Innovative Algorithm to Solve Axisymmetric Displacement and Stress Fields in Multilayered Pavement Systems

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Abstract

This paper presents an innovative algorithm to calculate the displacement and stress fields within a multilayered pavement system using Layered Elastic Theory and Hankel and Laplace integral transforms. In particular, a recurrence relationship, which links the Hankel transform of displacements and stresses at any point $P(r, z)$ within a multilayered pavement system with those at the surface point $Q(r, 0)$, is systematically derived. The Hankel transforms of displacements and stresses at any point within a multilayered pavement system can be explicitly determined using the derived recurrence relationships, and the subsequent inverse Hankel transforms give the displacements and stresses at the point of interest. Theoretical and computational verification of the proposed algorithm justify its correctness. The proposed algorithm does not use a numerical linear system solver employed in the traditional approach to solve the axisymmetric problems in multilayered pavement systems. Due to the explicitly-derived recurrence

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relationships for displacements and stresses, the proposed algorithm provides a more rapid solution time than the stress-function-based approach utilized in existing layered elastic theory programs.

Introduction

Layered Elastic Theory (LET) has been widely used to develop numerous programs to analyze multilayered pavement systems throughout the world, such as BISAR [2], JULEA [18], DIPLOMAT [6], Kenlayer [9] , LEAF [8], etc. The displacement and stress fields, generated from vertical, circular surface loads, are the main quantities to be determined in the analysis of a multilayered pavement system. There are two major classes of methods for calculating the displacements and stresses within a multilayered pavement using LET. The traditional one is based on the classical solution of an axisymmetric problem via stress function approach [12], and they are expressed in the forms of inverse Hankel transforms of certain functions [12]. For example, Burmister solved displacements and stresses in two- and three-layered soil systems by using an ingenious stress function [3, 4, 5]. Matsui, Maina and Inoue calculated the displacements and stresses in multilayered pavement systems subject to interface slips using Michell function [13]. By using Michell and Boussinesq functions, Maina and Matsui solved elastic responses of a pavement structure due to vertical and horizontal surface loading [14].

There are $4N - 2$ constants of integration in a N-layered pavement system, which are usually approximated using a numerical linear system solver for each given Hankel parameter in the stress-function-based method. Although some solution strategies can be manipulated so that only two equations need solving, but this requires successive matrix multiplications to be performed in order to obtain those two equations [9]. Furthermore, linear systems of four equations need to be solved successively using inter-layer contact condition in order to determine all the constants of integration for each Hankel parameter [9].

This traditional approach is a straight forward, but time-consuming since a large number of Hankel parameters are required to evaluate displacements and stresses at a point within a multilayered pavement system. This problem becomes more evident in developing sophisticated flexible pavement design tools based on LET, such as the Interim AASHTO Mechanistic-Empirical Design Guide, since tremendously large numbers of displacement and stress calculations are required in order to fully simulate the responses of a flexible pavement during its entire design life [1, 7, 11]. Recently, Khazanovich and Wang proposed several approaches aimed to specifically speed up the numerical evaluation of the inverse Hankel transforms in the layered elastic solutions [11].

Another class of solution methods is based on integral transform techniques, such as Laplace and Hankel transforms. These methods directly deal with the governing partial differential equations (PDEs) via appropriate integral transforms with respect to various independent variables, transforming complicated PDEs into easily handled equations without resorting to certain stress functions. Furthermore, corresponding inverse integral transforms give rise to the desired displacements and stresses in the form of integral equations, which can be resolved analytically for certain problems or numerically for complex ones [19]. The integral transformation approach has been previously used to solve multilayered pavement system problems in China [20, 24]. Additionally, Wong and Zhong applied an integral transform method to calculate the thermal stresses due to temperature variation in multilayered pavement systems [22]. One prominent advantage of these integral transform approaches is the constants of integration involved in the integral equations can be explicitly solved using boundary and inter-layer contact conditions, making the use of a numerical linear system solver unnecessary and thus the solution process should be more efficient.

This paper presents an innovative algorithm to calculate displacement and stress fields within a multilayered pavement system based on Hankel and Laplace integral transforms similar to the work proposed by Zhong et al. [24], but with a explicitly-defined recurrence relationship linking the Hankel transform of displacements and stresses at any point $P(r, z)$ with those at surface point $Q(r, 0)$.

This paper is organized in the following manner. Firstly, the underlying problems and assumptions used in this paper are introduced. Secondly, displacements and stresses in a homogeneous half-space are derived for an axisymmetric problem using integral transform techniques. Thirdly, the extension of the solutions for homogeneous half-space to those for the multi-layer case is systematically presented. Fourthly, theoretical justification of proposed algorithm for homogeneous half-space under concentrated and uniformly circular loading are given respectively, and computational justification for a three-layer system under uniformly circular loading is performed. Finally, discussion on the proposed algorithm and summary of this paper are given.

General Multilayered Pavement System and Assumptions

The displacement and stress fields in a multilayered pavement system can be reasonably assumed to be axisymmetric provided the external loading is axisymmetric [3]. A typical $N-$Layer pavement system is shown in Figure 1, where integer $n \geq 2$, h_i, ν_i and G_i are thickness, Poisson's ratio and the modulus of rigidity of the ith layer (thickness of the last layer is assumed to be infinite, i.e. $h_n = \infty$), respectively; p and δ are uniform pressure and radius of circular loading, respectively. The cylindrical coordinate system is used in this paper.

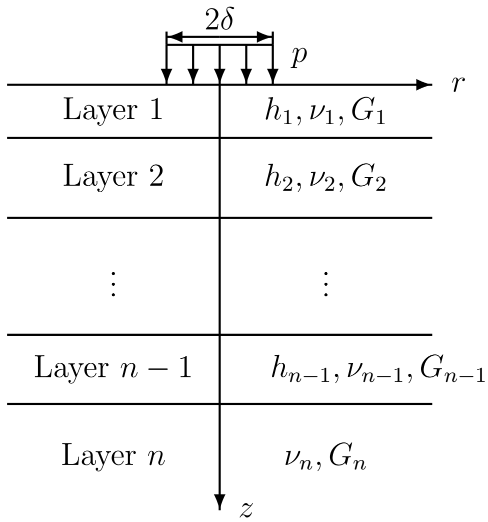

Figure 1: Multilayered Pavement System

The mathematical model to solve the displacements and stresses in a multilayered pavement system is based on LET, and the basic assumptions used in LET are as follows:

- Each layer is assumed to be linear elastic, homogeneous and isotropic.

- Each layer has infinite extent in r- direction. The thickness of each layer is finite except for the last layer.
- Fully bonded inter-layer contact conditions are assumed.
- Body forces are ignored.
- Only vertical traffic loading is considered and assumed to be axisymmetric about the z-axis.
- Strains and displacements are assumed to be small.
- All stress and displacement components vanish as $r \to \infty$ and $z \to \infty$.

In view of the above assumptions, the displacement and stress fields are axisymmetric about z-axis. To facilitate the presentation of proposed algorithm for solving the axisymmetric problems in multilayered pavement systems, displacement and stress fields in the homogeneous half-space are first determined.

Displacements and Stresses in a Homogeneous Half-space

Governing Equations

For a homogeneous half-space, the material parameters are defined as $\nu_i = \nu, G_i = G$ for $i = 1, \cdots, n$ in Figure 1. We denote the displacement along the radial r- direction as u, and vertical z- direction as w. The normal stress components are denoted as $\sigma_r, \sigma_\theta, \sigma_z$, shear stress components as $\tau_{zr}, \tau_{r\theta}, \tau_{\theta z}$, normal strain components as $\varepsilon_r, \varepsilon_\theta, \varepsilon_z$, and shear strain components as $\gamma_{zr}, \gamma_{r\theta}, \gamma_{\theta z}$, respectively. Due to axisymmetric assumption, the displacement along the circumferential θ- direction, shear stress components $\tau_{r\theta}, \tau_{\theta z}$ and shear strain components $\gamma_{r\theta}, \gamma_{\theta z}$ all vanish.

By virtue of the classical theory of linear elasticity [12, pp.274], the governing equations for the axisymmetric problem are as follows:

- Equations of equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1}$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zr}}{\partial r} + \frac{\tau_{zr}}{r} = 0 \tag{2}$$

- Stress-strain relationships

$$\begin{aligned} \varepsilon_r &= \frac{1}{E}\left[\sigma_r - \nu\left(\sigma_\theta + \sigma_z\right)\right] \\ \varepsilon_\theta &= \frac{1}{E}\left[\sigma_\theta - \nu\left(\sigma_r + \sigma_z\right)\right] \\ \varepsilon_z &= \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_r + \sigma_\theta\right)\right] \\ \gamma_{zr} &= \frac{\tau_{zr}}{G} \end{aligned} \tag{3}$$

- Strain-displacement relationships

$$\begin{aligned} \varepsilon_r &= \frac{\partial u}{\partial r} \\ \varepsilon_\theta &= \frac{u}{r} \\ \varepsilon_z &= \frac{\partial w}{\partial z} \\ \gamma_{zr} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{aligned} \tag{4}$$

The equations of equilibrium can be further written in terms of u and w

$$G\left(\nabla^2 u - \frac{u}{r^2}\right) + (\lambda + G)\frac{\partial e}{\partial r} = 0 \quad (5)$$

$$G\nabla^2 w + (\lambda + G)\frac{\partial e}{\partial z} = 0 \quad (6)$$

where the modulus of rigidity $G = \frac{E}{2(1+\nu)}$, $E =$ the modulus of elasticity, $\nu =$ Poisson's ratio, Lame constant $\lambda = \frac{2\nu G}{1-2\nu}$, the first strain invariant $e = \varepsilon_r + \varepsilon_\theta + \varepsilon_z$, and Laplacian for the axisymmetric problem in cylindrical coordinate system $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$.

Furthermore, the stress components can be expressed in terms of u and w by substituting Eq. (4) into Eq. (3)

$$\sigma_r = 2G\frac{\partial u}{\partial r} + \lambda e \quad (7)$$

$$\sigma_\theta = 2G\frac{u}{r} + \lambda e \quad (8)$$

$$\sigma_z = 2G\frac{\partial w}{\partial z} + \lambda e \quad (9)$$

$$\tau_{zr} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \quad (10)$$

Integral Transform Techniques

Hankel and Laplace integral transforms are employed to derive displacement and stress fields in a homogeneous half-space. To derive the solutions, some useful formulas involving these integral transforms are presented next:

Let $\widetilde{\phi}^m(\xi)$ be the Hankel transform of order m of a function $\phi(r)$, where m is zero or half of positive integer and ξ is the Hankel parameter. In view of reference [15, pp. 299],

$$\widetilde{\phi}^m(\xi) = \int_0^\infty r\phi(r)J_m(\xi r)dr \tag{11}$$

and $\phi(r)$ can be represented as the inverse Hankel transform of $\widetilde{\phi}^m(\xi)$

$$\phi(r) = \int_0^\infty \xi\widetilde{\phi}^m(\xi)J_m(\xi r)d\xi \tag{12}$$

Also, from reference [15, pp. 310-311]

$$\int_0^\infty r\frac{d\phi}{dr}J_1(\xi r)dr = -\xi\widetilde{\phi}^0(\xi) \tag{13}$$

$$\int_0^\infty r\left[\left(\frac{d}{dr}+\frac{1}{r}\right)\phi(r)\right]J_0(\xi r)dr = \xi\widetilde{\phi}^1(\xi) \tag{14}$$

$$\int_0^\infty r\left[\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\phi(r)\right]J_0(\xi r)dr = -\xi^2\widetilde{\phi}^0(\xi) \tag{15}$$

$$\int_0^\infty r\left[\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)\phi(r)\right]J_1(\xi r)dr = -\xi^2\widetilde{\phi}^1(\xi) \tag{16}$$

Denote $\overline{f}(s)$ as the Laplace transform of function $f(z)$, where s is the Laplace parameter. Referring to reference [15, pp. 136],

$$\overline{f}(s) = \int_0^\infty f(z)e^{-sz}dz \qquad (z > 0)$$

and its inverse Laplace transform gives

$$f(z) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\overline{f}(s)e^{sz}ds \qquad (\beta > 0, z > 0)$$

where i = pure imaginary number with $i^2 = -1$.

The following outlines the main steps involved in using integral transformation techniques to solve an elastic half-space problem

- Applying Hankel transform of order one to the both sides of Eq. (5) with respect to r and in conjunction with Eqs. (13) and (16), and order zero to Eq. (6) in conjunction with Eqs. (14) and (15) yields, respectively

$$
\begin{aligned}
(1-2\nu)\frac{\partial^2 \widetilde{u}^1}{\partial z^2} - \xi\frac{\partial \widetilde{w}^0}{\partial z} - 2(1-\nu)\xi^2\widetilde{u}^1 &= 0 \qquad (17)\\
2(1-\nu)\frac{\partial^2 \widetilde{w}^0}{\partial z^2} + \xi\frac{\partial \widetilde{u}^1}{\partial z} - (1-2\nu)\xi^2\widetilde{w}^0 &= 0 \qquad (18)
\end{aligned}
$$

- Applying Laplace transform to both sides of Eqs. (17) and (18) with respect to z yields, respectively

$$
\begin{aligned}
\left[(1-2\nu)s^2 - 2(1-\nu)\xi^2\right]\overline{\widetilde{u}^1} - \xi s\overline{\widetilde{w}^0} &= (1-2\nu)s\widetilde{u}^1(\xi,0) - \xi\widetilde{w}^0(\xi,0) + \\
&\quad (1-2\nu)\frac{\partial \widetilde{u}^1}{\partial z}(\xi,0) \qquad (19)\\
\xi s\overline{\widetilde{u}^1} + \left[2(1-\nu)s^2 - (1-2\nu)\xi^2\right]\overline{\widetilde{w}^0} &= \xi\widetilde{u}^1(\xi,0) + 2(1-\nu)s\widetilde{w}^0(\xi,0) + \\
&\quad 2(1-\nu)\frac{\partial \widetilde{w}^0}{\partial z}(\xi,0) \qquad (20)
\end{aligned}
$$

- Applying Hankel transform of order zero to the both sides of Eq. (9) with respect to r , and order one to Eq. (10) gives, respectively

$$
\begin{aligned}
\frac{\partial \widetilde{w}^0}{\partial z}(\xi,z) &= \frac{1}{1-\nu}\left[\frac{1-2\nu}{2G}\widetilde{\sigma_z}^0(\xi,z) - \nu\xi\widetilde{u}^1(\xi,z)\right] \qquad (21)\\
\frac{\partial \widetilde{u}^1}{\partial z}(\xi,z) &= \frac{1}{G}\widetilde{\tau_{zr}}^1(\xi,z) + \xi\widetilde{w}^0(\xi,z) \qquad (22)
\end{aligned}
$$

- Setting $z=0$ in Eqs. (21) and (22) yields $\frac{\partial \widetilde{w}^0}{\partial z}(\xi,0)$ and $\frac{\partial \widetilde{u}^1}{\partial z}(\xi,0)$, respectively. Substituting $\frac{\partial \widetilde{u}^1}{\partial z}(\xi,0)$ into Eq. (19), $\frac{\partial \widetilde{w}^0}{\partial z}(\xi,0)$ into Eq. (20), respectively, leads to a linear system of two equations involving two unknowns $\overline{\widetilde{u}^1}(\xi,s)$ and $\overline{\widetilde{w}^0}(\xi,s)$, which can be easily determined in terms of $\widetilde{u}^1(\xi,0), \widetilde{w}^0(\xi,0), \widetilde{\tau_{zr}}^1(\xi,0)$ and $\widetilde{\sigma_z}^0(\xi,0)$ as follows

$$\left\{\begin{array}{c} \overline{\widetilde{u}^1}(\xi,s) \\ \\ \\ \overline{\widetilde{w}^0}(\xi,s) \end{array}\right\} = \left[\begin{array}{cccc} P_{11} & P_{12} & P_{13} & P_{14} \\ \\ \\ P_{21} & P_{22} & P_{23} & P_{24} \end{array}\right] \left\{\begin{array}{c} \widetilde{u}^1(\xi,0) \\ \widetilde{w}^0(\xi,0) \\ \widetilde{\tau_{zr}}^1(\xi,0) \\ \widetilde{\sigma_z}^0(\xi,0) \end{array}\right\} \tag{23}$$

where $P_{ij}, i = 1,2$ and $j = 1,2,3,4$ are given in Appendix A.

- Performing inverse Laplace transforms of $\overline{\widetilde{u}^1}(\xi,s)$ and $\overline{\widetilde{w}^0}(\xi,s)$ in Eq. (23) with respect to s yield $\widetilde{u}^1(\xi,z)$ and $\widetilde{w}^0(\xi,z)$. Furthermore, $\widetilde{\sigma_z}^0(\xi,z)$ and $\widetilde{\tau_{zr}}^1(\xi,z)$ are ready to be obtained using Eqs. (21) and (22). We list $\widetilde{u}^1(\xi,z), \widetilde{w}^0(\xi,z), \widetilde{\tau_{zr}}^1(\xi,z)$ and $\widetilde{\sigma_z}^0(\xi,z)$ using the following vector-matrix form

$$\left\{\begin{array}{c} \widetilde{u}^1(\xi,z) \\ \widetilde{w}^0(\xi,z) \\ \widetilde{\tau_{zr}}^1(\xi,z) \\ \widetilde{\sigma_z}^0(\xi,z) \end{array}\right\} = e^{\xi z} \left[\begin{array}{cccc} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \\ G_{41} & G_{42} & G_{43} & G_{44} \end{array}\right] \left\{\begin{array}{c} \widetilde{u}^1(\xi,0) \\ \widetilde{w}^0(\xi,0) \\ \widetilde{\tau_{zr}}^1(\xi,0) \\ \widetilde{\sigma_z}^0(\xi,0) \end{array}\right\} \tag{24}$$

where $G_{ij}, i,j = 1,2,3,4$ are given in Appendix A.

- Since only vertical traffic loading applied on top of the half-space is considered, i.e. $\sigma_z(r,0)$ is known and $\tau_{zr}(r,0) = 0$, it follows that $\widetilde{\tau_{zr}}^1(\xi,0) = 0$ and $\widetilde{\sigma_z}^0(\xi,0)$ is also known. Furthermore, the last assumption used in LET above, i.e. all stress and displacement components vanish as $r \to \infty$ and $z \to \infty$ implies

$$\lim_{z\to\infty} \widetilde{u}^1(\xi,z) = 0 \qquad \text{and} \qquad \lim_{z\to\infty} \widetilde{w}^0(\xi,z) = 0. \tag{25}$$

Hence, $\widetilde{u}^1(\xi,0)$ and $\widetilde{w}^0(\xi,0)$ can be solved using the first two equations in Eq. (24) in conjunction with Eq. (25).

- Next, $\widetilde{u}^1(\xi,z)$, $\widetilde{w}^0(\xi,z)$, $\widetilde{\tau_{zr}}^1(\xi,z)$ and $\widetilde{\sigma_z}^0(\xi,z)$ are ready to be solved using Eq. (24), and $u(r,z), w(r,z), \tau_{zr}(r,z)$ and $\sigma_z(r,z)$ can be formulated using the appropriate inverse Hankel transforms of Eq. (24) as follows

$$\begin{aligned}
u(r,z) &= \int_0^\infty \xi \widetilde{u}^1(\xi,z) J_1(\xi r) d\xi \\
w(r,z) &= \int_0^\infty \xi \widetilde{w}^0(\xi,z) J_0(\xi r) d\xi \\
\tau_{zr}(r,z) &= \int_0^\infty \xi \widetilde{\tau_{zr}}^1(\xi,z) J_1(\xi r) d\xi \\
\sigma_z(r,z) &= \int_0^\infty \xi \widetilde{\sigma_z}^0(\xi,z) J_0(\xi r) d\xi
\end{aligned} \tag{26}$$

- Combining the first two equations in Eq. (26) and the first three equations in Eq. (4) with Eqs. (7) and (8) gives $\sigma_r(r,z)$ and $\sigma_\theta(r,z)$, respectively.

Displacements and Stresses in a Multilayered Pavement System

Referring to Figure 1, according to LET, the governing equations for the i-th layer in a multilayered pavement system are the same as those for a half-space case except that in the constitutive law, i.e. equation (3), E, ν and G should be replaced by E_i, ν_i and G_i , respectively, in order to distinguish between material properties of different layers. Let h_i be the thickness of ith layer, where $i = 1, \cdots, n-1$. Define

$$\begin{aligned}
H_0 &= 0 \\
H_i &= \sum_{k=1}^{i} h_k
\end{aligned}$$

Let Q(r, z) be the point where the displacements and stresses are evaluated, and assume that Q is located in the ith layer, where z is measured from the upper boundary of the i-th layer. Let

$$\widetilde{\Phi}_i(\xi, z) = \left[\widetilde{u}_i^1(\xi, z), \widetilde{w}_i^0(\xi, z), \widetilde{\tau_{zr}}_i^1(\xi, z), \widetilde{\sigma_z}_i^0(\xi, z)\right]^T$$

where the subscript i indicates that the point Q is located in the i-th layer and the superscript T stands for transposition of matrix. Let $[G(\xi, z)]_i$ denote the 4 x 4 matrix in equation (24), where subscript i indicates E_i, ν_i & G_i replacing $E, \nu,$ & G respectively in the Appendix A. The following illustrates how to generalize the results for a half-space problem to a multilayered system

- In view of Eq. (24), the relationship between Hankel transform of displacements and stresses at point with coordinate (r, z) and those at point with coordinate $(r, 0)$ can be expressed as

$$\widetilde{\Phi}_i(\xi, z) = e^{\xi z} \left[G(\xi, z)\right]_i \widetilde{\Phi}_i(\xi, 0) \tag{27}$$

where $i = 1, 2, \cdots, n$.

- Fully bonded conditions at the layer interfaces imply that

$$\widetilde{\Phi}_i(\xi, 0) = \widetilde{\Phi}_{i-1}(\xi, h_{i-1}) \tag{28}$$

where $i = 2, 3, \cdots, n$.

- Substituting Eq. (28) into Eq. (27) yields

$$\widetilde{\Phi}_i(\xi, z) = e^{\xi z} \left[G(\xi, z)\right]_i \widetilde{\Phi}_{i-1}(\xi, h_{i-1}) \tag{29}$$

where $i = 2, 3, \cdots, n$.

- Repeatedly applying the recurrence relation in Eq. (29) for $i = n-1, n-2, \cdots, 1$ and $z = h_{n-1}, h_{n-2}, \cdots, h_1$ yields

$$\widetilde{\Phi}_i(\xi, h_i) = e^{\xi H_i}\,[F(\xi, h_i)]\,\widetilde{\Phi}_1(\xi, 0) \tag{30}$$

where $i = 1, 2, \cdots, n-1$, and $[F(\xi, h_i)]$ is a 4 x 4 matrix whose components $[F(\xi, h_i)]_{kl}$, $k, l = 1, 2, 3, 4$ are

$$[F(\xi, h_i)]_{kl} = \begin{cases} ([G(\xi, h_1)]_1)_{kl} & \text{if } i = 1 \\ \sum_{j=1}^{4} ([G(\xi, h_i)]_i)_{kj} \cdot [F(\xi, h_{i-1})]_{jl} & \text{otherwise} \end{cases} \tag{31}$$

where $k, l = 1, 2, 3, 4$.

- Setting $i = i-1$ in Eq. (30), then substituting $\widetilde{\Phi}_{i-1}(\xi, h_{i-1})$ into Eq. (29) yields

$$\widetilde{\Phi}_i(\xi, z) = e^{\xi(H_{i-1}+z)}\,[G(\xi, z)]_i \cdot [F(\xi, h_{i-1})]\widetilde{\Phi}_1(\xi, 0) \tag{32}$$

- Since $\sigma_{z1}(r, 0)$ is given and $\tau_{zr1}(r, 0) = 0$, Hankel integral transforms of $\sigma_{z1}(r, 0)$, $\tau_{zr1}(r, 0)$ yields $\widetilde{\sigma}^0_{z1}(\xi, 0)$ and $\widetilde{\tau}^1_{zr1}(\xi, 0)$ respectively, where $\widetilde{\tau}^1_{zr1}(\xi, 0) = 0$. Analogous to the homogeneous half-space case, $\widetilde{u}^1_1(\xi, 0)$ and $\widetilde{w}^0_1(\xi, 0)$ can be determined by using the first two equations in Eq. (32) with $i = n$ in conjunction with

$$\lim_{z\to\infty} \widetilde{u}^1_n(\xi, z) = 0 \qquad \text{and} \qquad \lim_{z\to\infty} \widetilde{w}^0_n(\xi, z) = 0 \tag{33}$$

The final solutions for $\widetilde{u}^1_1(\xi, 0)$ and $\widetilde{w}^0_1(\xi, 0)$ are

$$\begin{aligned} \widetilde{u}^1_1(\xi, 0) &= \frac{T_{12}}{\Delta}\widetilde{\sigma}_{z1}^{\,0}(\xi, 0) \\ \widetilde{w}^0_1(\xi, 0) &= \frac{T_{22}}{\Delta}\widetilde{\sigma}_{z1}^{\,0}(\xi, 0) \end{aligned} \tag{34}$$

where Δ, T_{12} and T_{22} are given in Appendix B.

- Since $\widetilde{\Phi}_1(\xi,0) = \left[\widetilde{u}_1^1(\xi,0), \widetilde{w}_1^0(\xi,0), \widetilde{\tau_{zr}}_1^1(\xi,0), \widetilde{\sigma_z}_1^0(\xi,0)\right]^T$ is known, we can use Eq. (32) to calculate $\widetilde{\Phi}_i(\xi,z)$ for $i = 1, 2, \cdots, n$. Note, $[F(\xi, h_{i-1})]$ will becomes a 4 x 4 identity matrix when Q is located in the first layer, i.e. $i = 1$. Furthermore when Q is located in the last layer, i.e. $i = n$, the last assumption used in LET, i.e., all stress and displacement components are vanished as $r \to \infty$ and $z \to \infty$, indicates that the coefficients of $e^{\xi z}$ terms in the propagating matrix $[G(\xi,z)]_n$ must vanish.

- The inverse Hankel transforms of components of $\widetilde{\Phi}_i(\xi,z)$ give rise to $u_i(r,z), w_i(r,z), \tau_{zri}(r,z)$ and $\sigma_{zi}(r,z)$, see Appendix C for the complete expressions.

- Finally, $\sigma_r(r,z)$ and $\sigma_\theta(r,z)$ can be determined using Eq. (7) and Eq. (8), respectively, see Appendix D for the complete expressions.

Theoretical and Computational Justifications of Proposed Algorithm

Theoretical Justification for Homogeneous Half-Space Under Concentrated Vertical Loading

When a concentrated vertical force P is applied on the boundary plane boundary, Hankel transform of order zero of $\sigma_z(r,0)$ and order one of $\tau_{zr}(r,0)$ with respect to r yields, respectively

$$\widetilde{\sigma_z}^0(\xi,0) = -\frac{P}{2\pi} \qquad \text{and} \qquad \widetilde{\tau_{zr}}^1(\xi,0) = 0 \tag{35}$$

Referring to the results for a half-space problem, we can now deduce that

$$\widetilde{u}^1(\xi,0) = -\frac{(1-2\nu)P}{4\pi\xi G} \qquad \text{and} \qquad \widetilde{w}^0(\xi,0) = \frac{(1-\nu)P}{2\pi\xi G} \tag{36}$$

Substituting Eq.s (35) and (36) into Eq. (24) gives rise to $\widetilde{u}^1(\xi,z)$, $\widetilde{w}^0(\xi,z)$, $\widetilde{\tau_{zr}}^1(\xi,z)$ and $\widetilde{\sigma_z}^0(\xi,z)$ whose inverse Hankel transforms generate the following classical Boussinesq solutions [17, pp.401-402]

$$\begin{aligned} u(r,z) &= \frac{(1+\nu)Pr}{2\pi ER}\left(\frac{z}{R^2} - \frac{1-2\nu}{R+z}\right) \\ w(r,z) &= \frac{(1+\nu)P}{2\pi ER}\left[2(1-\nu) + \frac{z^2}{R^2}\right] \\ \sigma_z(r,z) &= -\frac{3Pz^3}{2\pi R^5} \\ \tau_{zr}(r,z) &= -\frac{3Pz^2r}{2\pi R^5} \end{aligned}$$

where $R = \sqrt{r^2+z^2}$.

Theoretical Justification for Homogeneous Half-Space Under Uniform Vertical Circular Loading

Consider circular loading with uniform pressure P and radius of δ (see Figure 1), i.e.

$$\sigma_z(r,0) = \begin{cases} -p & \text{if } r < \delta \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \tau_{zr}(r,0) = 0$$

then

$$\widetilde{\sigma_z}^0(\xi,0) = -\frac{P\delta J_1(\xi\delta)}{\xi} \qquad \text{and} \qquad \widetilde{\tau_{zr}}^1(\xi,0) = 0$$

Following the derivation in solving a half-space problem above, we have

$$\widetilde{u}^1(\xi,0) = -\frac{(1-2\nu)P\delta J_1(\xi\delta)}{2G\xi^2} \qquad \text{and} \qquad \widetilde{w}^0(\xi,0) = \frac{(1-\nu)P\delta J_1(\xi\delta)}{G\xi^2}$$

and

$$
\begin{aligned}
\sigma_z(r,z) &= -p\int_0^{\infty}(1+\frac{z}{\delta}x)e^{-\frac{z}{\delta}x}J_1(x)J_0(\frac{r}{\delta}x)dx \\
\sigma_r(r,z) &= p\int_0^{\infty}\left[(1-2\nu-\frac{z}{\delta}x)\frac{\delta}{rx}J_1(\frac{r}{\delta}x)-(1-\frac{z}{\delta}x)J_0(\frac{r}{\delta}x)\right]e^{-\frac{z}{\delta}x}J_1(x)dx \\
w(r,z) &= \frac{p}{2G}\int_0^{\infty}\left[z+\frac{2(1-\nu)\delta}{x}\right]e^{-\frac{z}{\delta}x}J_1(x)J_0(\frac{r}{\delta}x)dx
\end{aligned}
$$

In particular, setting $r = 0$ in these formulas and using the following results of infinite integrals [21, pp. 386]

$$
\begin{aligned}
\int_0^{\infty} e^{-\frac{z}{\delta}x}J_1(x)dx &= 1-\frac{z}{\delta}\left(1+\frac{z^2}{\delta^2}\right)^{-\frac{1}{2}} \\
\int_0^{\infty} xe^{-\frac{z}{\delta}x}J_1(x)dx &= \left(1+\frac{z^2}{\delta^2}\right)^{-\frac{3}{2}} \\
\int_0^{\infty} \frac{1}{x}e^{-\frac{z}{\delta}x}J_1(x)dx &= \left(1+\frac{z^2}{\delta^2}\right)^{\frac{1}{2}}-\frac{z}{\delta}
\end{aligned}
$$

We recover the following special formulas [9, pp. 50]

$$
\begin{aligned}
\sigma_z(0,z) &= -p\left[1-\frac{z^3}{\sqrt{(z^2+\delta^2)^3}}\right] \\
\sigma_r(0,z) &= -\frac{p}{2}\left[1+2\nu-\frac{2(1+\nu)z}{\sqrt{z^2+\delta^2}}+\frac{z^3}{\sqrt{(z^2+\delta^2)^3}}\right] \\
w(0,z) &= \frac{p}{2G}\left[\frac{\delta^2}{\sqrt{z^2+\delta^2}}+(1-2\nu)\left(\sqrt{z^2+\delta^2}-z\right)\right]
\end{aligned}
$$

Computational Justification for Three-Layer Elastic Systems Under Uniform Vertical Circular Loading

The three-layer elastic system is shown in Figure 2.

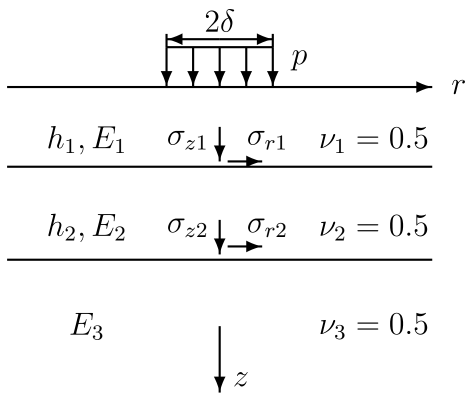

Figure 2: Three-layer Pavement System

Due to the complexities of integrands involved in the infinite integrals for solutions of stresses and displacements in three-layer elastic systems, the closed-form solutions are rarely available. Instead, numerical integration technologies, such as Gaussian quadrature formulas [16] can be applied to evaluate the infinite integrals.

In this case, the stress values calculated using the proposed algorithms are compared with the widely used Jones' Tables of stresses in three-layer elastic system [10]. As shown in Figure 2, the vertical and radial stresses in the bottom of the first and second layer on the axis of symmetry ($r = 0$), denoted by σ_{z1}, σ_{r1}, σ_{z2} and σ_{r2}, respectively, are calculated.

The following parameters are used in Jones' Table:

$$k_1 = \frac{E_1}{E_2} \qquad k_2 = \frac{E_2}{E_3} \qquad a_1 = \frac{\delta}{h_2} \quad \text{and} \qquad H = \frac{h_1}{h_2}$$

Table 1 lists the calculated stress values corresponding to different parameters using the proposed algorithms and those values from Jones' Table[1], suggesting that at least three significant digits are agreeable between our results and those values from Jones' Table.

[1]The stress sign convention used in Jones' Table is applied, i.e. compression stress is positive.

Table 1: Calculated stress values using the proposed algorithm and those in Jones' Table printed in brackets (stress values being expressed as a fraction of the applied vertical loading P).

a_1	H	k_1	k_2	σ_{z1}	σ_{r1}	σ_{z2}	σ_{r2}
0.1	0.125	2	2	4.294983e-1 (4.2950e-1)	-2.765433e-1 (-2.7672e-1)	8.960207e-3 (8.96e-3)	-8.199295e-3 (-8.2e-3)
0.2	0.25	2	2	4.246671e-1 (4.2462e-1)	-2.466141e-1 (-2.4653e-1)	7.061373e-3 (7.06e-3)	-1.000422e-1 (-1.0004e-1)
0.4	0.5	20	20	1.144855e-1 (1.1448e-1)	-2.080734e0 (-2.08072e0)	9.882485e-3 (9.88e-3)	-1.312787e-1 (-1.3128e-1)
0.8	1	200	20	1.235930e-2 (1.236e-2)	-4.249169e0 (-4.24864e0)	4.361452e-3 (4.36e-3)	-3.389900e-2 (-3.389e-2)
1.6	2	2	2	3.663678e-1 (3.6644e-1)	-3.843012e-1 (-3.8443e-1)	2.014516e-1 (2.0145e-1)	-1.536974e-1 (-1.5370e-1)
3.2	4	20	20	3.257526e-2 (3.258e-2)	-3.077159e0 (-3.07722e0)	2.061185e-2 (2.061e-2)	-1.884397e-1 (-1.8845e-1)

Discussion and Summary

Discussion

As shown in solving axisymmetric problems in a multilayered pavement system, the main unknowns $\widetilde{u}_1^1(\xi,0), \widetilde{w}_1^0(\xi,0)$ are solved explicitly using the recurrence relationship defined in Eq. (32). Once $\widetilde{u}_1^1(\xi,0), \widetilde{w}_1^0(\xi,0)$ are known, $\widetilde{u}_i^1(\xi,z)$, $\widetilde{w}_i^0(\xi,z)$, $\widetilde{\tau_{zr}}_i^1(\xi,z)$ and $\widetilde{\sigma_z}_i^0(\xi,z)$ are ready to be obtained. Appropriate inverse Hankel transforms of those quantities give the desired solutions for u, w, τ_{zr} and σ_z at any point within a multilayered pavement system, and solutions for σ_r and σ_θ can be easily solved using Eqs. (7) and (8). In the paper authored by Zhong, Wang and Guo [24], $\widetilde{u}_1^1(\xi,0)$ and $\widetilde{w}_1^0(\xi,0)$ are solved numerically using a quadratic equation which is obtained by successively performing numerical matrix multiplications. This is the main difference between Reference [24] and this paper.

Since all the integrands in the inverse Hankel transforms, which give the desired displacements and stresses, are determined explicitly using the proposed algorithm and all the variables are presented in the appendices of this paper, researchers and engineers can write a computer code to implement, check, and expand this layered elastic theory algorithm.

Summary

In this paper, application of integral transform techniques in solving axisymmetric problem in multilayered pavement systems is introduced. The desired displacements and stresses evaluated at any point within a N-layered pavement system are formulated in terms of appropriate inverse Hankel transforms. Let $Q(r,z)$ be an evaluation point located in the ith layer, a vector $\widetilde{\Phi}_i(\xi,z)$ consisting of appropriate Hankel transforms of $u_i(r,z), w_i(r,z), \tau_{zri}(r,z)$ and

$\sigma_{zi}(r, z)$ is formed. Furthermore, $\widetilde{\Phi}_i(\xi, z)$ and $\widetilde{\Phi}_1(\xi, 0)$ are related using inter-layer contact conditions, and explicit expressions for $\widetilde{u}_1^1(\xi, 0)$ and $\widetilde{w}_1^0(\xi, 0)$ are successfully derived in terms of the solvable quantities $\widetilde{\tau_{zr}}_1^1(\xi, 0)$ and $\widetilde{\sigma_{z}}_1^0(\xi, 0)$ using the assumptions for LET. Theoretical and computational verifications demonstrate the correctness of the proposed algorithm. The proposed algorithm does not use a numerical linear system solver employed in the traditional approach to solving these problems, which should produce faster solutions times.

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A P_{ij} in Eq. (23) and G_{ij} in Eq. (24), $i, j = 1, 2, 3, 4$.

$$
\begin{aligned}
P_{11} &= \frac{s}{\left(s^2-\xi^2\right)^2}\left(s^2+\frac{\nu\xi^2}{1-\nu}\right) \\
P_{12} &= \frac{\xi}{\left(s^2-\xi^2\right)^2}\left(s^2+\frac{\nu\xi^2}{1-\nu}\right) \\
P_{13} &= \frac{1}{2\left(s^2-\xi^2\right)^2 G}\left[2s^2-\frac{(1-2\nu)\xi^2}{1-\nu}\right] \\
P_{14} &= \frac{\xi s}{2(1-\nu)\left(s^2-\xi^2\right)^2 G} \\
P_{21} &= -\frac{\xi}{\left(s^2-\xi^2\right)^2}\left[\xi^2+\frac{\nu s^2}{1-\nu}\right] \\
P_{22} &= \frac{s}{\left(s^2-\xi^2\right)^2}\left[s^2-\frac{(2-\nu)\xi^2}{1-\nu}\right] \\
P_{23} &= -P_{14} \\
P_{24} &= -\frac{1}{2\left(s^2-\xi^2\right)^2 G}\left[2\xi^2-\frac{(1-2\nu)s^2}{1-\nu}\right] \\
G_{11} &= \frac{1}{4(1-\nu)}\left\{2(1-\nu)+\xi z+\left[2(1-\nu)-\xi z\right]e^{-2\xi z}\right\} \\
G_{12} &= \frac{1}{4(1-\nu)}\left[1-2\nu+\xi z-(1-2\nu-\xi z)\,e^{-2\xi z}\right] \\
G_{13} &= \frac{1}{8(1-\nu)\xi G}\left[3-4\nu+\xi z-(3-4\nu-\xi z)\,e^{-2\xi z}\right] \\
G_{14} &= \frac{z}{8(1-\nu)G}\left(1-e^{-2\xi z}\right) \\
G_{21} &= \frac{1}{4(1-\nu)}\left[1-2\nu-\xi z-(1-2\nu+\xi z)\,e^{-2\xi z}\right] \\
G_{22} &= \frac{1}{4(1-\nu)}\left\{2(1-\nu)-\xi z+\left[2(1-\nu)+\xi z\right]e^{-2\xi z}\right\} \\
G_{23} &= -G_{14} \\
G_{24} &= \frac{1}{8(1-\nu)\xi G}\left[3-4\nu-\xi z-(3-4\nu+\xi z)\,e^{-2\xi z}\right]
\end{aligned}
$$

$$
\begin{aligned}
G_{31} &= \frac{\xi G}{2(1-\nu)}\left[1+\xi z-(1-\xi z)\,e^{-2\xi z}\right] \\
G_{32} &= \frac{\xi^2 G z}{2(1-\nu)}\left(1-e^{-2\xi z}\right) \\
G_{33} &= G_{11} \\
G_{34} &= -G_{21} \\
G_{41} &= -G_{32} \\
G_{42} &= \frac{\xi G}{2(1-\nu)}\left[1-\xi z-(1+\xi z)e^{-2\xi z}\right] \\
G_{43} &= -G_{12} \\
G_{44} &= G_{22}
\end{aligned}
$$

B Δ, T_{12} & T_{22} in Eq. (34)

$$
\begin{aligned}
\Delta &= a_{11}a_{22}-a_{12}a_{21}+\frac{1-2\nu_n}{2\xi G_n}\,(a_{12}a_{31}+a_{21}a_{42} \\
&\quad -a_{11}a_{32}-a_{22}a_{41})+\frac{1-\nu_n}{\xi G_n}\,(a_{11}a_{42}+a_{22}a_{31} \\
&\quad -a_{12}a_{41}-a_{21}a_{32})+\frac{3-4\nu_n}{4\xi^2(G_n)^2}\,(a_{31}a_{42}-a_{32}a_{41}) \\
T_{12} &= a_{12}a_{24}-a_{14}a_{22}+\frac{1-2\nu_n}{2\xi G_n}\,(a_{22}a_{44}+a_{14}a_{32} \\
&\quad -a_{12}a_{34}-a_{24}a_{42})+\frac{1-\nu_n}{\xi G_n}\,(a_{12}a_{44}+a_{24}a_{32} \\
&\quad -a_{14}a_{42}-a_{22}a_{34})+\frac{3-4\nu_n}{4\xi^2(G_n)^2}\,(a_{32}a_{44}-a_{34}a_{42}) \\
T_{22} &= a_{14}a_{21}-a_{11}a_{24}+\frac{1-2\nu_n}{2\xi G_n}\,(a_{11}a_{34}+a_{24}a_{41} \\
&\quad -a_{21}a_{44}-a_{14}a_{31})+\frac{1-\nu_n}{\xi G_n}\,(a_{14}a_{41}+a_{21}a_{34} \\
&\quad -a_{11}a_{44}-a_{24}a_{31})+\frac{3-4\nu_n}{4\xi^2(G_n)^2}\,(a_{34}a_{41}-a_{31}a_{44})
\end{aligned}
$$

where $a_{ij} = [F(\xi, h_{n-1})]_{ij}$.

C $u(r,z), w(r,z), \tau_{zr}(r,z)$ & $\sigma_z(r,z)$ when Q is located in the i-th layer

- When $i = 1, 2, \cdots, n-1$

$$
\begin{aligned}
u &= \int_0^\infty \xi e^{\xi(H_{i-1}+z)} \left[G_{11}\ G_{12}\ G_{13}\ G_{14}\right]_i \cdot [F(\xi, h_{i-1})] \widetilde{\Phi}_1(\xi, 0) J_1(\xi r) d\xi \\
w &= \int_0^\infty \xi e^{\xi(H_{i-1}+z)} \left[G_{21}\ G_{22}\ G_{23}\ G_{24}\right]_i \cdot [F(\xi, h_{i-1})] \widetilde{\Phi}_1(\xi, 0) J_0(\xi r) d\xi \\
\tau_{zr} &= \int_0^\infty \xi e^{\xi(H_{i-1}+z)} \left[G_{31}\ G_{32}\ G_{33}\ G_{34}\right]_i \cdot [F(\xi, h_{i-1})] \widetilde{\Phi}_1(\xi, 0) J_1(\xi r) d\xi \\
\sigma_z &= \int_0^\infty \xi e^{\xi(H_{i-1}+z)} \left[G_{41}\ G_{42}\ G_{43}\ G_{44}\right]_i \cdot [F(\xi, h_{i-1})] \widetilde{\Phi}_1(\xi, 0) J_0(\xi r) d\xi
\end{aligned}
$$

- When $i = n$

$$
\begin{aligned}
u &= \int_0^\infty \xi e^{\xi(H_{n-1}-z)} \left[b_{11}\ b_{12}\ b_{13}\ b_{14}\right] \cdot [F(\xi, h_{n-1})] \widetilde{\Phi}_1(\xi, 0) J_1(\xi r) d\xi \\
w &= \int_0^\infty \xi e^{\xi(H_{n-1}-z)} \left[b_{21}\ b_{22}\ b_{23}\ b_{24}\right] \cdot [F(\xi, h_{n-1})] \widetilde{\Phi}_1(\xi, 0) J_0(\xi r) d\xi \\
\tau_{zr} &= \int_0^\infty \xi e^{\xi(H_{n-1}-z)} \left[b_{31}\ b_{32}\ b_{33}\ b_{34}\right] \cdot [F(\xi, h_{n-1})] \widetilde{\Phi}_1(\xi, 0) J_1(\xi r) d\xi \\
\sigma_z &= \int_0^\infty \xi e^{\xi(H_{n-1}-z)} \left[b_{41}\ b_{42}\ b_{43}\ b_{44}\right] \cdot [F(\xi, h_{n-1})] \widetilde{\Phi}_1(\xi, 0) J_0(\xi r) d\xi
\end{aligned}
$$

where

$$
\begin{aligned}
b_{11} &= \frac{2(1-\nu_n)-\xi z}{4(1-\nu_n)} \\
b_{12} &= -\frac{1-2\nu_n-\xi z}{4(1-\nu_n)} \\
b_{13} &= -\frac{3-4\nu_n-\xi z}{8(1-\nu_n)\xi G_n} \\
b_{14} &= -\frac{z}{8(1-\nu_n)G_n} \\
b_{21} &= -\frac{1-2\nu_n+\xi z}{4(1-\nu_n)} \\
b_{22} &= \frac{2(1-\nu_n)+\xi z}{4(1-\nu_n)} \\
b_{23} &= -b_{14} \\
b_{24} &= -\frac{3-4\nu_n+\xi z}{8(1-\nu_n)\xi G_n} \\
b_{31} &= -\frac{(1-\xi z)\xi G_n}{2(1-\nu_n)} \\
b_{32} &= -\frac{\xi^2 z G_n}{2(1-\nu_n)} \\
b_{33} &= b_{11} \\
b_{34} &= -b_{21} \\
b_{41} &= -b_{32} \\
b_{42} &= -\frac{(1+\xi z)\xi G_n}{2(1-\nu_n)} \\
b_{43} &= -b_{12} \\
b_{44} &= b_{22}
\end{aligned}
$$

D $\sigma_r(r,z)$ & $\sigma_\theta(r,z)$ when Q is located in the i-th layer

- When $i = 1, 2, \cdots, n-1$

$$
\begin{aligned}
\sigma_r &= 2G_i \int_0^\infty \xi e^{\xi z} \left[\Omega_i J_0(\xi r) - \frac{\Lambda_i}{r} J_1(\xi r) \right] d\xi \\
\sigma_\theta &= 2G_i \int_0^\infty \xi e^{\xi z} \left[\Theta_i J_0(\xi r) + \frac{\Lambda_i}{r} J_1(\xi r) \right] d\xi
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_i &= e^{\xi H_{i-1}} [G_{11}\ G_{12}\ G_{13}\ G_{14}]_i \cdot [F(\xi, h_{i-1})] \widetilde{\Phi}_1(\xi, 0) \\
\Omega_i &= e^{\xi H_{i-1}} [\psi_1\ \psi_2\ \psi_3\ \psi_4]_i \cdot [F(\xi, h_{i-1})] \widetilde{\Phi}_1(\xi, 0) \\
\Theta_i &= e^{\xi H_{i-1}} [\omega_1\ \omega_2\ \omega_3\ \omega_4]_i \cdot [F(\xi, h_{i-1})] \widetilde{\Phi}_1(\xi, 0)
\end{aligned}
$$

The components in $[\psi_1\ \psi_2\ \psi_3\ \psi_4]_i$ are

$$
\begin{aligned}
\psi_1 &= \frac{\xi}{4(1-\nu_i)} \left[2 + \xi z + (2 - \xi z) e^{-2\xi z} \right] \\
\psi_2 &= \frac{\xi}{4(1-\nu_i)} \left[1 + \xi z - (1 - \xi z) e^{-2\xi z} \right] \\
\psi_3 &= \frac{1}{8(1-\nu_i)G_i} \left[3 - 2\nu_i + \xi z - (3 - 2\nu_i - \xi z) e^{-2\xi z} \right] \\
\psi_4 &= \frac{1}{8(1-\nu_i)G_i} \left[2\nu_i + \xi z + (2\nu_i - \xi z) e^{-2\xi z} \right]
\end{aligned}
$$

The components in $[\omega_1\ \omega_2\ \omega_3\ \omega_4]_i$ are

$$
\begin{aligned}
\omega_1 &= \frac{\nu_i \xi}{2(1-\nu_i)} \left(1 + e^{-2\xi z} \right) \\
\omega_2 &= \frac{\nu_i \xi}{2(1-\nu_i)} \left(1 - e^{-2\xi z} \right) \\
\omega_3 &= \frac{\nu_i}{4(1-\nu_i)G_i} \left(1 - e^{-2\xi z} \right) \\
\omega_4 &= \frac{\nu_i}{4(1-\nu_i)G_i} \left(1 + e^{-2\xi z} \right)
\end{aligned}
$$

- When $i = n$

$$\sigma_r = 2G_n \int_0^\infty \xi e^{\xi z} \left[\Omega_n J_0(\xi r) - \frac{\Lambda_n}{r} J_1(\xi r) \right] d\xi$$
$$\sigma_\theta = 2G_n \int_0^\infty \xi e^{\xi z} \left[\Theta_n J_0(\xi r) + \frac{\Lambda_n}{r} J_1(\xi r) \right] d\xi$$

where

$$\Lambda_n = e^{\xi H_{n-1}} [G_{11}\ G_{12}\ G_{13}\ G_{14}]_n \cdot [F(\xi, h_{n-1})] \widetilde{\Phi}_1(\xi, 0)$$

$$\Omega_n = e^{\xi H_{n-1}} [\psi_1\ \psi_2\ \psi_3\ \psi_4]_n \cdot [F(\xi, h_{n-1})] \widetilde{\Phi}_1(\xi, 0)$$

$$\Theta_n = e^{\xi H_{n-1}} [\omega_1\ \omega_2\ \omega_3\ \omega_4]_n \cdot [F(\xi, h_{n-1})] \widetilde{\Phi}_1(\xi, 0)$$

The components in $[G_{11}\ G_{12}\ G_{13}\ G_{14}]_n$ are

$$G_{11} = \frac{2(1-\nu_n) - \xi z}{4(1-\nu_n)}$$
$$G_{12} = -\frac{1 - 2\nu_n - \xi z}{4(1-\nu_n)}$$
$$G_{13} = -\frac{3 - 4\nu_n - \xi z}{8(1-\nu_n)\xi G_n}$$
$$G_{14} = -\frac{z}{8(1-\nu_n)G_n}$$

The components in $[\psi_1\ \psi_2\ \psi_3\ \psi_4]_n$ are

$$\psi_1 = \frac{\xi(2 - \xi z)}{4(1-\nu_n)}$$
$$\psi_2 = -\frac{\xi(1 - \xi z)}{4(1-\nu_n)}$$
$$\psi_3 = -\frac{3 - 2\nu_n - \xi z}{8(1-\nu_n)G_n}$$
$$\psi_4 = \frac{2\nu_n - \xi z}{8(1-\nu_n)G_n}$$

The components in $[\omega_1 \; \omega_2 \; \omega_3 \; \omega_4]_n$ are

$$\begin{aligned}
\omega_1 &= \frac{\nu_n \xi}{2(1-\nu_n)} \\
\omega_2 &= -\omega_1 \\
\omega_3 &= -\frac{\nu_n}{4(1-\nu_n)G_n} \\
\omega_4 &= -\omega_3
\end{aligned}$$