2\sigma, 2\alpha + 2\sigma - 5) \). By assumption and the change of variables, the quantity \( \frac{\tilde{r} \theta}{\lambda_0} \) is large, so
the bracket is unnecessary. We now have,

\[ (2.33) \lesssim \lambda_0^{\frac{q+1}{2}} \int_{|s| \geq 1} \left[ \int_{S^4} \int_{|\hat{r}| \geq |s|} (\hat{r} + s)^{-20} \hat{r}^{-q} d\hat{r} d\theta \right]^{\frac{1}{2}} ds \]

\[ \lesssim \lambda_0^{\frac{q+1}{2}} \int_{|s| \geq 1} |s|^{-\frac{q}{2}} \left[ \int_{S^4} \int_{|\hat{r}| \geq |s|} (\hat{r} + s)^{-20} d\hat{r} d\theta \right]^{\frac{1}{2}} ds \]

\[ \lesssim \lambda_0^{\frac{q+1}{2}} \int_{|s| \geq 1} |s|^{-\frac{q}{2}} ds \lesssim \lambda_0^{\frac{q+1}{2}}. \]

In the last step we used that \( q > 2 \), thus the \( s \) integral is bounded. Further, since \( \lambda_0 \) is a small quantity, we bound with \( \lambda_0 \) as desired.

\[ \square \]

**Remark 2.17.** We state estimates involving the cut-off \( \chi_0(\lambda) = \chi\left(\frac{1}{\lambda_0}\right) \). The same estimates hold for the function \( \chi_1(\lambda) = \chi\left(\frac{1}{\lambda_0^2}\right) \). We introduce the second cut-off function when expanding \( \tilde{B}_5^+ \) in a Neumann series. We also note that the proofs for \( B_5^- \) are identical. Since we integrate over \( \mathbb{R} \) in the proofs and break the integrals into regions based on \( |u| \), not taking into account the sign of \( u \), the proofs work for both functions.

We now present an estimate on the inverse Fourier transform of the derivative. This is estimate and its proof are a refinement and generalization of Lemma 16 in [27].

**Lemma 2.18.** The inverse Fourier transform of \( \chi_0 \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+ \) in \( \lambda \) satisfies

\[ \int_{-\infty}^{\infty} \left\| \chi_0 \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+(u) \right\|_{HS(\sigma, -\alpha)} du < C < \infty \]

uniformly as \( \lambda_0 \rightarrow 0 \) if \( \sigma, \alpha > 2 \), and \( \sigma + \alpha > \frac{9}{2} \).

**Proof.** First note that by Lemma 1.4,

\[ \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+ = CR_3^+(\lambda^2) = C e^{i\lambda|x-y|} \frac{1}{|x-y|}. \]

It now follows that, up to a constant multiplier,

\[ \left[ \chi_0 \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+ \right]^\vee (u) = \left[ \chi_0 \frac{e^{i\lambda|x-y|}}{|x-y|} \right]^\vee (u) \]

\[ = \chi_0^\vee (u + |x-y|) \frac{1}{|x-y|}. \]

We again first consider when \( |u| \leq \frac{2}{\lambda_0} \). By scaling of the inverse Fourier transform and the fact that \( \chi \in \mathcal{S}(\mathbb{R}), \)
in this case we have

\[
\left\| \left[ \frac{\chi_0}{\lambda} \frac{d}{d\lambda} B_5^+ \right] (u) \right\| \lesssim \frac{\lambda_0}{|x - y|}.
\]

It is clear, as in the proofs of Proposition 2.10 and Lemma 2.16, that \( \sigma, \alpha > \frac{3}{2}, \sigma + \alpha > \frac{9}{2} \) is more than sufficient to establish

\[
\left\| \left[ \frac{\chi_0}{\lambda} \frac{d}{d\lambda} B_5^+ \right] (u) \right\|_{HS(\sigma, -\alpha)} \lesssim \lambda_0.
\]

Thus,

\[
\int_{|u| \leq \frac{2}{\lambda_0}} \left\| \left[ \frac{\chi_0}{\lambda} \frac{d}{d\lambda} B_5^+ \right] (u) \right\|_{HS(\sigma, -\alpha)} du \lesssim \int_{|u| \leq \frac{2}{\lambda_0}} \lambda_0 du \lesssim 1.
\]

Now if \(|u| \geq \frac{2}{\lambda_0}\), we have

\[
\left\| \left[ \frac{\chi_0}{\lambda} \frac{d}{d\lambda} B_5^+ \right] (u) \right\| \lesssim \frac{\lambda_0}{|x - y|} (\lambda_0 (u + |x - y|))^{-10}.
\]

We can further bound by

\[
\left\| \left[ \frac{\chi_0}{\lambda} \frac{d}{d\lambda} B_5^+ \right] (u) \right\| \lesssim \begin{cases} \frac{1}{\lambda_0 |u|^{10} |x - y|} & |x - y| \leq \frac{1}{2} |u| \\ \frac{1}{\lambda_0 |u|^{10} |x - y|} & |x - y| \geq 2 |u| \\ \frac{\lambda_0}{|u|} (\lambda_0 (u + |x - y|))^{-10} & \frac{1}{2} |u| < |x - y| < 2 |u| \end{cases}.
\]

The first two regions, the estimates follow from the triangle inequality. The integral of the Hilbert-Schmidt norm is bounded as in Proposition 2.10 and Lemma 2.16,

\[
\int_{|u| \geq \frac{2}{\lambda_0}} \frac{1}{\lambda_0^8 |u|^{10}} \| |x - y|^{-1} \|_{HS(\sigma, -\alpha)} du \lesssim \frac{1}{\lambda_0^8} \int_{|u| \geq \frac{2}{\lambda_0}} |u|^{-10} du \\
\lesssim \lambda_0.
\]

We now need only bound the annular part of the function. This procedure is essentially the same as in the estimation of (2.32) in Lemma 2.16. We wish to bound the following,

\[
\lambda_0 \int_{|u| \geq \frac{2}{\lambda_0}} \frac{1}{|u|} \left[ \int_{|x-y|=|u|} (x)^{-2\alpha} (\lambda_0 (u + |x - y|))^{-20} (y)^{-2\alpha} dxdy \right]^{\frac{1}{2}} du.
\]
Again we change the $y$ integral to polar coordinates centered about $x$ and perform the change of variables on the polar coordinates to scale $\lambda_0$ out of the limits of integration. As such, we wish to bound

$$ (2.35) \lesssim \lambda_0^{\frac{2}{3}} \int_{|s| \geq 1} \frac{1}{|s|} \left[ \int_{S^4} \int_{|\tilde{r}| \approx |s|} \int_{\mathbb{R}^5} \langle x \rangle^{-2\sigma} (s + \tilde{r})^{-20} \left\langle x + \frac{\tilde{r} \theta}{\lambda_0} \right\rangle^{-2\alpha} \tilde{r}^4 \, d\tilde{r} \, d\theta \right]^{\frac{1}{2}} \, ds. \quad (2.36) $$

Again, we note (2.34) to bound the $x$ integral.

$$ (2.36) \lesssim \lambda_0^{\frac{2}{3}} \int_{|s| \geq 1} \frac{1}{|s|} \left[ \int_{S^4} \int_{|\tilde{r}| \approx |s|} \tilde{r}^{4-\eta} (s + \tilde{r})^{-20} \, d\tilde{r} \, d\theta \right]^{\frac{1}{2}} \, ds \lesssim \lambda_0^{\frac{2}{3}} $$

We note that $q = \min(2\alpha, 2\sigma, 2\alpha + 2\sigma - 5) > 4$ by assumption. We use that $\frac{q}{2} - 1 > 1$ to ensure the $s$ integral converges. Further, the positive power of $\lambda_0$ is uniformly bounded as $\lambda_0 \to 0$ as desired.

**Lemma 2.19.** The inverse Fourier transform of $\chi_0 \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+ \in \lambda$ satisfies

$$ \int_{-\infty}^{\infty} \left\| \chi_0 \frac{d}{d\lambda} \left( \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+ \right) \right\|_{HS((3+, -3-)} \, du < C < \infty $$

uniformly as $\lambda_0 \to 0$.

**Proof.** One can prove this in identical form to that of Lemma 15 in [27]. The need for larger Hilbert-Schmidt weights is a consequence of the ambient space being $\mathbb{R}^5$ instead of $\mathbb{R}^3$. We instead differ slightly. We note that by Lemma 1.4, up to a constant multiplier

$$ \frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+ (\lambda) = \frac{d}{d\lambda} R_3^+ (\lambda^2) = e^{i\lambda|x-y|}. $$

So, we have

$$ \left\| \chi_0 e^{i\lambda|x-y|} \right\|^{\vee} (u) = |\chi_0^{\vee} (u + |x + y|)| \lesssim \lambda_0 \langle \lambda_0 (u + |x - y|) \rangle^{-10}. $$

Again, this is a consequence of scaling of the inverse Fourier transform and the fact that $\chi \in \mathcal{S}(\mathbb{R})$. As in the previous proofs of Lemma 2.16 and 2.18, on $|u| \leq \frac{2}{\lambda_0}$, the integral of the Hilbert-Schmidt norm is clearly bounded. As such we concentrate on when $|u| \gtrsim \frac{1}{\lambda_0}$.

When $|u| < \frac{1}{2} |x - y|$ or $|u| > 2|x - y|$, the triangle inequality yields that $|u + |x - y|| \gtrsim |u|$. As such, we
need only bound
\[ \int_{|u| \gtrsim \frac{1}{\lambda_0}} \lambda_0 \langle \lambda_0 u \rangle ^{-20} \| e^{i \lambda |x-y|} \|_{H^S(3+, -3-)} d\lambda. \]

This is a bounded quantity as \( \langle x \rangle ^{-6} \langle y \rangle ^{-6} \) is integrable over \( \mathbb{R}^5 \times \mathbb{R}^5 \).

We now need only consider the annular region on which \( |u| \approx |x-y| \). We thus wish to bound
\[ \lambda_0 \int_{|u| \gtrsim \frac{1}{\lambda_0}} \left[ \int \int_{|x-y| \approx |u|} \langle x \rangle ^{-6} \langle \lambda_0 (u + |x-y|) \rangle ^{-20} \langle y \rangle ^{-6} dxdy \right]^\frac{1}{2} du. \tag{2.37} \]

As in Lemmas 2.16 and 2.18, we make the change of coordinates from \( y \) to polar centered about \( x \) and again scale the \( \lambda_0 \) out of the limits of integration. Taking into account all the scaling, we need to bound
\[ \lambda_0^{-\frac{5}{2}} \int_{|s| \gtrsim 1} \left[ \int_{\mathbb{S}^4} \int_{|\tilde{r}| \approx |s|} \int_{\mathbb{R}^5} \langle x \rangle ^{-6} \langle s + \tilde{r} \rangle ^{-20} \left( x + \frac{\tilde{r} \theta}{\lambda_0} \right) ^{-6} \tilde{r}^4 d\tilde{r} d\theta \right]^\frac{1}{2} ds. \tag{2.38} \]

Again, we use (2.34) with \( \sigma = \alpha = 3+ \) to bound the \( x \) integral.

\[ (2.38) \lesssim \lambda_0^{\frac{1}{2}} \int_{|s| \gtrsim 1} \left[ \int_{\mathbb{S}^4} \int_{|\tilde{r}| \approx |s|} \tilde{r}^{-2-} \langle s + \tilde{r} \rangle ^{-20} d\tilde{r} d\theta \right]^\frac{1}{2} ds \]
\[ \lesssim \lambda_0^{\frac{1}{2}} \int_{|s| \gtrsim 1} |s|^{-1-} \left[ \int_{\mathbb{S}^4} \int_{\mathbb{R}} \langle s + \tilde{r} \rangle ^{-20} d\tilde{r} d\theta \right]^\frac{1}{2} ds \lesssim \lambda_0^{\frac{1}{2}}. \]

This establishes the desired bound.

Note that until Section 2.2.4, we handle the cases in which the \( \lambda \) derivatives do not act on the cut-offs \( \chi_0 \) or \( \chi_1 \).

### 2.2.1 No derivatives act on \( \tilde{B}_5^+ \)

If no derivatives act on \( \tilde{B}_5^+ (\lambda) \), they must act on the leading and trailing \( A(\lambda, \cdot) \) terms in (2.31). We note the fact that
\[ |\langle f, g \rangle| \leq \| f \|_{L^2, \sigma} \| g \|_{L^2, -\sigma} \]

This can be seen by the fact that these weighted \( L^2 \) spaces are each other’s dual spaces, or using Cauchy-Schwartz and using \( 1 = \langle x \rangle^{\sigma} \langle x \rangle^{-\sigma} \) factored to the appropriate side of the inner product. From Lemma 2.15,
and the above fact, we see that we must establish

\[ V S_0^{-1} [\chi_0 B_5^+]^\dag : L^{2,\gamma-3-} \to L^{2,\gamma+} \]

for \( \frac{1}{2} \leq \gamma \leq \frac{5}{2} \). This is because, depending on where the \( \lambda \) derivatives act, we need to map \( L^{2,-\frac{1}{2}-} \to L^{2,\frac{1}{2}+} \), if both derivatives act on the first \( A(\lambda, \cdot) \) term. \( L^{2,-\frac{1}{2}-} \to L^{2,\frac{1}{2}+} \) if both derivatives act on the second \( A(\lambda, \cdot) \) term, and \( L^{2,-\frac{1}{2}-} \to L^{2,\frac{1}{2}+} \) if one derivative acts on each \( A(\lambda, \cdot) \) term.

From Proposition 2.9, we need only establish that

\[ [\chi_0 B_5^+]^\dag : L^{2,\gamma-3-} \to L^{2,\gamma-4-}. \]

We define \( \tilde{B}_5^+ \) as a convergent Neumann series

\[ \tilde{B}_5^+ (\lambda) = [I + B_5^+ (\lambda) V S_0^{-1}]^{-1} = \sum_{n=0}^{\infty} (-B_5^+ (\lambda) V S_0^{-1})^n. \] (2.39)

We define \( \chi_1(\lambda) = \chi(\frac{\lambda}{1+\lambda_0}) \) so that \( \chi_1^n \chi_0 = \chi_0 \) for any \( n \geq 0 \). We use this and (2.39) to define \( \chi_0 B_5^+ \) as a Neumann series.

\[ \chi_0 \tilde{B}_5^+ (\lambda) = \chi_0 [I + B_5^+ (\lambda) V S_0^{-1}]^{-1} = \chi_0 \sum_{n=0}^{\infty} (-\chi_1 B_5^+ (\lambda) V S_0^{-1})^n. \] (2.40)

Upon applying the inverse Fourier transform to (2.40), we note that as in the scalar case multiplication of operator-valued functions yields convolution of their inverse Fourier transforms. We can bound the \( L^1 \) norm of the repeated convolutions by the product of the \( L^1 \) norms of each piece provided the the range of each operator is contained in the domain of the operator following it. This follows from Young’s inequality, or repeated application of Tonelli’s theorem if you like. We now have

\[ \| [\chi_0 B_5^+]^\dag \|_{L^1(L^2,\gamma-3-,L^2,\gamma-3-)} \leq \| \chi_0^\dag I \|_{L^1(L^2,\gamma-3-,L^2,\gamma-3-)} + \sum_{n=1}^{\infty} \| [\chi_0 B_5^+]^\dag V S_0^{-1} \|_{L^1(HS(\gamma-3-,\gamma-3-))}. \]

In view of Proposition 2.9 and Lemma 2.16 with \( \sigma = 1+\gamma+ \) and \( \alpha = \gamma-3- \), we see that the sum converges for \( \lambda_0 \) chosen small enough. The identity operator is clearly bounded on weighted \( L^2 \) spaces and by scaling of the inverse Fourier transform and \( \chi \in \mathcal{S}(\mathbb{R}) \), \( \| \chi_0^\dag \|_1 < \infty \). 

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We must take some care when the $\gamma = \frac{5}{2}$. Here we use that

$$[\chi_0 B_5^{\pm}]^\vee \in L^1(\text{HS}(\frac{7}{2}^+, -1)), \quad [\chi_0 B_5^{\pm}]^\vee \in L^1(\text{HS}(3^+, -1)),$$

in addition to $V S_0^{-1} : L^{2,-\alpha} \to L^{2,-\alpha+4^+}$.

### 2.2.2 All derivatives act on $\tilde{B}_5^+$

If all the derivatives act on $\tilde{B}_5^+(\lambda)$, we have that both the leading at the trailing $A(\lambda, \cdot)$ terms are in $L^{2,-\frac{3}{2}^-}$. We need to show that

$$\frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} [V S_0^{-1} \chi_0 \tilde{B}_5^+(\lambda)] : L^{2,-\frac{3}{2}^-} \to L^{2,\sigma^-}.$$

From Proposition 2.9, we need only show

$$\frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} [\chi_0 \tilde{B}_5^+(\lambda)] : L^{2,-\frac{3}{2}^-} \to L^{2,\sigma^-}$$

for some $\sigma \geq -\frac{7}{2}$.

We defined $\tilde{B}_5^+$ as a convergent Neumann series in (2.39), we now consider the action of derivatives on this series. If both derivatives act on $\tilde{B}_5^+$, and for the time being we assume the derivatives do not act on the cut-off $\chi_0$, we have the following Neumann series to consider.

$$\sum_{n=1}^{\infty} \sum_{m=0}^{n-1} (-1)^n \left[ (\chi_1(\lambda) B_5^+(\lambda) V S_0^{-1})^m \left( \chi_0(\lambda) \frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+(\lambda) V S_0^{-1} \right) \right] \left( \chi_1(\lambda) B_5^+(\lambda) V S_0^{-1} \right)^n (m+1) (2.41)$$

$$+ \sum_{j=0}^{m-1} (\chi_1(\lambda) B_5^+(\lambda) V S_0^{-1})^j \left( \chi_0(\lambda) \frac{d}{d\lambda} B_5^+(\lambda) V S_0^{-1} \right) \left( \chi_1(\lambda) B_5^+(\lambda) V S_0^{-1} \right)^m (j+1)$$

$$\times \left( \chi_1(\lambda) \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+(\lambda) V S_0^{-1} \right) \left( \chi_0(\lambda) \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+(\lambda) V S_0^{-1} \right)^n (m+1) (2.42)$$

$$+ (\chi_1(\lambda) B_5^+(\lambda) V S_0^{-1})^m \left( \chi_0(\lambda) \frac{1}{\lambda} \frac{d}{d\lambda} B_5^+(\lambda) V S_0^{-1} \right)$$

$$\times \sum_{j=0}^{n-m-2} (\chi_1(\lambda) B_5^+(\lambda) V S_0^{-1})^j \left( \chi_0(\lambda) \frac{d}{d\lambda} B_5^+(\lambda) V S_0^{-1} \right) \left( \chi_1(\lambda) B_5^+(\lambda) V S_0^{-1} \right)^n (j-m+1) (2.43)$$

Three subcases arise from the Neumann series above. We first present the subcase of (2.41), then the analysis for (2.42). We note that the analysis of (2.43) is similar to that of (2.42).

As in the case when no derivatives act on $\tilde{B}_5^+$, we evaluate the Neumann series in the $L^1$ norm, but now we must take values in different Hilbert-Schmidt spaces. We note that the terms of the series in (2.41) are
controlled by Lemmas 2.16 and 2.19. From right to left, we map between the following spaces.

\[(\chi_1 B^+_5)^\vee V S_0^{-1} \in L^1\left(HS\left(-\frac{1}{2} -, -1 -\right)\right),\]
\[\left(\frac{d}{d\lambda} \frac{d}{d\lambda} B^+_5\right)^\vee V S_0^{-1} \in L^1\left(HS\left(-1 -, -3 -\right)\right),\]
\[(\chi_1 B^+_5)^\vee \in L^1(HS(-3 -, -3 -)).\]

Finally, one uses \(VS^{-1}_0 : L^{2,-3-} \to L^{2,\frac{1}{2}+}\). We note that

\[\|\text{(2.41)}\|_{L^1(HS(-\frac{1}{2} -, \frac{1}{2} +))} \leq \sum_{n=1}^{\infty} (C\lambda_0)^{n-1}.\]

This series for (2.41) converges for \(\lambda_0\) small enough.

For the series in (2.42), we apply the same process. This time we use Lemmas 2.16 and 2.18 to control the various terms.

\[(\chi_1 B^+_5)^\vee V S_0^{-1} \in L^1\left(HS\left(-\frac{1}{2} -, -1 -\right)\right),\]
\[\left(\frac{d}{d\lambda} \frac{d}{d\lambda} B^+_5\right)^\vee \in L^1\left(HS\left(-1 -, -2 -\right)\right),\]
\[(\chi_1 B^+_5)^\vee \in L^1(HS(-2 -, -3 -)).\]

Now, one uses that \(VS^{-1}_0 : L^{2,-3-} \to L^{2,1+}\). Note that

\[\|\text{(2.42)}\|_{L^1(HS(-\frac{1}{2} -, \frac{1}{2} +))} \leq \sum_{n=1}^{\infty} (C\lambda_0)^{n-1}\]

Again, the series for (2.42) converges for \(\lambda_0\) small enough. The series for (2.43) works via the same mapping estimates, and is bound identically to (2.42).

### 2.2.3 One derivative acts on \(\widetilde{B}^+_5\)

If one derivative acts on the \(\widetilde{B}^+_5\), one derivative must have acted on either the first \(A(\lambda, \cdot)\) or the trailing \(A(\lambda, \cdot)\). According, we must show that

\[
\frac{1}{\lambda} \frac{d}{d\lambda} [V S_0^{-1} \chi_0 \widetilde{B}^+_5(\lambda)] : L^{2,-\frac{1}{2} -} \to L^{2,\frac{1}{2} +},
\]
\[
\frac{1}{\lambda} \frac{d}{d\lambda} [V S_0^{-1} \chi_0 \widetilde{B}^+_5(\lambda)] : L^{2,-\frac{3}{2} } \to L^{2,\frac{1}{2} +}.
\]
Using Proposition 2.9, we need only establish that
\[
\frac{1}{\lambda} \frac{d}{d\lambda} \left[ \tilde{\chi}_0 B_5^\pm (\lambda) \right] : L^{2,-\frac{1}{2}} \to L^{2,-\frac{3}{2}}, \\
\frac{1}{\lambda} \frac{d}{d\lambda} \left[ \tilde{\chi}_0 B_5^\pm (\lambda) \right] : L^{2,-\frac{3}{2}} \to L^{2,-\frac{5}{2}}.
\]

As in the other cases, we expand in a Neumann series, which we show converges in appropriate Hilbert-Schmidt norms.

\[
\chi_0(\lambda) \frac{1}{\lambda} \frac{d}{d\lambda} \left[ B_5^\pm (\lambda) \right] = \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} (-1)^n ((\chi_1 B_5^\pm(\lambda) V S_0^{-1})^m (\chi_0(\lambda) \frac{1}{\lambda} \frac{d}{d\lambda} B_5^\pm (\lambda) V S_0^{-1}((\chi_1 B_5^\pm(\lambda) V S_0^{-1})^n \left. \right|_{x-y})
\]

Upon applying the inverse Fourier transform to the series, multiplication becomes convolution, and we again bound by a product of the \(L^1\) norms. As such, we need Lemmas 2.16 and 2.18.

We note that

\[
[(\chi_1 B_5^\pm) V S_0^{-1}]^\vee \in L^1 \left( HS \left( -\frac{3}{2},-\frac{3}{2} \right) \right), \\
\left[ \chi_0(\lambda) \frac{1}{\lambda} \frac{d}{d\lambda} B_5^\pm V S_0^{-1} \right]^\vee \in L^1 \left( HS \left( -\frac{3}{2},-2 \right) \right), \\
[(\chi_1 B_5^\pm) V S_0^{-1}]^\vee \in L^1 \left( HS \left( -2,-2 \right) \right).
\]

Which is what we needed for the case when the \(\lambda\) derivative acted on the leading \(A(\lambda,\cdot)\).

When the \(\lambda\) derivative acts on the trailing \(A(\lambda,\cdot)\), we note

\[
[(\chi_1 B_5^\pm) V S_0^{-1}]^\vee \in L^1 \left( HS \left( -\frac{3}{2},-\frac{3}{2} \right) \right), \\
\left[ \chi_0(\lambda) \frac{1}{\lambda} \frac{d}{d\lambda} B_5^\pm V S_0^{-1} \right]^\vee \in L^1 \left( HS \left( -\frac{3}{2},-2 \right) \right), \\
[(\chi_1 B_5^\pm) V S_0^{-1}]^\vee \in L^1 \left( HS \left( -2,-2 \right) \right).
\]

2.2.4 All derivatives act on the cut-off

Lemma 2.20. For any function \(f \in S(\mathbb{R})\) there exists a value \(c\) between \(\xi\) and \(\xi + |x-y|\) so that

\[
[B_5^\pm f]^\vee(\xi) = C \frac{1}{|x-y|} \frac{\partial^2}{\partial \xi^2} f^\vee(c).
\]
Proof. We compute, up to a constant multiplier, the inverse Fourier transform,

\[
[B_5^+ f]^\vee (\xi) = \frac{1}{|x - y|^3} \left( e^{i|\lambda| |x - y|} (-i|\lambda| |x - y| + 1) f \right)^\vee (\xi)
\]

\[
= \frac{1}{|x - y|^3} \left[ -|x - y| \frac{\partial}{\partial \xi} f^\vee (\xi + |x - y|) + f^\vee (\xi + |x - y|) - f^\vee (\xi) \right]
\]

Now, as \( f \in \mathcal{S}(\mathbb{R}) \) and the inverse Fourier transform maps the Schwartz space to itself, \([B_5^+ f]^\vee\) can be assumed to be infinitely differentiable. As such, we can apply Taylor’s Theorem to see

\[
f^\vee (\xi) = f^\vee (\xi + |x - y|) + |x - y| \frac{\partial}{\partial \xi} f^\vee (\xi + |x - y|) + |x - y|^2 \frac{\partial^2}{\partial \xi^2} f^\vee (c)
\]

Substituting this into the inverse Fourier transform to obtain

\[
[B_5^+ f]^\vee (\xi) = \frac{1}{|x - y|^3} \left[ -|x - y| \frac{\partial}{\partial \xi} f^\vee (\xi + |x - y|) + f^\vee (\xi + |x - y|)
\]

\[
- \left( f^\vee (\xi + |x - y|) + |x - y| \frac{\partial}{\partial \xi} f^\vee (\xi + |x - y|) + |x - y|^2 \frac{\partial^2}{\partial \xi^2} f^\vee (c) \right) \right]
\]

\[
= \frac{1}{|x - y|^3} \frac{\partial^2}{\partial \xi^2} f^\vee (c).
\]

\[
\square
\]

Corollary 2.21.

\[
\left[ \left( \frac{d}{d\lambda_0} \frac{d}{d\lambda} \chi_{0} \right) B_5^+ \right]^\vee (\xi) = \frac{\lambda_0^2}{|x - y|} \psi(\lambda_0 c)
\]

Where \( \psi \in \mathcal{S}(\mathbb{R}) \) and \( c \) is between \( \xi \) and \( \xi + |x - y| \).

Proof. The proof follows from scaling considerations and Lemma 2.20. We write

\[
\frac{d}{d\lambda_0} \frac{d}{d\lambda} \chi_{0} = \lambda_0^{-3} \left( \frac{d}{d\lambda_0} \frac{d}{d\lambda} \chi \right) \circ d_{\lambda_0^{-1}},
\]

to see that

\[
\left( \frac{d}{d\lambda_0} \frac{d}{d\lambda} \chi_{0} \right)^\vee (\xi) = \lambda_0^{-3} \left( \left( \frac{d}{d\lambda_0} \frac{d}{d\lambda} \chi \right) \circ d_{\lambda_0^{-1}} \right)^\vee (\xi)
\]

\[
= \lambda_0^{-2} \left( \frac{d}{d\lambda} \frac{d}{d\lambda} \chi \right)^\vee (\lambda_0 \xi)
\]

By the chain rule, each \( \xi \) derivative will add an additional power of \( \lambda_0 \), and each \( |x - y| \) that arrives from
the “$(x - a)$” terms in the Taylor expansion, with this scaling are really $\lambda_0(|x - y| + \xi) - \lambda_0 \xi = \lambda_0 |x - y|$. Thus, the second derivative term comes with a $\lambda_0^2$.

As $\chi^{(j)}(\lambda)$ is supported on the annulus where $|\lambda| \approx 1$, $\frac{d}{d\lambda} \frac{d}{d\lambda} \chi \in S(\mathbb{R})$.

\begin{proposition}
The inverse Fourier transform of $(\frac{d}{d\lambda} \frac{d}{d\lambda} \chi_0)B_5^+$ in $\lambda$ satisfies

$$
\int_{-\infty}^{\infty} \left\| \left( \frac{d}{d\lambda} \frac{d}{d\lambda} \chi_0 \right) B_5^+ \right\|_{HS(\sigma, -\alpha)}^\vee (u) \, du < C\lambda_0^{-1}(1 + \lambda_0^{\frac{3}{2}})
$$

if $\sigma, \alpha > 2$, and $\sigma + \alpha > \frac{9}{2}$.
\end{proposition}

\begin{proof}
From Lemma 2.20, we have the form of the inverse Fourier transform. We first consider when $|u| \leq \frac{2}{\lambda_0}$. Here one can see,

$$
\int_{|u| \leq \frac{2}{\lambda_0}} \left\| \left( \frac{d}{d\lambda} \frac{d}{d\lambda} \chi_0 \right) B_5^+ \right\|_{HS(\sigma, -\alpha)}^\vee (u) \, du = \int_{|u| \leq \frac{2}{\lambda_0}} \frac{\lambda_0^2}{|x - y|} \psi(\lambda_0 c) \left\| B_5^- \right\|_{HS(\sigma, -\alpha)} \, du 
\lesssim \lambda_0^2 \int_{|u| \leq \frac{2}{\lambda_0}} |||x - y||^{-1} \, du 
\lesssim \lambda_0.
$$

Now, assume $|u| > \frac{2}{\lambda_0}$, we now divide into three subcases based on the size of $|u|$. First, we assume that $|u| < \frac{1}{2} |x - y|$. Here we do not use the Taylor expansion and cancellation, but instead note that

$$
\left\| \left( \frac{d}{d\lambda} \frac{d}{d\lambda} \chi_0 \right) B_5^+ \right\|_{HS(\sigma, -\alpha)}^\vee (u) \lesssim \frac{\lambda_0^{-2}}{|x - y|^3} [\lambda_0 (\lambda_0 (u + |x - y|))^{-10} + (\lambda_0 u)^{-10}].
$$

By the triangle inequality, $|u + |x - y| | \approx |x - y| \gtrsim |u|$. Thus,

$$
(2.44) \lesssim \frac{\lambda_0^{-2} + \lambda_0^{-1}}{|x - y|^3} \frac{1}{\lambda_0^{10} |u|^{10}} \lesssim \frac{1 + \lambda_0}{|x - y| \lambda_0^{12} |u|^{12}}.
$$

As $|x - y| \in HS(\sigma, -\alpha)$ if $\sigma, \alpha > 2$ and $\sigma + \alpha > 4$, as shown previously in Lemma 2.16 for instance. We have,

$$
\int_{|u| \gtrsim \frac{2}{\lambda_0}} \left\| (2.44) \right\|_{HS(\sigma, -\alpha)} \, du \lesssim \lambda_0^{-1} + 1.
$$

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The second region we consider is when \(|u| > 2|x - y|\). Here we note that \(|c| \approx |u|\). So that

\[
\left| \left[ \left( \frac{d}{d\lambda} \frac{d}{d\chi_0} \right) B_5^+ \right]^\vee(u) \right| \lesssim \frac{\lambda_0^2}{|x - y|} (\lambda_0 u)^{-10} \lesssim \frac{1}{\lambda_0^8 |u|^{10} |x - y|}.
\]

So that,

\[
\int_{|u| \geq \frac{2}{\sqrt{2}}} \left[ \left[ \left( \frac{d}{d\lambda} \frac{d}{d\chi_0} \right) B_5^+ \right]^\vee(u) \right]_{HS(\sigma, -\alpha)} du \lesssim \int_{|u| \geq \frac{2}{\sqrt{2}}} \frac{1}{\lambda_0^8 |u|^{10}} du \lesssim \lambda_0.
\]

The final region we consider is when \(\frac{1}{2} |x - y| \leq |u| \leq 2|x - y|\). As in the proofs of Lemmas 2.16, 2.18, and 2.19, we wish to bound the following quantity.

\[
\lambda_0^2 \int_{|u| \geq \frac{2}{\sqrt{2}}} \left[ \int \int_{|x - y| \approx |u|} \frac{\langle x \rangle^{-2\sigma} \langle y \rangle^{-2\alpha}}{|x - y|^2} dx dy \right] \frac{1}{2} du.
\]

By changing the \(y\) variable to polar coordinates centered about \(x\) and scaling the \(\lambda_0\) out of the limits of integration, we must bound

\[
(2.45) \lesssim \lambda_0^{-\frac{1}{2}} \int_{|s| \geq 1} \left[ \int_{S^4} \int_{|r| \approx |s|} \langle \tilde{r} \rangle^{-2q} \langle \tilde{r}^2 d\tilde{r} d\theta \rangle^{-\frac{2\alpha}{q}} dx d\tilde{r} d\theta \right] \frac{1}{2} ds.
\]

Using (2.34) to bound the \(x\) integral with \(q = \min(2\alpha, 2\sigma, 2\alpha + 2\sigma - 5)\), we have

\[
(2.46) \lesssim \lambda_0^{-\frac{q-1}{2}} \int_{|s| \geq 1} \left[ \int_{S^4} \int_{|r| \approx |s|} \tilde{r}^{2-q} d\tilde{r} d\theta \right] \frac{1}{2} ds
\]

\[
\lesssim \lambda_0^{-\frac{q-1}{2}} \int_{|s| \geq 1} |s|^{1-\frac{q}{2}} ds \lesssim \lambda_0^{-\frac{q-1}{2}}.
\]

As \(\frac{q}{2} > 2\), the \(s\) integral converges. This establishes the desired bound.

\[\square\]

The following corollary is not used here, but is needed in Section 2.2.5.

**Corollary 2.23.** The following bounds hold

\[
\int_{-\infty}^{\infty} \left\| \left( \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0 \right) B_5^+ \right\|_{HS(\sigma, -\alpha)}^\vee(u) \left\| du < C(1 + \lambda_0^{2+}), \right.
\]

\[
\int_{-\infty}^{\infty} \left\| \left( \frac{d}{d\lambda} \chi_0 \right) B_5^+ \right\|_{HS(\sigma, -\alpha)}^\vee(u) \left\| du < C\lambda_0(1 + \lambda_0^{2+}), \right.
\]

if \(\sigma, \alpha > 2\) and \(\sigma + \alpha > \frac{q}{2}\).
Proof. The proof of the inequalities is identical to the proof of Proposition 2.22, but with a different scaling. Note that

\[
\frac{1}{\lambda} \frac{d}{d\lambda} \chi_0 = \lambda_0^{-2} \frac{1}{(\lambda/\lambda_0)} \frac{d}{d(\lambda/\lambda_0)} \chi_0(\lambda) = \lambda_0^{-2} \left[ \frac{1}{\lambda} \frac{d}{d\lambda} \chi \right] \circ d_{\lambda_0^{-1}}(\lambda) = \lambda_0^{-2} \psi \circ d_{\lambda_0^{-1}}(\lambda).
\]

Where \( \psi \in \mathcal{S}(\mathbb{R}) \). Thus, the result corresponding to Corollary 2.21 has an extra factor of \( \lambda_0 \) for the first bound.

For the second bound, note that

\[
\frac{d}{d\lambda} \chi_0 = \lambda_0^{-1} \left[ \frac{d}{d\lambda} \chi \right] \circ d_{\lambda_0^{-1}}(\lambda) = \lambda_0^{-1} \tilde{\psi} \circ d_{\lambda_0^{-1}}(\lambda).
\]

Where \( \tilde{\psi} \in \mathcal{S}(\mathbb{R}) \). This accounts for the extra factor of \( \lambda_0^2 \).

Proposition 2.22 along with Lemma 2.16 establishes the dispersive estimate for the tail in this case. The Neumann series in this case takes the form

\[
\frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0(\lambda) \left[ B_\frac{5}{5}^+(\lambda) \right] = \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} (-1)^n \left[ (\chi_1 B_\frac{5}{5}^+(\lambda)V S_0^{-1})^m \left[ \left( \frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0(\lambda) \right) B_\frac{5}{5}^+(\lambda)V S_0^{-1} \right] \left( (\chi_1 B_\frac{5}{5}^+(\lambda)V S_0^{-1})^n \right) - (m+1) \right]
\]

We again recount the mappings required.

\[
[(\chi_1 B_\frac{5}{5}^+(\lambda)V S_0^{-1})^\vee] \in L^1 \left( HS \left( -\frac{1}{2}, -1, -1 \right) \right),
\]

\[
\left[ \left( \frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0(\lambda) \right) B_\frac{5}{5}^+(\lambda)V S_0^{-1} \right] \in L^1 \left( HS \left( -1, -1, -2 \right) \right),
\]

\[
[(\chi_1 B_\frac{5}{5}^+(\lambda)V S_0^{-1})^\vee] \in L^1 \left( HS \left( -2, -1, -2 \right) \right).
\]

Where we apply Proposition 2.22 with \( \sigma = 3+ \) and \( \alpha = 2+ \).

### 2.2.5 One derivative acts on the cut-off

In this case, we have to take another derivative of a Neumann Series similar to (2.47), but in which the second cut-off does not act on the the same cut-off function. We have the following Neumann series to

\[
\frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0(\lambda) \left[ B_\frac{9}{9}^+(\lambda) \right] = \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} (-1)^n \left[ (\chi_1 B_\frac{9}{9}^+(\lambda)V S_0^{-1})^m \left[ \left( \frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0(\lambda) \right) B_\frac{9}{9}^+(\lambda)V S_0^{-1} \right] \left( (\chi_1 B_\frac{9}{9}^+(\lambda)V S_0^{-1})^n \right) - (m+1) \right]
\]

We again recount the mappings required.

\[
[(\chi_1 B_\frac{9}{9}^+(\lambda)V S_0^{-1})^\vee] \in L^1 \left( HS \left( -\frac{1}{2}, -1, -1 \right) \right),
\]

\[
\left[ \left( \frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0(\lambda) \right) B_\frac{9}{9}^+(\lambda)V S_0^{-1} \right] \in L^1 \left( HS \left( -1, -1, -2 \right) \right),
\]

\[
[(\chi_1 B_\frac{9}{9}^+(\lambda)V S_0^{-1})^\vee] \in L^1 \left( HS \left( -2, -1, -2 \right) \right).
\]

Where we apply Proposition 2.22 with \( \sigma = 3+ \) and \( \alpha = 2+ \).
consider,

\[
\sum_{n=1}^{\infty} \sum_{m=0}^{n-1} (-1)^n ((\chi_1 B^+_5 + 5)^{-1} V S_0^{-(n-1)})^m \left[ \left( \frac{d}{d\lambda} \chi_0(\lambda) \right)^{-1} \frac{1}{\lambda} \frac{d}{d\lambda} B^+_5 (\lambda) V S_0^{-1} \right] ((\chi_1 B^+_5 + 5)^{-1} V S_0^{-(n-1)})^{-(m+1)} \tag{2.48}
\]

\[
+ \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} (-1)^n \frac{d}{d\lambda} ((\chi_1 B^+_5 + 5)^{-1} V S_0^{-(n-1)})^m \left[ \left( \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0(\lambda) \right) B^+_5 (\lambda) V S_0^{-1} \right] ((\chi_1 B^+_5 + 5)^{-1} V S_0^{-(n-1)})^{-(m+1)} \tag{2.49}
\]

\[
+ \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} (-1)^n ((\chi_1 B^+_5 + 5)^{-1} V S_0^{-(n-1)})^m \left[ \left( \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0(\lambda) \right) B^+_5 (\lambda) V S_0^{-1} \right] \frac{d}{d\lambda} ((\chi_1 B^+_5 + 5)^{-1} V S_0^{-(n-1)})^{-(m+1)} \tag{2.50}
\]

The \(\lambda\) derivatives in the second and third lines can act on either the cut-offs \(\chi_1\) or the perturbed resolvent \(B^+_5\). As such, we need the following estimates.

**Lemma 2.24.** The inverse Fourier transform of \((\frac{d}{d\lambda} \chi_0(\lambda)) \frac{1}{\lambda} \frac{d}{d\lambda} B^+_5\) in \(\lambda\) satisfies

\[
\int_{-\infty}^{\infty} \left\| \left[ \frac{d}{d\lambda} \chi_0(\lambda) \right] \frac{1}{\lambda} \frac{d}{d\lambda} B^+_5 \right\|_{HS(\sigma,-\alpha)} < C \lambda^{-1} (1 + \lambda_0^{1+})
\]

if \(\sigma > 2\) and \(\sigma + \alpha > \frac{9}{2}\).

**Proof.** This proof proceeds in the same manner as that of Lemma 2.18. We first note that

\[
\frac{d}{d\lambda} \chi_0(\lambda) = \lambda^{-1} \left[ \frac{d}{d\lambda} \chi_0(\lambda) \right] \circ d_{\lambda^{-1}}(\lambda) = \lambda^{-1} \psi \circ d_{\lambda^{-1}}(\lambda),
\]

where \(\psi \in \mathcal{S}(\mathbb{R})\). The calculations now proceed identically to the proof of Lemma 2.18, except for the constant factor of \(\lambda_0^{-1}\).

\[\square\]

We are now ready to control the terms of the Neumann series given in (2.48), (2.49), and (2.50). Consider first (2.48). We note the following mapping properties due to Lemmas 2.16 and 2.24.

\[
((\chi_1 B^+_5 V S_0^{-1})^\vee) \in L^1 \left( HS \left( -\frac{1}{2}, -1, -1 \right) \right),
\]

\[
\left[ \left( \frac{d}{d\lambda} \chi_0(\lambda) \right) \frac{1}{\lambda} \frac{d}{d\lambda} B^+_5 V S_0^{-1} \right]^\vee \in L^1 \left( HS \left( -1, -2, -2 \right) \right),
\]

\[
((\chi_1 B^+_5 (\lambda) V S_0^{-1})^\vee) \in L^1 \left( HS \left( -2, -2 \right) \right).
\]

Now, (2.48) is bounded in the same way as (2.47) up to a factor of \(\lambda_0^{-1}\).

Now, let us consider (2.49). By examining Proposition 2.22 and Lemma 2.18, if the \(\lambda\) derivative acts on the cut-off or the \(B^+_5\), the mappings are on the same Hilbert-Schmidt spaces. Using Lemma 2.16 and
When one derivative acts on a leading or trailing \( A \) or \( L \), From Lemma 2.15, we see that we need the inverse Fourier transform of this series to map \( L \) Neumann series. Section 2.2.3, we assumed the derivative did not act on the cut-off. As such, we must consider the following
\[
\left( \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0(\lambda) \right) B_5^+ (\lambda) V S_0^{-1} \bigg|^{\lambda} \in L^1 \left( HS \left( -\frac{1}{2}, -1 \right) \right),
\]
\[
\left( \frac{d}{d\lambda} (\chi_1 B_5^+) (\lambda) V S_0^{-1} \right)^{\lambda} \in L^1 \left( HS \left( -1, -2 \right) \right),
\]
\[
\left( \chi_1 B_5^+ (\lambda) V S_0^{-1} \right)^{\lambda} \in L^1 \left( HS \left( -2, -2 \right) \right),
\]
\[
\left( \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0(\lambda) \right) B_5^+ (\lambda) V S_0^{-1} \bigg|^{\lambda} \in L^1 \left( HS \left( -\frac{5}{2}, -\frac{5}{2} \right) \right).
\]

Where in the second term we used that \( V S_0^{-1} : L^{2,-\frac{1}{2}} \to L^{2,\frac{3}{2}} \) and Corollary 2.23 with \( \sigma = \frac{7}{2} + \) and \( \alpha = \frac{3}{2} + \). This allows us to bound the Neumann series for \( \lambda_0 \) small enough.

Finally, consider (2.50). This relies on the results of Corollary 2.23, Lemma 2.16 and Lemma 2.18.
\[
\left( \chi_1 B_5^+ (\lambda) V S_0^{-1} \right)^{\lambda} \in L^1 \left( HS \left( -\frac{1}{2}, -1 \right) \right),
\]
\[
\left( \frac{d}{d\lambda} (\chi_1 B_5^+) (\lambda) V S_0^{-1} \right)^{\lambda} \in L^1 \left( HS \left( -1, -2 \right) \right),
\]
\[
\left( \chi_1 B_5^+ (\lambda) V S_0^{-1} \right)^{\lambda} \in L^1 \left( HS \left( -2, -2 \right) \right),
\]
\[
\left( \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0(\lambda) \right) B_5^+ (\lambda) V S_0^{-1} \bigg|^{\lambda} \in L^1 \left( HS \left( -\frac{5}{2}, -\frac{5}{2} \right) \right).
\]

For completeness, we must state the other case in which only one derivative acts on a cut-off function. When one derivative acts on a leading or trailing \( A(\lambda, \cdot) \) and one derivative acts on \( B_5^+ \), as presented in Section 2.2.3, we assumed the derivative did not act on the cut-off. As such, we must consider the following Neumann series.
\[
\sum_{n=1}^{\infty} \sum_{m=0}^{n-1} (-1)^n \left( (\chi_1 B_5^+ (\lambda) V S_0^{-1} \right)^{\lambda} \left( \frac{1}{\lambda} \frac{d}{d\lambda} \chi_0(\lambda) B_5^+ (\lambda) V S_0^{-1} \right)^{\lambda} (\chi_1 B_5^+ (\lambda) V S_0^{-1} \right)^{\lambda} \right)
\]

From Lemma 2.15, we see that we need the inverse Fourier transform of this series to map \( L^{2,-\frac{1}{2}} \to L^{2,\frac{3}{2}}, \) or \( L^{2,-\frac{1}{2}} \to L^{2,\frac{3}{2}} \), depending on whether the derivative acts on the leading or trailing \( A(\lambda, \cdot) \). For the first
case, we use Lemma 2.16 and Corollary 2.23 to see

\[
((\chi B_0^+)\lambda VS_0^{-1})^\vee \in L^1\left(\mathcal{HS}\left(-\frac{1}{2},-1\right)\right),
\]

\[
\left[\left(1 + \frac{d}{d\lambda}\chi_0(\lambda)\right)B_0^+(\lambda)VS_0^{-1}\right]^\vee \in L^1\left(\mathcal{HS}\left(-1,-2\right)\right),
\]

\[
((\chi B_0^+)\lambda VS_0^{-1})^\vee \in L^1\left(\mathcal{HS}\left(-2,-2\right)\right).
\]

For the second case, we need that

\[
((\chi B_0^+)\lambda VS_0^{-1})^\vee \in L^1\left(\mathcal{HS}\left(-\frac{3}{2},-\frac{3}{2}\right)\right),
\]

\[
\left[\left(1 + \frac{d}{d\lambda}\chi_0(\lambda)\right)B_0^+(\lambda)VS_0^{-1}\right]^\vee \in L^1\left(\mathcal{HS}\left(-\frac{3}{2},-2\right)\right),
\]

\[
((\chi B_0^+)\lambda VS_0^{-1})^\vee \in L^1\left(\mathcal{HS}\left(-2,-2\right)\right).
\]

**Remark 2.25.** In our analysis we have not accounted for the powers of \(\lambda\) that arise from terms of the leading and trailing resolvents, \(A(\lambda, \cdot)\). Each contributes a sum of terms \(\lambda^2 + \lambda + 1\). Our analysis concentrated only on when the zero order \(\lambda\) term arose. To handle higher \(\lambda\) powers, one notes that the estimates considered in Lemmas 2.16, 2.18, 2.19, Proposition 2.22, and Corollary 2.21 gain positive powers of \(\lambda_0\) for each power of \(\lambda\) that occurs due to scaling considerations. That is, adding the \(\lambda\) powers to a cut-off function, we scale as

\[
\lambda^k\chi_0(\lambda) = \lambda_0^k\left(\frac{\lambda}{\lambda_0}\right)^k\chi\left(\frac{\lambda}{\lambda_0}\right) = \lambda_0^k[\lambda^k\chi] \circ d_{\lambda_0}^{-1}(\lambda)
\]

This is the opposite of the situation in Section 1.3.3, where negatives powers of \(\lambda\) help scaling considerations, here negative powers of \(\lambda\) are bad. This is because in Section 1.3.3, we are bounded below, \(\lambda \geq 1\), and the \(L^1\) norm of the inverse Fourier transform of \(\lambda^j\chi_L\) scales like \(L^j\), which grows unbounded when taking a supremum. In the small energy considerations, \(\lambda\) can get arbitrarily close to zero, and the \(L^1\) norm of the inverse Fourier transform would become unbounded due to singularities as \(\lambda \to 0\) if not for the use of Lemma 1.4 or derivatives on the cut-off restricting to an annulus away from zero eliminating such singularities.

We have now established that Theorem 2.1 holds for \(\lambda < \lambda_0\). This combined with the result of Section 2.1 prove Theorem 2.1 and then along with Theorem 1.2 imply Theorem 2.2.
Chapter 3

Exponential decay of dispersion managed solitons

3.1 Introduction

In non-linear optics, special solutions to Maxwell’s equations describing the evolution of a pulse moving through a fiber optic cable with group velocity of a carrier signal, are described by the non-linear Schrödinger equation as a first-order approximation, see e.g. [59]. In particular, one obtains the one-dimensional non-linear Schrödinger equation with periodically varying dispersion coefficient,

\[ iu_t + d(t)u_{xx} + |u|^2u = 0. \]  

(3.1)

In this case \( t \) is the distance along the cable and \( x \) denotes the (retarded) time and \( d(t) \) is the dispersion along the waveguide which one assumes to be piece-wise constant. Physically, \( d(t) \) arises by doping the cable with rare earth metals such as erbium.

In data transmission applications, soliton pulses are extremely desirable. Stable soliton pulses that keep their shape, even being sent over distances on the inter-continental scale, prevent data loss or corruption. Using soliton pulses to transmit binary information, the presence of a pulse being a binary ’1’ and the absence a binary ’0’ is an effective transmission mechanism. Further, fast decay in the pulses is desirable as the fast decay diminishes interference between pulses.

The non-linear Schrödinger will admit soliton solutions due to a fine balance between the linear dispersive regime and the phenomenon of finite time blow-up for non-linear solutions. We examine the so-called strong dispersion management regime, where the linear evolution dominates and the non-linearity provides a small correction.

The dispersion management technique, that of varying positive and negative dispersion along the cable, has been enormously fruitful. The presence of soliton solutions in glass-fiber cable was first predicted in 1980 by Lin, Kogelnik and Cohen in [51]. The connection between the evolution of the amplitude and the porous media equation was studied by Bronski and Kutz in [7]. The dispersion management technique has been
enormously successful in practice. Record-breaking data transmission rates of over one Terabit per second have been achieved on large distance scales. Much progress in applications of this technique has been made in the last several years, see [1, 11, 12, 22, 23, 45, 47, 48, 51, 54, 56, 57].

Rigorous mathematical analysis of the dispersion managed soliton phenomenon has recently yielded results. In Section 3.2, we reduce the problem of finding so-called dispersion management soliton solutions to (3.1) to a non-linear, non-local eigenvalue problem. In particular, we seek weak solutions to this eigenvalue problem of the form

$$\omega\langle f, g \rangle = -d_{av}\langle f_x, g_x \rangle + Q(g, f, f, f),$$

(3.2)

for all $g \in H^1(\mathbb{R})$ if $d_{av} \neq 0$ or for all $g \in L^2(\mathbb{R})$ if $d_{av} = 0$, and $Q$ is a quadra-linear, non-local averaging operator we define in the later sections.

In this vein, we offer a brief survey of the results known in the strong dispersion regime. Lushnikov, in [52, 53] made non-rigorous heuristic arguments that the dispersion managed solitons solutions to (3.1) should decay exponentially.

**Conjecture** (Lushnikov (2004)). The asymptotic form of the dispersion-managed soliton solution to (3.2) should be of the form

$$f_{\text{asympt}} \sim A \cos(a_0 x^2 + a_1 x + a_2) e^{-b|x|} \quad \text{as} \ |x| \to \infty$$

(3.3)

for some constants $A, a_j$, and $b > 0$.

There have been numerous numerical studies that corroborate this conjecture, [1, 52, 77].

By considering the energy functional related to (3.2),

$$H(f) = \frac{d_{av}}{2} \int_{\mathbb{R}} |f'(x)|^2 \, dx - \frac{1}{4} Q(f, f, f, f)$$

leads one to the following constrained minimization problems.

$$P_{\lambda}^{d_{av}} := \inf \{ H(f) : f \in H^1(\mathbb{R}), \|f\|_2^2 = \lambda \}$$

(3.4)

for $d_{av} > 0$, and

$$P_{\lambda}^0 := \inf \left\{ -\frac{1}{4} Q(f, f, f, f) : f \in L^2(\mathbb{R}), \|f\|_2^2 = \lambda \right\}$$

(3.5)
if \( d_{av} = 0 \).

There has been much study of these variational formulations of the dispersion management problem. In 2001 Zharnitsky, et. al. showed that there is a solution to (3.4) with \( f \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}) \), [88]. In 2004 Kunze considered (3.5) and showed that there is a solution \( f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) when \( d_{av} = 0 \), [46]. In 2005 Stanislavova showed that \( f \in C^\infty(\mathbb{R}) \) for Kunze’s solution to (3.5), [71].

In 2008, Hundertmark and Lee proved super-polynomial decay of the soliton solutions to (3.1) in both space and frequency.

**Theorem** (Hundertmark and Lee (2008)). The solutions to the weak dispersion managed soliton equation (3.2) are Schwartz functions, that is they are arbitrarily often differentiable and the solution and its derivatives decay faster than polynomially at infinity. That is if \( f \) is a solution to (3.2), then

\[
\sup_x |x|^m |f^{(n)}(x)| < \infty \quad \forall m, n \in \mathbb{N}_0.
\]

Hundertmark and Lee also worked on the discrete analog, so-called diffraction managed solitons. In [35], they proved the following

**Theorem** (Hundertmark and Lee (2008)). The diffraction managed solitons in the discrete analog of (3.2) decay exponentially.

Later in 2008, Erdoğan, Hundertmark and Lee were able to prove the conjectured exponential decay for the dispersion managed solitons for vanishing average dispersion, even further proving exponential decay in frequency.

**Theorem** (Erdoğan, Hundertmark and Lee (2008)). If \( f \in L^2(\mathbb{R}) \) is a solution to the dispersion management soliton equation (3.2) with \( d_{av} = 0 \), then there exists a constant \( \mu > 0 \) such that

\[
|f(x)| \lesssim e^{-\mu|x|}, \quad |\hat{f}(\xi)| \lesssim e^{-\mu|\xi|}.
\]

The rest of this chapter is devoted to proving the analogue of this theorem in the case when \( d_{av} > 0 \).

### 3.2 Reduction to eigenvalue problem

We consider when the dispersion is of the form

\[
d(t) = \frac{1}{\epsilon} d_0 \left( \frac{t}{\epsilon} \right) + d_{av}.
\]
Here we have a periodic, mean zero, piecewise constant term which physically corresponds to large, fast oscillation, and the average dispersion term \( d_{av} \). We now make the substitutions \( \xi = t/\epsilon \) and we choose \( w(\xi, x) \) so that \( u(t, x) = w(\xi, x) \). Then

\[
i\partial_t u(t, x) = \frac{i}{\epsilon} \partial_\xi w(\xi, x). \tag{3.7}
\]

Substituting (3.7) into (3.6), we have

\[
i \frac{1}{\epsilon} \partial_\xi w(\xi, x) = - \left[ \frac{1}{\epsilon} d_0 \left( \frac{t}{\epsilon} \right) + d_{av} \right] w_{xx} - |w|^2 w. \tag{3.8}
\]

In a slight abuse of notation, we multiply by \( \epsilon \) and denote \( \xi = t \) and \( u = w \). As this only rescales (3.1), it is not a serious concern. We now have

\[
i \partial_t u(t, x) = - d_0(t) u_{xx}(t, x) - \epsilon (d_{av} u_{xx}(t, x) + |w|^2 u(t, x)). \tag{3.9}
\]

In the strong dispersion regime, that is when \( \epsilon \ll 1 \), the linear evolution dominates and the nonlinearity provides a small correction. We wish to have soliton solutions, the dispersive nature of the linear evolution and the tendency of the non-linear evolution to blow-up in finite time must be in a delicate balance.

We define the following,

\[
D(t) = \int_0^t d_0(s) \, ds,
\]

\[
T_t = e^{iD(t) \partial_{xx}}.
\]

It is easy to see that \( T_t \) commutes with \( \partial_{xx} = \Delta \). For any function of \( f(x) \), which independent of \( t \). We note that

\[
i \partial_t T_t f(x) = i \partial_t \left( e^{iD(t) \partial_{xx}} f(x) \right) = i [i \partial_t D(t) \partial_{xx}] e^{iD(t) \partial_{xx}} f(x) = -d_0(t) \Delta (T_t f(x)).
\]

Where the last line follows from the Fundamental Theorem of Calculus. We make the ansatz

\[
u(t, x) = (T_t v(t, \cdot))(x).
\]
This ansatz yields the following equation.

\[ i\partial_t u(t, x) = i(\partial_t T_t)v(t, x) + iT_t(\partial_t v(t,x)) = -d_0(t)\Delta(T_t v(t, x)) + iT_t(v_t(t, x)) \]

Which, when equating the above expression for \( i\partial_t u(t, x) \) with that from (3.9) yields

\[ -d_0(t)\Delta(T_t v(t, x)) + iT_t(v_t(t,x)) = -d_0(t)\Delta(T_t v(t, x)) - \epsilon[d_{av}u_{xx}(t, x) + |u|^2u(t, x)] \]

Cancelling out the common factor of \(-d_0(t)\Delta(T_t v(t, x))\), we have

\[ iT_t(v_t(t, x)) = -\epsilon[d_{av}\Delta(T_t v(t, x)) + |T_t v|^2T_t v(t, x)] \]

Now, applying \( T_t^{-1} \) to both sides and noting that \( T_t \) and \( \Delta \) commute, we obtain

\[ i\partial_t v = -\epsilon d_{av}v_{xx} - \epsilon(T_t^{-1}[|T_t v|^2T_t v]) \]

Following the approach suggested by Gabitov and Tuirtsyn in [22, 23], we look at the “averaged” solution \( v \) by replacing \( T_t^{-1}[|T_t v|^2T_t v] \) with its average over one period, \( \int_0^1 T_t^{-1}[|T_t v|^2T_t v] \, dr \). We now examine the following equation

\[ iv_t = -\epsilon[d_{av}v_{xx} + Q(v, v, v)], \quad \text{where} \]

\[ Q(v_1, v_2, v_3) := \int_0^1 T_t^{-1}(T_t v_1 T_t v_2 T_t v_3) \, dr. \]  

This is similar to the treatment of the unstable pendulum done in [49]. The averaging procedure is rigorously justified in by Zharnitsky et. al. in [88]. Specifically, it is shown that the averaged solution stays \( \epsilon \) close to the exact solution in certain suitable Sobolev norms on long distances up to scale \( \epsilon^{-1} \). A variational study of (3.4) is done by Hundertmark and Lee in [34] that avoids heavy machinery such as Lions concentration compactness.

The averaging procedure uses an image measure based on \( D(r) \), the measure is defined by \( \mu(B) = \frac{1}{2} \int_{-1}^1 \chi_B(D(r)) \, dr \), where \( B \subset \mathbb{R} \) is any measurable set and \( \chi_B \) is the indicator function of the set \( B \). Noting that \( \mu(B) \geq 0 \), as the indicator function is non-negative and \( \mu(\mathbb{R}) = \frac{1}{2} \int_{-1}^1 \chi_B(D(r)) \, dr = \frac{1}{2} \int_{-1}^1 dr = 1 \). We can now see that \( \mu \) is a probability measure. The most commonly studied case is when \( d_0 = \chi_{[-1,0]} - \chi_{[0,1]} \),
and this is the case considered in this Chapter as seen in (3.11).

We study the stationary solutions via the ansatz \( v(t, x) = e^{i\omega t} f(x) \) in (3.10) for \( \omega > 0 \). This yields the time-independent eigenvalue equation

\[
-\omega f = -d_{av} f_{xx} - Q(f, f, f),
\]  

which describes the so-called dispersion managed solitons. This follows by equating the two lines below and cancelling out the common factor of \( \epsilon e^{i\omega t} \),

\[
i\partial_t e^{i\omega t} f(x) = -\epsilon \omega e^{i\omega t} f(x),
-\epsilon d_{av} e^{i\omega t} f(x) - \epsilon e^{i\omega t} \int_0^1 T_r^{-1} [\lvert T_r e^{i\omega t} f(x)\rvert^2 T_r f(x)] \, dr = \epsilon e^{i\omega t} [d_{av} f(x) - \int_0^1 T_r^{-1} [\lvert T_r f\rvert^2 T_r f] \, dr].
\]

As mentioned in the introduction, the case of \( d_{av} = 0 \) was handled by Erdoğan, Hundertmark and Lee in [17]. In this chapter we show how to extend to the case of \( d_{av} > 0 \), specifically proving the following.

**Theorem 3.1.** Let \( f \in H^1(\mathbb{R}) \) be a weak solution of (3.12) with \( d_{av} > 0 \). There there exists a \( \mu > 0 \) such that

\[
\lvert f(x) \rvert \lesssim e^{-\mu \langle x \rangle}, \quad \lvert \hat{f}(\xi) \rvert \lesssim e^{-\mu \langle \xi \rangle},
\]

where \( \hat{f} \) is the Fourier transform of \( f \).

As in [17], we have the following immediate corollary.

**Corollary 3.2.** Let \( f \in H^1(\mathbb{R}) \) be a weak solution of (3.12) with \( d_{av} > 0 \). Then both \( f \) and \( \hat{f} \) are analytic in a strip containing the real line.

This corollary follows from the exponential decay allowing for analytic continuation in a strip in the complex plane. These results will follow from the estimates on the multi-linear operators set forth in [17] and some adaptations of their arguments to the positive dispersion case.
3.3 Positive dispersion

We seek weak solutions $f \in H^1(\mathbb{R})$ to the dispersion management soliton equation, (3.12). That is, for all $g \in H^1(\mathbb{R}),$

$$\omega(g, f) = -d_{av}\langle g', f' \rangle + \langle g, Q(f, f) \rangle. \quad (3.13)$$

Here $(f, g)$ is the standard $L^2$ pairing defined by $\langle g, f \rangle = \int_{\mathbb{R}} g(x)f(x) \, dx$. Further, we define the quadra-linear operator $Q$ by

$$Q(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}} T_t f_1(x)T_t f_2(x)T_t f_3(x)T_t f_4(x) \, dx \, dt. \quad (3.14)$$

**Lemma 3.3.** The function $Q$ as defined in (3.14) is well-defined on $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Further, for $f_j \in L^2(\mathbb{R})$ and $j = 1, 2, 3, 4,$

$$|Q(f_1, f_2, f_3, f_4)| \lesssim \prod_{j=1}^4 \|f_j\|_2.$$

**Proof.** By Hölder’s inequality, we have

$$\|h_1 h_2 h_3 h_4\|_1 \leq \prod_{j=1}^4 \|h_j\|_4.$$

Take $h_i = T_t f_i$ or its complex conjugate as in (3.14). Now we use Hölder with respect to the product measure $dx \, dt$ to see

$$|Q(f_1, f_2, f_3, f_4)| \leq \prod_{j=1}^4 \left( \int_0^1 \int_{\mathbb{R}} |T_t f_j(x)|^4 \, dx \, dt \right)^{\frac{1}{4}} \quad (3.15)$$

We now note that

$$\int_0^1 \int_{\mathbb{R}} |T_t f(x)|^{3+1} \, dx \, dt \leq \left( \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^6 \, dx \, dt \right)^{\frac{1}{3}} \left( \int_0^1 \int_{\mathbb{R}} |T_t f(x)|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq \|f\|_2 \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |T_t f(x)|^6 \, dx \, dt \right)^{\frac{1}{2}}. \quad (3.16)$$

Where we used Cauchy-Schwarz and the uniticy of $T_t$ on $L^2(\mathbb{R})$. We now use the one-dimensional Strichartz
inequality with the admissible pair \((q,r) = (6,6)\). So that
\[
\left( \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |T_{r}f(x)|^{6} \, dr \, dx \right)^{\frac{1}{6}} \right)^{3} \lesssim \|f\|_{2}^{3}.
\] (3.17)

We combine (3.15), (3.16), and (3.17) to see that
\[
|Q(f_{1}, f_{2}, f_{3}, f_{4})| \lesssim \prod_{j=1}^{4} \|f_{j}\|_{2}^{4} \lesssim 1.
\]

We state the well-known, immediate corollary on the constrained minimization problem (3.4).

**Corollary 3.4.** If \(d_{av} \geq 0\), then for the constrained minimization problem in (3.4), \(P^{d_{av}} > -\infty\).

To obtain the decay of dispersion managed solitons, we use the ‘twisted’ dispersion management functionals
\[
Q_{\mu,\epsilon}(h_{1}, h_{2}, h_{3}, h_{4}) := Q(e^{F_{\mu,\epsilon}(X)}h_{1}, e^{-F_{\mu,\epsilon}(X)}h_{2}, e^{-F_{\mu,\epsilon}(X)}h_{3}, e^{-F_{\mu,\epsilon}(X)}h_{4}),
\]
(3.18)
\[
\tilde{Q}_{\mu,\epsilon}(h_{1}, h_{2}, h_{3}, h_{4}) := Q(e^{\tilde{F}_{\mu,\epsilon}(P)}h_{1}, e^{-\tilde{F}_{\mu,\epsilon}(P)}h_{2}, e^{-\tilde{F}_{\mu,\epsilon}(P)}h_{3}, e^{-\tilde{F}_{\mu,\epsilon}(P)}h_{4}).
\]
(3.19)

Where \(X\) denotes multiplication by \(x\) and \(P = -i\partial_{x}\) is the one-dimensional momentum operator. Similar to [17], define
\[
F_{\mu,\epsilon}(x) = \frac{\mu\langle x \rangle}{1 + \epsilon\langle x \rangle},
\] (3.20)

for \(\mu, \epsilon > 0\). We use \((\cdot) = (1 + |\cdot|^{2})^{1/2}\) instead of \(|\cdot|\) in the weight to avoid the non-differentiability of the absolute value at \(x = 0\).

We state the necessary estimates from [17] here.

**Theorem 3.5.** There exists a constant \(C\) such that the bounds
\[
Q_{\mu,\epsilon}(h_{1}, h_{2}, h_{3}, h_{4}) \leq C \prod_{j=1}^{4} \|h_{j}\|_{2}
\]
\[
\tilde{Q}_{\mu,\epsilon}(h_{1}, h_{2}, h_{3}, h_{4}) \leq C \prod_{j=1}^{4} \|h_{j}\|_{2}
\]

**Proof.** This is Theorem 2.2 in [17].
Theorem 3.6. There exists a constant $C$ such that if $\tau = \text{dist} (\text{supp}(h_\ell), \text{supp}(h_k)) \geq 1$ for some $\ell, k \in \{1, 2, 3, 4\}$, then

$$Q_{\mu, \epsilon}(h_1, h_2, h_3, h_4) \leq \frac{C}{\sqrt{\tau}} \prod_{j=1}^{4} \|h_j\|_2$$

for all $\mu, \epsilon \geq 0$. Moreover, if $\tau = \text{dist}(\text{supp}(\hat{h}_\ell), \text{supp}(\hat{h}_k)) \geq 1$, then also

$$\tilde{Q}_{\mu, \epsilon}(h_1, h_2, h_3, h_4) \leq \frac{C}{\sqrt{\tau}} \prod_{j=1}^{4} \|h_j\|_2$$

Proof. This is Theorem 2.3 in [17].

If we consider the Fourier transform of (3.12), then for all $g \in H^1(\mathbb{R})$ we have

$$\omega \langle \hat{g}, \hat{f} \rangle = -d_{av} \langle k\hat{g}, k\hat{f} \rangle + Q(\hat{g}, \hat{f}, \hat{f}, \hat{f}) + \tilde{Q}(e^{F_{\mu, \epsilon}} \hat{f}, \hat{f}, \hat{f}, \hat{f}).$$  \hspace{1cm} (3.21)

Choosing $g$ so that $\hat{g}(k) = e^{F_{\mu, \epsilon}(k)} \hat{f}(k)$, that is $g(x) = e^{F_{\mu, \epsilon}(P)} g$, (3.21) becomes

$$\omega \|\hat{f}\|_{\mu, \epsilon}^2 = -d_{av} \|kf\|_{\mu, \epsilon}^2 + \tilde{Q}(e^{F_{\mu, \epsilon}} \hat{f}, \hat{f}, \hat{f}, \hat{f})$$

Now, we simply use the inequality

$$\omega \|\hat{f}\|_{\mu, \epsilon} \leq |\tilde{Q}(e^{F_{\mu, \epsilon}} \hat{f}, \hat{f}, \hat{f}, \hat{f})|$$

Now, it follows with the same proof as the $d_{av} = 0$ case in [17] that $|\hat{f}(k)| \lesssim e^{\mu |k|}$ for some $\mu > 0$. We present the argument for exponential decay for $f$ which is similar, but does not reduce to the vanishing average dispersion case.

Consider a smooth real-valued function $\xi : \mathbb{R} \to \mathbb{R}$. If we choose $g = \xi^2 f$ using density of compactly supported continuous functions in $H^1(\mathbb{R})$, $g \in L^2(\mathbb{R})$ for all bounded $\xi$. Then (3.13) becomes

$$\omega \|\xi f\|_2^2 = -d_{av} \langle (\xi^2 f)', f' \rangle + Q(\xi^2 f, f, f, f).$$  \hspace{1cm} (3.22)

Notice that the left-hand side of the equation is real valued. Thus, we need only consider the real part of
the right-hand side of the equation.

\[ \Re[(\xi f)\', f')] = \frac{1}{2}[(\xi^2 f)\', f'] + \langle f', (\xi^2 f)' \rangle. \]  \hspace{1cm} (3.23)

Now,

\[ \langle (\xi^2 f)' , f' \rangle = \langle (2\xi \xi' f + \xi^2 f') , f' \rangle = (2\xi \xi' f , f') + \langle \xi^2 f', f' \rangle = -\langle (2\xi')^2 f + 2\xi \xi'' f + 2\xi \xi' f', f \rangle + \langle \xi^2 f', f' \rangle = -\| \xi' f \|^2 - \langle \xi^2 f' , f \rangle - \| \xi f' \|^2 \]  \hspace{1cm} (3.24)

Combining (3.23) and (3.24) we have,

\[ \Re[(\xi f)\', f')] = \frac{1}{2}[(\xi^2 f)\', f'] + \langle f', (\xi^2 f)' \rangle = \frac{1}{2} \left[ \| \xi f' \|^2 - \| \xi' f \|^2 - \langle 2\xi \xi'' f , f \rangle \right] \]  \hspace{1cm} (3.25)

Substituting (3.25) into (3.22), we have

\[ \omega \| \xi f \|^2 = -\frac{d_{\text{av}}}{2} \left[ \| \xi f' \|^2 - \| \xi' f \|^2 - \langle 2\xi \xi'' f , f \rangle \right] + Q(\xi^2 f , f , f , f) \]  \hspace{1cm} (3.26)

This is the inequality which we study as we have only terms involving \( f \), with no derivatives of \( f \).

We cannot use just the exponential weight as in the zero average dispersion case. Instead consider a function \( \psi \in C^\infty(\mathbb{R}) \) so that for some number \( R > 1 \), which we will choose later, that satisfies the following
i. \( \psi(x) = 0 \) on \( |x| < R \),

ii. \( \psi(x) = 1 \) on \( |x| > 2R \),

iii. \( 0 \leq \psi(x) \leq 1 \),

iv. \( |\psi'(x)| \lesssim R^{-1} \), and

v. \( |\psi''(x)| \lesssim 1 \).

Noting that for any \( k \geq 1 \), \( \text{supp}(\psi^{(k)}(x)) = \{ R < |x| < 2R \} \), we will choose our real-valued weight \( \xi = e^{F_{\mu,\epsilon}(x)} \psi(x) \).

Then, simple calculations show that

\[
|\xi'| \lesssim e^{F_{\mu,\epsilon}(x)} |\psi'| + \mu \psi \tag{3.27}
\]

\[
|\xi''| \lesssim e^{F_{\mu,\epsilon}(x)} [|\psi''| + (\mu + \mu \epsilon) \psi + \mu^2 \psi] \tag{3.28}
\]

We can now estimate the terms in (3.26).

\[
\|\xi' f\|_2^2 \lesssim \|e^{F_{\mu,\epsilon}(x)} |\psi'| + \mu \psi\| f\|_2^2 \tag{3.29}
\]

\[
\lesssim \frac{e^{8 \mu R}}{R^2} \|f_R\|_2^2 + \mu \frac{e^{4 \mu R}}{R} \|f_R\|_2 \|f_\mu\|_{\mu,\epsilon} + \mu^2 \|f_\mu\|_{\mu,\epsilon}^2
\]

where \( f_\mu = f \psi, f_R = f \chi_{\{ R < |x| < 2R \}} \) and \( \|f\|_{\mu,\epsilon} = \|e^{F_{\mu,\epsilon}} f\|_2 \). We also have

\[
|\langle \xi \xi', f \rangle| \lesssim \|e^{F_{\mu,\epsilon}} [|\psi'| + (\mu + \mu \epsilon) \psi + \mu^2 \psi] f, e^{F_{\mu,\epsilon}} f\|
\]

\[
\lesssim e^{4 \mu R} \|f_R\|_2 \|f_\mu\|_{\mu,\epsilon} + \mu (\epsilon + 1) \|f_\mu\|_{\mu,\epsilon}^2 + \mu^2 \|f_\mu\|_{\mu,\epsilon}^2 \tag{3.30}
\]

We choose \( \mu = \frac{\ln 2}{4 R^2} \) so that we have

\[
(3.29) \lesssim \frac{1}{R^2} \left( \|f_R\|_2^2 + \|f_R\|_2 \|f_\mu\|_{\mu,\epsilon} + \|f_\mu\|_{\mu,\epsilon}^2 \right) \tag{3.31}
\]

\[
(3.30) \lesssim \|f_R\|_2 \|f_\mu\|_{\mu,\epsilon} + \left( \frac{\epsilon + 1}{R} + \frac{1}{R^2} \right) \|f_\mu\|_{\mu,\epsilon}^2 \tag{3.32}
\]
Similar to [17], we define

\[ f_\psi = f \psi f \]
\[ f_\sim = f \sim f_\psi \]
\[ f_\sim = f \chi_{\{x|<R/3\}} \]
and note that \( f = f_\psi + f_\sim = f_\psi + f_\sim + f_\sim \). Further we let \( h = e^{F_\mu} f \).

Using the estimates of [17], specifically their equation (3.3), our estimates in (3.31) and (3.32) and following their technique of proof, we have

\[
\omega \|h_\psi\|^2 \lesssim d_{av} \left[ \frac{1}{R^2} \left( \|f_R\|^2 + \|f_R\|\|h_\psi\| + (\epsilon + 1)\|h_\psi\|^2 \right) + \left( \frac{1}{R} + \|f_R\| \right) \|h_\psi\| \right] 
+ \|h_\psi\|^4 + \|h_\psi\|^3\|h_\sim\| + \frac{1}{\sqrt{R}} \|h_\psi\|^2\|h_\sim\|^2 \|h_\sim\| + \|h_\psi\|^2\|h_\sim\|^2\|f_\sim\|
+ \frac{1}{\sqrt{R}} \|h_\psi\|\|h_\sim\|^2 \|h_\sim\| + \|h_\psi\|\|h_\sim\|^2\|f_\sim\|. \tag{3.33}
\]

Using that \( h_\sim, h_\sim \leq e^{8R}f \), and without loss of generality taking \( \|f\| = 1 \), and denoting \( \|h_\psi\| = x \),

\[
\omega x^2 \lesssim d_{av} \left[ \frac{1}{R^2} \|f_R\|^2 \right] + \left( \|f_\sim\| + \frac{1}{\sqrt{R}} + \frac{d_{av}}{R^2} \|f_R\| \right) x
+ \left( \frac{d_{av}(\epsilon + 1)}{R^2} + \frac{1}{\sqrt{R}} + \|f_\sim\| \right) x^2 + x^3 + x^4. \tag{3.34}
\]

This differs slightly from the polynomial in [17], as we now have a quartic instead of a cubic polynomial inequality. However, we notice that we can make the linear term arbitrarily small.

Now, if we restrict to \( \epsilon \leq 1 \), (3.34) is equivalent to

\[
\left( \omega - C \left[ \frac{d_{av}}{R^2} + \frac{1}{\sqrt{R}} + \|f_\sim\| \right] \right) x^2 - C \left( \|f_\sim\| + \frac{1}{\sqrt{R}} + \frac{d_{av}}{R^2} \|f_R\| \right) x
- C x^3 - C x^4 \leq C \left( \frac{d_{av}}{R^2} \|f_R\|^2 \right). \tag{3.35}
\]

Consider the class of functions \( G_\delta(x) = \frac{x}{2} x^2 - \delta x - C x^3 - C x^4 \). It is clear that \( G_\delta(x) \) has a positive maxima at some point \( x_m > 0 \). Consider the derivatives, \( G'_\delta(x) = \omega x - \delta - 3C x^2 - 4C x^3 \) and \( G''_\delta(x) = x - 6C x - 12C x^2 \). As these functions are all continuous, and the roots of polynomials depend continuously on the coefficients, see for example [78], there exists a \( \delta_0 > 0 \) so that \( G_\delta(x) \) has a positive local maximum \( x'_m > 0 \) whenever \( 0 < \delta \leq \delta_0 \).

Define \( \nu = \frac{1}{2} x'_m \), and pick \( R > 1 \) so that
\[ C \left( \| f \| + \frac{1}{\sqrt{R}} + d_{ae} \frac{1}{R^2} \left( \| f_R \| + \| f \| \right)^2 \right) \leq \min \left( \delta_0, \frac{\omega}{2}, G(\nu) \right), \]

ii. \( \| f \| \leq \nu/2 \).

With this choice, we rewrite (3.35) as

\[ G_{\delta_0}(\| f \|_{\mu, \epsilon}) \leq G_{\delta_0}(\nu). \]  

valid for any \( 0 < \epsilon \leq 1 \). This, and our choice of \( \mu \),

\[ \| f \|_{\mu, 1} \leq \| e^{\epsilon \frac{(x)}{1+\epsilon}} \|_\infty \| f \| \leq e^\mu \nu/2 \leq \nu. \]  

This implies exponential decay in the \( L^2 \) sense, \( e^{\mu |x|} f \in L^2(\mathbb{R}) \). We already have \( f \in H^p(\mathbb{R}) \) for \( p > 0 \) from
the $L^2$ exponential decay of the Fourier transform.

\[
e^{\mu |x|} |f(x)|^2 = e^{\mu |x|} \left| \int_x^\infty \frac{d}{dx} |f(s)|^2 \, ds \right|
\leq 2 \int_x^\infty e^{\mu |s|} |f(s)| |f'(s)| \, dx \leq 2 \|e^{\mu |\cdot|} f\|_2 \|f'\|_2 < \infty.
\]

This and the observations after (3.21) prove Theorem 3.1.
References


[50] Laugesen, R. *Harmonic analysis lecture notes*.


