Cosserat fluids and the continuum mechanics of turbulence: a generalized Navier–Stokes-α equation with complete boundary conditions

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(Received)

We here develop a continuum-mechanical formulation and generalization of the Navier–Stokes-α equation based on a general framework for fluid-dynamical theories involving gradient dependencies (Fried & Gurtin 2005). That generalization entails two additional material length scales: one of energetic origin, the other of dissipative origin. In contrast to Lagrangian averaging, our formulation delivers boundary conditions — involving yet another material length scale — and a complete framework based on thermodynamics applied to an isothermal system. As an application, we consider the classical problem of turbulent flow in a plane, rectangular channel with fixed, impermeable, slip-free walls and make comparisons with results obtained from direct numerical simulations. For this problem, only one of the material length scales involved in the flow equation enters the final solution. When the additional material length scale associated with the boundary conditions is signed to ensure satisfaction of the second law at the channel walls the theory delivers solutions that agree neither quantitatively nor qualitatively with observed features of plane channel flow. On the contrary, we find excellent agreement when the sign of the additional material parameter associated with the boundary conditions violates the second law. We discuss the implication of this result.

1. Introduction

The Lagrangian averaged Navier–Stokes-α model for (statistically homogeneous and isotropic) turbulent flow yields a governing equation for the fluid velocity $\mathbf{v}$ that can be written in the form

$$\rho\dot{\mathbf{v}} = -\mathrm{grad}\, p + \mu(1-\alpha^2\Delta)\Delta\mathbf{v} + 2\rho\alpha^2 \mathrm{div}\, \overset{\circ}{\mathbf{D}}; \tag{1.1}$$

as is customary, we refer to (1.1) as the *Navier–Stokes-α equation*. In this equation $\mathbf{v}$ is subject to the incompressibility constraint

$$\mathrm{div}\,\mathbf{v} = 0, \tag{1.2}$$

$\dot{\mathbf{v}}$ (often written as $\mathrm{D}\mathbf{v}/\mathrm{D}t$) is the material time derivative of $\mathbf{v}$, p is the pressure, Δ is the Laplace operator, $\mathbf{D} = \frac{1}{2}(\mathrm{grad}\,\mathbf{v} + (\mathrm{grad}\,\mathbf{v})^{\top})$ is the stretch-rate,

$$\overset{\circ}{\mathbf{D}} = \dot{\mathbf{D}} + \mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}, \tag{1.3}$$

with $\mathbf{W} = \frac{1}{2}(\operatorname{grad}\mathbf{v} - (\operatorname{grad}\mathbf{v})^{\top})$ the spin, is the corotational rate of $\mathbf{D}$. The Lagrangian averaged Euler equation, which is (1.1) with $\mu = 0$, was first derived by Holm, Marsden & Ratiu (1998a, 1998b). Subsequently, Chen, Foias, Holm, Olson, Titi & Wynne (1998, 1999a, 1999b) added the viscous term to the Lagrangian averaged Euler equation, giving (1.1). Derivations of (1.1) via Lagrangian averaging were provided by Shkoller (2001) and Marsden & Shkoller (2003).

Aside from the density ρ and the shear viscosity μ of the fluid, the flow equation (1.1) involves an additional material parameter $\alpha > 0$ carrying dimensions of length. Within the framework of Lagrangian averaging, α is the statistical correlation length of the excursions taken by a fluid particle away from its phase-averaged trajectory. More intuitively, α can be interpreted as the characteristic linear dimension of the smallest eddies that the model is capable of resolving. Like equations arising from Reynolds averaging, the Navier–Stokes-α equation provides an approximate model that resolves motions only above some critical scale, while relying on filtering to approximate effects at smaller scales. A synopsis of properties and advantages of the Navier–Stokes-α equation is provided by Holm, Jeffrey, Kurien, Livescu, Taylor & Wingate (2005).

The structure of (1.1) is formally suggestive of a conservation law expressing the balance of linear momentum, and one might ask whether there is a complete continuum mechanical framework in which the Navier–Stokes-α equation is embedded along with suitable boundary conditions. Based on experience with theories for plates, shells, and other structured media, the presence of a term involving the fourth-order spatial gradient of the velocity indicates that any such framework should involve a hyperstress in addition to the classical stress. Within the context of turbulence theory, a hyperstress might be viewed as providing a means to account for interactions across disparate length scales.

To see the need for an additional hyperstress assume an inertial frame, neglect noninertial body forces, and note first that the weak form of the classical momentum balance

$$\operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{0}, \tag{1.4}$$

with inertial force

$$\mathbf{b} = -\varrho\dot{\mathbf{v}} \tag{1.5}$$

treated for convenience as a body force, has the form

$$\underbrace{\int_{\partial R} \mathbf{t}(\mathbf{n})\cdot\boldsymbol{\phi}\,\mathrm{d}a + \int_{R} \mathbf{b}\cdot\boldsymbol{\phi}\,\mathrm{d}v}_{\mathcal{W}_{\mathrm{ext}}(R,\boldsymbol{\phi})} = \underbrace{\int_{R} \mathbf{T}:\operatorname{grad}\boldsymbol{\phi}\,\mathrm{d}v}_{\mathcal{W}_{\mathrm{int}}(R,\boldsymbol{\phi})}, \tag{1.6}$$

with

$$\mathbf{t}_{\mathbf{n}} = \mathbf{T}\mathbf{n} \tag{1.7}$$

the classical surface-traction of Cauchy. Granted smoothness, (1.6) holds for all *virtual velocities* (i.e., test fields) $\boldsymbol{\phi}$ and all control volumes R if and only if the balance (1.4) is satisfied at all points in the fluid and the traction condition (1.7) is satisfied — for any choice of the unit vector $\mathbf{n}$ — at all points in the fluid. Moreover, the requirement of frame-indifference applied to (1.6) yields the symmetry of the stress $\mathbf{T}$.

When $\boldsymbol{\phi}$ represents the velocity $\mathbf{v}$ of the fluid, the weak balance (1.6) is a physical balance

$$\underbrace{\int\limits_{\partial R} \mathbf{t}_{\mathbf{n}} \cdot \mathbf{v} \, \mathrm{d}a + \int\limits_{R} \mathbf{b} \cdot \mathbf{v} \, \mathrm{d}v}_{\mathcal{W}_{\mathrm{ext}}(R)} = \underbrace{\int\limits_{R} \mathbf{T} : \operatorname{grad} \mathbf{v} \, \mathrm{d}v}_{\mathcal{W}_{\mathrm{int}}(R)} \tag{1.8}$$

between:

(i) the external power $\mathcal{W}_{\mathrm{ext}}(R)$, which represents

(a) power expended on R by tractions acting on ∂R, and

(b) power expended by the inertial force $\mathbf{b}$ directly on the interior points of R;

(ii) the internal power $\mathcal{W}_{\mathrm{int}}(R)$, the integrand of which represents the classical stress power $\mathbf{T}:\operatorname{grad}\mathbf{v}$ expended within R by the stress field $\mathbf{T}$.

Here and in what follows, we write $\mathcal{W}_{\mathrm{ext}}(R)$ for the external power associated with an actual flow and $\mathcal{W}_{\mathrm{ext}}(R, \boldsymbol{\phi})$ for the (virtual) external power associated with a virtual velocity field $\boldsymbol{\phi}$. Note that, by (1.5), the negative inertial power is the kinetic energy rate.

The balance (1.6) represents a nonstandard form of the classical principle of virtual power (Gurtin 2001). This nonstandard form has been generalized by Fried & Gurtin (2006) to develop a gradient theory for liquid flows at small length scales and, when combined with suitable constitutive relations, results in a partial differential equation slightly more general than (1.1) but with the term involving the corotational rate of $\mathbf{D}$ removed. Conventional versions of this principle are formulated for the fluid region as a whole rather than for control volumes and as such generally involve particular boundary conditions. Here the principle of virtual power is used instead as a basic tool in determining the structure of the tractions and of the local force balances. As such, conditions on the external boundary play a role no different from those on the boundary of any control volume. Basic to this view is the premise, central to all of continuum mechanics, that any basic law for the body should hold also for all subregions of the body. On a more pragmatic note, the nonstandard formulation allows for the derivation of the associated angular momentum balance. (See Antman & Osborn (1979) for a rigorous treatment of the classical virtual-work principle.)

To capture the internal power associated with the formation of eddies during turbulent flow, we generalize the classical theory by including, in the internal power, a term linear in the *vorticity* gradient $\operatorname{grad}\boldsymbol{\omega} = \operatorname{grad}\operatorname{curl}\mathbf{v}$. Specifically, we introduce a second-order tensor-valued *hyperstress* $\mathbf{G}$ via an internal power expenditure of the form $\mathbf{G}:\operatorname{grad}\boldsymbol{\omega}$ and rewrite the power expended within R in the form

$$\mathcal{W}_{\mathrm{int}}(R) = \int\limits_{R} \big(\mathbf{T}:\operatorname{grad}\mathbf{v} + \mathbf{G}:\operatorname{grad}\boldsymbol{\omega}\big) \, \mathrm{d}v. \tag{1.9}$$

In conjunction with the internal power expenditure (1.9), we introduce a corresponding external power expenditure

$$\mathcal{W}_{\mathrm{ext}}(R) = \int\limits_{\mathcal{S}} \Big(\mathbf{t}_{\mathcal{S}} \cdot \mathbf{v} + \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \mathbf{v}}{\partial n}\Big) \, \mathrm{d}a + \int\limits_{R} \mathbf{b} \cdot \mathbf{v} \, \mathrm{d}v, \tag{1.10}$$

in which $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$ represent tractions on the bounding surface $\mathcal{S} = \partial R$ of R, while $\mathbf{b}$ represents the inertial body force(1.5). Here the term

$$\mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \mathbf{v}}{\partial n}, \tag{1.11}$$

which is not present in classical theories, is needed to balance the effects of the internal-power term $\mathbf{G}:\mathrm{grad}\,\boldsymbol{\omega}$, which involves the second gradient of $\mathbf{v}$.

The *principle of virtual power* replaces $\mathbf{v}$ by $\boldsymbol{\phi}$ and (hence) $\boldsymbol{\omega}$ by $\mathrm{curl}\,\boldsymbol{\phi}$ and is based on the requirement that

$$\mathcal{W}_{\mathrm{ext}}(R,\boldsymbol{\phi}) = \mathcal{W}_{\mathrm{int}}(R,\boldsymbol{\phi}) \tag{1.12}$$

for all control volumes R and any choice of the virtual velocity field $\boldsymbol{\phi}$. Consequences of the virtual power principle and the requirement that the internal power expenditure be frame-indifferent are that:

(i) The classical macroscopic balance $\rho\dot{\mathbf{v}} = \mathrm{div}\,\mathbf{T}$ must be replaced by the balance

$$\rho\dot{\mathbf{v}} = \mathrm{div}\,\mathbf{T} + \mathrm{curl}\,\mathrm{div}\,\mathbf{G}, \tag{1.13}$$

with $\mathbf{T}$ symmetric as in the classical theory.

(ii) Cauchy's classical condition $\mathbf{t}_{\mathbf{n}} = \mathbf{T}\mathbf{n}$ for the traction across a surface $\mathcal{S}$ with unit normal $\mathbf{n}$ must be replaced by the conditions

$$\left.\begin{aligned} \mathbf{t}_{\mathcal{S}} &= \mathbf{T}\mathbf{n} + \mathrm{div}_{\mathcal{S}}(\mathbf{G}\mathbf{n}\times) + \mathbf{n}\times(\mathrm{div}\,\mathbf{G} - 2K\mathbf{G}\mathbf{n}),\\ \mathbf{m}_{\mathcal{S}} &= \mathbf{n}\times\mathbf{G}\mathbf{n}, \end{aligned}\right\} \tag{1.14}$$

in which $\mathrm{div}_{\mathcal{S}}$ is the divergence operator on $\mathcal{S}$ and K is the mean curvature of $\mathcal{S}$.

Within the framework of finite deformations of an *elastic solid* with couple-stress, the balance (1.13) was first derived by the Cosserats (1909); see, also, Toupin (1962, 1964), Mindlin & Tiersten (1962), and Green & Naghdi (1968). The traction conditions (1.14) are special cases of traction conditions derived variationally by Toupin (1962, 1964) for the boundary of the elastic solid.

The balance (1.13) is identical to — and the traction conditions special cases of — equations (5.11) and (5.12) of Fried & Gurtin (2006), whose theory replaces $\mathrm{curl}\,\boldsymbol{\omega}$ in the internal power with the full second gradient $\mathrm{grad}^2\mathbf{v}$ and $\mathbf{G}$ by an analogous third-order hyperstress. After submitting our paper to press and after completing the derivation of the results presented here, we discovered work of Bluestein & Green (1967), who discuss second-gradient fluids based on the multipolar theory of second-gradient materials due to Green & Rivlin (1964). This theory results in redundant boundary conditions, which Bluestein & Green (1967) reduce using arguments of an ad hoc nature.

When supplemented by constitutive equations for the stress and hyperstress, the balance (1.13) yields a flow equation. Restricting attention to incompressible fluids, we invoke the standard decomposition

$$\mathbf{T} = \mathbf{S} - p\mathbf{1}, \qquad \mathrm{tr}\,\mathbf{S} = 0, \tag{1.15}$$

of the stress into a traceless extra stress $\mathbf{S}$ and a powerless pressure p and, motivated by the form of the Navier–Stokes-α equation (1.1), take the extra stress to be of the form

$$\mathbf{S} = 2\mu\mathbf{D} + 2\lambda\mathring{\mathbf{D}}, \qquad \mu > 0, \qquad \lambda > 0, \tag{1.16}$$

familiar from the theory of Rilvin–Ericksen fluids; cf. Rivlin & Ericksen, 1955; Truesdell & Noll, 1965, §119; Dunn & Fosdick, 1974. Further, we take the hyperstress to be of the simple linear form

$$\mathbf{G} = \zeta\,\mathrm{grad}\,\boldsymbol{\omega} + \xi(\mathrm{grad}\,\boldsymbol{\omega})^{\top}, \tag{1.17}$$

with $\zeta > 0$ and $-\zeta \le \xi \le \zeta$ to ensure non-negative dissipation.

Using (1.15)–(1.17) in (1.13) and assuming that the moduli μ, λ, ζ, and ξ are constant, we arrive at the flow equation

$$\rho\dot{\mathbf{v}} = -\mathrm{grad}\, p + \mu\Delta\mathbf{v} - \zeta\Delta\Delta\mathbf{v} + 2\lambda\mathrm{div}\,\mathring{\mathbf{D}}, \tag{1.18}$$

which, for the particular choices

$$\lambda = \rho\alpha^2 \qquad \text{and} \qquad \zeta = \mu\alpha^2 \tag{1.19}$$

of λ and ζ, specializes to the Navier–Stokes-α equation (1.1).

We develop counterparts of the classical notions of a free surface and a fixed surface without slip (we tacitly assume throughout that such boundaries are impermeable). Our results hinge on rewriting the external power expenditure (3.4) for the entire fluid body B and focusing on that portion of this expenditure associated with tractions. In this regard, we derive boundary force and moment balances

$$\mathbf{t}_{\mathcal{S}} = \mathbf{t}^{\mathrm{env}}_{\partial B} + 2\sigma K\mathbf{n} \qquad \text{and} \qquad \mathbf{m}_{\mathcal{S}} = \mathbf{m}^{\mathrm{env}}_{\partial B} \tag{1.20}$$

giving the tractions $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$ in terms of their environmental counterparts $\mathbf{t}^{\mathrm{env}}_{\partial B}$ and $\mathbf{m}^{\mathrm{env}}_{\partial B}$, and use these balances to express the power expended by tractions in the form

$$\int\limits_{\partial B} \Big((\mathbf{t}^{\mathrm{env}}_{\partial B} + 2\sigma K\mathbf{n}) \cdot \mathbf{v} + \mathbf{m}^{\mathrm{env}}_{\partial B} \cdot \mathbf{P}\frac{\partial \mathbf{v}}{\partial n} \Big)\, da. \tag{1.21}$$

We assume that the mean curvature K of — and the surface tension σ at — the boundary ∂B are known; (1.21) then suggests that reasonable boundary conditions might, at each point of ∂B, consist of

(i) a prescription of $\mathbf{t}^{\mathrm{env}}_{\partial B}$ or $\mathbf{v}$, or a relation between $\mathbf{t}^{\mathrm{env}}_{\partial B}$ and $\mathbf{v}$; and

(ii) a prescription of $\mathbf{m}^{\mathrm{env}}_{\partial B}$ or $\mathbf{P}\partial\mathbf{v}/\partial n$, or a relation between $\mathbf{m}^{\mathrm{env}}_{\partial B}$ and $\mathbf{P}\partial\mathbf{v}/\partial n$.

Consistent with this, we consider specific boundary conditions in which a portion $\mathcal{S}_{\mathrm{free}}$ of ∂B is a free surface and the remainder $\mathcal{S}_{\mathrm{nslp}}$ is a fixed surface without slip. On $\mathcal{S}_{\mathrm{free}}$, the environmental tractions $\mathbf{t}^{\mathrm{env}}_{\partial B}$ and $\mathbf{m}^{\mathrm{env}}_{\partial B}$ vanish and the classical condition $\mathbf{T}\mathbf{n} = \sigma K\mathbf{n}$ is replaced by the conditions

$$\mathbf{T}\mathbf{n} + \mathrm{div}_{\mathcal{S}}(\mathbf{G}\mathbf{n}\times) = \sigma K\mathbf{n} \qquad \text{and} \qquad \mathbf{n}\times\mathbf{G}\mathbf{n} = \mathbf{0}. \tag{1.22}$$

To describe the conditions on $\mathcal{S}_{\mathrm{nslp}}$, we first note that, if $\mathbf{v} = \mathbf{0}$ on $\mathcal{S}_{\mathrm{nslp}}$, then

$$\mathbf{P}\frac{\partial \mathbf{v}}{\partial n} = \boldsymbol{\omega}\times\mathbf{n} \tag{1.23}$$

with $\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{v}$ the vorticity. Based on this identity, we take, as boundary condition on $\mathcal{S}_{\mathrm{nslp}}$, the classical condition

$$\mathbf{v} = \mathbf{0} \tag{1.24}$$

supplemented by a condition of the form

$$\mathbf{n}\times\mathbf{G}\mathbf{n} = \mathbf{m}^{\mathrm{env}}_{\partial B} \tag{1.25}$$

with $\mathbf{m}^{\mathrm{env}}_{\partial B} = -\mu\ell\,\partial\mathbf{v}/\partial n = -\mu\ell\,\boldsymbol{\omega}\times\mathbf{n}$, where $|\ell|$ represents a material length scale. Thus we are led to the boundary condition

$$\mathbf{n}\times(\mathbf{G}\mathbf{n} - \mu\ell\boldsymbol{\omega}) = \mathbf{0}. \tag{1.26}$$

We refer to (1.26) as the *wall-eddy condition* and to ℓ as the *wall-eddy modulus*.

To display some of the central features of the theory, we consider the classical problem of steady, turbulent flow in a plane channel. We invoke the kinematical assumptions standard for plane Couette flow. Further, we assume that the channel walls are fixed and without slip. The flow equation (1.18) and boundary conditions (1.24) and (1.26) yield a fourth-order boundary-value-problem for the downstream component of the velocity as a function of the coordinate normal to the channel walls. Experiments and DNS simulations of channel flow show that, for suitably normalized laminar and turbulent velocity profiles, the slopes of the turbulent profiles at the channel walls have magnitudes greater than their laminar counterparts (Pope 2000). A central result of our work is that the solution of the channel problem is consistent with this "wall-slope requirement" only for negative values of the wall-eddy modulus ℓ:

$$\ell < 0 \tag{1.27}$$

Interestingly, such values of ℓ imply that

- $\mu\ell|\mathbf{n}\times\boldsymbol{\omega}|^2$ — a term that would usually be thought of as *dissipative* — is *negative*!

We show that $\mu\ell|\mathbf{n}\times\boldsymbol{\omega}|^2$ is also strictly negative in a solution of the Navier–Stokes-α equation — subject to boundary conditions in which the velocity vanishes on the channel walls and the shear component of the traction is given — presented previously by Chen, Foias, Holm, Olson, Titi & Wynne (1999) (granted that their wall modulus is converted to ours).

Assuming that ℓ is negative, we use the method of least-squares to fit our solution of the channel flow problem to the mean downstream velocity for turbulent channel flow predicted by the direct numerical simulations (DNS) of Kim, Moin & Moser (1987) and Moser, Kim & Mansour (1999) for three values of the friction Reynolds number. We find that the velocity profile shows good agreement and that the Reynolds shear stress in the downstream plane of the channel agrees with the numerical results only outside the viscous wall region. This discrepancy might be attributed to statistically inhomogeneous and/or anisotropic fluctuations within the viscous wall region. As would be expected from the sentence containing (1.27), for positive ℓ the calculated velocity profile differs both qualitatively and quantitatively from the DNS simulations.

Marsden & Shkoller (2001) recently established well-posedness results for the Navier–Stokes-α equation on bounded domains based on the simple boundary conditions $\mathbf{v} = \mathbf{0}$ and $\Delta\mathbf{v} = \mathbf{0}$. Such boundary conditions do not fit within the virtual-power framework used here, nor do they seem capable of characterizing turbulent behavior near a fixed surface: even so, the results of Marsden & Shkoller (2001) would seem to indicate the value of the Navier–Stokes-α equation itself, devoid of questions regarding boundary conditions. In this regard, it would seem interesting to see if analogous results hold for the boundary conditions $\mathbf{v} = \mathbf{0}$ and $\mathbf{n}\times\mathbf{G}\mathbf{n} = \mathbf{0}$, the latter condition being natural with respect to the virtual-power framework.

The question of whether initial-boundary-value problems for the Navier–Stokes-α equation are well-posed when boundary conditions appropriate to turbulence are imposed remains open, as does the question of whether solutions to initial-boundary-value problems for the Navier–Stokes-α equation converge to solutions of the classical Navier–Stokes equation. Because of the seemingly unstable structure of boundary conditions, at least those proposed here for a fixed surface without slip, it seems likely that answers to these questions will require novel analytical approaches.

2. Preliminaries

To simplify our calculations, we use direct notation. However, for clarity, we also present key definitions and results in component form.

2.1. *Notation*

We find it most convenient to work spatially; i.e., to use what is commonly called an Eulerian description. We write $\rho(\mathbf{x},t)$ for the *mass density*, $\mathbf{v}(\mathbf{x},t)$ for the velocity,

$$\mathbf{D} = \tfrac{1}{2}\big(\operatorname{grad}\mathbf{v} + (\operatorname{grad}\mathbf{v})^{\top}\big) \qquad \text{and} \qquad \mathbf{W} = \tfrac{1}{2}\big(\operatorname{grad}\mathbf{v} - (\operatorname{grad}\mathbf{v})^{\top}\big) \tag{2.1}$$

for the *stretching* and *spin*, and

$$\boldsymbol{\omega} = \operatorname{curl}\mathbf{v} \tag{2.2}$$

for the *vorticity*. We use a superposed dot for the *material time-derivative*; e.g., for $\varphi(\mathbf{x},t)$ a scalar field

$$\dot{\varphi} = \frac{\partial\varphi}{\partial t} + \mathbf{v}\cdot\operatorname{grad}\varphi.$$

Balance of mass is then the requirement that

$$\dot{\rho} + \rho\operatorname{div}\mathbf{v} = 0. \tag{2.3}$$

2.2. *Control volume R. Differential geometry on ∂R*

We denote by R an arbitrary region, fixed in time, that is contained in the region of space occupied by the body over some time interval. We refer to R as a *control volume* and write

$$\mathcal{S} = \partial R$$

for the boundary of R and $\mathbf{n}$ for the outward unit normal on $\mathcal{S}$, which we assume to be smooth. We let $\mathbf{P}$ denote the projection onto the plane normal to $\mathbf{n}$:

$$\mathbf{P} = \mathbf{1} - \mathbf{n}\otimes\mathbf{n} \qquad (P_{ij} = \delta_{ij} - n_i n_j). \tag{2.4}$$

The operator $\operatorname{grad}_{\mathcal{S}}$ defined on any vector field $\mathbf{g}$ by

$$\operatorname{grad}_{\mathcal{S}}\mathbf{g} = (\operatorname{grad}\mathbf{g})\mathbf{P} \qquad \big((\operatorname{grad}_{\mathcal{S}}\mathbf{g})_{ij} = g_{i,j} - g_{i,k}n_k n_j\big)$$

is the *surface gradient*;

$$\operatorname{div}_{\mathcal{S}}\mathbf{g} = \operatorname{tr}(\operatorname{grad}_{\mathcal{S}}\mathbf{g}) = \mathbf{P}:\operatorname{grad}\mathbf{g} = \operatorname{div}\mathbf{g} - \mathbf{n}\cdot(\operatorname{grad}\mathbf{g})\mathbf{n} = g_{i,i} - g_{i,k}n_i n_k$$

defines the *surface divergence*; $\partial/\partial n$ defined by

$$\frac{\partial\mathbf{g}}{\partial n} = (\operatorname{grad}\mathbf{g})\mathbf{n}$$

is the *normal derivative*. Then

$$\operatorname{grad}\mathbf{g} = \operatorname{grad}_{\mathcal{S}}\mathbf{g} + \frac{\partial\mathbf{g}}{\partial n}\otimes\mathbf{n}, \qquad \operatorname{div}\mathbf{g} = \operatorname{div}_{\mathcal{S}}\mathbf{g} + \frac{\partial\mathbf{g}}{\partial n}\cdot\mathbf{n}. \tag{2.5}$$

The *surface divergence* of a tensor field $\mathbf{A}$ is the vector field defined by

$$\operatorname{div}_{\mathcal{S}}\mathbf{A} = (\operatorname{grad}\mathbf{A})\mathbf{P} \qquad \big((\operatorname{div}_{\mathcal{S}}\mathbf{A})_i = A_{ij,k}P_{kj}\big). \tag{2.6}$$

Note that the domains of $\operatorname{grad}_{\mathcal{S}}\mathbf{g}$ $\operatorname{div}_{\mathcal{S}}\mathbf{g}$, $\partial\mathbf{g}/\partial n$, and $\operatorname{div}_{\mathcal{S}}\mathbf{A}$ are restricted to the surface $\mathcal{S}$.

Granted a smooth extension of the unit normal $\mathbf{n}$, $\operatorname{grad}\mathbf{n}$ is defined in a neighborhood of $\mathcal{S}$ and the field

$$\mathbf{K} = -\operatorname{grad}_{\mathcal{S}}\mathbf{n} = -(\operatorname{grad}\mathbf{n})\mathbf{P};$$

is the *curvature tensor* of $\mathcal{S}$; as is well known,

$$\mathbf{K} = \mathbf{K}^{\top}, \qquad \mathbf{K}\mathbf{n} = \mathbf{0}. \tag{2.7}$$

The scalar field

$$K = \tfrac{1}{2}\mathrm{tr}\,\mathbf{K} = -\tfrac{1}{2}\mathrm{div}_{\mathcal{S}}\mathbf{n}$$

is the *mean curvature* of $\mathcal{S}$.

Let $\mathbf{A}$ be a (second-order) tensor field and let $\mathbf{g}$ be a vector field. We make considerable use of the identities

$$\left.\begin{aligned}\mathrm{div}_{\mathcal{S}}(\mathbf{AP}) &= \mathrm{div}_{\mathcal{S}}\mathbf{A} + 2K\mathbf{An},\\ \mathrm{div}_{\mathcal{S}}(\mathbf{A}^{\top}\mathbf{g}) &= \mathbf{g}\cdot\mathrm{div}_{\mathcal{S}}\mathbf{A} + \mathbf{A}:\mathrm{grad}_{\mathcal{S}}\mathbf{g}.\end{aligned}\right\} \tag{2.8}$$

and, in particular, their specializations

$$\mathrm{div}_{\mathcal{S}}\mathbf{P} = 2K\mathbf{n}, \qquad \mathrm{div}_{\mathcal{S}}(\mathbf{A}^{\top}\mathbf{n}) = \mathbf{n}\cdot\mathrm{div}_{\mathcal{S}}\mathbf{A} - \mathbf{A}:\mathbf{K}, \tag{2.9}$$

which arise, respectively, on choosing $\mathbf{A} = \mathbf{1}$ in $(2.8)_1$ and $\mathbf{g} = \mathbf{n}$ in $(2.8)_2$.

We close this section by establishing an identity important to what follows. Assume that on a subsurface $\mathcal{S}_0$ of $\mathcal{S}$,

$$\mathbf{v} = \mathbf{0}. \tag{2.10}$$

Then $\mathbf{v}\cdot\mathbf{n} = 0$, so that $\mathrm{grad}_{\mathcal{S}}(\mathbf{v}\cdot\mathbf{n}) = \mathbf{0}$ and

$$(\mathrm{grad}_{\mathcal{S}}\mathbf{v})^{\top}\mathbf{n} = \mathbf{P}(\mathrm{grad}\,\mathbf{v})^{\top}\mathbf{n} = \mathbf{0};$$

hence $\boldsymbol{\omega}\times\mathbf{n} = \mathbf{P}(\boldsymbol{\omega}\times\mathbf{n}) = 2\mathbf{PWn} = (\mathbf{P}\mathrm{grad}\,\mathbf{v})\mathbf{n} = \mathbf{P}(\partial\mathbf{v}/\partial n)$ and the desired identity,

$$\mathbf{P}\,\frac{\partial\mathbf{v}}{\partial n} = \boldsymbol{\omega}\times\mathbf{n} \quad \text{on} \quad \mathcal{S}_0, \tag{2.11}$$

follows.

3. Power expenditures

Throughout this section R — with boundary $\mathcal{S}$ and outward unit normal $\mathbf{n}$ — is an arbitrary control volume.

3.1. *Internal power*

In discussing the manner in which power is expended internally, bear in mind that our goal is a theory that accounts, not only for the velocity gradient, but also for the gradient, $\mathrm{grad}\,\boldsymbol{\omega}$, of the *vorticity* $\boldsymbol{\omega}$. To accomplish this we generalize the classical theory — which has internal power of the form

$$\int\limits_R \mathbf{T}:\mathrm{grad}\,\mathbf{v}\,\mathrm{d}v = \int\limits_R T_{ij}v_{i,j}\,\mathrm{d}v$$

with $\mathbf{T}$ the *stress* and $\mathbf{T}:\mathrm{grad}\,\mathbf{v}$ the stress power — by introducing a *hyperstress* $\mathbf{G}$ with associated hyperstress power $\mathbf{G}:\mathrm{grad}\,\boldsymbol{\omega}$, and therefore write the *internal power* in the form

$$\mathcal{W}_{\mathrm{int}}(R) = \int\limits_R (\mathbf{T}:\mathrm{grad}\,\mathbf{v} + \mathbf{G}:\mathrm{grad}\,\boldsymbol{\omega})\,\mathrm{d}v = \int\limits_R (T_{ij}v_{i,j} + G_{ij}\omega_{i,j})\,\mathrm{d}v. \tag{3.1}$$

The fields $\mathbf{T}$ and $\mathbf{G}$ are defined over the deformed body for all time. Since $\mathrm{tr}\,(\mathrm{grad}\,\boldsymbol{\omega}) = 0$, we may, without loss in generality, require that $\mathbf{G}$ be *traceless*:

$$\mathrm{tr}\,\mathbf{G} = 0. \tag{3.2}$$

3.2. *External power*

Conventionally, power is expended on a control volume R by surface tractions acting on $\mathcal{S} = \partial R$ and body forces acting over R, and each of these force fields expends power (pointwise) over the velocity $\mathbf{v}$. Conventional continuum mechanics is based on a classical hypothesis of Cauchy asserting that the surface traction at a point $\mathbf{x}$ on $\mathcal{S}$ and time t be a function $\mathbf{t}_{\mathbf{n}}(\mathbf{x}, t)$ of the normal $\mathbf{n}(\mathbf{x}, t)$. Here, as we shall see, *it is necessary to abandon this hypothesis* and assume instead that for each control volume R and each time t there is a *surface-traction* field $\mathbf{t}_{\mathcal{S}}$ defined over $\mathcal{S} = \partial R$ such that $\mathbf{t}_{\mathcal{S}}$ gives the surface force, per unit area, on $\mathcal{S}$.

As is classical, we assume that the body force is given by a field $\mathbf{b}$, and that both $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{b}$ are power conjugate to the velocity $\mathbf{v}$. Further:

(i) we stipulate that $\mathbf{b}$ *account for inertia*;

(ii) we assume that the underlying frame is inertial;

(iii) *we neglect non-inertial body forces.*

It then follows that

$$\mathbf{b} = -\rho\dot{\mathbf{v}}. \tag{3.3}$$

The external power expended on the boundary of the body sets the stage for the formulation of boundary conditions; this power should therefore be based on kinematical fields that — when restricted to the boundary — may be specified independently. Further, since the *internal* power depends on $\operatorname{grad}^2\mathbf{v}$, through $\operatorname{grad}\operatorname{curl}\mathbf{v}$, the *external power* should include a boundary expenditure involving $\operatorname{grad}\mathbf{v}$. But the fields $\mathbf{v}$ and $\operatorname{grad}\mathbf{v}$ are kinematically coupled on $\mathcal{S}$, since a knowledge of $\mathbf{v}$ on $\mathcal{S}$ implies a knowledge of the tangential derivatives of $\mathbf{v}$ on $\mathcal{S}$; thus the *tangential* part of $\operatorname{grad}\mathbf{v}$ cannot be specified independently of $\mathbf{v}$. Bearing this in mind, we consider a (vectorial) *hypertraction* $\mathbf{m}_{\mathcal{S}}(\mathbf{x}, t)$ that expends power over the *normal* part $\partial\mathbf{v}/\partial n$ of the velocity gradient. Based on this discussion, we assume that the power expended externally on an arbitrary control volume R has the form

$$\mathcal{W}_{\mathrm{ext}}(R) = \int\limits_{\mathcal{S}} \Big(\mathbf{t}_{\mathcal{S}} \cdot \mathbf{v} + \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \mathbf{v}}{\partial n}\Big)\,\mathrm{d}a + \int\limits_{R} \mathbf{b} \cdot \mathbf{v}\,\mathrm{d}v. \tag{3.4}$$

Note that by (2.11), on any subsurface of $\mathcal{S}$ for which $\mathbf{v} = \mathbf{0}$ and $\mathbf{m}_{\mathcal{S}} \cdot \mathbf{n} = \mathbf{0}$ (cf. (4.19)), the integral over $\mathcal{S}$ in (3.4) takes the form

$$\int\limits_{\mathcal{S}} (\mathbf{n} \times \mathbf{m}_{\mathcal{S}}) \cdot \boldsymbol{\omega}\,\mathrm{d}a, \tag{3.5}$$

showing that in this special but important case *the power expended at the boundary by tractions is due solely to vorticity.*

4. Principle of virtual power

Most commonly, the principle of virtual power is used to generate weak formulations of boundary-value problems. In this form, the principle is stated for the body B as a whole and is contingent upon the provision of particular conditions on the boundary ∂B of B. Here, we use the principle of virtual power to *determine the structure of the tractions and of the local force balances.* This involves a *nonstandard* formulation in which the principle is stated for an arbitrary control volume R as opposed to the body as a whole. As such, conditions on ∂B play a role no different from those on the boundary ∂R of any control volume R.

4.1. *Statement of the principle of virtual power*

To state this principal, assume that, at some arbitrarily chosen but *fixed time*, the region occupied by the body is known, as are the tractions $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$, the body force $\mathbf{b}$, and the stresses $\mathbf{T}$ and $\mathbf{G}$, and consider the velocity $\mathbf{v}$ as a virtual field $\boldsymbol{\phi}$ that may be specified *independently of the actual evolution of the body*. Then, writing

$$\left.\begin{aligned} \mathcal{W}_{\rm ext}(R,\boldsymbol{\phi}) &= \int\limits_{\mathcal{S}} \Big(\mathbf{t}_{\mathcal{S}}\cdot\boldsymbol{\phi} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\Big)\,{\rm d}a + \int\limits_{R} \mathbf{b}\cdot\boldsymbol{\phi}\,{\rm d}v, \\ \mathcal{W}_{\rm int}(R,\boldsymbol{\phi}) &= \int\limits_{R} (\mathbf{T}:{\rm grad}\,\boldsymbol{\phi} + \mathbf{G}:{\rm grad}\,{\rm curl}\,\boldsymbol{\phi})\,{\rm d}v, \end{aligned}\right] \tag{4.1}$$

respectively, for the external and internal expenditures of *virtual power*, the *principle of virtual power* is the requirement that *the external and internal powers be balanced*: given any control volume R,

$$\mathcal{W}_{\rm ext}(R,\boldsymbol{\phi}) = \mathcal{W}_{\rm int}(R,\boldsymbol{\phi}) \quad \text{for all virtual velocities } \boldsymbol{\phi}. \tag{4.2}$$

4.2. *Consequences of the principle of virtual power*

To determine the consequence of this principle, we first consider the individual terms in the internal power. Using the divergence theorem, we obtain

$$\int\limits_{R} \mathbf{T}:{\rm grad}\,\boldsymbol{\phi}\,{\rm d}v = -\int\limits_{R} {\rm div}\,\mathbf{T}\cdot\boldsymbol{\phi}\,{\rm d}v + \int\limits_{\mathcal{S}} \mathbf{T}\mathbf{n}\cdot\boldsymbol{\phi}\,{\rm d}a. \tag{4.3}$$

Similarly, the divergence theorem applied twice yields

$$\begin{aligned} \int\limits_{R} \mathbf{G}:{\rm grad}\,{\rm curl}\,\boldsymbol{\phi}\,{\rm d}v &= -\int\limits_{R} ({\rm div}\,\mathbf{G})\cdot({\rm curl}\,\boldsymbol{\phi})\,{\rm d}v + \int\limits_{\mathcal{S}} \mathbf{G}\mathbf{n}\cdot{\rm curl}\,\boldsymbol{\phi}\,{\rm d}a \\ &= -\int\limits_{R} ({\rm curl}\,{\rm div}\,\mathbf{G})\cdot\boldsymbol{\phi}\,{\rm d}v + \int\limits_{\mathcal{S}} (\mathbf{G}\mathbf{n}\cdot{\rm curl}\,\boldsymbol{\phi} + (\mathbf{n}\times{\rm div}\,\mathbf{G})\cdot\boldsymbol{\phi})\,{\rm d}a. \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{W}_{\rm int}(R,\boldsymbol{\phi}) = -\int\limits_{R} ({\rm div}\,\mathbf{T} + {\rm curl}\,{\rm div}\,\mathbf{G})\cdot\boldsymbol{\phi}\,{\rm d}v \\ + \int\limits_{\mathcal{S}} (\mathbf{G}\mathbf{n}\cdot{\rm curl}\,\boldsymbol{\phi} + (\mathbf{T}\mathbf{n} + \mathbf{n}\times{\rm div}\,\mathbf{G})\cdot\boldsymbol{\phi})\,{\rm d}a. \end{aligned} \tag{4.4}$$

Further, by $(2.5)_1$,

$${\rm grad}\,\boldsymbol{\phi} = {\rm grad}_{\mathcal{S}}\boldsymbol{\phi} + \frac{\partial\boldsymbol{\phi}}{\partial n}\otimes\mathbf{n};$$

therefore, letting

$$\mathbf{g} = \mathbf{G}\mathbf{n} \tag{4.5}$$

it follows that

$$\begin{aligned} \mathbf{g}\cdot{\rm curl}\,\boldsymbol{\phi} &= -(\mathbf{g}\times):{\rm grad}\,\boldsymbol{\phi} = -(\mathbf{g}\times):{\rm grad}_{\mathcal{S}}\boldsymbol{\phi} - (\mathbf{g}\times):\Big(\frac{\partial\boldsymbol{\phi}}{\partial n}\otimes\mathbf{n}\Big) \\ &= -(\mathbf{g}\times):{\rm grad}_{\mathcal{S}}\boldsymbol{\phi} + (\mathbf{n}\times\mathbf{g})\cdot\frac{\partial\boldsymbol{\phi}}{\partial n} \end{aligned}$$

and (4.4) becomes

$$\mathcal{W}_{\text{int}}(R,\boldsymbol{\phi}) = -\int\limits_R (\operatorname{div}\mathbf{T} + \operatorname{curl}\operatorname{div}\mathbf{G})\cdot\boldsymbol{\phi}\,\mathrm{d}v - \int\limits_{\mathcal{S}} (\mathbf{g}\times)\,\mathbf{:}\,\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi}\,\mathrm{d}a + \int\limits_{\mathcal{S}} \Big((\mathbf{Tn} + \mathbf{n}\times\operatorname{div}\mathbf{G})\cdot\boldsymbol{\phi} + (\mathbf{n}\times\mathbf{g})\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\Big)\,\mathrm{d}a. \quad (4.6)$$

Our next step is to establish an important identity for the integral $\int_{\mathcal{S}}(\mathbf{g}\times)\,\mathbf{:}\,\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi}\,\mathrm{d}a$; specifically, letting $\mathbf{A} = -(\mathbf{g}\times)$, we now show that

$$\int\limits_{\mathcal{S}} \mathbf{A}\,\mathbf{:}\,\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi}\,\mathrm{d}a = -\int\limits_{\mathcal{S}} (\operatorname{div}_{\mathcal{S}}\mathbf{A} + 2K\mathbf{An})\cdot\boldsymbol{\phi}\,\mathrm{d}a. \quad (4.7)$$

The verification of (4.7) is based on the *surface divergence theorem*: let $\boldsymbol{\tau}$ be a *tangential* vector field on $\mathcal{S}$ and let $\mathcal{T}$ be a subsurface of $\mathcal{S}$ with $\boldsymbol{\nu}$ the outward unit normal to the boundary curve $\partial\mathcal{T}$ (so that $\boldsymbol{\nu}$ is tangent to $\mathcal{S}$, normal to $\partial\mathcal{T}$, and directed outward from $\mathcal{T}$); then

$$\int\limits_{\partial\mathcal{T}} \boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds = \int\limits_{\mathcal{T}} \operatorname{div}_{\mathcal{S}}\boldsymbol{\tau}\,\mathrm{d}a. \quad (4.8)$$

To establish (4.7) note that

$$\boldsymbol{\tau} \stackrel{\text{def}}{=} \mathbf{PA}^{\top}\boldsymbol{\phi}$$

represents a tangential vector field, so that, by (4.8),

$$\int\limits_{\partial\mathcal{T}} \boldsymbol{\tau}\cdot\boldsymbol{\nu}\,ds = \int\limits_{\mathcal{T}} \operatorname{div}_{\mathcal{S}}(\mathbf{PA}^{\top}\boldsymbol{\phi})\,\mathrm{d}a. \quad (4.9)$$

Further, by $(2.8)_2$ — with $\mathbf{A}$ replaced by $\mathbf{AP}$ — and $(2.8)_1$,

$$\operatorname{div}_{\mathcal{S}}(\mathbf{PA}^{\top}\boldsymbol{\phi}) = (\mathbf{AP})\,\mathbf{:}\,\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi} + \boldsymbol{\phi}\cdot\operatorname{div}_{\mathcal{S}}(\mathbf{AP}) = \mathbf{A}\,\mathbf{:}\,\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi} + \boldsymbol{\phi}\cdot(\operatorname{div}_{\mathcal{S}}\mathbf{A} + 2K\mathbf{An});$$

hence (4.9) takes the form

$$\int\limits_{\partial\mathcal{T}} \boldsymbol{\tau}\cdot\boldsymbol{\nu}\,\mathrm{d}s = \int\limits_{\mathcal{T}} (\mathbf{A}\,\mathbf{:}\,\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi} + (\operatorname{div}_{\mathcal{S}}\mathbf{A} + 2K\mathbf{An})\cdot\boldsymbol{\phi})\,\mathrm{d}a. \quad (4.10)$$

Finally, if we take $\mathcal{T} = \mathcal{S}$, then $\partial\mathcal{T}$ is empty and the left side of (4.10) vanishes; thus (4.7) is satisfied; thus

$$-\int\limits_{\mathcal{S}} (\mathbf{g}\times)\,\mathbf{:}\,\operatorname{grad}_{\mathcal{S}}\boldsymbol{\phi}\,\mathrm{d}a = \int\limits_{\mathcal{S}} (\operatorname{div}_{\mathcal{S}}(\mathbf{g}\times) - 2K\mathbf{n}\times\mathbf{g})\cdot\boldsymbol{\phi}\,\mathrm{d}a. \quad (4.11)$$

and therefore

$$\mathcal{W}_{\text{int}}(R,\boldsymbol{\phi}) = -\int\limits_R (\operatorname{curl}\operatorname{div}\mathbf{G} + \operatorname{div}\mathbf{T}\cdot\boldsymbol{\phi}\,\mathrm{d}v + \int\limits_{\mathcal{S}} \Big((\mathbf{Tn} + \operatorname{div}_{\mathcal{S}}(\mathbf{g}\times) - 2K\mathbf{n}\times\mathbf{g} + \mathbf{n}\times\operatorname{div}\mathbf{G})\cdot\boldsymbol{\phi} + (\mathbf{n}\times\mathbf{g})\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\Big)\,\mathrm{d}a. \quad (4.12)$$

We are now in a position to apply the virtual-power balance (4.2): by $(4.1)_1$ and (4.12),

$$\int\limits_{\mathcal{S}} \Big(\mathbf{t}_{\mathcal{S}} \cdot \boldsymbol{\phi} + \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \boldsymbol{\phi}}{\partial n}\Big)\, \mathrm{d}a + \int\limits_{R} \mathbf{b} \cdot \boldsymbol{\phi}\, \mathrm{d}v = -\int\limits_{R} (\operatorname{div} \mathbf{T} + \operatorname{curl} \operatorname{div} \mathbf{G}) \cdot \boldsymbol{\phi}\, \mathrm{d}v$$
$$+ \int\limits_{\mathcal{S}} \Big((\mathbf{Tn} + \operatorname{div}_{\mathcal{S}}(\mathbf{g}\times) - 2K\mathbf{n} \times \mathbf{g} + \mathbf{n} \times \operatorname{div} \mathbf{G}) \cdot \boldsymbol{\phi} + (\mathbf{n} \times \mathbf{g}) \cdot \frac{\partial \boldsymbol{\phi}}{\partial n}\Big)\, \mathrm{d}a; \quad (4.13)$$

using (4.5) and rearranging (4.13), we have the "only if" implication in the next result.

(#) *Given any virtual velocity* $\boldsymbol{\phi}$ *and any control volume* R*, the virtual balance*

$$\underbrace{\int\limits_{\mathcal{S}} \Big(\mathbf{t}_{\mathcal{S}} \cdot \boldsymbol{\phi} + \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \boldsymbol{\phi}}{\partial n}\Big)\, \mathrm{d}a + \int\limits_{R} \mathbf{b} \cdot \boldsymbol{\phi}\, \mathrm{d}v}_{\mathcal{W}_{\mathrm{ext}}(R,\boldsymbol{\phi})} = \underbrace{\int\limits_{R} (\mathbf{T} : \operatorname{grad} \boldsymbol{\phi} + \mathbf{G} \,\vdots\, \operatorname{grad} \operatorname{curl} \boldsymbol{\phi})\, \mathrm{d}v}_{\mathcal{W}_{\mathrm{int}}(R,\boldsymbol{\phi})} \quad (4.14)$$

is satisfied if and only if

$$\int\limits_{\mathcal{S}} (\mathbf{t}_{\mathcal{S}} - \mathbf{Tn} - \operatorname{div}_{\mathcal{S}}(\mathbf{Gn}\times) + 2K\mathbf{n} \times \mathbf{Gn} - \mathbf{n} \times \operatorname{div} \mathbf{G}) \cdot \boldsymbol{\phi}\, \mathrm{d}a$$
$$+ \int\limits_{\mathcal{S}} (\mathbf{m}_{\mathcal{S}} - \mathbf{n} \times \mathbf{Gn}) \cdot \frac{\partial \boldsymbol{\phi}}{\partial n}\, \mathrm{d}a = -\int\limits_{R} (\operatorname{div} \mathbf{T} + \operatorname{curl} \operatorname{div} \mathbf{G} + \mathbf{b}) \cdot \boldsymbol{\phi}\, \mathrm{d}v. \quad (4.15)$$

The reverse implication, that (4.15) implies (4.14), follows upon reversing the argument leading to (4.14).

4.3. *Local force balance and traction conditions*

Since the control volume R and the virtual field $\boldsymbol{\phi}$ in (4.15) *may be arbitrarily chosen*, we may appeal to the the fundamental lemma of the calculus of variations and arrive at the *local force balance*

$$\operatorname{div} \mathbf{T} + \operatorname{curl} \operatorname{div} \mathbf{G} + \mathbf{b} = \mathbf{0} \qquad (T_{ij,j} + \varepsilon_{ikr} G_{rj,jk} + b_i = 0) \quad (4.16)$$

and — bearing in mind that, since $\boldsymbol{\phi}$ is arbitrary, $\boldsymbol{\phi}$ and $\partial \boldsymbol{\phi}/\partial n$ may be arbitrarily chosen independent of one another on $\mathcal{S}$ (cf. the paragraph containing (3.4)) — the *traction conditions*

$$\left.\begin{aligned} \mathbf{t}_{\mathcal{S}} &= \mathbf{Tn} + \operatorname{div}_{\mathcal{S}}(\mathbf{Gn}\times) + \mathbf{n} \times (\operatorname{div} \mathbf{G} - 2K\mathbf{Gn}), \\ \mathbf{m}_{\mathcal{S}} &= \mathbf{n} \times \mathbf{Gn}. \end{aligned}\right\} \quad (4.17)$$

In view of (3.3), the local force balance becomes the *local momentum balance*

$$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \operatorname{curl} \operatorname{div} \mathbf{G} \qquad (\rho \dot{v}_i = T_{ij,j} + \varepsilon_{ikr} G_{rj,jk}). \quad (4.18)$$

Note that, as a consequence of $(4.17)_2$, the hypertraction is tangent to the boundary:

$$\mathbf{m}_{\mathcal{S}} \cdot \mathbf{n} = \mathbf{0}. \quad (4.19)$$

4.4. *Digression: virtual power balance with Cosserat stresses*

Assume that the stress $\mathbf{T}$, hyperstress $\mathbf{G}$, and body force $\mathbf{b}$ are consistent with the local force balance (4.16) and moment balance $\mathbf{T} = \mathbf{T}^{\top}$. Then, by (4.4), whose derivation is based on the explicit form (4.1) of the internal power $\mathcal{W}_{\mathrm{int}}(R, \boldsymbol{\phi})$,

$$\mathcal{W}_{\mathrm{int}}(R, \boldsymbol{\phi}) = \int\limits_{\mathcal{S}} (\mathbf{Gn} \cdot \operatorname{curl} \boldsymbol{\phi} + (\mathbf{Tn} + \mathbf{n} \times \operatorname{div} \mathbf{G}) \cdot \boldsymbol{\phi})\, \mathrm{d}a + \int\limits_{R} \mathbf{b} \cdot \boldsymbol{\phi}\, \mathrm{d}v, \quad (4.20)$$

and hence, appealing to (4.1), we arrive at the following integral identity:

$$\int_{\mathcal{S}} (\mathbf{Gn} \cdot \operatorname{curl} \boldsymbol{\phi} + (\mathbf{Tn} + \mathbf{n} \times \operatorname{div} \mathbf{G}) \cdot \boldsymbol{\phi}) \, \mathrm{d}a + \int_R \mathbf{b} \cdot \boldsymbol{\phi} \, \mathrm{d}v = \int_R (\mathbf{T} : \operatorname{grad} \boldsymbol{\phi} + \mathbf{G} : \operatorname{grad} \operatorname{curl} \boldsymbol{\phi}) \, \mathrm{d}v. \quad (4.21)$$

If we define symmetric and skew Cosserat stresses $\mathbf{C}_{\mathrm{S}}$ and $\mathbf{C}_{\mathrm{A}}$ by

$$\mathbf{C}_{\mathrm{S}} \overset{\text{def}}{=} \mathbf{T} \qquad \text{and} \qquad \mathbf{C}_{\mathrm{A}} \overset{\text{def}}{=} -(\operatorname{div} \mathbf{G})\times,$$

and a total Cosserat stress $\mathbf{C}$ by

$$\mathbf{C} = \mathbf{C}_{\mathrm{S}} + \mathbf{C}_{\mathrm{A}},$$

then (4.21) becomes

$$\int_{\mathcal{S}} (\mathbf{Cn} \cdot \boldsymbol{\phi} + \mathbf{Gn} \cdot \operatorname{curl} \boldsymbol{\phi}) \, \mathrm{d}a + \int_R \mathbf{b} \cdot \boldsymbol{\phi} \, \mathrm{d}v = \int_R (\mathbf{C}_{\mathrm{S}} : \operatorname{grad} \boldsymbol{\phi} + \mathbf{G} : \operatorname{grad} \operatorname{curl} \boldsymbol{\phi}) \, \mathrm{d}v, \quad (4.22)$$

which is the form the virtual-power balance would take within the Cosserat framework. (See Mindlin & Tiersten (1962): in that study the right side of (1.13) gives the internal power; the right side of (1.4) subject to (1.6) and (1.7) gives the external power.)

The virtual balance (4.22) gives a sense in which the "traction" $\mathbf{Gn}$ *is conjugate to the vorticity* and hence a sense in which the $\mathbf{G}$ represents a couple-stress. Unfortunately, this balance is of little use in virtual power arguments: a knowledge of $\boldsymbol{\phi}$ on $\mathcal{S}$ implies a knowledge of the tangential derivatives of $\boldsymbol{\phi}$ on $\mathcal{S}$, and so $\boldsymbol{\phi}$ and $\operatorname{curl} \boldsymbol{\phi}$ cannot generally be varied independently. For that reason a portion of the power expenditure $\mathbf{Gn} \cdot \operatorname{curl} \boldsymbol{\phi}$ should be explicitly accounted for in the term conjugate to $\boldsymbol{\phi}$ and the terms $\operatorname{div}_{\mathcal{S}}(\mathbf{Gn}\times) - 2K\mathbf{n} \times \mathbf{Gn}$ appear in the relation $(4.17)_1$ for the traction $\mathbf{t}_{\mathcal{S}}$.

5. Balance laws for forces and moments

5.1. *Consequences of frame-indifference*

Frame-indifference requires that the theory be invariant under all changes in frame. In accord with this principle we require that the internal power be invariant under transformations of the form

$$\tilde{\mathbf{v}}(\mathbf{x}, t) \mapsto \tilde{\mathbf{v}}(\mathbf{x}, t) + \underbrace{\boldsymbol{\alpha}(t) + \boldsymbol{\Omega}(t)\mathbf{x}}_{\mathbf{w}(\mathbf{x},t)}, \quad (5.1)$$

where $\boldsymbol{\alpha}(t)$ is an arbitrary scalar and $\boldsymbol{\Omega}(t)$ an arbitrary skew tensor, at each t. It then follows, as a consequence of the virtual balance (4.2), that *the external power is automatically consistent with frame-indifference.*

Consider the internal power. By (5.1), the velocity gradient transforms according to $\operatorname{grad} \mathbf{v}(\mathbf{x}, t) \mapsto \operatorname{grad} \mathbf{v}(\mathbf{x}, t) + \boldsymbol{\Omega}$; we may therefore conclude from (3.1) that for the internal power to be frame-indifferent we must have

$$\int_R \mathbf{T} : \boldsymbol{\Omega} \, \mathrm{d}v = 0$$

for all skew tensors $\mathbf{\Omega}$ and all control volumes R; hence the stress $\mathbf{T}$ is *symmetric*:

$$\mathbf{T} = \mathbf{T}^{\top}. \tag{5.2}$$

A consequence of (5.2) is that the *stress power* $\mathbf{T}\colon\operatorname{grad}\mathbf{v}$ in the internal power (3.1) may equally well be written in the form $\mathbf{T}\colon\mathbf{D}$, with $\mathbf{D}$ the stretching defined in $(2.1)_1$.

We now turn to the external power (3.4), which is automatically frame-indifferent: invariance under (5.1) implies that

$$\left.\begin{aligned} \int\limits_{\mathcal{S}} \mathbf{t}_{\mathcal{S}}\,\mathrm{d}a + \int\limits_{R} \mathbf{b}\,\mathrm{d}v = \mathbf{0}, \\ \int\limits_{\mathcal{S}} (\mathbf{x}\times\mathbf{t}_{\mathcal{S}} + \mathbf{n}\times\mathbf{m}_{\mathcal{S}})\,\mathrm{d}a + \int\limits_{R} \mathbf{x}\times\mathbf{b}\,\mathrm{d}v = \mathbf{0}, \end{aligned}\right\} \tag{5.3}$$

which bear comparison to their classical conterparts in which $\mathbf{t}_{\mathcal{S}} = \mathbf{T}\mathbf{n}$ and $\mathbf{m}_{\mathcal{S}} = \mathbf{0}$. When combined with (3.3), (5.3) represent balances for linear and angular momentum. The term $\mathbf{n}\times\mathbf{m}_{\mathcal{S}}$ represents a distribution of couples on $\mathcal{S}$.

Our formulation of the virtual power principle ensures that the classical balances (5.3) are satisfied automatically.

5.2. *Locality of the tractions. Action-reaction principle*

A consequence of (4.17) is that the *tractions are local*: at any point $\mathbf{x}$ on $\mathcal{S}$, $\mathbf{t}_{\mathcal{S}}(\mathbf{x})$ depends on $\mathcal{S}$ through a dependence on the normal $\mathbf{n}(\mathbf{x})$ and curvature tensor $\mathbf{K}(\mathbf{x})$ at $\mathbf{x}$, while $\mathbf{m}_{\mathcal{S}}(\mathbf{x})$ depends on $\mathcal{S}$ through $\mathbf{n}(\mathbf{x})$ (where for convenience we have suppressed the argument t). Thus, writing $\mathbf{t}_{(\mathbf{n},\mathbf{K})}$ and $\mathbf{m}_{\mathbf{n}}$ for the corresponding functions, we obtain

$$\mathbf{t}_{\mathcal{S}} = \mathbf{t}_{(\mathbf{n},\mathbf{K})}, \qquad \mathbf{m}_{\mathcal{S}} = \mathbf{m}_{\mathbf{n}} \tag{5.4}$$

i.e., e.g., $\mathbf{t}_{\mathcal{S}}(\mathbf{x}) = \mathbf{t}_{(\mathbf{n}(\mathbf{x}),\mathbf{K}(\mathbf{x}))}(\mathbf{x})$. Then, letting $-\mathcal{S}$ denote for the surface $\mathcal{S}$ oriented by $-\mathbf{n}$ (which has curvature tensor $-\mathbf{K}$), we see that, by (4.17),

$$\mathbf{t}_{\mathcal{S}} = -\mathbf{t}_{-\mathcal{S}}, \qquad \mathbf{m}_{\mathcal{S}} = \mathbf{m}_{-\mathcal{S}}; \tag{5.5}$$

(5.5) represents an *action-reaction principle* for oppositely oriented surfaces that touch and are tangent at a point.

Consider an *arbitrary surface* $\mathcal{S}$ with orientation $\mathbf{n}$ and define the plus side of $\mathcal{S}$ as the side into which $\mathbf{n}$ points and the minus side as the other side. In the definition (3.4) of the external power the quantity $\mathcal{W}_{\mathrm{surf}}(\mathcal{S})$ defined by

$$\mathcal{W}_{\mathrm{surf}}(\mathcal{S}) = \int\limits_{\mathcal{S}} \Big(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\Big)\,\mathrm{d}a \tag{5.6}$$

represents the power expended on the boundary of a control volume. However, because the tractions are local, this definition is also meaningful for an arbitrary surface $\mathcal{S}$ with orientation $\mathbf{n}$. In this instance, $\mathcal{W}_{\mathrm{surf}}(\mathcal{S})$ represents the power expended *by* the material on the plus side of $\mathcal{S}$ *on* the material on the minus side of $\mathcal{S}$, so that, by (5.5) (and since $\partial\mathbf{v}/\partial n = \mathbf{n}\cdot\operatorname{grad}\mathbf{v}$) we have the *power balance*

$$\mathcal{W}_{\mathrm{surf}}(\mathcal{S}) = -\mathcal{W}_{\mathrm{surf}}(-\mathcal{S}). \tag{5.7}$$

5.3. *Environmental tractions. Balance of forces and moments at the boundary*

Let $B(t)$ denote the region of space occupied by the liquid at an arbitrarily chosen time and let $\mathbf{n}(\mathbf{x},t)$ denote the outward unit normal to $\partial B(t)$. We assume that $\partial B(t)$ is smooth.

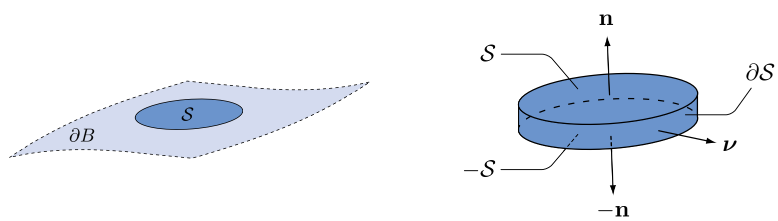

FIGURE 1. Pillbox corresponding to a subsurface $\mathcal{S}$ of the boundary ∂B of the region B of space occupied by the body. Only a portion of ∂B is depicted. Whereas $\mathbf{n}$ is oriented into the environment, $-\mathbf{n}$ is oriented into the fluid. The outward unit normal on the lateral face $\partial\mathcal{S}$ of the pillbox is denoted by $\boldsymbol{\nu}$.

We denote by $\mathbf{t}^{\rm env}_{\partial B}$ and $\mathbf{m}^{\rm env}_{\partial B}$ the *environmental tractions* so that, given any subsurface $\mathcal{S}$ of ∂B,

$$\int\limits_{\mathcal{S}} \mathbf{t}^{\rm env}_{\partial B}\,{\rm d}a \qquad \text{and} \qquad \int\limits_{\mathcal{S}} \mathbf{x}\times\mathbf{t}^{\rm env}_{\partial B}\,{\rm d}a + \int\limits_{\mathcal{S}} \mathbf{n}\times\mathbf{m}^{\rm env}_{\partial B}\,{\rm d}a \tag{5.8}$$

represent the net force and moment exerted on $\mathcal{S}$ by the environment.

We let σ denote the *surface tension* of the fluid at the boundary and assume, for convenience, that σ is *constant.*

Consider an *arbitrary* evolving subsurface $\mathcal{S}(t)$ of $\partial B(t)$. We view $\mathcal{S}$ as a *boundary pillbox* of infinitesimal thickness containing a portion of the boundary, a view that allows us to isolate the physical processes in the material on the two sides of the boundary. The geometric boundary of $\mathcal{S}$ consists of its boundary curve $\partial\mathcal{S}$. But $\mathcal{S}$ viewed as pillbox has a *pillbox boundary* consisting of (Figure 1):

- a surface $\mathcal{S}$ with unit normal $\mathbf{n}$; $\mathcal{S}$ is viewed as lying in the *environment* at the interface of the fluid and the environment;
- a surface $-\mathcal{S}$ with unit normal $-\mathbf{n}$; $-\mathcal{S}$ is viewed as lying in the fluid adjacent to the boundary;
- a "lateral face" represented by $\partial\mathcal{S}$.

The outward unit normal on the lateral face $\partial\mathcal{S}$ of the pillbox is denoted by $\boldsymbol{\nu}$.

To derive force and moment balances for the boundary we first note that, by the net power expended on the pillbox is given by (2.4),

$$\int\limits_{\partial\mathcal{S}} \sigma\boldsymbol{\nu}\,{\rm d}s = \int\limits_{\partial\mathcal{S}} \sigma\mathbf{P}\boldsymbol{\nu}\,{\rm d}s \qquad \text{and} \qquad \int\limits_{\partial\mathcal{S}} \sigma\mathbf{x}\times\boldsymbol{\nu}\,{\rm d}s = \int\limits_{\partial\mathcal{S}} \sigma\mathbf{x}\times\mathbf{P}\boldsymbol{\nu}\,{\rm d}s.$$

represent the force and moment exerted by the fluid on the lateral face of the pillbox by surface tension. Further, by (5.5), the force and moment exerted by the fluid on the pillbox surface $-\mathcal{S}$ are

$$-\int\limits_{\mathcal{S}} \mathbf{t}_{\mathcal{S}}\,{\rm d}a \qquad \text{and} \qquad -\int\limits_{\mathcal{S}} \mathbf{x}\times\mathbf{t}_{\mathcal{S}}\,{\rm d}a - \int\limits_{\mathcal{S}} \mathbf{n}\times\mathbf{m}_{\mathcal{S}}\,{\rm d}a, \tag{5.9}$$

while $(5.8)_1$ and $(5.8)_2$ represent the force and moment exerted by the fluid on the pillbox surface $\mathcal{S}$. Thus the force and moment balances for the pillbox have the form

$$\left.\begin{aligned}\int\limits_{\mathcal{S}} \mathbf{t}^{\text{env}}_{\partial B}\,\mathrm{d}a - \int\limits_{\mathcal{S}} \mathbf{t}_{\mathcal{S}}\,\mathrm{d}a + \int\limits_{\partial\mathcal{S}} \sigma\mathbf{P}\boldsymbol{\nu}\,\mathrm{d}s = \mathbf{0},\\ \int\limits_{\mathcal{S}} (\mathbf{x}\times\mathbf{t}^{\text{env}}_{\partial B} + \mathbf{n}\times\mathbf{m}^{\text{env}}_{\partial B})\,\mathrm{d}a - \int\limits_{\mathcal{S}} (\mathbf{x}\times\mathbf{t}_{\mathcal{S}} + \mathbf{n}\times\mathbf{m}_{\mathcal{S}})\,\mathrm{d}a + \int\limits_{\partial\mathcal{S}} \sigma\,\mathbf{x}\times\mathbf{P}\boldsymbol{\nu}\,\mathrm{d}s = \mathbf{0}.\end{aligned}\right\} \tag{5.10}$$

We now localize these balances, starting with the force balance. The counterpart of (4.8) for tensor fields $\mathbf{A}$ that satisfy $\mathbf{A}\mathbf{n} = \mathbf{0}$ is

$$\int\limits_{\partial\mathcal{S}} \mathbf{A}\boldsymbol{\nu}\,ds = \int\limits_{\mathcal{S}} \mathrm{div}_{\mathcal{S}}\mathbf{A}\,\mathrm{d}a. \tag{5.11}$$

Thus, since surface tension σ is constant, we may use $(2.9)_1$ to conclude that

$$\int\limits_{\partial\mathcal{S}} \sigma\mathbf{P}\boldsymbol{\nu}\,\mathrm{d}s = \int\limits_{\mathcal{S}} 2\sigma K\mathbf{n}\,\mathrm{d}a$$

and $(5.10)_1$ becomes

$$\int\limits_{\mathcal{S}} (\mathbf{t}^{\text{env}}_{\partial B} - \mathbf{t}_{\mathcal{S}} + 2\sigma K\mathbf{n})\,\mathrm{d}a = \mathbf{0};$$

Since $\mathcal{S}$ is an arbitrary subsurface of ∂B, we have the *local force balance for the boundary*:

$$\mathbf{t}_{\mathcal{S}} = \mathbf{t}^{\text{env}}_{\partial B} + 2\sigma K\mathbf{n}. \tag{5.12}$$

A slightly more complicated analysis results in the *local torque balance for the boundary*:

$$\mathbf{n}\times\mathbf{m}_{\mathcal{S}} = \mathbf{n}\times\mathbf{m}^{\text{env}}_{\partial B}; \tag{5.13}$$

cf. the paragraph containing (5.12) in Anderson, Cermelli, Fried, Gurtin, and McFadden (2005).

The hypertraction $\mathbf{m}^{\text{env}}_{\partial B}$ enters the theory through the torque balance (5.13) and since the normal part of $\mathbf{m}^{\text{env}}_{\partial B}$ is irrelevant to this balance, we assume without loss in generality that

$$\mathbf{n}\cdot\mathbf{m}^{\text{env}}_{\partial B} = \mathbf{0}. \tag{5.14}$$

Thus, by (4.19), we may replace (5.13) by

$$\mathbf{m}_{\mathcal{S}} = \mathbf{m}^{\text{env}}_{\partial B}. \tag{5.15}$$

Finally, by (4.17), the balances (5.12) and (5.15) expressed in terms of the stress $\mathbf{T}$ and hyperstress $\mathbf{G}$ have the form

$$\left.\begin{aligned}\mathbf{T}\mathbf{n} + \mathrm{div}_{\mathcal{S}}(\mathbf{G}\mathbf{n}\times) + \mathbf{n}\times(\mathrm{div}\,\mathbf{G} - 2K\mathbf{G}\mathbf{n}) = \mathbf{t}^{\text{env}}_{\partial B} + 2\sigma K\mathbf{n},\\ \mathbf{n}\times\mathbf{G}\mathbf{n} = \mathbf{m}^{\text{env}}_{\partial B}.\end{aligned}\right\} \tag{5.16}$$

6. Energetics. Dissipation

6.1. *Free energy imbalance. Dissipation inequality*

Let $\mathcal{R}(t)$ be an arbitrary region that convects with the body. We restrict attention to a purely mechanical theory based on the requirement that

(#) *the temporal increase in free energy of $\mathcal{R}(t)$ be less than or equal to the power expended on $\mathcal{R}(t)$.*

Precisely, letting ψ denote the *specific free energy*, this requirement takes the form of a free energy imbalance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{R}(t)} \rho\psi\,\mathrm{d}v \le \mathcal{W}_{\mathrm{ext}}(\mathcal{R}(t)). \tag{6.1}$$

The imbalance (6.1) is consistent with standard continuum thermodynamics based on balance of energy and an entropy imbalance (the Clausius–Duhem inequality): in that more general framework, granted isothermal conditions with temperature ϑ_0, (6.1) would be satisfied with right side minus left side equal to ϑ_0 times the entropy production. (Cf., e.g., (2.9) of Anderson, Cermelli, Fried, Gurtin & McFadden (2005); taking $\vartheta = \vartheta_0 =$ constant in $(2.9)_2$ of Anderson, Cermelli, Fried, Gurtin & McFadden (2005) and substracting the resulting equation from $(2.9)_1$ of Anderson, Cermelli, Fried, Gurtin & McFadden (2005) yields (6.1).)

Balance of mass implies that $(\mathrm{d}/\mathrm{d}t)\int_{\mathcal{R}(t)}\rho\psi\,\mathrm{d}v = \int_{R(t)}\rho\dot{\psi}\,\mathrm{d}v$; since $\mathcal{W}_{\mathrm{ext}}(\mathcal{R}(t)) = \mathcal{W}_{\mathrm{int}}(\mathcal{R}(t))$, we may therefore use the expression (3.1) for the internal power $\mathcal{W}_{\mathrm{int}}(\mathcal{R}(t))$ in conjunction with the symmetry of $\mathbf{T}$, to localize (6.1); the result is the *local free energy imbalance*

$$\rho\dot{\psi} - \mathbf{T}\colon\mathbf{D} - \mathbf{G}\colon\mathrm{grad}\,\boldsymbol{\omega} \le 0 \qquad (\rho\dot{\psi} - T_{ij}D_{ij} - G_{ij}\omega_{i,j} \le 0), \tag{6.2}$$

where $\mathbf{D}$ is the stretching defined in $(2.1)_1$. The difference

$$\Gamma \overset{\mathrm{def}}{=} \mathbf{T}\colon\mathbf{D} + \mathbf{G}\colon\mathrm{grad}\,\boldsymbol{\omega} - \rho\dot{\psi} \ge 0 \tag{6.3}$$

represents the *bulk dissipation* and allows us to rewrite (6.1) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{R}(t)} \rho\psi\,\mathrm{d}v - \mathcal{W}_{\mathrm{ext}}(\mathcal{R}(t)) = -\int_{\mathcal{R}(t)} \Gamma\,\mathrm{d}v \le 0. \tag{6.4}$$

Note that, by (4.20) and the power balance $\mathcal{W}_{\mathrm{ext}}(\mathcal{R}(t)) = \mathcal{W}_{\mathrm{int}}(\mathcal{R}(t))$ we can rewrite the free energy imbalance equivalently as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{R}(t)} \rho\psi\,\mathrm{d}v - \underbrace{\Bigg[\int_{\mathcal{S}(t)} \big((\mathbf{T}\mathbf{n} + \mathbf{n}\times\mathrm{div}\,\mathbf{G})\cdot\mathbf{v} + \mathbf{G}\mathbf{n}\cdot\boldsymbol{\omega}\big)\,\mathrm{d}a + \int_{R} \mathbf{b}\cdot\mathbf{v}\,\mathrm{d}v\Bigg]}_{\text{internal power expenditure in Cosserat form}} = -\int_{\mathcal{R}(t)} \Gamma\,\mathrm{d}v \le 0. \tag{6.5}$$

6.2. *Imbalance of free and kinetic energy*

The power expended by the body force has the form

$$\mathbf{b}\cdot\mathbf{v} = -\tfrac{1}{2}\rho\overline{|\mathbf{v}|^2}^{\boldsymbol{\cdot}},$$

and we may rewrite the external power expenditure as the sum of a non-inertial expenditure minus a kinetic-energy rate:

$$\mathcal{W}_{\rm ext}(\mathcal{R}(t)) = \underbrace{\int\limits_{\partial\mathcal{R}(t)} \Big(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\Big)\,{\rm d}a}_{\text{noninertial power expenditure}} - \underbrace{\frac{\rm d}{{\rm d}t}\int\limits_{\mathcal{R}(t)} \tfrac{1}{2}\rho|\mathbf{v}|^2\,{\rm d}v}_{\text{kinetic energy}}\,. \tag{6.6}$$

By (6.6), the free energy imbalance (6.1) — for a control volume $\mathcal{R}(t)$ that convects with the fluid — takes the form of an imbalance of free and kinetic energy

$$\underbrace{\frac{\rm d}{{\rm d}t}\int\limits_{\mathcal{R}(t)} \rho(\psi + \tfrac{1}{2}|\mathbf{v}|^2)\,{\rm d}v}_{\text{net energy rate}} - \int\limits_{\partial\mathcal{R}(t)} \Big(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\Big)\,{\rm d}a = -\underbrace{\int\limits_{\mathcal{R}(t)} \Gamma\,{\rm d}v}_{\text{net dissipation}} \le 0. \tag{6.7}$$

Further, using standard continuum mechanics, we may rewrite (6.7) as an imbalance for a control volume R; precisely, (6.7) is satisfied for all regions $\mathcal{R}(t)$ that convect with the body if and only if

$$\frac{\rm d}{{\rm d}t}\int\limits_{R} \rho(\psi + \tfrac{1}{2}|\mathbf{v}|^2)\,{\rm d}v + \int\limits_{\mathcal{S}} \rho(\psi + \tfrac{1}{2}|\mathbf{v}|^2)\,\mathbf{v}\cdot\mathbf{n}\,{\rm d}a - \int\limits_{\mathcal{S}} \Big(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\Big)\,{\rm d}a = -\int\limits_{R} \Gamma\,{\rm d}v \le 0. \tag{6.8}$$

for all control volumes R.

7. Application to turbulent flow

7.1. *Simple constitutive equations for an incompressible fluid*

We assume that the fluid is incompressible, so that

$$\rho = \text{constant} \qquad \text{and} \qquad \operatorname{div}\mathbf{v} = \operatorname{tr}\mathbf{D} = 0. \tag{7.1}$$

Without loss in generality, we may then suppose that

$$\mathbf{T} = \mathbf{S} - p\mathbf{1}, \quad \operatorname{tr}\mathbf{S} = 0 \qquad (T_{ij} = S_{ij} - p\delta_{ij}, \quad S_{kk} = 0), \tag{7.2}$$

where the *pressure* p is a constitutively indeterminate field that does not affect the internal power (3.1); the field $\mathbf{S}$ represents the *extra stress*. Then, by $(7.1)_2$,

$$\mathbf{T}\colon\mathbf{D} = \mathbf{S}\colon\mathbf{D}, \tag{7.3}$$

and the local free-energy imbalance (6.3) reduces to the *dissipation inequality*

$$\Gamma = \mathbf{S}\colon\mathbf{D} + \mathbf{G}\colon\operatorname{grad}\boldsymbol{\omega} - \rho\dot{\psi} \ge 0. \tag{7.4}$$

Guided by the presence of the term involving the corotational rate $\overset{\circ}{\mathbf{D}} = \dot{\mathbf{D}} + \mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}$ of the stretching tensor $\mathbf{D}$ in the Navier–Stokes-α equation, we suppose that the specific free energy and the extra stress are given by constitutive equations of the form

$$\rho\psi = \lambda|\mathbf{D}|^2 \qquad \text{and} \qquad \mathbf{S} = 2\mu\mathbf{D} + 2\lambda\overset{\circ}{\mathbf{D}}, \tag{7.5}$$

with λ and μ constant. These choices are familiar from the theory of Rivlin–Ericksen fluids; cf. Rivlin & Ericksen, 1955; Truesdell & Noll, 1965, §119; Dunn & Fosdick, 1974.

To ensure that the specific free energy has a strict minimum when $\mathbf{D}=\mathbf{0}$, we assume that

$$\lambda > 0. \tag{7.6}$$

Futher, based on a result of Mindlin & Tiersten (1962) for an elastic solid, we assume that the hyperstress is given by a constitutive equation of the form

$$\mathbf{G} = \zeta\,\mathrm{grad}\,\boldsymbol{\omega} + \xi(\mathrm{grad}\,\boldsymbol{\omega})^{\top}, \tag{7.7}$$

with ζ and ξ constant. Note that, consistent with (3.2), $\mathbf{G}$ as determined by (7.7) is traceless. We conjecture that (7.7) is the most general linear, isotropic relation possible between $\mathbf{G}$ and $\mathrm{grad}\,\boldsymbol{\omega}$.

With the choices (7.5) and (7.7), the dissipation inequality (7.4) reduces to

$$\Gamma = 2\mu|\mathbf{D}|^2 + (\zeta+\xi)|\mathbf{A}|^2 + (\zeta-\xi)|\mathbf{Z}|^2 \geq 0, \tag{7.8}$$

where we have introduced

$$\mathbf{A} = \tfrac{1}{2}\big(\mathrm{grad}\,\boldsymbol{\omega} + (\mathrm{grad}\,\boldsymbol{\omega})^{\top}\big) \qquad \text{and} \qquad \mathbf{Z} = \tfrac{1}{2}\big(\mathrm{grad}\,\boldsymbol{\omega} - (\mathrm{grad}\,\boldsymbol{\omega})^{\top}\big). \tag{7.9}$$

Conditions that are both necessary and sufficient that (7.8) be satisfied for all $\mathbf{D}$, $\mathbf{A}$, and $\mathbf{Z}$ are $\mu \geq 0$, $\zeta+\xi \geq 0$, and $\zeta-\xi \geq 0$; equivalently, $\mu \geq 0$, $\zeta \geq 0$, and $-\zeta \leq \xi \leq \zeta$, and we are led to the *moduli conditions*:

$$\mu \geq 0, \qquad \zeta \geq 0, \qquad -\zeta \leq \xi \leq \zeta. \tag{7.10}$$

7.2. *Flow equation*

Bearing in mind $(7.1)_2$, (7.2), $(7.5)_2$, and the assumption that λ and μ are constant,

$$\mathrm{div}\,\mathbf{T} = \mathrm{div}\,\mathbf{S} - \mathrm{grad}\,p = \mu\Delta\mathbf{v} + 2\lambda\mathrm{div}\,\mathring{\mathbf{D}} - \mathrm{grad}\,p; \tag{7.11}$$

similarly, in view of $(7.1)_2$, (7.7), and the assumption that ζ and ξ are constant,

$$\begin{aligned}(\mathrm{curl}\,\mathrm{div}\,\mathbf{G})_i &= \varepsilon_{ikr}G_{rj,jk} = \varepsilon_{ikr}(\zeta\varepsilon_{rpq}v_{q,pjjk} + \xi\varepsilon_{jpq}v_{q,prjk}) \\ &= \zeta(\delta_{ip}\delta_{kq} - \delta_{iq}\delta_{kp})v_{q,pjjk} + \xi(\varepsilon_{ikr}\varepsilon_{jpq})v_{q,prjk} \\ &= \zeta(\underbrace{v_{q,qijj}}_{=0} - v_{i,ppjj}) + \xi\varepsilon_{jpq}\underbrace{\varepsilon_{ikr}v_{q,krjp}}_{=0} = -\zeta(\Delta\Delta\mathbf{v})_i,\end{aligned}$$

so that

$$\mathrm{curl}\,\mathrm{div}\,\mathbf{G} = -\zeta\Delta\Delta\mathbf{v}. \tag{7.12}$$

Using (7.11) and (7.12) in the local momentum balance (4.18), we arrive at the *flow equation*

$$\rho\dot{\mathbf{v}} = -\mathrm{grad}\,p + \mu\Delta\mathbf{v} - \zeta\Delta\Delta\mathbf{v} + 2\lambda\mathrm{div}\,\mathring{\mathbf{D}}, \tag{7.13}$$

which, in components, has the equivalent form

$$\rho\dot{v}_i = -p_{,i} + \mu v_{i,jj} - \zeta v_{i,jjkk} + 2\lambda\mathring{D}_{ij,j}. \tag{7.14}$$

With the special choices

$$\lambda = \rho\alpha^2 \qquad \text{and} \qquad \zeta = \mu\alpha^2, \tag{7.15}$$

the flow equation (7.13) reduces to the *Navier–Stokes-α equation*:

$$\rho\dot{\mathbf{v}} = -\mathrm{grad}\,p + \mu(1-\alpha^2\Delta)\Delta\mathbf{v} + 2\rho\alpha^2\mathrm{div}\,\mathring{\mathbf{D}}. \tag{7.16}$$

The parameter α carries dimensions of length. Viewed within the context of Lagrangian averaging, α is the statistical correlation length of the excursions taken by a fluid particle away from its phase-averaged trajectory. More intuitively, α can be interpreted as the characteristic linear dimension of smallest eddy resolvable by the Navier–Stokes-α model.

Returning to the general flow equation (7.13), we may identify two characteristic length scales

$$L_e = \sqrt{\frac{\lambda}{\rho}} \qquad \text{and} \qquad L_d = \sqrt{\frac{\zeta}{\mu}}. \tag{7.17}$$

Whereas L_e is of energetic origin, L_d is dissipative in nature. In the standard development of the Navier–Stokes-α equation, the term involving $\Delta\Delta\mathbf{v}$ is added, ad hoc, to the equation subsequent to the Lagrangian averaging of the Euler equation. Lagrangian averaging yields the term involving the corotational rate of $\mathbf{D}$. The length scale α of the Navier–Stokes-α theory is therefore both energetic and dissipative. We see no a priori justification for the choices (7.15).

7.3. *Free energy imbalance revisited*

Next, we may use (7.8) to write the free energy imbalance (6.8) (for a control volume R) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_R (\lambda|\mathbf{D}|^2 + \tfrac{1}{2}\rho|\mathbf{v}|^2)\,\mathrm{d}v + \int\limits_{\mathcal{S}} (\lambda|\mathbf{D}|^2 + \tfrac{1}{2}\rho|\mathbf{v}|^2)\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}a - \int\limits_{\mathcal{S}} \Big(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\Big)\,\mathrm{d}a$$
$$= -\int\limits_R (2\mu|\mathbf{D}|^2 + (\zeta+\xi)|\mathbf{A}|^2 + (\zeta-\xi)|\mathbf{Z}|^2)\,\mathrm{d}v \le 0, \tag{7.18}$$

with $\mathbf{A}$ and $\mathbf{Z}$ the symmetric and skew parts of $\operatorname{grad}\boldsymbol{\omega}$ as defined in (7.9).

Turbulence is often studied assuming spatial periodicity and restricting attention to a control volume R consisting of a single cubic cell. We now derive the form of the free-energy imbalace (7.18) for a cubic cell in a spatial periodic flow. Since each face of such a cell must have bulk fields $\mathbf{v}$, $\operatorname{grad}\mathbf{v}$, … each equal to its value on the opposing face, while the outward normals on the two faces are equal and opposite, we may conclude, using (4.17), that

$$\int\limits_{\mathcal{S}} (\lambda|\mathbf{D}|^2 + \tfrac{1}{2}\rho|\mathbf{v}|^2)\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}a = 0, \qquad \int\limits_{\mathcal{S}} \Big(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\Big)\,\mathrm{d}a = 0. \tag{7.19}$$

(The fact that K is undefined at each corner is not a problem: simply replace each corner with a spherical cap of radius ϵ; then, since the area of each cap is $\mathrm{O}(\varepsilon^2)$, while $K = \epsilon^{-1}$, the integral over each cap tends to zero as $\epsilon \to 0$.) Further, since $\operatorname{div}\mathbf{v} = 0$,

$$\int\limits_R \operatorname{grad}\mathbf{v} : (\operatorname{grad}\mathbf{v})^\top\,\mathrm{d}v = \int\limits_R \operatorname{div}\big((\operatorname{grad}\mathbf{v})\mathbf{v}\big)\,\mathrm{d}v = \int\limits_{\mathcal{S}} \mathbf{n}\cdot(\operatorname{grad}\mathbf{v})\mathbf{v}\,\mathrm{d}v = 0;$$

hence

$$2\int\limits_R |\mathbf{D}|^2\,\mathrm{d}v = 2\int\limits_R |\mathbf{W}|^2\,\mathrm{d}v\,\mathrm{d}v = \int\limits_R |\boldsymbol{\omega}|^2\,\mathrm{d}v, \tag{7.20}$$

and a similar argument yields

$$2\int_R |\mathbf{A}|^2\,\mathrm{d}v = 2\int_R |\mathbf{Z}|^2\,\mathrm{d}v = \int_R |\mathrm{grad}\,\boldsymbol{\omega}|^2\,\mathrm{d}v \tag{7.21}$$

Thus, for R for a cubic cell in a spatial periodic flow the free-energy imbalace (6.8) has the simple form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_R \tfrac{1}{2}(\lambda|\boldsymbol{\omega}|^2 + \rho|\mathbf{v}|^2)\,\mathrm{d}v = -\int_R (\mu|\boldsymbol{\omega}|^2 + \zeta|\mathrm{grad}\,\boldsymbol{\omega}|^2)\,\mathrm{d}v \le 0. \tag{7.22}$$

and yields the conclusion that the integral $\int_R(\lambda|\boldsymbol{\omega}|^2 + \rho|\mathbf{v}|^2)\,\mathrm{d}v$ decreses with time.

8. Boundary conditions

In this section we develop counterparts of the classical notions of a free surface and a fixed surface without slip; that is, a surface at which the fluid abuts and adheres to a motionless, nondeformable environment. For convenience, when discussing free surfaces we neglect the pressure of the environment.

As in §5.3, we let $\mathbf{t}^{\mathrm{env}}_{\partial B}$ and $\mathbf{m}^{\mathrm{env}}_{\partial B}$ denote the traction and hypertraction exerted by the environment on the boundary ∂B of the region occupied by the fluid, and we let $\mathbf{n}$ denote the outward unit normal on ∂B.

8.1. *Weak formulation of the flow equation and boundary conditions at a prescribed time when a portion of the boundary is a free surface and the remainder a fixed surface without slip*

Because we work within a framework based on the principle of virtual power, it is fairly straightforward to derive a weak (variational) formulation of the flow equation and the boundary conditions discussed above. We begin by rewriting the virtual balance (4.14) with $R = B$ and with the tractions $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$ specified, via the boundary force and moment balances

$$\mathbf{t}_{\mathcal{S}} = \mathbf{t}^{\mathrm{env}}_{\partial B} + 2\sigma K\mathbf{n} \qquad \text{and} \qquad \mathbf{m}_{\mathcal{S}} = \mathbf{m}^{\mathrm{env}}_{\partial B}$$

(cf. (5.12) and (5.15)), in terms of their environmental counterparts $\mathbf{t}^{\mathrm{env}}_{\partial B}$ and $\mathbf{m}^{\mathrm{env}}_{\partial B}$:

$$\int_{\partial B}\Big((\mathbf{t}^{\mathrm{env}}_{\partial B} + 2\sigma K\mathbf{n})\cdot\boldsymbol{\phi} + \mathbf{m}^{\mathrm{env}}_{\partial B}\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\Big)\,\mathrm{d}a + \int_B \mathbf{b}\cdot\boldsymbol{\phi}\,\mathrm{d}v = \int_B (\mathbf{T}\colon\mathrm{grad}\,\boldsymbol{\phi} + \mathbf{G}\colon\mathrm{grad}\,\mathrm{curl}\,\boldsymbol{\phi})\,\mathrm{d}v. \tag{8.1}$$

Consider the boundary integral in (8.1), but with $\boldsymbol{\phi} = \mathbf{v}$. Bearing in mind (4.19), this term has the form

$$\int_{\partial B}\Big((\mathbf{t}^{\mathrm{env}}_{\partial B} + 2\sigma K\mathbf{n})\cdot\mathbf{v} + \mathbf{m}^{\mathrm{env}}_{\partial B}\cdot\mathbf{P}\frac{\partial\mathbf{v}}{\partial n}\Big)\,\mathrm{d}a, \tag{8.2}$$

with $\mathbf{P}$ the projection onto the plane normal to ∂B defined by (2.4).

We assume that the mean curvature K of — and the surface tension σ at — the boundary are known. Then the form of the integal (8.2) and experience with the principle of virtual power suggest that reasonable boundary conditions might, at each point of ∂B, consist of a prescription of

(i) $\mathbf{t}^{\mathrm{env}}_{\partial B}$ or $\mathbf{v}$, or a relation between $\mathbf{t}^{\mathrm{env}}_{\partial B}$ and $\mathbf{v}$; and
(ii) $\mathbf{m}^{\mathrm{env}}_{\partial B}$ or $\mathbf{P}\partial\mathbf{v}/\partial n$, or a relation between $\mathbf{m}^{\mathrm{env}}_{\partial B}$ and $\mathbf{P}\partial\mathbf{v}/\partial n$.

Consistent with this, we consider specific *boundary conditions* in which a portion $\mathcal{S}_{\rm free}$ of ∂B is a *free surface* and the remainder $\mathcal{S}_{\rm nslp}$ is a *fixed surface without slip*:

(I) On $\mathcal{S}_{\rm free}$ the environmental tractions vanish ($\mathbf{t}^{\rm env}_{\partial B} = \mathbf{m}^{\rm env}_{\partial B} = \mathbf{0}$), so that

$$\mathbf{t}_{\mathcal{S}} \equiv \mathbf{Tn} + {\rm div}_{\mathcal{S}}(\mathbf{Gn}\times) = \sigma K\mathbf{n} \quad \text{and} \quad \mathbf{m}_{\mathcal{S}} \equiv \mathbf{n}\times\mathbf{Gn} = \mathbf{0} \quad \text{on} \quad \mathcal{S}_{\rm free} \tag{8.3}$$

(cf. (5.16); $(8.3)_1$ follows from $(5.16)_1$ and $(8.3)_2$).

(II) On $\mathcal{S}_{\rm nslp}$ the fluid velocity vanishes and the hypertraction $\mathbf{m}^{\rm env}_{\partial B}$ is prescribed, so that

$$\mathbf{v} = \mathbf{0} \quad \text{and} \quad \mathbf{m}_{\mathcal{S}} \equiv \mathbf{n}\times\mathbf{Gn} = \mathbf{m}^{\rm env}_{\partial B} \quad \text{on} \quad \mathcal{S}_{\rm nslp}; \tag{8.4}$$

cf. $(5.16)_2$. Here $\mathbf{m}^{\rm env}_{\partial B}$ may be prescribed as a field on ∂B or constitutively as function of such fields; in §8.2 we consider the specific condition $\mathbf{m}^{\rm env}_{\partial B} = -\mu\ell\mathbf{P}\partial\mathbf{v}/\partial n = -\mu\ell\boldsymbol{\omega}\times\mathbf{n}$ (cf. (2.11)) with ℓ constant.

As is customary when discussing boundary conditions of the form $(8.4)_1$, we restrict attention to virtual velocity fields $\boldsymbol{\phi}$ that are *kinematically admissible* in the sense that

$$\boldsymbol{\phi} = \mathbf{0} \quad \text{on} \quad \mathcal{S}_{\rm nslp}. \tag{8.5}$$

Given such a field, granted the boundary conditions (8.3) and (8.4), the virtual-power balance (8.1) yields the *virtual balance*:

$$\int_{\mathcal{S}_{\rm free}} \sigma K\mathbf{n}\cdot\boldsymbol{\phi}\,{\rm d}a + \int_{\mathcal{S}_{\rm nslp}} \mathbf{m}^{\rm env}_{\partial B}\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\,{\rm d}a + \int_B \mathbf{b}\cdot\boldsymbol{\phi}\,{\rm d}v = \int_B (\mathbf{T}\,{:}\,{\rm grad}\,\boldsymbol{\phi} + \mathbf{G}\,{:}\,{\rm grad}\,{\rm curl}\,\boldsymbol{\phi})\,{\rm d}v. \tag{8.6}$$

The result (#) on page 12 then implies that, given any kinematically admissible $\boldsymbol{\phi}$, (8.6) *is equivalent to* (4.15) and hence to

$$\int_{\mathcal{S}_{\rm free}} \big(\sigma K\mathbf{n} - (\mathbf{Tn} + {\rm div}_{\mathcal{S}}(\mathbf{Gn}\times) - 2K\mathbf{n}\times\mathbf{Gn})\big)\cdot\boldsymbol{\phi}\,{\rm d}a - \int_{\mathcal{S}_{\rm free}} (\mathbf{n}\times\mathbf{Gn})\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\,{\rm d}a + \int_{\mathcal{S}_{\rm nslp}} (\mathbf{m}^{\rm env}_{\partial B} - \mathbf{n}\times\mathbf{Gn})\cdot\frac{\partial\boldsymbol{\phi}}{\partial n}\,{\rm d}a = -\int_B ({\rm div}\,\mathbf{T} + {\rm curl}\,{\rm div}\,\mathbf{G} + \mathbf{b})\cdot\boldsymbol{\phi}\,{\rm d}v. \tag{8.7}$$

Thus, arguing as in the steps leading to (4.16) and (4.17), we see that, granted $\mathbf{b}$ is given by (3.3), the momentum balance (4.18) is satisfied in B, while the conditions (8.3) and (8.4) are satisfied on ∂B.

Conversely, (4.18), (8.3), and (8.4) imply that (8.7) and (hence) (8.6) are satisfied for all kinematically admissible $\boldsymbol{\phi}$. Finally, as is clear from the discussion in §7, granted the constitutive equations $(7.5)_2$ and (7.7), the momentum balance is equivalent to the flow equation (7.13). We have therefore established a *weak formulation* of the flow equation and the boundary conditions (8.3) and (8.4):

- Granted (3.3) and the constitutive equations $(7.5)_2$ and (7.7), the virtual balance (8.6) is satisfied for all kinematically admissible virtual fields $\boldsymbol{\phi}$ if and only if:
  (i) the flow equation (7.13) is satisfied within the fluid;
  (ii) the conditions (8.3) and (8.4) are satisfied on the boundary of the fluid.

8.2. *The vorticity condition*

Consider the boundary condition (8.4) for a fixed surface without slip. By $(8.4)_1$, $\mathbf{v} = \mathbf{0}$ on $\mathcal{S}_{\rm nslp}$; thus, since $\mathbf{m}_{\mathcal{S}} = \mathbf{m}^{\rm env}_{\partial B}$, we may use (2.11) and (8.2) with ∂B replaced by $\mathcal{S}_{\rm nslp}$; the result is

$$\int\limits_{\mathcal{S}_{\rm nslp}} \mathbf{m}^{\rm env}_{\partial B} \cdot \mathbf{P}\,\frac{\partial \mathbf{v}}{\partial n}\,{\rm d}a = \int\limits_{\mathcal{S}_{\rm nslp}} \mathbf{m}^{\rm env}_{\partial B} \cdot (\boldsymbol{\omega} \times \mathbf{n})\,{\rm d}a, \tag{8.8}$$

and, guided by this relation, we consider a constitutive equation for $\mathbf{m}^{\rm env}_{\partial B}$ of the form

$$\mathbf{m}^{\rm env}_{\partial B} = -\mu\ell\boldsymbol{\omega} \times \mathbf{n} \quad \text{on} \quad \mathcal{S}_{\rm nslp}, \tag{8.9}$$

with ℓ a constitutive modulus whose magnitude $|\ell|$ represents a material length scale. At this point ℓ may be positive or negative. Finally, since $\mathbf{m}^{\rm env}_{\partial B} = \mathbf{n} \times \mathbf{G}\mathbf{n}$ on $\mathcal{S}_{\rm nslp}$ (cf. (8.4)), we arrive at the boundary condition

$$\mathbf{n} \times (\mathbf{G}\mathbf{n} - \mu\ell\boldsymbol{\omega}) = \mathbf{0} \quad \text{on} \quad \mathcal{S}_{\rm nslp}. \tag{8.10}$$

We refer to (8.10) as the *vorticity condition* and to ℓ as the *wall-eddy modulus.*

8.3. *Free energy imbalance, dissipation, and the sign of the wall-eddy modulus* ℓ

In a recent work (Fried & Gurtin 2006) we have given a general discussion of the use of an energy imbalance for a boundary pillbox to develop constitutive relations describing the interaction of the fluid and its environment. We here sketch the corresponding analysis, but only as it applies to the boundary conditions $(8.4)_1$ and (8.10). Let $\mathcal{S}$ denote a fixed (i.e. time-independent) subsurface of $\mathcal{S}_{\rm nslp}$ with $\mathcal{S}$ viewed as a fixed *boundary pillbox* of infinitesimal thickness (cf. §5.3).

Let $\psi^{\rm x}$ denote the *excess free energy* of the fluid at the surface $\mathcal{S}_{\rm nslp}$, measured per unit area, so that

$$\int\limits_{\mathcal{S}} \psi^{\rm x}\,{\rm d}a$$

represents the net free energy of the pillbox. Since $\mathbf{v} = \mathbf{0}$, it is clear from (2.11) and the paragraph containing (5.6) that

$$\mathcal{W}_{\rm surf}(-\mathcal{S}) = -\int\limits_{\mathcal{S}} \mathbf{m}_{\mathcal{S}} \cdot \frac{\partial \mathbf{v}}{\partial n}\,{\rm d}a = -\int\limits_{\mathcal{S}} \mathbf{m}_{\mathcal{S}} \cdot (\boldsymbol{\omega} \times \mathbf{n})\,{\rm d}a \tag{8.11}$$

represents the power expended by the fluid on the pillbox surface $-\mathcal{S}$. We assume that the power expended by the environment on the pillbox surface $\mathcal{S}$ vanishes and hence that the environment is *passive.* (One might wonder how an environment with $\mathbf{m}^{\rm env}_{\partial B} \neq \mathbf{0}$ can be passive. As is clear from the first paragraph of §8, because $\mathcal{S}_{\rm nslp}$ abuts a motionless, nondeformable environment, the environmental tractions $\mathbf{t}^{\rm env}_{\partial B}$ and $\mathbf{m}^{\rm env}_{\partial B}$ must be indeterminate and hence incapable of expending power.) The power expended by the fluid on the lateral face of the pillbox by surface tension vanishes, because the boundary curve $\partial\mathcal{S}$ is stationary. Thus, since, by (5.15), $\mathbf{m}_{\mathcal{S}} = \mathbf{m}^{\rm env}_{\partial B}$, the net power expended on the pillbox is given by

$$-\int\limits_{\mathcal{S}} \mathbf{m}_{\mathcal{S}} \cdot (\boldsymbol{\omega} \times \mathbf{n})\,{\rm d}a = -\int\limits_{\mathcal{S}} \mathbf{m}^{\rm env}_{\partial B} \cdot (\boldsymbol{\omega} \times \mathbf{n})\,{\rm d}a. \tag{8.12}$$

Consider the quantity $D(\mathcal{S})$ defined by

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{S}} \psi^{\mathrm{x}}\,\mathrm{d}a}_{\text{free energy rate}} \; - \underbrace{\int\limits_{\mathcal{S}} \big(-\mathbf{m}^{\mathrm{env}}_{\partial B}\cdot(\boldsymbol{\omega}\times\mathbf{n})\big)\,\mathrm{d}a}_{\text{power expenditure}} = -D(\mathcal{S}). \tag{8.13}$$

Were we to parallel the development in bulk, as discussed in §6.1, with the requirement that the temporal increase in free energy of $\mathcal{S}$ be less than or equal to the power expended on $\mathcal{S}$, then $D(\mathcal{S}) \geq 0$ would represent the *energy dissipated* within the pillbox. Assuming that ψ^{x} is constant and recalling that $\mathcal{S}$ is fixed, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{S}} \psi^{\mathrm{x}}\,\mathrm{d}a = 0,$$

we would find, as a consequence of (8.13), that

$$D(\mathcal{S}) = -\int\limits_{\mathcal{S}} \mathbf{m}^{\mathrm{env}}_{\partial B}\cdot(\boldsymbol{\omega}\times\mathbf{n})\,\mathrm{d}a \geq 0.$$

Thus

$$-\mathbf{m}^{\mathrm{env}}_{\partial B}\cdot(\boldsymbol{\omega}\times\mathbf{n}) \tag{8.14}$$

would represent the *dissipation per unit area*, so that, by (8.9),

$$\int\limits_{\mathcal{S}} \mu\ell|\boldsymbol{\omega}\times\mathbf{n}|^2\,\mathrm{d}a \geq 0. \tag{8.15}$$

Thus, since $\mathcal{S}$ was arbitrarily chosen, we would conclude that

$$\ell \geq 0. \tag{8.16}$$

However, as we shall see, for flow in a channel with the boundary conditions

$$\mathbf{v} = \mathbf{0} \qquad \text{and} \qquad \mathbf{n}\times(\mathbf{G}\mathbf{n} - \mu\ell\boldsymbol{\omega}) = \mathbf{0} \quad \text{on} \;\; \mathcal{S}_{\mathrm{nslp}} \tag{8.17}$$

(cf. $(8.4)_1$, (8.10)), our theory with $\ell \geq 0$ delivers solutions that agree *neither quantitatively nor qualitatively* when compared to the direct numerical simulations of Kim, Moin & Moser (1987) and Moser, Kim & Mansour (1999) and the experimental results of Wei & Wilmarth (1989); on the other hand there is *excellent agreement* when

$$\ell < 0. \tag{8.18}$$

Interestingly, such values of ℓ imply that $\mu\ell\,|\mathbf{n}\times\boldsymbol{\omega}|^2$ — a term which would usually be termed *dissipative* — is *negative*! (Cf. the sentence containing (8.15).)

In this regard it is of interest to revisit the free energy imbalance (7.18) applied to the body itself, with the entire boundary ∂B a fixed surface without slip. In this case the non-slip boundary conditions $(8.4)_1$ and (8.10) are satisfied on the entire boundary ∂B, so that, by (8.9), (8.11), and (8.12),

$$\int\limits_{\partial B} \Big(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\Big)\,\mathrm{d}a = \int\limits_{\partial B} \mu\ell|\boldsymbol{\omega}\times\mathbf{n}|^2\,\mathrm{d}a$$

and, restricting attention to the Navier–Stokes–α theory so that $\lambda = \rho\alpha^2$, (7.18) takes

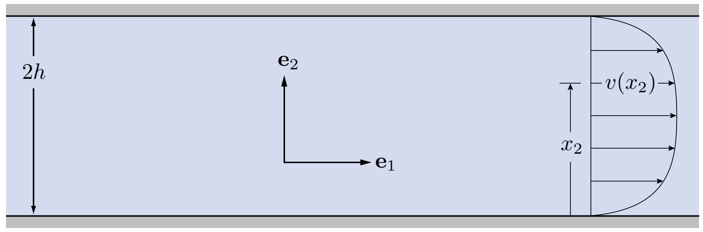

FIGURE 2. Schematic for the problem of flow in a channel of gap $2h$. The coordinates in the directions downstream and out of the plane are x_1 and x_3.

the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_B \rho(\alpha^2|\mathbf{D}|^2 + \tfrac{1}{2}|\mathbf{v}|^2)\,\mathrm{d}v = \begin{cases} -\mu\ell\displaystyle\int_{\partial B} |\boldsymbol{\omega}\times\mathbf{n}|^2\,\mathrm{d}a - \int_B \Gamma\,\mathrm{d}v & \text{for } \ell > 0, \\ \mu\,|\ell|\displaystyle\int_{\partial B} |\boldsymbol{\omega}\times\mathbf{n}|^2\,\mathrm{d}a - \int_B \Gamma\,\mathrm{d}v & \text{for } \ell < 0. \end{cases} \tag{8.19}$$

with $\Gamma \geq 0$, given by (7.8) and (7.9), the bulk dissipation per unit volume. The result (8.19), which is quite striking, demonstrates the destabilizing effect of the non-slip boundary conditions when $\ell < 0$. Indeed, when $\ell > 0$ the right side of (8.19) is negative and the total energy

$$\int_B \rho(\alpha^2|\mathbf{D}|^2 + \tfrac{1}{2}|\mathbf{v}|^2)\,\mathrm{d}v$$

decreases with time. But, when $\ell < 0$, the boundary term $\mu|\ell|\int_{\partial B} |\boldsymbol{\omega}\times\mathbf{n}|^2\,\mathrm{d}a$ is positive and hence *destabilizing*, because its influence on the total energy is to make it increase with time. For $\ell < 0$ the boundary term cannot represent dissipation; instead it represents a *production of energy due to the formation of eddies at the boundary*.

These conclusions render Navier–Stokes–α theory with $\ell < 0$ *incompatible with thermodynamics* as embodied in a free energy imbalance. While we know of no succesful continuum mechanical theory for which experiments yield moduli of signs opposite to those imposed by thermodynamics, one might argue that continuum thermodynamics is inapplicable to a discussion of turbulence when applied at a fixed boundary without slip. Indeed, turbulent eddies generated at such boundaries might render the state of the fluid there sufficiently far removed from equilibrium that standard continuum thermodynamical laws might no longer be valid. In this regard it is interesting to note that the free energy imbalance applied in bulk delivers moduli of signs ($\mu, \zeta \geq 0$) consistent with those of the Navier–Stokes–α equation; cf. (7.10) and (7.13).

For these reasons, we do not dismiss the Navier–Stokes–α theory based on incompatibity with the boundary free energy imbalance, but leave it to others who might better understand the physics of turbulence to either accept or reject our arguments.

9. Flow in a rectangular channel

We now consider the problem of a steady, turbulent flow through an infinite, rectangular channel formed by two parallel walls separated by a gap $2h$ (Figure 2). We suppose that channel walls are fixed and without slip in the sense that the no-slip and wall-eddy conditions (8.17) hold. Further, we assume that the environmental hypertraction is of the particular form (8.9). This simple model problem allows us to investigate the affects of the length scales L_d and ℓ and to make comparisons with numerical and experimental results. As noted in the paragraph including (8.18), to obtain agreement with the observed features of turbulent flow, we find that it is necessary to take $\ell < 0$.

9.1. *Solution of the channel problem*

Employing the notation of Figure 2, we assume that the fluid velocity $\mathbf{v}$ has the form

$$\mathbf{v}(\mathbf{x}) = v(x_2)\mathbf{e}_1; \tag{9.1}$$

$\mathbf{v}$ is therefore consistent with $\operatorname{div}\mathbf{v} = 0$ and obeys $\dot{\mathbf{v}} = \mathbf{0}$. In view of (9.1), the flow equation (7.13) gives

$$-\zeta v'''' + \mu v'' = \frac{\partial p}{\partial x_1}, \qquad 2\lambda v' v'' = \frac{\partial p}{\partial x_2}, \qquad \frac{\partial p}{\partial x_3} = 0, \tag{9.2}$$

while the conditions $(8.4)_1$ and (8.10) for a fixed boundary without slip give

$$v(0) = 0, \quad v(2h) = 0, \quad \mu\ell v'(0) = \zeta v''(0), \quad \mu\ell v'(2h) = -\zeta v''(2h), \tag{9.3}$$

where a prime is used to denote differentiation with respect to the downstream coordinate.

Since v depends only on x_2, (9.2) implies that

$$p(x_1, x_2) = -\beta x_1 + \lambda |v'(x_2)|^2, \quad \text{with} \quad \beta = \text{constant}. \tag{9.4}$$

We assume, without loss of generality, that the pressure decreases with increasing x. It then follows that

$$\beta > 0. \tag{9.5}$$

Further, in view of (9.2), (9.3), and (9.4), v can be expressed as

$$v(x_2) = \frac{\beta h^2}{2\mu}\left\{1 - \left(1 - \frac{x_2}{h}\right)^2 - \frac{2\left(\frac{L_d}{h} + \frac{\ell}{L_d}\right)}{\frac{h}{L_d}\left(1 + \frac{\ell}{L_d}\tanh\frac{h}{L_d}\right)}\left(1 - \frac{\cosh\frac{h}{L_d}\left(1 - \frac{x_2}{h}\right)}{\cosh\frac{h}{L_d}}\right)\right\}, \tag{9.6}$$

with L_d the dissipative length scale defined (7.17), and, to ensure that (9.6) is nonsingular, ℓ assumed consistent with

$$1 + \frac{\ell}{L_d}\tanh\frac{h}{L_d} \neq 0. \tag{9.7}$$

Importantly, the condition (9.7) allows for both positive and negative values of ℓ.

9.2. *Behavior at the wall. Sign of the wall-eddy modulus*

Experiments and DNS simulations of channel flow show that, for suitably normalized laminar and turbulent velocity profiles, the slopes of the turbulent profiles at the channel walls have magnitudes greater than their laminar counterparts (Pope 2000). We now show that the theory exhibits this feature of turbulent flow only when the wall-eddy modulus ℓ is negative.

We begin by normalizing v by its maximum value to yield

$$V(x_2) = \frac{v(x_2)}{v(h)}. \tag{9.8}$$

For comparison, we introduce

$$V_c(x_2) = 1 - \left(1 - \frac{x_2}{h}\right)^2, \tag{9.9}$$

which is the analogous normalization of the laminar solution to the plane channel problem. To embody the observed features of turbulent flow, the slope of V at the base of the channel must exceed that of V_c. A calculation shows that $V'(0) > V_c'(0)$ if and only if

$$\frac{\left(\frac{L_d}{h} + \frac{\ell}{L_d}\right)\left(2 + \sinh\frac{h}{L_d} - 2\cosh\frac{h}{L_d}\right)}{\frac{h}{L_d}\left(1 + \frac{\ell}{L_d}\tanh\frac{h}{L_d}\right)\cosh\frac{h}{L_d}} < 0. \tag{9.10}$$

Since $L_d > 0$ and $h > 0$ it follows that v as defined by (9.6) captures the observed features of turbulent channel flow only if the wall-eddy modulus obeys the inequalities

$$\ell < 0 \tag{9.11}$$

and, in addition to (9.7),

$$\frac{L_d}{h} < \frac{|\ell|}{L_d}. \tag{9.12}$$

Later, we find that $|\ell|/L_d \sim 1$. This being the case, the inequality (9.12) can be interpreted as requiring that the dissipative length scale L_d must be less than the characteristic dimension of the channel. If L_d is identified with the parameter α of the Navier–Stokes-α model, then the requirement that $L_d < h$ is consistent with standard practice in simulations, where α is commonly taken to be a small fraction of the characteristic linear dimension of the flow domain.

The requirement (9.11) that the wall-eddy modulus be negative implies, as we note in the §8.3, that the free energy imbalance (8.15) at the boundary is not satisfied. To obtain agreement with observed behavior at the channel walls, we must therefore violate this form of the second law. (Cf. the discussion in the paragraph containing (6.1).)

Hereafter, we therefore assume that ℓ is negative, so that $|\ell| > 0$, and obeys (9.12). For simplicity, we introduce

$$b = \frac{\frac{|\ell|}{L_d} - \frac{L_d}{h}}{1 - \frac{|\ell|}{L_d}\tanh\frac{h}{L_d}} > 0 \tag{9.13}$$

and write v as

$$v(x_2) = \frac{\beta h^2}{2\mu}\left\{1 - \left(1 - \frac{x_2}{h}\right)^2 + \frac{2b}{\frac{h}{L_d}}\left(1 - \frac{\cosh\frac{h}{L_d}\left(1 - \frac{x_2}{h}\right)}{\cosh\frac{h}{L_d}}\right)\right\}. \tag{9.14}$$

9.3. *Comparison with the solution of Chen, Foias, Holm, Olson, Titi & Wynne* (1999)

These authors use the Navier–Stokes-α theory to study flow in a channel subject to the boundary conditions

$$v(0) = 0, \quad v(2h) = 0, \quad \mu v'(0) = \tau_{\mathrm{w}}, \quad \mu v'(2h) = -\tau_{\mathrm{w}}. \tag{9.15}$$

where $\tau_{\mathrm{w}} > 0$ denotes the *wall shear stress*. Despite the differences between $(9.15)_{3,4}$ and $(9.3)_{3,4}$, the solution to this problem can be obtained directly from (9.14) on setting

$$L_d = \alpha \qquad \text{and} \qquad \frac{|\ell|}{L_d} = \frac{1-\theta}{\tanh\frac{h}{L_d}} + \frac{\theta}{\frac{h}{L_d}}, \tag{9.16}$$

with

$$\theta = \frac{\beta h}{\tau_{\rm w}} \qquad \text{and} \qquad 0 < \theta < 1. \tag{9.17}$$

Importantly, we must have $\ell < 0$ to achieve this correspondence. Moreover, noting that for the channel problem with $\mathbf{v}(\mathbf{x}) = v(x_2)\mathbf{e}_1$,

$$\mathbf{m}_{\mathcal{S}} = \mathbf{e}_2 \times \mathbf{G}\mathbf{e}_2 = \zeta v''\mathbf{e}_1 \qquad \text{and} \qquad \frac{\partial \mathbf{v}}{\partial n} = (\mathbf{n}\cdot\mathbf{e}_2)v'\mathbf{e}_1, \tag{9.18}$$

and using the expression (8.14) for the dissipation and bearing in mind sign difference between the normals on the channel walls, we find that

$$-\mathbf{m}^{\rm env}_{\partial B}\cdot\frac{\partial \mathbf{v}}{\partial n}\Big|_{x_2=0} = -\mathbf{m}^{\rm env}_{\partial B}\cdot\frac{\partial \mathbf{v}}{\partial n}\Big|_{x_2=2h} = L_d^2\tau_{\rm w}v''(0) \tag{9.19}$$

Futher, using (9.16) and $(9.17)_1$ in (9.6) and bearing in mind $(9.17)_2$, we have

$$v''(0) = -\frac{\beta}{\mu}\left(1 + \frac{\frac{h}{\alpha}(1-\theta)}{\theta\tanh\frac{h}{\alpha}}\right) < 0;$$

since $\tau_{\rm w} > 0$, it therefore follows that, for the boundary conditions (9.15),

$$-\mathbf{m}^{\rm env}_{\partial B}\cdot\frac{\partial \mathbf{v}}{\partial n}\Big|_{x_2=0} = -\mathbf{m}^{\rm env}_{\partial B}\cdot\frac{\partial \mathbf{v}}{\partial n}\Big|_{x_2=2d} < 0. \tag{9.20}$$

Thus, as is the case for our solution (9.6) when the wall-eddy modulus obeys $\ell < 0$, the solution of Chen, Foias, Holm, Olson, Titi & Wynne (1999) — which agrees well with experimental and numerical predictions of turbulence — violates the conventional notion of free energy imbalance at the boundary.

9.4. *Comparison with numerical and experimental results*

Assuming that wall-eddy modulus ℓ is negative and consistent with (9.11), we now compare the solution v to the problem for channel flow to the mean downstream velocity for turbulent channel flow as predicted by the direct numerical simulations (DNS) of Kim, Moin & Moser (1987) and Moser, Kim & Mansour (1999).

To make proper comparisons, we employ the standard definitions

$$v_\tau = \sqrt{\frac{\tau_{\rm w}}{\rho}}, \qquad {\rm Re}_\tau = \frac{\rho v_\tau}{\mu}, \qquad \text{and} \qquad y^+ = \frac{{\rm Re}_\tau}{h}x_2 \tag{9.21}$$

for the *friction velocity* v_τ, *friction Reynolds number* ${\rm Re}_\tau$, and the *viscous length* y^+. (Throughout this section, we employ the terminology and notation of Pope (2000).) In addition, corresponding to v as defined by (9.6), we introduce a dimensionless velocity φ via

$$\begin{aligned}\varphi(y^+) &= \frac{1}{v_\tau}v\left(\frac{h}{{\rm Re}_\tau}y^+\right)\\ &= \frac{{\rm Re}_\tau\theta}{2}\left\{1 - \left(1 - \frac{y^+}{{\rm Re}_\tau}\right)^2 + \frac{2b}{\frac{h}{L_d}}\left(1 - \frac{\cosh\frac{h}{L_d}\left(1 - \frac{y^+}{{\rm Re}_\tau}\right)}{\cosh\frac{h}{L_d}}\right)\right\}.\end{aligned} \tag{9.22}$$

Like Chen, Foias, Holm, Olson, Titi & Wynne (1999), we consider the Reynolds shear stress $\langle u_1u_2\rangle$ in the downstream plane of the channel. On identifying v with the mean downstream velocity in turbulent channel flow and writing the velocity field as $v\mathbf{e}_1 + \mathbf{u}$, with $\mathbf{u}$ the *fluctuating velocity*, the downstream component of the ensemble-averaged

Re_τ	h/L_d	$\lvert\ell\rvert/L_d$	θ
180	16.6	0.957	0.0583
395	34.8	0.974	0.0336
590	48.1	0.980	0.0239

TABLE 1. Values of h/L_d, $|\ell|/L_d$, and θ determined by fitting φ to the DNS data of Kim, Moin & Moser (1987) and Moser, Kim & Mansour (1999) for the nominal values $\mathrm{Re}_\tau = 180$, $\mathrm{Re}_\tau = 395$, and $\mathrm{Re}_\tau = 590$ of the friction Reynolds number.

Navier–Stokes equations is

$$\mu v'' - \rho\langle u_1 u_2\rangle' = \frac{\partial p}{\partial x_1}\,, \tag{9.23}$$

where

$$\langle u_1 u_2\rangle(0) = \langle u_1 u_2\rangle(2h) = 0 \tag{9.24}$$

and $\partial p/\partial x_1$ is constant. Integration of (9.23) yields

$$-\rho\langle u_1 u_2\rangle(x_2) = \mu v'(0)\left(1 - \frac{x_2}{h}\right) - \mu v'(x_2). \tag{9.25}$$

To make appropriate comparisons, we introduce the dimensionless Reynolds shear stress

$$\begin{aligned} r(y^+) &= -\frac{1}{\tau_{\mathrm{w}}}\langle u_1 u_2\rangle\left(\frac{d}{\mathrm{Re}_\tau}y^+\right) \\ &= \varphi'(0)\left(1 - \frac{y^+}{\mathrm{Re}_\tau}\right) - \varphi'(y^+). \end{aligned} \tag{9.26}$$

We use the nonlinear least-squares method to fit φ as defined by (9.22) to the DNS data of Kim, Moin & Moser (1987) and Moser, Kim & Mansour (1999) for the nominal values $\mathrm{Re}_\tau = 180$, $\mathrm{Re}_\tau = 395$, and $\mathrm{Re}_\tau = 590$ of the friction Reynolds number. The values of the parameters h/L_d, $|\ell|/L_d$ (note from (9.13) that b depends on both h/L_d and $|\ell|/L_d$), and θ determined by these fits are listed in Table 1. Plots of φ and r corresponding to these fits are shown, along with the DNS data, in Figure 3. While the plots of φ exhibit very close agreement with the data, the plots of r agree with the data only for $y^+ \gtrsim 70$ — that is, for y^+ outside the viscous wall region. As Chen, Foias, Holm, Olson, Titi & Wynne (1999) note, this discrepancy might be attributed to the presence of statistically inhomogeneous and/or anisotropic fluctuations within the viscous wall region.

Interestingly, despite the fairly wide disparity between the friction Reynolds numbers considered, the fitted values of $|\ell|/L_d$ listed in Table 1 are all very close. This suggests that the magnitude of the wall-eddy modulus ℓ should remain close to the viscous length L_d.

Let Φ denote the dimensionless version of the solution of Chen, Foias, Holm, Olson, Titi & Wynne (1999). Since Φ involves only two independent dimensionless parameters, h/α and θ, it should not be surprising that φ provides superior fits. Generally, fits based on Φ lie noticably above the DNS data for $1.2 \lesssim y^+ \lesssim 40$ and very slightly below the DNS data for $y^+ \gtrsim 70$ (Chen, Foias, Holm, Olson, Titi & Wynne 1999). This observation is consistent with Figure 4, which shows the relative difference $(\varphi - \Phi)/\varphi$ between the fits corresponding to φ and Φ for $\mathrm{Re}_\tau = 395$. In contrast, the plots of the dimensionless Reynolds shear stresses corresponding to φ and Φ show negligible differences over the

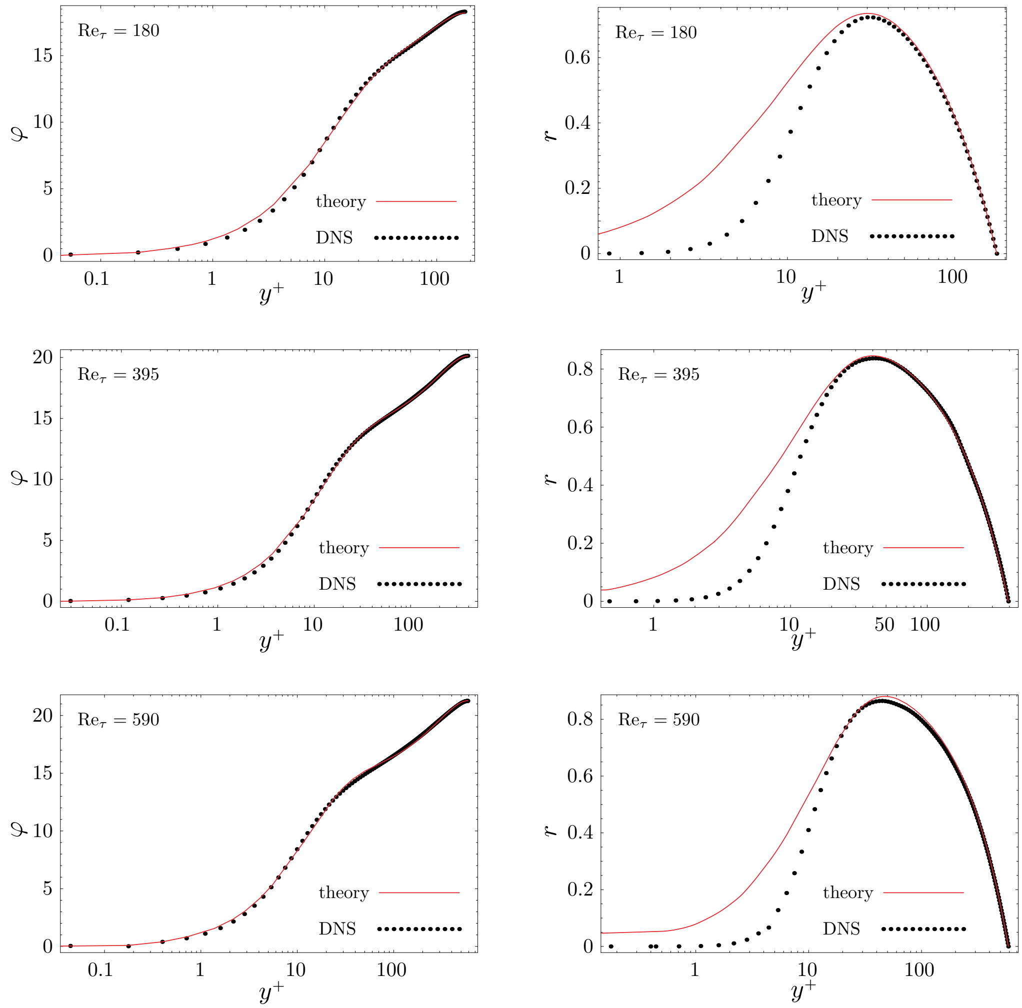

FIGURE 3. Comparison of the dimensionless velocity profile ϕ and the dimensionless Reynolds shear stress r with the data arising from the DNS simulations of Kim, Moin & Moser (1987) and Moser, Kim & Mansour (1999) for the nominal values $\mathrm{Re}_\tau = 180$, $\mathrm{Re}_\tau = 395$, and $\mathrm{Re}_\tau = 590$ of the friction Reynolds number.

range ($y^+ \gtrsim 70$) of viscous lengths where they agree with the DNS data. Within the viscous wall region, the match provided by r provides only a marginal improvement over its counterpart associated with Φ.

If only two of the parameters h/L_d, $|\ell|/L_d$, and θ are independent, a fit based on φ will not necessarily improve upon one based on Φ. In particular, if we assume that $|\ell|/L_d$ is determined in terms of h/L_d and θ via $(9.16)_2$ it then follows from (9.13) that

$$\varphi(y^+) = \frac{\mathrm{Re}_\tau \theta}{2}\left\{1 - \left(1 - \frac{y^+}{\mathrm{Re}_\tau}\right)^2 + \frac{2(1-\theta)}{\theta \frac{h}{L_d} \tanh \frac{h}{L_d}}\left(1 - \frac{\cosh \frac{h}{L_d}\left(1 - \frac{y^+}{\mathrm{Re}_\tau}\right)}{\cosh \frac{h}{L_d}}\right)\right\}. \tag{9.27}$$

Using the values of h/L_d and θ listed in Table 1 in (9.16) yields 0.950, 0.970, and 0.980 for the values of $|\ell|/L_d$ corresponding to $\mathrm{Re}_\tau = 180$, $\mathrm{Re}_\tau = 395$, and $\mathrm{Re}_\tau = 590$, respectively. Comparison with the values, 0.957, 0.974, and 0.980 of $|\ell|/L_d$ listed in

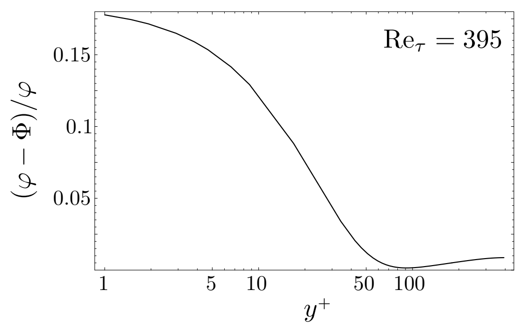

FIGURE 4. Relative difference $(\varphi - \Phi)/\varphi$ between the fits obtained for φ and Φ for the nominal value $\mathrm{Re}_\tau = 395$ of the friction Reynolds number.

Table 1 suggests that the difference between fits based on φ and Φ should decrease as Re_τ increases.

Within the context of plane channel flow, the sole distinction between the full theory involving the energetic and dissipative length scales L_e and L_d defined in (7.17) and the Navier–Stokes-α theory arises in the expression (9.4) for the pressure. Due to the presence of λ, the appropriately nondimensionalized version of (9.4) yields a dependence upon L_e^2. This dependence allows for an additional degree of freedom — and, thus, superior results — when fitting the pressure.

The theory can be used to determine a drag law relating the friction Reynolds number Re_τ to the actual Reynolds number

$$\mathrm{Re} = \frac{\rho}{2\mu h}\int_0^{2h} v(x_2)\,\mathrm{d}x_2 = \int_0^{\mathrm{Re}_\tau} \varphi(y^+)\mathrm{d}y^+. \tag{9.28}$$

Specifically, using the expression (9.22) for φ in (9.28), we find that

$$\mathrm{Re} = \mathrm{Re}_\tau^2\theta\left\{\frac{1}{3} + \frac{b}{\frac{h}{L_d}}\Big(1 - \frac{L_d}{h}\tanh\frac{h}{L_d}\Big)\right\}. \tag{9.29}$$

Recalling the definition (9.17) of θ and defining τ_w via

$$\tau_\mathrm{w} = \mu v'(0) = \beta h\Big(1 + b\tanh\frac{h}{L_d}\Big), \tag{9.30}$$

we find from (9.29) that the theory predicts a resitance law

$$\mathrm{Re}_\tau = c\sqrt{\mathrm{Re}}, \tag{9.31}$$

where c depends only on h/L_d and $|\ell|/L_d$ and has the particular form

$$c = \frac{\sqrt{1 - \frac{|\ell|}{L_d}\tanh\frac{h}{L_d}}}{\sqrt{1 - \frac{L_d}{h}\tanh\frac{h}{L_d}}\sqrt{\frac{1}{3} + \frac{b}{\frac{h}{L_d}}\Big(1 - \frac{L_d}{h}\tanh\frac{h}{L_d}\Big)}}. \tag{9.32}$$

In view of (9.31), the theory underpredicts, by a factor of $\mathrm{Re}^{3/8}$ the scaling $\mathrm{Re}_\tau \sim \mathrm{Re}^{7/8}$ exhibited by the empirical Blasius resistance law (Blasius 1914; Dean 1978; Pope 2000). Agreement with this classical scaling would be achieved if one were to follow Chen, Foias, Holm, Olson, Titi & Wynne (1999) (who allow α to depend on Re) and take

L_d or ℓ to depend on Re. However, constitutive parameters — including L_d and ℓ — should be independent of flow parameters such as the Reynolds number. Alternatively, allowing L_d or ℓ to depend on a frame-indifferent variable such as $|\mathrm{grad}\,\boldsymbol{\omega}|$ might lead to a resistance law consistent with experimental observations. We leave the investigation of this possibility for future study.

10. Addendum: vortex kinetic energy

The term $\frac{1}{2}\lambda|\boldsymbol{\omega}|^2$ appearing in the integrand on the left side of the equality (7.22) is suggestive of a *vortex kinetic energy*, even though it arises from the *free energy*. However, the inequality (7.22) holds only for a spatially periodic flow. We now develop an extended theory that has a vortex kinetic energy in all flows.

10.1. *Gradient kinetic energy. Inertial power balance*

We now consider a kinetic energy, per unit volume of the form

$$\tfrac{1}{2}(\rho|\mathbf{v}|^2 + \kappa|\boldsymbol{\omega}|^2),$$

in which κ, like ρ, is constant, so that the kinetic energy of any region $\mathcal{R}(t)$ that *convects with the fluid* is given by

$$\mathcal{K}(\mathcal{R}(t)) = \int\limits_{\mathcal{R}(t)} \tfrac{1}{2}(\rho|\mathbf{v}|^2 + \kappa|\boldsymbol{\omega}|^2)\,\mathrm{d}v \tag{10.1}$$

and that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(\mathcal{R}(t)) = \int\limits_{\mathcal{R}(t)} (\rho\dot{\mathbf{v}}\cdot\mathbf{v} + \kappa\dot{\boldsymbol{\omega}}\cdot\boldsymbol{\omega})\,\mathrm{d}v. \tag{10.2}$$

Our goal is to determine inertial components of the body force $\mathbf{b}$ and the traction $\mathbf{t}_{\mathcal{S}}$ appropriate to the kinetic energy (10.1). In particular, recalling that $\mathbf{b}$ is assumed to be purely inertial, we first seek a body force $\mathbf{b}$ and an *inertial traction* $\mathbf{t}_{\mathcal{S}}^{\mathrm{in}}$ with the following properties:

(i) There is a *noninertial* traction such that

$$\mathbf{t}_{\mathcal{S}} = \mathbf{t}_{\mathcal{S}}^{\mathrm{in}} + \mathbf{t}_{\mathcal{S}}^{\mathrm{ni}}. \tag{10.3}$$

(ii) The kinetic-energy rate (10.2) is balanced by the *negative* of the power expended by $\mathbf{b}$ and $\mathbf{t}_{\mathcal{S}}^{\mathrm{in}}$ in the sense of the *inertial power balance*:

$$\int\limits_{R} (\rho\dot{\mathbf{v}}\cdot\mathbf{v} + \kappa\dot{\boldsymbol{\omega}}\cdot\boldsymbol{\omega})\,\mathrm{d}v = -\int\limits_{R} \mathbf{b}\cdot\mathbf{v}\,\mathrm{d}v - \int\limits_{\mathcal{S}} \mathbf{t}_{\mathcal{S}}^{\mathrm{in}}\cdot\mathbf{v}\,\mathrm{d}a, \tag{10.4}$$

where in writing (10.4) we have, without loss of generality, replaced $\mathcal{R}$ by an arbitrary control volume R and $\partial\mathcal{R}$ by $\partial R = \mathcal{S}$. (See Podio-Guidugli (1996), who bases his discussion of (classical) kinetic energy on a balance between the rate of kinetic energy and the power expended by an inertial body force.)

Actually, we shall work within a virtual-power framework more stringent than that expressed in (ii).

10.2. *Inertial virtual-power balance. Inertial body force and surface traction*

With a view toward making use of experience gained in the virtual power analysis of §3 and §4, we define (tensorial and vectorial) *momentum-rate forces* $\mathbf{p}$ and $\mathbf{q}$ via

$$\mathbf{p} = \rho\dot{\mathbf{v}} \qquad \text{and} \qquad \mathbf{q} = \kappa\dot{\boldsymbol{\omega}}, \tag{10.5}$$

an we rewrite (10.4) in the form

$$\int\limits_R (\mathbf{p}\cdot\mathbf{v} + \mathbf{q}\cdot\boldsymbol{\omega})\,\mathrm{d}v = -\int\limits_R \mathbf{b}\cdot\mathbf{v}\,\mathrm{d}v - \int\limits_{\mathcal{S}} \mathbf{t}^{\mathrm{in}}_{\mathcal{S}}\cdot\mathbf{v}\,\mathrm{d}a. \tag{10.6}$$

Guided by our discussion of virtual power in §4 (in particular, in the paragraph containing (4.1)) and comparing (10.6) to the virtual power relation defined by (4.1) and (4.2), we assume that, at some arbitrarily chosen but *fixed time*, the region occupied by the fluid is known, as are the inertial traction $\mathbf{t}^{\mathrm{in}}_{\mathcal{S}}$, the inertial body force $\mathbf{b}^{\mathrm{in}}$, and the momentum-rate forces $\mathbf{p}$ and $\mathbf{q}$, and consider the velocity field $\mathbf{v}$ as a virtual field $\boldsymbol{\phi}$ that may be specified *independently of the actual evolution of the fluid*:

$$\int\limits_R (\mathbf{p}\cdot\boldsymbol{\phi} + \mathbf{q}\cdot\mathrm{curl}\,\boldsymbol{\phi})\,\mathrm{d}v = -\int\limits_R \mathbf{b}\cdot\boldsymbol{\phi}\,\mathrm{d}v - \int\limits_{\mathcal{S}} \mathbf{t}^{\mathrm{in}}_{\mathcal{S}}\cdot\boldsymbol{\phi}\,\mathrm{d}a. \tag{10.7}$$

This paradigm represents an intrinsic method of decomposing the rate of kinetic energy into the negative of a power expenditure by a body force field $\mathbf{b}$ and a traction field $\mathbf{t}^{\mathrm{in}}_{\mathcal{S}}$, with each of these fields uniquely determined. As a bonus, this treatment of inertia guarantees a variational framework for the resulting partial differential equation.

Writing $\mathbf{n}$ for the outward unit normal to $\mathcal{S} = \partial R$, if we integrate the term in (10.7) involving $\mathbf{q}\cdot\mathrm{curl}\,\boldsymbol{\phi}$ by parts we find that

$$\int\limits_R (\mathbf{b} + \mathbf{p} + \mathrm{curl}\,\mathbf{q})\cdot\boldsymbol{\phi}\,\mathrm{d}v + \int\limits_{\mathcal{S}} (\mathbf{t}^{\mathrm{in}}_{\mathcal{S}} + \mathbf{q}\times\mathbf{n})\cdot\boldsymbol{\phi}\,\mathrm{d}a = 0; \tag{10.8}$$

since this is to hold for all virtual fields $\boldsymbol{\phi}$, we arrive at explicit expressions for $\mathbf{b}$ and $\mathbf{t}^{\mathrm{in}}_{\mathcal{S}}$:

$$\left.\begin{aligned} \mathbf{b} &= -\mathbf{p} - \mathrm{curl}\,\mathbf{q}, \\ \mathbf{t}^{\mathrm{in}}_{\mathcal{S}} &= -\mathbf{q}\times\mathbf{n}. \end{aligned}\right\} \tag{10.9}$$

Next, using (2.2), (10.3), (10.5), and (10.10), we may express the net body force $\mathbf{b}$ and the net surface traction $\mathbf{t}_{\mathcal{S}}$ in the forms

$$\left.\begin{aligned} \mathbf{b} &= -\rho\dot{\mathbf{v}} - \kappa\,\mathrm{curl}\,\dot{\boldsymbol{\omega}} = -\rho\dot{\mathbf{v}} - \kappa\,\mathrm{curl}\,\dot{\overline{\mathrm{curl}\,\mathbf{v}}}, \\ \mathbf{t}_{\mathcal{S}} &= -\kappa\dot{\boldsymbol{\omega}}\times\mathbf{n} + \mathbf{t}^{\mathrm{ni}}_{\mathcal{S}} = -\kappa\dot{\overline{\mathrm{curl}\,\mathbf{v}}}\times\mathbf{n} + \mathbf{t}^{\mathrm{ni}}_{\mathcal{S}}, \end{aligned}\right\} \tag{10.10}$$

where $\mathbf{t}^{\mathrm{ni}}_{\mathcal{S}}$ is the *noninertial surface traction*. With a view toward writing (10.10) in terms of the acceleration $\dot{\mathbf{v}}$, we note that

$$\begin{aligned} \big(\dot{\overline{\mathrm{curl}\,\mathbf{v}}}\big)_i &= \epsilon_{ijk}\Big(\frac{\partial v_{k,j}}{\partial t} + v_{k,jl}v_l\Big) \\ &= \epsilon_{ijk}\Big(\frac{\partial v_k}{\partial t} + v_{k,l}v_l\Big)_{,j} - \epsilon_{ijk}v_{k,l}v_{l,j} \\ &= \epsilon_{ijk}(\dot{v}_k)_{,j} - \epsilon_{ijk}v_{k,l}v_{l,j} \end{aligned}$$

and, hence, that

$$\big(\operatorname{curl}\overline{\operatorname{curl}\mathbf{v}}^{\boldsymbol{\cdot}}\big)_i = -\Delta\dot{v}_i + v_{i,k}\Delta v_k + v_{i,kj}v_{k,j} + \operatorname{grad} q, \tag{10.11}$$

with

$$q = \tfrac{1}{2}\operatorname{div}\big(\operatorname{div}(\mathbf{v}\otimes\mathbf{v})\big), \tag{10.12}$$

and that

$$\big(\overline{\operatorname{curl}\mathbf{v}}^{\boldsymbol{\cdot}}\times\mathbf{n}\big)_i = (\dot{v}_i)_{,j}n_j - v_{i,l}v_{l,j}n_j. \tag{10.13}$$

On defining

$$\big((\operatorname{grad}^2\mathbf{v})\operatorname{grad}\mathbf{v}\big)_i \overset{\text{def}}{=} v_{i,kj}v_{k,j}$$

we may use direct notation to write the identities (10.11) and (10.13) as:

$$\left.\begin{aligned}\operatorname{curl}\overline{\operatorname{curl}\mathbf{v}}^{\boldsymbol{\cdot}} &= -\Delta\dot{\mathbf{v}} + (\operatorname{grad}\mathbf{v})\Delta\mathbf{v} + (\operatorname{grad}^2\mathbf{v})\operatorname{grad}\mathbf{v},\\ \overline{\operatorname{curl}\mathbf{v}}^{\boldsymbol{\cdot}}\times\mathbf{n} &= (\operatorname{grad}\dot{\mathbf{v}} - (\operatorname{grad}\mathbf{v})\operatorname{grad}\mathbf{v})\mathbf{n}.\end{aligned}\right\} \tag{10.14}$$

Thus, by $(10.10)_1$, we find that

$$\left.\begin{aligned}\mathbf{b} &= -\kappa\operatorname{grad} q - (\rho-\kappa\Delta)\dot{\mathbf{v}} - \kappa\big((\operatorname{grad}\mathbf{v})\Delta\mathbf{v} + (\operatorname{grad}^2\mathbf{v})\operatorname{grad}\mathbf{v}\big),\\ \mathbf{t}_{\mathcal{S}} &= -\kappa(\operatorname{grad}\dot{\mathbf{v}} - (\operatorname{grad}\mathbf{v})\operatorname{grad}\mathbf{v})\mathbf{n} + \mathbf{t}_{\mathcal{S}}^{\text{ni}}.\end{aligned}\right\} \tag{10.15}$$

10.3. *The flow equation and free energy imbalance with vortex kinetic energy*

We continue to work within the constitutive framework set out in §7, so that the force balance (4.16) remains valid. Thus, by $(10.15)_1$, we have the *flow equation*

$$\rho\dot{\mathbf{v}} - \kappa(\Delta\dot{\mathbf{v}} - (\operatorname{grad}\mathbf{v})\Delta\mathbf{v} - (\operatorname{grad}^2\mathbf{v})\operatorname{grad}\mathbf{v}) = -\operatorname{grad} P + \mu\Delta\mathbf{v} - \zeta\Delta\Delta\mathbf{v} + 2\lambda\operatorname{div}\mathring{\mathbf{D}}, \tag{10.16}$$

with

$$P = p + \kappa q \tag{10.17}$$

the *effective pressure*.

Next, arguing as in the steps leading up to (7.18) we find, using (10.4), that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_R \big(\lambda|\mathbf{D}|^2 + \tfrac{1}{2}(\rho|\mathbf{v}|^2 + \kappa|\operatorname{grad}\mathbf{v}|^2)\big)\,\mathrm{d}v &\\ + \int\limits_{\mathcal{S}} \big(\lambda|\mathbf{D}|^2 + \tfrac{1}{2}(\rho|\mathbf{v}|^2 + \kappa|\operatorname{grad}\mathbf{v}|^2)\big)\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}a - \int\limits_{\mathcal{S}}\Big(\mathbf{t}_{\mathcal{S}}^{\text{ni}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\Big)\,\mathrm{d}a &\\ = -\int\limits_R (2\mu|\mathbf{D}|^2 + (\zeta+\xi)|\mathbf{A}|^2 + (\zeta-\xi)|\mathbf{Z}|^2)\,\mathrm{d}v \le 0.&\end{aligned} \tag{10.18}$$

Thus for a *periodic flow* with R a unit cell, we find as a consequence of (7.20) and (7.21) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_R \tfrac{1}{2}(\rho|\mathbf{v}|^2 + (\lambda+\kappa)|\operatorname{grad}\boldsymbol{\omega}|^2)\,\mathrm{d}v = -\int\limits_R(\mu|\boldsymbol{\omega}|^2 + \zeta|\operatorname{grad}\boldsymbol{\omega}|^2)\,\mathrm{d}v \le 0. \tag{10.19}$$

Therefore, granted a periodic flow, the energy imbalance (10.19) for a cubic control volume in the theory with vortex kinetic energy is exactly the same as its counterpart (7.22) for the theory with conventional kinetic energy provided that we replace λ in the conventional theory by $\lambda + \kappa$.

10.4. *Plane Poiseuille flow revisited*

Granted that the fluid velocity is of the form (9.1) assumed for the problem of plane Poiseuille flow, kinematics alone yields the conclusion that

$$\overline{\operatorname{curl}\mathbf{v}}^{\boldsymbol{\cdot}} = \mathbf{0}. \tag{10.20}$$

Thus, when gradient kinetic energy is accounted for and inertial body forces are neglected, it follows from (10.14) that the flow equation (10.16) reduces to the form (7.13) of the theory without gradient kinetic energy. In addition, the non-slip boundary condition (8.17) is unaltered. The problem of plane Poiseuille flow for the theory with gradient kinetic energy therefore reduces to the problem considered in Section 9 and the resulting solutions are therefore unchanged.

Acknowledgments

This work was supported by the U. S. Department of Energy. We thank Bob Moser for providing us with the data from the DNS simulations of Kim, Moin & Moser (1987) and Moser, Kim & Mansour (1999), Timothy Wei for providing us with the data from the experiments of Wei & Willmarth (1989), and Xuemei Chen for her careful reading and of various drafts of this paper.

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1067	Chen, Y.-C., and E. Fried	Uniaxial nematic elastomers: Constitutive framework and a simple application—*Proceedings of the Royal Society of London A*, in press (2005)	June 2005
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1069	Gioia, G., and X. Dai	Surface stress and reversing size effect in the initial yielding of ultrathin films—*Journal of Applied Mechanics*, in press (2005)	Aug. 2005
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1073	Fried, E., and S. Sellers	Soft elasticity is not necessary for striping in nematic elastomers—*Nature Physics* (submitted)	Sept. 2005
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1078	Riahi, D. N.	Nonlinear compositional convection in mushy layers—*Journal of Fluid Mechanics* (submitted)	Dec. 2005
1079	Bhattacharjee, P., and D. N. Riahi	Mathematical modeling of flow control using magnetic fluid and field—In *The Euler Volume* (Abington, UK: Taylor and Francis), in press (2005)	Dec. 2005
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1082	Riahi, D. N.	Inertial and Coriolis effects on oscillatory flow in a horizontal dendrite layer—*Transport in Porous Media* (submitted)	Jan. 2006
1083	Wu, Y., and K. T. Christensen	Population trends of spanwise vortices in wall turbulence—*Journal of Fluid Mechanics* (submitted)	Jan. 2006
1084	Natrajan, V. K., and K. T. Christensen	The role of coherent structures in subgrid-scale energy transfer within the log layer of wall turbulence—*Physics of Fluids* (submitted)	Jan. 2006
1085	Wu, Y., and K. T. Christensen	Reynolds-stress enhancement associated with a short fetch of roughness in wall turbulence—*AIAA Journal* (submitted)	Jan. 2006
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