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Spectral geometry of hyperbolic 3-manifolds

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SPECTRAL GEOMETRY OF HYPERBOLIC 3-MANIFOLDS

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ABSTRACT

This thesis uses techniques from spectral geometry and builds from the work of Schoen, Culler and Shalen, Meyerhoff, and others to obtain various estimates and inequalities involving geometric data of hyperbolic 3-manifolds. These give numerical relationships between quantities like volume, length of geodesics, area of embedded surfaces, isoperimetric constants, eigenvalues of the Laplacian, and Margulis numbers for hyperbolic 3-manifolds.

DEDICATION

This thesis is dedicated to my family, in particular my wife and daughter for all their help and support, my parents for their constant support and for raising me the way I would have raised myself, and my grandparents for their accumulated wisdom and kindness. I thank them for making all things possible — *concordia discors*.

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Chapter 1

Introduction

ὤμοι, πέπληγμαι καιρίαν πληγὴν ἔσω.

Agamemnon, Aeschylus

A quintessential mathematical problem is the classification of manifolds. This taxonomic nightmare has engaged topologists for the last hundred years. The classification of 2-manifolds was one of the great early triumphs of modern mathematics at the end of the 19th century. In the beginning of the 20th century it was not clear how one would deal with manifolds of dimension three, let alone higher dimensions. As it turned out, the high dimensional manifolds (dimensions five and higher) were handled by the 1960's. We should at this point say something about classification. In two dimensions, closed oriented manifolds are determined up to homeomorphism by the Euler characteristic. This invariant is easy to compute and can be calculated using a wide variety of techniques, from homological methods to the Gauss-Bonnet theorem. This is a very satisfactory classification in that it gives a complete description of all the possible 2-manifolds and a practical way of identifying them. It is known that such a classification cannot exist for manifolds of dimension four or more. In high dimensions this deficiency has been made up for by powerful techniques like surgery and the h-cobordism theorem which can be used to understand and work with the structure of high dimensional manifolds.

Another difference between low dimensions and high dimensions is the distinctness of the categories of topological, piecewise linear, and smooth manifolds. In dimensions five and higher these three categories are all distinct. Furthermore a topological manifold can possess inequivalent PL structures, and a PL manifold can possess inequivalent smooth structures.

In dimension four there has been much current research activity. Topologically, 4-manifolds are similar to high dimensional manifolds, namely surgery and cobordism theorems hold as they do in high dimensions. As for smooth 4-manifolds (every PL 4-manifold has a unique smoothing) matters are completely different. For one, the smooth versions of the surgery theorems from high dimensions fail in four dimensions. For another, there are compact closed 4-manifolds which possess an infinite number of distinct smooth structures. Additionally, there are exotic $\mathbf{R}^4$'s, i.e. manifolds homeomorphic to the standard Euclidean four-space but not diffeomorphic to it. Thus, 4-dimensional manifolds are quite different than manifolds in any other dimension.

In dimensions two and three every topological manifold has a unique smooth structure. This allows for an exciting interplay between the topology and the geometry of the manifolds. For example, the uniformization of Riemann surfaces can be thought of as a geometric classification of topological 2-manifolds. This is an important viewpoint because it can be generalized to three dimensions. Thurston has formulated a Geometrization Conjecture which states that every 3-manifold can be canonically decomposed into geometric pieces. These geometric pieces are all of the form X/Γ where X is one of eight model geometric spaces and Γ is a discrete subgroup of the group of isometries of X. Of these eight model geometries, seven have been classified in the sense that there is a list of all possible types of manifolds and a practical set of invariants for identifying them. Six of these geometries correspond to the Seifert fibered spaces which were classified by Seifert in the 1930's. The one geometry which remains somewhat intractable is hyperbolic geometry. Hyperbolic manifolds are those manifolds which admit a Riemannian metric of constant curvature of -1.

In the 1960's it was proved that every closed oriented 3-manifold was obtained by surgery on a link in the 3-sphere. In the early 1980's it was proved that on most links almost every surgery yields a hyperbolic 3-manifold. Thus, in some sense most 3-manifolds admit hyperbolic structures. Additionally, by the Mostow Rigidity theorem, a hyperbolic manifold has a unique hyperbolic structure, thus the topology uniquely determines the geometry. In this thesis we will investigate the geometry of hyperbolic 3-manifolds. In particular we would like to understand the "shape" of hyperbolic 3-manifolds. There are many numerical quantities associated with a Riemannian manifold which describe the shape of the manifold. For hyperbolic 3-manifolds, these quantities all become topological invariants. Several geometric invariants have already been studied in some detail like volume and the Chern-Simons invariant. In this thesis we will look more closely at two other numbers: the first eigenvalue and the isoperimetric constant. These two quantities are closely related and tend to occur in the realm of spectral geometry, hence the title of this thesis.

In the next chapter we will consider lower bounds for the first eigenvalue and the isoperimetric constant. In the third chapter we will look at upper bounds. These bounds will take the form of inequalities depending on geometric quantities like volume, diameter, and injectivity radius. In the fourth chapter we will apply these results by combining the various bounds. In the last chapter we will look at a few explicit examples of hyperbolic 3-manifolds and their numerical invariants.

Chapter 2

Lower bounds for the first eigenvalue and the isoperimetric constant

Introduction. There is an extensive body of literature concerning the study of low eigenvalues of Riemannian manifolds of negative curvature. Most of these works emphasize the case of Riemann surfaces. One of the reasons for this has been the paucity of examples and constructions of higher dimensional manifolds. In recent years this lacuna has been filled, at least in dimension three, by the work of Thurston, Jorgensen, Riley, and others. Now that we have an abundance of examples to work with, we hope to emulate that great work which has been done for surfaces and establish a better understanding of the relationship between the spectral data of a hyperbolic three-manifold and its geometry and topology.

The specific object we will consider here is λ_1 the low or first eigenvalue. More specifically, let M be a finite volume hyperbolic (constant curvature of -1) n-manifold, and let Δ be the Laplace-Beltrami operator (Δ = -div grad) on M. Then λ_1 is defined to be the first positive element in the discrete spectrum for the equation $\Delta f = \lambda f$.

In the two-dimensional case it was shown by [SWY] that for a compact hyperbolic surface M there exists a constant c>0 depending only on the genus of M, such that $\lambda_1 \geq c\ell$ where ℓ is the total length of the shortest collection of simple closed geodesics separating M. There are examples due to Buser [B] of compact hyperbolic surfaces of a fixed genus which have arbitrarily small first eigenvalue.

The two-dimensional case is often very different from the higher dimensions due to Mostow's Rigidity Theorem which implies that the geometry of a hyperbolic n-manifold for n>2 is uniquely determined by its topology. Thus, Schoen suspected that the phenomenon observed by Buser for surfaces would be a purely two-dimensional one, and he went on to show that if M is a compact hyperbolic n-manifold with n>2, then there exists a constant c>0, depending only on n, such that $\lambda_1 \geq c/V^2$ where V is the volume of M. For hyperbolic manifolds of dimension greater than two, the volume tends to play the role which the genus plays for surfaces. Schoen's result implies that if we fix the volume, then λ_1 must be bounded away from zero. Therefore, there are no higher dimensional analogs to Buser's examples.

What we plan to do in this section is to look more closely at Schoen's lower bound in the case of hyperbolic three-manifolds and use recent results which are specifically three-dimensional to establish stronger bounds in this case.

The Margulis Lemma

One of the most important facts concerning the structure of hyperbolic manifolds is the Margulis Lemma. We will first define what is meant by a *thick-thin decomposition.*

Definition 2.1: Let M be a complete hyperbolic manifold (not necessarily compact or of finite volume). Given an $\varepsilon > 0$, we can decompose M into two pieces, $M = M_{thin(\varepsilon)} \cup M_{thick(\varepsilon)}$ where

$$M_{thin(\varepsilon)} = \{x \in M : inj(x) < \varepsilon\}$$

and

$$M_{thick(\varepsilon)} = \{x \in M : inj(x) \geq \varepsilon\}$$

Here inj(x) denotes the injectivity radius of x.

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$$\mu(M) > \tfrac{1}{2}\log 3 > .5045$$

Note that there are examples of hyperbolic 3-manifolds with Margulis numbers less than 1, thus $\log 3$ is not the Margulis constant for general hyperbolic 3-manifolds.

Some lemmas

Following the basic outline of Schoen's proof we begin with a few preliminary lemmas. Unless stated otherwise, M will be a compact hyperbolic three-manifold. We will also use the convention that $Vol(X)$ denotes the n-dimensional volume where n is equal to the dimension of the object X.

Lemma 2.4: (cf. [S],[B2]) Let Σ be a closed surface in M which bounds a region Ω in M. Suppose that Σ has constant mean curvature $|H| \geq 1$. Then

$$Vol(\Sigma) \geq 2Vol(\Omega).$$

Proof: We will use the following inequality of Heintze-Karcher [HK]:

$$Vol(\Omega) \leq \left(\int_0^\infty \left(\cosh r - \left(\min_\Sigma |H| \right) \sinh r \right)_+^{n-1} dr \right) Vol(\Sigma)$$

where $(\ )_+$ indicates to take the positive part of a function. Using our hypotheses on the mean curvature and the dimension we have the following:

$$Vol(\Omega) \leq \left(\int_0^\infty (\cosh r - \sinh r)_+^2 \, dr \right) Vol(\Sigma)$$
$$Vol(\Omega) \leq \left(\int_0^\infty e^{-2r} dr \right) Vol(\Sigma)$$
$$Vol(\Omega) \leq \tfrac{1}{2} Vol(\Sigma)$$

from which we conclude the lemma.

It would be nice if we could prove that such an inequality holds even when the mean curvature of the surface is between 0 and 1. Unfortunately, we are unable to do so at this time, thus we must use a different approach. Instead of obtaining an inequality of an isoperimetric form like above we will use a lemma of Schoen to give a lower bound of the volume of an embedded surface which depends on the injectivity radius of the manifold. We will use the following lemma:

Lemma 2.5: [S] For a point $P \in M$, let $i(P)$ denote the injectivity radius of M at P. Suppose the sectional curvature of M satisfies $K_M \leq -\kappa^2$ for some $\kappa \geq 0$, and let Σ be a hypersurface in M with mean curvature H satisfying $|H| \leq \Lambda$. Suppose also that $Vol(\Sigma) < \infty$ and $\mathcal{H}^{n-2}(\overline{\Sigma} - \Sigma) = 0$ where $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure. Then for every point $P \in \overline{\Sigma}$ we have:

$$i(P)e^{-i(P)(\Lambda-\kappa)_+} \leq \left[\omega_{n-1}^{-1} Vol(\Sigma)\right]^{1/n-1}$$

where ω_n denotes the volume of the unit ball in Euclidean n-space.

From this more general lemma we obtain the following:

Lemma 2.6: Let Σ be a closed surface in a hyperbolic 3-manifold M. Suppose that Σ has constant mean curvature H satisfying $0 \leq |H| \leq 1$. Then for every point $P \in \Sigma$ we have:

$$Vol(\Sigma) \geq \pi\, inj(P)^2$$

Proof: This inequality is merely a special case of Lemma 2.5. We use that the mean curvature is bounded and that $\kappa = 1$ since M is hyperbolic.

Now in anticipation of using the Margulis Lemma, which was be discussed earlier, we will consider the case where the surface is contained in a tubular neighborhood around a closed geodesic. Let γ be a simple closed geodesic in M. Let $T_{\gamma,R}$ denote an embedded tube of radius R around γ, i.e. $T_{\gamma,R} = \{x \in M \mid d(x,\gamma) \leq R\}$. In this setting we obtain the following:

Lemma 2.7: Let Σ be a closed surface in hyperbolic tube $T_{\gamma,R}$. Suppose that $\Sigma = \partial\Omega$ in $T_{\gamma,R}$. Then we have the following:

$$Vol(\Sigma) \geq 2Vol(\Omega).$$

Proof: Since T=$T_{\gamma,R}$ is embedded we can use *Fermi coordinates* (see [B] or [G]) for points in T. Namely, by fixing a point on γ and a vector normal to γ we use the triple $\{\rho,\theta,t\}$ to specify the point a distance ρ along the normal vector making an angle of θ with the selected normal based a distance t along γ from the selected point. Basically, these coordinates should be thought of as cylindrical coordinates about a geodesic. Stokes theorem applied to the radial function gives

$$\iint_\Sigma \nabla\rho \cdot n \, d\Sigma = \iiint_\Omega \Delta\rho \, d\Omega$$

Using the comparison that $\Delta\rho \geq 2$ we get that

$$\begin{aligned} Vol(\Sigma) &= \iint_\Sigma 1 \, d\Sigma = \iint_\Sigma \nabla\rho \cdot n \, d\Sigma \\ &= \iiint_\Omega \Delta\rho \, d\Omega \geq \iiint_\Omega 2 \, d\Omega = 2Vol(\Omega) \end{aligned}$$

which gives the desired conclusion.

Now that we have the above inequalities we can use Cheeger's inequality to obtain a lower bound for the first eigenvalue.

Isoperimetric Constants

The classical isoperimetric inequality says that in the Euclidean plane the standard circle encloses the most area for a given perimeter. In an arbitrary manifold, the ratio of the volume of a domain to the volume of its boundary can have more complicated behavior. For a compact manifold M we can define the *isoperimetric constant* $h=h(M)$ as follows:

$$h(M) = \inf_{\Omega} \frac{Vol(\partial\Omega)}{Vol(\Omega)}$$

where Ω runs through all domains with volume less than half the volume of M. Note that the isoperimetric constant is sometimes given the equivalent definition of

$$h(M) = \inf_{\Sigma} \frac{Area(\Sigma)}{\min(Vol(A), Vol(B))}$$

where Σ runs through all hypersurfaces which split M into exactly two components A and B.

The isoperimetric constant encodes a certain amount of global geometric information about the manifold. In the next section we will see how h affects the spectral geometry of the manifold. Note that if a 3-manifold has a very small isoperimetric constant, there must be some small smooth embedded surface which bounds a very large domain. This situation is easy to visualize in two dimensions where we can have a narrow "neck" like a sphere with a pinched equator. We will later show that there exists compact hyperbolic 3-manifolds with arbitrarily small isoperimetric constant. We will also consider universal upper bounds for the isoperimetric constant. For now we will mention the following observation:

Lemma 2.8: There exists a universal upper bound for the isoperimetric constant of a compact hyperbolic 3-manifold.

Proof: The volume of a hyperbolic ball of radius r is equal to $\pi(\sinh(2r)-2r)$. The surface of a hyperbolic ball of radius r is equal to $4\pi\sinh^2 r$. Thus, if a hyperbolic 3-manifold has a isometrically embedded ball B of radius r then

$$h(M)=\inf_{\Omega}\frac{Vol(\partial\Omega)}{Vol(\Omega)}\leq\frac{Vol(\partial B)}{Vol(B)}=\frac{4\pi\sinh^2 r}{\pi(\sinh(2r)-2r)}$$

In [M2] Myerhoff proved that in every compact hyperbolic 3-manifold there exists an isometrically embedded ball of radius at least .052. Thus, we calculate that

$$h(M)\leq\frac{4\pi\sinh^2(.052)}{\pi(\sinh(.104)-.104)}=57.7131$$

If we consider those hyperbolic 3-manifolds with first Betti number greater than 2, then by the work of Culler and Shalen (see [CS1]), $\frac{1}{2}\log 3$ is a Margulis number for M, thus M must have an isometrically embedded ball of radius $\frac{1}{2}\log 3$. This gives us an upper bound for the isoperimetric constant of M:

$$h(M)\leq 5.6805$$

Note: This is a very rough bound which is by no means sharp. We will see later that as the diameter or volume of a hyperbolic 3-manifold gets large, the isoperimetric constant of the manifold approaches 2. Thus a manifold with maximal isoperimetric constant must be sufficiently "small". It would be interesting to find explicit examples of hyperbolic 3-manifolds with large isoperimetric constant.

Cheeger's Inequality

Cheeger gave a lower bound for the first eigenvalue of a compact Riemannian manifold in 1970.

Cheeger's Inequality. [C] Let M be a compact Riemannian manifold. Let $\lambda_1 = \lambda_1(M)$ denote the first eigenvalue of M. Let $h = h(M)$ denote the isoperimetric constant of M. Then we have that:

$$\lambda_1 \geq \tfrac{1}{4}h^2$$

We will now use Cheeger's inequality to give a lower bound for the first eigenvalue of a compact hyperbolic 3-manifold. Here we will follow the basic outline of Schoen and make use of an existence theorem from minimal surface theory.

Existence Theorem. For any v with $0 \leq v \leq \frac{1}{2}Vol(M)$, there exits an open set $\Omega_v \subset M$ with $Vol(\Omega_v) = v$, and a smooth embedded hypersurface Σ_v with the property that $\overline{\Sigma_v} = \partial\Omega_v, \mathcal{H}^s(\overline{\Sigma_v} - \Sigma_v) = 0$ for $s > n-8$, and Ω_v has the extremal property

$$Vol(\Sigma_v) = \inf\{Vol(\partial\Omega) : \Omega \subset M \text{ with } Vol(\Omega) = v\}$$

Moreover, the mean curvature vector H of Σ_v satisfies $|H| \equiv H_v$ for a constant $H_v \geq 0$ as well as the property that H points everywhere into or out of Ω_v.

Proof: See [F].

The extremal properties of Σ_v and Ω_v give us the inequality:

$$h(M) \geq \inf\left\{\frac{Vol(\Sigma_v)}{Vol(\Omega_v)} : 0 \leq v \leq \tfrac{1}{2} Vol(M)\right\}$$

Now we can apply the previous lemmas to get a lower bound on *h(M)*.

Case 1: If $H_v \geq 1$ then by Lemma 2.4 we have that $Vol(\Sigma_v) \geq 2Vol(\Omega_v)$ so $\frac{Vol(\Sigma_v)}{Vol(\Omega_v)} \geq 2$.

Case 2: If $\mu(M)$ is the Margulis number of M and $inj(P) < \mu(M)$ for every $P \in \Sigma_v$ then by the Margulis lemma $\Sigma_v \subset M_{thin}$. Thus, $\Sigma_v \subset T_{\gamma,R}$ for some hyperbolic tube of radius R around a geodesic γ. Therefore we can apply Lemma 2.7 to get the inequality $\frac{Vol(\Sigma_v)}{Vol(\Omega_v)} \geq 2$.

Case 3. If $0 \leq H_v \leq 1$ and $inj(P) \geq \mu(M)$ for some point $P \in \Sigma_v$ then we can apply Lemma 2.6 to get $Vol(\Sigma_v) \geq \pi inj(P)^2$, so $Vol(\Sigma_v) \geq \pi\mu(M)^2$. This gives us the inequality for $\mu(M)$. Since we are only considering those v such that $0 \leq v \leq \frac{1}{2}Vol(M)$, we have $\frac{Vol(\Sigma_v)}{Vol(\Omega_v)} \geq \frac{2\pi\mu(M)^2}{Vol(M)}$.

Combining the above cases gives the following:

Theorem 2.9: Let M be a compact hyperbolic 3-manifold. Then the isoperimetric constant of M satisfies:

$$h(M) \geq \min\left\{\frac{2\pi\mu(M)^2}{Vol(M)}, 2\right\}$$

Later we will contrast this with an upper bound for *h(M)*. Note here that since there is a universal lower bound for $\mu(M)$ the only way for *h(M)* to approach 0 is if the volume of M approaches infinity.

We can use Theorem 2.9 to get a lower bound for the first eigenvalue of M:

Theorem 2.10: Let M be a compact hyperbolic 3-manifold. Then the first eigenvalue of M satisfies:

$$\lambda_1(M) \geq \min\left\{\frac{\pi^2\mu(M)^4}{Vol(M)^2}, 1\right\}$$

Proof: This follows immediately from Cheeger's Inequality $\lambda_1 \geq \frac{1}{4}h^2$.

Before moving on to upper bounds, we will consider another lower bound for *h(M)* and thus for $\lambda_1(M)$. This one follows from a theorem of Croke:

Theorem 2.11: (see [Cr]) Let M be a compact n-manifold without boundary and Ricci curvature bounded below by (n-1)k. If D is the diameter of M and α_n is the volume of the unit n-sphere then

\

$$h(M) \geq \frac{\pi}{\alpha_n} \frac{Vol(M)}{D\int_0^D (\sqrt{-1/k}\sinh\sqrt{-k}r)^{n-1}dr}$$

If we restrict to the case of constant -1 curvature and dimension 3 we get the following

Theorem 2.12: Let M be a compact hyperbolic 3-manifold of diameter D. Then the isoperimetric constant of M satisfies

$$h(M) \geq \frac{2}{\pi} \frac{Vol(M)}{D(\sinh(2D) - 2D)}$$

Note that since there are examples of hyperbolic 3-manifolds with bounded volume, yet unbounded diameter, this inequality still allows for the possibility of arbitrarily small isoperimetric constants.

We can apply the following inequality of Meyerhoff to get lower bounds on the isoperimetric constant that depend on different geometric data.

Lemma 2.13: In a closed hyperbolic 3-manifold M whose shortest geodesic has length ℓ, the diameter D of M is bounded as follows:

$$D \leq \frac{Vol(M)}{\pi \sinh^2(\ell/4)}$$

Proof: see [M2]

Corollary 2.14: Let M be a compact hyperbolic 3-manifold whose shortest geodesic has length ℓ. Then the isoperimetric constant of M satisfies the following inequality:

$$h(M) \geq \frac{2\sinh^2(\ell/4)}{(\sinh(2D) - 2D)}$$

Corollary 2.15: Let M be a compact hyperbolic 3-manifold whose shortest geodesic has length ℓ. Then the isoperimetric constant of M satisfies the following inequality:

$$h(M) \geq \frac{4\sinh^2(\ell/4)}{\left(\sinh(2Vol(M)(\pi\sinh^2(\ell/4))^{-1}) - 2Vol(M)(\pi\sinh^2(\ell/4))^{-1}\right)}$$

Although neither of these inequalities is sharp, they do reflect some interesting relationships between the various geometric data of the manifold. By the work of Meyerhoff in [M1] it follows that as the length of the shortest geodesic approaches zero, the diameter of the manifold must approach infinity. Thus the quantities involved in the inequalities above are not completely independent.

We can use Croke's inequality and Cheeger's inequality to get one more lower bound for the first eigenvalue:

Theorem 2.16: Let M be a compact hyperbolic 3-manifold with diameter D. Then the first eigenvalue of M satisfies the following inequality:

$$\lambda_1(M) \geq \frac{4Vol(M)^2}{\pi^2 D^2(\cosh(2D) - 4D - 1)^2}$$

This lower bound is remarkably different than Schoen's lower bound. The reciprocal dependence of the two bounds with respect to *Vol(M)* is quite surprising. It is not known if one is generally better than the other. One interesting difference is that Schoen's lower bound does not give any information if the eigenvalue is out of the official "small" range from zero to one, whereas the inequality in Theorem 2.9 could possibly give lower bounds for λ_1 which were greater than 1. There are no known examples of hyperbolic 3-manifolds which can be shown to have arbitrarily large first eigenvalue.

Chapter 3

Upper bounds for the first eigenvalue and the isoperimetric constant

One of the most generally applicable techniques for getting upper bounds on the first eigenvalue is to use the *Rayleigh quotient.*

The Rayleigh Quotient: (see [C] for more details) Let M be a compact manifold without boundary. Let $f \in C^1(M)$ such that $f \neq 0$ and $\int_M f = 0$ then we have:

$$\lambda_1(M) \leq \frac{\int_M \|\nabla f\|^2}{\int_M f^2}$$

with equality if and only if f is an eigenfunction.

Theorem 3.1: (see [Be]) Let M be an n-manifold without boundary whose Ricci curvature is bounded from below by (n-1)k. Let D denote the diameter of M then the first eigenvalue of M satisfies the following inequality:

$$\lambda_1(M) \leq \lambda(k, D/2)$$

where $\lambda(k,r)$ denotes the first eigenvalue for the Dirichlet eigenvalue problem for a ball of radius r in a simply connected space of constant curvature equal to k.

The proof of this uses Bishop's inequality.

To use this theorem we will need to bound the first eigenvalue of a ball in hyperbolic space. In Chavel's book [C] he uses the Rayleigh quotient to derive to following bound:

Lemma 3.2: Let $\lambda(r)$ denote the first eigenvalue for the Dirichlet problem for a ball of radius r in hyperbolic n-space. Then we have the following:

$$\lambda(r) \leq \left(\frac{n-1}{2}(\coth\frac{r}{2} - 1) + \sqrt{\left(\frac{(n-1)^2}{4} + \frac{4\pi^2}{r^2} + \frac{(n-1)^2}{4}(\coth\frac{r}{2} - 1)^2 \right)} \right)^2$$

We can now restrict this and apply to the above theorem to get an upper bound on the first eigenvalue in terms of the diameter.

Theorem 3.3: Let M be a compact hyperbolic 3-manifold of diameter D. Then the first eigenvalue of M satisfies the following inequality:

$$\lambda_1(M) \leq \left((\coth\frac{D}{4} - 1) + \sqrt{\left(1 + \frac{16\pi^2}{D^2} + (\coth\frac{D}{4} - 1)^2 \right)} \right)^2$$

Note that $\lim_{D \to \infty} \left((\coth\frac{D}{4} - 1) + \sqrt{\left(1 + \frac{16\pi^2}{D^2} + (\coth\frac{D}{4} - 1)^2 \right)} \right)^2 = 1$. Thus, as the length of the shortest geodesic goes to zero, the upper bound for the first eigenvalue goes to one. Later we will look more closely at how this relates to the lower bounds for λ_1.

Here is an upper bound due to Buser which depends on the isoperimetric constant of the manifold:

Theorem 3.4: (see [B2]) If the Ricci curvature of a compact n-manifold M without boundary is bounded below by $-(n-1)\delta^2$ $(\delta \geq 0)$ then:

$$\lambda_1(M) \leq 2\delta(n-1)h(M) + 10h(M)^2$$

where $h(M)$ is the isoperimetric constant of M.

We will consider the following immediate corollary:

Corollary 3.5: If M is a compact hyperbolic 3-manifold with isoperimetric constant h then the first eigenvalue is bounded as follows:

$$\lambda_1(M) \leq 4h + 10h^2$$

This inequality is not sharp.

We will now consider some upper bounds on the isoperimetric constant of hyperbolic 3-manifolds. Since the isoperimetric constant is defined an infinum over all dividing surfaces, the most direct way of get an upper bound is to consider specific cases. The simplest is that of an embedded ball.

Let B(r) be an isometrically embedded ball of radius r. Then we have already shown that

$$h(M) \leq \frac{4\sinh^2 r}{\sinh(2r) - 2r}$$

Since a manifold M with Margulis number $\mu(M)$ must have an isometrically embedded ball of radius μ, we can rephrase this as:

Corollary 3.6: Let M be a hyperbolic 3-manifold with Margulis number μ. Then the isoperimetric constant of M is bounded above as follows:

$$h(M) \le \frac{4\sinh^2\mu}{\sinh(2\mu) - 2\mu}$$

Now let us consider a tubular neighborhood about a simple closed geodesic. Using the Fermi coordinates it is easy to compute that the volume of a solid torus of radius R around a geodesic of length ℓ is

$$\begin{aligned} V &= \int_0^R \int_0^{2\pi} \int_0^{\ell} \sinh\rho \cosh\rho \, dt \, d\theta \, d\rho \\ &= \pi\ell \sinh^2 R \end{aligned}$$

Similarly, we get the area of the boundary of the tubular neighborhood to be

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^{\ell} \sinh R \cosh R \, dt \, d\theta \\ &= 2\pi\ell \sinh R \cosh R \end{aligned}$$

Note that when we look at the isoperimetric quotient for the hyperbolic tube, the length cancels out, thus the quotient is only dependent on the radius of the embedded tube. Meyerhoff has shown in [M1] that as the length goes to zero the radius of the embedded tube goes to infinity.

Lemma 3.7: Let M be a hyperbolic 3-manifold with an embedded tube of radius R. Then the isoperimetric constant of M is bounded above as follows:

$$h(M) \leq \frac{2\pi\ell \sinh R \cosh R}{\pi\ell \sinh^2 R} = 2\coth R$$

Note that as the radius of the embedded tube goes to infinity, the right hand side approaches 2, which is the isoperimetric constant for hyperbolic 3-space.

We will end this section with one last upper bound for the isoperimetric constant. This one comes from Theorem 3.3 by applying Cheeger's inequality $\frac{1}{4}h^2 \leq \lambda_1$.

Theorem 3.8: Let M be a compact hyperbolic 3-manifold with diameter D. Then the isoperimetric constant of M is bounded above as follows:

$$h(M) \leq 2\left((\coth\frac{D}{4} - 1) + \sqrt{\left(1 + \frac{16\pi^2}{D^2} + (\coth\frac{D}{4} - 1)^2 \right)} \right)$$

Chapter 4

Applications of the bounds

In this section we will consider various combinations and applications of the bounds obtained in the previous two chapters. The initial reason for investigating these bounds was to get upper and lower bounds for the first eigenvalue of a hyperbolic 3-manifold in terms of various geometric data which could be combined so that the spectral data, namely the first eigenvalue, would drop out leaving an inequality involving only geometric data. In particular we hoped to obtain an improved lower bound for the volume of a compact hyperbolic 3-manifold. Unfortunately, the bounds we have obtained were not sharp enough to do this. However, we do obtain some interesting inequalities concerning other geometric quantities.

Schoen's lower bound for the first eigenvalue of a hyperbolic manifold is quite general, and not very sharp. Explicitly he gets:

Theorem 4.1: [S]. The first eigenvalue of a compact hyperbolic 3-manifold satisfies the inequality:

$$\lambda_1 \geq \frac{8.76 \times 10^{-15}}{Vol(M)^2}$$

Adding a topological hypothesis we obtain:

Theorem 4.2: The first eigenvalue of a compact hyperbolic 3-manifold with first Betti number greater than two satisfies the following inequality:

$$\lambda_1 \geq \frac{.2246}{Vol(M)^2}$$

Proof: Since we are assuming that the first Betti number is greater than two, we can apply the result of Culler and Shalen that $\mu(M) \geq \frac{1}{2}\log 3$. Then we apply Theorem 2.10.

We will now consider the possible values for the isoperimetric constant of a hyperbolic 3-manifold.

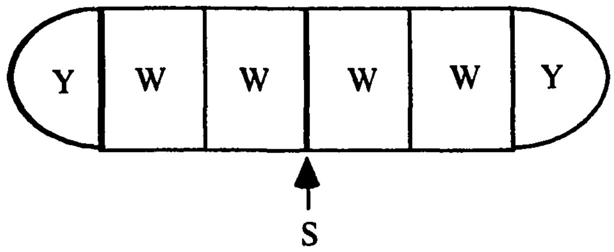
Theorem 4.3: There exists compact hyperbolic 3-manifolds with arbitrarily small isoperimetric constant.

Proof: Kojima and Miyamoto [KM] proved that the smallest hyperbolic 3-manifold with totally geodesic boundary has volume 6.452...and show how to construct explicit examples of hyperbolic 3-manifolds with totally geodesic boundary. Let Y be a hyperbolic 3-manifold with a totally geodesic boundary which is a surface of genus 2. Let W be a hyperbolic 3-manifold with totally geodesic boundary consisting of two disjoint surfaces of genus 2. Let all of these boundary surfaces be isometric. Now for each $k \geq 1$ consider the manifold M_k:

$$M_k = Y \cup_{\partial} \underbrace{W \cup_{\partial} \cdots \cup_{\partial} W}_{2k} \cup_{\partial} Y$$

Where the boundaries are identified by isometries so that M_k possesses a hyperbolic structure. Now consider the surface S in the manifold which divides M into two equal pieces, as in the figure below:

M=



Since S is a totally geodesic surface of genus 2, the area of S is equal to 4π. By the result of Kojima and Miyamoto mentioned above the volume of Y and the volume of W must be greater then, say, 6. Thus the volume of a domain in M_k bounded by S is at least $6k+6$. So we get the following:

$$h(M_k) \le \frac{Vol(S)}{\frac{1}{2}Vol(M_k)} \le \frac{4\pi}{6k+6} = \frac{2\pi}{3k+3}$$

As k goes to infinity, $h(M_k)$ approaches zero. Therefore, there are hyperbolic 3-manifolds with arbitrarily small isoperimetric constant. Note that, as must happen, the volumes of these manifolds must approach infinity. Applying Buser's inequality we get the following:

Corollary 4.4: There exist compact hyperbolic 3-manifolds with arbitrarily small first eigenvalue.

Thus, there are no nonzero universal lower bounds for neither the isoperimetric constant nor the first eigenvalue of a compact hyperbolic 3-manifold. Now we will consider universal upper bounds for these quantities. We have already shown that universal upper bounds exist, so now we will look more closely at these bounds.

We have already shown that for all hyperbolic 3-manifolds the isoperimetric constant is bounded above by 57.713. It follows from Theorem 3.8 that the isoperimetric constant of a compact hyperbolic 3-manifold is bounded from above by the diameter of the manifold. We can use this to prove the following:

Theorem 4.5: Let M be a compact hyperbolic 3-manifold. If $h(M) \geq 50$ then the diameter D of M is bounded from above:

$$D \leq .65$$

Proof: By applying theorem 3.8 we get:

$$50 \leq h(M) \leq 2\left((\coth\frac{D}{4} - 1) + \sqrt{\left(1 + \frac{16\pi^2}{D^2} + (\coth\frac{D}{4} - 1)^2\right)}\right)$$

which, since the right hand side is monotone decreasing, implies that $D \leq .65$.

Note the hyperbolic 3-manifold with the smallest known diameter is the manifold obtained by (5,1)-surgery on the figure-8 knot. This manifold has diameter no less than .7485.

We get better bounds by considering the injectivity radius. It is conjectured that the smallest hyperbolic 3-manifold is the Weeks' manifold which has volume .9472. If it could be shown that this manifold had the smallest possible maximal injectivity radius then the upper bound could be considerably lower. We will formulate the following:

Proposition 4.6: If the Weeks' manifold has smallest maximal injectivity radius among hyperbolic 3-manifolds then for any compact hyperbolic 3-manifold M, we have the following:

$$h(M) \leq 5.987$$

Proof: Using calculations from the computer program SnapPea, we find that the Weeks' manifold has a maximal injectivity radius of .51916. Computing the isoperimetric quotient for a ball of radius .51916 gives us the upper bound.

Note that there is an algorithm for finding the hyperbolic 3-manifold of smallest diameter and volume. Thurston has a theorem which says that there exists a constant β such that every compact hyperbolic 3-manifold of volume at most V can be obtained by doing hyperbolic Dehn filling on a manifold composed of at most βV ideal tetrahedra (see [T]). There are only a finite number of manifolds obtained by gluing a finite number of tetrahedra. Each of these manifolds admits only a finite number of fillings with diameter less than a given constant. Therefore, one could enumerate all of the finite number of possibilities. Unfortunately, the numbers grow exponentially and the constant β is currently very large, so this algorithm is currently not of practical value. If this algorithm could be made practical then we could calculate the universal upper bound for the isoperimetric constant of a hyperbolic 3-manifold.

Chapter 5

Some calculations of examples

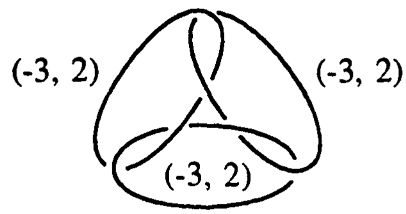
In this last section we will look at some specific examples. Weeks' computer program SnapPea has been evolving since its conception in the early 1980's. Weeks and Hodgson [WH] and many others have been doing extensive computer investigations of hyperbolic 3-manifolds. We will use computations from SnapPea to calculate various geometric data. We will then use this data to give explicit bounds on the isoperimetric constant and the first eigenvalue of these examples.

There are only two known examples of hyperbolic 3-manifolds with volume less than one. The smallest is the Weeks' manifold which was discovered and is discussed by Weeks in his Ph.D. thesis at Princeton. The next smallest is the manifold obtained by (5,1) surgery on the figure-8 knot.

The following tables include a description of the manifold as surgery on a link in the 3-sphere. The volume, Chern-Simons invariant, first homology, and isometry group of the manifold are given. Next, the length spectrum, which is the list of complex lengths (i.e. real or translation lengths plus torsion or rotation angles) of all geodesics of real length less than two with multiplicities. Then we give the maximum and minimum values for the injectivity radius of the manifold, followed by the diameter. This collecton of data is then used in the various bounds discussed earlier to give the best known ranges for the first eigenvalue and isoperimetric inequality. Lastly, we include a picture of the Dirichlet domain of the manifold. Much of this information is discussed in the paper [HW2].

The Weeks' Manifold:

Surgery description:



Vol = 0.942707362776928   CS = 0.060043066678727

H_1 = $Z_5 \oplus Z_5$ Isom = D_6

length spectrum:

multiplicity	length		torsion	
3	0.5846036850179870	+	i 2.4953704555604684	*
3	0.7941346629918657	-	i 2.3048568167434712	
3	1.2898511615162902	+	i 2.4839246176639380	

Injectivity Radius:
Min: .292301   Max: .519162

Diameter: $.75247 \le D \le 1.5049$

Spectral Data:

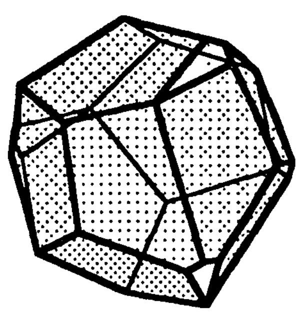
First Eigenvalue:

$$.8064 \le \lambda_1 \le 382.223$$

Isoperimetric Constant:

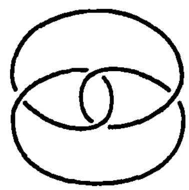
$$1.796 \le h \le 5.987$$

Dirichlet domain:



(5,1) Surgery on the Figure-8 Knot:

Surgery description:



Vol = 0.981368828892232 CS = 0.077038180263770
H_1 = Z_5 Isom = D_2
length spectrum:

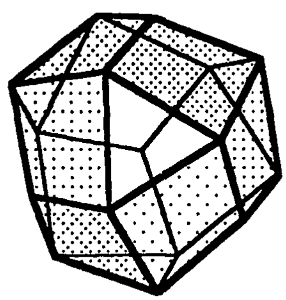
multiplicity	length		torsion	
1	0.5780824355318555	+	i 2.1324306361541782	
1	0.7215683663144467	-	i 1.1512129898755381	*
2	0.8894429972125508	+	i 2.9418590470227377	
2	0.9983251885367372	-	i 2.9210177857853122	
1	1.0403151250687664	+	i 0.9823718902812068	
2	1.7938008425243408	-	i 1.5568710482226091	
1	1.8222796995026176	-	i 2.4135390256883349	

Injectivity Radius:
Min: .289041 Max: .535437

Diameter: $.748537 \le D \le 1.49707$

Spectral Data:
First Eigenvalue:
$$.8421 \le \lambda_1 \le 361.624$$
Isoperimetric Constant:
$$1.8354 \le h \le 5.8168$$

Dirichlet Domain:



The Seifert-Weber Dodecahedral Manifold:

Surgery description:

(None Known)

Vol = 11.199064740814610 CS = ???

$H_1 = Z_5 \oplus Z_5 \oplus Z_5$ Isom $= S_5$

length spectrum:

multiplicity	length		torsion
15	1.7456695380470224	-	i 2.5578465804258377
12	1.9927689956946324	-	i 1.8849555921538761
10	2.6186693070705335	+	i 0.6951772170489633

Injectivity Radius:

Min: .872889 Max: 1.054

Diameter: $1.5179 \le D \le 3.0358$

Spectral Data:

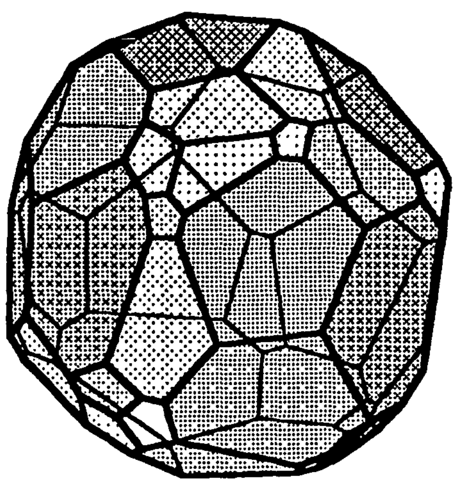
First Eigenvalue:

$.1146 \le \lambda_1 \le 119.51$

Isoperimetric Constant:

$.67727 \le h \le 3.2686$

Dirichlet domain:



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Vita

Patrick James Callahan was born in Rochester, New York on the 18th of November in the year 1968. He spent most of his childhood in Coronado, California. He entered the University of California at San Diego in 1986 and graduated three years later, eager to continue his mathematical education at the University of Illinois at Urbana-Champaign. A few days before driving to Illinois he married Susan A. Callahan. The next year he had earned his Masters of Science degree. Four years later he finished up his Doctor of Philosophy degree in Mathematics. A few months before finishing his degree, Patrick and Susan had their first daughter, Cassandra Aurelia Callahan. In August of 1994, he will start his professional career at the University of Texas at Austin where he will hold the R.H. Bing Professorship.