J. Fluid Mech. (2006), *vol.* 553, *pp.* 389–400.
doi:10.1017/S0022112006009037 Printed in the United Kingdom

Nonlinear oscillatory convection in rotating mushy layers

By D. N. RIAHI

Department of Theoretical and Applied Mechanics, 216 Talbot Laboratory, 104 South Wright Street, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA

(Received 2 May 2005 and in revised form 13 October 2005)

We consider the problem of nonlinear oscillatory convection in a horizontal mushy layer rotating about a vertical axis. Under a near-eutectic approximation and the limit of large far-field temperature, we determine the stable and unstable oscillatory solutions of the weakly nonlinear problem by using perturbation and stability analyses. It was found that depending on the values of the parameters, supercritical simple travelling modes of convection in the form of hexagons, squares, rectangles or rolls can become stable and preferred, provided the value of the rotation parameter τ is not too small and is below some value, which can depend on the other parameter values. Each supercritical form of the oscillatory convection becomes subcritical as τ increases beyond some value, and each subcritical form of the oscillatory convection is unstable. In contrast to the non-rotating case, qualitative properties of the left-travelling modes of convection are different from those of the right-travelling modes, and such qualitative difference is found to be due to the interactions between the local solid fraction and the Coriolis term in the momentum-Darcy equation.

1. Introduction

Riahi (2003) extended the steady problem of convection in rotating mushy layers treated by Guba (2001) by following Anderson & Worster (1995) in assuming a much wider range $\varepsilon \ll \delta$ for the amplitude of convection, taking into account the interactions between the local solid fraction and the convection associated with the Coriolis term in the momentum-Darcy equation and carrying out stability analysis of the finite-amplitude steady solutions. It was found, in particular that over most of the range of the parameter values, subcritical down-hexagons with down-flow at the cell centres and up-flow at the cell boundaries can be preferred over up-hexagons, where flow is upward at the cell centres and downward at the cell boundaries.

Riahi (2002) extended the linear oscillatory problem of convection in mushy layers and in the absence of rotation due to Anderson & Worster (1996) by employing weakly nonlinear and stability analyses to determine the stable finite-amplitude oscillatory solutions. He found, in particular, that depending on the values of the parameters, only supercritical simple travelling modes of convection in the form of either right-travelling rolls (where the phase velocity of the rolls is in the direction of the component of the position vector along the wavenumber vector) or left-travelling rolls, (where the phase velocity of the rolls is in the direction opposite to that of the component of the position vector along the wavenumber vector) or supercritical standing rolls can be stable. The weakly nonlinear and stability properties of the right-travelling mode were found to be the same as those of the left-travelling mode. The author is grateful to the editor and two referees for pointing out an error due to the inadequate number of decimals being taken for π in the computation, which existed in Riahi (2002) as well as in an earlier draft version of this paper, which was then eliminated in the later version of the present paper. The main qualitative results stated in Riahi (2002) were not affected by that error.

The present study, which can also be considered as a rotating extension of the work due to Riahi (2002), leads to significant results that were found to be sharply different from the corresponding ones in Riahi (2002). In particular, we found that over a significant range of the parameter values, three-dimensional simple travelling modes of convection in the form of hexagons, squares and rectangles can be preferred over the oscillatory rolls, provided the rotation rate is not too small, and the qualitative features of the right-travelling modes differ, in general, from those of the left-travelling modes for the same types of flow pattern.

The only other studies on weakly nonlinear oscillatory convection in a rotating mushy layer known to the author are those by Guba & Boda (1998) and Govender & Vadasz (2002). Guba & Boda (1998) studied the effect of uniform rotation on linear problem for convection in a mushy layer, but their investigation did not take into account the interaction between the local solid fraction and the flow associated with the Coriolis term. These authors determined the critical value of the Rayleigh number at the onset of convection and the frequency of the oscillatory mode as functions of each of the parameters under the condition of given values of all the other parameters of the problem. Their main result was that depending on the values of the parameters, the oscillatory mode can be more critical than the stationary mode or vice versa. Govender & Vadasz (2002) considered the problem of two-dimensional oscillatory convection in a rotating mushy layer and employed a near-eutectic approximation and the limit of large far-field temperature. The momentum-Darcy equation was extended only to include the time derivative and the Coriolis terms. The authors did not take into account the presence of the interactions between the local solid fraction and the flow associated with the Coriolis term, and their weakly nonlinear analysis was based on the zero-order limit of the mushy-layer thickness. The main result of the study was that two-dimensional oscillatory flow was supercritical.

It should be noted that the linear part of the present investigation can be considered an extension of the linear model treated by Guba & Boda (1998) in the sense that we fully took into account the interaction between the local solid fraction and the Coriolis term.

With regard to the motivation of the present study of the effect of rotation in the context of oscillatory convection in mushy layers, it should be noted that it has been of interest to investigate whether an external constraint of rotation can enhance or otherwise modify the oscillatory mode of convection discovered by Anderson & Worster (1996). As these authors noted, an oscillatory mode may be the most critical one in some cases, and our investigation of the effect of rotation on nonlinear oscillatory convection in mushy layers was aimed at further understanding how rotation can affect the oscillatory flow features in the mushy layers and, in particular, reduce the flow tendency for chimney formation, which is important for production of higher-quality crystals in industrial processes.

2. Formulation

We consider a binary alloy melt that is cooled from below and is solidified at a constant speed V_0. Following Amberg & Homsy (1993) and Anderson & Worster (1995), we consider a mushy layer of thickness d adjacent and above the solidification front to be physically isolated from the overlying liquid and the underlying solid zones. The overlying liquid is assumed to have a composition $C_0 > C_e$ and temperature $T_\infty > T_L(C_0)$ far above the mushy layer, where C_e is the eutectic composition, $T_L(\tilde{C})$ is the liquidus temperature of the alloy and $\tilde{C}$ is the composition. It is then assumed that the horizontal mushy layer, which is treated as a porous layer obeying Darcy's law, is bounded from above and below by rigid and isothermal boundaries. The solidifying system is assumed to be rotating at a constant speed Ω in the vertical direction anti-parallel to the gravity vector. We consider the solidifying system in a moving frame of reference $o\tilde{x}\tilde{y}\tilde{z}$, whose origin lies on the solidification front, translating at the speed V_0 with the solidification front in the positive $\tilde{z}$-direction and rotating with the speed Ω along the $\tilde{z}$-axis.

Next, we consider the non-dimensional form of the equations for momentum-Darcy, continuity, heat and solute for the flow of melt in the mushy layer in the already described moving frame. These equations, as well as the corresponding boundary conditions, are given below:

$$K(\tilde{\phi})\tilde{u} = -\nabla\tilde{P} - \tilde{R}\tilde{\theta}\boldsymbol{z} + \tau\tilde{\boldsymbol{u}} \times \boldsymbol{z}/(1-\tilde{\phi}), \tag{1a}$$

$$\nabla\cdot\tilde{\boldsymbol{u}} = 0, \tag{1b}$$

$$(\partial/\tilde{\partial}t - \partial/\partial\tilde{z})(\tilde{\theta} - S_t\tilde{\phi}) + \tilde{\boldsymbol{u}}\cdot\nabla\tilde{\theta} = \nabla^2\tilde{\theta}, \tag{1c}$$

$$(\partial/\partial\tilde{t} - \partial/\partial\tilde{z})[(1-\tilde{\phi})\tilde{\theta} + C_r\tilde{\phi}] + \tilde{\boldsymbol{u}}\cdot\nabla\tilde{\theta} = 0, \tag{1d}$$

$$\tilde{\theta} + 1 = \tilde{w} = 0 \quad \text{at } \tilde{z} = 0, \tag{1e}$$

$$\tilde{\theta} = \tilde{w} = \tilde{\phi} = 0 \quad \text{at } \tilde{z} = \delta, \tag{1f}$$

where $\tilde{\boldsymbol{u}} = \tilde{u}\boldsymbol{x} + \tilde{v}\boldsymbol{y} + \tilde{w}\boldsymbol{z}$ is the volume flux vector per unit area, $\tilde{u}$ and $\tilde{v}$ are the horizontal components of $\tilde{\boldsymbol{u}}$ along the horizontal $\tilde{x}$- and $\tilde{y}$-directions, respectively, $\boldsymbol{x}$ and $\boldsymbol{y}$ are unit vectors along the positive $\tilde{x}$- and $\tilde{y}$-directions, $\tilde{w}$ is the vertical component of $\tilde{\boldsymbol{u}}$ along the $\tilde{z}$-direction, $\boldsymbol{z}$ is a unit vector along the positive $\tilde{z}$-direction, $\tilde{P}$ is the modified pressure, $\tilde{\theta} = [\tilde{T} - T_L(C_0)]/\Delta T = (\tilde{C} - C_0)/\Delta C$ is the non-dimensional temperature, $\tilde{t}$ is the time variable, $\tilde{\phi}$ is the local solid fraction, $R = \beta\Delta C g\prod(0)/(V_0\nu)$ is the Rayleigh number, $\prod(0)$ is the reference value at $\tilde{\phi} = 0$ of the permeability $\prod(\tilde{\phi})$ of the porous medium, ν is the kinematic viscosity, g is acceleration due to gravity, $K(\tilde{\phi}) \equiv \prod(0)/\prod(\tilde{\phi})$, $S_t = L/(C_L\Delta T)$ is the Stefan number, C_L is the specific heat per unit volume, L is the latent heat of solidification per unit volume, $C_r = (C_s - C_0)/\Delta C$ is a concentration ratio, C_s is the composition of the solid-phase forming the dendrites, $\tau = 2\Omega\prod(0)/\nu$ is the Coriolis parameter, which is the square root of a Taylor number and $\delta = dV_0/k$ is a growth Péclet number representing the dimensionless depth of the mushy layer. It should be noted that the interaction between the local solid fraction and the Coriolis term referred to in the previous section, is the deviation of the last term in the right-hand side of equation (1a) from $\tau\boldsymbol{u}\times\boldsymbol{z}$. Equation (1$d$) is based on the limit of sufficiently large value of the Lewis number k/k_s, where k_s is the solute diffusivity.

Following earlier work (Amberg & Homsy 1993; Anderson & Worster 1995), we assume the following rescaling in the limit of sufficiently small δ:

$$C_r = C/\delta,\ S_t = S/\delta,\ \varepsilon \ll \delta \ll 1,\ (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = (x, y, z, t\delta)\delta, \quad R^2 = \delta\tilde{R}, \tag{2a}$$

$$(\tilde{\theta}, \tilde{\phi}, \tilde{\boldsymbol{u}}, \tilde{P}) = [\theta_B(z) + \varepsilon\theta, \phi_B(z) + \varepsilon\phi, 0 + (\varepsilon R/\delta)\boldsymbol{u}, \quad RP_B(z) + R\varepsilon P], \tag{2b}$$

where C and S are order-one quantities as $\delta \to 0$, and the quantities with subscript 'B' are those for the basic motionless state. The small deviation of each dependent variable from its basic quantity is measured by perturbation amplitude ε and can vary, in general, with respect to x, y, z and t variables. The assumption of $S_t \rightarrow \infty$ in the limit $\delta \rightarrow 0$ used here is relevant in the present study, and, in fact, it allowed Anderson & Worster (1996) to detect an oscillatory instability from their linear model.

The rescaling (2*a*)–(2*b*) is then used in the governing system (1*a*)–(1*f*). This system admits a motionless basic state, which is steady and horizontally uniform. The basic state solution is already known (Anderson & Worster 1995) and will not be repeated here. However, as was explained in Riahi (2002), a new parameter $G \equiv 1 + S/C$ is found to be suitable for use in the basic state solution instead of S. Since ϕ_B is found to be small and of order δ, then $\tilde{\phi}$ is expected to be small and the following expansion for $K(\tilde{\phi})$ is implemented in the governing system:

$$K(\tilde{\phi}) = 1 + K_1\tilde{\phi} + K_2\tilde{\phi}^2 + \cdots, \tag{3}$$

where K_1 and K_2 are constants.

For the analysis to be presented in the next section, it was found convenient to use the representation $\boldsymbol{u} = \nabla \times \nabla \times zV + \nabla \times z\psi$ for the divergent-free vector field $\boldsymbol{u}$, where V and ψ are the poloidal and toroidal functions for $\boldsymbol{u}$, respectively (Chandrasekhar 1961). Taking the vertical components of the curl and double curl of the Darcy-momentum equation and using the continuity equation, we find the system for V, ψ, θ and ϕ, which is of the form given in Riahi (2003) and, thus, will not be repeated here.

3. Analysis

Here we seek oscillatory solutions of the already described nonlinear system by applying a weakly nonlinear analysis, based on a double-series expansions in powers of two small parameters δ and ε for the dependent variables, R and the frequency ω of the oscillatory modes of the type used in Riahi (2002). In the following analyses, the coefficient of $f \equiv (V, \psi, \theta, \phi, R, \omega)$ in the order $\varepsilon^m \delta^n$ of such double-series expansions is designated by f_{mn}.

3.1. *Linear problem*

Considering the system to the lowest order in ε, we find the linear problem. At order ε^0/δ, the system yields $\omega_{00} = 0$. At order $\varepsilon^0\delta^0$, the same system yields the following results:

$$V_{00} = [(\pi^2 + a^2)/(GR_{00}a^2)]\sin(\pi z)\sum_{n=-N}^{N} U_n, \quad U_n \equiv (A_n^+W_n^+ + A_n^-W_n^-), \tag{4a}$$

$$\psi_{00} = \pi\tau[(\pi^2 + a^2)/(GR_{00}a^2)]\cos(\pi z)\sum_{n=-N}^{N} U_n, \tag{4b}$$

$$\theta_{00} = -\sin(\pi z)\sum_{n=-N}^{N} U_n, \tag{4c}$$

$$\left.\begin{aligned} &\phi_{00} = \left\{-(\pi^2 + a^2)\pi/\left[GC\left(\pi^2 - \omega_{01}^2\right)\right]\right\}\sum_{n=-N}^{N}[f_n(z)U_n^+ + f_n^*(z)U_n^-],\\ &f_n(z) \equiv \{(i\omega_{01}S_n/\pi)\sin(\pi z) + \cos(\pi z) + \exp[i\omega_{01}S_n(z-1)]\},\\ &U_n^{\pm} \equiv A_n^{\pm}W_n^{\pm},\ W_n^{\pm} \equiv \exp[i(\boldsymbol{a}_n \cdot \boldsymbol{r} \pm S_n\omega t)],\end{aligned}\right\} \tag{4d}$$

$$R_{00}^2 = (\pi^2 + a^2)(\pi^2 + a^2 + \pi^2\tau^2)/(a^2G), \tag{4e}$$

where

$$S_n \equiv 1 \quad \text{for } n>0 \quad \text{and } -1 \quad \text{for } n<0. \tag{4f}$$

Here i is the pure imaginary number ($\mathrm{i}=\sqrt{-1}$), subscript 'n' takes only non-zero integer values from $-N$ to N, N is a positive integer representing the number of distinct modes, $\boldsymbol{r}$ is the position vector, and the horizontal wavenumber vectors $\boldsymbol{a}_n$ satisfy the properties

$$\boldsymbol{a}_n\cdot \boldsymbol{z}=0, \quad |\boldsymbol{a}_n|=a, \quad \boldsymbol{a}_{-n}=-\boldsymbol{a}_n. \tag{5}$$

The coefficients A_n^+ and A_n^- are constants and satisfy the conditions

$$\sum_{n=-N}^{N}(A_n^+A_n^{+*}+A_n^-A_n^{-*})=2, \quad A_n^{\pm *}=A_{-n}^{\pm}, \tag{6}$$

where the asterisk indicates the complex conjugate. For left- (right-)travelling-wave solutions, $A_n^-=0(A_n^+=0)$, while for standing-wave solutions $A_n^-=A_n^+$ (Riahi 2002). It should be noted that the result (4a)–(4e) in the limit of a single mode agree with the corresponding ones given in Guba & Boda (1998). Minimizing the expression for R_{00} given in (4e), with respect to the wavenumber a, we find

$$R_{00c}=\pi[1+(1+\tau^2)^{0.5}]/\sqrt{G}, \quad a_c=\pi(1+\tau^2)^{0.25}. \tag{7}$$

Here, R_{00c} is the minimum value of R_{00} achieved at $a=a_c$.

As the results to be discussed in the next section indicate, the magnitude of some of the presumed order-one coefficients R_{nm} can become too large for large values of τ, and so the validity of the present model may become questionable for large values of τ. Hence, we shall assume that the value of τ can be at most of order unity. Also, due to the complexity of the present rotating and oscillatory flow investigation, we consider a simplifying assumption by following Riahi (2002) and focusing on a limiting case where K_1 is small, of order ε, so that we can write $K_1=\varepsilon K_c$, where K_c is a constant of order one.

Considering the governing system in the order $\varepsilon^0\delta^1$, eliminating ψ_{01} and ϕ_{01} and applying the existence condition of the type carried out in Riahi (2002), we find that the real and imaginary parts of this condition yield

$$\begin{aligned}(R_{01c}/R_{00c})=G_tG\left[1/4+\pi^2(1+\cos\omega_{01})/\left(\pi^2-\omega_{01}^2\right)^2\right]\\ +\pi^2\tau^2/\left[2C\left(\pi^2+a_c^2+\pi^2\tau^2\right)\right],\end{aligned} \tag{8a}$$

$$\left.\begin{aligned}\omega_{01}\left\{1+G_t\left[\left(\pi^2+a_c^2\right)/\left(\pi^2-\omega_{01}^2\right)\right]\left[1-2\pi^2\sin\omega_{01}/\left(\omega_{01}\pi^2-\omega_{01}^3\right)\right]\right\}=0,\\ G_t\equiv(G-1)/(CG^2).\end{aligned}\right\} \tag{8b}$$

It should be noted from (8b) that ω_{01} depends on the rotation parameter τ only through a_c, and (8b) is equivalent to the non-rotating case of Anderson & Worster (1996), where $a_c=\pi$.

Hence, the critical Rayleigh number R_c for the linear system can be written as

$$R_c=R_{00c}+\delta R_{01c}+O(\delta^2). \tag{9}$$

3.2. *Nonlinear and stability problem*

Next, we analyse the nonlinear problem for the oscillatory convection. At order ε/δ, we find $\omega_{10}=0$. Following Riahi (2002) to determine the solvability conditions for the system in order ε, we find $R_{10}=0$, which is expected for the oscillatory modes as explained in Riahi (2002). By the same reasoning, the solvability conditions for the system at order $\varepsilon\delta^m$ also yield $R_{1m}=\omega_{1m}=0\,(m>0)$. The solutions to the order ε-system are then found. These solutions are functions of independent variables and non-dimensional parameters as well as on $\Phi_{lp}\equiv \boldsymbol{a}_l\cdot\boldsymbol{a}_p/a^2$ and $\Psi_{lp}\equiv\boldsymbol{a}_l\times\boldsymbol{a}_p/a^2$ $(l,p=-N,\ldots,-1,1,\ldots,N)$.

We now consider the system in order ε^2. The solvability conditions for the system in the order ε^2 yield the expression for R_{20}, which together with (6), were used to study the oscillatory solutions. Similar to the earlier studies (Riahi 2002, 2003), we restrict our analyses to the regular or semi-regular solutions where (6) yield, for each n

$$\left.\begin{array}{ll} |A_n^+|^2=|A_n^-|^2=1/(2N), & |A_n^+|^2=1/N(|A_n^-|=0),\\ |A_n^-|^2=1/N(|A_n^+|=0), & (|A_n^+|^2,|A_n^-|^2)=[(0.5-b),0.5+b)]/N, \end{array}\right\} \qquad (10)$$

for standing, left-travelling, right-travelling and general travelling waves, respectively, where a given value of the parameter b in its appropriate range $|b|<0.5$ (Riahi 2002) provides particular general travelling waves. In later references on a semi-regular solution in the form of rectangles, an angle γ, which is less than 90°, is defined to be the angle between two adjacent wavenumber vectors of any rectangular cell.

To distinguish the physically realizable solution among all the possible oscillatory solutions, the stability of the finite-amplitude solution is investigated by superposing on the solutions perturbations of infinitesimal amplitude and with addition of a time dependence of the form $\exp(\sigma t)$, where σ is the growth rate. Following Riahi (2002, 2003), we obtain the stability system for the disturbances, which was solved by an expansion similar to that for the finite-amplitude solutions.

The present stability analysis is of the same type as given in Riahi (2003) and, thus, will not be repeated here. Instead, we briefly state the key aspects and the results of the analysis. The growth rate to the lowest order in ε of the most critical disturbances is zero. Similar to the result for R_{10} presented in the previous subsection, we found that the solvability conditions for the disturbance system in the order ε yield $\sigma_{10}=0$. Application of this procedure to the disturbance systems at orders $\varepsilon\delta^m$ $(m>0)$ then implies that $\sigma_{1m}=0$. Next, the solvability conditions in the order ε^2 determine the expression for σ_{20}.

4. Results and discussion

In this section, the results are presented in terms of the physical parameters S, C, K_2 and τ, even though the analysis was presented more simply in terms of G, G_t, K_2 and τ.

4.1. *Linear problem*

The linear system and its eigenvalue problem, which led to the results (4)–(9), are, in general, functions of the parameters S, C and τ. Here and hereinafter value of $\delta=0.2$ is chosen to evaluate R_c and other quantities whose values may depend on δ. The well-known stabilizing effect of the Coriolis force on convection (Chandrasekhar 1961) can be seen from the expressions for R_c and a_c, which can be found from (7), (8*a*) and (9). However, a_c is unaffected with respect to the variations of S and C, while R_c depends strongly on the ratio S/C. The results for the frequency ω_{01} of the oscillatory motion as function of C and for several given values of τ are presented in figure 1. Just like the non-rotating case (Anderson & Worster 1996; Riahi 2002), ω_{01} was found to vary only with respect to C if the ratio S/C is kept

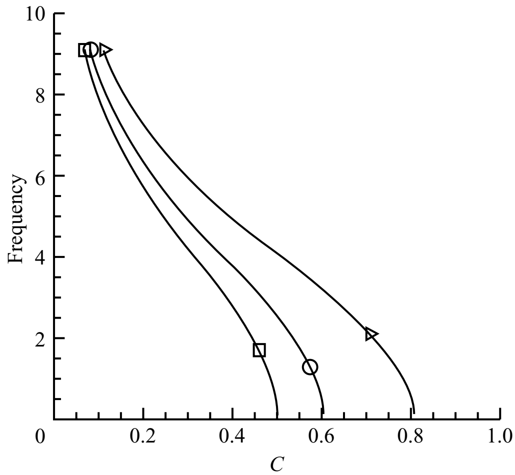

FIGURE 1. The frequency ω_{01} versus C. □, $\tau = 0$; ○, 1.0; △, 2.0.

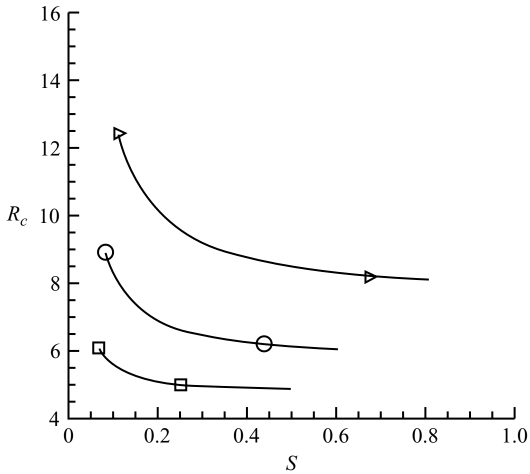

FIGURE 2. R_c against S for $C = S$. □, $\tau = 0$; ○, 1.0 △, 2.0.

fixed. It can be concluded from the results presented in this figure that the period of the oscillatory mode decreases with increasing τ and C. It should be noted that in most of the results of the analysis presented in this paper, G has the value of 2.0, which implies $S = C$. Using the expression for G_t given in (8*b*), we then find that $G_t = 0.25/C$. Hence, an increase in G_t corresponds to a decrease in C (or S) and vice versa. The results for R_c as a function of S and for several values of τ are presented in figure 2 for $C = S$. It can be seen clearly from (7) and (9) that, in general, S and C are strongly destabilizing and stabilizing, respectively, if these two later parameters are kept independent of one another. It can be concluded from the results presented in figure 2 that S is destabilizing for a given τ, while τ is stabilizing for a given S. The rate of destabilization with respect to S is higher at higher values of τ. Thus, in general, for the linear system, the effects of increasing τ or C are stabilizing, while the effect of increasing S is destabilizing. These results are understandable with respect to the physical interpretation of the parameters τ, S and C since the Coriolis effect is known to be stabilizing at the onset of motion for the linear convective systems (Chandrasekhar 1961), while according to the physical interpretation given in the non-rotating case (Riahi 2002), the scaled Stefan number S is destabilizing for a given C, and the scaled compositional ratio C is stabilizing for a given S. When $S = C$, then the destabilizing effect of S appears to dominate over the stabilizing effect of C, as the results shown in the figure 2 indicate.

Note that in the expression (8a) for R_{01c}, the second term on the right-hand side, which contains the factor (τ^2/C), is due to the interaction between the leading term in the basic state of the local solid fraction and the Coriolis term in the momentum-Darcy equation. This interaction term is associated with the basic state solid fraction, owing to the difference between the local fluid velocity and the local volume flux of inter-dendritic fluid, as the momentum equation is formulated in terms of Darcy, rather than the local, velocity. If this interaction term is not taken into account, then we find that the value of R_c is reduced for non-zero rotation cases. Hence, presence of such an interaction term is stabilizing in the present oscillatory flow problem as far as the linear system is concerned, while in the steady case (Riahi 2003), such an interaction term was found to be destabilizing for the linear system.

4.2. *Nonlinear and stability problem*

In this paper, an important quantity due to the nonlinear effects is the coefficient R_{20}. Since $R_{1n} = 0 (n = 0, 1, \ldots)$ in the present problem, the coefficient R_{20} represents leading contributions to the change in R required to obtain finite amplitude ε for a nonlinear solution. In terms of this coefficient, the amplitude of convection is of order

$$|\varepsilon| = [(R - R_c)/R_{20}]^{0.5}. \tag{11}$$

Here the sign of R_{20} determines whether the oscillatory solution exists for values of R above or below R_c. For supercritical convection, where $R > R_c$, the amplitude of convection is largest, provided the value of R_{20} is smallest among all the solutions to the nonlinear problem. Here, R_{20} is due to the linear and nonlinear interactions between the local solid fraction and the Coriolis term in the momentum-Darcy equation, the nonlinear convective terms in the temperature equation and the nonlinear interactions between the flow velocity and the non-uniform and nonlinear permeability associated with the perturbation to the basic state solid fraction.

We calculated the expression for R_{20} for convection with different plan forms such as rolls, rectangles, squares and hexagons. In all the cases that were studied, it was found that for given values of the parameters and for $\tau \neq 0$, the values of R_{20} for the right-travelling waves were generally different from the values of R_{20} for the left-travelling rolls. This result is in contrast to the corresponding result for the non-rotating case (Riahi 2002) where for any given parameter values, the value of R_{20} was found to be the same for the right-travelling and left-travelling waves. We detected that such a difference in the R_{20} values for the left-travelling and right-travelling waves in the rotating system was due to the interactions between the local solid fraction and the flow associated with the Coriolis force in the momentum-Darcy equation. In particular, since according to (4d) the z-dependence of the linear solution for the right-travelling mode of the local solid fraction is the complex conjugate of that for the left-travelling mode, the interactions of the solid fraction with the Coriolis term carries such a difference in the z-dependence behaviour of the left- and right-travelling modes to the order-ε solutions for the flow velocity and concentration and subsequently to the solvability condition in the order ε^2, which leads to different values of R_{20} for these modes. The present results for the oscillatory modes as well as

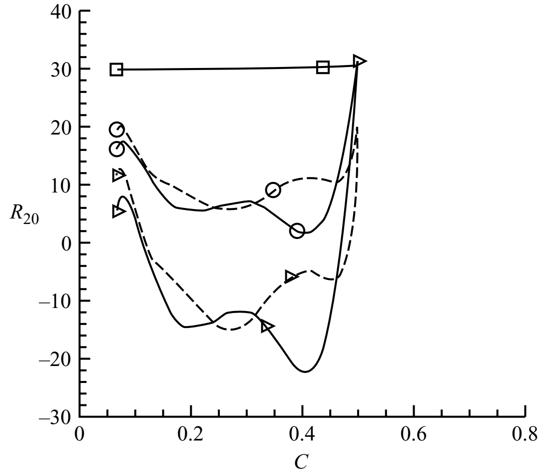

FIGURE 3. R_{20} against C for left- (solid lines) and right- (dash lines) travelling hexagons. Here, $S = C$ and $K_2 = 0$. □, $\tau = 0.001$; ○, 0.004; △, 0.005.

those for the stationary modes (Riahi 2003) indicate that in the rotating system, the porosity of the mushy layer affects the amplitude of the finite-amplitude solutions. In the present oscillatory regime, the amounts affected by the local porosity on the amplitude of the right-travelling mode differ from that for the left-travelling mode.

The coefficients R_{20} for oscillatory convection in the form of hexagons ($N = 3$), square cells ($N = 2$, $\gamma = 90°$), rectangles ($N = 2$, $\gamma < 90°$) and rolls ($N = 1$) were computed for various values of τ, S, C and K_2. The main results that are given below about the variation of R_{20} with respect to the parameters for each type of oscillatory mode, were found to be qualitatively the same, regardless of the particular form of the flow pattern chosen. It was found that, primarily depending on the value of τ and secondarily depending on the range of values of the other parameters, R_{20} can be positive or negative and, thus, both supercritical and subcritical oscillatory solutions can be possible. For sufficiently small values of τ, like $\tau < 0.001$, $R_{20} > 0$, S/C is destabilizing, while K_2 is stabilizing, which is consistent with the stabilizing effect of decreasing the permeability, and the value of R_{20} is almost the same for right-travelling and left-travelling modes. Variation of R_{20} with respect to C for $S = C$ is non-monotonic for $\tau \neq 0$. Such non-monotonic behaviour was found to increase in intensity with increasing rotational effect. For values of τ that are not too small, such as $\tau \geqslant 0.001$, effects of K_2, C and S remain qualitatively the same as in the case of $0 < \tau < 0.001$, but the value of R_{20} for the right-travelling mode is now generally different from the corresponding one for the left-travelling mode. For larger values of τ such as $\tau \geqslant 0.005$, K_2 and S/C are again stabilizing and destabilizing, respectively, and there is an intermediate range in C where $R_{20} < 0$, provided S/C or K_2 is not too large. Depending on the range of the parameter values, R_{20} for either left- or right-travelling mode of supercritical convection in the form of hexagons, squares, rectangles or rolls, can have the smallest value among all the solutions that have been determined.

To provide some graphical results for the qualitative nonlinear features shared by different forms of finite-amplitude solutions, we consider the cases of particular oscillatory hexagons. Some typical results about the variation of R_{20} with respect to C are presented in figure 3 for a left-travelling hexagons (solid lines) and the corresponding right-travelling hexagons (dash lines) for $\tau = 0.001$, 0.004 and 0.005.

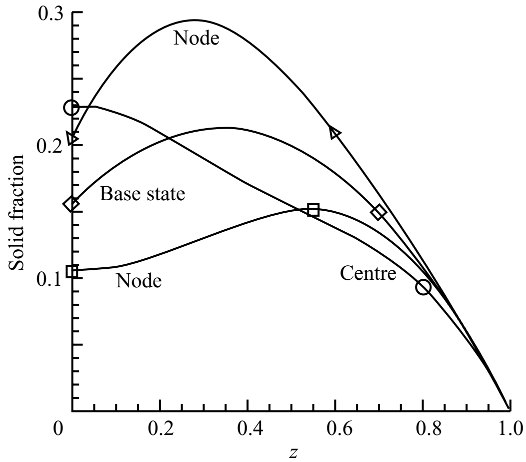

FIGURE 4. Solid fraction for right-travelling up-hexagons versus z for $C = 0.248$, $S = C$, $\tau = 0.07$ and $K_2 = 0$. ○, △, □ and ◇ show, respectively, $\tilde{\phi}(x = 0,\ y = 0,\ z, t = 0)$ at a cell's centre, $\tilde{\phi}(x = 1.33,\ y = 0,\ z, t = 0)$ at a cell's node initially, $\tilde{\phi}(x = 1.33,\ y = 0,\ z, t = 0.635)$ at the same cell's node, but at half of the period of oscillation and the basic solid fraction ϕ_B.

Here $S = C$ and $K_2 = 0$. It is seen from this figure that C is stabilizing for relatively larger values of C, and such stabilization is aided by the rotation. If τ is not too large, then the supercritical regime can be realized in the particular range of values for C. The supercritical regime can also be realized if C is sufficiently large. The effect of τ is, in general, destabilizing. The differences between the left- and right-travelling hexagons can be seen clearly from this figure. These differences can be significant only for significant values of τ if C is not large, which implies significant values of the rotation rate for small values of S. Our generated data indicated that K_2 is generally stabilizing, while S/C is generally destabilizing. If the interactions between the local solid fraction and the Coriolis term are not taken into account, then it was found that R_{20} can be positive over a wider range in C, and its value is generally affected by those interactions. However, K_2 is still stabilizing and S/C or τ is still destabilizing. Hence, those interactions appear to enhance the destabilization effect of S for given S/C.

We also examined the vertical distribution of solid fraction at different locations in the horizontal direction and in time for the oscillatory hexagons. Some typical results are presented in figures 4 and 5 for the vertical distribution of the basic state and total solid fraction at a centre and at a node of a right-travelling up-hexagon ($\varepsilon > 0$) and a right-travelling down-hexagon ($\varepsilon < 0$) for $\tau = 0.07$. In these calculations $\delta = 0.2$, $C = 0.248$, $S = C$, $K_2 = 0$, and the value $|\varepsilon| = 0.002$ is chosen, which is the maximum value of $|\varepsilon|$ beyond which the solid fraction becomes negative and subsequently physically unrealistic. We have chosen zero values for K_2 in these calculations since $\tilde{\phi}$ is found to be much less sensitive with respect to K_2 at such a small value of $|\varepsilon|$. Two cases of a node, corresponding to full and half periods of oscillation are shown in these figures, which provide significant differences between qualitative behaviour of the node at two instants in time. It can be seen from the figure 4 that for the up-hexagonal case and after completing any period of oscillation, there are greater tendencies for crystal formation at the node near the lower boundary and melt generation at the centre near the upper boundary; but just completing about half a period of oscillation, there is a greater tendency for melt formation at the

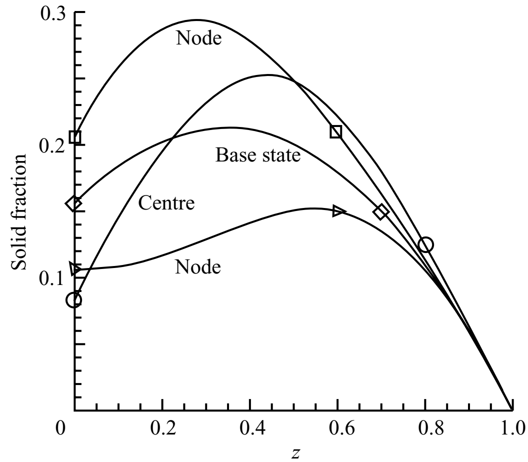

FIGURE 5. Solid fraction for right-travelling down-hexagons versus z for $C = 0.248$, $S = C$, $\tau = 0.07$ and $K_2 = 0$. ○, △, □ and ◇ show, respectively, $\tilde{\phi}(x = 0,\ y = 0,\ z, t = 0)$ at a cell's centre, $\tilde{\phi}(x = 1.33,\ y = 0,\ z, t = 0)$ at a cell's node initially, $\tilde{\phi}(x = 1.33,\ y = 0,\ z, t = 1.0)$ at the same cell's node, but at half of the period of oscillation and the basic solid fraction ϕ_B.

node near the lower boundary. However, it can be seen from figure 5 that for the down-hexagonal case and right after completing any period of oscillation, there are greater tendencies for crystal formation at the centre near the upper boundary and chimney formation at the node near the lower boundary; but just completing about half a period of oscillation, there is a greater tendency for crystal formation at the node near the lower boundary.

Similar to the stability procedure carried out in Riahi (2003), the growth rate σ_{20} of the disturbances acting on the finite-amplitude oscillatory solutions was computed for different integers N and various values of the quantities Φ_{nm} ($|\Phi_{nm}| \leqslant 1$) and Ψ_{nm} ($|\Psi_{nm}| \leqslant 1$). If τ is too small, such as $\tau < 0.004$, the stability results are essentially the same as those for the zero-rotation case (Riahi 2002). If τ is not too small, then the stability results indicate that only right- or left-travelling supercritical solutions in the form of either rolls, rectangles, squares or hexagons are stable in a particular range of values of the parameters, provided the corresponding R_{20} has the smallest positive value among those for all the finite-amplitude flow solutions.

5. Conclusion

We investigated the problem of nonlinear oscillatory convection in a rotating mushy layer during alloy solidification. We analysed the two- and three-dimensional oscillatory modes of convection in the rotating mushy layer using the model due to Amberg & Homsy (1993). We performed a weakly nonlinear analysis to determine the oscillatory solutions admitted by the nonlinear problem and employed stability analysis to determine the solutions that can be stable with respect to arbitrary three-dimensional disturbances in different ranges of the parameter values. We found that for a rotation rate that is not too small, and depending on the range of values of the other parameters, a simple travelling supercritical solution in the form of rolls, rectangles, squares, down- or up-hexagons can possibly be stable. Subcritical oscillatory flow solutions were found to exist, but are unstable based on the present theory. The effect of rotation was found to be stabilizing in the linear regime and mainly destabilizing in the nonlinear regime of the present theory.

A notable result of the present study, which is qualitatively independent of the type of solution and its pattern and is also shown in the figure 3 as an example, is the destabilizing effect of rotation in the weakly nonlinear regime and the non-monotonic dependence of R_{20} with respect to the variation of C with fixed S/C. Thus, in order for C to increase, the value of S must be decreased accordingly. The non-monotonic behaviour of R_{20} indicates that as C increases, either the destabilizing effect of the Stefan number or the stabilizing effect of the concentration ratio dominates, depending on their dominating range of values and the rotational constraint tends to enhance such behaviour.

In contrast to the steady-rotating case (Riahi 2003) where subcritical down-hexagons could be stable, subcritical hexagons were found to be unstable here in the oscillatory-rotating case, and the qualitative differences between the right- and left-travelling modes were found to be due to the interactions between the local solid fraction and the Coriolis term. A notable conclusion for the oscillatory convection in the rotating mushy-layer systems is that the porosity affects the amplitude of the preferred flow solutions, and such porosity effect on the right-travelling solution is, in general, different from that on the left-travelling solution.

The author would like to thank the editor and two referees for pointing out an error in an earlier version, which was due to an inadequate number of decimals being used for π in the computation.

REFERENCES

Amberg, G. & Homsy, G. M. 1993 Nonlinear analysis of buoyant convection in binary solidification with application to channel formation. *J. Fluid Mech.* **252**, 79–98.

Anderson, D. M. & Worster, M. G. 1995 Weakly nonlinear analysis of convection in mushy layers during the solidification of binary alloys. *J. Fluid Mech.* **302**, 307–331.

Anderson, D. M. & Worster, M. G. 1996 A new oscillatory instability in a mushy layer during the solidification of binary alloys. *J. Fluid Mech.* **307**, 245–267.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Clarendon Press.

Govender, S. & Vadasz, P. 2002 Weak non-linear analysis of moderate Stefan number oscillatory convection in rotating mushy layers. *Transport Porous Media* **48**, 353–372.

Guba, P. 2001 On the finite-amplitude steady convection in rotating mushy layers. *J. Fluid Mech.* **437**, 337–365.

Guba, P. & Boda, J. 1998 The effect of uniform rotation on convective instability of a mushy layer during binary alloy solidification. *Stud. Geophys. Geod.* **42**, 289–296.

Riahi, D. N. 2002 On nonlinear convection in mushy layers. Part 1. Oscillatory modes of convection. *J. Fluid Mech.* **467**, 331–359.

Riahi, D. N. 2003 Nonlinear steady convection in rotating mushy layers. *J. Fluid Mech.* **485**, 279–306.