MANY BODY TOPICS IN CONDENSED MATTER PHYSICS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Physics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2010

Urbana, Illinois

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Abstract

Two different problems involving many-body systems are presented. A hydrodynamic version of the Calogero system of one-dimensional particles interacting on the line is derived using a classical field formalism, and the results are contrasted to a derivation starting from first quantum mechanical principles. This new classical approach is shown to help in understanding subtleties occurring in the latter, such as the conditions for chiral motion, the decomposition of the Hamiltonian in terms of chiral currents and the nature of the physical velocity and density operators. Explicit collective solitonic excitations in the linear and non-linear limits are also presented. Additionally, we overview the possibility of expanding this formalism to the study of the Fractional Quantum Hall Effect.

The second problem involves a simple two-dimensional model of a $p_x + ip_y$ superfluid in which the mass flow that gives rise to the intrinsic angular momentum is easily calculated by numerical diagonalization of the Bogoliubov-de Gennes operator. The results confirm theoretical predictions such as the Thomas-Fermi approximation and the Ishikawa formula, in which the mass flow at zero-temperature and for a constant director $\mathbf{l}$ follows $j_{\text{mass}} = \frac{1}{2}\text{curl}(\rho\hbar\mathbf{l}/2)$.

To Daniela and Alexander.

Acknowledgments

I would like to thank my wife Daniela for her support through all these years and for the sacrifices she has made along the way to help me finish this thesis. Most importantly, for being fun and beautiful in every sense of the word. My son, Alexander, for being just a joy to look at every single time and bringing constant happiness to my life. I also wish to deeply thank Michael Stone for giving me the chance to work with him, and specially for having an infinite patience when my productivity was barely an ϵ above zero. Many thanks to John Stack for always being helpful and understanding, and to Wendy Wimmer for being the best secretary on Earth. Lastly, thanks to Paula for trusting and caring for me when I started this journey.

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1 Introduction

In this thesis we look at several analytical properties of two different and well known many-body systems in condensed matter physics. The first system in question is the Calogero model, an integrable system of particles interacting on the line in one dimension, which shares many of the properties of systems exhibiting the Fractional Quantum Hall Effect (FQHE), such as fractional statistics, chiral currents and more. We will present a new approach based on classical fields that yields a hydrodynamical description of the Calogero model, contrasting it to results obtainable through a direct quantum mechanical route. A review of the literature showing a mapping between the Calogero model and a matrix model of the FQHE will be provided, suggesting the future use of the classical approach to study the latter.

The second system under study is that of a $p+ip$ superfluid, in which a system of fermions in two dimensions embedded in a harmonic trap interacts through a pair gap potential. We will look at the question of the angular momentum in such a system by numerically solving the exact Hamiltonian and obtaining the mass and angular momentum distributions of the fluid. The results will be shown to be in agreement with both the previous numerical estimates and some of the theoretical predictions.

1.1 Structure of this Thesis

We start the next chapter by introducing the Calogero model and some of its properties, followed by a review of the work found in [1] showing a Quantum Mechanical approach to the hydrodynamic Calogero model. A few calculations not shown originally in [1] have been added to help anyone reviewing this material and wishing to expand on it. In particular, the need to enforce the Sutherland inner product in the non-chiral case of section 2.8.1 is presented in a slightly different way to better suit the taste of this thesis' author. The reduction of the non-chiral Euler equation to the Benjamin-Ono equation in section 2.9 using the chirality condition on the currents is also included in full in the Appendix A.8.

The results from the second chapter come into play on the third one, where we review an alternative approach to derive the hydrodynamic Calogero model based on the work of Abanov and Wiegmann [2] and Stone et. al. [3]. Employing a classical theory that starts with the Benjamin-Ono equation, this approach sheds light on some questions that are obscure or without an easy physical interpretation in the Quantum mechanical case. This new framework also provides an easier way to compute exact solutions, presented in section 3.4, allowing us to visualize solitonic collective excitations to the hydrodynamic model and their interactions.

The fourth chapter is mostly a quick and condensed review of known results in the literature pertaining the FQHE. A simple fluid model of the FQHE originally introduced by Susskind [4] is presented, followed by a non-abelian matrix extension by Polychronakos [5] that improves on the first and exhibits the properties of the FQHE more rigorously. The mapping from this model to the Calogero one, obtained in [5], is then introduced. The chapter ends suggesting the use of the Calogero formalism previously developed in chapter three to the FQHE, by reverting the existing mapping and lifting the Calogero model to the FQHE matrix model.

The last chapter presents a numerical calculation of the angular momentum and mass current distributions of a $p + ip$ superfluid based on a simple Bogoliubov-de Gennes (BdG) Hamiltonian, as shown in [6]. We review the basics of the BdG approach and suitable forms for the eigenstates of the harmonic oscillator, and in section 5.4 look at the matrices that need to be diagonalized to calculate the mass and angular momentum distributions. Along the way in the derivation, we fix an erratum overlooked in the original paper, laying out the logic needed to justify the use of such matrices. The results, which yield an angular momentum of $\hbar/2$ per particle, are contrasted to the theoretical predictions given by the Thomas-Fermi approximation and the Ishikawa mass current calculation and found to be in agreement. We include brief derivations of these in Appendix C.1,C.2, the latter based on previous work by Stone and Roy [7].

Throughout this thesis, the author has tried to present the material from his point of view where possible, including some remarks and clarifications that are most likely a bit naive and redundant for the seasoned theoretician, but that hopefully serve to help a starting graduate student wishing to understand and expand on these topics.

2 Quantum Mechanical Approach to the Calogero-Sutherland Model

2.1 General Introduction to the CS Model

The Calogero (Sutherland) model [8] in its simplest form is a system of identical particles interacting on the line (circle) through an inverse-square potential. Its Hamiltonian is given by

$$H_{cal} = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i<j} \frac{\lambda(\lambda-1)}{(x_i - x_j)^2}, \quad H_{sut} = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{8} \sum_{i \neq j} \frac{\lambda(\lambda-1)}{\sin^2(\frac{\theta_i - \theta_j}{2})}, \quad (2.1)$$

where the parameter λ controls the statistics of the particles. This innocently looking system has many rich properties that were discovered between 1970 and 1980, its spectrum being first solved exactly by Sutherland [9] in 1970, and has been reviewed several times in the literature [10, 11]. The system is integrable [12], which implies that its Hamiltonian possesses N conserved quantities. This property is also shared by an equation originally describing the hydrodynamics of stratified fluids, the Quantum Benjamin-Ono (QBO) equation [13, 14],

$$\dot{u} + u u_x = \frac{1}{2} \lambda [u_{xx}]_H, \quad \text{where} \quad f_H(x) \equiv \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{1}{x - \xi} f(\xi) d\xi. \qquad (2.2)$$

When equation (2.2) is extended to the complex plane, the field u becomes complex, and will be shown to essentially encode the Calogero-Sutherland model dynamics.

Another property of interest of the Calogero model is the support of soliton solutions in the continuum hydrodynamic limit. In this limit, the dynamics of the system can be described by just two fields, a density $\rho(x)$ and a velocity $v(x)$. Furthermore, these two fields can be encoded into a single complex valued field $u(z)$ that satisfies an extension of equation (2.2) to the complex plane, which supports soliton solutions. Some of our work has involved understanding and expanding the relationship between this field and the Calogero model. These results are presented in chapter 3.

The key property of the Calogero-Sutherland (CS) model that is perhaps currently the most exciting is its connection to a matrix model theory of the fractional quantum Hall effect. Polychronakos has shown [5] that there exists a mapping between the coordinates of said matrix model and the CS model. The 2D evolution of the matrix model in the lowest Landau level can be projected down into a 1D system, whose evolution is dictated by the CS model equations of motion (EOM).

In this chapter, we review Stone and Gutman's work [1], who derive a hydrodynamic model for the Calogero model using a Quantum mechanical framework. Although this approach successfully derives the Hamiltonian and its underlying equations of motion, it is not completely satisfactory, since it leaves a few points obscured without much understanding as to why certain definitions work. For these reasons, in chapter 3 we will take a look at an alternative classical approach based on the original work of Abanov et. al [2] and successive contributions by Stone, Anduaga and Lei [3]. This approach is based on an extension of the QBO equation as the starting point, and will provide us with insight into the CS model, which ultimately may lead to an application in the FQHE context.

2.2 Overview of the Quantum Mechanical Formulation

In the rest of this chapter we review an approach that derives the dynamics of the Calogero-Sutherland model in the continuum limit, starting from a basic quantum-mechanical discrete Hamiltonian of fermions interacting via an inverse square potential on the line, the so called Calogero model:

$$H_{\text{Calogero}} \equiv \sum_{i}^{N} \left(-\frac{1}{2}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{j,j\neq i}^{N} \frac{\lambda(\lambda-1)}{(x_i - x_j)^2} \right). \tag{2.3}$$

We will follow the steps in [1], with a few changes in the way the results are presented. Furthermore, a lot of the underlying algebra not shown in the original paper will be included, part in the main text and the rest in the appendix, in the hope of helping readers wanting to work through the calculations. Although the derivations are not overly difficult conceptually (except perhaps those in section 2.7), they are fairly lengthy and do have a few subtleties that require careful consideration. At the heart of the problem lies the existence of two different inner-products, one which is the natural (canonical) inner product for two functions in real space,

$$\langle \Phi_1 | \Phi_2 \rangle_{\text{Sutherland}} = \frac{1}{N!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} |\Delta|^{2\lambda} \Phi_1^* \Phi_2, \tag{2.4}$$

where the factor $|\Delta|^{2\lambda} = \prod_{i<j}(x_i - x_j)^{2\lambda}$ is just a weight that arises from working in a transformed basis $\Psi = \Delta^\lambda \Phi$. This inner product is referred to as the Sutherland inner product, due to the fact that the Φ functions are the eigenfunctions of the Sutherland Model (the compact version of the Calogero model on a circle), which obeys the Hamiltonian

$$H_{\text{Sutherland}} = -\frac{1}{2}\sum_i \frac{\partial^2}{\partial \theta_i^2} + \frac{1}{4}\sum_{i<j} \frac{\lambda(\lambda-1)}{\sin^2(\theta_i - \theta_j)/2}. \tag{2.5}$$

The second inner product arises from the theory of symmetric functions and can be defined as

$$\langle F(p) | G(p) \rangle_{\text{Jack}} = \int \prod_{n=1}^{\infty} \left(\lambda \frac{d^2 p_n}{\pi n} \right) [F(p)]^* G(p) \exp \left\{ -\lambda \sum_{n=1}^{\infty} \frac{1}{n} p_n^* p_n \right\}, \tag{2.6}$$

where the p_n variables are related to the angular variables $z_i = \exp(i\theta)$ by $p_n = \sum_i z_i^n$. The details of this inner product, called "λ-Jack", will be provided later. It is important to note that these definitions are for a discrete number of particles. In the continuum limit, the two inner products differ by a weight (see section 2.7), which is the one precisely needed to make sense of the results obtained. In section 2.8, we will see that the physical fields of the continuum limit require us to enforce Hermiticity wrt. the Sutherland inner product and not the λ-Jack one. But this is the end of the story, we must first look at the basics of the model.

2.3 The Calogero Model on the Line

Starting from the Hamiltonian in (2.3), we wish to diagonalize this operator and find the groundstate of the system. To achieve this, we try a block-decomposition, with each block equal to the product of two operators, $Q^\dagger Q$, where

$$Q_i \equiv \frac{\partial}{\partial x_i} - \sum_{j\neq i} \frac{\lambda}{x_i - x_j}, \qquad Q_i^\dagger \equiv -\frac{\partial}{\partial x_i} - \sum_{j\neq i} \frac{\lambda}{x_i - x_j}. \tag{2.7}$$

In terms of these new operators, we have

$$\sum_i Q_i^\dagger Q_i = \sum_i \left(-\frac{\partial^2}{\partial x_i^2} + \sum_{j\neq i} \frac{-\lambda}{(x_i - x_j)^2} + \sum_{j\neq i} \frac{\lambda}{x_i - x_j} \sum_{k\neq i} \frac{\lambda}{x_i - x_k} \right). \tag{2.8}$$

The sum of the last term can be regrouped as

$$\sum_{k,j\neq i} \frac{\lambda^2}{(x_i - x_j)(x_i - x_k)} = \sum_{\substack{j\neq i\\ k=j}} \frac{\lambda^2}{(x_i - x_j)^2} + \sum_{\substack{k,j\neq i\\ j\neq k}} \frac{\lambda^2}{(x_i - x_j)(x_i - x_k)}. \tag{2.9}$$

We can use the algebraic identity

$$\frac{1}{(x_j - x_i)(x_i - x_k)} + \frac{1}{(x_i - x_k)(x_k - x_j)} + \frac{1}{(x_k - x_j)(x_j - x_i)} = 0 \tag{2.10}$$

to show that the last term, which can be written as

$$\begin{aligned} -2 \sum_{j<i<k} \lambda^2 \Bigg(\frac{1}{(x_j - x_i)(x_i - x_k)} &+ \frac{1}{(x_i - x_k)(x_k - x_j)} \\ &+ \frac{1}{(x_k - x_j)(x_j - x_i)} \Bigg) = 0, \end{aligned} \tag{2.11}$$

becomes zero. Putting together the leftover terms we can write

$$\frac{1}{2}\sum_i Q_i^\dagger Q^i = \frac{1}{2}\sum_i \left(-\frac{\partial^2}{\partial x_i^2} + \sum_{j\neq i} \frac{\lambda(\lambda-1)}{(x_i-x_j)^2} \right) \equiv H_{\text{Calogero}} \tag{2.12}$$

Now that we have expressed the Calogero Hamiltonian in terms of simpler operators, we seek its ground state, which should be annihilated by all the Q_i's. In analogy to the fermionic case, where we know the ground state wavefunction for $\lambda = 1$ (the Vandermonde determinant), we try the simplest antisymmetric state that we can form with the coordinates of the particles, namely

$$\Delta^\lambda \equiv \prod_{i<j}(x_i - x_j)^\lambda. \tag{2.13}$$

The proof that Δ^λ is indeed the ground state of (2.12) would follow trivially if we could show that

$$Q_k = \Delta^\lambda \frac{\partial}{\partial x_k}\frac{1}{\Delta^\lambda}. \tag{2.14}$$

To achieve this, we consider

$$\begin{aligned} \Delta^\lambda \frac{\partial}{\partial x_k}\frac{1}{\Delta^\lambda} &= \frac{\partial}{\partial x_k} - \frac{1}{\Delta^\lambda}\frac{\partial \Delta^\lambda}{\partial x_k} \\ &= \frac{\partial}{\partial x_k} - 1/(\prod_{i\neq k}(x_i - x_k)^\lambda)\frac{\partial}{\partial x_k}\prod_{i\neq k}(x_i-x_k)^\lambda \\ &= \frac{\partial}{\partial x_k} - \sum_{i\neq k}\frac{\lambda}{(x_i - x_k)} = Q_k. \end{aligned} \tag{2.15}$$

This way, we have shown that $Q_k\Delta^\lambda = 0$ for all integers k, and established Δ^λ as the groundstate of the system with energy $E = 0$. From here on, we could proceed to study the properties of the excitations over the groundstate. However, we wish to do so on the circle instead of on the line, thus we need to turn to the analogous Sutherland model.

2.4 The Calogero Model on the Circle: The Sutherland Model

The physical system in consideration is the same as the Calogero model but in a unit circle instead of the infinite line. Because the circle is compact, we will see that the new ground state has a non-zero energy. Instead of starting directly from (2.5) and obtaining the ground state just like we did for the Calogero model, we can work backwards and reconstruct (2.5) based on what we know already for the Calogero model. Since the Sutherland model is the same as the Calogero one just in different coordinates, the function Δ which is the product of the distances between particles should still be the groundstate of the system. Thus, we can write the groundstate Δ^λ in polar coordinates and then impose that this be the groundstate of the Sutherland Hamiltonian. The distance between 2 points in the unit circle is given by

$$|z_i - z_j| = \sqrt{2 - 2\cos(\theta_i - \theta_j)} = 2\sin\left(\frac{\theta_i - \theta_j}{2}\right) \tag{2.16}$$

Thus, we have

$$\Delta = \prod_{i<j}(x_i - x_j) \rightarrow \prod_{i<j} 2\sin\left(\frac{\theta_i - \theta_j}{2}\right) \tag{2.17}$$

In cartesian coordinates Δ^λ was the groundstate of the Calogero model and was annihilated by Q's. Furthermore, (2.14) tells us how the Q's should look like in terms of Δ. We can use this to our advantage to write

$$H = \frac{1}{2}\sum_i^N Q_i^\dagger Q_i \equiv \frac{1}{2}\sum_i \left(-\frac{1}{\Delta^\lambda}\frac{\partial}{\partial\theta_i}\Delta^\lambda\right)\left(\Delta^\lambda\frac{\partial}{\partial\theta_i}\frac{1}{\Delta^\lambda}\right) \tag{2.18}$$

We need to expand the sum in order to obtain a functional expression for the Hamiltonian. After some mind-numbing algebra found in appendix A.1, we find

$$\begin{aligned} H &= -\frac{1}{2}\sum_i \frac{\partial^2}{\partial\theta_i^2} + \frac{1}{4}\sum_{i<j}\frac{\lambda(\lambda-1)}{\sin^2\left(\frac{\theta_i-\theta_j}{2}\right)} - \frac{\lambda^2}{24}N(N^2-1) \\ &= H_{\text{Sutherland}} - \frac{\lambda^2}{24}N(N^2-1) \end{aligned} \tag{2.19}$$

Note that the ground energy is not zero anymore but has a constant value. Now that we have the Hamiltonian in θ-coordinates and know it's ground-state, we can study its excitations, which we will assume to be of the form $\Psi = \Delta^\lambda \Phi(z_1, \ldots, z_N)$. Since the Δ term carries all the fermionic statistics, the Φ function must necessarily be symmetric. Making a change of basis on the above Hamiltonian[1] using the ground state of the system, $H\Delta^\lambda\Phi = E\Delta^\lambda\Phi$, we define a Hamiltonian operator that absorbs the transformation $H'\Phi \equiv \Delta^{-\lambda}H\Delta^\lambda\Phi = E\Phi$. To obtain the algebraic expression of this new Hamiltonian, we need a bit of algebra,

$$\begin{aligned} H' &= -\frac{1}{2}\sum_i^N \frac{1}{\Delta^{2\lambda}}\frac{\partial}{\partial\theta_i}\Delta^{2\lambda}\frac{\partial}{\partial\theta_i} = -\frac{1}{2}\sum_i\left(\frac{\partial^2}{\partial\theta_i^2} + \frac{1}{\Delta^{2\lambda}}\left(\frac{\partial^2}{\partial\theta_i^2}\Delta^{2\lambda}\right)\Delta^{2\lambda}\frac{\partial}{\partial\theta_i}\right) \\ &= -\frac{1}{2}\sum_i^N \frac{\partial}{\partial\theta_i} - \frac{\lambda}{2}\sum_{\substack{i,j \\ i\neq j}} \text{cotg}\left(\frac{\theta_i-\theta_j}{2}\right). \end{aligned} \tag{2.20}$$

Splitting the sum into $i > j$ and $i < j$ terms and relabeling the indices yields

$$H' = -\frac{1}{2}\sum_i^N \frac{\partial}{\partial\theta_i} - \frac{\lambda}{2}\sum_{i<j}\text{cotg}\left(\frac{\theta_i-\theta_j}{2}\right)\left(\frac{\partial}{\partial\theta_i} - \frac{\partial}{\partial\theta_j}\right). \tag{2.21}$$

We define $D_i = z_i\frac{\partial}{\partial z_i} = -i\frac{\partial}{\partial\theta_i}$, where $z_i = \exp(i\theta_i)$, and use the trigonometric identity

$$\text{cotg}\left(\frac{\theta_i-\theta_j}{2}\right) = i\frac{z_i+z_j}{z_i-z_j}, \tag{2.22}$$

to write the excitation Hamiltonian as

$$H' = \frac{1}{2}\sum_i D_i^2 + \frac{\lambda}{2}\sum_{i<j}\frac{z_i+z_j}{z_i-z_j}(D_i - D_j). \tag{2.23}$$

[1]We use the Heisenberg picture of Quantum dynamics, in which the change of basis is absorbed by the operators and the wavefunctions remain unchanged.

Note that the $\{\Phi\}$ functions satisfy the Sutherland canonical inner-product defined previously in (2.4). For simplicity, we will initially only consider $\Phi(z_1, \ldots, z_N)$ that are symmetric polynomials in the z_i (i.e. we are discarding any powers containing $\bar{z}_i$). This corresponds to counter-clockwise moving excitations with positive L_z angular momentum, in order words, we are constrained to chiral excitations only. The non-chiral extensions will be covered in the next section.

Since we need the Φ to be symmetric, it is best to move to a set of variables that is automatically symmetric under the exchange of particles. Thus, we go from $\{z_1, \ldots, z_N\}$ to $\{p_1, \ldots, p_N\}$ variables, where $p_n = \sum_i z_i^n$ are called power sums [15]. We have to see how the Hamiltonian in (2.23) changes under these new definitions. The new angular momentum operator D_i becomes

$$D_i = z_i \frac{\partial}{\partial z_i} = z_i \sum_{n=1}^{N} \frac{\partial p_n}{\partial z_i} \frac{\partial}{\partial p_n} = \sum_{n=1}^{N} n z_i^n \frac{\partial}{\partial p_n}. \tag{2.24}$$

We must substitute this into (2.23), and carry out the algebraic operations, which requires a bit of work. We refer the reader to appendix A.2 for the full details of the calculation, including here only two main results,

$$\begin{aligned} \sum_i^N D_i^2 &= \sum_{m=1}^{N} m^2 p_m \frac{\partial}{\partial p_m} + \sum_{m,n=1}^{N} mn p_{n+m} \frac{\partial}{\partial p_n} \frac{\partial}{\partial p_m}, \\ (z_i + z_j) \frac{z_i^N - z_j^N}{z_i - z_j} &= z_i^N + 2 z_i^{N-1} z_j + \ldots + 2 z_i z_j^{N-1} + z_j^N, \end{aligned} \tag{2.25}$$

that leads to the expression

$$\begin{aligned} 2H' &= \sum_m m^2 p_m \frac{\partial}{\partial p_m} + \sum_{n,m} nm p_{n+m} \frac{\partial}{\partial p_n} \frac{\partial}{\partial p_m} \\ &\quad + \lambda \sum_{m,n=1}^{n+m \leq N} (m+n) p_m p_n \frac{\partial}{\partial p_{n+m}} + \lambda \sum_{l=1}^{N} l(N-l) p_l \frac{\partial}{\partial p_l}. \end{aligned} \tag{2.26}$$

Since we are interested in the collective field formalism where $N \to \infty$, we will ignore the $n+m \leq N$ constraint and allow the sums to extend to infinity.

The resulting excitation Hamiltonian is

$$
\begin{aligned}
2H' &= \sum_{n=1}^{N} \left((1-\lambda)n^2 + \lambda n N\right) p_n \frac{\partial}{\partial p_n} \\
&\quad + \sum_{n,m=1}^{N} \left(nm p_{n+m} \frac{\partial}{\partial p_n}\frac{\partial}{\partial p_m} + \lambda(n+m) p_n p_m \frac{\partial}{\partial p_{n+m}} \right) \quad (2.27)
\end{aligned}
$$

We have not yet discussed the Hermiticity of the above Hamiltonian. We already know that H is Hermitian under the Sutherland inner product (2.4). Additionally, if we define the adjoint of the functions $\{p_n\}$ as

$$p_n^\dagger = \frac{n}{\lambda}\frac{\partial}{\partial p_n} \quad (2.28)$$

then H' is also Hermitian wrt. the λ-Jack inner product defined in (2.6). Note that both inner products coincide when $\lambda = 1$ or when $N \to \infty$ in the current chiral case [1]. To get a better physical picture, we now wish to express the Hamiltonian in (2.27) in terms of physical quantities. We thus identify the particle-density operator with a new operator j, whose positive components[2] are the $\{p_n\}$ and its negative components are determined by Hermiticity (i.e. they are not independent) in the current chiral case. We define

$$j_n \equiv p_n, \quad j_{-n} \equiv j_n^\dagger = p_n^\dagger = \frac{n}{\lambda}\frac{\partial}{\partial p_n} \quad \text{for } n > 0. \quad (2.29)$$

In position space, the above becomes

$$j(\theta) = \frac{1}{2\pi}\sum_{n=-N}^{N} j_n e^{-in\theta}, \quad \text{with } j_0 = \frac{N}{2}, \quad (2.30)$$

[2]Note that $p_n = \int_0^{2\pi} e^{in\theta}\rho(\theta) d\theta$ is the positive momentum fourier component of the density, where $\rho(\theta) = \sum_i^N \delta(\theta - \theta_i)$.

which satisfies the right going filling-fraction ν chiral algebra

$$\begin{aligned}
[j(\theta), j(\theta')] &= \frac{1}{4\pi}\sum_{n=-N}^{N}\sum_{l=-N}^{N}[j_n, j_l]\mathrm{e}^{-i(n\theta+l\theta')} \\
&= \frac{1}{4\pi}\sum_{n,l}\nu n\delta_{n+l,0}\mathrm{e}^{-i(n\theta+l\theta')} \\
&= \frac{1}{4\pi}\frac{1}{-i}\frac{d}{d\theta}\underbrace{\sum_{n=-N}^{N}\mathrm{e}^{-in(\theta-\theta')}}_{\to 2\pi\delta(\theta-\theta')} \\
&= \frac{i\nu}{2\pi}\delta^{'}(\theta-\theta') \qquad (2.31)
\end{aligned}$$

Our Hamiltonian in terms of the new j_n becomes

$$\begin{aligned}
2H' &= \lambda^2\sum_{n,m=1}^{N}[j_{n+m}j_{-n}j_{-m} + j_nj_mj_{-n-m}] \\
&\quad +\lambda\sum_{n=1}^{N}[(1-\lambda)n+\lambda N]j_nj_{-n}. \qquad (2.32)
\end{aligned}$$

To obtain an expression of the above in terms of the real-space density $j(\theta)$ requires some algebra, since the sums go from 1 to N instead of from $-N$ to N as in $j(\theta)$. We look at expressions for the first and second terms separately. The first term can be condensed into

$$\begin{aligned}
\int :j(\theta)^3: d\theta &= \frac{1}{(2\pi)^3}\int\sum_{n=-N}^{N}\sum_{k,l=-N}^{N}j_nj_kj_le^{-i(n+k+l)\theta}d\theta \\
&= \frac{1}{(2\pi)^2}\sum_{n,k=-N}^{N}j_nj_kj_{-(n+k)} \\
&= \frac{1}{(2\pi)^2}3\sum_{n,k}^{N}\left[j_nj_kj_{-(n+k)} + j_{n+k}j_{-n}j_{-k}\right]. \qquad (2.33)
\end{aligned}$$

If we define j_+ as the part of $j(\theta)$ with $j_n, n>0$, and similarly j_- as j_n with

$n < 0$, the second term can be extracted from

$$
\begin{aligned}
\int_0^{2\pi} j\partial_\theta(j_+ - j_-)d\theta &= \int_0^{2\pi} j\partial_\theta j_+ d\theta - \int_0^{2\pi} j\partial_\theta j_- d\theta \\
&= \frac{1}{4\pi^2}\int_0^{2\pi}\sum_{n=-N}^{N} j_n e^{-in\theta}\partial_\theta \sum_{k=1}^{N} j_k e^{-ik\theta} d\theta \\
&\quad -\frac{1}{4\pi^2}\int_0^{2\pi}\sum_{n=-N}^{N} j_n e^{-in\theta}\partial_\theta \sum_{k=1}^{N} j_{-k} e^{ik\theta} d\theta \\
&= \frac{-i}{4\pi^2} : \sum_{k=1}^{N}\sum_{n=-N}^{N}\int_0^{2\pi} j_n j_k k e^{-i(k+n)\theta} : \\
&\quad +\frac{-i}{4\pi^2} : \sum_{k=1}^{N}\sum_{n=-N}^{N}\int_0^{2\pi} j_n j_{-k} k e^{-i(n-k)\theta} : \\
&= \frac{-i}{4\pi^2}\sum_{k=1}^{N} : (j_{-k}j_k + j_k j_{-k}) : k \\
&= \frac{-i}{\pi}\sum_{k=1}^{N} k j_{-k} j_k, \qquad (2.34)
\end{aligned}
$$

which allows us to write

$$
-\lambda(\lambda-1)\sum_{k=1}^{N} j_{-k}j_k = -\underbrace{\frac{\lambda(\lambda-1)}{4\pi}}_{\mathrm{a}} 4\pi^2 i : \int_0^{2\pi} j\partial_\theta(j_+ - j_-)d\theta : . \qquad (2.35)
$$

Putting together (2.33) and (2.35) yields the chiral form of the excitation Hamiltonian in terms of physical currents,

$$
\begin{aligned}
2H' &= 4\pi^2\int_0^{2\pi}\left[\frac{\lambda^2}{3} : j(\theta)^3 : d\theta - ia : j(\theta)\partial_\theta(j_+(\theta) - j_-(\theta)) : d\theta\right] \\
&= 4\pi^2\int_0^{2\pi}\left\{\frac{\lambda^2}{3}j^3 - aj\partial_\theta j_{\mathrm{H}}\right\} d\theta. \qquad (2.36)
\end{aligned}
$$

Here $a \equiv \lambda(\lambda-1)/4\pi$. The resulting classical eom. (where $\lambda(\lambda-1) \to \lambda^2$) is of the Benjamin-Ono form (2.2),

$$
\partial_\tau j + j\partial_\theta j - \beta\partial^2_{\theta\theta} j_H = 0. \qquad (2.37)
$$

where $\tau = 2\pi\lambda t$, $\beta = 1/4\pi$, and H denotes the Hilbert transform as defined in (2.2). As we will see in chapter 3, this equation can be used as the starting point to derive the mechanics of the Calogero-Sutherland model, obtaining the same results that the Quantum mechanical approach yields but from a different point of view. The Benjamin-Ono equation on the infinite line has a right-going solution

$$j(x,t) - \langle j \rangle \equiv j(x,t) - \frac{\rho_0}{2} = \frac{4U}{\beta^{-}2U^2[x-U\tau]^2+1}. \tag{2.38}$$

The excess charge carried by the soliton is

$$\int_{\infty}^{\infty} (j(x,t) - \langle j \rangle)dx - 4\pi\beta = 1. \tag{2.39}$$

Thus, the Hamiltonian (2.36) does indeed provide collective mode excitations for our system, which are unit-charged solitons. We will investigate the interaction of these solitons and their dynamics in chapter 3.

2.5 Non-Chiral Extension

We now turn our attention back to the excitation Hamiltonian (2.23), with the purpose of including non-chiral excitations in addition to the chiral ones of the previous section. We can accomplish this if we extend the definition of the p_n variables to negative integers, $p_{-|n|} = \sum_i z_i^{-|n|} = \sum_i \bar{z}_i^n$, which is equivalent to working with functions Φ such that $\Phi = \Phi(z_1, \ldots, z_N, \bar{z}_1, \ldots, \bar{z}_N)$ (i.e. the functions are not analytic anymore).

Since the scope of the sums in the definitions of each p_n has been redefined, we need to extend the D_i operators (this new set depends on $2N$ variables). We have

$$\begin{aligned} D_i &= z_i \frac{\partial}{\partial z_i} = z_i \sum_{n=-\infty}^{\infty} \frac{\partial p_n}{\partial z_i}\frac{\partial}{\partial p_n} = \sum_{n=-\infty}^{-1} (-|n|) z_i^{-|n|} \frac{\partial}{\partial p_n} + \sum_{n=0}^{\infty} n z_i^n \frac{\partial}{\partial p_n} \\ &= \sum_{-\infty}^{\infty} n z_i^n \frac{\partial}{\partial p_n} \end{aligned} \tag{2.40}$$

This is analogous to the result obtained for the chiral chase, with the addition of a contribution coming from the $\bar{z}$ terms in the form of $\bar{z}_i^n \frac{\partial}{\partial p_{-n}}$. In order to write the Hamiltonian in terms of the p_i variables, we need to compute $\sum D_i^2$. We show the details of this simple calculation in the appendix A.3, which yields

$$\sum_{i=1}^{N} D_i^2 = \sum_{m=-\infty}^{\infty} m^2 p_m \frac{\partial}{\partial p_m} + \sum_{n,m=-\infty}^{-\infty} nmp_{n+m} \frac{\partial}{\partial p_m}\frac{\partial}{\partial p_n}. \tag{2.41}$$

The next step is the calculation of the second term of the Hamiltonian,

$$\begin{aligned} \frac{\lambda}{2}\sum_{i<j}\frac{z_i+z_j}{z_i-z_j}(D_i-D_j) &= \frac{\lambda}{2}\sum_{i<j}\frac{z_i+z_j}{z_i-z_j}\sum_{l=-N}^{N} l(z_i^l - z_j^l)\frac{\partial}{\partial p_l} \\ &= \frac{\lambda}{2}\sum_{i<j}\sum_{l=1}^{N}\frac{z_i+z_j}{z_i-z_j}(z_i^l - z_j^l) l \frac{\partial}{\partial p_l} \\ &\quad + \frac{\lambda}{2}\sum_{i<j}\sum_{l=1}^{N}\frac{z_i+z_j}{z_i-z_j}(z_i^{-l} - z_j^{-l})(-l)\frac{\partial}{\partial p_{-l}}, \end{aligned} \tag{2.42}$$

which breaks down into two big sums. Luckily, we've already calculated the first term for the chiral case, which is given by (A.21). For the second term, we must use the identity

$$\frac{z_i+z_j}{z_i-z_j}(z_i^{-n} - z_j^{-n}) = -(z_i^{-n} + 2z_i^{-n+1}z_j^{-1} + \ldots + 2z_i^{-1}z_j^{-n+1} + z_j^{-n}), \tag{2.43}$$

which can be recycled from the one in (2.25) (see appendix A.4 for details). After completing the second term calculation, and adding the first one, we arrive at

$$\begin{aligned} \frac{\lambda}{2}\sum_{i<j}\frac{z_i+z_j}{z_i-z_j}(D_i-D_j) = \frac{\lambda}{2}\Bigg(&\sum_{l=-\infty}^{\infty} |l|(N-|l|)p_l \frac{\partial}{\partial p_l} \\ &+ \sum_{n,m=1}^{\infty}(n+m)(p_n p_m \frac{\partial}{\partial p_{(n+m)}} p_{-n}p_{-m}\frac{\partial}{\partial p_{-(n+m)}})\Bigg) . \end{aligned} \tag{2.44}$$

The full non-chiral excitation Hamiltonian, which is the sum of the above plus (2.41) is given by

$$
\begin{aligned}
2H' &= \sum_{n,m=-\infty}^{\infty} mnp_{n+m}\frac{\partial}{\partial p_n}\frac{\partial}{\partial p_m} + \sum_{m=-\infty}^{\infty}\left[(1-\lambda)m^2 + \lambda N|m|p_m\frac{\partial}{\partial p_m}\right] \\
&\quad +\lambda\sum_{n,m=1}^{\infty}(n+m)\left(p_n p_m\frac{\partial}{\partial p_{n+m}} + p_{-n}p_{-m}\frac{\partial}{\partial p_{-n-m}}\right). \qquad (2.45)
\end{aligned}
$$

This Hamiltonian is the non-chiral extension of (2.23) and accounts for the energy of fermions with negative angular momentum. To find the Hermitian properties of this new Hamiltonian, we need to understand how the λ-Jack inner product changes in the non-chiral case. The difference between the two cases is that in the non-chiral case the wavefunction depends on both p_n and p_{-n} (since we need to allow for both chiralities), thus new terms come up when we try to find the adjoints of the p_n. The calculation to find the new adjoint relations can be done in cartesian coordinates (as in [1]) or by staying in polar coordinates, which is what we show below.

The starting point is the inner product in (2.6), where now $\Phi = \Phi(p_n, p_{-n})$. In terms of the real and imaginary parts of p_n, we have that $p_n = r_n + is_n$, and by its definition, $p_{-n} = r_n - is_n$. We can use this to write

$$
d^2p_n = dr_n ds_n = \frac{dp_n + dp_{-n}}{2} \wedge \frac{dp_n - dp_{-n}}{2i} = \frac{i}{2}dp_n dp_{-n}, \qquad (2.46)
$$

from which the inner product reads

$$
\langle \Phi_1, \Phi_2\rangle = \int \prod_{n=1}^{\infty}\left[\frac{\lambda}{n\pi}\frac{i}{2}dp_n dp_{-n}\right]\Phi_1\Phi_2^* \exp\left(-\lambda\sum_{n=1}^{\infty}\frac{1}{n}p_n p_{-n}\right). \qquad (2.47)
$$

From here, the calculation of the adjoint operator is very simple,

$$
\begin{aligned}
\langle \Phi_1, \frac{\partial}{\partial p_n}\Phi_2\rangle &= \int\prod \ldots \Phi_1\left(\frac{\partial}{\partial p_n}\Phi_2\right)^* \exp(\ldots) \\
&= \int\prod \ldots \Phi_1\frac{\partial}{\partial p_{-n}}\Phi_2^* \exp(\ldots) \\
&= -\int\prod \ldots \frac{\partial}{\partial p_{-n}}\left(\Phi_1 \exp(-\lambda\sum_{k=1}^{\infty}\frac{1}{k}p_k p_{-k})\right)\Phi_2^* \\
&= -\langle \frac{\partial}{\partial p_{-n}}\Phi_1, \Phi_2\rangle + \langle \frac{p_n\lambda}{n}\Phi_1, \Phi_2\rangle \qquad (2.48)
\end{aligned}
$$

This and the analogous calculation for $n < 0$ lead to the relation

$$\left(\frac{\partial}{\partial p_n}\right)^\dagger = -\frac{\partial}{\partial p_{-n}} + \frac{\lambda}{n}\mathrm{sgn(n)}p_n \qquad \forall n \in Z \tag{2.49}$$

The key difference between this result and the chiral one in (2.29) is that the latter satisfied $p_n^\dagger = \frac{n}{\lambda}\frac{\partial}{\partial p_n}$, or equivalently $(\frac{\partial}{\partial p_n})^\dagger = \frac{\lambda}{n}p_n$, where the former contains an extra term coupling both chiral sectors. It is good to note that this new inner-product, which following [1] we call Jack', satisfies

$$\begin{aligned} p_n^\dagger &= \frac{n}{\lambda\ \mathrm{sgn(n)}}\left(\frac{\partial}{\partial p_n} + \left(\frac{\partial}{\partial p_{-n}}\right)^\dagger\right) \\ &= \frac{n}{\lambda\ \mathrm{sgn(n)}}\left(\frac{\lambda\ \mathrm{sgn(n)}}{n}p_{-n}\right) = p_{-n} \end{aligned} \tag{2.50}$$

which can be seen at once by looking at $\langle \Phi_1, p_n\Phi_2\rangle$, or alternatively by inverting the 2x2 equation system in (2.29). Thus, creating an excitation with index n (clockwise) is equivalent to destroying one excitation with opposite chirality. More importantly, we can now look at the Hamiltonian in (2.45) and see that it is Hermitian wrt. this new Jack' inner product. We will come back to this point in section 2.8.

2.6 Collective Fields in the Non-Chiral Case

Our next objective involves finding an expression for the non-chiral Hamiltonian (2.45) in terms of collective density and velocity fields, analogous to the result found by Polychronakos in [16]. We can define the density ρ and its canonical conjugate field $\Pi = \frac{\delta}{\delta\rho}$ as

$$\begin{cases} \rho(\theta) \equiv \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} p_n e^{-in\theta} \implies p_n = \int_0^{2\pi}\rho(\theta)e^{in\theta}d\theta \\ \Pi(\theta) \equiv \sum_{n=-\infty}^{\infty} e^{in\theta}\frac{\partial}{\partial p_n} \implies \frac{\partial}{\partial p_n} = \frac{1}{2\pi}\int_0^{2\pi}\Pi(\theta)e^{-in\theta}d\theta. \end{cases} \tag{2.51}$$

Before we substitute the old variables $\{p_n, \partial/\partial p_n\}$ by the $\{\rho, \Pi\}$ fields into the Hamiltonian 2.45, it is helpful to massage the latter (see appendix A.6

for details) and rewrite it as

$$
\begin{aligned}
2H' &= \sum_{n,m=-\infty}^{\infty} mnp_{n+m}\frac{\partial}{\partial p_n}\frac{\partial}{\partial p_m} + \sum_{m=-\infty}^{\infty}(1-\lambda)m^2 p_m \frac{\partial}{\partial p_m} \\
&\quad +\lambda \sum_{n,m=-\infty}^{\infty} p_n p_m (n+m)\mathrm{sgn}(m)\frac{\partial}{\partial p_{n+m}}. \qquad (2.52)
\end{aligned}
$$

We can now use the field definitions and substitute them into the above, which leads us to three different terms,

$$
\begin{aligned}
&2H' = \\
&\underbrace{\sum_{n,m=-\infty}^{\infty} nm \int_0^{2\pi} \rho(\theta)e^{i(n+m)\theta}d\theta \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\Pi(\theta')\Pi(\theta'')e^{-in\theta'}e^{-im\theta''}d\theta' d\theta''}_{A} \\
&+\underbrace{\lambda \sum_{n,m=1}^{\infty}\frac{n+m}{2\pi}\int_0^{2\pi}\rho(\theta)\rho(\theta')e^{in\theta}e^{im\theta'}\mathrm{sgn}(m)\Pi(\theta'')e^{-i(n+m)\theta''}\theta'' d\theta d\theta' d\theta''}_{B} \\
&+\underbrace{\sum_{n=-\infty}^{\infty}\frac{1-\lambda}{2\pi}n^2\int_0^{2\pi}\int_0^{2\pi}\rho(\theta)e^{in\theta}\Pi(\theta')e^{-in\theta'}d\theta d\theta'}_{C}, \qquad (2.53)
\end{aligned}
$$

which we have labeled A, B and C. The calculation of each term is straightforward, and we lay out the details in appendix A.6. The results for each term are

$$
\begin{aligned}
A &= -\int_0^{2\pi}\rho(\theta)\left(\frac{\partial \Pi(\theta)}{\partial\theta}\right)^2 d\theta, && (2.54)\\
B &= \int \rho(\theta)\cot\left(\frac{\theta-\theta'}{2}\right)\rho(\theta')\partial_{\theta'}\Pi(\theta')d\theta d\theta', && (2.55)\\
C &= \int_0^{2\pi}(1-\lambda)\partial_\theta\rho(\theta)\partial_\theta\Pi(\theta)d\theta. && (2.56)
\end{aligned}
$$

Putting all terms together, we can write the Sutherland non-chiral excitation Hamiltonian as

$$
\begin{aligned}
2H' &= \int d\theta \left(-\rho\left(\partial_\theta\Pi\right)^2 + (1-\lambda)\partial_\theta\rho(\theta)\partial_\theta\Pi(\theta)\right. \\
&\quad \left. + \lambda\rho(\theta)\int \cot\left(\frac{\theta-\theta'}{2}\right)\rho(\theta')\partial_{\theta'}\Pi(\theta')d\theta'\right) \qquad (2.57)
\end{aligned}
$$

Looking at the first term, which resembles the kinetic energy, we might be tempted to identify the field Π as the velocity potential, with $\mathrm{v} = \partial_\theta \Pi$. This, however, is not entirely correct, since it turns out that the velocity defined this way is unphysical (i.e. does not satisfy the continuity equation). The distinction between the Π field and the physical velocity will be covered in section 2.8. The problem lies in the fact that the Hamiltonian above is not explicitly Hermitian wrt. the Sutherland inner product. Once we explicitly account for the weight of the Sutherland inner product by including it in the Hamiltonian, we will recover the physical velocity. We will also see that, were we to do the same for the Jack' inner product, the resulting velocity would still not satisfy the continuity equation (i.e. it's missing a term).

2.7 Inner Products in the Continuum Limit

Before we proceed further, we must go back and look at the two inner products and establish the relationship between them in the continuum limit. We begin by looking at how much the weights of the Sutherland (2.4) and λ-Jack (2.6) inner products differ in the large-N chiral case. We have

$$\begin{aligned}\exp\left\{-\lambda\sum_{n=i}^{\infty}\frac{1}{n}p_n^* p_n\right\} &= \exp\left\{-\lambda\sum_{n=1}^{\infty}\frac{1}{n}\sum_j z^{*n}_j\sum_k z_k^n\right\} \\ &= \exp\left\{-\lambda\left(\sum_{n,j}|z_j|^{2n} + \sum_n\frac{1}{n}\sum_{j\neq k}(\bar{z}_j z_k)^n\right)\right\}.\end{aligned} \tag{2.58}$$

If we assume for a moment that $|z| < 1$, we can use the logarithm expansion

$$\ln(1-z) = -\left(0 + \frac{z}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \dots\right) = -\sum_{n=1}^{\infty}\frac{z^n}{n}, \tag{2.59}$$

which is valid for $|z| < 1$. We can then write

$$
\begin{aligned}
\exp\left\{-\lambda\sum_{n=i}^{\infty}\frac{1}{n}p_n^*p_n\right\} &= \exp\sum_j \lambda\underbrace{\left(-\sum_n \frac{1}{n}(|z_j|^2)^n\right)}_{\ln(1-|z_j|^2)}\exp\sum_{j\neq k}\underbrace{-\lambda\sum_n\frac{(\bar{z}_jz_k)^n}{n}}_{\lambda\ln(1-\bar{z}_jz_k)} \\
&= \exp\left\{\lambda\sum_j \ln(1-|z|^2)\right\}\exp\left\{2\lambda\sum_{j<k}(1-\bar{z}_jz_k)\right\} \\
&= \prod_{j=1}^{\infty}(1-|z_j|^2)^{\lambda}\prod_{j<k}(1-\bar{z}_jz_k)^{2\lambda} \qquad (2.60)
\end{aligned}
$$

Our real $\{z_i\}$ satisfy $|z_i| = 1$. We can still use the above expression, however, if we introduce a convergence factor μ that satisfies $|\mu| < 1$ and write $z_i = \lim_{\mu\to 1_-}\mu z_i$. Doing so give us

$$
\begin{aligned}
\exp\left\{-\lambda\sum_{n=i}^{\infty}\frac{1}{n}p_n^*p_n\right\} &= \prod_{j=1}(1-\mu)^{\lambda}\prod_{j<k}\left(1-\frac{\mu z_k}{z_j}\right)^{\lambda}\left(1-\frac{\bar{z}_j\mu}{\bar{z}_k}\right)^{\lambda} \\
&= (1-\mu)^{2N-1}\prod_{j<k}(z_j-\mu z_k)^{\lambda}(\bar{z}_j-\mu\bar{z}_k)^{\lambda} \\
&\to |\Delta(z)|^{2\lambda}(1-\mu)^{2N-1}, \quad \text{as } \mu\to 1_-. \qquad (2.61)
\end{aligned}
$$

From here we see that the explicit weights of the inner products are proportional, but the proportionality constant diverges to zero for the physical limit $\mu \to 1$. We would like to find an expression for the two inner products in terms of the particle density field ρ, the anti-fourier transform of the p_n. This will require two things: Finding the weights of both inner products as a function of the density, and finding the Jacobian of the Sutherland inner product when going from the θ_i variables to the density ρ.

We start by considering the weight of the λ-Jack inner product. We will make use of the identity

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{|n|} e^{in\theta} = \sum_{-\infty}^{-1} \frac{1}{|n|} e^{in\theta} + \sum_{1}^{\infty} \frac{1}{|n|} e^{in\theta} = \sum_{1}^{\infty} \frac{1}{n} e^{-in\theta} + \sum_{1}^{\infty} \frac{1}{n} e^{in\theta}$$
$$= -\ln\left(1-e^{-i\theta}\right) - \ln\left(1-e^{i\theta}\right) = -\ln\left|1-e^{i\theta}\right|^2 = -2\ln\left|1-e^{i\theta}\right|$$
$$= -2\ln\left(2\left|\sin\frac{\theta}{2}\right|\right), \tag{2.62}$$

which is valid for any value of θ. The sums don't include the $n=0$ term. To simplify the notation, we call $W \equiv \exp(-\lambda \sum \frac{1}{n} p_n p_{-n})$. Since we also know that $p_n = \int_0^{2\pi} \rho(\theta)\exp(in\theta)d\theta$, we can replace p_n by the density field in W, obtaining

$$\mathrm{W} = \exp\left\{-\lambda \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{2\pi} \rho(\theta) e^{in\theta} d\theta \int_0^{2\pi} \rho(\theta') e^{-in\theta'} d\theta'\right\}$$
$$= \exp\left\{-\lambda \sum_{n=1}^{\infty} \frac{1}{n} \int\int \rho(\theta)\rho(\theta') e^{in(\theta-\theta')} d\theta d\theta'\right\}. \tag{2.63}$$

In order to decouple the θ variables from the θ' ones, we make the change of variables $u = \theta - \theta'$ and $v = \theta + \theta'$, which implies the Jacobian $dudv = 2d\theta d\theta'$. In this new variables, we have (see appendix A.5 for the complete steps)

$$\mathrm{W} = \exp\left\{-\frac{\lambda}{2} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{4\pi} \int_{-2\pi}^{2\pi} \rho\left(\frac{u+v}{2}\right) \rho\left(\frac{v-u}{2}\right) e^{inu} dudv\right\}$$
$$= \exp\left\{-\frac{\lambda}{2} \int_0^{4\pi} \int_0^{2\pi} \rho\left(\frac{u+v}{2}\right) \rho\left(\frac{v-u}{2}\right) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{|n|} e^{inu} dudv\right\}$$
$$= \exp\left\{\lambda \int_0^{4\pi} \int_0^{2\pi} \rho\left(\frac{u+v}{2}\right) \rho\left(\frac{v-u}{2}\right) \ln\left|2\sin\frac{u}{2}\right| dudv\right\}. \tag{2.64}$$

Extending the range of integration from $[0, 2\pi]$ to $[-2\pi, 2\pi]$ in the u variable (ρ and $\sin u$ are both 2π-periodic in u), and changing variables back to our original θ, θ', we get

$$\exp\left\{-\lambda \sum \frac{1}{n} p_n p_{-n}\right\} =$$
$$= \exp\left\{\lambda \int_0^{2\pi} \int_0^{2\pi} \rho(\theta)\rho(\theta') \ln\left|2\sin\frac{\theta-\theta'}{2}\right| d\theta d\theta'\right\}. \tag{2.65}$$

This expression gives us the weight W of the λ-Jack inner product in terms of the density fields. From (2.61) we already know that the λ-Jack and the Sutherland weights are related, however, something must be done about the diverging term. It turns out that the weight of the Sutherland inner product contains an additional counterterm [17], yielding

$$|\Delta(z)|^{2\lambda} \simeq C \exp\left(\lambda \int_0^{2\pi} \rho(\theta) \ln \rho(\theta) + \right.$$
$$\left. + \lambda \int_0^{2\pi} \int_0^{2\pi} \rho(\theta)\rho(\theta') \ln \left|2 \sin\left(\frac{\theta-\theta'}{2}\right)\right| d\theta d\theta'\right). \quad (2.66)$$

We have obtained an expression for the weights of the inner products but we still need to find the Jacobian arising from the $\{z_i\}$ to the $\{p_n\}$ transformation (we will call it $P(p_1, p_2, \ldots)$). This mapping is subtle, since we go from N variables to $2N$. According to [1], as the z_i move on their unit circles, each p_n moves as the endpoint of an N-step random walk in the complex plane. Thus, the large-N image of the z_n is dense on the C^N domain. Jevicki [18] shows that the Jacobian $P(p_1, p_2, \ldots)$ is an expansion in powers of $1/N$ of the density:

$$P(p_1, p_2, \ldots) \sim \exp\left\{\int_0^{2\pi} \left[-\frac{1}{2\rho_0}\rho'^2 + \frac{1}{6\rho_0^2}\rho'^3 - \frac{1}{12\rho_0^3}\rho'^4 + \ldots\right] d\theta\right\}. \quad (2.67)$$

Here $\rho' = \rho - \rho_0$ and $\rho_0 = N/2\pi$. Taking this result, [1] finds

$$\int_0^{2\pi} \left[-\frac{1}{2\rho_0}\rho'^2 + \frac{1}{6\rho_0^2}\rho'^3 + \ldots\right] d\theta =$$
$$= \int_0^{2\pi} [\rho_0 \ln \rho_0 - (\rho_0 + \rho') \ln(\rho_0 + \rho')] d\theta, \quad (2.68)$$

which allows the Jacobian to be written as

$$P(p_1, p_2, \ldots) \sim \exp\left\{-\int_0^{2\pi} \rho \ln \rho d\theta\right\}. \quad (2.69)$$

This Jacobian is of the same form as the counterterm needed to complete the Sutherland inner-product weight. We know that in the collective field formalism, the inner product integrals are replaced by functional integrals over the particle density,

$$\langle \Phi_1, \Phi_2 \rangle_{\text{Sutherland}} = \int \prod_i^N \frac{d\theta_i}{2\pi} \Delta^{2\lambda} \Phi_1^* \Phi_2 \quad \overrightarrow{n\to\infty} \int \delta\rho J[\rho] \Phi_1^*[\rho] \Phi_2[\rho]. \quad (2.70)$$

Combining (2.66) and (2.69), we find the weight $J[\rho]$ to be [19–21]

$$\begin{aligned} J[\rho] &= \exp\left((\lambda - 1) \int_0^{2\pi} \rho(\theta) \ln \rho(\theta) d\theta \right. \\ &\left. + \lambda \int_0^{2\pi} \int_0^{2\pi} \rho(\theta)\rho(\theta') \ln \left| 2 \sin\left(\frac{\theta - \theta'}{2} \right) \right| d\theta d\theta' \right). \quad (2.71) \end{aligned}$$

Only the second term is present in the collective-field formalism under the Jack inner product. We will refer to its weight as $K(\rho)$.

2.8 The Non-Chiral Hydrodynamic Model

The non-chiral hydrodynamic Hamiltonian of equation (2.57) is expressed in terms of the collective fields ρ and Π. However, we still don't know how the physical velocity relates to the field Π. From the form of the Hamiltonian, it looks as if the field $\partial_\theta \Pi$ would indeed be the velocity, but unfortunately $\partial_\theta \Pi$ does not satisfy the continuity equation. The seed of the problem lies in the fact that the inner product in our Hilbert space is weighted, both in terms of the $J[\rho]$ Sutherland or $K[\rho]$ Jack' inner product weights. Employing the Heisenberg picture of Quantum Mechanics, one can make a change of basis so as to absorb the weight of the inner product into the Hamiltonian. After this, the Hamiltonian will look different in this new weightless inner-product case, depending on whether we considered the Sutherland or the Jack' to be the meaningful inner product. Although the end results are the same, the order and logic of reasoning presented below differs slightly from that in [1].

We will see that there are two ways to find the physical velocity, and both are necessary to get a clear picture of the problem. The first method is to impose explicit Hermiticity of the Hamiltonian under the Sutherland inner product. Doing this yields a modified velocity operator which satisfies the continuity equation. The question remains, however, whether the same could be possible by explicitly enforcing the Jack' inner product. The answer is no, and we need the second method to understand why.

This alternative method involves ignoring the inner product issue for a moment, and calculating the velocity field purely from the continuity equation. After doing so, we can compare the Hamiltonians found through this method and the first one, and discover that we obtain the same results when we enforce the Sutherland inner product and not the Jack' one. Let's proceed and show the details.

2.8.1 Enforcing Sutherland's Inner Product

We make a change of basis in the Heisenberg picture, where the wavefunctions (i.e. basis vectors) remain unchanged but the operators change according to the transformation. Considering the Sutherland inner product weight $J(\rho)$ as the transformation, we have the following transformation on the Hamiltonian of (2.57):

$$H \to \sqrt{J} H (\sqrt{J})^{-1} \quad \Longrightarrow \quad \begin{cases} \Pi & \to \quad \sqrt{J}\Pi(\sqrt{J})^{-1} \\ \rho & \to \quad \sqrt{J}\rho(\sqrt{J})^{-1} \quad = \rho. \end{cases} \tag{2.72}$$

Thus, only the Π operator changes. Using its definition, we can calculate the new transformed operator Π' as

$$\begin{aligned} \Pi \to \Pi' &= \sqrt{J}\Pi(\sqrt{J})^{-1} = \sqrt{J}\left(-\frac{1}{2}\frac{1}{J^{3/2}}\frac{\partial J}{\partial \rho} + \frac{1}{\sqrt{J}}\frac{\partial}{\partial \rho}\right) \\ &= \Pi - \frac{1}{2}\frac{\partial \ln J}{\partial \rho}. \end{aligned} \tag{2.73}$$

The new Hamiltonian contains the operator $\partial_\theta \Pi'$, which satisfies

$$\partial_\theta \Pi \to \partial_\theta \Pi' = \partial_\theta \Pi - \frac{1}{2}\partial_\theta \frac{\partial \ln J}{\partial \rho}. \tag{2.74}$$

Leaving the complete details of the calculation to appendix A.7, we only outline the main results. We have

$$\frac{\partial}{\partial_\theta}\frac{\partial \ln J}{\partial \rho} = (\lambda-1)\frac{\partial_\theta \rho}{\rho} + \lambda \int_0^{2\pi} \rho(\theta') \cot\left(\frac{\theta-\theta'}{2}\right) d\theta', \qquad (2.75)$$

so we can write the transformed operator as

$$\partial_\theta \Pi \rightarrow \partial_\theta \Pi - \frac{(\lambda-1)}{2}\frac{\partial_\theta \rho}{\rho} + \frac{\lambda}{2}\int_0^{2\pi} \rho(\theta') \cot\left(\frac{\theta-\theta'}{2}\right) d\theta'. \qquad (2.76)$$

We will define the velocity (which will turn out to be the physical one) as $v = \partial_\theta \Pi$. We need to introduce this into the Hamiltonian (2.57). Doing so, we obtain the known [16] excitation Hamiltonian for the non-chiral case in terms of the physical velocity v and the density ρ,

$$\begin{aligned} H &= \int \left(\frac{1}{2}\rho v^2 + \frac{\pi^2\lambda^2}{6}\rho^3 + \frac{(\lambda-1)^2}{8}\frac{(\partial_\theta \rho)^2}{\rho}\right) d\theta \\ &\quad + \frac{\lambda(\lambda-1)}{8}\iint \frac{[\rho(\theta)-\rho(\theta')]^2}{2\sin^2(\theta-\theta')/2} d\theta d\theta'. \end{aligned} \qquad (2.77)$$

This is the real-space Hamiltonian of the Sutherland model in the continuum limit, which is now Hermitian under the usual canonical Quantum Mechanical inner product (i.e. weightless) since the Sutherland weight has been absorbed. The Hamiltonian looks as a fluid in terms of the field operators ρ and v. The first term is just the kinetic energy of the fluid, the second term can be seen as a statistical term in nature[3]. The third term comes from a Pitaevski like non-linear Schroedinger equation, and the last term is a Coulomb-type interaction term except that it has been regularized [16].

2.8.2 Physical Velocity and the Continuity Equation

An alternative derivation of the physical Hamiltonian (2.77) involves finding the physical velocity through the continuity equation applied to the Hamiltonian (2.57). The Heisenberg equation of motion for the density field tells

[3]For $\lambda = 1$, this term is the usual potential energy coming from Fermi statistics and the fact that electrons can't occupy same energy levels, relating a given density to the magnitude of the Fermi energy.

us that

$$
\begin{aligned}
\dot{\rho} = i[H,\rho] = i\int d\theta' \Bigg(& -\frac{\rho(\theta')}{2}[(\partial_{\theta'}\Pi)^2\,,\rho(\theta)] \\
& +\frac{(1-\lambda)}{2}\partial_{\theta'}\rho(\theta')[\partial_\theta \Pi(\theta'),\rho(\theta)] \\
& + \lambda\rho(\theta')\int \cot\left(\frac{\theta'-\theta''}{2}\right)\rho(\theta'')[\partial_{\theta'}\Pi(\theta'),\rho(\theta)]d\theta''\Bigg).
\end{aligned}
\tag{2.78}
$$

The following commutators are useful in our calculation:

$$
\left\{
\begin{array}{lcl}
[\partial_{\theta'}\Pi\partial_{\theta'}\Pi,\rho(\theta)] & = & 2\partial_{\theta'}\delta(\theta-\theta')\partial_{\theta'}\frac{\delta}{\delta\rho(\theta')} \\
{[\partial_{\theta'}\Pi,\rho(\theta)]} & = & \partial_{\theta'}\delta(\theta-\theta').
\end{array}
\right.
\tag{2.79}
$$

This allows us to write

$$
\begin{aligned}
\dot{\rho} &= i\int d\theta' \left(-\rho(\theta')\partial_{\theta'}\delta(\theta-\theta')\partial_{\theta'}\frac{\delta}{\delta\rho(\theta')} + \frac{(1-\lambda)}{2}\partial_{\theta'}\rho(\theta')\partial_{\theta'}\delta(\theta-\theta') \right. \\
&\quad \left. + \frac{\lambda}{2}\rho(\theta')\int \cot\left(\frac{\theta'-\theta''}{2}\right)\rho(\theta'')\partial_{\theta''}\delta(\theta''-\theta)d\theta''\right) \\
&\quad - \frac{i\lambda}{2}\int \rho(\theta')\partial_{\theta''}\left[\cot\left(\frac{\theta'-\theta''}{2}\right)\rho(\theta'')\right]\delta(\theta''-\theta)d\theta' d\theta''\Bigg) \\
&= i\partial_\theta\left(\rho\partial_\theta\frac{\delta}{\delta\rho}\right) + \frac{i(\lambda-1)}{2}\partial_\theta^2\rho + \frac{i\lambda}{2}\partial_\theta\Bigg(\rho\underbrace{\int\cot\left(\frac{\theta-\theta'}{2}\right)\rho(\theta')d\theta'}_{\rho_{\mathrm{H}}}\Bigg) \\
&= i\nabla\left[\rho(\partial_\theta\frac{\delta}{\delta\rho} + \frac{\lambda-1}{2}\partial_\theta\ln\rho + \frac{\lambda}{2}\rho_{\mathrm{H}})\right].
\end{aligned}
\tag{2.80}
$$

Thus, the continuity equation takes the form

$$
\nabla\left(\rho i\left(\partial_\theta\frac{\delta}{\delta\rho} + \frac{\lambda-1}{2}\partial_\theta\ln\rho + \frac{\lambda}{2}\rho_{\mathrm{H}}\right) + \rho v\right) = 0 \tag{2.81}
$$

from where we can read the physical velocity

$$
v_{\mathrm{phys}} \equiv v = -i\partial\frac{\delta}{\delta\rho} + -i\frac{\lambda}{2}\rho_{\mathrm{H}} + i\frac{(1-\lambda)}{2}\nabla\ln\rho, \tag{2.82}
$$

which coincides with the one defined in [1]. Knowing the relationship between the physical velocity and the density, we can use the last equation to write

$$
\partial_\theta\Pi = iv_{\mathrm{phys}} - \frac{\lambda}{2}\rho_{\mathrm{H}} - \frac{(\lambda-1)}{2}\nabla\ln\rho. \tag{2.83}
$$

Note that this equation is the same as (2.76), which we had derived by imposing Sutherland Hermiticity. Had we imposed Jack' Hermiticity by using the $K[\rho]$ inner-product weight, the resulting velocity would be missing a term, and thus would not be physical. Thus, we confirm that the Sutherland inner product is the one that must be enforced. As a consistency check, if we use (2.83) and recycle the previous calculation from equation (2.76) onwards, we arrive at the same Hamiltonian in (2.77).

2.9 Non-Chiral Current Decomposition?

We would like to obtain an expression analogous to equation (2.36) for the non-chiral case, with the left and right going modes decoupled. After all, the "lifted" FQHE 2D system can be separated in terms of left and right going currents on the edges. However, we encounter an unsolvable problem: If we define the chiral currents j_R, j_L in terms of a velocity operator that is Hermitian under the Jack' inner product,

$$j_{R,L} = \frac{1}{2}\left(\rho \pm \frac{v}{\lambda\pi}\right), \tag{2.84}$$

where

$$v(\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \underbrace{2\pi\left(-n\frac{\partial}{\partial p_{-n}} + \frac{\lambda}{2}\mathrm{sgn}(n)p_n\right)e^{-in\theta}}_{v_n}, \tag{2.85}$$

we find that

$$\begin{aligned} [j_{R,n}, j_{R,m}] &= m\nu\delta_{m+n,0}, &(2.86)\\ [j_{L,n}, j_{L,m}] &= -m\nu\delta_{m+n,0}, &(2.87)\\ [j_{L,n}, j_{R,m}] &= 0, &(2.88)\end{aligned}$$

and so the left and right currents are cleanly separated. However, when we try to express the Hamiltonian in terms of decoupled left and right chiral sectors,

$$H = 4\pi^2 \int_0^{2\pi} \left\{ \frac{\lambda^2}{6}(j_R^3 + j_L^3) - \frac{\lambda(\lambda-1)}{8\pi}\left(j_R\partial(j_R)_H + j_L\partial(j_L)_H\right) \right\} d\theta, \quad (2.89)$$

we fail (see Stone et. al [1] page 19). This is not surprising, though, since we've already seen that the "good" inner product is the Sutherland one. If we try the alternative route, and consider a definition of the current in terms of the physical velocity found before, we can write the Hamiltonian in the form of equation (2.89). Unfortunately, these new currents are non-commuting, which implies that an initially zero J_L does not remain zero. Thus, it seems it is not possible to decompose the Hamiltonian in terms of decoupled left and right going blocks for the non-chiral case. We will get a second shot at this problem through a different perspective when we consider a classical approach to the Sutherland model (chapter 3), and although the outcome will remain the same, we will gain a better understanding as to why this occurs.

A different question is the condition that the collective fields must satisfy in order to make the current chiral. This should be a solvable problem, since one could experimentally produce a chiral current and then find the properties and relationships that the fields satisfy. Rightly so, Bettelheim, Abanov and Wiegmann [22] have found the right-going chiral constraint to be

$$v_{\text{physical}} = \pi\lambda\rho - \frac{1}{2}(\lambda-1)\partial_\theta(\ln\rho)_H. \quad (2.90)$$

This constraint allows one to recover the chiral EOM starting from the non-chiral one. To show this, we start from the hydrodynamic EOMs that are satisfied by the non-chiral collective field Hamiltonian in (2.77):

$$\begin{cases} \dot\rho + \frac{\kappa}{2}\left(\rho^2 + \frac{a}{2\pi}\rho\nabla(\log\rho)_{\mathrm{H}}\right) = 0 & \text{(Continuity equation)}, \\ \dot v + \nabla\left(\frac{v^2}{2} + w\right) = 0 & \text{(Euler equation)}, \end{cases} \quad (2.91)$$

where

$$w = \frac{\kappa^2}{(2\pi)^2}\frac{\delta(\rho\epsilon)}{\delta\rho}, \quad \rho\epsilon = \frac{\pi^2}{6}\rho^3 + \frac{a^2}{8}\frac{(\nabla\rho)^2}{\rho} + \frac{1}{2}\pi a\rho\nabla\rho_H. \quad (2.92)$$

Using the chiral constraint condition on the velocity together with the continuity equation, it can be shown that the Euler equation for the non-chiral case collapses to the BO equation (2.37) of the chiral case after linearization. This requires some considerable algebra and the use of the Tricomi's version

$$(\phi_1(\phi_2)_H)_H + ((\phi_1)_H\phi_2)_H = (\phi_1)H(\phi_2)_H - \phi_1\phi_2 \tag{2.93}$$

of the Poincare-Bertrand identity [23]. Since the details of this calculation are not shown in [22], we have included them in full in Appendix A.8 for instructional purposes.

To sum up the results reviewed so far, under the Quantum Mechanical approach we can derive a hydrodynamic model of the Calogero-Sutherland model. We find that in the general non-chiral case, out of the two inner products available only the Sutherland one provides us with physical variables. Unlike the chiral case, it seems it is not possible to decompose the Hamiltonian into decoupled sectors. However, it is possible to find a condition on the fields for pure chiral evolution and recover the results of the chiral case. Although correct, this condition is rather mysterious and unintuitive. The next chapter will help us understand this condition in a much simpler way.

3 Classical Approach to the Calogero-Sutherland Model

The answers to some questions under the Quantum Mechanical approach are not simple. For example, it is not obvious how we could define left and right currents so that we can decompose the Hamiltonian in a form such as the chiral one of (2.36), or whether this is possible at all. Additionally, in the Quantum approach, the condition on the density and velocity fields for pure left and right-going chiral motion comes from the need to convert operators that are Hermitian with respect to one of the two natural inner products of the Calogero-Sutherland Hilbert space, into operators that are Hermitian with respect to the other [1]. This, although technically satisfactory, doesn't provide a physical picture as to why it happens.

A different point of view will be obtained by studying a classical model based on the Quantum Benjamin-Ono equation. Its explicit solutions will reveal a direct connection to the Calogero-Sutherland model, both at the equation of motion and the Hamiltonian level, and the understanding of these solutions will allow us to obtain the Quantum Mechanical results in an arguably simpler way. The following reviews the work presented in [3], with some added comment and proofs to compliment the original work.

3.1 Toy Model: Classical 1D Fermi Gas

In order to understand the motivation behind the classical Calogero hydrodynamic approach, it's best if we start by looking at the well known one-dimensional gas of spinless, unit-mass fermions. This system can be represented by the simplest Galilean-invariant Hamiltonian

$$H_{\text{fluid}} = \int \left\{ \frac{1}{2}\rho v^2 + \frac{\lambda^2 \pi^2}{6}\rho^3 \right\} dx. \tag{3.1}$$

Here v is the fluid velocity and λ is a parameter that generalizes the model for later comparison (for fermions, $\lambda = 1$). We can satisfy the continuity equation if we introduce a Poisson bracket in our function space,

$$\{\theta(x,t), \rho(x',t)\} = \delta(x - x'), \tag{3.2}$$

where θ is the velocity potential $v = \partial_x \theta$. The Hamiltonian can be rewritten as

$$H_{\text{fluid}} = \frac{1}{2\lambda\pi} \int \left\{ \frac{1}{6}(v + \lambda\pi\rho)^3 - \frac{1}{6}(v - \lambda\pi\rho)^3 \right\} dx, \tag{3.3}$$

and the equations of motion $\dot{\rho} = \{H_{\text{fluid}}, \rho\}$ and $\dot{v} = \{H_{\text{fluid}}, v\}$ can be similarly rearranged as

$$\partial_t(v + \lambda\pi\rho) + (v + \lambda\pi\rho)\partial_x(v + \lambda\pi\rho) = 0 \tag{3.4}$$

$$\partial_t(v - \lambda\pi\rho) + (v - \lambda\pi\rho)\partial_x(v - \lambda\pi\rho) = 0. \tag{3.5}$$

We see that the Hamiltonian can be decomposed into two non-interacting sectors by defining the right-going and left-going Riemann invariants $I_{R,L}(x) = (v \pm \lambda\pi\rho)$, which are proportional to the chiral currents $j_{R,L} = 1/2(\rho + v/\lambda\pi)$. The Riemann invariants have vanishing crossed Poisson commutators

$$\{I_{R,L}(x), I_{R,L}(x')\} = \pm 2\pi\lambda\partial_x\delta(x - x') \tag{3.6}$$

$$\{I_R(x), I_L(x')\} = 0. \tag{3.7}$$

Until here, it seems we have gained little from our analysis. However, the novelty comes from noting that (3.4) and (3.5) can be combined and rewritten in terms of a complex-valued field u that satisfies discontinuous boundary conditions on the real axis,

$$\partial_t u + u\partial_z u = 0, \quad \text{with} \quad \lim_{\epsilon \to 0} u(x \pm i\epsilon) = v \pm \lambda\pi\rho, \tag{3.8}$$

where the boundary conditions on the real x-axis relate u to the physical fields ρ and v. Note the similarity between this equation and (2.37) of the Quantum chiral Calogero model, this free fermion model being a simplified version of the Calogero one. An extension of (3.8) will be obtained in the next section that represents the Calogero model instead of the free Fermi gas.

3.2 The CS - BO Mapping

Unlike the quantum mechanical treatment, in which we start from the discrete CS Hamiltonian of equation (2.1) and work to obtain the continuum limit expressions, here the starting point will be the QBO equation (2.2), which for simplicity we reproduce below:

$$\dot{u} + uu_x = \frac{1}{2}\lambda[u_{xx}]_H, \quad \text{where} \quad f_H(x) \equiv \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{1}{x-\xi} f(\xi) d\xi. \tag{3.9}$$

This is a non-linear and non-local partial differential equation, where the non-locality is given by the Hilbert transform on the real line. Although it is important on its own right, it can also be viewed as an extension of (3.8) of the classical toy model presented above. Before attempting to find its solutions, we can introduce a Poisson bracket similar to the previously introduced in (3.2),

$$\{u(x), u(x')\} = 2\lambda\pi\partial_x\delta(x - x'). \tag{3.10}$$

This allows us to write (3.9) as the usual Heisenberg time evolution for the field u[1], that is, $\dot{u}(x,t) = \{H_{\mathrm{BO}}, u(x,t)\}$, where H_{BO} is a Hamiltonian given

[1]This is a general property of any Poisson bracket. It can be shown that $\{,\}$ as defined above is indeed a Poisson bracket by satisfing the necessary Jacobi identity $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ and antisymmetry property.

by [2]

$$H_{\mathrm{BO}} = \frac{1}{2\pi\lambda}\int_{-\infty}^{\infty}\left(\frac{1}{6}u^3 - \frac{\lambda}{4}u(u_x)_H\right)dx. \tag{3.11}$$

This Hamiltonian, together with the Poisson bracket, is equivalent to the QBO equation and will prove useful later on.

3.3 Pole Ansatz Solutions

To obtain solutions to the QBO equation, we try an ansatz solution that is a sum of poles

$$u(x,t) = \sum_{j=1}^{N}\frac{i\lambda}{x - a_j(t)} - \sum_{j=1}^{M}\frac{i\lambda}{x - b_j(t)}, \tag{3.12}$$

where the poles at $a_j(t), j = 1, \ldots, N$ lie below the real axis, and the poles $b_j(t), j = 1, \ldots, M$ lie above it. Note that N and M do not need to be equal, and that the $\{a_j,\, b_j\}$ are not necessarily real for the real line Hilbert transform contour in 3.9 (unlike Chen et al. [24], we do not set $b_i = a_i^*$ since we are not interested in a real $u(x,t)$ but instead in real $a_i(t)$). Under this contour, the poles satisfy the Hilbert transform property

$$H\left(\frac{1}{z-a_i}\right) = \frac{-i}{z-a_i} \quad \text{and} \quad H\left(\frac{1}{z-b_i}\right) = \frac{i}{z-b_i}, \tag{3.13}$$

for a_i above the real axis and b_i below (H denotes the Hilbert transform). Plugging the ansatz into (3.9) (see Appendix B.1 for full derivation), the a_j

[2]The QBO equation is known to be infinitely integrable, and the method of Lax pairs provides a mechanism for retrieving any of the integrable constants of motion. The first of these is the momentum, while the second is the energy, or Hamiltonian.

and b_j can be shown to satisfy,

$$i\dot{a}_j = \sum_{k\neq j}^{N} \frac{\lambda}{a_k - a_j} - \sum_{k=1}^{M} \frac{\lambda}{b_k - a_j}, \tag{3.14}$$

$$i\dot{b}_j = \sum_{k\neq j}^{N} \frac{\lambda}{b_j - b_k} - \sum_{k=1}^{M} \frac{\lambda}{b_j - a_k}. \tag{3.15}$$

We see that the velocities depend on the values of both a_j and b_j. The acceleration can be computed from the above using straightforward algebra (details in Appendix B.2), yielding

$$\ddot{a}_j = \sum_{k\neq j} \frac{2\lambda^2}{(a_j - a_k)^3} \qquad \forall j = 1, \ldots, N \tag{3.16}$$

Remarkably, we find that the dynamics of the $\{a_j\}$ are self-determined, and the only role of the $\{b_j\}$ is to set the initial complex velocities $\dot{a}_j(0)$ of the poles. The N equations in (3.16) determining the accelerations are a complex version of the Calogero model equations of motion. For the a_i-real case, they can be derived from the Calogero Hamiltonian (2.1), or alternatively from the many-body Calogero Lagrangian

$$L_{\text{Calogero}} = \frac{1}{2}\sum_{i=1}^{N} \dot{a}_i^2 - \sum_{i<j} \frac{\lambda^2}{(a_i - a_j)^2}. \tag{3.17}$$

The same procedure can be repeated when searching for periodic solutions. These can be obtained by making an infinity train of poles spaced by 2π. The resulting acceleration is

$$\ddot{a}_j = -\frac{\partial}{\partial a_j}\left\{\frac{1}{4}\sum_{k\neq j} \frac{\lambda^2}{\sin^2(a_j - a_k)/2}\right\}, \tag{3.18}$$

which is again the same EOM derived from the Sutherland model Hamiltonian H_{sut} (2.1).

The main difference between the $\{a_j\}$ poles introduced in our ansatz solution and the $\{x_j\}$ of the Calogero model is that the former can be complex valued, while the latter are necessarily real. We must remember that the QBO equation is incomplete without a specification of the Hilbert transform contour. Different Hilbert transform contours than the real line one used in (3.9) will change the relationship between $\{a_j\}$ and $\{b_j\}$, and thus ultimately the evolution of the $\{a_j\}$. We need a contour that allows us to have $\{a_j(0)\} \in \Re$ while at the same time satisfying the Hilbert transform property (3.13), necessary to derive the $\{b_j\}$-independent acceleration for the $\{a_j\}$. Following Abanov and Wiegmann [2], the contour we need is shown in Figure 3.1, which results

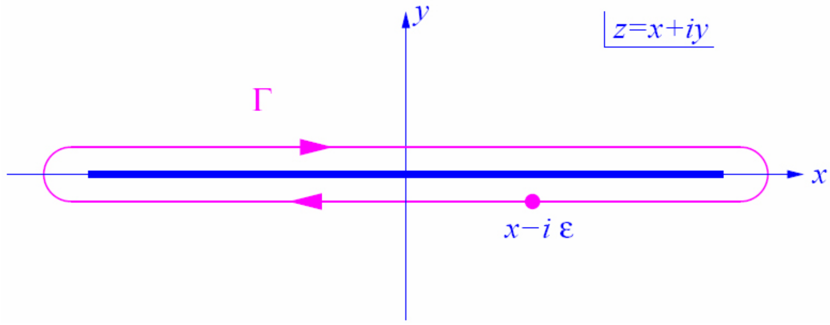

Figure 3.1: The Γ contour wraps around the $\{a_i\}$ poles that represent the Calogero particles and lie on the real axis. The soliton shepherd poles $\{b_i\}$ are outside of the contour.

in the following Hilbert transform:

$$u_\Gamma(z) \equiv \frac{\mathrm{P}}{\pi} \oint_\Gamma \frac{1}{z-\psi} d\psi. \tag{3.19}$$

Here Γ is the simple closed and positively oriented contour upon which z lies, as shown in Figure 3.1. This way, if we can put the $a_j(0)$ on the x-axis and distribute the b_j over the rest of the complex plane so that $\dot{a}_j(0) \in \Re$, then the Calogero evolution given by (3.16) ensures that the $\{a_j\}$ stay real. Thus, we arrive at the realization that a complex field $u(z,t)$ obeying the BO equation on the Γ contour can provide real-axis Calogero-Sutherland dynamics.

3.4 Excitations in the Continuum Limit

So far, we have managed to represent a discrete set of Calogero particles as the pole solutions of the complex-valued field $u(z)$ that satisfies the BO equation. However, in order to reproduce the results found in the Quantum mechanical treatment, we need to have a continuum of Calogero particles. The easiest way is to modify the velocity equations in (3.15) for the $\{a_j, b_j\}$ poles and upgrade them to continuous fields. However, it is not immediately obvious how to do this. Following [3], we will look at the simplest "naive" approach and study its consequences, only to realize that it is not sufficient. We will then review how this can be improved upon to turn it into a consistent model.

3.4.1 The Simplest Continuum Extension

We start by making the simplest extension in the velocities of (3.15),

$$i\dot{a}(x,t) = P\int_{-\infty}^{\infty} \frac{\lambda}{\xi - x}\rho(\xi,t)d\xi - \sum_k \frac{\lambda}{b_k - x}, \tag{3.20}$$

$$i\dot{b}_j(t) = \sum_{k;k\neq j} \frac{\lambda}{b_j - b_k} - \int_{-\infty}^{\infty} \frac{\lambda}{b_j - \xi}\rho(\xi,t)d\xi, \tag{3.21}$$

where $\dot{a}(x,t) \equiv v(x,t)$ is the velocity of the pole at x and represents our physical field. We have chosen to keep a discrete number of $\{b_j\}$ poles, which we are free to do since the N and M indexes are independent, as it turns out that this is sufficient to shepherd the $a(x,t)$ poles properly (i.e. keep their velocities real). Note that these equations are linear in the densities, so naturally a superposition of solutions will remain a solution.
Since we know from the Quantum mechanical treatment that the Calogero model supports solitonic solutions, it is not unreasonable to expect the same for our new fields. We consider a density profile of the form

$$\rho(x,0) = \rho_0 + \rho_1(x), \qquad \text{with} \quad \rho_1(x) = \left(\frac{A}{\pi}\right)\frac{1}{x^2 + A^2}, \tag{3.22}$$

where $\rho_1(x)$ corresponds to an excess of one particle near $x = 0$ (ρ_1, the Lorentzian density integrates to unity over the entire real axis). Under this proposed density, the velocity evolution of the Calogero poles, given by (3.21), becomes

$$i\dot{a}(x,t) = -\frac{\lambda x}{x^2 + A^2} - \sum_k \frac{\lambda}{b_k - x}. \tag{3.23}$$

Note that the density fluctuation tends to push $a(x)$ off the x-axis, so we must compensate for this by choosing a suitable combination of b poles. It turns out that a single b-pole at $b = iA$ is enough:

$$i\dot{a}(x,t) = -\frac{\lambda x}{x^2 + A^2} + \frac{\lambda}{x - iA} = \frac{iA\lambda}{x^2 + A^2}. \tag{3.24}$$

Thus, the a poles have a purely real initial velocity

$$v_{\text{pole}}(x,0) = \dot{a}(x,0) = \frac{A\lambda}{x^2 + A^2}, \tag{3.25}$$

and from what we've seen previously in (3.18), remains real over time. On the other hand, the evolution of the $b = iA$ pole can be found also by using (3.21),

$$\begin{aligned} i\dot{b} &= -\int_{-\infty}^{\infty} \frac{\lambda}{b - \psi} \left(\frac{A}{\pi}\right) \frac{1}{\psi^2 + A^2} d\psi - \int_{-\infty}^{\infty} \frac{\lambda \rho_0}{b - \psi} d\psi \\ &= -\frac{\lambda}{b + iA} + i\lambda\pi\rho_0 \\ &= \frac{i\lambda}{2A} + i\lambda\pi\rho_0. \end{aligned} \tag{3.26}$$

Thus, the b-pole velocity is

$$\dot{b} = \frac{\lambda}{2A} + \lambda\pi\rho_0, \tag{3.27}$$

which is also purely real. Thus, we can interpret the b-poles as a shepherd [3]: they move parallel to the real axis and track the soliton excitation. We conclude that the proposed density profile (3.22) is a constant-shape soliton with motion

$$\rho(x,t) = \rho_0 + \left(\frac{A}{\pi}\right) \frac{1}{(x - v_{\text{soliton}} t)^2 + A^2}, \tag{3.28}$$

and speed $v_{\text{soliton}} = \dot{b}$. Note that the amplitude and direction of the soliton are controlled by the magnitude and sign of the imaginary part of the b

shepherd pole respectively.
Since the pole evolution in (3.21) is linear, we might wonder how to represent multi-soliton configurations. Before we tackle this question, it is worth to note that the evolution of the soliton (3.28) is completely characterized by the position of the shepherd pole b. Moreover, the evolution of the shepherd pole was obtained by using the continuum velocities in (3.21). However, the same end result for the b-pole evolution can be achieved if we consider the *discrete* system in (3.18), whose velocities are given by (3.15), and take the a-pole distribution to be $a = b^*$:

$$\begin{aligned} i\dot{b}_j &= \sum_{k\neq j}^{N} \frac{\lambda}{b_j - b_k} - \sum_{k=1}^{M} \frac{\lambda}{b_j - a_k} = -\frac{\lambda}{b_j - b_j^*} \\ &= -\frac{\lambda}{2\mathrm{Im}(b_j)} = \frac{i\lambda}{2A}, \end{aligned} \tag{3.29}$$

where we set $\{a_j\} = a_j = b_j^*$ and also set $b = iA$ on the second line. This way, we see that the continuum chiral soliton problem coincides with the Benjamin-Ono conventional multisoliton solutions[3]. Thus, the recipe to establish a right-chiral multisoliton system is to place for every soliton a shepherd pole b above the real axis, where $\mathrm{Im}(b) = A$, and the real part of b locates the center of the soliton.
Unfortunately, our solutions are not entirely satisfactory, since they fail to satisfy the continuity equation, given by

$$\dot{\rho} + \partial_x(\rho v_{\text{pole}}) = 0. \tag{3.30}$$

To show this, we use the fact that the soliton density is a wave, $\rho = \rho(x - v_{\text{soliton}}t)$ to relate the time and space derivatives,

$$\partial_t \rho = -v_{\text{soliton}} \partial_x \rho, \tag{3.31}$$

which can be applied to the continuity equation, yielding the condition

$$\partial_x\{\rho(v_{\text{pole}} - v_{\text{soliton}})\} = 0 \qquad \Longrightarrow \qquad \rho(v_{\text{pole}} - v_{\text{soliton}}) = C, \tag{3.32}$$

[3]This fact is employed to run numerical simulations showing interactions between solitons, since it's much easier to numerically solve (3.15) than (3.21), even though the results are equivalent.

where C is a constant. Because this relationship must be valid for the entire x-axis, we can consider large distances, where $\rho \to \rho_0$ and $v_{\text{pole}} \to 0$, to find that $C = \rho_0 v_{\text{soliton}}$, which leads to the necessary condition

$$v_{\text{pole}} = \frac{\rho - \rho_0}{\rho} v_{\text{soliton}}. \tag{3.33}$$

We had previously seen from the explicit solutions that the density and the a-pole velocity are linked by $v_{\text{pole}}(x) = \lambda\pi(\rho(x) - \rho_0)$. However, this would require that $\lambda\pi = v_{\text{soliton}}/\rho$, which is not correct since ρ varies with position. Thus, our solutions don't satisfy the continuity equation. We will see how this can be solved on the next section.

3.4.2 The Non-Linear Correction

The problem with the above approximation lies in the naive continuum approximations (3.20, 3.21). According to [3], a non-linear correction must be introduced into the latter to correctly describe the continuum limit of (3.15), namely

$$\sum_{k;k\neq j} \frac{\lambda}{a_k - a_j} \sim P\int_{-\infty}^{\infty} \frac{\lambda}{\xi - a_j}\rho(\xi,t)d\xi - \frac{\lambda}{2}\partial_x \ln \rho(x)|_{x=a_j}. \tag{3.34}$$

The $\partial_x \ln \rho$ term is but the first order correction in an asymptotic series in gradients of ρ based on the Euler-Maclaurin expansion (see Appendix in [3] for a derivation). This first order correction is enough to produce self-consistent solutions that satisfy the continuity equation. Unfortunately, the non-linear nature of the new velocity equations prevents us from finding multisoliton solutions by superposition as done in the previous section. It is possible however, to find simple soliton solutions, just as in the linear case. If we start with the previously studied density in (3.22), and look at the pure a-pole contribution to the $\dot{a}$ velocity, we find

$$\begin{aligned} P\int_{-\infty}^{\infty} \frac{\lambda}{\psi - x}\rho(\psi,0)d\psi - \frac{\lambda}{2}\partial_x \ln \rho &= -\frac{\lambda x}{x^2 - A^2 + A/(\pi\rho_0)} \\ &= -\frac{\lambda x}{x^2 + B^2}, \end{aligned} \tag{3.35}$$

where

$$B = \sqrt{A^2 + \frac{A}{\pi \rho_0}}. \tag{3.36}$$

Thus, the a-pole velocity expression is analogous to the previous section, with the b-pole needing to be placed at $b = iB$ instead of $b = iA$ in order to keep the $\{\dot{a}\}$ real. The a-pole velocity, including the new b-pole contribution, thus becomes

$$i\dot{a}(x) = P\int_{-\infty}^{\infty} \frac{\lambda}{\psi - x}\rho(\psi, 0)d\psi - \frac{\lambda}{2}\partial_x \ln \rho + \frac{\lambda}{x - iB} = \frac{iB\lambda}{x^2 + B^2}. \tag{3.37}$$

The new pole velocity, which is the extension of (3.25) for the non-linear case, reads

$$v_{\text{pole}}(x) = \dot{a}(x) = \frac{B\lambda}{x^2 + B^2}. \tag{3.38}$$

We should revisit the continuity equation with our new solution. Since the form of ρ is unchanged, the condition in (3.33) remains. However, using the new v_{pole} explicit solution allows us to write $v_{\text{soliton}} = \rho_0 \lambda \pi (B/A)$. This velocity is x-independent, as it should. Lastly, the b-pole velocity $\dot{b}$ can be calculated and show to be equal to v_{soliton}. Thus, the picture of the b-pole as a shepherd pole that tracks the physical soliton remains valid in the non-linear case.

3.4.3 Rederiving The Hydrodynamic Model

So far under the classical formulation, we have found that a field u satisfying the QBO equation 3.9 can be written in terms of new fields a,b that satisfy the Calogero-Sutherland equation. In the continuum limit, we have found specific excitation solutions as a coherent superposition of these fields and made them consistent through a non-linear correction. However, we still haven't recovered a Hamiltonian, or an equation of motion, in terms of the physical density and velocity fields. As a guide to how we should proceed, we can look back at the toy model of section 3.1, in which starting from the system's Hamiltonian, arrived at an equation of motion for a field u on the complex plane, akin to the QBO equation together with certain boundary conditions.

The starting step is thus to find the boundary conditions that the field u satisfies in terms of the density and velocity fields. For this, and following [2], we decompose the field u into two parts, depending on which eigenvalue the part has wrt. the Hilbert transform on Γ, just as we did when we found the solutions in the previous section. The decomposition reads

$$u(z,t) = u_+(z,t) + u_-(z,t), \tag{3.39}$$

$$u_-(z,t) = \sum_{j=1}^{N} \frac{i\lambda}{z - a_j(t)}, \tag{3.40}$$

$$u_+(z,t) = \sum_{j=1}^{M} \frac{-i\lambda}{z - b_j(t)}. \tag{3.41}$$

The $u_\pm(z,t)$ are eigenfunctions of the Γ-contour Hilbert transform with eigenvalues $\pm i$ respectively. In the hydrodynamic limit, u_- becomes

$$u_-(z,t) = i\int_{-\infty}^{\infty} \rho(\xi,t)\frac{\lambda}{z-\xi}d\xi, \tag{3.42}$$

which, by its definition, is discontinuous on the real axis[4]:

$$u_-(x+i\epsilon) - u_-(x-i\epsilon) = 2\lambda\pi\rho(x). \tag{3.43}$$

This yields the boundary condition

$$u(x \pm i\epsilon) = \lambda\pi(i\rho_H \pm \rho). \tag{3.44}$$

With the local correction included, the a-pole velocity $v \equiv \dot{a}(x,t)$ reads $v = i\lambda\pi\rho_H + i\lambda/2\partial_x \ln\rho + u_+$, from which we get the second boundary condition

$$u_+(x,t) = v - i\lambda\pi\rho_H - \frac{i\lambda}{2}\partial_x \ln\rho \tag{3.45}$$

[4]The integrals for the upper and lower plane limits can be calculated using Cauchy's theorem over semicircles with counterclockwise and clockwise orientations respectively.

Thus, the field u has boundary conditions

$$\lim_{\epsilon\to 0} u(x \pm i\epsilon, t) = v \pm \lambda\pi\rho - \frac{i\lambda}{2}\partial_x \ln\rho. \tag{3.46}$$

This is a generalization of (3.8) from the free fermion toy model to the CS model. The next question is, can we go further, and provide a Hamiltonian involving the ρ, v fields through the field $u(z,t)$? The answer is satisfactory. If we consider the complex-plane extended version of equation (3.9) on the Γ contour of Figure 3.1, the Hamiltonian (3.11) becomes

$$H_\gamma = \frac{1}{2\lambda\pi}\oint_\Gamma \left\{ \frac{1}{6}u^3 - \frac{\lambda}{4}u(\partial_z u)_\Gamma \right\} dz. \tag{3.47}$$

If we deform the Γ contour towards the real axis, and use the boundary conditions satisfied by $u(x \pm i\epsilon, t)$ in (3.46), after some algebra the Hamiltonian becomes the same as the one in (2.77), that is,

$$H_{\text{hydro}} = \int_{-\infty}^{\infty} \left\{ \frac{1}{2}\rho v^2 + \frac{\lambda^2\pi^2}{6}\rho^3 - \frac{\lambda^2\pi}{2}\rho(\partial_x\rho)_H + \frac{\lambda^2}{8}\frac{(\partial_x\rho)^2}{\rho} \right\} dx. \tag{3.48}$$

Thus, we see that this classical formulation is able to recover the hydrodynamic model obtained through the Quantum mechanical approach[5]. However, we have not tackled the remaining questions that the latter approach posed. For example, is it possible to decompose the Hamiltonian (3.47) as the sum of non-interacting currents? The $u(x \pm i\epsilon, t)$ play the same role as the Riemann invariants of the chiral model, defined in (3.7), since they have vanishing Poisson brackets. At first sight, it might seem that this Hamiltonian already provides a suitable decomposition into these two currents, since the integral over Γ is naturally divided into an upper-plane and lower-plane part. However, the $(\partial_z u)_\Gamma$ factor breaks the illusion, since it couples the two branches dynamically [6]. Just like in the Quantum mechanical formalism, we see that an initially zero $u(x - i\epsilon)$ does not remain so in time. Therefore, we conclude that the Hamiltonian (3.47) is the closest we can get to a real

[5]Polychronakos states [16] that it is possible to derive the classical hydrodynamic limit by "simply" taking the limit $\hbar \to 0$ from the Quantum mechanical collective field formalism instead of converting the classical discrete fields to continuum fields through the Euler-McLaurin expansion.

[6]Dynamically coupled in this context means that the time evolution of $u(x + ie, t)$ will contain both $u(x + i\epsilon, t)$ and $u(x - i\epsilon, t)$ factors, which come out from computing $\dot{u}(x + i\epsilon) = i[u(x + i\epsilon), H_\Gamma]$.

chiral current decomposition.

We can, however, find the chirality condition for pure left or right going motion. Unlike the Quantum mechanical approach, the physical picture provides an easy answer: we must demand that all the shepherd b poles lie on either the upper or lower half complex plane, so that all the a-solitons have the same direction. For the right going case, we demand that

$$(u_+)_H = iu_+, \tag{3.49}$$

which, due to the property in equation (3.13) property that Γ satisfies, can only be true if the b-poles are all in the upper plane. The above can be satisfied by imposing [1, 2]

$$v = \lambda\pi(\rho - \rho_0) - \frac{1}{2}\lambda(\partial_x \ln \rho)_H. \tag{3.50}$$

This is the condition between current and velocity for pure chiral motion, which is the same result obtained in (2.90) under the Quantum mechanical approach. After looking at these results, we can conclude that the field $u(z,t)$, together with the boundary conditions (3.46) and the Quantum Benjamin-Ono equation 3.9 provide a continuum hydrodynamic model for the Calogero-Sutherland model analogous to the one obtained from the Quantum Mechanical treatment, albeit from a different perspective. It provides an arguably simplified framework from which it is easier to derive physical properties.

4 The FQHE-CS Model Connection

4.1 Introduction

As we mentioned before, Polychronakos has shown in [5] the existance of a mapping between a matrix model of the FQHE and the Calogero-Sutherland model. In the following, we wish to review said mapping and the matrix model formulation leading to it. To accomplish this, we should first become familiar with the basis of the Quantum Hall effect.

It is beyond the scope of this thesis to aboard a comprehensive explanation of the Quantum Hall effect. On a basic experimental level, we can broadly characterize it as an effect that appears when one considers a two-dimensional sample in the presence of a perpendicular magnetic field. Under this conditions, the sample contains an off-diagonal conductivity[1] σ that is quantized,

$$\sigma_{xy} = \frac{ne^2}{\hbar}, \tag{4.1}$$

where e is the charge of the electron, and n is an integer or a fraction. When the electron-impurity interactions or scattering dominate, n can be shown to be an integer, and we have the so-called Integer Quantum Hall Effect (IQHE). When the electron-electron interactions dominate, n can take fractional values, and we are under the Fractional Quantum Hall Effect (FQHE). There are different models to explain each of these effects, depending on the level of rigorosity and detail needed. The IQHE is the simplest to understand, and can be modeled by looking at the energy level structure of non-interacting electrons in two dimensions under a magnetic field. The eigenstates for an

[1] The conductivity σ is defined as the inverse of the resistivity, that is, $I = \sigma V$. Actual measurements of the Hall conductivity have been found to be integer or fractional multiples of e^2/h to nearly one part in a billion.

individual electron are given by

$$\psi_{n,m}(z,\bar{z}) = z^m L_n^m(z,\bar{z}) e^{\frac{-|z|^2}{4l_0^2}}, \tag{4.2}$$

where $L_n^m(z,\bar{z})$ are Laguerre polynomials and $l_0 = \sqrt{\hbar/eB}$ is the magnetic length, which controls the spread of the Gaussian factor. These states are degenerate, with one index n indicating the different energies (referred to as Landau levels in the literature) and the other index m characterizing the angular momentum. When one considers a finite-sized system, it turns out that for each Landau level n, only a certain number of m values are possible[2]. It can be shown that for fixed electron density there can only be a maximum of Φ/Φ_0 electrons per Landau level, where Φ is the magnetic flux piercing the sample, and $\Phi_0 = h/e$ is the flux quantum. Now, in any given real physical system, the electronic density will fill only $p-1$ Landau levels exactly, leaving the last p-level partially filled. Based on an argument by Laughlin [25], it can be proved that, in the presence of impurities, the contributions from the partially filled level is the same as one in which the whole band is filled. Thus, the off-diagonal conductivity quantization (4.1) holds. This is also reflected in the conductivity vs. magnetic field curve in Figure 4.1

The plateaus indicate that within them, the magnetic field is not strong enough to start filling the next band, thus the conductivity remains constant. Once this is overcome, the next Landau level begins filling up and consequently we see a jump in the conductivity.

The FQHE is more complicated to model, with the rigorous formulations requiring the use of field theory [26]. A simplified model exists, based on the use of Laughlin wavefunctions that replaces the real multi-electron Coulomb interaction by a toy interaction in the angular momentum of electron pairs. This model contains all the basic properties that appear in the FQHE, namely charge quantization for the excitations, and therefore non-abelian statistics, and fractional quantization of the conductivity. The reader is referred to Chetan Nayak's notes [27] for a "user-friendly" exposition of the topic. In the following, we will review the work of Susskind [4] and Polychronakos [5] that establish a connection to the Calogero Model. The former introduced a simplified fluid model for describing the FQHE, while the latter extended it

[2]This is easily seen by calculating $\langle r\rangle_{0,m} = l_0\sqrt{2(m+1)}$.

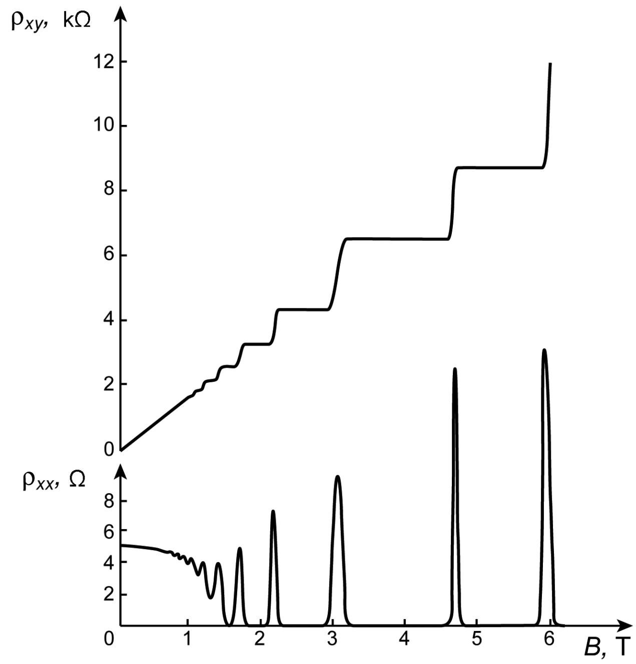

Figure 4.1: Top: Transversal resistivity as a function of the applied magnetic field. The resistivity depends on the total number of partially or completely filled Landau bands. Each plateau in the resistivity represent a partial filling of the last Landau band. When this partially-filled Landau band becomes full, the resistivity jumps to the next plateau, repeating the same process. Bottom: Diagonal resistivity as a function of the applied magnetic field.

through a non-abelian matrix-theory to provide more rigorous results.

4.2 Fluid Model Description of the FQHE

We will roughly follow the steps in [4] to arrive at a fluid description of the FQHE. We start by considering a classical 2D fluid system with N particles in the absence of a magnetic field. The simplest description is given by the Lagrangian

$$L = \sum_a \frac{m}{2} \dot{x}_a^2 - U(x), \tag{4.3}$$

where $U(x)$ is an interaction potential, and the index a runs over all the electrons in our system. In the continuum limit, the particle labels become pairs of real numbers, $a \to (y_1, y_2)$, and thus the physical position of each particle is determined by the map $x_i = x_i(y_1, y_2)$[3]. Consequently, the real space density is related to the particle label density by the Jacobian,

$$\rho(x) = \rho_0 \left| \frac{\partial y}{\partial x} \right| . \tag{4.4}$$

Thus, the Lagrangian of the fluid in the continuum limit reads

$$L = \int d^2y \rho_0 \left[\frac{m}{2} \dot{x}_a^2 - V(\rho_0 \left| \frac{\partial y}{\partial x} \right|) \right] . \tag{4.5}$$

The most important property of this Lagrangian is the invariance under area-preserving diffeomorphisms (APD), which is to be expected since reshuffling the particle labels should not affect the Lagrangian. Note that this is a much broader symmetry than the permutation group that one would have in standard quantum mechanics. It is important to know what this implies for the fluid, since Noether's theorem states that for each symmetry there's a conserved quantity. We will see that in this case, the conserved quantity is

[3] The y coordinates are the particle-label space and the x coordinates are the real-space ones. For example, $\vec{x} = 2\vec{y}$ would represent particles in real space that are stretched out by 1/2 in each direction, with a corresponding density of 1/4 the density in y-space. The y-space density can always be taken to be unity.

the vorticity of the fluid.

In general, if a Lagrangian is invariant under a transformation δx (on the physical space), then the quantity $\int d^2y \Pi_a \delta x_a$ is conserved, where $\Pi_a \sim \dot{x}_a$. Since the factor δx_a depends in our case on the APD done on the y-space particle labels, we need to consider an infinitesimal APD transformation $y'_i = y_i + f_i(y)$ and estimate the induced transformation δx. To first order, we have the expansion $|\frac{\partial y'}{\partial y}| = 1 + \vec{\nabla} \vec{f}$, which implies that the APD transformation must satisfy the condition

$$f_i = \epsilon_{ij} \frac{\partial \Lambda(y)}{\partial y_j}. \tag{4.6}$$

The physical coordinates transform as $x_a(y) \to x_a(y_i + f_i(y))$. Using the above, we can compute the physical transformation as

$$\delta x_a = \frac{\partial x_a}{\partial y_i} = \epsilon_{ij} \frac{\partial \Lambda(x_a)}{\partial y_i} \frac{\partial \Lambda(y)}{\partial y_j}. \tag{4.7}$$

The conserved quantity can now be directly computed,

$$\frac{d}{dt} [\underbrace{\frac{\partial}{\partial y_j} \left(\epsilon_{ij} \dot{x}_a \frac{\partial x_a}{\partial y_j} \right)}_{\text{vorticity}}] = 0 \quad \Longrightarrow \quad \frac{d}{dt} \int_\Gamma \dot{x}_a dx_a = 0, \tag{4.8}$$

which shows that the vorticity, or circulation of the fluid, is conserved. In particular, the vorticity-free fluid satisfies the condition

$$\frac{\partial}{\partial y_j} \left(\epsilon_{ij} \dot{x}_a \frac{\partial x_a}{\partial y_j} \right) = 0. \tag{4.9}$$

We can think of the distortion field δx as the excitations over the background, much like the A field in electromagnetism indicates the electrical charges. We can parameterize the small perturbations in real space around the uniform density ρ_0, where we assume the potential has a minimum, by A_j:

$$x_i = y_i + \epsilon_{ij} \frac{A_j}{2\pi \rho_0}. \tag{4.10}$$

Here the condition $\delta A_j = 2\pi \rho_0 \frac{\partial \Lambda}{\partial y_j}$ must be satisfied[4] in order for the transformation to be an APD. We assume a parabolic potential of the form

[4]Up to linear order in Λ.

$V = \mu(\rho - \rho_0)^2$, which has a minimum at ρ_0. The density to first order in the perturbation is $\rho = \rho_0 - \frac{1}{2\pi}(\nabla \times A)$. Combining these results, the Lagrangian in terms of the perturbation field A becomes

$$L = \frac{1}{g^2} \int d^2 y \frac{1}{2} \left[\dot{A}^2 - \frac{2\mu\rho^2}{m} (\nabla \times \vec{A})^2 \right], \quad \text{with} \quad g = (2\pi)^2 \frac{\rho_0}{m}. \tag{4.11}$$

The vortex free condition in the linearized approximation becomes the Gauss law constraint $\nabla E = \nabla \dot{A} = 0$. Thus, we see that the vortices in the fluid theory are equivalent to charges in the electromagnetic theory.

So far we have reviewed a model for a classical fluid with a symmetry group and found the conditions for free vorticity and conservation of vorticity. However, we need to introduce a magnetic field to characterize the FQHE. Thus, we embed the fluid, which we now assume is charged, in a magnetic field. The Lagrangian (4.3) acquires an extra term given by $\frac{eB}{2}\epsilon_{ab}\dot{x}_a x_b$, which represents the usual Lorentz force $F_{mag} = q\vec{v}x\vec{B}$. In the continuum limit, the magnetic contribution becomes

$$L_{\text{mag}} = \frac{eB}{2} \int \rho_0 d^2 y \epsilon_{ab} \dot{x}_a x_b = \frac{eB}{2\pi^2 \rho_0} \int \rho_0 d^2 y \epsilon_{ab} \dot{A}_a A_b, \tag{4.12}$$

which must be added to the previous Lagrangian. In the above, we used (4.10) to perturb around the equilibrium position. Although the complete Lagrangian contains all the terms in both (4.5) and (4.12), only the latter dominates the long-range behavior. Thus, we can discard the so called Maxwell term (4.5) and concentrate on the Chern-Simmons term (4.12). We know that the Lagrangian must be invariant under the APD symmetry, so we would like to write it in a way that explicitly shows this fact. The APD symmetry yields the following conserved gauge generator:

$$\frac{1}{2}\frac{\partial}{\partial y_j}\left(\epsilon_{ij}\epsilon_{ab} x_b \frac{\partial x_a}{\partial y_i}\right) = \frac{1}{2}\epsilon_{ij}\epsilon_{ab}\frac{\partial x_b}{\partial y_j}\frac{\partial x_a}{\partial y_i} = \left|\frac{\partial y_j}{\partial x_a}\right|^{-1} = \frac{\rho_0}{\rho}. \tag{4.13}$$

So we see that the density is time independent. Fixing the value of this generator is equivalent to fixing the vortex distribution of the system (again, equivalent to fixing the charge distribution in the electromagnetic theory). In the absence of vortices, we can set the value of the generator to 1.[5] We

[5] In the presence of vortices, we need to add a δ function to represent the vortices, as

have

$$\frac{1}{2}\epsilon_{ij}\epsilon_{ab}\frac{\partial x_b}{\partial y_j}\frac{\partial x_a}{\partial y_i} - 1 = 0. \tag{4.14}$$

We can write our Lagrangian plus the generator condition in a compact way by introducing a Lagrange multiplier A_0, yielding

$$L' = \frac{eB\rho_0}{2}\epsilon_{ab}\int d^2y \left[\left(\dot{x}_a - \frac{1}{2\pi\rho_0}\{x_a, A_0\}\right)x_b + \frac{\epsilon_{ab}}{2\pi\rho_0}A_0\right]. \tag{4.15}$$

Here $\{,\}$ are Poisson brackets, defined by $\{F(y), G(y)\} \equiv \epsilon_{ij}\partial_i F \partial_j G$. This is the Lagrangian for a fermionic fluid in the presence of a magnetic field that automatically satisfies the APD symmetry through the A_0 multiplier. It contains enough structure to represent the main features of the FQHE, namely excitations with fractional charge that satisfy fractional statistics. To see this, we can use (4.10) to analyze small perturbations of the fluid around its equilibrium position, yielding a field A that supports vortices solutions, which satisfy

$$\nabla \times A = 2\pi\rho_0 q\delta^2(y) \quad \Longrightarrow \quad A_i = q\rho_0\epsilon_{ij}\frac{y_j}{y^2}. \tag{4.16}$$

Taking this solution and looking again at (4.10), we see[6] that the Chern-Simmons vortex represents a real space lack or surplus of density at the center of the vortex, and since the density carries charge, therefore charge. The magnitude in lack/excess is

$$e_{qp} = \rho_0 q e. \tag{4.17}$$

Thus, the charged vortex is equivalent to the Laughlin quasiparticle in the Laughlin wavefunction formalism. In a more rigorous argument, since the potential A diverges near the vortex solution, to correctly understand the near-vortex behavior we have to include a source in the conserved quantity

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done in equation (4.18).

[6]The physical coordinates x_i are related to the vortex perturbation by $x_i = y_i(1 + \frac{q}{2\pi|y|^2})$.

(4.14),

$$\frac{1}{2}\epsilon_{ij}\epsilon_{ab}\frac{\partial x_b}{\partial y_j}\frac{\partial x_a}{\partial y_i} - 1 = q\delta^2(y), \tag{4.18}$$

and abandon the small perturbation approximation. This gives a well behaved solution given by

$$x_i = y_i\sqrt{1+\frac{q}{\pi|y|^2}} \quad \Longrightarrow \quad \lim_{|y|\to 0} x_i = \sqrt{\frac{q}{\pi}} = x_i(0). \tag{4.19}$$

The point $y = 0$ is mapped to a circle of radius $\sqrt{\frac{q}{\pi}}$, leaving an empty hole in the center (of area q). Defining $\nu = \frac{2\pi\rho_0}{eB}$ (the filling fraction in Laughlin's picture) we see that the quasiparticle charge is

$$e_{qp} = e\nu. \tag{4.20}$$

Under this model, it can be shown heuristically that the statistics of the quasiparticles are fractional and that the filling fraction is quantized.

4.3 Improving the Fluid Model: The Matrix Model

The above theory is based on the APD group symmetry, which is too "detailed" to capture the granular character of the electron. We must remember that in reality the electrons are governed by the permutation symmetry group, the APD arising as an approximation in the fluid model description. One way to make the APD theory granular is by making $[y_1, y_2]$ non-commutative, since then, by the uncertainty principle, the phase-space area of the electron is increased. Thus, the correct theory must be based on a non-commutative space with a discrete indivisible unit of y-space area which is identified with the electron. The Lagrangian that generalizes (4.15) is [5]

$$L' = \frac{eB}{2}\epsilon_{ab}Tr\left[(\dot{X}_a - i[X_a, \hat{A}_0]_m)X_b + eB\theta\hat{A}_0\right], \tag{4.21}$$

which is a matrix theory in a background magnetic field (on an infinite plane, as we shall see below). In this new Lagrangian, the quantities that represent the electron are the eigenvalues of the X_1 and X_2 matrices, and the APD symmetry group has been replaced by the U(N) group. The multiplier A_0 encodes the non-commutativity of space through its EOM,

$$[X_a, X_b] - i\theta\epsilon_{ab} = 0. \tag{4.22}$$

Note that, for a general distribution of vortices, the above constraint, which is equivalent to Gauss law, reads

$$[X_a, X_b] - i\theta\epsilon_{ab} = \text{sources}, \tag{4.23}$$

where the right side sources are the vortices or quasiholes, which are the equivalent of charges in the electromagnetic theory. Polychronakos [5] shows that a point source at the origin can be represented by

$$[X_1, X_2] = i\theta(1 + q|0\rangle\langle 0|), \tag{4.24}$$

where $\{|n\rangle\}, n = 0, 1, \ldots$ is an oscillator basis for the Hilbert space, and $|0\rangle$ has the minimal spatial spread. A solution for this source is found to be

$$X_1 + iX_2 = \sqrt{2\theta} \sum_{i=1}^{\infty} \sqrt{n+q}|n-1\rangle\langle n|, \tag{4.25}$$

which represents a quasihole. Quantization of the charge q, together with fractional statistics for the quasiholes can be shown starting from the above solution. We have a problem, however, when we consider a finite-sized system, since the constraint (4.22) doesn't support any finite-sized matrix solutions (a commutator of finite matrices is always traceless). Thus, the Lagrangian (4.21) represents a system with an infinite number of electrons. In order to work with a finite, bounded system, we follow Polychronakos [5] and introduce two extra terms, which yields the matrix model Lagrangian

$$L' = \frac{eB}{2}\epsilon_{ab} Tr\left[(\dot{X}_a - i[X_a, \hat{A}_0]_m)X_b + 2\theta\hat{A}_0 - \omega X_a^2\right] + \Psi^\dagger(i\dot{\Psi} - A_0\Psi). \tag{4.26}$$

The first term, proportional to ω, is a harmonic potential necessary to bound the system, and the second term Ψ is a complex N-dimensional vector that acts as a source, as can be seen by the new gauge constraint

$$-iB[X_1, X_2] - B\theta = \Psi\Psi^\dagger. \tag{4.27}$$

The classical EOM arising from the above Lagrangian is

$$\dot{X}_a + \omega\epsilon_{ab}X_b = 0, \quad \text{with} \quad [X_1, X_2] = i\theta(1 - N|v\rangle\langle v|), \tag{4.28}$$

where $|v\rangle$ is an N-vector of the form $(0, \ldots, 0, 1-N)$. We can get an idea of how the solutions look at a fixed time by choosing a basis in which X_1 is diagonal. The instantaneous matrices can then be parameterized in terms of the N eigenvalues of X_1, which we call x_n, and the N diagonal elements of X_2 called y_n. These results in

$$(X_1)_{mn} = x_n\delta_{mn}, \qquad (X_2)_{mn} = y_n\delta_{mn} + \frac{i\theta}{x_m - x_n}(1 - \delta_{mn}) \tag{4.29}$$

This is a snapshot in time that shows us how the two matrices are related. From (4.28), we have $\dot{X}_1 = -\omega X_2$, so infinitesimally we can write

$$X_1(t) = X_1(0) - t\omega X_2(0). \tag{4.30}$$

Thus, to see how the eigenvalues x_n evolve in time, we can use second order perturbation theory on (4.30), taking $-t\omega X_2$ as the perturbation. The time evolution of the eigenvalues is thus

$$x_n^{(2)} = x_n - t\omega\langle n|x_2|n\rangle + t^2\omega^2 \sum_{i\neq j} \frac{|\langle i|X_2|j\rangle|^2}{x_i - x_j}. \tag{4.31}$$

The last term can be explicitly calculated using (4.29), together with the identity $\langle i|X_2|j\rangle = \frac{i\theta}{x_i - x_j}$, to obtain

$$\begin{aligned} \ddot{x}_n &= 2\omega^2\theta^2 \sum_{i;i\neq n} \frac{1}{(x_n - x_i)^3} && (4.32) \\ \dot{x}_n &= -\omega y_n. && (4.33) \end{aligned}$$

Comparing this to the EOM of the Calogero model (3.16) presented in the previous chapter, we realize that we can identify the position of the Calogero particles x_n as the eigenvalues of X_1 and the conjugate momenta p_n as the diagonal values of X_2, ωy_n. This is the mapping that links the matrix model of the FQHE and the Calogero-Sutherland model.

4.4 Extending The Mapping?

In chapter 3, we saw that the introduction of a complex-valued field $u(x,t)$ that satisfies the QBO equation provides new insight into the properties of the hydrodynamic Calogero-Sutherland model, such as explicit left and right soliton solutions, the condition for pure chiral evolution in terms of the density and velocity fields and the separation of the Hamiltonian in terms of chiral currents. Furthermore, in this chapter we have reviewed the fundamentals of the known connection between the 1D Calogero-Sutherland model and the 2D matrix model of the FQHE. Thus, a natural continuation is to try to incorporate the complex-valued field formalism into the matrix model theory by reverting the CS-FQHE mapping, with the purpose of gaining a new perspective into the FQHE.

Although at the moment it is not clear whether this goal can be achieved, the richness of the Calogero model and its connection to the FQHE provides a degree of comfort. For instance, Bernevig has studied [28] the solutions of the Laplace-Beltrani operator (2.23) for a negative parameter λ, called Jack-polynomials (the Calogero model that we considered satisfies the same operator with positive λ). Certain Jacks that obey a so-called "(k,r,N)-admissible" property span a basis of polynomials that vanish when k particles coincide. This is the necessary condition to represent non-abelian states in the FQHE, and it turns out that these special Jack polynomials represent non-abelian FQHE states. Thus, a connection at the wavefunction level also exists, albeit with a "wrong-signed" λ.

Another interesting result concerns the calculation of the entanglement entropy. Katsura et al. [29] have calculated the entanglement entropy in the ground state of the Calogero model. It seems interesting to study the possibility of "lifting" this calculation to the FQHE using the mapping between the two models. The difficulty here among other things lies in finding proper wavefunctions for the matrix model of the FQHE.

5 Harmonic Trapped $p + ip$ Fermions

5.1 Introduction

The calculation of the intrinsic angular momentum of the A-phase of superfluid ^{3}He has a long history, with different results depending on what theoretical framework is employed. If the superfluid is regarded as a Bose condensate of Cooper pairs where each pair possesses angular momentum $\hbar$, the total angular momentum of a spatially uniform system of N particles is $\frac{1}{2}N\hbar$ [30]. An alternative argument adds an attenuating factor of Δ/ϵ_F to the above [31], based on the fact that when $\Delta \ll \epsilon_F$, the opening of the energy gap Δ affects only states lying within an energy range of a few Δ about the fermi surface at $E = \epsilon_F$.

The difficulty in establishing the correct answer lies in the fact that one can only obtain approximate analytical solutions once spatial variations of the fluid density or order parameter are allowed, with different approaches leading to different answers [32]. In the following, we will review the work of Stone and Anduaga [6] that uses numerical solutions of the BdG equation to estimate the angular momentum and associated mass flow of a two-dimensional model of $p_x + ip_y$ superfluid fermions confined in a harmonic trap. This work is in a sense a continuation of previous results obtained by Stone and Roy [7], only this time an exact harmonic-trap model is solved numerically instead of a rigid wall model analytically. The results show that, when solved exactly, the BdG equation produces results entirely consistent with the Cooper-pair wave function approach: there is no Δ/ϵ_F suppression, and the zero-temperature intrinsic angular momentum is $\frac{1}{2}\hbar$ per particle. This is also in agreement with previous numerical computations of the angular momentum in a cylindrical container obtained by Kita [33].

Sections 5.2 and 5.3 present some general properties of the Bogoliubov de Gennes (BdG) formalism and the two-dimensional harmonic oscillator. In section 5.4, the task of finding eigenstates of the Hamiltonian is tackled, along with finding suitable matrices to diagonalize numerically. At this point we introduce a clarification of an error in the original paper, where some negative signs were omitted in the resulting tridiagonal matrices. The last section covers the actual numerical results and their comparison to theoretical models such as the Thomas-Fermi approximation and the Ishikawa mass current calculation.

5.2 The BdG Equation in a Pair-Interaction Hamiltonian

Before we tackle the problem of finding specific solutions of the harmonic-trapped fermionic Hamiltonian, it is useful to look at the bigger picture and review the properties of the BdG approach, since many results involving operator mean-values can be expressed in terms of components of the eigenvector solutions independently of the fine details of the Hamiltonian considered.

Suppose that H_{ij} is an N-by-N matrix representing a one-particle (i.e. first quantized) Hamiltonian H. In general, when dealing with a multi-particle system, it is useful to switch into a formulation in terms of occupation numbers, also known as second quantization. The second-quantized many-body Hamiltonian corresponding to the above is

$$\hat{H} = \hat{a}_i^\dagger H_{ij} \hat{a}_j, \tag{5.1}$$

where a sum over repeated indices is understood. The label i includes all the necessary quantum numbers needed to specify a state of the one-particle Hamiltonian, for example angular momentum l, quantum number n, and spin number σ when H_{ij} is a two-dimensional harmonic oscillator. In the case of a continuum basis, the label incorporates both the space-coordinates and any spin index. Since we want to represent a system of fermions, the

$\hat{a}_i^\dagger$ and $\hat{a}_i$ are anti-commuting second-quantization creation and annihilation operators, that is,

$$\{\hat{a}_i, \hat{a}_j\} = \{\hat{a}_i^\dagger, \hat{a}_j^\dagger\} = 0, \quad \{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}. \tag{5.2}$$

By definition, the operator $\hat{a}_i^\dagger$ creates a particle in a state i, and the commutation relations indicate that $\hat{a}_i$ destroys such a particle.

The above second-quantized Hamiltonian represents a system of non-interacting particles. Since we want to represent a condensate of Cooper pairs, we must allow for two-particle interaction terms, which in their most general form can be written as

$$\hat{H}_{\text{Bogoliubov}} = \hat{a}_i^\dagger H_{ij} \hat{a}_j + \frac{1}{2}\Delta_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger + \frac{1}{2}\Delta_{ij}^\dagger \hat{a}_i \hat{a}_j. \tag{5.3}$$

This so-called Bogoliubov Hamiltonian contains a parameter that regulates the particle-pair energy, the gap function matrix Δ_{ij}. The corresponding one for an $l = 1$ angular momentum per pair is skew-symmetric. Naturally, different forms of Δ_{ij} describe different condensates and result in different patterns of symmetry breaking. The following work assumes a given non-zero Δ_{ij}, which can be taken to be externally imposed, without worrying about its origin.

The particle-number non-conserving Bogoliubov Hamiltonian can also be written as

$$\hat{H}_{\text{Bogoliubov}} = \frac{1}{2}(\hat{a}_i^\dagger \quad \hat{a}_i) \begin{pmatrix} H_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & -H_{ij}^T \end{pmatrix} \begin{pmatrix} \hat{a}_j \\ \hat{a}_j^\dagger \end{pmatrix} + \frac{1}{2}\mathrm{tr} H, \tag{5.4}$$

where H^T denotes the transpose of the Hermitian matrix H. We would like to diagonalize this Hamiltonian, and subsequently find the relationship between the new operators that create the eigenstates and our old $\{\hat{a}_i\}$ operators in which we know the observables. To do this, we start by assuming we know the solutions of the single-particle Bogoliubov-de Gennes eigenvalue problem

$$\begin{pmatrix} H & \Delta \\ \Delta^\dagger & -H^T \end{pmatrix} \begin{pmatrix} \vec{u}_m \\ \vec{v}_m \end{pmatrix} = E_m \begin{pmatrix} \vec{u}_m \\ \vec{v}_m \end{pmatrix}. \tag{5.5}$$

Since the complete $\{\vec{u}_m, \vec{v}_m\}$ pairs are eigenvectors, they can always be normalized to satisfy $|\vec{u}_m|^2 + |\vec{v}_m|^2 = 1$. The dimension of the each $\vec{u}_m, \vec{v}_m$ is equal to the single-particle Hamiltonian dimension N. A property of the BdG equation (5.5) is that given an eigenvector with energy E_m, under conjugation we obtain a new eigenvector with energy $-E_m$. This can be seen by expanding the matrix equation,

$$\hat{H}\vec{u}_m + \Delta\vec{v}_m = E_m\vec{u}_m, \quad \hat{\Delta}^\dagger\vec{u_m} - H^T\vec{v_m} = E_m\vec{v_m}. \tag{5.6}$$

If we use the properties $H^T = H^*$, $\Delta^T = -\Delta$, and $\Delta^* = -\Delta^T$, we can conjugate the above to obtain,

$$\hat{H^T}\vec{u^*}_m - \Delta^\dagger\vec{v^*}_m = E_m\vec{u^*}_m, \quad -\hat{\Delta}\vec{u^*_m} - H\vec{v^*_m} = E_m\vec{v^*_m}, \tag{5.7}$$

which can be written in matrix form as

$$\begin{pmatrix} H & -\Delta \\ \Delta^\dagger & -H^T \end{pmatrix} \begin{pmatrix} \vec{v^*_m} \\ \vec{u^*_m} \end{pmatrix} = -E_m \begin{pmatrix} \vec{v^*_m} \\ \vec{u^*_m} \end{pmatrix}. \tag{5.8}$$

so the BdG eigenvalues come in $\pm$ pairs (i.e. the spectrum is symmetric around $E = 0$). If we look at the structure of the above matrix eigenvalue equation, it is easy to see that in the gapless $\Delta = 0$ case, the upper branch of the eigenvalue spectrum is given by solutions of the form $\vec{u}^i_m = \delta_{m,i}$, $\vec{v} = \vec{0}$, while the lower branch solutions have reversed roles for $\vec{u}_m, \vec{v}_m$. The upper branch solutions are associated with mainly positive-energy eigenvalues, while the lower branch has mainly negative-energy ones. For example, if we consider the one-dimensional non-relativistic gas with energy $E(k) = k^2/2 - \mu$, only a few of the upper branch eigenvalues will dip into the negative region ($k < \sqrt{2\mu}$) and the rest will have positive energies, while similarly some states of the lower energy branch will have positive energies. The same holds approximately true in the presence of a gap $\Delta \neq 0$, with $|\vec{u}_m| \approx 1$ and $|\vec{v}_m| \approx 0$ for the upper branch region, and $|\vec{u}_m| \approx 0$, $|\vec{v}_m| \approx 1$ for the lower branch. This is graphically represented in Figure 5.1. We now turn our attention back to the solutions of the BdG single-particle operator. In linear algebra it is customary, after finding a set of eigenvectors for a given operator, to change bases and work in the new eigenvector basis. Let us define a set of new annihilation operators $\{b_j, b_j^\dagger\}$ that create the eigenvectors

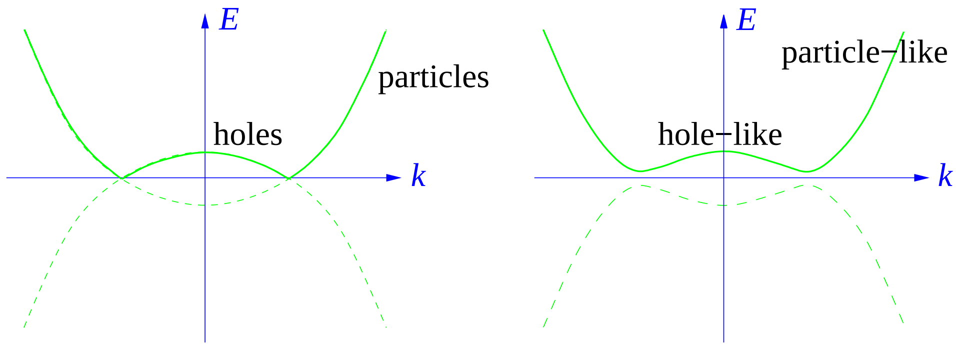

Figure 5.1: The BdG operator spectrum for a one-dimensional non-relativistic gas with $E(k) = k^2/2 - \mu$. Left: the spectrum with $\Delta = 0$. Right: the spectrum after a gap is opened by a non-zero Δ. In each case the part of the spectrum with BdG eigenvalue $E > 0$ is shown as a solid line, and the part with $E < 0$ is shown dashed. For $\Delta = 0$, the solid branch of the spectrum has $|\vec{u}| = 1$, $|\vec{v}| = 0$, and opposite for the dashed branch. For $\Delta \neq 0$, the $E > 0$ "particle-like" region has $|\vec{v}| \approx 0$ and $|\vec{u}| \approx 1$, and the "hole-like" region has $|\vec{u}| \approx 0$ and $|\vec{v}| \approx 1$.

of H. We can re-express the Hamiltonian (5.4) in terms of these as

$$\hat{H}_{\text{Bogoliubov}} = \frac{1}{2}(\hat{b}_i^\dagger \quad \hat{b}_i)\, \underbrace{I \begin{pmatrix} H_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & -H_{ij}^T \end{pmatrix} I^\dagger}_{\text{diag}(E_1,\ldots,E_{2N})} \begin{pmatrix} \hat{b}_j \\ \hat{b}_j^\dagger \end{pmatrix} + \frac{1}{2}\text{tr}H, \tag{5.9}$$

where $I^\dagger$ is the matrix that changes from the old basis where the $\{a_i, a_i^\dagger\}$ are defined, to the new eigenvector basis. From elementary linear algebra we know that this matrix is just the list of all eigenvectors arranged by columns[1], resulting in

$$I^\dagger = \begin{pmatrix} u_1 & \ldots & u_N & u_{N+1} & \ldots & u_{2N} \\ \vdots & & \vdots & \vdots & & \vdots \\ v_1 & \ldots & v_N & v_{N+1} & \ldots & v_{2N} \\ \vdots & & \vdots & \vdots & & \vdots \end{pmatrix}, \tag{5.10}$$

where we have a total of $2N$ columns. Using the conjugation symmetry from (5.8), we see that the second half of this matrix is not independent from the first, allowing us to write $u_{N+i} = v_i^*, v_{N+i} = u_i^*$. This way, the change of

[1] Note that this matrix satisfies $I^\dagger(1, 0, \ldots, 0)^T = (u_1, \ldots, u_N, v_1, \ldots, v_N)^T$, which is what we seek.

basis matrix becomes

$$I^\dagger = \begin{pmatrix} u_1 & \dots & u_N & v_1^* & \dots & v_N^* \\ \vdots & & \vdots & \vdots & & \vdots \\ v_1 & \dots & v_N & u_1^* & \dots & u_N^* \\ \vdots & & \vdots & \vdots & & \vdots \end{pmatrix}. \tag{5.11}$$

Here the left half eigenvectors have eigenvalues $E_n > 0$, while the second have eigenvalues $E_{N+n} = -E_n < 0$. This way, the old and new operators, which are related through the $I^\dagger$ matrix as $I^\dagger(b_j, b_j^\dagger)^T = (a_j, a_j^\dagger)^T$, can be written as

$$a_j = u_{jn} b_n + v_{jn}^* b_n^\dagger \tag{5.12}$$
$$a_j^\dagger = u_{jn}^* b_n^\dagger + v_{jn} b_n, \tag{5.13}$$

where the sum over repeated indices is implied. The mutual orthogonality and completeness of the eigenvectors $(\vec{u}_m, \vec{v}_m)^T$ ensures that the new operators $\hat{b}_m, \hat{b}^\dagger$ have the same anti-commutation relations as the old ones. To find the expression of the Hamiltonian in terms of these new operators, we only need to look at (5.9) and realize that if we absorb the change of basis matrices $I, I^\dagger$ into the Hamiltonian, the latter becomes diagonal. Thus, we can write

$$\begin{aligned} \hat{H}_{\text{Bogoliubov}} &= \frac{1}{2}(\hat{b}_i^\dagger \quad \hat{b}_i) \begin{pmatrix} \{E\} & 0 \\ 0 & \{-E\} \end{pmatrix} \begin{pmatrix} \hat{b}_j \\ \hat{b}_j^\dagger \end{pmatrix} + \frac{1}{2}\text{tr}H \\ &= \frac{1}{2}\sum_{n=1}^{N} E_n (b_n^\dagger b_n - b_n b_n^\dagger) + \frac{1}{2}\text{Tr}H \\ &= \sum_{n=1}^{N} E_n b_n^\dagger b_n - \frac{1}{2}\sum_{n=1}^{N} E_n + \frac{1}{2}\sum_{n=1}^{N} E_i^{(0)}. \end{aligned} \tag{5.14}$$

Here $E_i^{(0)}$ are the eigenvalues of H_{ij}, which unlike the E_m can be of either sign[2]. The usefulness of the new operators b_j surfaces when we analyze the ground state of the system and compute mean values of operators acting on it. By definition[3], the ground state $|0\rangle_b$ is such that it is annihilated by all the

[2]Note that by construction, the E_m are positive.

[3]One can ask about the relationship between the new ground state $|0\rangle_b$ defined as the state annihilated by all the $\hat{b}_m$ operators, and the older one of the $\hat{a}_m$ operators. It

$\hat{b}_m$ (we must remember that $b_m^\dagger$ creates the m-th quasiparticle excitation). Our physical operators are defined in terms of the original physical fermionic operators $\{a_i, a_i^\dagger\}$. If we consider an operator of the form $\hat{O} = \hat{a}_i^\dagger O_{ij} \hat{a}_j$, its ground-state expectation value can be written as

$$\begin{aligned}
\langle \hat{O} \rangle &= {}_b\langle 0|\hat{a}_i^\dagger O_{ij} \hat{a}_j|0\rangle_b \\
&= {}_b\langle 0|(v_{im}\hat{b}_m + u_{im}^*\hat{b}_m^\dagger) O_{ij} (u_{jn}\hat{b}_n + v_{jn}^*\hat{b}_n^\dagger)|0\rangle_b \\
&= v_{im} O_{ij} v_{jm}^* \\
&= \sum_{m=1}^{N} \vec{v}_m^T O \vec{v}_m^*. \qquad (5.15)
\end{aligned}$$

Note that the last sum involves the sum over the eigenstates with positive energy. Using the $E \leftrightarrow -E$, $\vec{u} \leftrightarrow \vec{v}^*$ symmetry, we could alternatively write this last expression as a sum over the negative eigenvalues of the Hamiltonian with $\vec{v}$ replaced by $\vec{u}$, that is, $\sum \vec{u}_m^\dagger O \vec{u}_m$. Equipped with the above expression, we can compute the number of particles in the ground state by finding the expectation value of the number operator $O_{ij} = \delta_{ij}$. This becomes

$$N = {}_b\langle 0|\hat{a}_i^\dagger \hat{a}_i|0\rangle_b = v_{im} v_{im}^* = \sum_{m=1}^{N} |\vec{v}_m|^2. \qquad (5.16)$$

We can get a better feeling for the expression if we again use the $\vec{u} \leftrightarrow \vec{v}^*$ symmetry and write

$$N = \sum_{\text{negative}E} |\vec{u}_m|^2. \qquad (5.17)$$

We had previously discussed that in the gapless case, $|\vec{u}_m| = 1$ and $|\vec{v}_m| = 0$ for the particle-like states (upper energy branch) and that some of these states would dip into negative energies as a consequence of a non-zero chemical potential. Therefore, when $\Delta = 0$, the above sum will be equal to the number of negative-energy eigenstates of H_{ij} (which is zero when $\mu = 0$).

We can also derive an expression for the mean values of space-dependent op-

turns out that the new ground state $|0\rangle_b$ can be written in terms of the old one through the coherent superposition of paired states $|0\rangle_b = C \exp\left(\frac{1}{2}\hat{a}_i^\dagger \hat{a}_j^\dagger \psi_{ij}\right)|0\rangle_a$, where $\psi_{ij} = v_{im}^*(u^{-1})_{mj}^*$ is the un-normalized Cooper-pair wavefunction.

erators in terms of the spatial eigenfunctions components similarly to (5.15). The expectation value of an operator of the form $\hat{O} = \int \hat{a}^\dagger(x) O(x,x') \hat{a}(x) d^2x d^2x'$ on the ground state of the system is given by

$$\langle \hat{O} \rangle \;=\; \sum_{i=1}^{N} \int v_i^T(\vec{x}) O(\vec{x},\vec{x}') v_i(x') d^2x d^2x' \tag{5.18}$$

For example, in the harmonic case using this expression together with the density operator $\hat{\rho}(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$ yields the mean value of the density in the ground state of the system:

$$N = \int \sum_{m,l=1}^{N} |v_{m,l}(r,\theta)|^2 d^2r \implies \rho(r,\theta) = \sum_{m,l}^{N} |v_{m,l}(r,\theta)|^2. \tag{5.19}$$

This analysis has shown us that we can compute mean values such as particle number and angular momentum if know the energy spectrum and the eigenvectors of the single-particle BdG Hamiltonian. This in itself is no easy task if we cannot find a way to simplify the way the states link to each other. At the very least, we should understand the form of the solutions of the single-particle Hamiltonian H, to then look at how the gap operator Δ couples these solutions. We must then turn our attention to the two-dimensional harmonic oscillator and review its spectrum and eigenvalues.

5.3 Reviewing the 2D Harmonic Oscillator

The model that we are considering involves fermions confined in a harmonic trap in the x-y plane. Therefore, the one-particle Hamiltonian of the previous section inside $H_{\text{Bogoliubov}}$ is that of the two-dimensional harmonic oscillator:

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + \omega^2(x^2+y^2)\right). \tag{5.20}$$

Here $p_x = -i\hbar\partial/\partial x$ is the usual momentum operator, with x, p being canonically conjugates, $[x, p_x] = i\hbar$. The mass of the particles has been taken to be unity. This Hamiltonian is invariant under rotations along the $\hat{z}$ axis, which implies that the angular momentum operator L_z must commute with it[4]. Therefore, it is possible to diagonalize simultaneously the Hamiltonian and the L_z operator, or equivalently stated, we can choose eigenvectors of the Hamiltonian that are also eigenvectors of L_z. The additional advantage of this approach is seen in the commutator algebra between L_z and the momentum operators $P = p_x - ip_y$ and $P^\dagger = p_x + ip_y$:

$$[L_z, p_x \pm ip_y] = [(\vec{r} \times \vec{p})_z, p_x \pm ip_y] = \pm\hbar(p_x \pm ip_y). \tag{5.21}$$

Thus, the $L_z, P^\dagger$ and P operators form an $SU(2)$ triplet, with $P^\dagger$ and P being the raising and lowering ladder operators respectively. Since we will see in the next section that the gap operator Δ is proportional to P, its action on the eigenvectors of L_z will be simple. Thus, we want to find ladder operators of L_z in terms of the usual harmonic-oscillator ladder operators

$$a_x = \sqrt{\frac{\omega}{2\hbar}}\left(x + i\frac{p_x}{\omega}\right), \quad a_x^\dagger = \sqrt{\frac{\omega}{2\hbar}}\left(x - i\frac{p_x}{\omega}\right), \tag{5.22}$$

which have bosonic commutation relationships and generate the eigenstates of the harmonic oscillator as

$$|n_x, n_y\rangle = \frac{(a_x^\dagger)^{n_x}}{\sqrt{n_x!}}\frac{(a_y^\dagger)^{n_y}}{\sqrt{n_y!}}|0,0\rangle. \tag{5.23}$$

To achieve this, we can first express the angular momentum operator in terms of the canonical operators,

$$L_z \equiv xp_y - yp_x = i\hbar(a_y^\dagger a_x - a_x^\dagger a_y). \tag{5.24}$$

Next, we propose a general ladder operator of the form

$$b_i^\dagger = C_1 a_1^\dagger + C_2 a_2^\dagger, \quad |C_1|^2 + |C_2|^2 = 1, \tag{5.25}$$

where the right equality ensures that $[b_i, b_i^\dagger] = 1$. To determine the coefficients, we must impose the L_z ladder operator condition

$$[L_z, b_i^\dagger] = \alpha\hbar b_i^\dagger, \tag{5.26}$$

[4]In general, if a field H is invariant under a transformation Θ, then the infinitesimal action of this transformation $\delta\Theta$ on the field must also be zero. Since this action can be represented in a Lie algebra by the commutator between the generator of the transformation and a matrix representation of the field, we must have that $[\delta\Theta, H] = 0$.

where α is an arbitrary parameter. Substituting the expressions for $b_i^\dagger$ into the above leads to two possible values for the parameter α, namely $\alpha = \pm 1$. These values yield the new operators

$$b_1^\dagger = \frac{1}{\sqrt{2}}(a_x^\dagger + i a_y^\dagger), \quad b_2^\dagger = \frac{1}{\sqrt{2}}(a_x^\dagger - i a_y^\dagger), \tag{5.27}$$

which obey the ladder operator relations we were seeking,

$$[L_z, b_1^\dagger] = \hbar b_1^\dagger, \quad [L_z, b_2^\dagger] = -\hbar b_2^\dagger. \tag{5.28}$$

In terms of these new operators, the Hamiltonian can be expressed as

$$H = \hbar\omega(b_1^\dagger b_1 + b_2^\dagger b_2 + 1), \tag{5.29}$$

and the eigenstates become

$$|n,l\rangle = \frac{(b_1^\dagger)^N}{\sqrt{N!}}\frac{(b_2^\dagger)^M}{\sqrt{M!}}|0,0\rangle, \quad n = N + M, \quad l = N - M. \tag{5.30}$$

Thus, the eigenstates of the Harmonic oscillator can be created by applying the ladder operators successively. The angular momentum of the state $|n,l\rangle$ is $\hbar l$ and its energy is $E_{n,l} = \hbar\omega(n+1)$. This way, the ladder operators move the states diagonally in n, l space. The relationship between the n, l and N, M integers tells us that the set of states with energy quantum number n is $(n+1)$-fold degenerate, with the angular momentum l ranging between $l = -n$ to $l = n$ in steps of two. This can be seen schematically in Figure 5.2.

The normalized real-space wavefunctions $\langle r, \theta | n, l\rangle$ corresponding to these eigenstates are generally written as

$$\begin{aligned} \psi_{N,l}(r,\theta) &\equiv \langle r,\theta | 2N + |l|, l\rangle \\ &= \omega^{|l|+1/2}\sqrt{\frac{N!}{\pi(N+|l|)!}} e^{il\theta} r^l e^{-\omega r^2/2} L_N^{|l|}(\omega r^2), \end{aligned} \tag{5.31}$$

where $L_n^{|m|}$ is the associated Legendre polynomial

$$L_n^{|m|}(x) = \frac{x^{-|m|}e^x}{n!}\frac{d^n}{dx^n}e^{-x}x^{n+|m|}. \tag{5.32}$$

The notation $\langle r, \theta | 2N + |l|, l \rangle$, which is employed to simplify the labeling of the Legendre polynomials, indicates that these states have energies

$$E_{N,l} = \hbar\omega(2N + |l| + 1). \tag{5.33}$$

Now that we know the structure of the harmonic oscillator eigenstates in a suitable basis, we must find the eigenstates for the whole BdG single-particle Hamiltonian (5.5).

5.4 Finding the BdG Eigenstates

The single-particle BdG Hamiltonian involves the gap parameter Δ, whose form depends on the specifics of the system considered. Here we consider a system of fermions characterized by an order parameter with a $p_x + ip_y$ symmetry. This can be, for example, a single atomic layer of ^{3}He in the A-phase and with the angular momentum director $\mathbf{l}$ of the Cooper pairs pointing in the $+\hat{z}$ direction. The Hamiltonian we consider is then the BdG operator

$$H_{\text{BdG}} = \begin{pmatrix} H - \mu & \Delta(p_x + ip_y) \\ \Delta(p_x - ip_y) & -(H - \mu) \end{pmatrix}. \tag{5.34}$$

Here H is the harmonic oscillator Hamiltonian that we studied in section 5.3, μ is a chemical potential that controls the number of particles in the trap, and Δ is a scalar parameter. The off-diagonal gap operator can be written in terms of the $b, b^\dagger$ by simply following the definitions in (5.22, 5.27) as

$$\Delta(p_x + ip_y) = i\Delta\sqrt{\hbar\omega}(b_1^\dagger - b_2). \tag{5.35}$$

We had previously seen in (5.21) that the above is a ladder operator of L_z, increasing the angular momentum of any eigenstate on which it acts by unity. This agrees with the Cooper pair condensate picture, in which each electron pair possesses an angular momentum $+\hbar\hat{z}$. Because of the structure of the above Hamiltonian, a state characterized by the angular momentum quantum number l of the form $(\sum_m a_m|m, l+1\rangle, \sum_m b_m|m, l\rangle)^T$ will remain invariant in the value of l under the action of H_{BdG}. Therefore, we will consider eigenstates of the form

$$\Psi_{m,l}(r,\theta) = \begin{pmatrix} iu_{m,l}(r,\theta) \\ v_{m,l}(r,\theta) \end{pmatrix} = \begin{pmatrix} i\sum_n u^n_{m,l}\langle r,\theta|n, l+1\rangle \\ \sum_n v^n_{m,l}\langle r,\theta|n,l\rangle \end{pmatrix} \tag{5.36}$$

with eigenvalues $E = E_{m,l}$. Thus, for a given value of l we have an invariant subspace of eigenstates labeled by the index m. The way these states are connected by the H_{BdG} Hamiltonian is schematically shown in Figure 5.2. Note that the states are labeled by the angular momentum of the lower

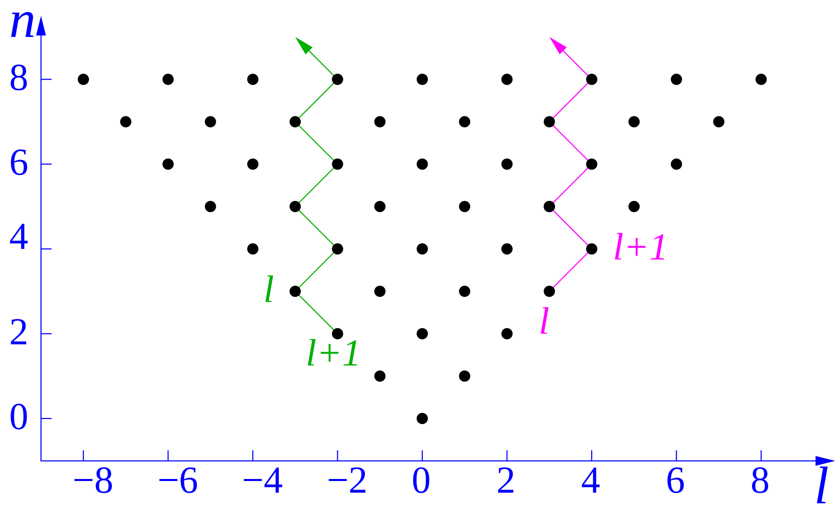

Figure 5.2: The low-lying part of the harmonic oscillator spectrum. The zig-zag paths indicate how states are coupled by the tridiagonal matrix $H^{(l)}$. For l positive, the first entry in the eigenvector will be a "v", and the second entry a "u", and so on. For l negative, the u's and v's are interchanged.

component v. Because of the unit off-set between the angular momentum of the u and the v's, the $E \leftrightarrow -E$ pairing is between eigenstates with angular-momentum label l and those with label $-(l+1) = -l-1$.

We need to find how the H_{BdG} Hamiltonian acts on the proposed eigenstates of (5.36). Before we present the results, we need to explain the approach we will take. Historically, the derivation in ([6]) contained an error, which was compensated by a discrepancy in the phase between the Legendre polynomials as defined in (5.32), and the actual Legendre polynomials used when running the simulation code in the Mathematica 6 software. Therefore, although the matrices shown in equations 32 and 33 in ([6]) are correct, there are two intermediate steps that need to be clarified to understand the results correctly. Therefore, we will look at how the H_{BdG} Hamiltonian acts on slightly more general states than those of (5.36).

Considering the most general form of the eigenstates Ψ_m, l, we have that

$$\Psi_{m,l} = \left[\begin{array}{c} \alpha \sum_n a^n u^n_{m,l} |n, l+1\rangle \\ \beta \sum_n b^n v^n_{m,l} |n, l\rangle \end{array} \right], \tag{5.37}$$

where the coefficients α, β, a, b are complex numbers. In particular, the case $a = b = 1, \beta = 1, \alpha = i$ reduces to the proposed eigenstates of (5.36) used in [6]. After some straightforward algebra, the action of the BdG Hamiltonian on this state can be written as

$$\begin{aligned} H\Psi^{\uparrow}_{m,l} &= \sum_{n \geq l+1} \alpha a^n \left[u^n_{m,l}\epsilon_n + i\Delta \frac{\beta}{\alpha}\left(\frac{b}{a}\right)^{n-1} \frac{1}{a}\sqrt{\frac{n+l+1}{2}} v^{n-1}_{m,l} \right. \\ &\quad \left. - i\frac{\beta}{\alpha}\left(\frac{b}{a}\right)^{n} b v^{n+1}_{m,l}\sqrt{\frac{n-l+1}{2}} \right] |n, l+1\rangle \end{aligned} \tag{5.38}$$

$$\begin{aligned} H\Psi^{\downarrow}_{m,l} &= \sum_{n \geq l} \beta b^n \left[-\epsilon_n v^n_{m,l} + i\Delta \frac{\alpha}{\beta}\left(\frac{a}{b}\right)^{n-1} \frac{1}{b}\sqrt{\frac{n-l}{2}} u^{n-1}_{m,l} \right. \\ &\quad \left. - i\frac{\alpha}{\beta}\left(\frac{a}{b}\right)^{n} a\sqrt{\frac{n+l+2}{2}} u^{n+1}_{m,l} \right] |n, l\rangle, \end{aligned} \tag{5.39}$$

where $l \geq 0$, and $\Psi^{\uparrow}_{m,l}, \Psi^{\downarrow}_{m,l}$ denote the upper and lower components of $\Psi_{m,l}$ respectively. The $l < 0$ case is similar to this one with the sums in n starting at $n \geq -(l+1)$ and $n > -l$ for the upper and lower terms. By looking at the full eigenvector equation $H\Psi_{m,l} = E_{m,l}\Psi_{m,l}$ we notice that since the $|n, l\rangle$ and $|n, l+1\rangle$ are independent, each term within the n-sums must satisfy the equality condition. Therefore, we have a series of equations involving coefficients $u^i_{m,l}, v^i_{m,l}$, which can be thought of as a matrix expressed in a basis of vectors composed of these u's and v's. Choosing an alternating basis $v^l, u^{l+1}, v^{l+2}, \ldots$, the above equations can be written as the eigenvalue problem of a matrix $H^{(l)}$, where the eigenvectors satisfy

$H^{(l)}(\alpha_1 v_l^1, \alpha_2 u_l^2, \alpha_3 v_l^3, \ldots)^T = E_{m,l}(\alpha_1 v_l^1, \alpha_2 u_l^2, \alpha_3 v_l^3, \ldots)^T$ ($l > 0$ case). This matrix reads

$$H^{(l\geq 0)} = \begin{pmatrix} -\epsilon_l & -i\Delta\frac{\alpha}{\beta}\left(\frac{a}{b}\right)^l a\sqrt{l+1} & 0 & \ldots \\ i\Delta\frac{\beta}{\alpha}\left(\frac{b}{a}\right)^l \frac{1}{a}\sqrt{l+1} & \epsilon_{l+1} & -i\frac{\beta}{\alpha}\left(\frac{b}{a}\right)^{l+1} b\sqrt{1} & 0 \\ 0 & i\frac{\alpha}{\beta}\left(\frac{a}{b}\right)^{l+1}\frac{1}{b}\sqrt{1} & -\epsilon_{l+2} & 0 \\ \vdots & 0 & \ddots & \ddots \end{pmatrix}. \tag{5.40}$$

When we choose $a = b = 1, \beta = 1, \alpha = i$, which leads to the same eigenstates considered in (5.36) and [6], the $(-|l|, -|l|+1)$-subspace BdG matrix in the $l \geq 0$ case becomes

$$H^{(l\geq 0)} = \begin{pmatrix} -\epsilon_l & \Delta\sqrt{l+1} & & & 0 & \ldots \\ \Delta\sqrt{l+1} & \epsilon_{l+1} & -\Delta\sqrt{1} & & & \\ & -\Delta\sqrt{1} & -\epsilon_{l+2} & \Delta\sqrt{l+2} & & \\ & & \Delta\sqrt{l+2} & \epsilon_{l+3} & -\Delta\sqrt{2} & \\ 0 & & & -\Delta\sqrt{2} & -\epsilon_{l+4} & \ddots \\ \vdots & & & & \ddots & \ddots \end{pmatrix}, \tag{5.41}$$

where the $l < 0$ case yields

$$H^{(l<0)} = \begin{pmatrix} \epsilon_{-l-1} & \Delta\sqrt{-l} & & & 0 & \ldots \\ \Delta\sqrt{-l} & -\epsilon_{-l} & -\Delta\sqrt{1} & & & \\ & -\Delta\sqrt{1} & \epsilon_{-l+1} & \Delta\sqrt{-l+1} & & \\ & & \Delta\sqrt{-l+1} & -\epsilon_{-l+2} & -\Delta\sqrt{2} & \\ 0 & & & -\Delta\sqrt{2} & \epsilon_{-l+3} & \ddots \\ \vdots & & & & \ddots & \ddots \end{pmatrix}. \tag{5.42}$$

As we mentioned before, these angular momentum restricted matrices are not, however, the ones appearing in equations 32 and 33 of [6], which do not contain the alternating sign in the off-diagonal elements. This is despite the fact that the starting definitions for the eigenstates are the same in both cases. We already mentioned that even though our matrices in (5.41) and (5.42) are correct, they become modified when we take into account the phase of the Laguerre polynomials used in the Mathematica 6 software.

The real-space wavefunctions already defined in (5.31) can be written in terms of the N, M quantum numbers of equation (5.30) as

$$|n,l\rangle = \left(\frac{\omega}{\hbar}\right)^{\frac{N+M}{2}} \frac{1}{2^{N+M}\sqrt{N!M!}} \left(z - \frac{\hbar}{\omega}\partial_z\right)^N \left(\bar{z} - \frac{\hbar}{\omega}\partial_{\bar{z}}\right)^M |0,0\rangle, \quad (5.43)$$

where $|0,0\rangle \simeq \exp(-z\bar{z})$. The Mathematica harmonic oscillator eigenstates are written like

$$\begin{aligned} &|n,l\rangle_{\text{Mathematica}} = \\ &= \left(\frac{\omega}{\hbar}\right)^{\frac{N+M}{2}} \frac{i^M}{2^{N+M}\sqrt{N!M!}} \left(z - \frac{\hbar}{\omega}\partial_z\right)^N \left(\bar{z} - \frac{\hbar}{\omega}\partial_{\bar{z}}\right)^M |0,0\rangle, \end{aligned} \quad (5.44)$$

which contain an extra $i^{(n-l)}$ phase from the definition of the Legendre polynomials. To make both pictures compatible, we must redefined the phase of our harmonic oscillator eigenstates to match the Mathematica 6 ones,

$$|n,l\rangle \to |n,l\rangle_{\text{Mathematica}} = i^{n-l}|n,l\rangle = (-1)^{\frac{n-l}{2}}|n,l\rangle. \quad (5.45)$$

Given this phase change, it is straightforward to calculate how (5.39) changes and arrive at the new corrected $H^{(l)}$ matrices. Alternatively, we can reuse our existing calculations if we find how the coefficients a, b, α, β of the generalized eigenstate (5.37) change upon this transformation. From their definitions, we see that $\alpha \to \alpha/i, \beta \to \beta/i, a \to ai, b \to bi$. Looking at the matrix (5.40), we see that the only changes with respect to the original results occur in the off-diagonal terms, which have alternating factors of the form $\beta/(\alpha a)$ and $\alpha/(\beta b)$. The former picks up a -1 phase, while the latter remains invariant. This is just what we need to absorb the alternating sign! Thus, the $H^{(l)}$ matrices compatible with the Mathematica 6 harmonic oscillator eigenstates become

$$H^{(l\geq 0)} = \begin{pmatrix} -\epsilon_l & \Delta\sqrt{l+1} & & & 0 & \dots \\ \Delta\sqrt{l+1} & \epsilon_{l+1} & \Delta\sqrt{1} & & & \\ & \Delta\sqrt{1} & -\epsilon_{l+2} & \Delta\sqrt{l+2} & & \\ & & \Delta\sqrt{l+2} & \epsilon_{l+3} & \Delta\sqrt{2} & \\ 0 & & & \Delta\sqrt{2} & -\epsilon_{l+4} & \ddots \\ \vdots & & & & \ddots & \ddots \end{pmatrix}, \quad (5.46)$$

where the $l < 0$ case yields

$$H^{(l<0)} = \begin{pmatrix} \epsilon_{-l-1} & \Delta\sqrt{-l} & & & & 0 & \dots \\ \Delta\sqrt{-l} & -\epsilon_{-l} & \Delta\sqrt{1} & & & & \\ & \Delta\sqrt{1} & \epsilon_{-l+1} & \Delta\sqrt{-l+1} & & & \\ & & \Delta\sqrt{-l+1} & -\epsilon_{-l+2} & \Delta\sqrt{2} & & \\ 0 & & & \Delta\sqrt{2} & \epsilon_{-l+3} & \ddots & \\ \vdots & & & & \ddots & \ddots & \end{pmatrix}. \qquad (5.47)$$

These are the matrices used in [6]. Through the previous argument we have justified their use when used in Mathematica 6 calculations. For each value of l, they provide us with a set of eigenvectors from which we can read the components $u^n_{m,l}$ and $v^n_{m,l}$, and since they are tri-diagonal, they are easy to diagonalize numerically. In the next section we explore the dependence of the gap parameter Δ on the spectrum.

5.5 Numerical Results

5.5.1 The Energy Spectrum

Using simple Mathematica 6 code [34], the spectrum and the eigenvectors $u^n_{m,l}$ and $v^n_{m,l}$ can be computed for a variety of values of μ and Δ. The value of $\hbar\omega$ has been set to unity in all plots, since changing ω servers only to rescale the energy and r.

In Figure 5.3, we see a plot of the eigenvalues for $\mu = 40.1$, $\Delta = 0.5$. Just as in Figure 5.1, we can see the interpenetrating wedges of $\pm E$ copies the harmonic oscillator spectrum, plus a non-zero gap lying symmetrically around $\epsilon = 0$. There is a special group of states crossing the gap from the upper continuum to the lower as l increases. These states are referred to as a chiral Majorana edge mode, their existence first pointed out by Volovik [35]. Their name originates from the fact that they live on the physical edge of the system, and also satisfy the Majorana equation [36] [7].

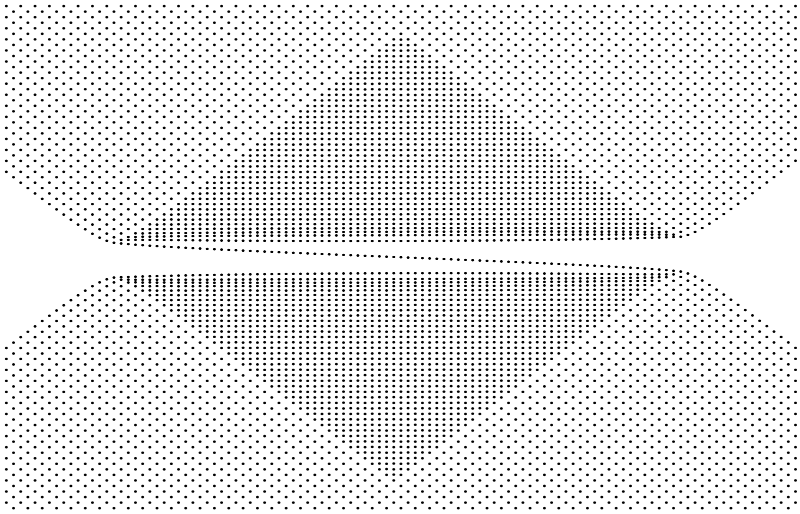

Figure 5.3: BdG operator eigenvalue spectrum for $\mu = 40.1$ and $\Delta = 0.5$. The axes have been suppressed. The angular momentum l ranges from -55 to $+55$ on the x-axis, while the energy eigenvalues $E_{m,l}$ are plotted vertically. Each column of eigenvalues is the result of diagonalizing a 100-by-100 tridiagonal matrix for each l. The number of particles in the system can be visually obtained by counting the number of dots from the upper branch that dip into negative energies.

5.5.2 Fluid Density and Angular Momentum Distributions

The fluid density, which we had expressed in terms of the eigenvectors of the BdG Hamiltonian in (5.19), has been plotted as a function of the radius in Figure 5.4. In order to check its validity, it is useful to compare it to that of the one obtained by the Thomas-Fermi approximation, which in its simplest form considers a set of non-interacting fermions in an external potential. The purpose of the approximation is to provide a relationship between the mass density and the external potential without the need to solve Schroedinger's equation, by generalizing the uniform-density results to the non-uniform case. The details of the model and its justification are laid out on Appendix C.1. The main result that concerns us is the expression of the mass density as a function of the external and chemical potentials,

$$\rho(r)_{\mathrm{TF}} = \frac{1}{2\pi\hbar^2} m^2(\mu - V(r)). \tag{5.48}$$

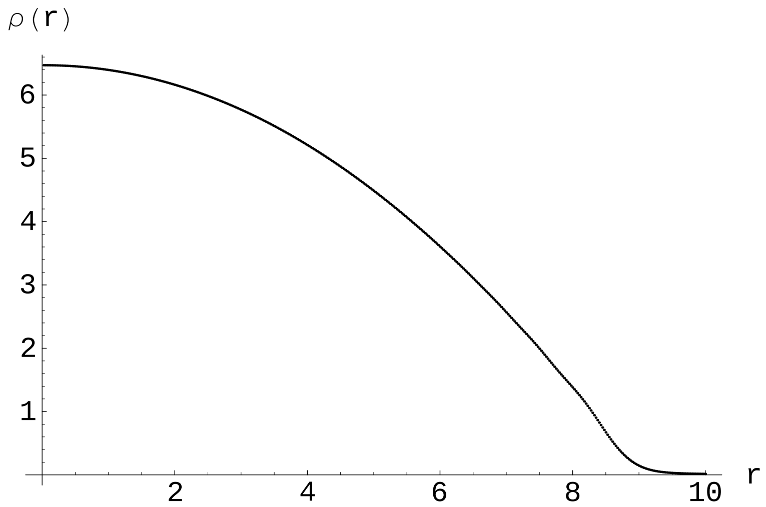

Figure 5.4: Fluid density $\rho(r)$ as a function of the radius for the case $N = 2\pi \int_0^\infty \rho(r) r dr = 840$, and $\Delta = 0.5$.

Considering the ground state of our system and the harmonic potential, and with the choice of $m = 1, \hbar = 1$, the Thomas-Fermi approximation predicts a mass density of the form

$$\rho_{\mathrm{TF}}(r) = \begin{cases} \frac{1}{2\pi}(\mu - \frac{1}{2}r^2) & r < \sqrt{2\mu} \\ 0 & r > \sqrt{2\mu} \end{cases}, \tag{5.49}$$

which follows closely the density in Figure 5.4. Under the same approximation, the particle number is estimated at

$$N_{\mathrm{TF}} = 2\pi \int_0^\infty \rho_{\mathrm{TF}}(r) r dr = \frac{1}{2}\mu^2. \tag{5.50}$$

This results in a total of 804 particles for $\mu = 40.1$. The actual particle number for the free-particle case $\Delta = 0$ is given by $N_0 = 1/2[\mu]([\mu]+1) = 820$, where $[\mu]$ indicates the integer part of μ. In our case, the non-zero value of Δ makes the particle number creep upwards compared to the gapless case, resulting in a particle number $N = 840$.

The angular momentum density, whose analytical form in terms of the BdG eigenvalues can be obtained from (5.18), is given by

$$L(r) = \langle \hat{L}_z \rangle = \sum_{l,m} v_{m,l} \left(-i \frac{\partial}{\partial \theta} \right) v_{m,l}^* = -\sum_{l,m} l |v_{m,l}(r,\theta)|^2. \tag{5.51}$$

This is plotted in Figure 5.5 as a function of the radius for the same parameters. We can see immediately that the distribution peaks towards the edge of the system, just opposite to the density. How the angular momentum per particle depends on the magnitude of the gap Δ can be seen in Figure 5.6.

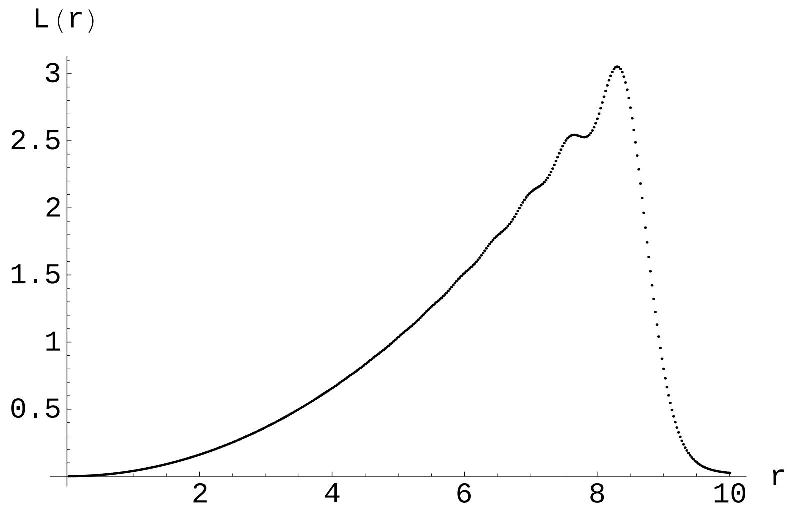

Figure 5.5: Angular momentum density $L(r)$ as a function of the radius for the same parameters as Figure 5.4. The total angular momentum is $L_{\text{tot}} = 2\pi \int_0^\infty L(r) r dr = 410$. Thus, for these parameters, $L_{\text{tot}}/N = 0.49$.

Except for very small Δ, we find invariably that

$$L_{\text{tot}} = 2\pi \int_0^\infty L(r) r dr \approx \frac{1}{2} N. \tag{5.52}$$

There is no simple identity lying behind this fact, and mathematically it results from a quite non trivial rearrangement of spectral weight between the positive and negative E eigenstates of any given l. We can, however, check the angular momentum distribution by relating it to the azimuthal mass flow distribution from which it arises. From Schroedinger's equation, we know that the mass current is given by $\vec{j} = \frac{\hbar}{2i}(\Psi^* \vec{\nabla} \Psi - \Psi \vec{\nabla} \Psi^*)$, where Ψ is the wavefunction of the system. Since the angular momentum satisfies $\vec{L} = \vec{r} \times \vec{j}_{\text{mass}}$, we can relate the angular component of mass flow to the $\hat{z}$-angular momentum component as

$$j_{\text{mass},\theta} = \frac{1}{2i} \sum_{m,l} \left\{ v_{m,l} \left(\frac{1}{r} \frac{\partial}{\partial \theta} v^*_{m,l} \right) - \left(\frac{1}{r} \frac{\partial}{\partial \theta} v_{m,l} \right) v^*_{m,l} \right\} = \frac{1}{r} L(r). \tag{5.53}$$

This quantity is plotted in Figure 5.7. The almost straight line mass flow re-

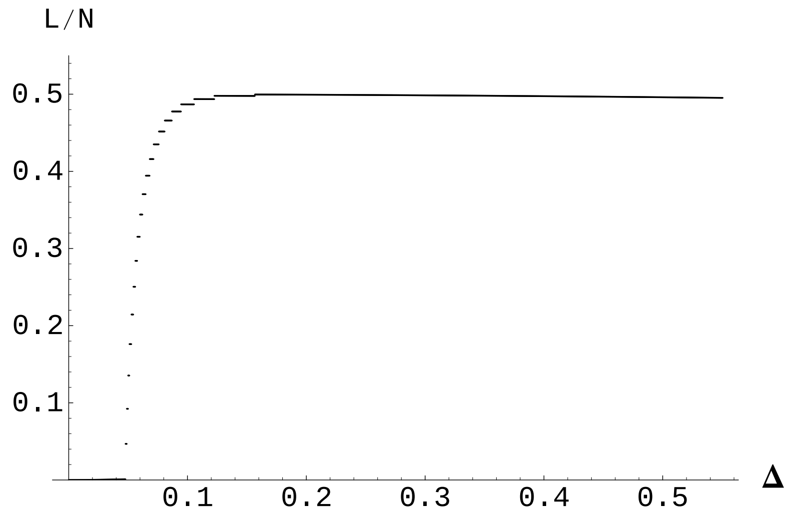

Figure 5.6: The ground state angular momentum per particle $L_{\rm tot}/N$ plotted as a function of Δ for $\mu = 40.1$ The discontinuous steps in the rising part of the graph are due to individual levels crossing $\epsilon = 0$, and changing their occupation. The $\Delta > 0.05$ ground state is therefore not adiabatically connected to the $\Delta = 0$ ground state.

veals that the mass current is well described by the Ishikawa-Mermin-Muzikar formula [37] [38], which predicts

$$\mathbf{j}_{\rm mass} = \frac{1}{2}\nabla \times \left(\frac{1}{2}\rho\hbar\mathbf{l}\right), \tag{5.54}$$

where l is the angular momentum unit vector. A review of the derivation of this formula following [7] is included in Appendix C.2. Using the Thomas-Fermi approximation for the density, the azimuthal mass flow inside the trap should be

$$j_{\rm TF,mass,\theta} = \frac{\hbar}{4}\nabla \times \left(\frac{1}{4\pi}r^2\hat{e}_z\right) = -\frac{\hbar}{8\pi}r\hat{e}_\theta. \tag{5.55}$$

Thus, the linearity of the azimuthal flow inside the trap confirms that the fluid has a near-parabolic particle density profile. The Ishikawa-Mermin-Muzikar formula also predicts a total angular momentum

$$L_{\rm tot} = \int (\mathbf{r} \times \mathbf{j}_{\rm mass}) d^3\mathbf{r} = \frac{1}{2}\hbar \int \rho \; \mathbf{l} \; d^3\mathbf{r} = \frac{1}{2}N\hbar\mathbf{l}, \tag{5.56}$$

which agrees with the numerical result in (5.52). The oscillations near the

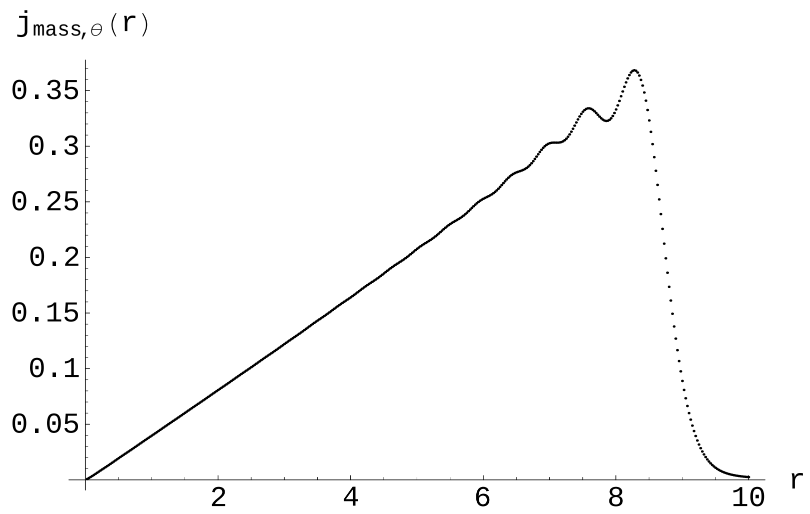

Figure 5.7: The ground state azimuthal mass flow $j_{\text{mass},\theta}(r) = L(r)/r$ corresponding to the angular momentum density in Figure 5.5.

abrupt drop of $j_{\text{mass},\theta}(r)$ to zero at the edge of the droplet of confined fluid are also seen in the numerical results of [33].

5.5.3 The Edge-Mode Contribution

It is interesting to ask how much of the mass flow and angular momentum is supplied by the chiral Majorana edge mode, since they play a special role in the BdG spectrum, providing the most striking $l \leftrightarrow -l$ asymmetry. Examination of Figure 5.3 shows that positive-energy states within the gap exist for each integer l in the range $-\mu$ to 0 [5]. We already know how to calculate the mean value of the angular momentum operator through (5.15), therefore we only need to find the coefficients $|\vec{v}|^2$ for each eigenstate.

The Majorana edge modes have been analytically found for a rigid-wall container [7], and shown that they are an equal superposition of particle and holes, that is, $|\vec{u}|^2 = |\vec{v}|^2 = \frac{1}{2}$. This has been verified numerically by Stone and Anduaga [6] as we will see below. Thus, the theoretical contribution of

[5] Remember that we are calculating operator mean values through equation (5.15), which involves the $\vec{v}_m$ branch of the positive energy eigenvalues. We could alternatively use the negative-energy eigenstates of the $\vec{u}_m$ branch, and thus use the gapped states that range from 0 to μ.

the edge modes to the angular momentum is

$$L_{\text{tot}}^{\text{edge-mode}} = \sum_{l=-\mu}^{0} \vec{v}_l^T \hat{L} \vec{v}_l^* = -\frac{1}{2} \sum_{l=-\mu}^{0} \hbar l = \frac{1}{4}\mu(\mu+1) \simeq \frac{1}{2} N\hbar. \tag{5.57}$$

This way, for the harmonic-trapped system the edge-mode contribution appears to be equal to total angular momentum of the system. We can now digress and compare this analytical result of the uniform fluid bounded by a rigid wall. The edge modes have been calculated in [7] and are given by

$$\xi_0(k_x) = \sqrt{\frac{\Delta}{2v_f}} e^{ik_f x cos\theta} \sin(k_f y \sin\theta) e^{-\Delta y/v_f} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{5.58}$$

In this case, the edge modes have a dispersion $E(k) = -\Delta(k/k_F)$ and merge into the continuum at $k = k_F$. The mass current for a given wavefunction Ψ is given by $\vec{j} = \frac{\hbar}{2i}(\Psi^* \vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*)$, which for these edge states reduces to $j = (k_F \cos\theta, k_F \sin\theta)$. Integrating over the wall, the boundary mass current yields

$$j_{\text{boundary}}^{\text{edge-mode}} = \frac{1}{2} \int_0^1 \frac{d(k_F \cos\theta)}{2\pi} k_F \cos\theta = \frac{\hbar}{8\pi} k_F^2 = \frac{\hbar}{2} \rho_{\text{bulk}}, \tag{5.59}$$

where $\rho_{\text{bulk}} = k_F^2/4\pi$ is the bulk fluid density. We can contrast this result to the harmonic-trap case by calculating its edge mass current. Since we have already seen that our numerical results agree with Ishikawa-Mermin-Muzikar formula, we can spatially integrate the mass current in (5.54) between the beginning of the edge modes and the end, which we call r_1 and r_2. Therefore, the density at these two points satisfies $\rho(r_1) = \rho_{\text{bulk}}$ and $\rho(r_2) = 0$. These allows us to write

$$j_{\text{boundary,ht}}^{\text{edge-mode}} = \frac{1}{2} \int_{r_1}^{r_2} -\frac{1}{2} \frac{\partial}{\partial r} \hbar \rho(r) dr = -\frac{\hbar}{4}(\rho(r_2) - \rho(r_1)) = \frac{\hbar}{4} \rho_{\text{bulk}}, \tag{5.60}$$

where we've added the label ht to avoid confusion with the rigid wall current. Comparing this result to the previous, we see that the rigid-wall edge modes oversupply angular momentum by a factor of two. It has been shown in [7] that this twice-too-large bound state angular momentum is reduced by contributions from the unbound continuum states, and that the resulting edge momentum density is exactly what is required to give the $N\hbar/2$ total angular momentum.

To investigate whether the edge modes are the source of the *entire* harmonic trap edge current, their contribution to the angular momentum density and the mass-flow current has been isolated. To do this, it is necessary to identify which of the within-the-gap eigenstates is indeed a bounded state. Figure 5.8 shows that this is easy to do simply by looking at the modulus of the v-branch eigenstates, since we know that the positive energy edge modes must have $|\vec{v}|^2 = 1/2$. When we sum the contributions to the angular momentum

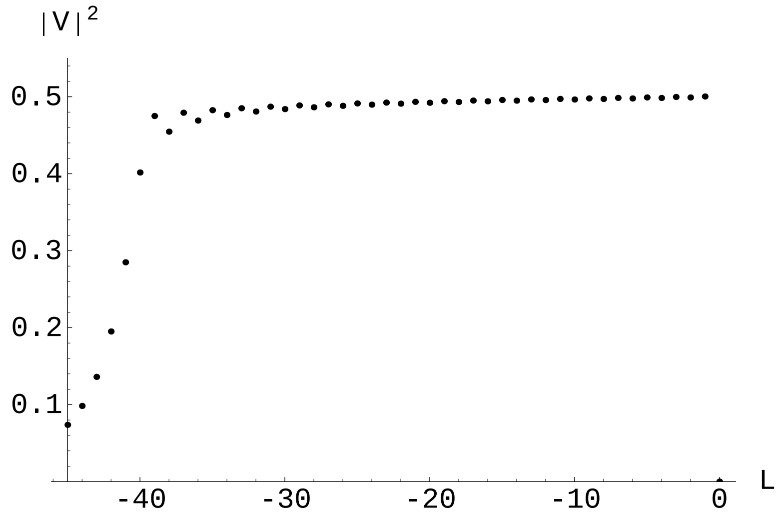

Figure 5.8: The coefficient $|\vec{v}|^2$ for the lowest positive energy modes within the gap. The point at which the edge-modes merge into the upper continuum is signaled by the sharp decrease in $|v|^2$ near $l = -40$, where the states transition into the particle u branch. The parameters are the same as those in Figure 5.4.

density and the mass flow from these states only, we obtain the results shown in Figures 5.9 and 5.10. As anticipated, the *total* angular momentum of the edge mode distribution is the same as the complete angular momentum distribution,

$$2\pi \int_0^\infty L^{\text{edge-mode}}(r) r dr \simeq 2\pi \int_0^\infty L(r) r dr. \tag{5.61}$$

However, both $L^{\text{edge-mode}}(r)$ and $j^{\text{edge-mode}}_{\text{mass},\theta}(r)$ are localized only near the boundary of the fluid, and differ substantially from $L(r)$ and $j_{\text{mass},\theta}$. It can thus be concluded that, as with the rigid wall model of [7], the continuum

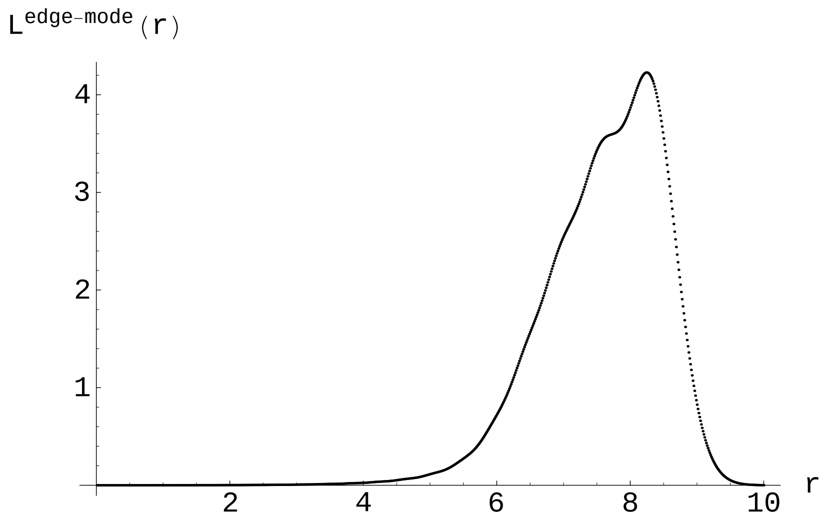

Figure 5.9: The edge mode contribution to the angular momentum density $L(r)$ for the same parameters as Figure 5.4. The contributions for $-41 \leq l \leq 0$ are included.

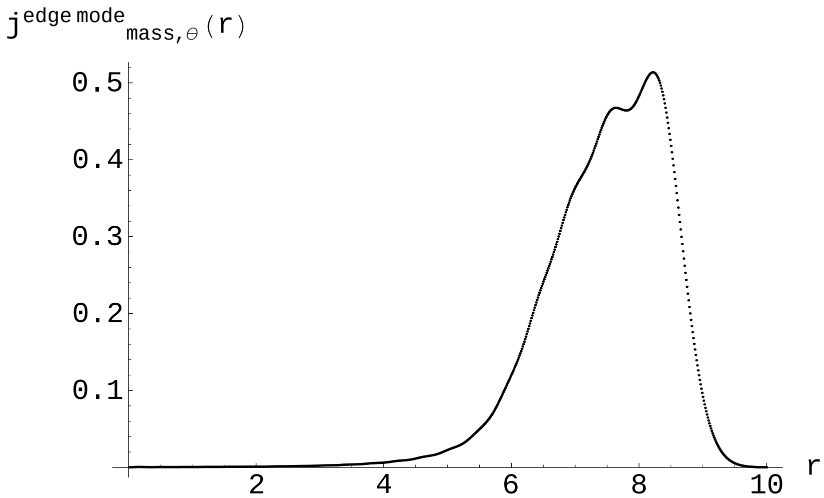

Figure 5.10: The $-41 \leq l \leq 0$ edge mode contribution to the mass flow for the same parameters as Figure 5.4.

modes provide an important component of the mass-flow. The bound-state angular momentum contribution being equal to the total angular momentum should therefore be regarded as a coincidence arising from the particular form of the harmonic trap density profile.

5.6 Discussion

We have seen that the numerical results of the previous section are in agreement with theoretical predictions such as the Ishikawa-Mermin-Muzikar formula and the rigid-wall model of Stone and Roy. Why is it, then, that other estimate of the angular momentum, such as [31], find results that are suppressed by powers of Δ/ϵ_F? The argument for this suppression is that the only contributions to the angular momentum come from the $O(\Delta/\epsilon_F)$ particle-hole asymmetry due to the curvature of the dispersion relation near the Fermi surface, all other contributions cancelling one-another. We can see why such argument is not correct by looking at the problem of calculating the mass current, which can be cast [39] into a weighted sum over k_z of the quantity

$$j_{k_z} = \lim_{s \to 0} \left\{ -\frac{1}{2} \sum_n \mathrm{sgn}(E_{n,k_z}) |E_{n,k_z}|^{-s} \right\}, \tag{5.62}$$

where E_{n,k_z} are the eigenvalues of the Dirac Hamiltonian

$$H_{\mathrm{Dirac}} = -i\sigma_3 \partial_x + \sigma_2 k_z + m(x)\sigma_1, \tag{5.63}$$

in which $m(x)$ changes sign as x changes passes through zero. Given the above, we can build an associated operator through the relation

$$Q = \sigma_2 H_{\mathrm{Dirac}} - k_z. \tag{5.64}$$

This new operator anticommutes with the Dirac Hamiltonian,

$$\{Q, H_{\mathrm{Dirac}}\} = 0. \tag{5.65}$$

This suggests that if Ψ is an eigenstate of H_{Dirac}, then $Q\Psi$ will also be one with the opposite energy (except for the $E = k_z$ topologically bound state which is annihilated by Q). If one looks at equation 5.62, then it is easy to conclude that all contributions to the mass current would cancel except for the bound state one. However, on closer analysis we realize that the pairing $(\Psi, Q\Psi)$ with energies $(E, -E)$ is illusory: In order to have a properly defined eigenvalue problem for H_{Dirac}, we must impose self-adjoint boundary conditions on the eigenfuctions. If Ψ obeys these boundary conditions, then, in general, $Q\Psi$ will not. Thus, there is no general cancellation between the different contributions to the total angular momentum.

5.7 Conclusions

In this chapter we have seen that, when exactly solved, the BdG formalism produces a mass flow and angular momentum that coincides to that obtained from the Cooper-pair wavefunction: The zero-temperature intrinsic angular momentum is $\frac{1}{2}\hbar$ per particle, and there is no Δ/ϵ_F suppression. This has been contrasted with analytical predictions found in previous work by Ishikawa [37], Mermin-Muzikar [38], and particularly in [7], arriving at the same value for the total angular momentum through direct numerical calculations.

A Quantum Mechanical Approach to the Calogero-Sutherland Model

A.1 Derivation of the Sutherland Hamiltonian

We start by expanding the terms in the Hamiltonian 2.5,

$$
\begin{aligned}
H\Psi &= \frac{1}{2}\sum_i \left(-\frac{1}{\Delta^\lambda}\frac{\partial}{\partial\theta_i}\Delta^\lambda\right)\left(\Delta^\lambda\frac{\partial}{\partial\theta_i}\frac{1}{\Delta^\lambda}\right)\Psi \\
&= \sum_i \frac{-1}{\Delta^\lambda}\frac{\partial}{\partial\theta_i}\left(\Delta^{2\lambda}\frac{\left(\Delta^\lambda\frac{\partial\Psi}{\partial\theta_i} - \Psi\frac{\partial}{\partial\theta_i}\Delta^\lambda\right)}{\Delta^{2\lambda}}\right) \\
&= \frac{1}{2}\left[\sum_i \frac{-1}{\Delta^\lambda}\frac{\partial}{\partial\theta_i}\left(\Delta^\lambda\frac{\partial}{\partial\theta_i} - \frac{\partial}{\partial\theta_i}\Delta^\lambda\right)\right]\Psi \\
&= \frac{1}{2}\left[\sum_i \frac{-1}{\Delta^\lambda}\left(\frac{\partial}{\partial\theta_i}\Delta^\lambda\frac{\partial}{\partial\theta_i} + \Delta^\lambda\frac{\partial^2}{\partial\theta_i^2} - \frac{\partial}{\partial\theta_i}\left(\frac{\partial}{\partial\theta_i}\Delta^\lambda\right)\right)\right]\Psi \\
&= \left[\frac{1}{2}\sum_i\left(-\frac{\partial^2}{\partial\theta_i^2} + \frac{1}{\Delta^\lambda}\frac{\partial^2\Delta^\lambda}{\partial\theta_i^2}\right)\right]\Psi. \qquad \text{(A.1)}
\end{aligned}
$$

We need to calculate the second derivative of Δ^λ. For this, we start with the first derivative,

$$
\begin{aligned}
\frac{\partial}{\partial\theta_i}\Delta^\lambda &= \frac{\partial}{\partial\theta_i}\left[\prod_{k<l} 2\sin\left(\frac{\theta_k-\theta_l}{2}\right)\right]^\lambda = \frac{\partial}{\partial\theta_i} e^{\lambda\sum_{k<l}\ln\left(2\sin\left(\frac{\theta_k-\theta_l}{2}\right)\right)} \\
&= e^{\lambda\sum_{k<l}\ln\left(2\sin\left(\frac{\theta_k-\theta_l}{2}\right)\right)}\frac{\partial}{\partial\theta_i}\left[\lambda\sum_{k<l}\ln\left(2\sin\left(\frac{\theta_k-\theta_l}{2}\right)\right)\right] \\
&= \Delta^\lambda\lambda\sum_{k<l}\frac{1}{2}\frac{\cos\left(\frac{\theta_k-\theta_l}{2}\right)\delta_{i,k}-\cos\left(\frac{\theta_k-\theta_l}{2}\delta_{i,l}\right)}{\sin\left(\frac{\theta_k-\theta_l}{2}\right)} \\
&= \Delta^\lambda\lambda\frac{1}{4}\sum_{\substack{k,l\\k\neq l}}\left(\mathrm{cotg}\left(\frac{\theta_k-\theta_l}{2}\right)\delta_{i,k}-\mathrm{cotg}\left(\frac{\theta_k-\theta_l}{2}\right)\delta_{i,l}\right) \\
&= \Delta^\lambda\lambda\frac{1}{4}\left(\sum_{\substack{l=1\\l\neq i}}\mathrm{cotg}\left(\frac{\theta_i-\theta_l}{2}\right)-\sum_{\substack{k=1\\k\neq i}}\mathrm{cotg}\left(\frac{\theta_k-\theta_i}{2}\right)\right) \\
&= \Delta^\lambda\frac{\lambda}{2}\sum_{\substack{l=1\\l\neq i}}\mathrm{cotg}\left(\frac{\theta_i-\theta_l}{2}\right). \qquad \text{(A.2)}
\end{aligned}
$$

Now we can proceed with the second derivative using (A.2). We get

$$
\frac{\partial^2\Delta^\lambda}{\partial\theta_i^2} = \Delta^\lambda\frac{\lambda^2}{2}\left(\sum_{\substack{j=1\\j\neq i}}\mathrm{cotg}\left(\frac{\theta_i-\theta_j}{2}\right)\right)^2 + \Delta^\lambda\frac{\lambda}{2}\sum_{\substack{j=1\\j\neq i}}\frac{\partial}{\partial\theta_i}\mathrm{cotg}\left(\frac{\theta_i-\theta_j}{2}\right) \qquad \text{(A.3)}
$$

This way, we can write the Hamiltonian in (A.1) as

$$
\begin{aligned}
2H &= \sum_i -\frac{\partial^2}{\partial\theta_i^2} + \underbrace{\frac{\lambda^2}{4}\sum_i\left(\sum_{\substack{j=1\\j\neq i}}\mathrm{cotg}\left(\frac{\theta_i-\theta_j}{2}\right)\right)^2}_{\mathrm{A}} \\
&\quad \underbrace{+\frac{\lambda}{2}\sum_i\sum_{\substack{j=1\\j\neq i}}\frac{\partial}{\partial\theta_i}\mathrm{cotg}\left(\frac{\theta_i-\theta_j}{2}\right)}_{\mathrm{B}}. \qquad \text{(A.4)}
\end{aligned}
$$

where we have labeled the right two terms A and B. To compute term A, the key is to split the sum in the j, l variables into "diagonal" and "non-diagonal" parts. We have

$$
\begin{aligned}
\mathrm{A} &= \frac{\lambda^2}{4}\sum_i \sum_{\substack{j,l \\ j\neq i \\ l\neq i}} \operatorname{cotg}\left(\frac{\theta_i-\theta_j}{2}\right)\operatorname{cotg}\left(\frac{\theta_i-\theta_l}{2}\right) \qquad (A.5)\\
&= \underbrace{\frac{\lambda^2}{4}\sum_{\substack{i,j \\ j\neq i}} \operatorname{cotg}^2\left(\frac{\theta_i-\theta_j}{2}\right)}_{A_1} + \underbrace{\frac{\lambda^2}{4}\sum_{\substack{i,j,l \\ j\neq i\neq l \\ l\neq j}} \operatorname{cotg}\left(\frac{\theta_i-\theta_j}{2}\right)\operatorname{cotg}\left(\frac{\theta_i-\theta_l}{2}\right)}_{A_2}.
\end{aligned}
$$

Using that $\operatorname{cotg}^2(\theta) = \sec^2(\theta) - 1$, we have

$$
\begin{aligned}
A_1 &= \frac{\lambda^2}{4}\sum_{\substack{i,j \\ j\neq i}} (\sec^2\left(\frac{\theta_i-\theta_j}{2}\right) - 1)\\
&= \frac{\lambda^2}{4}\sum_{\substack{i,j \\ j\neq i}} \frac{1}{\sin^2\left(\frac{\theta_i-\theta_j}{2}\right)} - N(N-1)\frac{\lambda^2}{4}. \qquad (A.6)
\end{aligned}
$$

To deal with the term A_2, we use the trigonometric identity

$$
\begin{aligned}
1 &= \operatorname{cotg}\left(\frac{\theta_j-\theta_i}{2}\right)\operatorname{cotg}\left(\frac{\theta_i-\theta_l}{2}\right) + \operatorname{cotg}\left(\frac{\theta_i-\theta_l}{2}\right)\operatorname{cotg}\left(\frac{\theta_l-\theta_j}{2}\right)\\
&\quad + \operatorname{cotg}\left(\frac{\theta_l-\theta_j}{2}\right)\operatorname{cotg}\left(\frac{\theta_j-\theta_i}{2}\right). \qquad (A.7)
\end{aligned}
$$

We have

$$
\begin{aligned}
A_2 &= -\frac{\lambda^2}{4}\sum_{j\neq i\neq l} \operatorname{cotg}\left(\frac{\theta_j-\theta_i}{2}\right)\operatorname{cotg}\left(\frac{\theta_i-\theta_l}{2}\right)\\
&= -\frac{\lambda^2}{4}2\sum_{j<i<l} \left[\operatorname{cotg}\left(\frac{\theta_j-\theta_i}{2}\right)\operatorname{cotg}\left(\frac{\theta_i-\theta_l}{2}\right)\right.\\
&\qquad \underbrace{\left. + \operatorname{cotg}\left(\frac{\theta_i-\theta_l}{2}\right)\operatorname{cotg}\left(\frac{\theta_l-\theta_j}{2}\right) + \ldots\right]}_{1}\\
&= -\frac{\lambda^2}{2}\sum_{j<i<l}^{N} 1 = -\frac{\lambda^2}{2}\frac{N}{6}(N-1)(N-2) \qquad (A.8)
\end{aligned}
$$

Putting together A_1 and A_2 we can write the final expression for A

$$\begin{aligned} A &= \frac{\lambda^2}{4} \sum_{\substack{i,j \\ j \neq i}} \frac{1}{\sin^2\left(\frac{\theta_i - \theta_j}{2}\right)} - \frac{\lambda^2}{2}\left(\frac{N(N-1)}{2} + \frac{N}{6}(N-1)(N-2)\right) \\ &= \frac{\lambda^2}{4} \sum_{\substack{i,j \\ j \neq i}} \frac{1}{\sin^2\left(\frac{\theta_i - \theta_j}{2}\right)} - \frac{\lambda^2}{12} N(N^2-1) \end{aligned} \tag{A.9}$$

The B term is simply given by

$$B = \frac{\lambda}{2} \sum_{\substack{j=1 \\ j \neq i}} \frac{(-1)}{2} \frac{1}{\sin^2\left(\frac{\theta_i - \theta_j}{2}\right)} = -\frac{\lambda}{4} \sum_{\substack{j=1 \\ j \neq i}} \frac{1}{\sin^2\left(\frac{\theta_i - \theta_j}{2}\right)} \tag{A.10}$$

Combining the final A and B terms, the Hamiltonian in (A.4) becomes

$$\begin{aligned} H &= -\frac{1}{2} \sum_i \frac{\partial^2}{\partial \theta_i^2} + \frac{1}{4} \sum_{i<j} \frac{\lambda(\lambda-1)}{\sin^2\left(\frac{\theta_i - \theta_j}{2}\right)} - \frac{\lambda^2}{24} N(N^2-1) \\ &= H_{\text{Sutherland}} - \frac{\lambda^2}{24} N(N^2-1). \end{aligned} \tag{A.11}$$

A.2 Change of Basis to Symmetric Variables

We want to apply the transformation given by (2.24) to the Hamiltonian (2.23). We have

$$\begin{aligned} \sum_i^N D_i^2 &= \sum_i z_i \frac{\partial}{\partial z_i} \left(\sum_{m=1}^N m z_i^m \frac{\partial}{\partial p_m} \right) \\ &= \sum_i^N \left[\sum_{m=1}^N m^2 z_i^m \frac{\partial}{\partial p_m} + \sum_{m=1}^N m z_i^m \left(z_i \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial p_m} \right] \\ &= \sum_{m=1}^N m^2 p_m \frac{\partial}{\partial p_m} + \sum_{i,m=1}^N m z_i^m \left(\sum_{n=1}^N n z_i^n \frac{\partial}{\partial p_n} \right) \frac{\partial}{\partial p_m} \\ &= \sum_{m=1}^N m^2 p_m \frac{\partial}{\partial p_m} + \sum_{m,n=1}^N mn p_{n+m} \frac{\partial}{\partial p_n} \frac{\partial}{\partial p_m}. \end{aligned} \tag{A.12}$$

To deal with the last term of the Hamiltonian (2.23), we must calculate the ratio of $(z_i^N - z_j^N)/(z_i - z_j)$, which we will do in two steps. First, we compute

$$
\begin{aligned}
z_i^N - z_j^N &= (z_i - z_j)(A_0 z_i^n - 1 + A_1 z_i^{n-2} z_j + \ldots + A_{N-1} z_j^{N-1}) \\
&= (A_0 z_i^N + A_1 z_i^{N-1} z_j + \ldots + A_{n-1} z_i z_j^{N-1}) \\
&\quad -(A_0 z_i^{N-1} z_j + A_1 z_i^{N-2} z_j^2 + \ldots + A_{n-2} z_i z_j^{N-1} + A_{N-1} z_j^N) \\
&= (A_0 z_i^N + (A_1 - A_0) z_i^{N-1} z_j + (A_2 - A_1) z_i^{N-2} z_j^2 + \ldots + \\
&\quad +(A_{N-1} - A_{N-2}) z_i z_j^{N-1} - A_{N-1} z_j^N. \qquad \text{(A.13)}
\end{aligned}
$$

Comparing both sides, we see that $A_N = 1, A_0 = 1$ and $A_{i+1} - A_i = 0$, which yields the identity

$$
z_i^N - z_j^N = (z_i - z_j)(z_i^{N-1} + z_i^{N-2} z_j + \ldots + z_i z_j^{N-2} + z_j^{N-1}). \qquad \text{(A.14)}
$$

Secondly, we look at

$$
\begin{aligned}
(z_i + z_j)\frac{z_i^N - z_j^N}{z_i - z_j} &= (z_i - z_j)(z_i^{N-1} + z_i^{N-2} z_j + \ldots + z_i z_j^{N-2} + z_j^{N-1}) \\
&= (z_i^N + z_i^{N-1} z_j + \ldots + z_i^2 z_j^{N-2} + z_i z_j^{N-1}) \\
&\quad +(z_i^{N-1} z_j + z_i^{N-2} z_j^2 + \ldots + z_i z_j^{N-1} + z_j^N). \qquad \text{(A.15)}
\end{aligned}
$$

It is clear that all the inner terms are repeated twice while z_i^N and z_j^N appear only once, thus we can write

$$
(z_i + z_j)\frac{z_i^N - z_j^N}{z_i - z_j} = z_i^N + 2 z_i^{N-1} z_j + 2 z_i^{N-2} z_j^2 + \ldots + 2 z_i z_j^{N-1} + z_j^N. \quad \text{(A.16)}
$$

We are now ready to write the second term from (2.23). We have

$$
\begin{aligned}
\frac{\lambda}{2} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j}(D_i - D_j) &= \frac{\lambda}{2} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \sum_{l=1}^{N} l(z_i^l - z_j^l)\frac{\partial}{\partial p_l} \qquad \text{(A.17)} \\
&= \frac{\lambda}{2}\frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} l(z_i^l + 2 z_i^{l-1} z_j + \ldots + 2 z_i z_j^{l-1} + z_j^l)\frac{\partial}{\partial p_l}.
\end{aligned}
$$

Because the condition $i \neq j$ would complicate things after we performed one of the sums (for example in the j variable), we avoid this by writing

$\sum_{i\neq j} = \sum_{i,j} - \sum_{i=j}$. Thus, we obtain

$$\frac{\lambda}{2}\sum_{i<j}\frac{z_i+z_j}{z_i-z_j}(D_i-D_j) = \underbrace{\frac{\lambda}{2}\frac{1}{2}\sum_{i,j,l}^{N} l(z_i^l+2z_i^{l-1}z_j+\ldots+2z_iz_j^{l-1}+z_j^l)\frac{\partial}{\partial p_l}}_{\text{A}}$$
$$-\frac{\lambda}{2}\frac{1}{2}\sum_{\substack{i=1\\l=1\\(i=j)}}^{N} l(z_i^l+2z_i^{l-1}z_i+\ldots+2z_iz_i^{l-1}+z_i^l)\frac{\partial}{\partial p_l} \tag{A.18}$$

The sum of all the z_i's in the second term is equal to $2(l-1)z_i^l + 2z_i^l = 2lz_i^l$ (the first contribution comes from the middle terms and the second one from the endpoints). For the first term, we perform the sum first over i and then over j. We have

$$\begin{aligned}
\text{A} &= \frac{\lambda}{2}\frac{1}{2}\sum_{j,l}^{N} l(p_l+2p_{l-1}z_j+\ldots+2p_1z_j^{l-1}+Nz_j^l)\frac{\partial}{\partial p_l} \\
&= \frac{\lambda}{2}\frac{1}{2}\sum_{l=1}^{N} l(Np_l+2p_{l-1}p_1+\ldots+2p_1p_{l-1}+Np_l)\frac{\partial}{\partial p_l} \\
&= \frac{\lambda}{2}\frac{1}{2}\left(\sum_{l=1}^{N} 2l(p_{l-1}p_1+\ldots+p_1p_{l-1})\frac{\partial}{\partial p_l}+\sum_{l=1}^{N}2Np_l\frac{\partial}{\partial p_l}\right).
\end{aligned} \tag{A.19}$$

Putting both terms together, we can write

$$\begin{aligned}
\frac{\lambda}{2}\sum_{i<j}\frac{z_i+z_j}{z_i-z_j}(D_i-D_j) &= \frac{\lambda}{2}\frac{1}{2}\left(\sum_{l=1}^{N} 2l(p_{l-1}p_1+\ldots+p_1p_{l-1})\frac{\partial}{\partial p_l}\right. \\
&\qquad \left.+\sum_{l=1}^{N}2Np_l\frac{\partial}{\partial p_l}\right) - \frac{\lambda}{2}\frac{1}{2}\sum_{l=1}^{N}2l^2p_l\frac{\partial}{\partial p_l} \\
&= \frac{\lambda}{2}\sum_{l=1}^{N} l(p_{l-1}p_1+\ldots+p_1p_{l-1})\frac{\partial}{\partial p_l} \\
&\quad +\frac{\lambda}{2}\sum_{l=1}^{N} l(N-l)p_l\frac{\partial}{\partial p_l}.
\end{aligned} \tag{A.20}$$

We can write the first term more conveniently in terms of a double sum in the n, m variables, such that $n + m = l$. This way, we have

$$
\begin{aligned}
\frac{\lambda}{2}\sum_{i<j}\frac{z_i+z_j}{z_i-z_j}(D_i-D_j) &= \frac{\lambda}{2}\sum_{m,n=1}^{n+m\leq N}(m+n)p_m p_n\frac{\partial}{\partial p_{n+m}} \\
&+\frac{\lambda}{2}\sum_{l=1}^{N} l(N-l)p_l\frac{\partial}{\partial p_l}. \qquad \text{(A.21)}
\end{aligned}
$$

Therefore, using the results in (A.12) and (A.21), we can write the Hamiltonian (2.23) as

$$
\begin{aligned}
2H' &= \sum_m m^2 p_m\frac{\partial}{\partial p_m}+\sum_{n,m} nmp_{n+m}\frac{\partial}{\partial p_n}\frac{\partial}{\partial p_m} \\
&+\lambda\sum_{m,n=1}^{n+m\leq N}(m+n)p_m p_n\frac{\partial}{\partial p_{n+m}}+\lambda\sum_{l=1}^{N} l(N-l)p_l\frac{\partial}{\partial p_l}. \qquad \text{(A.22)}
\end{aligned}
$$

Putting together the terms that have the same structure (and dropping the $n + m \leq N$ restriction, since we're interested in the limit $n \to \infty$), the Hamiltonian in the new coordinates becomes

$$
\begin{aligned}
2H' &= \sum_{n=1}^{N}\left((1-\lambda)n^2+\lambda nN\right)p_n\frac{\partial}{\partial p_n} \\
&+\sum_{n,m=1}^{N}\left(nmp_{n+m}\frac{\partial}{\partial p_n}\frac{\partial}{\partial p_m}+\lambda(n+m)p_n p_m\frac{\partial}{\partial p_{n+m}}\right). \qquad \text{(A.23)}
\end{aligned}
$$

A.3 Derivation of the Non-Chiral Extension

Starting from the D_i operators in (2.40), we can write

$$
\begin{aligned}
\sum_{i=1}^{N} D_i^2 &= \sum_{i=1}^{N} z_i\frac{\partial}{\partial z_i}\left(\sum_{-\infty}^{\infty} nz_i^n\frac{\partial}{\partial p_n}\right) \\
&= \sum_{i=1}^{N} z_i\frac{\partial}{\partial z_i}\left(\underbrace{\sum_{-\infty}^{-1}-|n|z_i^{-|n|}\frac{\partial}{\partial p_{-|n|}}}_{\text{A}}+\underbrace{\sum_{1}^{\infty} nz_i^n\frac{\partial}{\partial p_n}}_{\text{B}}\right). \qquad \text{(A.24)}
\end{aligned}
$$

The B term has been already calculated in the chiral case in equation (A.12), with the exception that the n index in the sums ranges from $-\infty$ to ∞). For term A, we have

$$\begin{aligned}
\mathrm{A} &= \sum_{i=1}^{N} z_i \left(\sum_{n=-\infty}^{-1} |n|^2 z_i^{-|n|-1} \frac{\partial}{\partial p_{-|n|}} + \sum_{n=-\infty}^{-1} -|n| z_i^{-|n|} \frac{\partial}{\partial z_i} \frac{\partial}{\partial p_{-|n|}} \right) \\
&= \sum_{i=1}^{N} \sum_{n=-\infty}^{-1} |n|^2 z_i^{-|n|} \frac{\partial}{\partial p_{-|n|}} + \sum_{i=1}^{N} \sum_{n=-\infty}^{-1} -|n| z_i^{-|n|} \left(z_i \frac{\partial}{\partial z_i} \right) \frac{\partial}{\partial p_{-|n|}} \\
&= \sum_{n=-\infty}^{-1} |n|^2 p_{-|n|} \frac{\partial}{\partial p_{-|n|}} + \sum_{n=-\infty}^{-1} \sum_{i=1}^{\infty} -|n| z_i^{-|n|} \left(\sum_{m=-\infty}^{\infty} m z_i^m \frac{\partial}{\partial p_m} \right) \frac{\partial}{\partial p_{-|n|}} \\
&= \sum_{n=-\infty}^{-1} n^2 p_n \frac{\partial}{\partial p_n} + \sum_{n=-\infty}^{-1} \sum_{m=-\infty}^{\infty} nm p_{n+m} \frac{\partial}{\partial p_m} \frac{\partial}{\partial p_n}. \qquad \text{(A.25)}
\end{aligned}$$

Putting these results together, we obtain (2.41)

$$\sum_{i=1}^{N} D_i^2 = \sum_{m=-\infty}^{\infty} m^2 p_m \frac{\partial}{\partial p_m} + \sum_{n,m=-\infty}^{-\infty} nm p_{n+m} \frac{\partial}{\partial p_m} \frac{\partial}{\partial p_n}. \qquad \text{(A.26)}$$

A.4 Extending a Useful Identity

Starting from the identity (2.25) and doing the substitution $z_i \to z_i^{-1}$, $z_j \to z_j^{-1}$, we get

$$\left(\frac{1}{z_i} + \frac{1}{z_j} \right) \frac{z_i^{-n} - z_j^{-n}}{\frac{1}{z_i} - \frac{1}{z_j}} = -\frac{z_i + z_j}{z_i - z_j} (z_i^{-n} - z_j^{-n}). \qquad \text{(A.27)}$$

From here, we see that the rule to derive the desired identity is to take the old result, make the above substitution and multiply by -1. This yields

$$\frac{z_i + z_j}{z_i - z_j} (z_i^{-n} - z_j^{-n}) = -(z_i^{-n} + 2 z_i^{-n+1} z_j^{-1} + \ldots + 2 z_i^{-1} z_j^{-n+1} + z_j^{-n}). \quad \text{(A.28)}$$

We can use this identity to calculate the B term of the Hamiltonian (2.42). We obtain

$$
\begin{aligned}
\mathrm{B} &= \frac{\lambda}{2}\sum_{i<j}\sum_{l=1}^{N}(+l)(z_i^{-l}+2z_i^{-l+1}z_j^{-1}+\ldots+2z_i^{-1}z_j^{-l+1}+z_j^{-l})\frac{\partial}{\partial p_{-l}} \\
&= \frac{\lambda}{2}\frac{1}{2}\left(\sum_{i,j}\sum_{l=1}^{N}l(z_i^{-l}+2z_i^{-l+1}z_j^{-1}+\ldots+2z_i^{-1}z_j^{-l+1}+z_j^{-l})\frac{\partial}{\partial p_{-l}}\right. \\
&\qquad \left.-\sum_{i=1}^{N}\sum_{l=1}^{N}l(2lz_i^{-l})\frac{\partial}{\partial p_{-l}}\right) \\
&= \frac{\lambda}{2}\frac{1}{2}\left(\sum_{i=1}^{N}\sum_{l=1}^{N}l(Nz_i^{-l}+2z_i^{-l+1}p_{-1}+\ldots+p_{-l})\frac{\partial}{\partial p_{-l}}\right. \\
&\qquad \left.-2\sum_{l=1}^{N}l^2p_{-l}\frac{\partial}{\partial p_{-l}}\right) \\
&= \frac{\lambda}{2}\frac{1}{2}\left(\sum_{l=1}^{N}l(2Np_{-l})\frac{\partial}{\partial p_{-l}}+\sum_{l=1}^{N}2l(p_{-l+1}p_{-1}+\ldots+p_{-1}p_{-l+1})\frac{\partial}{\partial p_{-l}}\right. \\
&\qquad \left.-2\sum_{l=1}^{N}l^2p_{-l}\frac{\partial}{\partial p_{-l}}\right) \\
&= \frac{\lambda}{2}\left(\sum_{l=1}^{N}l(N-l)p_{-l}\frac{\partial}{\partial p_{-l}}+\sum_{n,m=1}^{N}(n+m)p_{-n}p_{-m}\frac{\partial}{\partial p_{-(n+m)}}\right). \qquad \text{(A.29)}
\end{aligned}
$$

A.5 Sutherland Inner-Product Weight

In order to decouple the θ variables from the θ' in (2.63), we make the change of variables $u = \theta - \theta'$ and $v = \theta + \theta'$. Since the Jacobian value is 2, we have $dudv = 2d\theta d\theta'$. In these new variables, the inner product weight (referred as

W) is

$$
\begin{aligned}
\mathrm{W} &= \exp\left\{-\frac{\lambda}{2}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{4\pi}\int_{-2\pi}^{2\pi}\rho(\frac{u+v}{2})\rho(\frac{v-u}{2})e^{inu}dudv\right\} \\
&= \exp\left\{-\frac{\lambda}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^{4\pi}\int_{-2\pi}^{0}\rho(\frac{u+v}{2})\rho(\frac{v-u}{2})e^{inu}dudv\right)\right. \\
&\quad \left. + \int_0^{4\pi}\int_0^{2\pi}\rho(\frac{u+v}{2})\rho(\frac{v-u}{2})e^{inu}dudv\right\}. \qquad (A.30)
\end{aligned}
$$

Making a change of variables in the first $[-2\pi, 0]$ integral, with $h = -u$ and $dh = -du$ yields

$$
\begin{aligned}
&\int_{-2\pi}^{0}\rho\left(\frac{u+v}{2}\right)\rho\left(\frac{v-u}{2}\right)e^{inu}dudv = \\
&= \int_{2\pi}^{0}\rho\left(\frac{v-h}{2}\right)+\rho\left(\frac{h+v}{2}\right)e^{-inh}(-dh)dv \\
&= \int_{0}^{2\pi}\rho\left(\frac{u+v}{2}\right)\rho\left(\frac{v-u}{2}\right)e^{-inu}dudv. \qquad (A.31)
\end{aligned}
$$

Inserting this result in the previous expression, we obtain

$$
\begin{aligned}
\mathrm{W} &= \exp\left\{-\frac{\lambda}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^{4\pi}\int_0^{2\pi}\rho\left(\frac{u+v}{2}\right)\rho\left(\frac{v-u}{2}\right)(e^{inu}+e^{-inu})dudv\right)\right\} \\
&= \exp\left\{-\frac{\lambda}{2}\int_0^{4\pi}\int_0^{2\pi}\rho\left(\frac{u+v}{2}\right)\right. \\
&\qquad \left.\rho\left(\frac{v-u}{2}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n}e^{inu}+\sum_{n=-\infty}^{-1}\frac{1}{|n|}e^{inu}\right)dudv\right\} \\
&= \exp\left\{-\frac{\lambda}{2}\int_0^{4\pi}\int_0^{2\pi}\rho\left(\frac{u+v}{2}\right)\rho\left(\frac{v-u}{2}\right)\underbrace{\sum_{n=-\infty}^{\infty}\frac{1}{|n|}e^{inu}}_{-2\ln|2\sin\frac{u}{2}|}dudv\right\} \\
&= \exp\left\{\lambda\int_0^{4\pi}\int_0^{2\pi}\rho\left(\frac{u+v}{2}\right)\rho\left(\frac{v-u}{2}\right)\ln\left|2\sin\frac{u}{2}\right|dudv\right\}. \qquad (A.32)
\end{aligned}
$$

We can extend the integral from $[0, 2\pi]$ to $[-2\pi, 2\pi]$ in u (ρ and $\sin u$ are both 2π periodic in u, $\rho((u+v)/2+(2\pi)/2) = -\rho((u+v)/2)$ and $|\sin(u+2\pi)/2| =$

$|\sin u/2|$), which gives us

$$\mathrm{W} = \exp\left\{\frac{\lambda}{2}\int_0^{4\pi}\int_{-2\pi}^{2\pi}\rho\left(\frac{u+v}{2}\right)\rho\left(\frac{v-u}{2}\right)\ln\left|2\sin\frac{u}{2}\right|dudv.\right\} \tag{A.33}$$

We can make the same transformation that we had done before to go back to our original $\{\theta,\theta'\}$ variables ($d\theta d\theta' = 1/2dudv$), where the domain of integration is the same as the original one, to finally obtain (2.65).

A.6 Hydrodynamic Fields in Non-Chiral Case

The Hamiltonian (2.45) can be rewritten into the form of equation (2.52) through the identity

$$\begin{aligned}
&\sum_{n,m=-\infty}^{\infty} p_n p_m (n+m) sgn(m)\frac{\partial}{\partial p_{n+m}} = \\
&= -\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{-1} p_n p_m (n+m)\frac{\partial}{\partial p_{n+m}} + \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} p_n p_m (n+m)\frac{\partial}{\partial p_{n+m}} \\
&= -\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} p_n p_{-|m|}(n-|m|)\frac{\partial}{\partial p_{n-|m|}} + \sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} p_n p_{|}m|(n+|m|)\frac{\partial}{\partial p_{n+|m|}} \\
&= \sum_{n,m=1}^{\infty} p_{-n}p_{-m}(n+m)\frac{\partial}{\partial p_{-n-m}} + \sum_{m=1}^{\infty} p_0 p_{-m}|m|\frac{\partial}{\partial p_{-m}} \\
&\quad + \sum_{m=1}^{\infty} p_0 p_m |m|\frac{\partial}{\partial p_m} + \sum_{n,m=1}^{\infty} p_n p_m (n+m)\frac{\partial}{\partial p_{n+m}} \\
&= \sum_{n,m=1}^{\infty}(n+m)\left(p_{-n}p_{-m}\frac{\partial}{\partial p_{-n-m}} + p_n p_m \frac{\partial}{\partial p_{n+m}}\right) + \sum_{m=-\infty}^{\infty} N p_m |m|\frac{\partial}{\partial p_m}.
\end{aligned} \tag{A.34}$$

Substituting the fields into (2.52) leads us to (2.53), which contains three different terms. The first term A is

$$\begin{aligned}
\mathrm{A} &= \sum_{n,m=-\infty}^{\infty} \frac{nm}{2\pi} \int \rho(\theta)\Pi(\theta')\Pi(\theta'')e^{in(\theta-\theta')}e^{im(\theta-\theta'')}d\theta d\theta' d\theta'' \\
&= \sum_{n,m=-\infty}^{\infty} \frac{1}{2\pi} \int \rho(\theta)\Pi(\theta')\Pi(\theta'')i\frac{\partial}{\partial\theta'}e^{in(\theta-\theta')}i\frac{\partial}{\partial\theta''}e^{im(\theta-\theta'')}d\theta d\theta' d\theta'' \\
&= -\sum_{n,m=-\infty}^{\infty} \frac{1}{2\pi} \int \rho(\theta)\frac{\partial\Pi(\theta')}{\partial\theta'}\frac{\partial\Pi(\theta'')}{\partial\theta''}e^{in(\theta-\theta')}e^{im(\theta-\theta'')}d\theta d\theta' d\theta'' \\
&= \frac{-1}{2\pi} \int \rho(\theta)\frac{\partial\Pi(\theta')}{\partial\theta'}\frac{\partial\Pi(\theta'')}{\partial\theta''}2\pi\delta(\theta-\theta')2\pi\delta(\theta-\theta'')d\theta d\theta' d\theta'' \\
&= -\int_0^{2\pi} \rho(\theta)\left(\frac{\partial\Pi(\theta)}{\partial\theta}\right)^2 d\theta. \qquad \text{(A.35)}
\end{aligned}$$

For the second term B, we find

$$\begin{aligned}
\mathrm{B} &= \sum_{n,m=-\infty}^{\infty} \frac{n+m}{2\pi} \int_0^{2\pi} \rho(\theta)\rho(\theta')\Pi(\theta'')\mathrm{sgn}(m)e^{in(\theta-\theta'')}e^{im(\theta'-\theta'')}d\theta d\theta' d\theta'' \\
&= \sum_{n,m=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \rho(\theta)\rho(\theta')\Pi(\theta'')\mathrm{sgn}(m)\left(-i\frac{\partial}{\partial\theta}+-i\frac{\partial}{\partial\theta'}\right) \\
&\qquad e^{in(\theta-\theta'')}e^{im(\theta'-\theta'')}d\theta d\theta' d\theta'' \\
&= \frac{i}{2\pi}\sum_{n,m=-\infty}^{\infty} \int_0^{2\pi} \partial_\theta\rho(\theta)\rho(\theta')\Pi(\theta'')\mathrm{sgn}(m)e^{in(\theta-\theta'')}e^{im(\theta'-\theta'')}d\theta d\theta' d\theta'' \\
&\quad +\frac{i}{2\pi}\sum_{n,m=-\infty}^{\infty} \int_0^{2\pi} \rho(\theta)\partial_{\theta'}\rho(\theta')\Pi(\theta'')\mathrm{sgn}(m)e^{in(\theta-\theta'')}e^{im(\theta'-\theta'')}d\theta d\theta' d\theta'' \\
&= i\sum_{m=-\infty}^{\infty} \mathrm{sgn}(m)\int_0^{2\pi} \partial_{\theta''}\rho(\theta'')\rho(\theta')\Pi(\theta'')e^{im(\theta'-\theta'')}d\theta' d\theta'' \\
&\quad +i\sum_{m=-\infty}^{\infty} \mathrm{sgn}(m)\int_0^{2\pi} \partial_{\theta'}\rho(\theta')\rho(\theta'')\Pi(\theta'')e^{im(\theta'-\theta'')}d\theta' d\theta'', \qquad \text{(A.36)}
\end{aligned}$$

where we summed over the n variable in the last line. From the geometric series, one can easily calculate the following identity:

$$\sum_{m=-\infty}^{\infty} \mathrm{sgn}(m)e^{im(\theta'-\theta'')} = \frac{1}{i}\cot\left(\frac{\theta'-\theta''}{2}\right). \qquad \text{(A.37)}$$

We can use this result to perform the sums in B, yielding

$$
\begin{aligned}
\mathrm{B} &= -\int \partial_{\theta''}\rho(\theta'')\rho(\theta')\Pi(\theta'')\cot\left(\frac{\theta'-\theta''}{2}\right)d\theta' d\theta'' \\
&\quad -\int \partial_{\theta'}\rho(\theta')\rho(\theta'')\Pi(\theta'')\cot\left(\frac{\theta'-\theta''}{2}\right)d\theta' d\theta'' \\
&= \int \rho(\theta'')\rho(\theta')(\partial_{\theta''}\Pi(\theta''))\cot\left(\frac{\theta'-\theta''}{2}\right)d\theta' d\theta'' \\
&\quad +\int \rho(\theta')\rho(\theta'')\Pi(\theta'')\cancel{\partial_{\theta''}\cot\left(\frac{\theta'-\theta''}{2}\right)d\theta' d\theta''} \\
&\quad +\int \rho(\theta')\rho(\theta'')\Pi(\theta'')\cancel{\partial_{\theta'}\cot\left(\frac{\theta'-\theta''}{2}\right)d\theta' d\theta''} \\
&= \int \rho(\theta)\cot\left(\frac{\theta-\theta'}{2}\right)\rho(\theta')\partial_{\theta'}\Pi(\theta')d\theta d\theta'. \qquad (A.38)
\end{aligned}
$$

The last term we need to work out is C. We have that

$$
\begin{aligned}
\mathrm{C} &= \sum_{n=-\infty}^{\infty}\frac{1-\lambda}{2\pi}n^2\int \rho(\theta)\Pi(\theta')e^{in(\theta-\theta')}d\theta d\theta' \\
&= \sum_{n=-\infty}^{\infty}\frac{1-\lambda}{2\pi}\int \rho(\theta)\Pi(\theta')\left(\partial_\theta\partial_{\theta'}\right)e^{in(\theta-\theta')}d\theta d\theta' \\
&= \int \frac{1-\lambda}{2\pi}\partial_\theta\rho(\theta)\partial_{\theta'}\Pi(\theta')\sum_{n=-\infty}^{\infty}e^{in(\theta-\theta')}d\theta d\theta' \\
&= \int_0^{2\pi}(1-\lambda)\partial_\theta\rho(\theta)\partial_\theta\Pi(\theta)d\theta. \qquad (A.39)
\end{aligned}
$$

This way, putting together the terms A, B and C, the Hamiltonian obtains the form in equation (2.57).

A.7 Enforcing the Sutherland Inner Product

Starting from 2.74, we calculate

$$
\begin{aligned}
\frac{\partial \ln J}{\partial \rho} &= (\lambda-1)\int 2\pi\left[\delta(\theta-\theta')\ln(\rho(\theta'))+\delta(\theta-\theta')\right]d\theta' \\
&\quad +\lambda\int\left[\delta(\theta-\theta')\rho(\theta'')+\rho(\theta')\delta(\theta-\theta'')\right]\ln|2\sin\left(\frac{\theta'-\theta''}{2}\right)|d\theta d\theta' \\
&= (\lambda-1)[\ln(\rho(\theta))+1]+\lambda\int\rho(\theta'')\ln|2\sin\left(\frac{\theta-\theta''}{2}\right)|d\theta'' \\
&\quad +\lambda\int\rho(\theta')\ln|2\sin\left(\frac{\theta-\theta'}{2}\right)|d\theta' \\
&= (\lambda-1)[\ln(\rho(\theta))+1]+2\lambda\int\rho(\theta')\ln|2\sin\left(\frac{\theta-\theta'}{2}\right)|d\theta'. \qquad \text{(A.40)}
\end{aligned}
$$

Taking the derivative of the above gives us

$$
\frac{\partial}{\partial_\theta}\frac{\partial \ln J}{\partial \rho} = (\lambda-1)\frac{\partial_\theta \rho}{\rho}+\lambda\int_0^{2\pi}\rho(\theta')\cot\left(\frac{\theta-\theta'}{2}\right)d\theta'. \qquad \text{(A.41)}
$$

Thus, the field has transformed as

$$
\partial_\theta \Pi \to \partial_\theta \Pi - \frac{(\lambda-1)}{2}\frac{\partial_\theta\rho}{\rho}+\frac{\lambda}{2}\int_0^{2\pi}\rho(\theta')\cot\left(\frac{\theta-\theta'}{2}\right)d\theta' \qquad \text{(A.42)}
$$

Now that we know how the fields change, we can calculate the transformed Hamiltonian. We introduce two shortcuts for notation

$$
\left\{\begin{array}{lcl} \alpha & \equiv & \frac{\lambda-1}{2}\frac{\partial_\theta\rho}{\rho} \\ \beta & \equiv & \frac{\lambda}{2}\int\cot\left(\frac{\theta-\theta'}{2}\right)\rho(\theta')d\theta', \end{array}\right. \qquad \text{(A.43)}
$$

and so under this notation, $\partial_\theta\Pi \to \partial_\theta\Pi - \alpha - \beta$. The Hamiltonian can then be written as

$$
\begin{aligned}
2H' &= \int d\theta\left[-\rho(\partial_\theta\Pi-\alpha-\beta)^2-2\alpha\rho(\partial_\theta\Pi-\alpha-\beta)\right. \\
&\quad \left.+\lambda\rho\int\cot\left(\frac{\theta-\theta'}{2}\right)\rho(\theta')(\partial_{\theta'}\Pi-\alpha'-\beta')\right], \qquad \text{(A.44)}
\end{aligned}
$$

where $\alpha' = \alpha(\theta')$. We use a trick to write the last term again in terms of α and β by relabeling $\theta \leftrightarrow \theta'$ in the integral,

$$\begin{aligned}
&\lambda \int\int \rho(\theta)\cot\left(\frac{\theta-\theta'}{2}\right)\rho(\theta')(\partial_{\theta'}\Pi - \alpha' - \beta')d\theta d\theta' \\
&= \int(\partial_\theta\Pi - \alpha - \beta)(-2\beta)d\theta. \qquad \text{(A.45)}
\end{aligned}$$

Using this and expanding the squared term in the Hamiltonian yields

$$\begin{aligned}
2H' &= \int d\theta(-\rho)\left[(\partial_\theta\Pi - (\alpha+\beta))^2\right. \qquad \text{(A.46)}\\
&\quad \left.+2\alpha\partial_\theta\Pi - 2\alpha^2 - 2\alpha\beta + 2\beta(\partial_\theta\Pi - \alpha - \beta)\right] \\
&= \int d\theta(-\rho)\Big[(\partial_\theta\Pi)^2 - 2\partial_\theta\Pi(\cancel{\alpha}+\cancel{\beta}) + \cancel{\alpha^2} + \cancel{2\alpha\beta} + \cancel{\beta^2} + \qquad \text{(A.47)}\\
&\quad \cancel{2\alpha\partial_\theta\Pi} - \cancel{2}\alpha^2 - \cancel{2\alpha\beta} + \cancel{2\beta\partial_\theta\Pi} - 2\alpha\beta - \cancel{2}\beta^2\Big] \\
&= \int(-\rho)\left[(\partial_\theta\Pi)^2 - \alpha^2 - 2\alpha\beta - \beta^2\right] \\
&= \int d\theta\Bigg[-\rho(\partial_\theta\Pi)^2 + \frac{(\lambda-1)^2}{4}\frac{(\partial_\theta\rho)^2}{\rho} + \\
&\qquad + \frac{\lambda-1}{2}\lambda\partial_\theta\rho\int\cot\left(\frac{\theta-\theta'}{2}\right)\rho(\theta')d\theta' \\
&\qquad + \frac{\lambda^2}{2}\rho\left[\int\cot\left(\frac{\theta-\theta'}{2}\right)\rho(\theta')d\theta'\right]^2\Bigg]. \qquad \text{(A.48)}
\end{aligned}$$

We need the identity

$$\begin{aligned}
&\int_0^{2\pi} d\theta\rho(\theta)\left(\int_0^{2\pi}\cot\left(\frac{\theta-\theta'}{2}\right)\rho(\theta')d\theta'\right)^2 \\
&= \frac{4\pi^2}{3}\int_0^{2\pi}\rho^3(\theta)d\theta - \frac{1}{3}\left(\int_0^{2\pi}\rho d\theta\right)^3, \qquad \text{(A.49)}
\end{aligned}$$

where the last term on the RHS is equal to $N^3/3$ in our case, and the trigonometric version of

$$\frac{\partial}{\partial t}\left\{P\int_{-\infty}^{\infty}\frac{\phi(x)}{x-t}dx\right\} = P\int_{-\infty}^{\infty}\frac{\phi(x)-\phi(t)}{(x-t)^2}dx, \qquad \text{(A.50)}$$

which applies to the last term below. Identifying $v = \partial_\theta \Pi$ as the velocity, we can write the final expression as

$$
\begin{aligned}
H &= \int \left(\frac{1}{2}\rho v^2 + \frac{\pi^2\lambda^2}{6}\rho^3 + \frac{(\lambda-1)^2}{8}\frac{(\partial_\theta\rho)^2}{\rho} \right) d\theta \\
&\quad - \frac{\lambda(\lambda-1)}{8} \iint \frac{[\rho(\theta)-\rho(\theta')]^2}{2\sin^2(\theta-\theta')/2} d\theta d\theta'. \qquad \text{(A.51)}
\end{aligned}
$$

A.8 Restricting the Non-Chiral Evolution: Chiral Constraint

Starting from the Euler equation in the non-chiral case (2.91), we want to arrive at the Euler equation for the chiral case, which becomes the Benjamin-Ono (2.37) in its linear approximation. We will use the continuity equation (2.91) and the right-going chiral constraint. We can explicitly carry out the differentiation inside w term in (2.92),

$$
\begin{aligned}
\frac{\delta(\rho\epsilon(\theta))}{\delta\rho(\theta')} &= \frac{\pi^2}{2}\rho^2\delta(\theta-\theta') + \frac{a^2}{8}\left[\frac{2\nabla\rho}{\rho}\nabla\delta(\theta-\theta') - \frac{(\nabla\rho)^2}{\rho^2}\delta(\theta-\theta') \right] \\
&\quad + \frac{\pi a}{2}\delta(\theta-\theta')\nabla\rho_H + \frac{\pi a}{2}\rho\frac{\delta}{\delta\rho}\nabla\rho_H. \qquad \text{(A.52)}
\end{aligned}
$$

The following identity

$$
\frac{\delta}{\delta\rho(\theta)} \int \mathrm{cotg}\left(\frac{\theta'-\theta''}{2}\right)\rho(\theta'')d\theta' = -\mathrm{cotg}\left(\frac{\theta-\theta'}{2}\right), \qquad \text{(A.53)}
$$

enables us to reexpress the above as

$$
\begin{aligned}
\frac{\delta(\rho\epsilon(\theta))}{\delta\rho(\theta')} &= \frac{\pi^2}{2}\rho^2\delta(\theta-\theta') + \frac{a^2}{8}\left[\frac{2\nabla\rho}{\rho}\nabla\delta(\theta-\theta') - (\nabla\log\rho)^2\delta(\theta-\theta') \right] \\
&\quad + \frac{\pi a}{2}\delta(\theta-\theta')\nabla\rho_H - \frac{\pi a}{2}\rho\nabla_{\theta'}\mathrm{cotg}\left(\frac{\theta-\theta'}{2}\right). \qquad \text{(A.54)}
\end{aligned}
$$

Since we can think of the above operators as distributions, we can integrate on both sides to find their form when acting on a function,

$$\begin{aligned}
\int \frac{\delta(\rho\epsilon(\theta))}{\delta\rho(\theta')}d\theta' &= \frac{\pi^2}{2}\rho^2 + \frac{a^2}{8}\left[-2\nabla^2\log\rho - (\nabla\log\rho)^2\right] \\
&\quad + \frac{\pi a}{2}\nabla\rho_H - \frac{\pi a}{2}\underbrace{\int \rho(\theta')\nabla_{\theta'}\text{cotg}\left(\frac{\theta-\theta'}{2}\right)d\theta'}_{-\nabla\rho_H} \\
&= \frac{\pi^2}{2}\rho^2 - \frac{a^2}{8}\left[2\nabla^2\log\rho + (\nabla\log\rho)^2\right] + \pi a\nabla\rho_H. \quad \text{(A.55)}
\end{aligned}$$

Thus, the w term can be written as

$$w = \frac{\kappa^2}{(2\pi)^2}\left[\frac{\pi^2}{2}\rho^2 - \frac{a^2}{8}\left[2\nabla^2\log\rho + (\nabla\log\rho)^2\right] + \pi a\nabla\rho_H\right]. \quad \text{(A.56)}$$

Using this result, the Euler equation 2.91 takes the form

$$\dot{v} + \nabla\left\{\frac{v^2}{2} + \frac{\kappa^2}{(2\pi)^2}\left[\frac{\pi^2}{2}\rho^2 - \frac{a^2}{8}\left[2\nabla^2\log\rho + (\nabla\log\rho)^2\right] + \pi a\nabla\rho_H\right]\right\} = 0. \quad \text{(A.57)}$$

We can now impose the right-going chiral condition (2.90). Introducing this into the above, the Euler equation takes the form

$$\begin{aligned}
&\frac{\kappa}{2}\dot{\rho} + \frac{a\kappa}{4\pi}\partial_t\nabla(\log\rho)_H + \nabla\left\{\frac{k^2}{8}\rho^2 + \frac{ak^2}{8\pi}\rho\nabla(\log\rho)_H + \frac{a^2k^2}{8(2\pi)^2}\left[\nabla(\log\rho)_H\right]^2\right. \\
&\left. + \frac{\kappa^2}{(2\pi)^2}\left[\frac{\pi^2}{2}\rho^2 - \frac{a^2}{8}\left[2\nabla^2\log\rho + (\nabla\log\rho)^2\right] + \pi a\nabla\rho_H\right]\right\} = 0. \quad \text{(A.58)}
\end{aligned}$$

The first three terms inside $\nabla\{\ldots\}$ are $v^2/2$. Dividing the whole equation by $\kappa/2$ and reordering terms, we get

$$\begin{aligned}
0 &= \frac{\kappa}{2}\nabla\left[\rho^2 + \frac{a}{2\pi}\rho\nabla(\log\rho)_H + \frac{a^2}{2(2\pi)^2}\right. \\
&\quad \left.\left\{\left[\nabla(\log\rho)_H\right]^2 - 2\nabla^2\log\rho - (\nabla\log\rho)^2 + \frac{8\pi}{a}\nabla\rho_H\right\}\right] \\
&\quad + \dot{\rho} + \frac{a}{2\pi}\partial_t\nabla(\log\rho)_H = 0, \quad \text{(A.59)}
\end{aligned}$$

which can be further rearranged into

$$\begin{aligned}0 \;=\; & \underbrace{\dot{\rho}+\frac{\kappa}{2}\nabla\left[\rho^2+\frac{a}{2\pi}\rho\nabla(\log\rho)_H\right]}_{\text{non-linear chiral Euler eq.}}+\frac{k}{2}\nabla\left[\frac{2}{k}\frac{a}{2\pi}\partial_t\nabla(\log\rho)_H+\frac{a^2}{2(2\pi)^2}\right.\\ & \left.\left\{\left[\nabla(\log\rho)_H\right]^2-2\nabla^2\log\rho-(\nabla\log\rho)^2+\frac{8\pi}{a}\nabla\rho_H\right\}\right]. \qquad \text{(A.60)}\end{aligned}$$

Since we want to show that this entire equation is equivalent to the chiral Euler equation (indicated in the above equation), we must necessarily have

$$\begin{aligned}&\frac{k}{2}\frac{a}{2\pi}\nabla\left[\frac{2}{k}\partial_t\nabla(\log\rho)_H+\frac{a}{4\pi}*\right.\\ &\left.*\left\{\left[\nabla(\log\rho)_H\right]^2-2\nabla^2\log\rho-(\nabla\log\rho)^2+\frac{8\pi}{a}\nabla\rho_H\right\}\right]=0, \qquad \text{(A.61)}\end{aligned}$$

which is satisfied if the inside of $\nabla[\ldots]$ can be equaled be a constant, that is, we must prove

$$\begin{aligned}&\underbrace{\frac{2}{k}\partial_t\nabla(\log\rho)_H}_{A}+\\ &+\underbrace{\frac{a}{4\pi}\left\{\left[\nabla(\log\rho)_H\right]^2-2\nabla^2\log\rho-(\nabla\log\rho)^2\right\}+2\nabla\rho_H}_{B}=C, \qquad \text{(A.62)}\end{aligned}$$

where C is an arbitrary constant. We will start by analyzing the A term and show that it cancels the B term. The key to prove this is to use the continuity equation together with the right-going chiral condition applied to the time derivative of the logarithm of term A. We have

$$\frac{2}{\kappa}\partial_t(\log\rho)_H=\frac{2}{\kappa}\int\frac{\partial_t\rho(\theta')}{\rho(\theta')}\mathrm{cotg}\left(\frac{\theta-\theta'}{2}\right)d\theta'. \qquad \text{(A.63)}$$

The continuity equation (2.91) together with the chirality condition (2.90) reads

$$\partial_t\rho(\theta')+\frac{\kappa}{2}\nabla_{\theta'}\left(\rho^2+\frac{a}{2\pi}\rho\nabla(\log\rho)_H\right)=0, \qquad \text{(A.64)}$$

which we can use to solve for $\partial_t\rho$ appearing in (A.63),

$$\partial_t\rho(\theta')=-\frac{\kappa}{2}\nabla_{\theta'}\left(\rho^2+\frac{a}{2\pi}\rho\nabla(\log\rho)_H\right). \qquad \text{(A.65)}$$

Substituting (A.65) into (A.63), we get

$$\begin{aligned} A &= \frac{2}{\kappa}\partial_t(\log\rho)_H \\ &= \int -\frac{2}{\kappa}\frac{\kappa}{2}\nabla_{\theta'}\left(\rho^2+\frac{a}{2\pi}\rho\nabla(\log\rho)_H\right)\frac{\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)}{\rho(\theta')}d\theta' \end{aligned} \tag{A.66}$$

We need to massage this result into something that looks more like the B term. Thus, we integrate by parts:

$$\begin{aligned} A &= \int\left(\rho^2+\frac{a}{2\pi}\rho\nabla(\log\rho)_H\right)\nabla_{\theta'}\left[\frac{\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)}{\rho(\theta')}\right]d\theta' \\ &= \int\left(\rho^2+\frac{a}{2\pi}\rho\nabla(\log\rho)_H\right)\left[\nabla_{\theta'}\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)\frac{1}{\rho(\theta')}\right. \\ &\quad\left.-\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)\frac{\nabla_{\theta'}\rho(\theta')}{\rho(\theta')^2}\right]d\theta' \\ &= \int\left(\rho+\frac{a}{2\pi}\nabla(\log\rho)_H\right)\nabla_{\theta'}\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)d\theta' \\ &\quad-\int\left(\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)\nabla_{\theta'}\rho+\frac{a}{2\pi}\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)\frac{\nabla\rho(\theta')}{\rho}\nabla(\log\rho)_H\right)d\theta'. \end{aligned} \tag{A.67}$$

For the first term, we use the identity $\nabla_{\theta'}\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)=-\nabla_\theta\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)$. Then the above becomes

$$\begin{aligned} A &= -\nabla_\theta\underbrace{\int\rho(\theta')\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)d\theta'}_{\rho_H} \\ &\quad-\frac{a}{2\pi}\nabla_\theta\underbrace{\int\nabla(\log\rho)_H\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)d\theta'}_{(\nabla(\log\rho)_H)_H} \\ &\quad+\underbrace{\int\nabla_{\theta'}\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)\rho(\theta')d\theta'}_{-\rho_H} \\ &\quad-\frac{a}{2\pi}\underbrace{\int\operatorname{cotg}\left(\frac{\theta-\theta'}{2}\right)(\nabla\log\rho(\theta'))\nabla(\log\rho)_H d\theta'}_{[\nabla\log\rho\nabla(\log\rho)_H]_H}. \end{aligned} \tag{A.68}$$

For the second term, we use the double Hilbert transform property

$$(\nabla(\log\rho)_H)_H=\nabla(\log\rho)_{HH}=-\nabla\log\rho. \tag{A.69}$$

We can then write term A as

$$A = -2\nabla\rho_H + \frac{a}{2\pi}\nabla^2\log\rho - \frac{a}{2\pi}\left[\nabla\log\rho\nabla(\log\rho)_H\right]_H. \quad \text{(A.70)}$$

The first two terms above appear also in B but with opposites signs. To deal with the last term, we can call $\phi \equiv \nabla\log\rho$ and then write it as $-\frac{a}{2\pi}[\phi\phi_H]_H$. The Hilbert transform satisfies the property

$$H[\phi_1 H[\phi_2] + \phi_2 H[\phi_1]] = H[\phi_1]H[\phi_2] - \phi_1\phi_2. \quad \text{(A.71)}$$

When $\phi_1 = \phi_2$, this collapses to

$$H[\phi_1 H[\phi_1]] = \frac{1}{2}H[\phi_1]^2 - \frac{1}{2}\phi_1^2. \quad \text{(A.72)}$$

We can use this property to rewrite the last term of (A.70) as

$$-\frac{a}{2\pi}\left[\nabla\log\rho\nabla(\log\rho)_H\right] = -\frac{a}{4\pi}\left[\nabla(\log\rho)_H\right]^2 + \frac{a}{4\pi}\left[\nabla\log\rho\right]^2. \quad \text{(A.73)}$$

At last, we arrive at the following expression for term A:

$$A = -2\nabla\rho_H - \frac{a}{4\pi}\left\{\left[\nabla(\log\rho)_H\right]^2 - 2\nabla^2\log\rho - \left[\nabla\log\rho\right]^2\right\} = -B. \quad \text{(A.74)}$$

Looking back at equation (A.62), we see that the condition is satisfied by simply setting $C = 0$. Thus, we've managed to show that the Euler equation in the non-chiral case combined with the continuity equation and the right going chiral condition reduces to

$$\dot{\rho} + \frac{\kappa}{2}\nabla\left[\rho^2 + \frac{a}{2\pi}\rho\nabla(\log\rho)_H\right]. \quad \text{(A.75)}$$

This equation coincides with the Benjamin-Ono equation by linearizing

$$\rho\partial_\theta(\ln\rho) \sim \partial_\theta\rho_H. \quad \text{(A.76)}$$

B Classical Approach to the Calogero-Sutherland Model

B.1 Pole Ansatz EOM

We start by considering the QBO equation

$$\dot{u} + uu_x = \frac{1}{2} H\left(u_{xx}\right), \tag{B.1}$$

and seek solutions of the form

$$u\left(x,t\right) = \sum_{j=1}^{N} \frac{i}{x - a_j\left(t\right)} - \sum_{j=1}^{M} \frac{i}{x - b_j\left(t\right)}, \tag{B.2}$$

where the $a_j\left(t\right)$ are below the real axis and the $b_j\left(t\right)$ above it. The number of poles N and M are two independent integers. Plugging the ansatz into the above yields three different terms

$$\begin{aligned}
&\underbrace{\left[\sum_{j=1}^{N} \frac{i}{x-a_j} - \sum_{j=1}^{M} \frac{i}{x-b_j}\right]_t}_{A} + \\
&+ \underbrace{\left[\sum_{j=1}^{N} \frac{i}{x-a_j} - \sum_{j=1}^{M} \frac{i}{x-b_j}\right]\left[\sum_{k=1}^{N} \frac{i}{x-a_k} - \sum_{k=1}^{M} \frac{i}{x-b_k}\right]_x}_{B} \\
&= \underbrace{\frac{1}{2} H\left(u_{xx}\right)}_{C}.
\end{aligned} \tag{B.3}$$

We indicate spatial and time derivatives by the lower labels x, t, whereas H represents the Hilbert transform operator. We have

$$A = \sum_{j=1}^{N} \frac{i\dot{a}_j}{\left(x-a_j\right)^2} - \sum_{j=1}^{M} \frac{i\dot{b}_j}{\left(x-b_j\right)^2} \tag{B.4}$$

$$\begin{aligned} B = & \sum_{j,k=1}^{N} \frac{1}{\left(x-a_j\right)\left(x-a_k\right)^2} - \sum_{j=1}^{N}\sum_{k=1}^{M} \frac{1}{\left(x-b_j\right)\left(x-a_k\right)^2} \\ & - \sum_{j=1}^{N}\sum_{k=1}^{M} \frac{1}{\left(x-a_j\right)\left(x-b_k\right)^2} + \sum_{j,k=1}^{M} \frac{1}{\left(x-b_j\right)\left(x-b_k\right)^2} \end{aligned} \tag{B.5}$$

$$\begin{aligned} C = & \frac{1}{2}\left[H(u)\right]_{xx} = \frac{1}{2}\left[\sum_{j=1}^{N} \frac{-i^2}{\left(x-a_j\right)^2} - \sum_{j=1}^{M} \frac{i^2}{\left(x-b_j\right)^2}\right]_{xx} \\ = & \sum_{j=1}^{N} \frac{1}{\left(x-a_j\right)^3} + \sum_{j=1}^{M} \frac{1}{\left(x-b_j\right)^3}. \end{aligned} \tag{B.6}$$

To compute C we have used the Hilbert transform property (3.13) and $H(\partial_t^k u) = \partial_t^k H(u)$. Substituting this back into the EOM, we get

$$\begin{aligned} & \sum_{j=1}^{N} \frac{i\dot{a}_j}{\left(x-a_j\right)^2} - \sum_{j=1}^{M} \frac{i\dot{b}_j}{\left(x-b_j\right)^2} + \sum_{j,k=1}^{N} \frac{1}{\left(x-a_j\right)\left(x-a_k\right)^2} \\ & - \sum_{j=1}^{N}\sum_{k=1}^{M} \frac{1}{\left(x-b_j\right)\left(x-a_k\right)^2} - \sum_{j=1}^{N}\sum_{k=1}^{M} \frac{1}{\left(x-a_j\right)\left(x-b_k\right)^2} \\ & + \sum_{j,k=1}^{M} \frac{1}{\left(x-b_j\right)\left(x-b_k\right)^2} = \sum_{j=1}^{N} \frac{1}{\left(x-a_j\right)^3} + \sum_{j=1}^{M} \frac{1}{\left(x-b_j\right)^3}. \end{aligned} \tag{B.7}$$

The expressions simplify if we split the limits in the sums that have only "diagonal terms" (i.e. terms not containing both a or b) into the terms $j = k$

and $j \neq k$,

$$
\begin{aligned}
&\sum_{j=1}^{N} \frac{i\dot{a}_j}{\left(x-a_j\right)^2} - \sum_{j=1}^{M} \frac{i\dot{b}_j}{\left(x-b_j\right)^2} + \cancel{\sum_{k=1}^{N} \frac{1}{\left(x-a_k\right)^3}} + \cancel{\sum_{k=1}^{M} \frac{1}{\left(x-b_k\right)^3}} \\
&+ \sum_{j \neq k}^{N} \frac{1}{\left(x-a_j\right)\left(x-a_k\right)^2} + \sum_{j \neq k}^{M} \frac{1}{\left(x-b_j\right)\left(x-b_k\right)^2} \\
&\underbrace{- \sum_{j=1}^{N} \sum_{k=1}^{M} \frac{1}{\left(x-b_j\right)\left(x-a_k\right)^2} - \sum_{j=1}^{N} \sum_{k=1}^{M} \frac{1}{\left(x-a_j\right)\left(x-b_k\right)^2}}_{D} \\
&= \cancel{\sum_{j=1}^{N} \frac{1}{\left(x-a_j\right)^3}} + \cancel{\sum_{j=1}^{M} \frac{1}{\left(x-b_j\right)^3}}.
\end{aligned}
\tag{B.8}
$$

We need to have x-independent coefficients (besides the $\frac{1}{x-a_i}$ or $\frac{1}{x-b_i}$ factors) so that we can compare the coefficients in the expansion, as one would do for example in a Laurant expansion. We have

$$
\begin{aligned}
D &= -\sum_{j=1}^{N} \sum_{k=1}^{M} \frac{1}{\left(x-b_j\right)\left(x-a_k\right)^2} - \sum_{j=1}^{N} \sum_{k=1}^{M} \frac{1}{\left(x-a_j\right)\left(x-b_k\right)^2} \\
&= -\sum_{j=1}^{N} \sum_{k=1}^{M} \left[\frac{1}{\left(x-b_j\right)\left(x-a_k\right)^2} + \frac{1}{\left(x-a_k\right)\left(x-b_k\right)^2}\right].
\end{aligned}
\tag{B.9}
$$

Using the algebraic identity (2.10), we can write

$$
\begin{aligned}
&\frac{1}{\left(x-b_j\right)\left(x-a_k\right)^2} + \frac{1}{\left(x-a_k\right)\left(x-b_j\right)^2} \\
&= \frac{1}{\left(a_k-b_j\right)\left(x-a_k\right)^2} + \frac{1}{\left(b_j-a_k\right)\left(x-b_j\right)^2}.
\end{aligned}
\tag{B.10}
$$

The expression for D now reads

$$
D = -\sum_{j=1}^{N} \sum_{k=1}^{M} \left[\frac{1}{\left(a_k-b_j\right)\left(x-a_k\right)^2} + \frac{1}{\left(b_j-a_k\right)\left(x-b_j\right)^2}\right]. \tag{B.11}
$$

Two of the remaining terms in (B.8) can be rewritten using a “trick” involving the identity (2.10),

$$
\begin{aligned}
\sum_{j\neq k}^{N}\frac{1}{(x-a_j)(x-a_k)^2} &= \frac{1}{2}\left[\sum_{j\neq k}^{N}\frac{1}{(x-a_j)(x-a_k)^2}+\right.\\
&\quad\left.+\sum_{j\neq k}^{N}\frac{1}{(x-a_k)(x-a_j)^2}\right]\\
&= \frac{1}{2}\left[\sum_{j\neq k}^{N}\frac{1}{(a_k-a_j)(x-a_k)^2}+\right.\\
&\quad\left.+\sum_{j\neq k}^{N}\frac{1}{(a_j-a_k)(x-a_j)^2}\right]\\
&= \sum_{j\neq k}^{N}\frac{1}{(a_j-a_k)(x-a_j)^2}, \qquad \text{(B.12)}\\
\sum_{j\neq k}^{M}\frac{1}{(x-b_j)(x-b_k)^2} &= \frac{1}{2}\left[\sum_{j\neq k}^{M}\frac{1}{(x-b_j)(x-b_k)^2}+\right.\\
&\quad\left.+\sum_{j\neq k}^{M}\frac{1}{(x-b_k)(x-b_j)^2}\right]\\
&= \frac{1}{2}\left[\sum_{j\neq k}^{M}\frac{1}{(b_k-b_j)(x-b_k)^2}+\right.\\
&\quad\left.+\sum_{j\neq k}^{M}\frac{1}{(b_j-b_k)(x-b_j)^2}\right]\\
&= \sum_{j\neq k}^{M}\frac{1}{(b_j-b_k)(x-b_j)^2}. \qquad \text{(B.13)}
\end{aligned}
$$

Using the above and the last expression for D, we can write (B.8) as

$$
\begin{aligned}
0 = &\sum_{j=1}^{N}\frac{1}{(x-a_j)^2}\left[i\dot{a}_j+\sum_{j\neq k}^{N}\frac{1}{a_j-b_k}-\sum_{k=1}^{M}\frac{1}{a_j-a_k}\right]\\
+ &\sum_{j=1}^{N}\frac{1}{(x-b_j)^2}\left[-i\dot{b}_j+\sum_{j\neq k}^{M}\frac{1}{b_j-b_k}-\sum_{k=1}^{M}\frac{1}{b_j-a_k}\right]. \qquad \text{(B.14)}
\end{aligned}
$$

Since a_j and b_j are independent variables, we arrive at the velocities EOM

$$i\dot{a}_j = \sum_{k=1}^{M} \frac{1}{a_j - b_k} - \sum_{k\neq j}^{N} \frac{1}{a_j - a_k}, \tag{B.15}$$

$$-i\dot{b}_j = \sum_{k=1}^{N} \frac{1}{b_j - a_k} - \sum_{j\neq k}^{M} \frac{1}{b_j - b_k}. \tag{B.16}$$

B.2 Deriving the Pole Accelerations

We start by computing the a-pole acceleration from the EOM (3.15),

$$E = -i\ddot{a}_j = -i\sum_{k=1}^{M} \frac{i\dot{a}_j - i\dot{b}_k}{(a_j - b_k)^2} - (-i)\sum_{k\neq j}^{N} \frac{i\dot{a}_j - i\dot{a}_k}{(a_j - a_k)^2}. \tag{B.17}$$

By using the EOM again, we can replace the velocities appearing above, yielding

$$\begin{aligned} E = & -\sum_{k=1}^{M} \frac{i}{(a_j - b_k)^2}\left[\sum_{l=1}^{M} \frac{1}{a_j - b_l} - \sum_{l\neq j}^{N} \frac{1}{a_j - a_l}\right. \\ & \left. \qquad + \sum_{l=1}^{N} \frac{1}{b_k - a_l} - \sum_{l\neq k}^{M} \frac{1}{b_k - b_l}\right] \\ & + \sum_{k\neq j}^{N} \frac{i}{(a_j - a_k)^2}\left[\sum_{l=1}^{M} \frac{1}{a_j - b_l} - \sum_{l\neq j}^{N} \frac{1}{a_j - a_l}\right. \\ & \left. \qquad - \sum_{l=1}^{M} \frac{1}{a_k - b_l} + \sum_{l\neq k}^{N} \frac{1}{a_k - a_l}\right]. \end{aligned} \tag{B.18}$$

There are three types of sums if considering the indices N and M: $N-N$, $M-M$ and mixed $N-M$ terms. We put together all the terms that have the same type of index limit, obtaining

$$
\begin{aligned}
E = \quad & \underbrace{i\left[\sum_{k,l}^{M}\frac{-1}{\left(a_j-a_k\right)^2\left(a_j-b_l\right)}-\sum_{k\neq l}^{M}\frac{-1}{\left(a_j-b_k\right)^2\left(b_k-b_l\right)}\right]}_{F} \\
+ \quad & \underbrace{i\left[\sum_{k,l\neq j}^{N}\frac{-1}{\left(a_j-a_k\right)^2\left(a_j-b_l\right)}-\sum_{\substack{k\neq l\\ k\neq j}}^{M}\frac{1}{\left(a_j-a_k\right)^2\left(a_k-a_l\right)}\right]}_{G} \\
+ \quad & i\left[\sum_{k=1}^{M}\sum_{l\neq j}^{N}\frac{1}{\left(a_j-b_k\right)^2\left(a_j-a_l\right)}-\sum_{k=1}^{M}\sum_{l=1}^{N}\frac{1}{\left(a_j-b_k\right)^2\left(b_k-a_l\right)}\right. \\
+ \quad & \left.\sum_{k\neq j}^{N}\sum_{l=1}^{M}\frac{1}{\left(a_j-a_k\right)^2\left(a_j-b_l\right)}-\sum_{k\neq j}^{N}\sum_{l=1}^{M}\frac{1}{\left(a_j-a_k\right)^2\left(a_k-b_l\right)}\right].
\end{aligned}
\tag{B.19}
$$

We group the last two lines and label them by H. We want to analyze the terms separately, starting with G. In this case, we have to put both terms in equal footing, so we extract the term $k = l$ from the first sum and the term $l = j$ from the second. We have

$$
\begin{aligned}
G \;=\; & i\sum_{k=1}^{N}\frac{-1}{\left(a_j-a_k\right)^2\left(a_j-a_k\right)}+\sum_{k\neq l\neq j}^{N}\frac{-1}{\left(a_j-a_k\right)^2\left(a_j-a_l\right)} \\
& +\sum_{k\neq l}^{N}\frac{-1}{\left(a_j-a_k\right)^2\left(a_k-a_j\right)}+\sum_{k\neq l\neq j}^{N}\frac{-1}{\left(a_j-a_k\right)^2\left(a_k-a_l\right)}.
\end{aligned}
$$

This can be written as

$$
\begin{aligned}
G &= i\sum_{k\neq j}^{N}\frac{-1}{\left(a_j-a_k\right)^2\left(a_j-a_k\right)}+\sum_{k\neq l\neq j}^{N}\frac{-1}{\left(a_j-a_k\right)^2\left(a_j-a_l\right)} \\
&\quad+\sum_{k\neq l}^{N}\frac{-1}{\left(a_j-a_k\right)^2\left(a_k-a_j\right)}+\sum_{k\neq l\neq j}^{N}\frac{-1}{\left(a_j-a_k\right)^2\left(a_k-a_l\right)} \\
&= -2i\sum_{k\neq j}^{N}\frac{1}{\left(a_j-a_k\right)^3}+\sum_{k\neq l\neq j}^{N}\left[\frac{-1}{\left(a_j-a_k\right)^2\left(a_j-a_l\right)}\right. \\
&\qquad\left.+\frac{1}{\left(a_j-a_k\right)^2\left(a_k-a_l\right)}\right]. \qquad \text{(B.20)}
\end{aligned}
$$

If we rewrite the last two terms using identity (2.10), we can see that they cancel out. Explicitly,

$$
\begin{aligned}
\sum_{k\neq l\neq j}^{N}\left[\frac{-1}{\left(a_j-a_k\right)^2\left(a_j-a_l\right)}+\frac{1}{\left(a_j-a_k\right)^2\left(a_k-a_l\right)}\right] &= \\
\sum_{k\neq l\neq j}^{N}\left[\frac{1}{\left(a_l-a_j\right)\left(a_k-a_l\right)^2}-\frac{1}{\left(a_k-a_j\right)\left(a_k-a_l\right)^2}\right] &= 0. \qquad \text{(B.21)}
\end{aligned}
$$

This leads to a simple expression for G, namely

$$
G = \sum_{k\neq j}^{N}\frac{-2i}{\left(a_j-a_k\right)^3}. \qquad \text{(B.22)}
$$

In order to compute F, we must extract the $k=l$ term and write out the rest. This yields

$$
\begin{aligned}
F &= (-i)\left[\sum_{k=l}^{N}\frac{1}{\left(a_j-b_k\right)^3}\right. \\
&\quad\left.+\sum_{k\neq l}^{M}\left(\frac{1}{\left(a_j-b_k\right)^2\left(a_j-b_l\right)}-\frac{1}{\left(a_j-b_k\right)^2\left(b_k-b_l\right)}\right)\right]. \qquad \text{(B.23)}
\end{aligned}
$$

Using identity (2.10) one more time just like we did for the G term, we can write

$$\begin{aligned}
\sum_{k\neq l}^{M}\left[\frac{1}{(a_j-b_k)^2(a_j-b_l)}-\frac{1}{(a_j-b_k)^2(b_k-b_l)}\right] &= \\
-\sum_{k\neq l}^{M}\left[\frac{1}{(b_k-b_l)^2(b_l-a_j)}-\frac{1}{(b_k-b_l)^2(b_k-a_j)}\right] &= 0. \qquad \text{(B.24)}
\end{aligned}$$

This results in the compact expression

$$F = \sum_{k\neq 1}^{N}\frac{-i}{(a_j-b_k)^3}. \qquad \text{(B.25)}$$

The only group of terms left to analyze is H. We need to group terms and relabel some indices in order to have all the sums in the same footing. We have

$$\begin{aligned}
H = i\Bigg[&\sum_{k=1}^{M}\sum_{l\neq j}^{N}\frac{1}{(a_j-b_k)^2(a_j-a_l)}-\sum_{k=1}^{M}\sum_{l=1}^{N}\frac{1}{(a_j-b_k)^2(b_k-a_l)} \\
&+\sum_{k=1}^{M}\sum_{l\neq j}^{N}\frac{1}{(a_j-a_l)^2(a_j-b_k)}-\sum_{k=1}^{M}\sum_{l\neq j}^{1}\frac{1}{(a_j-a_l)^2(a_l-b_k)}\Bigg]. \qquad \text{(B.26)}
\end{aligned}$$

Now that the terms are in the same footing, we can extract from the second sum the term $l=j$ (since none of the other 3 sums have it). This reads

$$\begin{aligned}
H &= i\sum_{k=1}^{M}\frac{1}{(a_j-b_k)^3}+i\sum_{k=1}^{M}\sum_{l\neq j}^{N}\Bigg[\frac{1}{(a_j-b_k)^2(a_j-a_l)} \\
&\quad +\frac{1}{(a_j-a_l)^2(a_j-b_k)}-\frac{1}{(a_j-b_k)^2(b_k-a_l)}-\frac{1}{(a_j-a_l)^2(a_l-b_k)}\Bigg] \\
&= i\sum_{k=1}^{M}\frac{1}{(a_j-b_k)^3} = -F. \qquad \text{(B.27)}
\end{aligned}$$

Thus, we see that $H + F = 0$. We can now go back to (B.17) and write the final expression for the acceleration of the a-poles,

$$E = -i\ddot{a}_j = -2i\sum_{k\neq j}^{N} \frac{1}{(a_j - a_k)^3} \implies \ddot{a}_j = \sum_{k\neq j}^{N} \frac{2}{(a_j - a_k)^3}. \qquad \text{(B.28)}$$

C Harmonic Trapped $p + ip$ Fermions

C.1 The Thomas-Fermi Approximation

The Thomas-Fermi approximation in its simplest form considers a set of non-interacting fermions in an external potential. Its purpose is to provide a relationship between the mass density and the external potential without the need to solve Schroedinger's equation. The starting point is to look at a uniform density system of fermions and express the density and the energy in terms of the fermi momentum. The calculation can be carried over in any spatial dimension (see [40, 41]), while here we will restrict the results to the two dimensional case. The free-particle-in-a-box quantum model in two dimensions yields an energy and momentum that are quantized and labeled by two quantum numbers,

$$E_{\vec{n}} = \frac{\hbar\pi^2}{2mL}|\vec{n}|^2, \quad \vec{k}_{\vec{n}} = \frac{2\pi}{L}\vec{n}, \quad \vec{n} = (n_x, n_y). \tag{C.1}$$

Here L is the dimension of the box and n_x, n_y positive integers. If we think of these free particles as being fermions, as we fill up the system each fermion will take the lowest possible energy level available. If we call k_f to the maximum momentum a fermion is allowed to have (i.e. fermi momentum), the total number of fermions we have is given by the volume of a hypersphere of radius k_f divided by the volume each fermion occupies, which is $(2\pi/L)^2$. There is no spin-degeneracy since we are in two dimensions. The relationship between the number-density n and the (dimensionless) fermi-momentum k_f is

$$N = \left(\frac{L}{2\pi}\right)^2 \pi k_f^2, \quad \implies n = \frac{k_f^2}{4\pi}, \tag{C.2}$$

where N is the total number of particles. We can also reverse the above, and use (C.1) to express the Fermi energy in terms of the density,

$$E_f = n\frac{2\pi\hbar^2}{m}. \tag{C.3}$$

A useful quantity is the density of states $g(E)$, defined as $g(E) \equiv dN/dE$, which counts how many states lie between energies E and $E + dE$. It can be easily calculated from the last equation, yielding

$$g(E) \equiv \frac{dN}{dE} = \frac{mL^2}{2\pi\hbar^2}. \tag{C.4}$$

The density of states allows us to calculate the total energy and the average energy per particle $\bar{E}$ in an easy way,

$$\begin{aligned} E_{\text{tot}} &= \int_0^{E_f} Eg(E)dE = \frac{mL^2}{2\pi\hbar^2}\frac{E_f^2}{2} = N\frac{E_f}{2}, \\ \bar{E} &= \frac{E_f}{2} = \frac{2\pi\hbar^2}{2m}n, \end{aligned} \tag{C.5}$$

obtaining an average energy per particle of 1/2 the Fermi energy. So far, the above expressions are exact for a uniform system. The Thomas-Fermi approximation consists of extending the above relations to a non-uniform system, that is, assume the same functional form for the density and energy in terms of the fermi momenta. This allows us to rewrite (C.2) and (C.5) as

$$n(r) = \frac{1}{4\pi}k_f^2(r), \tag{C.6}$$

$$E_{\text{tot}} = \int \bar{E}(r)n(r)d^2r = \int \frac{2\pi\hbar^2}{2m}n^2(r)d^2r. \tag{C.7}$$

We have thus far linked the total energy to the number density of the system. However, we still have no information about how this relates to any external potentials. The evolution criteria comes from requiring that the total energy of the system is minimized, without altering the N total number of particles in the system. That is, the functional

$$\frac{\delta}{\delta n}\left(E_{\text{tot}} + \int n(r)V(r)d^2r - \mu\int n(r)d^2r\right) = 0 \tag{C.8}$$

must be minimized. Here, the chemical potential μ is introduced as a Lagrange multiplier that ensures that the total number of particles in the system is N, and the middle term is just the external potential energy. This calculation can be carried over easily in 2 dimensions (see [40] for the calculation in N-dimensions), yielding

$$\int \left(\frac{4\pi\hbar^2}{2m} n(r) + V(r) - \mu \right) d^2 r = 0. \tag{C.9}$$

This gives us the relationship between the mass density and the external potential, the key result of the Thomas-Fermi approximation,

$$\rho(r) = mn(r) = \frac{1}{2\pi\hbar^2} m^2 (\mu - V(r)). \tag{C.10}$$

C.2 Derivation of the Ishikawa Mass Current

The Ishikawa mass current formula in 5.54 can be derived in several ways. Mermin and Muzikar [38] found that for a slowly varying density, the mass current satisfies

$$\mathbf{j}_{\text{mass}} = \rho \mathbf{v}_s + \frac{1}{4} \text{curl} \rho \hbar \mathbf{l} - \frac{1}{2} c_0 \mathbf{l} \left(\mathbf{l} \cdot \text{curl} \hbar \mathbf{l} \right), \tag{C.11}$$

where c_0 is a number that in the BCS limit $\Delta \ll \epsilon_F$ is close to ρ, but goes to zero in the limit of tightly-bound Cooper pairs, and v_s is the velocity of the fluid. This reduces to the Ishikawa formula for a constant field $\mathbf{l}$ and $\mathbf{v}_s = 0$. An alternative derivation is provided by Stone and Roy [7], based on minimally coupling the particle-number current operator to an Abelian gauge field in the fermionic Lagrangian, and enforcing the $U(1)$ symmetry corresponding to particle-number conservation. The following is a review of this result.

Starting from the action of a fermionic system in the presence of a gap,

$$S = \int d^2xdt \left(\frac{1}{2} \Psi^\dagger (i\partial_t - \hat{H})\Psi \right), \tag{C.12}$$

with $\Psi^\dagger = [\psi^\dagger, \psi]$ a 2-component spinor and $\hat{H}$ our usual Bogoliubov Hamiltonian, we can minimally couple the particle-number current to an Abelian gauge field $(A_0, \mathbf{A})$, where A_0 is the time component and $\mathbf{A} \equiv (A_1, A_2)$ are the in-plane components of the externally imposed field. The minimal coupling can be achieved either the direct route by adding a term $(A_0, \vec{A}) \cdot (\rho_{\text{num}}, \vec{j}_{\text{num}})$ to the action, or by converting the timespace derivatives into covariant derivatives, that is, $(\partial_t, \nabla) \to (\partial_t - iA_0, \nabla - i\vec{A})$. To write the Bogoliubov Hamiltonian, we also need the general form of the gap operator, given by [7]

$$\hat{\Delta} = \frac{1}{2} \left(\frac{\Delta}{k_f} \right) e^{i\Phi/2} \left\{ \hat{\Sigma}, \hat{P} \right\} e^{i\Phi/2}. \tag{C.13}$$

Here, $\{,\}$ denotes an anticommutator, Δ is the magnitude of the induced gap in the quasiparticle spectrum, and Φ is the overall phase of the order parameter. The spin part Σ is a symmetric 2x2 matrix that for our purposes becomes diagonal, $\Sigma = iI$, when the spin vector is chosen along the $\hat{y}$ direction. The orbital dependence is encoded in $\hat{P}$, the usual angular momentum ladder operator $\hat{P} = -i(\hat{p}_x + i\hat{p}_y)$ corresponding to Cooper pairs with their angular momentum $l = 1$ pointing in the $\hat{z}$ direction.

Changing the derivatives into covariant ones yields the minimally coupled Bogoliubov Hamiltonian

$$\hat{H}(A, \Phi) = \begin{bmatrix} -\frac{1}{2m}(\nabla - i\vec{A})^2 - A_0 & i\left(\frac{\Delta}{k_f}\right) e^{i\Phi/2} \hat{P} e^{i\Phi/2} \\ -i\left(\frac{\Delta}{k_f}\right) e^{-i\Phi/2} \hat{P}^\dagger e^{-i\Phi/2} & \frac{1}{2m}(\nabla + i\vec{A})^2 + A_0 \end{bmatrix}. \tag{C.14}$$

A bit of algebra shows that this can be written as the uncoupled Hamiltonian plus the minimal coupling $(A_0, \vec{A}) \cdot (\Psi^\dagger\Psi, i/2m(\Psi^\dagger\nabla\Psi - \Psi\nabla\Psi^\dagger))$. In order to preserve invariance under the local $U(1)$ gauge transformation

$$\begin{bmatrix} \psi \\ \psi^\dagger \end{bmatrix} \to \begin{bmatrix} e^{i\phi}\psi \\ e^{-i\phi}\psi^\dagger \end{bmatrix}, \tag{C.15}$$

the fields must transform as

$$\begin{array}{rcl} \Phi & \rightarrow & \Phi + 2\phi \\ \vec{A} & \rightarrow & \vec{A} + \nabla\phi \\ A_0 & \rightarrow & A_0 + \partial_t\phi \end{array} \tag{C.16}$$

Thus, the gauge field A behaves like the electromagnetic one. In order to be left out with an action in terms of A, Φ only, the Ψ, $\Psi^\dagger$ fields must be integrated out from the partition function. Thus the effective action, defined by

$$iS_{\text{eff}}(A,\Phi) = \ln\left\{\int d[\Psi]d[\Psi^\dagger]\exp[iS(A,\Phi,\Psi,\Psi^\dagger)]\right\}, \tag{C.17}$$

becomes [42–44]

$$S_{\text{eff}}(A,\Phi) = \int d^2x dt \left\{\frac{\rho_0}{2m}\left[\frac{1}{c_s^2}\left(\frac{\partial\Phi/2}{\partial_t} - A_0\right)^2 - (\nabla\Phi/2 - \vec{A})^2\right]\right. \tag{C.18}$$

$$\left. -\sigma_{xy}\left(\frac{\partial\Phi/2}{\partial_t} - A_0\right)(\nabla\times\vec{A})_z - \rho_0\left(\frac{\partial\Phi/2}{\partial_t} - A_0\right)\right\}. \tag{C.19}$$

For a 2+1 dimensional Galilean invariant system of particles with mass m, we have $\sigma_{xy} = \frac{1}{8\pi}$, $c_s = v_f/2$ and $\rho_0 = m/2\pi\epsilon_f$, where c_s is the speed of sound and ρ_0 is the equilibrium number density. Equipped with the above action, the 4-component particle-number current can finally be calculated:

$$\vec{j}_{\text{num}} \equiv \frac{\delta S_{\text{eff}}}{\delta\vec{A}} = \rho_0\vec{v}_s + \sigma_{xy}(\hat{z}\times\nabla)\left(\frac{\partial\Phi/2}{\partial_t} - A_0\right), \tag{C.20}$$

$$\rho_{\text{num}} \equiv \frac{\delta S_{\text{eff}}}{\delta A_0} = \rho_0 - \frac{\rho_0}{mc_s^2}\left(\frac{\partial\Phi/2}{\partial_t} - A_0\right) + \sigma_{xy}\nabla\times\vec{A}. \tag{C.21}$$

The spatial current takes a simplified form when expressed in terms of the above number density,

$$\vec{j}_{\text{num}} = \rho_{\text{num}}\vec{v}_s - \frac{1}{4m}(\hat{z}\times\nabla)(\rho - \sigma_{xy}B_z). \tag{C.22}$$

Since we are interested in the case where there are no external fields, we can discard the magnetic field contribution, yielding a mass flux

$$\vec{j} = m\vec{j}_{\text{num}} = \frac{1}{4}\nabla\times\rho\hat{z}. \tag{C.23}$$

This is the planar analogue of the Ishikawa current in equation (5.54).

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