MEGA-BIMODULES OF TOPOLOGICAL POLYNOMIALS:
SUB-HYPERBOLICITY AND THURSTON OBSTRUCTIONS

BY

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DISSERTATION

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Abstract

In 2006, Bartholdi and Nekrashevych solved a decade-old problem in holomorphic dynamics by creatively applying the theory of self-similar groups. Nekrashevych expanded this work in 2009 to define what we refer to as *mega-bimodules* which capture the topological data of Hurwitz classes of topological polynomials. He also showed that proving that these mega-bimodules are sub-hyperbolic will have two important implications: that all iterated monodromy groups of topological polynomials are contracting and that the Hubbard-Schliecher spider algorithm for complex polynomials generalizes to topological polynomials. We prove sub-hyperbolicity in the simplest non-trivial case and apply these mega-bimodules to holomorphic dynamics to prove a partial converse to the Berstein-Levy Theorem proved in 1985.

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Chapter 1

Introduction

1.1 Self-similar groups

A group of automorphisms of an infinite rooted d-ary tree is said to be *self-similar* if the restriction of the action of any group element on the subtree below any vertex (which is isomorphic to the entire tree) is another element of the group. This element is called the *restriction* of the original group element at that vertex. Equivalently: a group is self-similar if it can be generated by a finite-state automaton.

The prototypical self-similar group is the Grigorchuk group, introduced in 1980 [Gri80]. This group is *contracting*; that is, if we fix a vertex sufficiently far from the root of our tree, we will have that the mapping that takes a group element to its restriction at the fixed vertex is decreasing in terms of the wordlength of the group. Grigorchuk used the contracting property to prove a variety of interesting results about his group. For instance, it was the first example of a group whose growth is superpolynomial and subexponential (called intermediate growth) [Gri84]. The study of self-similar groups grew out of the power of the techniques Grigorchuk employed.

A decade after Grigorchuk introduced his group, Fabrykowski and Gupta defined their own group with intermediate growth [FG91]. Bartholdi and Grigorchuk studied the Schreier graphs of the action of this group on the levels of the tree and found that these graphs converge to a fractal set [BG00]. This work helped to inspire Nekrashevych to define the limit set of a contracting self-similar group. He then related these self-similar limit sets to the fractal Julia sets of post-critically finite complex rational maps by defining the *iterated monodromy group* (IMG) of a such a map. The iterated monodromy group encodes the backwards dynamics of rational and similar maps as the action of lifts of loops with a basepoint on preimages of that basepoint. Nekrashevych showed that the limit set of the IMG of a rational map is homeomorphic to the map's Julia set [Nek05], which is especially interesting since the Julia set is constructed using the *forward* dynamics of the map. Earlier, Pilgrim had also considered a monodromy action by an absolute Galois group on the set of Belyi polynomials [Pil00].

Iterated monodromy groups have proved to be a rich source of interesting groups. Grigorchuk and Żuk

have studied the properties of $IMG(z^2-1)$, also known as the Basilica group [GŻ02a], [GŻ02b]. Their work, along with a result of Bartholdi and Virág, shows that this group is an example of a new kind of amenability [BV05]. Bux and Pérez have studied the properties of $IMG(z^2+i)$ and shown it to have intermediate growth like Grigorchuk's group [BP06]. Additionally, the Fabrykowsi-Gupta group is, in fact, an iterated monodromy group of a cubic polynomial [Nek05].

1.2   Complex dynamics

In view of the connection between iterated monodromy groups and self-similar groups, holomorphic dynamics now also serves as a source for building interesting groups. The modern theory of complex rational maps began with the work of Fatou and Julia during World War I [Jul18], [Fat19], [Fat20]. Their theory (exposited well in [Bla84]) shows that for f a complex rational function of degree $d \geq 2$, the Riemann sphere splits into two disjoint sets (now known as the Fatou and Julia sets of f) based on the forward dynamics of f. In the 1970s, Mandelbrot used computers to produce remarkable images of these sets and of the separating curves between them [Man82]. Mandelbrot particularly studied the Julia sets of quadratic complex polynomials, and he characterized when these sets are connected (this yielded the famous Mandelbrot set).

The early 1980s saw an explosion of research in this area, due in part to Mandelbrot's work. Douady and Hubbard made important advances in the theory of complex polynomials, especially quadratics [DH82], [DH84]. To this day, quadratic polynomials remain the best understood in this theory (see e.g. [BS02]). However, the work of Bielefeld, Fisher, and Hubbard on preperiodic polynomials along with the work of Poirer on periodic polynomials have greatly improved our understanding outside the quadratic case [BFH92], [Poi09]. Non-polynomial rational maps have proved more difficult to study; most results deal only with those maps with low degree and small post-critical set (see e.g. [B$^+$00]).

In the 1980s, Douady and Hubbard employed a procedure now know as *mating* to combine pairs of quadratic polynomials to produce quadratic rational maps. This would allow them to apply to rational functions their machinery for polynomials. Interestingly, they found that mating some pairs of quadratic polynomials does not produce rational maps, and so the question arose as to when two quadratic polynomials are 'mateable'.

To answer the mateablity question, Thurston considered a family of branched covering maps from the sphere to itself that topologically resemble complex rational maps whose critical points have finite forward orbits (we call the finite forward orbits the set of *post-critical* points). These maps became commonly referred to as *Thurston maps*, and Thurston characterized when these maps are equivalent to complex rational maps

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by the existence or non-existence of obstructing multicurves called *Thurston obstructions* (see Theorem 2.3.5 in this paper) [DH93]. Researchers have also considered *topological polynomials*, which are Thurston maps that behave like complex polynomials with finite post-critical set.

The mateability of quadratic complex polynomials has since been solved by Rees, and others have contributed to this general area [Ree86] [Tan92] [Shi00]. However, we remain interested in Thurston's theory partially because of its implications outside of complex dynamics. In fact, the concepts in the preceding paragraph have analogues in the theory of three-manifolds. For instance, a Thurston map admitting an obstruction is analogous to a compact, oriented, irreducible three-manifold having a nonperipheral incompressible torus. For more details regarding this connection, see the survey papers of McMullen: [McM91] [McM94]. Thurston proved his characterization and rigidity theorem using Teichmüller theory, so naturally links exist there as well.

Much about Thurston obstructions remains mysterious. While producing an obstruction for a specific example may not be difficult, no one has yet discovered an algorithm for determining the existence or non-existence of an obstructing multicurve in the general setting (although Pilgrim has found that if an obstruction exists, it must be of a canonical form) [Pil01]. The Berstein-Levy theorem for hyperbolic topological polynomials (see Theorem 2.3.12) remains the best result for the *non*-existence of an obstruction: If the forward orbit of every critical point contains a critical point, then the topological polynomial is equivalent to a complex polynomial [Lev85]. More recently, Kameyama and Pilgrim have established algebraic criteria for Thurston equivalence of rational maps, but a general algorithm remains elusive [Kam01] [Pil03a].

Topological polynomials which admit obstructions have not seen much study. Usually, researchers use Thurston's characterization to construct complex polynomials using purely topological methods. However, Haïssinsky and Pilgrim [HP09] and Bonk and Meyer [BM] have used obstructed topological polynomials to define metrics on the sphere that are not quasisymmetric to the standard sphere. Such metrics interest analysts who seek geometric criteria for quasisymmetric equivalence to the standard sphere. The motivation for this problem comes from Cannon's Conjecture (see e.g. [BK02]).

Much about the dynamics of a topological polynomial can be determined by looking only on how the polynomial acts on the set of its critical and post-critical points (recall that this is a finite set). We call this the *mapping scheme* of the polynomial. If a cycle in the mapping scheme of a polynomial contains a critical point, then that critical point will be attracting. For this reason, we call such a cycle an *attractor*. With this language, we can interpret the Bernstein-Levy result as: Any polynomial whose mapping scheme has all cycles attractors is equivalent to a complex polynomial. Naturally, the question arises as to how many non-attractor cycles (and what kinds) are necessary to support an obstructed topological polynomial.

Our main result serves as a partial converse to the Berstein-Levy Theorem:

Theorem 1.2.1. *Suppose that a polynomial mapping scheme satisfies one of the following conditions:*

1. *at least one (non-attractor) period of length at least two and not containing critical values,*

2. *at least two (non-attractor) periods not containing critical values,*

3. *at least two non-attractor periods both of length at least two, or*

4. *at least four non-attractor periods.*

Then this scheme is realized by a topological polynomial that is not equivalent to any complex polynomial.

This result covers most of the mapping schemes not satisfying the hypotheses of the Berstein-Levy Theorem. Our methods work for more mapping schemes than those explicitly listed in Theorem 1.2.1; in fact, we can show that most mapping schemes with three non-attractor periods are realized by topological polynomials not equivalent to any complex polynomial. We discuss this more in Section 4.3.

1.3 Mega-bimodules of topological polynomials

To prove Theorem 1.2.1, we use new tools that Bartholdi and Nekrashevych recently developed to tackle questions of Thurston equivalence and Thurston obstructions [BN06].

Using the iterated monodromy groups of topological polynomials satisfying a particular mapping scheme, Bartholdi and Nekrashevych defined a new self-similar group of actions of the pure mapping class group on these polynomials and proved that it is contracting. Since the restriction map on this group leaves the Thurston equivalence class invariant, the contracting property allowed them to algebraically determine the Thurston equivalence class of these topological polynomials.

Nekrashevych continued this work to provide a description of topological polynomials and their iterated monodromy groups in terms of twisted kneading automata [Nek09]. These automata encode all the topological data needed to determine the Thurston equivalence class of the topological polynomial. Further, every self-similar group generated by a twisted kneading automaton is isomorphic to the iterated monodromy group of some topological polynomial. This characterization allows us to translate topological and dynamical questions about these polynomials into algebraic questions which we can answer with explicit computations in these groups.

We utilize this characterization of topological polynomials to prove Theorem 1.2.1 and produce many examples of obstructed topological polynomials realizing these mapping schemes. We determine whether

multicurves are obstructions to a polynomial f by considering the Dehn twists about the curves as elements of the self-similar group constructed from the iterated monodromy groups of polynomials with isomorphic mapping scheme to f. If the restriction map acts cyclically on these Dehn twists, then the multicurve must be an obstruction.

These algebraic calculations take place in Nekrashevych's generalization of the self-similar group of actions of the pure mapping class group on topological polynomials developed in [BN06]. These actions are of a supergroup of the pure mapping class group: the pure symmetric automorphism group of a free group. This group itself has been the subject of much study (see e.g. [McC86], [GK98], [Bar04]) and is the group of motions of the trivial n-link in 3-space. From the self-similar action of a quotient of this group, Nekrashevych obtained what we will refer to as the *mega-bimodule* $\mathfrak{G}_f$ of a topological polynomial f.

Nekrashevych constructed these mega-bimodules to provide a combinatorial version of an analytic algorithm of Hubbard and Schleicher known as the spider algorithm [HS94]. This spider algorithm analyzes complex polynomials by considering their actions on a family of curves that meet at infinity: a so-called spider. These actions on spiders are analogous with the kneading automata in our setting. The combinatorial spider algorithm of Nekrashevych (if it terminates) takes a twisted kneading automaton of a topological polynomial and produces a simpler one. It would also allow some proofs related to iterated monodromy groups to avoid analytic arguments that morally should not be necessary. He asked:

Question 1. [Nek09]

For f a topological polynomial, is the mega-bimodule $\mathfrak{G}_f$ sub-hyperbolic?

If the answer to Question 1 is positive, then the combinatorial spider algorithm will terminate in all cases. This question is also interesting from a group-theoretic perspective to prove the following conjecture:

Conjecture 1.3.1. *[Nek09]*

The iterated monodromy group of a topological polynomial is contracting.

Once we have the contracting property in these groups, we will obtain many useful and interesting results by applying methods similar to those used by Grigorchuk in the study of his group. For example, there exists an algorithm that solves the word problem in a contracting group in polynomial time.

Proposition 1.3.2. *[Nek09]*

For f a topological polynomial, if $\mathfrak{G}_f$ is sub-hyperbolic, then $IMG(f)$ is contracting.

We give a positive answer to Question 1 in the simplest non-trivial case with the following result:

Theorem 1.3.3. *If f is a quadratic topological polynomial that is periodic with period length $n = 4$, then $\mathfrak{G}_f$ is sub-hyperbolic.*

To prove Theorem 1.3.3, we use a generating set of the pure symmetric group defined in [MM96] and [GK98] and develop an augmented length function on this generating set. After describing the restriction maps of the self-similar action of this generating set in terms of Boolean operations on finite sets, we use a simple computer program to implement thousands of computations and obtain the result.

While Theorem 1.3.3 only deals with the period length $n = 4$ case, we prove that this generating set is closed under the restriction maps for *all* periodic quadratic topological polynomials. This property lends itself much better to a proof of contraction than with the traditional generating set, since the restriction maps can often be badly increasing in terms of wordlength in the traditional generating set.

1.4   Future work

While Theorem 1.2.1 does not provide a complete categorization of mapping schemes in terms of their realizability by obstructed topological polynomials, we do discuss some aspects of mappings schemes requiring further study in order to prove such a characterization.

We do not provide any criteria for the *non*-realizability by obstructed polynomials. Researchers in holomorphic dynamics have sought related results for decades. Nekrashevych has found such a criterion for a topological polynomial whose iterated monodromy group is generated by a kneading automaton (see Sections 2.1 and 2.5) [Nek05]. We would like to generalize his criterion and connect it with the methods used here.

We are in the process of extending Theorem 1.3.3 to the quadratic period five case and some cubic periodic cases. Our goal is to generalize from the nuclei of these mega-bimodules to a conjectured description of the nuclei for other mega-bimodules.

We would also like to study the mating of polynomials using the mega-bimodule technology. The mating of polynomials into a Thurston map induces injections on the level of iterated monodromy groups. We believe that this algebraic information can help determine when mating different pairs of polynomials results in the same Thurston map. Also, recent work by Meyer has shown that expanding Thurston maps without periodic critical points can always be realized as matings of topological polynomials [Mey], so a better understanding of matings could help produce a characterization of the bimodules of expanding Thurston maps.

Finally, Pilgrim has recently employed similar methods to obtain results on the backwards iteration of polynomials on simple closed curves [Pil]. Pilgrim, Thurston, and others believe that the dynamics of simple closed curves under the backwards iteration of Thurston maps could provide an alternative characterization of equivalence to complex maps. We would like to use the techniques described in this paper to duplicate

and extend Pilgrim's results.

Chapter 2

Preliminaries

2.1   Groups, automata, and bimodules

In this section we give some of the standard definitions and results in the theory of self-similar groups (see [Nek05] for a complete introduction to this theory) and introduce some of the key concepts from [Nek09].

For T an infinite, rooted, d-ary tree and v a vertex of the tree, we write T_v for the subtree of T below v. Note that $T_v \simeq T$. For $g \in Aut(T)$, we write $g|_v$ for the action of g from T_v to $T_{g(v)}$.

Definition 2.1.1. A faithful action of a group G on a tree T by automorphisms is *self-similar* if for any vertex $v \in T$, the action of $g|_v$ from T_v to $T_{g(v)}$ is equal to the action of some $h \in G$ when we identify T_v and $T_{g(v)}$ with the entire tree T.

A group with a self-similar action on a tree is said to be a *self-similar group*. Equivalently, we can also define a self-similar group as one generated by an automaton

Definition 2.1.2. An *automaton* A over an alphabet X is given by a set of input states A, a set of output states B, and a transition map $\tau : A \times X \to X \times B$.

If $\tau(a, x) = (y, b)$, then we write $a \cdot x = y \cdot b$ and use notation $y = a(x)$ and $b = a|_x$. We say that b is the *restriction* of a at x.

Definition 2.1.3. We say that an automaton is a *group automaton* if for every $a \in A$ the map $x \mapsto a(x)$ is a permutation of the alphabet (we assume for group automatons that there exists trivial state $\mathtt{I} \in A, B$ such that $\mathtt{I} \cdot x = x \cdot \mathtt{I}$ for all $x \in X$).

We represent a group automaton as a labeled directed graph called an *abbreviated Moore diagram* with vertex set equal to the states $A \cup B$ and with a directed edge from a to b labeled by x if and only if $a \cdot x = y \cdot b$ for some $y \in X$. We also label the states by the permutations they induce on X. For simplicity, we omit the trivial state $\mathtt{I}$.

In Figure 2.1 we give an example of the abbreviated Moore diagram of the group automaton associated with Grigorchuk's group. This automaton has $A = B = \{\mathtt{I}, a, b, c, d\}$, $X = \{0, 1\}$, and transition function:

$$a \cdot 0 = 1 \cdot \mathtt{I} \qquad a \cdot 1 = 0 \cdot \mathtt{I}$$

$$b \cdot 0 = 0 \cdot a \qquad b \cdot 1 = 1 \cdot c$$

$$c \cdot 0 = 0 \cdot a \qquad c \cdot 1 = 1 \cdot d$$

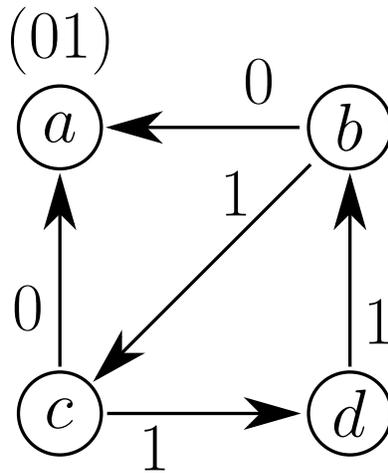$$d \cdot 0 = 0 \cdot \mathtt{I} \qquad d \cdot 1 = 1 \cdot b$$

Figure 2.1: Example abbreviated Moore diagram

A convenient way to describe the action of each of the states of a group automaton is with *wreath recursive notation*. For alphabet $X = \{0, 1, ..., d-1\}$, we represent a state a by $\pi_a(a|_0, a|_1, ..., a|_{d-1})$ where the restrictions are as defined above and π_a is the element of the symmetric group of X induced by the action of a (i.e. $a(x) = \pi_a(x)$ for all $x \in X$). We will omit π_a when it is trivial (we say such a state is *inactive*) and we omit the restrictions if they are all the trivial state $\mathtt{I}$. Here is the wreath recursive notation for the generators of the Grigorchuk group:

$$a = (01) \qquad b = (a, c) \qquad c = (a, d) \qquad d = (1, b)$$

We can think of X^* (the set of words in the finite alphabet X) as an infinite, rooted, d-ary tree (where d is the size of the alphabet). The root vertex is the empty word, the first level of vertices are the letters of X, and each word $w \in X^*$ is adjacent to the d vertices in the level below it of the form wx for $x \in X$. We present the beginning of the binary tree for $X = \{0, 1\}$ in Figure 2.2.
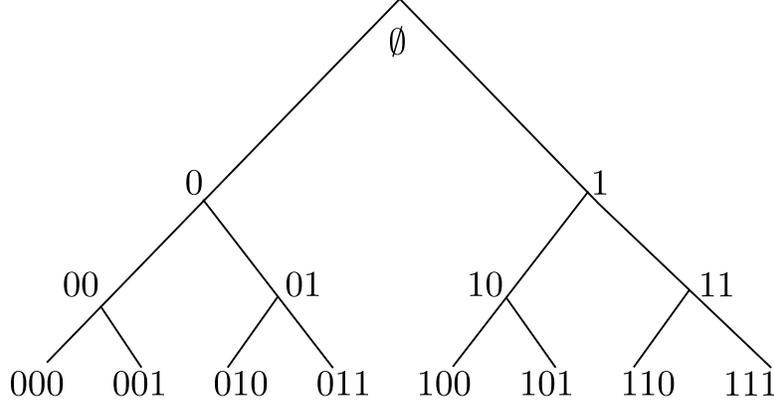
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Figure 2.2: The first three levels of the binary tree

When $B \subset F(A)$ (the free group on the input states A), we have an action of $F(A)$ on the tree X^* by graph automorphisms as follows. For $a \in A$, we have the action on the word $w = xw' \in X^*$ given by $a(xw') = a(x)a|_x(w')$. Notice that $a|_x \in B \subset F(A)$ and thus can be written as a word in $A^{\pm 1}$, so the action of $a|_x$ on w' is well-defined. The wreath recursion $a = \pi_a(a|_0, a|_1, ..., a|_{d-1})$ describes the automorphism of X^* in the following way: π_a gives the action of a on the first level of the tree, and the restrictions $a|_i$ describe the actions on the subtrees (which are isomorphic to the entire tree).

We can compute the wreath recursive notation for the product (as elements of $F(A)$) of $a, b \in A$ where $a = \pi_a(a|_0, a|_1, ..., a|_{d-1})$ and $b = \pi_b(b|_0, b|_1, ..., b|_{d-1})$ by:

$$ab = \pi_a \pi_b(a|_{\pi_b(0)} b|_0, a|_{\pi_b(1)} b|_1, ..., a|_{\pi_b(d-1)} b|_{d-1}).$$

While automata provide nice, compact descriptions for self-similar actions, a different context makes computations easier. We use algebraic objects called bimodules (sometimes called bisets) to encode the automata.

Definition 2.1.4. For G and H groups, a *permutational (G-H)-bimodule* is a set $\mathfrak{M}$ with a left action of G and a right action of H which commute. A *covering bimodule* has free right action with only finitely many orbits. We call a $(G\text{-}G)$-bimodule simply a G-bimodule.

We say that two $(G\text{-}H)$-bimodules are isomorphic if there exists a bijection between them that agrees with the the actions (that is, a bijective map F such that for all $g \in G, h \in H$ and x in the domain bimodule we have $F(g \cdot x \cdot h) = g \cdot F(x) \cdot h$).

For $\mathfrak{M}_1$ a $(G_1\text{-}G_2)$-bimodule and $\mathfrak{M}_2$ a $(G_2\text{-}G_3)$-bimodule, we may form the *tensor product* $\mathfrak{M}_1 \otimes \mathfrak{M}_2$

which is the $(G_1\text{-}G_3)$-bimodule equal to the quotient of $\mathfrak{M}_1 \times \mathfrak{M}_2$ by the equivalence

$$(x_1 \cdot g_2) \otimes x_2 = x_1 \otimes (g_2 \cdot x_2)$$

for all $g_2 \in G_2, x_1 \in \mathfrak{M}_1, x_2 \in \mathfrak{M}_2$. The actions are defined in the natural way: $g_1 \cdot (x_1 \otimes x_2) \cdot g_3 = (g_1 \cdot x_1) \otimes (x_2 \cdot g_3)$ for all $g_1 \in G_1, g_3 \in G_3, x_1 \in \mathfrak{M}_1, x_2 \in \mathfrak{M}_2$.

It is straightforward to show that the tensor product of bimodules is an associative operation and that the tensor product of covering bimodules is again a covering bimodule.

For $\mathfrak{M}$ a covering $(G\text{-}H)$-bimodule, a *basis* of $\mathfrak{M}$ is an orbit transversal X to the right action. So if X is a basis, every element of $\mathfrak{M}$ can be written uniquely as $y \cdot h$ for some $y \in X, h \in H$. Thus, for every $g \in G, x \in X$, we have that there exists a unique pair $y \in X, h \in H$ such that $g \cdot x = y \cdot h$. We write $\mathfrak{M}^{\otimes n}$ for the tensor product of n copies of $\mathfrak{M}$. It is easy to show that if X is a basis of $\mathfrak{M}$, then X^n is a basis for $\mathfrak{M}^{\otimes n}$.

Notice that if we have a covering $(G\text{-}H)$-bimodule $\mathfrak{M}$ with basis X, we may construct an abstract automaton with set of input states G and set of output states H over the alphabet X, where for any $g \in G, x \in X$ and $y \in X, h \in H$ the unique pair such that $g \cdot x = y \cdot h$, we set $g(x) = y$ and $g|_x = h$. We call this the *complete automaton* for the bimodule. We can similarly define an automaton using as input states a generating set of G; we say that such an automaton *generates* $\mathfrak{M}$. Likewise, given a finite group automaton with $B \subset F(A)$, we may define the $F(A)$-bimodule that it generates via the action on X^* described previously.

The bimodule machinery makes easier many computations for these groups, which allows us to prove things like the contracting property.

Definition 2.1.5. A self-similar action of a group G on a tree T is *contracting* if here exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists $m \in \mathbb{N}$ such that for all vertices $v \in T$ at least m levels down from the root, then $g|_v \in \mathcal{N}$. We say that the corresponding G-bimodule is *hyperbolic*.

Definition 2.1.6. For G acting on a tree T, if the faithful quotient of this action (i.e. the quotient of G by the kernel of the action) is self-similar, then we say that the associated G-bimodule is *sub-hyperbolic*. Note that in this case, G itself is not a self-similar group.

The contracting property is very strong and allows us to construct algorithms to determine whether two automata generate the same group of automorphisms and to solve the word problem in the self-similar group in polynomial time.

We will be interested in groups generated by particular kinds of group automata, called kneading automata. We now develop the terminology necessary to understand these objects.

Definition 2.1.7. For $A = (a_i)_{i \in I}$ a sequence of permutations of a finite set X (we do not use a set of permutations since we wish to allow for repeated elements), we define the *cycle diagram* of A to be the oriented 2-dimensional CW-complex $D(A)$ with 0-cells the elements of X and for every cycle $(x_1, x_2, ..., x_n)$ of every permutation of A, we attach a 2-cell to the vertices $x_1, x_2, ..., x_n$ so that their order around the boundary of the 2-cell matches the order in the cycle. Two different 2-cells do not have any 1-cells in common.

Figure 2.3 shows three example cycle diagrams. The first is for the permutations $(1234), (12)(34)$, the second is for the permutations $(123), (134)$, and the third is for the permutations $(12)(34), (14)$.
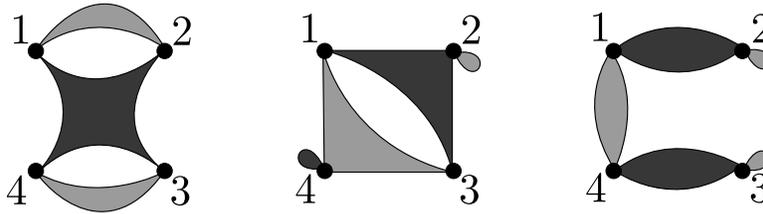

Figure 2.3: Example cycle diagrams

We say a sequence $A = (a_i)_{i \in I}$ of permutations of a finite set X is *dendroid* if its cycle diagram $D(A)$ is contractible. Notice that a dendroid sequence must be transitive, cannot have any non-trivial cycles appear more than once, and must have that any two cycles are disjoint or share only one element.

In the examples in Figure 2.3, the first two are not dendroid, but the third is.

Alternatively, we may define a hypergraph on the vertices X where each cycle of length at least two defines a hyperedge (i.e. a set containing at least two vertices). The sequence of permutations is dendroid if and only if this hypergraph is connected with no cycles.

Definition 2.1.8. We say that a group automaton with alphabet X, set of input states A, and set of output states B is *dendroid* if all three of the following conditions hold:

1. The sequence of permutations on X defined by elements of A is dendroid.

2. For every non-trivial $b \in B$, there exists a unique pair $a \in A, x \in X$ such that $a \cdot x = y \cdot b$ for some $y \in X$.

3. For every element $a \in A$ and every cycle $(x_1, x_2, ..., x_n)$ of the action of a on X, we have that $a|_{x_i} = \mathtt{I}$ for all but at most one index i.

Notice that the Grigorchuk automaton in Figure 2.1 is *not* dendriod becuase it violates condition (2) above (the state a has two incoming arrows). In Figure 2.4 we give two examples of abbreviated Moore diagrams of dendroid automata on the binary alphabet.
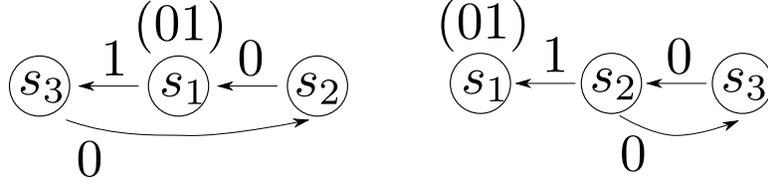

Figure 2.4: Example dendroid automata

If we place a cyclic ordering $(a_1, a_2, ..., a_n)$ on the input set of states A of a dendroid automaton, we get an induced cyclic ordering $(b_1, b_2, ..., b_m)$ on the output set of states B. For a cyclic ordering $(a_1, ..., a_n)$ on the input set A of an automaton acting on the alphabet X with $\#X = d$, then we obtain the induced ordering on the output set B by computing the restriction of $(a_n a_{n-1}...a_2 a_1)^d|_x$ for some $x \in X$. The choice of x does not affect the *cyclic* order of the resulting restriction, which will be a word using every element of B exactly once.

Using the left dendroid automaton in Figure 2.4, we see that $s_1 s_2 s_3 s_1 s_2 s_3 \cdot 0 = s_1 s_2 s_3 s_1 \cdot 0 \cdot s_1 s_2 = s_1 s_2 s_3 \cdot 1 \cdot s_1 s_2 = s_1 \cdot 1 \cdot s_1 s_2 = 0 \cdot s_3 s_1 s_2$. So the induced cyclic ordering is the same and the original ordering.

In the right dendroid automaton in Figure 2.4, we see that $s_1 s_2 s_3 s_1 s_2 s_3 \cdot 0 = s_1 s_2 s_3 s_1 \cdot 0 \cdot s_3 s_2 = s_1 s_2 s_3 \cdot 1 \cdot s_3 s_2 = s_1 \cdot 1 \cdot s_1 s_3 s_2 = 0 \cdot s_1 s_3 s_2$. So the induced cyclic ordering is the opposite of the original ordering.

The braid group on n strands, B_n acts naturally on cyclic orders of n elements. The action of the generators $\sigma_i \in B_n$ is by $(a_1, ..., a_i, a_{i+1}, ...a_n)^{\sigma_i} = (a_1, ..., a_{i+1}, a_i^{a_{i+1}}, ..., a_n)$ where $a_i^{a_{i+1}} = a_{i+1}^{-1} a_i a_{i+1}$.

Definition 2.1.9. A dendroid automaton is called a *twisted kneading automaton* if it has cyclically ordered input set $A = (a_1, a_2, ..., a_n)$ and output set B equal to conjugates of elements of A with induced cyclic ordering $(a_1, a_2, ..., a_n)^\alpha$ for some $\alpha \in B_n$, the braid group on n strands.

Twisted kneading automata with trivial twist are examples of *kneading* automata.

Definition 2.1.10. A group automaton is *kneading* if it is dendroid with set of output states B equal to set of input states A.

The automata in Figure 2.4 are kneading. The left automaton is also twisted kneading with trivial twist, but the right automaton is not twisted kneading.

In Figure 2.5 we give examples of twisted kneading automata with non-trivial twist.
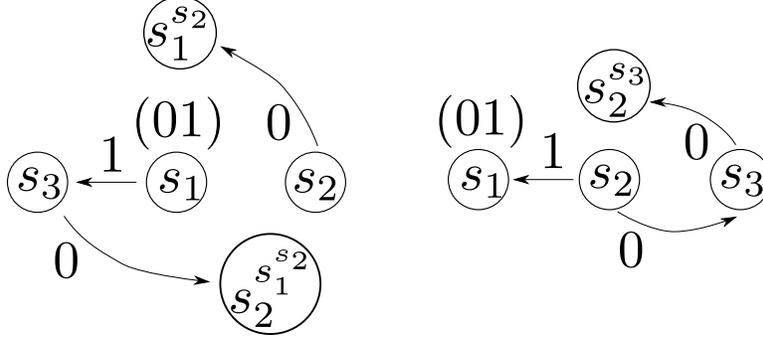
Figure 2.5: Example twisted kneading automata

We will refer to a bimodule generated by a (twisted) kneading automaton as a (twisted) kneading bimodule.

2.2 Iterated mondromy groups

In this section we give some general definitions and results as found in [Nek05].

Definition 2.2.1. For $p : \mathcal{M}_1 \to \mathcal{M}$ a covering of a path-connected and locally path-connected space $\mathcal{M}$ by a path-connected subspace $\mathcal{M}_1 \subseteq \mathcal{M}$, we define the *tree of preimages* T of a basepoint $t \in \mathcal{M}$ in the following way:

- The n-th level of T is the set $p^{-n}(t)$ of the preimages of t under the n-th iteration of p.

- Every vertex $z \in p^{-n}(t)$ is connected by an edge with the vertex $f(z) \in p^{-(n-1)}(t)$ in the previous level.

If d is the number of sheets in the covering p, then T will be a d-regular rooted tree.

Definition 2.2.2. For $p : \mathcal{M}_1 \to \mathcal{M}$ a covering of a path-connected and locally path-connected space $\mathcal{M}$ by a path-connected subspace $\mathcal{M}_1 \subseteq \mathcal{M}$, we define the *iterated monodromy action* of $\pi_1(\mathcal{M}, t)$ on the tree of preimages T (with basepoint t) as follows:

For $\gamma \in \pi_1(\mathcal{M}, t)$ and $z \in p^{-n}(t)$ a vertex, there is a unique lift $\tilde{\gamma}_z$ of γ under p^n that begins at z. The action of γ takes the vertex z to the other endpoint of the path $\tilde{\gamma}_z$, which is another element of $p^{-n}(t)$.

Notice that an element $\gamma \in \pi_1(\mathcal{M})$ acts by a permutation on each level $p^{-n}(t)$. Given a p^n-lift $\tilde{\gamma}_z$ with starting point z and terminal point z', since $p(\tilde{\gamma}_z)$ is a p^{n-1}-lift of γ with starting point $p(z)$ and endpoint $p(z')$, then this permutation action on the levels of the tree is, in fact, an automorphism of T.

One can easily show that this action does not depend (up to conjugacy) on the choice of basepoint t.

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Definition 2.2.3. For $p : \mathcal{M}_1 \to \mathcal{M}$ a covering of a path-connected and locally path-connected space $\mathcal{M}$ by a path-connected subspace $\mathcal{M}_1 \subseteq \mathcal{M}$, we define the *iterated monodromy group* $IMG(p)$ to be the quotient of $\pi_1(\mathcal{M})$ by the kernel of the iterated monodromy action.

To see that $IMG(p)$ is self-similar, we will construct the $\pi_1(\mathcal{M})$-bimodule $\mathfrak{M}_p$ associated with the iterated monodromy action.

As usual let $p : \mathcal{M}_1 \to \mathcal{M}$ be a covering of a path-connected and locally path-connected space $\mathcal{M}$ by a path-connected subspace $\mathcal{M}_1 \subseteq \mathcal{M}$. We define the bimodule $\mathfrak{M}_p$ to be the set of homotopy classes of paths starting at our basepoint t and ending at any point in $p^{-1}(t)$. The right action by an element $\gamma \in \pi_1(\mathcal{M}, t)$ is by appending the loop γ to the beginning of the path. The left action by an element $\gamma \in \pi_1(\mathcal{M}, t)$ is by appending the unique p-lift of γ to the end of the path.

Notice that the right action is free and the orbits of this action consist of all paths with a common terminal point. So if we choose paths $X = \{x_0, ..., x_{d-1}\}$ from t to each of its preimages under p, we have that X is a basis for $\mathfrak{M}_p$. The isomorphism class of $\mathfrak{M}_p$ does not depend on choice of basepoint t or basis X.

Theorem 2.2.4. *[Nek05]*

For $p : \mathcal{M}_1 \to \mathcal{M}$ a covering of a path-connected and locally path-connected space $\mathcal{M}$ by a path-connected subspace $\mathcal{M}_1 \subseteq \mathcal{M}$, T the tree of preimages, and $\mathfrak{M}_p$ as defined above, if we choose a basis X, then the action of $\pi_1(\mathcal{M})$ on X^* (recall X^n is a basis for $\mathfrak{M}_p^{\otimes n}$) is conjugate with the action of $\pi_1(\mathcal{M})$ on T.

So we can use the bimodule $\mathfrak{M}_p$ to understand the self-similar action of $IMG(p)$ on T as follows: For $\gamma \in \pi_1(\mathcal{M}, t)$ and $t_i \in p^{-1}(t)$, let $x_i \in X$ be the basis element corresponding to a path from t to t_i. We have $\gamma \cdot x_i = \gamma(x_i) \cdot \gamma|_{x_i}$ in $\mathfrak{M}_p$ for some $\gamma(x_i) \in p^{-1}(t)$ and some $\gamma|_{x_i} \in \pi_1(\mathcal{M}, t)$. So for g the image of γ in $IMG(p)$, the restriction of the action of g to the subtree below vertex t_i is the image in $IMG(p)$ of $\gamma|_{x_i}$.

Since the bimodule $\mathfrak{M}_p$ will play a crucial role in our study of topological polynomials, in Section 2.5 we describe the specific construction in this case. For now, we give an example for the complex polynomial $f(z) = z^2 - 1$.

In Figure 2.6, we see $f(z) = z^2 - 1$ acting as a double cover $f : S^2 \setminus \{\infty, 0, -1, 1\} \to S^2 \setminus \{\infty, 0, -1\}$. We include the generators s_1 and s_2 of $\pi_1(S^2 \setminus \{\infty, 0, -1\}, t)$ (where $t = \frac{1-\sqrt{5}}{2}$ is a fixed point) and their lifts under f.

The basepoint t has two preimages: itself and $-\frac{1-\sqrt{5}}{2}$, which we label as t_0 and t_1 respectively. For a basis X, we choose the constant path x_0 from t to t_0 and the path x_1 from t to t_1 shown in Figure 2.6. So we have the following action of the generators in $\mathfrak{M}_f$:
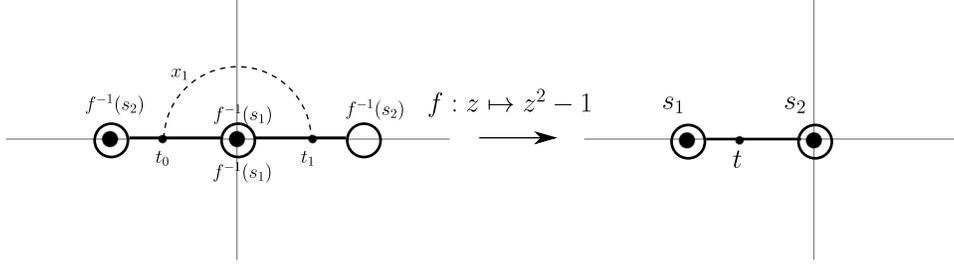
Figure 2.6: The polynomial $f(z) = z^2 - 1$

$$s_1 \cdot x_0 = x_1 \cdot s_2$$

$$s_1 \cdot x_1 = x_0 \cdot \mathtt{I}$$

$$s_2 \cdot x_0 = x_0 \cdot s_1$$

$$s_2 \cdot x_1 = x_1 \cdot \mathtt{I}$$

The abbreviated Moore diagram for the generating automaton for $\mathfrak{M}_f$ is given in Figure 2.7.
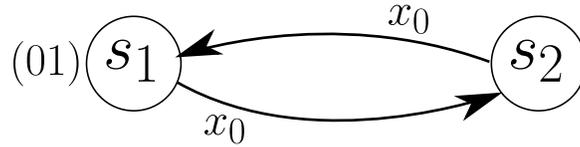

Figure 2.7: The automaton associated with $f(z) = z^2 - 1$

So $IMG(z^2 - 1)$ (also known as the *Basilica group*) has two generators: s_1, s_2, and their self-similar actions on the infinite, rooted, binary tree can be described with the wreath-recursive formulae:

$$s_1 = (01)(s_2, \mathtt{I})$$

$$s_2 = (s_1, \mathtt{I})$$

2.3   Thurston equivalence of topological polynomials

In this section, we give the standard definitions and results regarding Thurston equivalence of topological polynomials. An interested reader may find a more thorough discussion in [Pil03a].

Definition 2.3.1. For $f : S^2 \to S^2$ a branched cover of the sphere and C_f the set of its critical (i.e.

branching) points, we define the *post-critical set* of f to be the forward orbits of points in C_f. That is:

$$P_f = \cup_{\omega \in C_f} \cup_{n \geq 1} f^{\circ n}(\omega),$$

where $f^{\circ n}$ is the composition of f with itself n times.

If f is post-critically finite (i.e. if P_f is a finite set), we say that f is a *Thurston map*.

Definition 2.3.2. A Thurston map is a *topological polynomial* if there exists some $\omega \in C_f$ such that $f^{-1}(\omega) = \{\omega\}$ (we will call this point ∞).

We say that the *degree* of a topological polynomial is the number of sheets of the covering.

Definition 2.3.3. Two Thurston maps f and g are said to be Thurston equivalent (henceforth, simply *equivalent*) if there exist orientation-preserving homeomorphisms $\phi_0, \phi_1 : S^2 \to S^2$ with $\phi_0(P_f) = \phi_1(P_f) = P_g$ that are isotopic relative to P_f such that $\phi_0 f = g\phi_1$. That is, if the following diagram commutes:

$$
\begin{array}{ccc}
(S^2, P_f) & \xrightarrow{\phi_0} & (S^2, P_g) \\
f \downarrow & & \downarrow g \\
(S^2, P_f) & \xrightarrow{\phi_1} & (S^2, P_g)
\end{array}
$$

In Figure 2.8, we give diagrams in the style of [BN06] of two different topological polynomials. For these diagrams, we choose a basepoint t near infinity and draw a loop around ∞ in the negative direction. By passing to a homotopic map, we may assume that the topological polynomial takes this loop to itself by a degree d mapping (where d is the degree of the polynomial) which fixes our basepoint t. We call this loop the *circle at infinity*. The point t has d preimages: $\{t = t_0, t_1, ..., t_{d-1}\}$, all on the circle at infinity. The preimages of our circle at infinity are subpaths of the circle, starting at some t_i and ending at t_{i+1} (adding mod d).

In Figure 2.8 we include the generators $\{s_1, s_2, s_3\}$ of $\pi_1(S^2 \backslash P, t)$ and their preimages to help demonstrate the mapping. Both topological polynomials fold the horizontal line at the finite critical point, which is the preimage of ω_1 (the post-critical point surrounded by s_1).

To a Thurston map f, we associate a topological orbifold $\mathcal{O}_f$ which has underlying space S^2 and weight $\nu(x)$ at $x \in S^2$ equal to the least common multiple of the local degree of f over all iterated preimages of x. The Euler characteristic

$$\chi(\mathcal{O}_f) = 2 - \sum_{\omega \in P_f} (1 - \frac{1}{\nu(\omega)})$$

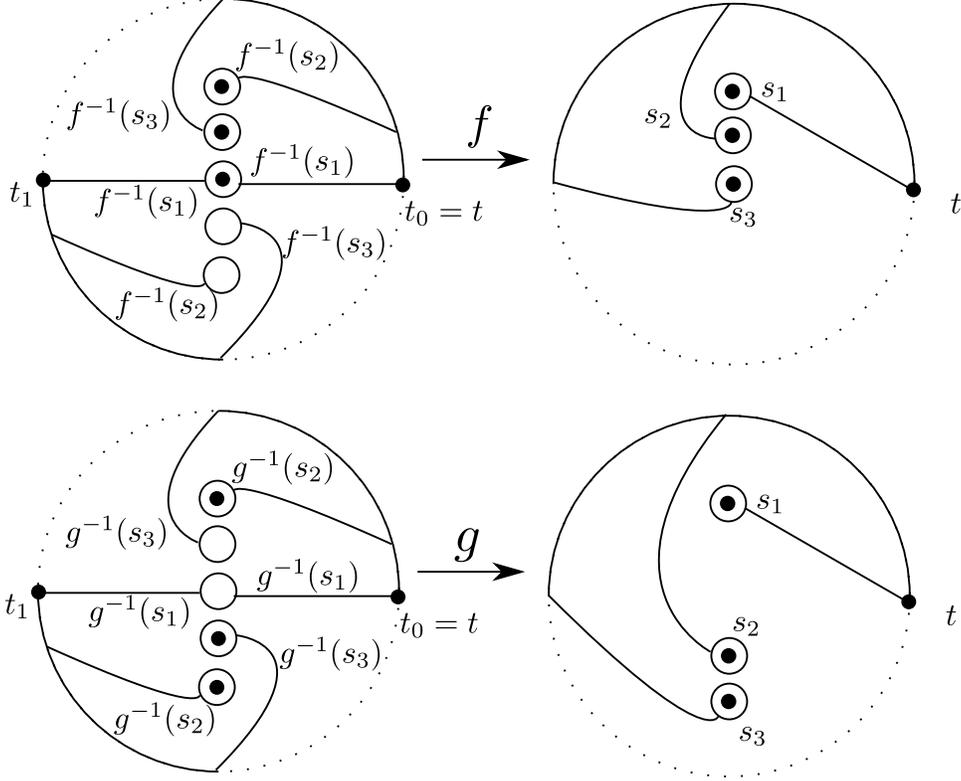of this orbifold is always non-positive. If it is negative, we say that the orbifold is *hyperbolic*.

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Figure 2.8: Example topological polynomials

Definition 2.3.4. A simple closed curve γ on $S^2 \setminus P$ is *non-peripheral* if both components of $S^2 \setminus \gamma$ contain at least two points in P.

A *multicurve* $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_n\}$ is a set of non-peripheral simple closed curves on $S^2 \setminus P_f$ that are disjoint and pairwise non-homotopic. We say that a multicurve Γ is *f-stable* if for all $\gamma \in \Gamma$, we have that every non-peripheral component of $f^{-1}(\gamma)$ is homotopic to some curve in Γ. For Γ stable under f, there is an induced map $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ given by:

$$f_\Gamma(\gamma_i) = \sum_{\delta \in f^{-1}(\gamma_i)} \frac{\delta}{\deg f|_\delta}.$$

By the Perron-Frobenius Theorem, there is a leading positive real eigenvalue $\lambda(f_\Gamma)$ that realizes the spectral radius of f_Γ.

We can now state Thurston's characterization and rigidity theorem:

Theorem 2.3.5. *[DH93] A Thurston map f with hyperbolic orbifold is equivalent to a rational function if and only if for any stable multicurve Γ, $\lambda(f_\Gamma) < 1$. In that case, the rational function is unique up to conjugation by a Möbius transformation.*

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Definition 2.3.6. A stable multicurve Γ such that $\lambda(f_\Gamma) \geq 1$ is called an *obstruction*.

Unfortunately, there is no known algorithm for determining whether a Thurston map admits an obstruction.

For topological polynomials, we can restate Thurston's theorem as follows:

Theorem 2.3.7. *[BFH92] A topological polynomial is equivalent to a complex polynomial if and only if it admits no obstructions. In that case, the complex polynomial is unique up to conjugation by an affine transformation.*

The obstructions to topological polynomials are better understood.

Definition 2.3.8. For f a topological polynomial and Γ an obstruction that it admits, then a *Levy cycle* is a set $\Gamma' = \{\gamma_0, \gamma_1, ..., \gamma_{n-1}\} \subseteq \Gamma$ such that each $f^{-1}(\gamma_i)$ has exactly one non-peripheral component $\tilde{\gamma}_{i-1}$ homotopic to a curve in Γ', $\tilde{\gamma}_{i-1}$ is homotopic to γ_{i-1}, and the map $f : \tilde{\gamma}_{i-1} \to \gamma_i$ has degree 1 (subtracting mod n).

Theorem 2.3.9. *[Lev85] For f a topological polynomial and Γ an obstruction that it admits, then Γ contains a Levy cycle.*

Note that a Levy cycle need not be stable (as in the definition in [BN06]). However, in this paper we will only consider stable Levy cycles, since they are easier to identify with our method.

The topological polynomial g in Figure 2.8 admits a Levy cycle consisting of a single curve Γ, as shown in Figure 2.9.
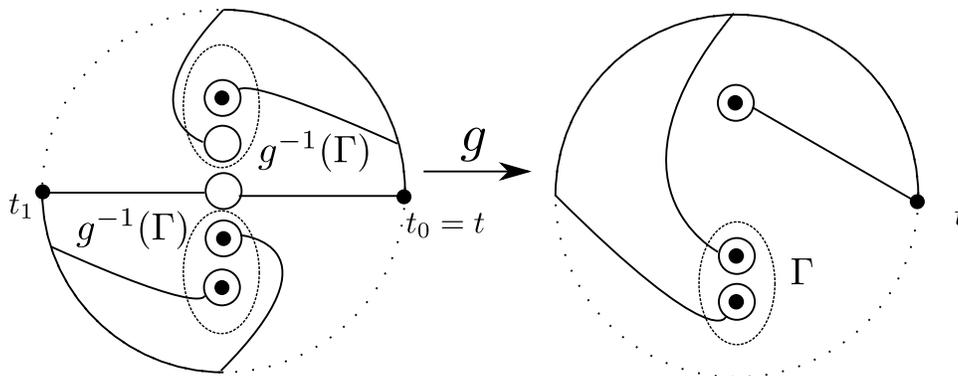

Figure 2.9: Example Levy cycle

So a topological polynomial is equivalent to a complex polynomial unless it admits a Levy cycle. But even for topological polynomials, we do not have an algorithm to determine whether an obstruction exists for a particular polynomial. However, some progress has been made.

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Definition 2.3.10. We say that a topological polynomial f is *hyperbolic* if for all $\omega \in C_f$ (recall C_f is the set of critical points), there exists some $k \geq 1$ such that $f^{\circ k}(\omega) \in C_f$.

Definition 2.3.11. We say that a topological polynomial f is *periodic* if $C_f \subset P_f$, and *preperiodic* otherwise.

Notice that a periodic polynomial is always hyperbolic.

In his thesis, Levy used a result of Berstein to prove the following:

Theorem 2.3.12. *[Lev85] If f is a hyperbolic topological polynomial, then f does not admit a Levy cycle, and so f is equivalent to a complex polynomial.*

The topological polynomial f in Figure 2.8 is periodic, and thus by Berstein-Levy must be equivalent to some complex polynomial.

2.4   Mapping schemes

Here we give the standard definitions for mapping schemes as found in [B$^+$00], but we also borrow some notation from [Koc07].

Definition 2.4.1. A *polynomial mapping scheme of degree d* is a tuple $S(C, P, \alpha, \nu)$ where:

- $Z = C \cup P$ is a finite set,

- $\alpha : Z \to P$ is surjective,

- $\nu : Z \to \mathbb{N}$ has $\nu^{-1}(\{n \geq 2\}) = C$,

such that the following conditions are satisfied:

- The Riemann-Hurwitz Formula:
$$d = \frac{\sum_{z \in Z}(\nu(z) - 1)}{2} + 1,$$

- Local degrees: $\forall z \in Z$,
$$\sum_{\alpha(x)=z} \nu(x) \leq d,$$

- Infinity: $\exists z \in C$, which we will call ∞, such that:

$$\alpha(z) = z, \nu(z) = d.$$

We will treat a mapping scheme as a finite directed graph with vertex set $Z = C \cup P$ and for each $z \in Z$, there are $\nu(z)$ directed edges from z to $\alpha(z)$.

Definition 2.4.2. A directed cycle in the graph of a mapping scheme is called an *attractor* if it contains an element of C .

Definition 2.4.3. A mapping scheme is *hyperbolic* if for all $z \in C$, there exists some $k \geq 1$ such that $\alpha^{\circ k}(z) \in C$ (i.e. if every directed cycle is an attractor).

Definition 2.4.4. A mapping scheme is *periodic* if $C \subset P$ and *preperiodic* otherwise.

Figure 2.10 shows two example mapping schemes of degree 2. The left one is hyperbolic and the right one is not.

$$\infty\circlearrowleft \quad \omega_3 \Longrightarrow \omega_1 \longrightarrow \omega_2 \qquad \infty\circlearrowleft \quad \omega_0 \Longrightarrow \omega_1 \longrightarrow \omega_2 \longrightarrow \omega_3$$

Figure 2.10: Example mapping schemes

Definition 2.4.5. We say that a topological polynomial f *realizes* a polynomial mapping scheme $S(C, P, \alpha, \nu)$ if there exists a bijection $\beta : C \cup P \to C_f \cup P_f$ such that for all $z \in C \cup P$, we have $f(\beta(z)) = \beta(\alpha(z))$ and the local degree of f at $\beta(z)$ equals $\nu(z)$.

Notice that the topological polynomials f and g from Figure 2.8 realize the mapping schemes in Figure 2.10 where s_i loops around ω_i and $\omega_0 = g^{-1}(\omega_1)$ (f realizes the hyperbolic mapping scheme and g the non-hyperbolic one).

One can easily show that equivalent topological polynomials have isomorphic mapping schemes.

A result of Thom gives the following:

Theorem 2.4.6. *[B$^+$00] For every polynomial mapping scheme, there is a topological polynomial which realizes it.*

So we can interpret the Berstein-Levy Theorem as a result about mapping schemes:

Theorem 2.4.7. *[Lev85] A hyperbolic polynomial mapping scheme is realizable only by topological polynomials that are equivalent to complex polynomials.*

In other words, a polynomial mapping scheme with every period being an attractor (i.e. containing an element of C) cannot be realized by an obstructed topological polynomial.

It is relatively easy to show that a polynomial mapping scheme with only a single finite period, and that period having length equal to one, cannot be realized by an obstructed topological polynomial.

So naturally we ask which other mapping schemes can be realized by obstructed topological polynomials.

In an unpublished result, Koch found that every unicritical (i.e. $\#(C \setminus \{\infty\}) = 1$) preperiodic polynomial mapping scheme with period length $n \geq 2$ *is* realized by an obstructed topological polynomial [Koch]. Extending these methods, one could establish topological arguments for cases 1 and 2 of our main result, which we restate below. However, we know of no topological constructions for cases 3 or 4.

Theorem 1.2.1. *Suppose that a polynomial mapping scheme satisfies one of the following conditions:*

1. *at least one (non-attractor) period of length at least two and not containing critical values,*

2. *at least two (non-attractor) periods not containing critical values,*

3. *at least two non-attractor periods both of length at least two, or*

4. *at least four non-attractor periods.*

   Then this scheme is realized by a topological polynomial that is not equivalent to any complex polynomial.

2.5 Nekrashevych's characterization of topological polynomials

In this section we continue to summarize the definitions and results of [Nek09].

For $f : S^2 \to S^2$ a topological polynomial with post-critical set $P_f = \{\infty, \omega_1, ..., \omega_n\}$, let $\{s_i\}_{i=1}^n$ be a planar generating set of $\pi_1(S^2 \setminus P_f, t)$, for $t \in S^2 \setminus P_f$ the basepoint on the circle at infinity (see Section 2.3). That is, on the closed disc that is a retraction of $S^2 \setminus \{\infty\}$, the generator s_i is a simple loop based at t going around ω_i in the positive direction, and the loops are cyclically ordered in the positive direction.

Figure 2.11 gives three examples of different planar generating sets with the same set of punctures.
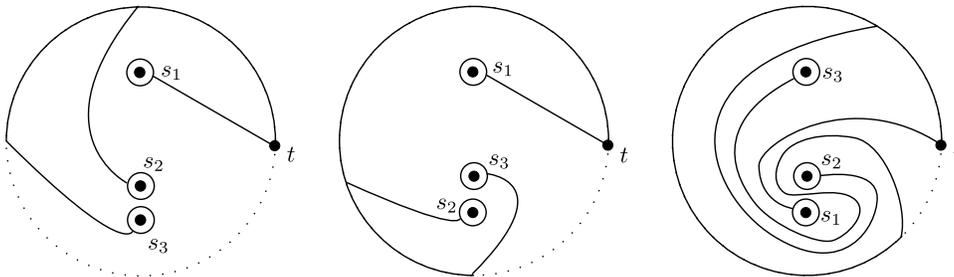

Figure 2.11: Example planar generating sets

Let $t_j \in f^{-1}(t) = \{t = t_0, t_1, ..., t_{d-1}\}$. Define $\mathfrak{M}_f$ to be the $(\pi_1(S^2 \setminus P_f, t) - \pi_1(S^2 \setminus P_f, t_j))$-bimodule of homotopy classes of paths in $S^2 \setminus P_f$ starting at t_j and ending at any point in $f^{-1}(t)$. The right action

of $\pi_1(S^2 \setminus P_f, t_j)$ is by concatenation of the loop at t_j to the beginning of the path, and the left action of $\pi_1(S^2 \setminus P_f, t)$ is by concatenation of the f-lift of the loop at t to the end of the path.

Up to isomorphism, $\mathfrak{M}_f$ does not depend on the choice of basepoints. If we choose a path from t to t_j, we may identify the fundamental groups in the usual way, and the isomorphism class of the $\pi_1(S^2 \setminus P_f)$-bimodule $\mathfrak{M}_f$ does not depend on the choice of this path. So we will choose $j = 0$ and so identify the fundamental groups by the constant path at $t = t_0$.

Notice that the right action is free and that two elements of $\mathfrak{M}_f$ belong to the same right orbit if and only if they have the same endpoints. So the number of right orbits equals the degree of f, and the bimodule is covering.

We make a canonical choice for a basis of $\mathfrak{M}_f$, $X = \{0, 1, ..., d-1\}$. The basis element k for the orbit associated with the endpoint t_k is the path in the positive direction along the circle at infinity from t to t_k.

A result of Nekrashevych states that the combinatorial data encoded in this bimodule completely describes the post-critical dynamics of the topological polynomial, up to Thurston equivalence.

Theorem 2.5.1. *[Nek09] The bimodule $\mathfrak{M}_f$ defined above is generated by a twisted kneading automaton, and the twisted kneading automaton associated to the topological polynomial f along with the cyclic order $(s_1, s_2, ..., s_n)$ of the generators of $F_n = \pi_1(S^2 \setminus P_f)$ uniquely determine the Thurston equivalence class of f.*

In fact, these bimodules can be described more explicitly.

For $F_n = F(s_1, s_2, ..., s_n)$ the free group on n generators, define $a_{i,j} \in Aut(F_n)$ by $a_{i,j}(s_i) = s_i^{s_j}$ and for all $k \neq i$, $a_{i,j}(s_k) = s_k$. Notice that $a_{j,j}$ and $[a_{i,j}, a_{k,j}]$ are trivial for all $1 \leq i, j, k \leq n$, $[a_{i,j}, a_{k,l}]$ is trivial for all i, j, k, l distinct, and for a fixed j_0 we have $\prod_{1 \leq i \leq n} a_{i,j_0} \in Inn(F_n)$. For $q : Aut(F_n) \rightarrow Out(F_n)$ the quotient map, let $P\Sigma O_n$ be the image under q of $P\Sigma_n = \langle a_{i,j} \rangle \leq Aut(F_n)$. We call $P\Sigma O_n$ the *pure symmetric outer automorphism group* of the free group of rank n. From now on, we will abuse notation and write $a_{i,j}$ for its image in $Out(F_n)$.

For G a group and $\alpha \in Aut(G)$, we define the associated G-bimodule $[\alpha]$ to be the set of expressions $\alpha \cdot g$ for $g \in G$ with the actions $h_l \cdot (\alpha \cdot g) \cdot h_r = \alpha \cdot \alpha(h_l) g h_r$ for all $h_l, h_r \in G$. It is easy to show that for $\alpha \in Inn(G)$ and $\mathfrak{M}$ any G-bimodule, $[\alpha] \otimes \mathfrak{M} \simeq \mathfrak{M} \simeq \mathfrak{M} \otimes [\alpha]$, so we can uniquely define the isomorphism class of the bimodule for $\alpha \in Out(G)$.

What Nekrashevych actually showed in the proof of Theorem 2.5.1 is that the bimodule $\mathfrak{M}_f$ is isomorphic to a twisted kneading bimodule of the form $\mathfrak{M}_{\mathfrak{K}(f)} \otimes [\phi]$ where $\mathfrak{M}_{\mathfrak{K}(f)}$ is a kneading bimodule and $\phi \in P\Sigma O_n$. He went on to define the $P\Sigma O_n$-bimodule $\mathfrak{G}_f = \{[\alpha] \otimes \mathfrak{M}_f \otimes [\beta] \mid \alpha, \beta \in P\Sigma O_n\}$, with the natural left and right actions. Nekrashevych proved the following:

Proposition 2.5.2. *[Nek09] Every twisted kneading automaton over F_n is associated with some topological polynomial.*

We can think of $\alpha, \beta \in P\Sigma O_n$ as acting on $S^2 \setminus P_f$ and the tensor operation as functional composition. Figure 2.12 demonstrates the action of $a_{i,j}$ on $S^2 \setminus P_f$ and the planar generating set $\{s_1, ..., s_n\}$.
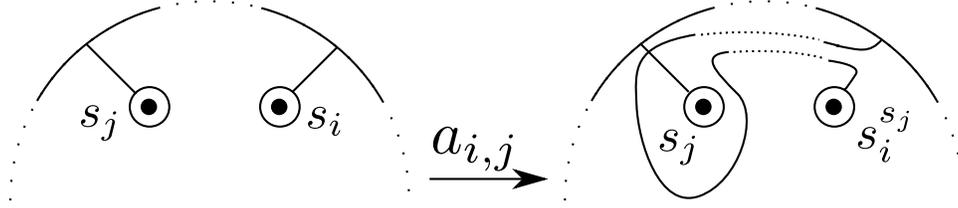

Figure 2.12: Action of $a_{i,j}$

Lemma 2.5.3. *The action of $a_{i+1,i}a_{i,i+1}$ is that of a Dehn Twist about a curve separating $\{\omega_i, \omega_{i+1}\}$ from the rest of P_f (ω_i is the puncture surrounded by s_i).*
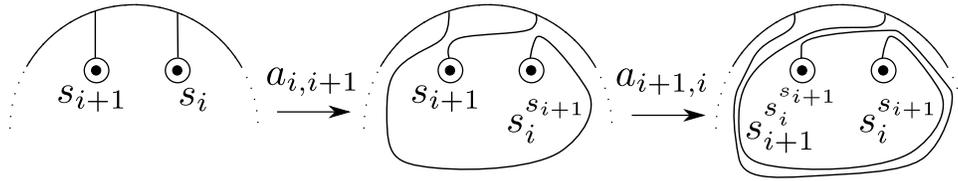
Proof. See Figure 2.13. □

Figure 2.13: Action of $a_{i+1,i}a_{i,i+1}$ by a Dehn twist

Further, we may represent a Dehn Twist about a curve separating $\{\omega_i, \omega_{i+1}, \omega_{i+2}\}$ from the rest of the post-critical set by the word $a_{i+2,i+1}a_{i+2,i}a_{i+1,i}a_{i+1,i+2}a_{i,i+2}a_{i,i+1} \in P\Sigma O_n$. And so on. Notice that Dehn twists take planar generating sets to planar generating sets with the same cyclic ordering of punctures.

In this way, we see that the pure mapping class group $PMod(S^2 \setminus P_f)$ is a subgroup of $P\Sigma O_n$. In fact, the $P\Sigma O_n$-bimodule $\mathfrak{G}_n$ was originally defined by Bartholdi and Nekrashevych as a $PMod(S^2 \setminus P_f)$-bimodule [BN06], and later extended by Nekrashevych [Nek09].

The twisted kneading bimodule $\mathfrak{M}_{\mathfrak{K}(f)} \otimes [\phi]$ encodes a topological description of the preimages of the planar generating set $\{s_i\}_{i=1}^n$ under the topological polynomial f. The kneading bimodule $\mathfrak{M}_{\mathfrak{K}(f)}$ is generated by a kneading automaton $\mathfrak{K}(f)$ determined (up to labels) by the mapping scheme of f, which we will denote by $S(f)$. The kneading automaton $\mathfrak{K}(f)$ has a state for each generator $s_i \in \pi_1(S^2 \setminus P_f)$. We have an arrow from state s_i to state s_j if and only if $f(\omega_j) = \omega_i$. So the unlabeled Moore diagram of this kneading automaton is the subgraph of $S(f)$ induced by the vertices in P_f, but with the arrows reversed.

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The labels are determined by numbering the lifts of the basepoint t. Recall the alphabet $X = f^{-1}(t) = \{0, 1, ..., d-1\}$ for $d = deg(f)$. We label the directed edge from s_i to s_j by k if ω_j is contained in the loop that goes from t to t_k along the circle at infinity in the positive direction, then follows the lift of s_i starting at t_k, and then travels back along the circle at infinity in the negative direction to t. The active states will be those associated with critical values of f. For ω_i a critical value of f, we have $s_i(k) = k'$ if the f-lift of s_i that starts at t_k ends at $t_{k'}$.

Notice that the permutations of an active state and the coordinates of the non-trivial restrictions are related by the type of critical point(s) with which the corresponding critical value is associated.

Lemma 2.5.4. *For ω_0 a critical value of topological polynomial f and $\{\omega_1, \omega_2, ..., \omega_k\} = f^{-1}(\omega_0) \cap C_f$ with the local degree of f at ω_i equal to d_i for $1 \leq i \leq k$, then:*

1. *The action of the state s_0 in the kneading automaton $\mathfrak{K}(f)$ will have k non-trivial cycles and their lengths will be given by the set $\{d_i\}_{i=1}^{k}$.*

2. *If $\omega_i \in P_f$, then the label of the arrow from state s_0 to s_i (the state corresponding with $\omega_i \in P_f$) will be one of the coordinates on which s_0 acts by a cycle of length d_i.*

3. *Any labels of edges in $\mathfrak{K}(f)$ from s_0 to states not associated with critical points will be of coordinates on which s_0 acts trivially.*

Proof. These properties follow more or less immediately from the definitions of $\mathfrak{M}_f$ and $\mathfrak{K}(f)$.

1. If s_0 is a small loop about ω_0 connected by an arc to t, then its preimages about ω_i, a critical point with local degree d_i will be a small loop about ω_i with d_i arcs connecting it to d_i preimages of t. Notice that this will give a cycle of these d_i preimages in $\mathfrak{K}(f)$.

2. Continuing as above, notice that one of the paths from t to its preimage, along the arc to the loop about ω_i, back along the next arc in the positive direction to a different preimage of t, and then back to t will contain ω_i in its interior, and starting at any other preimage will not contain ω_i.

3. Unlike in our first observation, if s_0 is a small loop about ω_0 connected by an arc to t, then its preimage about a non-critical post-critical point ω will also consist of a small loop about ω connected by an arc, but to a preimage of t. Thus, the preimage of s_0 which starts at this preimage of t (which is the coordinate on which s_0 will restrict to the state associated with ω) also terminates at this preimage of t.

$\square$

Now a lift of s_i might not be homotopic to a generator; it might even surround multiple post-critical points. Expressing these lifts as elements of $F(s_1, ..., s_n)$ allows us to determine the twist $\phi \in P\Sigma O_n$ associated with the polynomial. This will not be important to our work, and so we will only give an example (see below) and refer the reader to [BN06], [Nek05], [Nek09] to understand the details of how to compute the element ϕ.

In Figures 2.14 and 2.15, we give two examples of starting with a topological polynomial (we use f and g from Figure 2.8), finding its mapping scheme, and then producing the associated kneading automaton. Here we explain how to find $\phi \in P\Sigma O_3$ such that $\mathfrak{M}_f \simeq \mathfrak{M}_{\mathfrak{K}(f)} \otimes [\phi]$:
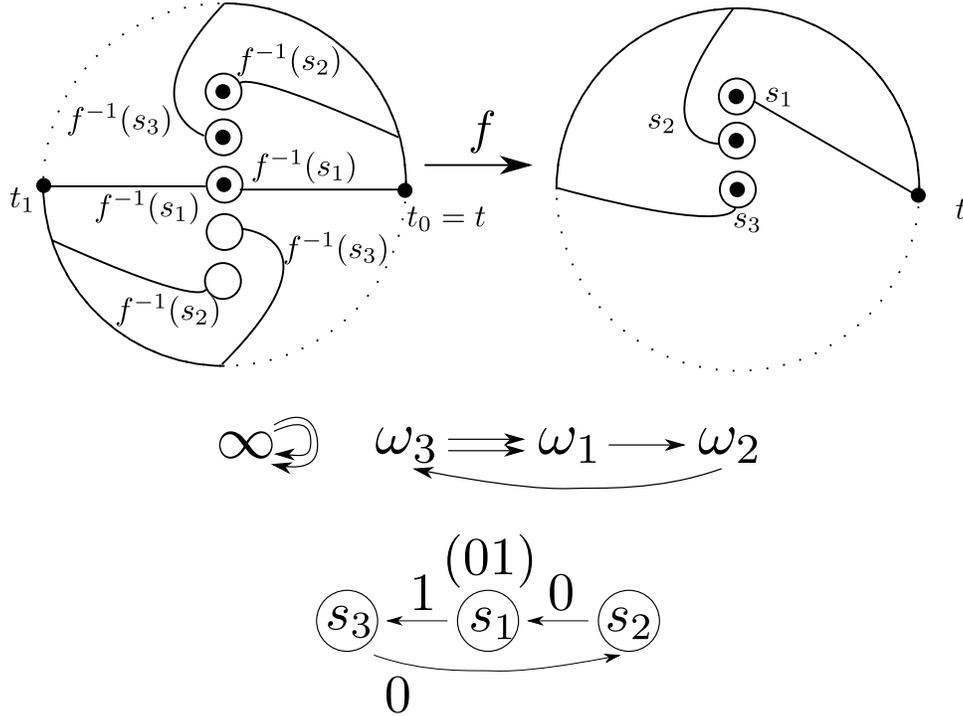

Figure 2.14: The topological polynomial f, its mapping scheme $S(f)$, and its kneading automaton $\mathfrak{K}(f)$

If we read off the automaton generating $\mathfrak{M}_f$ from the map f at the top of Figure 2.14, we get the following wreath recursion:

$$s_1 = (01)(s_2^{-1}s_1^{-1}, s_1s_2s_3)$$

$$s_2 = (s_1, \mathtt{I})$$

$$s_3 = (s_2, \mathtt{I})$$

Notice that this automaton is not kneading or twisted kneading. Let $\alpha \in Aut(F_3)$ be the conjugation by s_2s_3. Since $\alpha \in Inn(F_3)$, we have that $[\alpha] \otimes \mathfrak{M}_f \simeq \mathfrak{M}_f$. To compute the automaton generating $[\alpha] \otimes \mathfrak{M}_f$,

we need only conjugate our wreath recursion for $\mathfrak{M}_f$ by $s_2 s_3 = (s_1 s_2, \mathtt{I})$. This gives us:

$$s_1 = (01)(\mathtt{I}, s_3)$$

$$s_2 = (s_1^{s_2}, \mathtt{I})$$

$$s_3 = (s_2^{s_1^{s_2}}, \mathtt{I})$$

This automaton *is* twisted kneading with kneading automaton $\mathfrak{K}(f)$ (see the bottom of Figure 2.14) and twist $\phi = a_{2,1} a_{1,2}$. The wreath recursion for $\mathfrak{K}(f)$ is just the previous one without the conjugations, since we have separated the action of the twist ϕ:

$$s_1 = (01)(\mathtt{I}, s_3)$$

$$s_2 = (s_1, \mathtt{I})$$

$$s_3 = (s_2, \mathtt{I})$$

The reader may wish to check understanding of this process by computing the twist for g in Figure 2.15 (the correct twist is $a_{2,3}$).
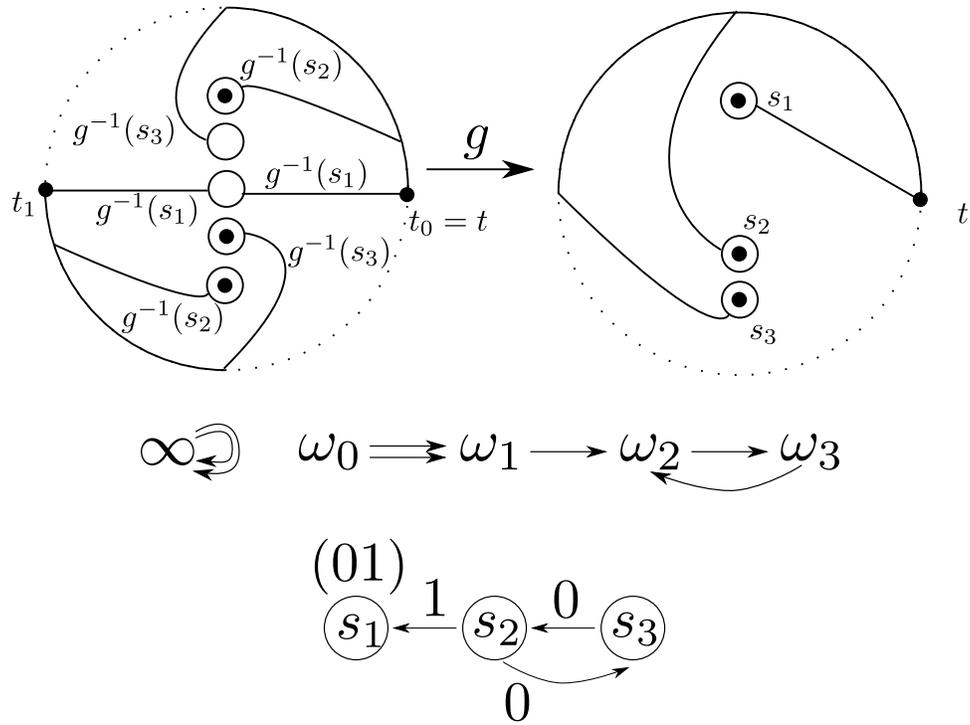

Figure 2.15: The topological polynomial g, its mapping scheme $S(g)$, and its kneading automaton $\mathfrak{K}(g)$

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2.6 The quadratic mega-bimodules

We will now describe the bimodule $\mathfrak{G}_f$ for $\deg(f) = 2$. In this case, f will either be periodic with period length n or preperiodic with preperiod $k \geq 1$ and period n. We will only consider those topological polynomials with period length $n \geq 2$. In other words, f will be a topological polynomial realizing one of the mapping schemes in Figure 2.16.
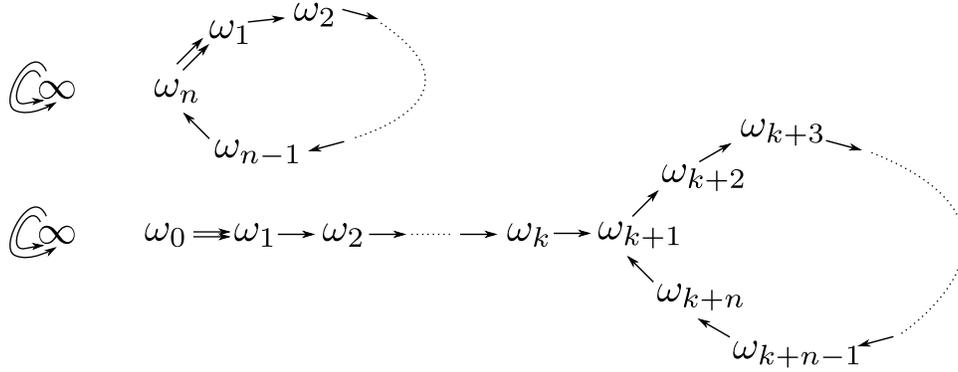

Figure 2.16: The mapping schemes of quadratic topological polynomials

First we describe the periodic case (with period length $n \geq 2$).

For $x_1 x_2 ... x_n \in \{0,1\}^n$, definte $\mathfrak{M}_{x_1 x_2 ... x_n}$ to be the F_n-bimodule generated by the automaton $\mathfrak{K}_{x_1 x_2 ... x_n}$ with states $\{\mathtt{I}, s_1, ... s_n\}$ and alphabet $\{0, 1\}$ defined by:

- $s_1 = (01)(\mathtt{I}, s_n)$ if $x_n = 1$, and $s_1 = (01)(s_n, \mathtt{I})$ if $x_n = 0$,

- and for all $1 \leq i < n$, $s_{i+1} = (s_i, \mathtt{I})$ if $x_i = 0$, and $s_{i+1} = (\mathtt{I}, s_i)$ if $x_i = 1$.

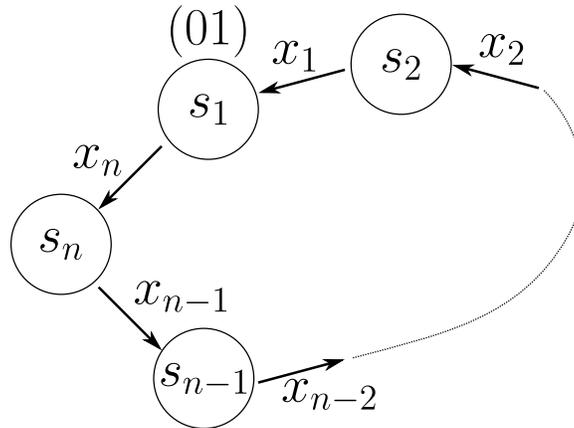See Figure 2.17 for the abbreviated Moore diagram of this automaton.

Figure 2.17: Moore diagram of the automaton $\mathfrak{K}_{x_1 x_2 ... x_n}$

The automata $\mathfrak{K}_{x_1 x_2 \ldots x_n}$ are precisely the periodic binary kneading automata with period $n \geq 2$ [Nek05].

Notice that $\mathfrak{M}_{x_1 x_2 \ldots x_n} \simeq \mathfrak{M}_{\overline{x_1 x_2} \ldots \overline{x_n}}$ by the map that switches 0 and 1, so we will assume that $x_n = 1$ (i.e. that $s_1 = (01)(\mathtt{I}, s_n)$).

We define the $P\Sigma O_n$-bimodules $\mathfrak{G}_n$ to be the set of all F_n-bimodules of the form $[\alpha] \otimes \mathfrak{M}_{x_1 x_2 \ldots x_{n-1} 1} \otimes [\beta]$ for $x_1 x_2 \ldots x_{n-1} 1 \in \{0, 1\}^n$ and $\alpha, \beta \in P\Sigma O_n$, with the natural left and right actions. We can easily compute the action of the generators $a_{i,j}$ on the kneading bimodules as below.

Proposition 2.6.1. *For $1 \leq i, j \leq n$, we have the following actions of the bimodules defined by $a_{i,j} \in P\Sigma O_n$ on the bimodules $\mathfrak{M}_{x_1 x_2 \ldots x_{n-1} 1}$, and by extension on the elements of $\mathfrak{G}_n$:*

For $1 < i \leq n$,

$$[a_{i,1}] \otimes \mathfrak{M}_{x_1 \ldots x_{i-2} 1 x_i \ldots x_{n-1} 1} = \mathfrak{M}_{x_1 \ldots x_{i-2} 0 x_i \ldots x_{n-1} 1}$$

$$[a_{i,1}] \otimes \mathfrak{M}_{x_1 \ldots x_{i-2} 0 x_i \ldots x_{n-1} 1} = \mathfrak{M}_{x_1 \ldots x_{i-2} 1 x_i \ldots x_{n-1} 1} \otimes [a_{i-1,n}]$$

For $1 < j \leq n$,

$$[a_{1,j}] \otimes \mathfrak{M}_{x_1 x_2 \ldots x_{n-1} 1} = \mathfrak{M}_{x_1 x_2 \ldots x_{n-1} 1} \otimes \left[a_{n,j-1} \prod_{x_{i-1} = \overline{x_{j-1}}, i-1 < n} a_{i-1,j-1} \right]$$

For $1 < i, j \leq n$,

$$[a_{i,j}] \otimes \mathfrak{M}_{x_1 \ldots x_{i-2} \overline{x_{j-1}} x_i \ldots x_{n-1} 1} = \mathfrak{M}_{x_1 \ldots x_{i-2} \overline{x_{j-1}} x_i \ldots x_{n-1} 1}$$

$$[a_{i,j}] \otimes \mathfrak{M}_{x_1 \ldots x_{i-2} x_{j-1} x_i \ldots x_{n-1} 1} = \mathfrak{M}_{x_1 \ldots x_{i-2} x_{j-1} x_i \ldots x_{n-1} 1} \otimes [a_{i-1,j-1}]$$

Proof. This follows from direct computation.

For example, the first computation is for $[a_{i,1}] \otimes \mathfrak{M}_{x_1 \ldots x_{i-2} 1 x_i \ldots x_{n-1} 1}$ for $1 < i \leq n$. We have $s_1 = (01)(\mathtt{I}, s_n)$ and since $x_{i-1} = 1$, we have that $s_i = (\mathtt{I}, s_{i-1})$. So $s_i^{s_1} = (01)(s_n^{-1}, \mathtt{I}) \cdot (\mathtt{I}, s_{i-1}) \cdot (01)(\mathtt{I}, s_n) = (s_{i-1}, \mathtt{I})$. So the resulting bimodule is $\mathfrak{M}_{x_1 \ldots x_{i-2} 0 x_i \ldots x_{n-1} 1}$ since the coordinates of the restrictions of s_i have switched.

But if we consider $[a_{i,1}] \otimes \mathfrak{M}_{x_1 \ldots x_{i-2} 0 x_i \ldots x_{n-1} 1}$ for $1 < i \leq n$ then $s_i = (s_{i-1}, \mathtt{I})$. So when we do the computation, we have $s_i^{s_1} = (01)(s_n^{-1}, \mathtt{I}) \cdot (s_{i-1}, \mathtt{I}) \cdot (01)(\mathtt{I}, s_n) = (\mathtt{I}, s_{i-1}^{s_n})$. Now not only have the coordinates of the restrictions switched, but we have also replaced s_{i-1} with $s_{i-1}^{s_n}$, so we gain the twist $a_{i-1,n}$. Thus, our resulting bimodules is $\mathfrak{M}_{x_1 \ldots x_{i-2} 1 x_i \ldots x_{n-1} 1} \otimes [a_{i-1,n}]$.

$\square$

Notice that since $\prod_{1<i\leq n} a_{i,1} \in Inn(F_n)$, we have that $\left[\prod_{1<i\leq n} a_{i,1}\right] \otimes \mathfrak{M} \simeq \mathfrak{M}$ for any F_n-bimodule $\mathfrak{M}$. So since $\left[\prod_{1<i\leq n} a_{i,1}\right] \otimes \mathfrak{M}_{x_1...x_{n-1}1} = \mathfrak{M}_{\overline{x_1}...\overline{x_{n-1}}1} \otimes \left[\prod_{x_{i-1}=0} a_{i-1,n}\right]$, we have that $\mathfrak{M}_{x_1...x_{n-2}01} \simeq \mathfrak{M}_{x_1...x_{n-2}11} \otimes \left[a_{n-2,n}\prod_{x_{i-1}=0} a_{i-1,n}\right]$. Thus, we need only consider kneading bimodules of the form $\mathfrak{M}_{x_1...x_{n-2}11}$ for our basis.

Recall that for f a periodic quadratic topological polynomial with period length $n \geq 2$, the $P\Sigma O_n$-bimodule $\mathfrak{G}_f$ is the set of all F_n-bimodules of the form $[\alpha] \otimes \mathfrak{M}_f \otimes [\beta]$ where $\alpha, \beta \in P\Sigma O_n$ with the natural $P\Sigma O_n$ right and left actions.

Proposition 2.6.2. *For f a periodic quadratic topological polynomial with period length $n \geq 2$, then $\mathfrak{G}_n = \mathfrak{G}_f$.*

Proof. By Theorem 2.5.1, $\mathfrak{M}_f$ is isomorphic to $\mathfrak{M}_{x_1x_2...x_{n-2}11} \otimes [\phi]$ for some $x_1x_2...x_{n-1}1 \in \{0,1\}^n$ and some $\phi \in P\Sigma O_n$. By Proposition 2.6.1, the left action of $P\Sigma O_n$ is transitive on the basis of kneading bimodules. Thus, the two bimodules are equal.

$\square$

Now we describe the preperiodic case (with preperiod length $k \geq 1$ and period length $n \geq 2$).

For $x_1x_2...x_k...x_{k+n} \in \{0,1\}^{k+n}$ such that $x_k = \overline{x_{k+n}}$, define $\mathfrak{M}_{x_1x_2...x_k,x_{k+1}...x_{k+n}}$ to be the F_{k+n}-bimodule generated by the automaton $\mathfrak{K}_{x_1x_2...x_k,x_{k+1}...x_{k+n}}$ with states $\{\mathtt{I}, s_1, ..., s_{k+n}\}$ and alphabet $\{0,1\}$ defined by:

- $s_1 = (01)$,

- $s_{k+1} = (s_{k+n}, s_k)$ if $x_k = 1$ and $x_{k+n} = 0$, and $s_{k+1} = (s_k, s_{k+n})$ if $x_k = 0$ and $x_{k+n} = 1$,

- and for all $1 \leq i < k+n, i \neq k+1$, $s_{i+1} = (s_i, \mathtt{I})$ if $x_i = 0$, and $s_{i+1} = (\mathtt{I}, s_i)$ if $x_i = 1$.

See Figure 2.18 for the abbreviated Moore diagram of this automaton.

The automata $\mathfrak{K}_{x_1x_2...x_k,x_{k+1}...x_{k+n}}$ with $x_k = \overline{x_{k+n}}$ are precisely the preperiodic binary kneading automata with preperiod $k \geq 1$ and period $n \geq 2$ [Nek05].

Notice that $\mathfrak{M}_{x_1x_2...x_k,x_{k+1}...x_{k+n}} \simeq \mathfrak{M}_{\overline{x_1x_2}...\overline{x_k},\overline{x_{k+1}}...\overline{x_{k+n}}}$ by the map that switches 0 and 1, so we will assume $x_k = 1, x_{k+n} = 0$ (i.e. that $s_{k+1} = (s_{k+n}, s_k)$).

We define the $P\Sigma O_{k+n}$-bimodule $\mathfrak{G}_{k,n}$ to be the set of all F_{k+n}-bimodules of the form $[\alpha] \otimes \mathfrak{M}_{x_1...x_{k-1}1,x_{k+1}...x_{k+n-1}0} \otimes [\beta]$ for $x_1x_2...x_{k-1}1, x_{k+1}...x_{k+n-1}0 \in \{0,1\}^{k+n}$ and $\alpha, \beta \in P\Sigma O_{k+n}$, with the natural left and right actions. We can easily compute the action of the generators $a_{i,j}$ on the kneading bimodules as below.
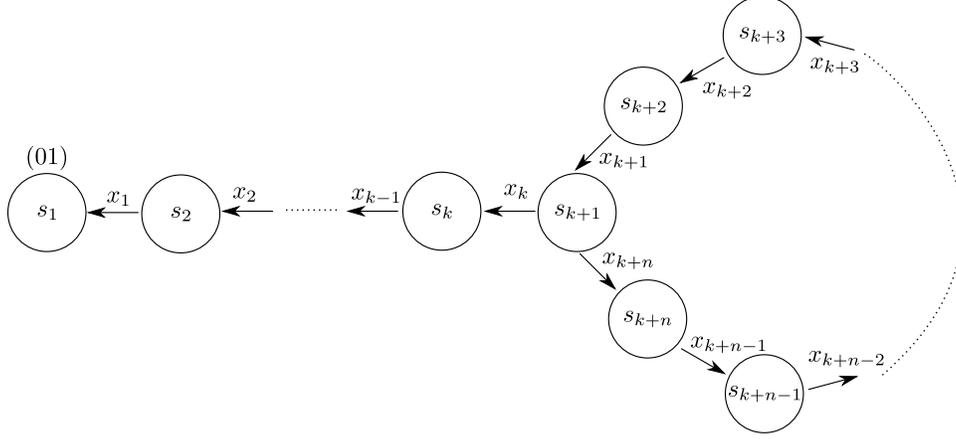
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Figure 2.18: Moore diagram of the automaton $\mathfrak{K}_{x_1 x_2 \ldots x_k, x_{k+1} \ldots x_{k+n}}$

Proposition 2.6.3. *For $1 \le i, j \le k+n$, we have the following actions of the bimodules defined by $a_{i,j} \in P\Sigma O_{k+n}$ on the bimodules $\mathfrak{M}_{x_1 x_2 \ldots x_{k-1} 1, x_{k+1} \ldots x_{k+n-1} 0}$, and by extension on the elements of $\mathfrak{G}_{k,n}$:*

$$[a_{k+1,1}] \otimes \mathfrak{M}_{x_1 x_2 \ldots x_{k-1} 1, x_{k+1} \ldots x_{k+n-1} 0} = \mathfrak{M}_{x_1 x_2 \ldots x_{k-1} 0, x_{k+1} \ldots x_{k+n-1} 1} \simeq$$

$$\simeq \mathfrak{M}_{\overline{x_1 x_2} \ldots \overline{x_{k-1}} 1, \overline{x_{k+1}} \ldots \overline{x_{k+n-1}} 0}$$

$$[a_{1,k+1}] \otimes \mathfrak{M}_{x_1 x_2 \ldots x_{k-1} 1, x_{k+1} \ldots x_{k+n-1} 0} =$$

$$= \mathfrak{M}_{x_1 x_2 \ldots x_{k-1} 1, x_{k+1} \ldots x_{k+n-1} 0} \otimes \left[\prod_{x_{i-1}=0} a_{i-1,k+n}^{-1} \prod_{x_{i-1}=1} a_{i-1,k}^{-1} \right] =$$

$$= \mathfrak{M}_{x_1 x_2 \ldots x_{k-1} 1, x_{k+1} \ldots x_{k+n-1} 0} \otimes \left[\prod_{x_{i-1}=1} a_{i-1,k+n} \prod_{x_{i-1}=0} a_{i-1,k} \right]$$

For $i \ne k+1$,

$$[a_{i,1}] \otimes \mathfrak{M}_{x_1 x_2 \ldots x_{i-2} x_{i-1} x_i \ldots x_{k+n-1} 0} = \mathfrak{M}_{x_1 x_2 \ldots x_{i-2} \overline{x_{i-1}} x_i \ldots x_{k+n-1} 0}$$

For $j \ne k+1$,

$$[a_{1,j}] \otimes \mathfrak{M}_{x_1 x_2 \ldots x_{k-1} 1, x_{k+1} \ldots x_{k+n-1} 0} = \mathfrak{M}_{x_1 x_2 \ldots x_{k-1} 1, x_{k+1} \ldots x_{k+n-1} 0} \otimes \left[\prod_{x_{i-1}=x_{j-1}} a_{i-1,j-1} \right]$$

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For $i \neq 1$,

$$[a_{i,k+1}] \otimes \mathfrak{M}_{x_1x_2...x_{i-2}0x_i...x_{k+n-1}0} = \mathfrak{M}_{x_1x_2...x_{i-2}0x_i...x_{k+n-1}0} \otimes [a_{i-1,k+n}]$$

$$[a_{i,k+1}] \otimes \mathfrak{M}_{x_1x_2...x_{i-2}1x_i...x_{k+n-1}0} = \mathfrak{M}_{x_1x_2...x_{i-2}1x_i...x_{k+n-1}0} \otimes [a_{i-1,k}]$$

For $j \neq 1$,

$$[a_{k+1,j}] \otimes \mathfrak{M}_{x_1x_2...x_{j-2}0x_j...x_{k+n-1}0} = \mathfrak{M}_{x_1x_2...x_{j-2}0x_j...x_{k+n-1}0} \otimes [a_{k+n,j-1}]$$

$$[a_{k+1,j}] \otimes \mathfrak{M}_{x_1x_2...x_{j-2}1x_j...x_{k+n-1}0} = \mathfrak{M}_{x_1x_2...x_{j-2}1x_j...x_{k+n-1}0} \otimes [a_{k,j-1}]$$

For $i,j \neq 1, k+1$,

$$[a_{i,j}] \otimes \mathfrak{M}_{x_1x_2...x_{i-2}\overline{x_{j-1}}x_i...x_{k+n-1}0} = \mathfrak{M}_{x_1x_2...x_{i-2}\overline{x_{j-1}}x_i...x_{k+n-1}0}$$

$$[a_{i,j}] \otimes \mathfrak{M}_{x_1x_2...x_{i-2}x_{j-1}x_i...x_{k+n-1}0} = \mathfrak{M}_{x_1x_2...x_{i-2}x_{j-1}x_i...x_{k+n-1}0} \otimes [a_{i-1,j-1}]$$

Proof. This follows from direct computation.

For example, the first computation is for $[a_{k+1,1}] \otimes \mathfrak{M}_{x_1x_2...x_{k-1}1,x_{k+1}...x_{k+n-1}0}$. Notice that in the kneading automaton generating $\mathfrak{M}_{x_1x_2...x_{k-1}1,x_{k+1}...x_{k+n-1}0}$ we have $s_1 = (01), s_{k+1} = (s_{k+n}, s_k)$. So $s_{k+1}^{s_1} = (01) \cdot (s_{k+n}, s_k) \cdot (01) = (s_k, s_{k+n})$. So the resulting bimodule is $\mathfrak{M}_{x_1x_2...x_{k-1}0,x_{k+1}...x_{k+n-1}1}$, since the coordinates of the restrictions of s_k have switched. As we noted above, this bimodule is isomorphic to $\mathfrak{M}_{\overline{x_1x_2}...\overline{x_{k-1}}1,\overline{x_{k+1}}...\overline{x_{k+n-1}}0}$.

Another example: $[a_{i,k+1}] \otimes \mathfrak{M}_{x_1x_2...x_{i-2}0x_i...x_{k+n-1}0}$ (for $i \neq 1$). Here we have $s_i = (s_{i-1}, \mathtt{I})$ and $s_{k+1} = (s_{k+n}, s_k)$. So $s_i^{s_{k+1}} = (s_{k+n}^{-1}, s_k^{-1}) \cdot (s_{i-1}, \mathtt{I}) \cdot (s_{k+n}, s_k) = (s_{i-1}^{s_{k+n}}, \mathtt{I})$. Since none of the coordinates have changed, the kneading sequence remains unaltered, but we have replaced s_{i-1} with $s_{i-1}^{s_{k+n}}$, so we gain the twist $a_{i-1,k+n}$. Thus, our resulting bimodule is $\mathfrak{M}_{x_1x_2...x_{i-2}0x_i...x_{k+n-1}0} \otimes [a_{i-1,k+n}]$.

$\square$

Proposition 2.6.4. *For f a preperiodic quadratic topological polynomial with preperiod length $k \geq 1$ and period length $n \geq 2$, then $\mathfrak{G}_{k,n} = \mathfrak{G}_f$.*

Proof. By Theorem 2.5.1, $\mathfrak{M}_f$ is isomorphic to $\mathfrak{M}_{x_1...x_{k-1}1,x_{k+1}...x_{k+n-1}0} \otimes [\phi]$ for some $x_1x_2...x_{k-1}1, x_{k+1}...x_{k+n-1}0 \in \{0,1\}^{k+n}$ and some $\phi \in P\Sigma O_{k+n}$. By Proposition 2.6.3, the left action of $P\Sigma O_{k+n}$ is transitive on the basis of kneading bimodules. Thus, the two bimodules are equal.

$\square$

Now for $1 \le i \le n$, define $\gamma_i \in P\Sigma O_{k+n}$ by:

$$\gamma_i = a_{k+i,k+i-1}a_{k+i,k+i-2}...a_{k+i,k+1}a_{k+i,k+n}a_{k+i,k+n-1}...a_{k+i,k+i+1}.$$

So for $n = 2$, we would have:

$$\gamma_1 = a_{k+1,k+2}$$

$$\gamma_2 = a_{k+2,k+1}$$

For $n = 3$, we would have:

$$\gamma_1 = a_{k+1,k+3}a_{k+1,k+2}$$

$$\gamma_2 = a_{k+2,k+1}a_{k+2,k+3}$$

$$\gamma_3 = a_{k+3,k+2}a_{k+3,k+1}$$

For $n = 4$:

$$\gamma_1 = a_{k+1,k+4}a_{k+1,k+3}a_{k+1,k+2}$$

$$\gamma_2 = a_{k+2,k+1}a_{k+2,k+4}a_{k+2,k+3}$$

$$\gamma_3 = a_{k+3,k+2}a_{k+3,k+1}a_{k+3,k+4}$$

$$\gamma_4 = a_{k+4,k+3}a_{k+4,k+2}a_{k+4,k+1}$$

Notice that the product $\gamma = \gamma_n\gamma_{n-1}...\gamma_2\gamma_1$ acts on $S^2 \setminus P_f$ by a Dehn Twist about a simple closed curve Γ that separates the points $\{\omega_{k+1}, ..., \omega_{k+n}\}$ from the rest of P_f. That is, Γ separates the period of the post-critical set from the preperiod and ∞. Since $k \ge 1$ and $n \ge 2$, the curve Γ is non-peripheral.

Lemma 2.6.5. *For g a pre-periodic quadratic topological polynomial with preperiod length $k \ge 1$ and period length $n \ge 2$ and $\gamma \in P\Sigma O_{k+n}$ acting by a Dehn twist T_Γ about a simple closed curve Γ that separates the period $\{\omega_{k+1}, ..., \omega_{k+n}\}$ from the rest of the post-critical set, if $[\gamma] \otimes \mathfrak{M}_g \simeq \mathfrak{M}_g \otimes [\gamma]$, then Γ is a g-stable Levy cycle of length 1, and g is not equivalent to any complex polynomial.*

Proof. First notice that both the bimodules $[\gamma] \otimes \mathfrak{M}_g$ and $\mathfrak{M}_g \otimes [\gamma]$ are twisted kneading since $\mathfrak{M}_g$ is twisted kneading and Dehn twists preserve the cyclic ordering of coordinates (this also follows from the fact that a toplogical polynomial composed with a Dehn twist yields another topological polynomial). By

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$[\gamma] \otimes \mathfrak{M}_g \simeq \mathfrak{M}_g \otimes [\gamma]$ we have $T_\Gamma \circ g$ homotopic to $g \circ T_\Gamma$, so $g(\Gamma)$ must be homotopic to Γ and g must map it by degree 1. Notice also that the other component of $g^{-1}(\Gamma)$ is peripheral about the point ω_k. Thus, Γ is a g-stable Levy cycle of length 1 for g.

$\square$

Using the computations given in Proposition 2.6.3, we can verify the following lemma:

Lemma 2.6.6. *For any $x_1...x_{k-1} \in \{0,1\}^{k-1}$, we have*

$$[\gamma_1] \otimes \mathfrak{M}_{x_1...x_{k-1}1,00...00} = \mathfrak{M}_{x_1...x_{k-1}1,00...00} \otimes [\gamma_n],$$

and for $1 < i \leq n$,

$$[\gamma_i] \otimes \mathfrak{M}_{x_1...x_{k-1}1,00...00} = \mathfrak{M}_{x_1...x_{k-1}1,00...00} \otimes [\gamma_{i-1}].$$

Proof. This follows directly from Proposition 2.6.3. $\square$

We are now ready to state our first result.

Theorem 2.6.7. *For f a pre-periodic topological polynomial of degree 2 with preperiod length $k \geq 1$ and period length $n \geq 2$, then there exists a topological polynomial g which has the same mapping scheme as f, but is not equivalent to any complex polynomial.*

Proof. Pick a planar generating set $\{s_1, ..., s_{k+n}\}$. Let $\mathfrak{M}_f$ be as defined earlier and let $\mathfrak{M}_f \simeq \mathfrak{M}_{\mathfrak{K}(f)} \otimes [\phi]$ be the twisted kneading bimodule representation of $\mathfrak{M}_f$ as in Theorem 2.5.1. We have that $\mathfrak{M}_{\mathfrak{K}(f)} = \mathfrak{M}_{x_1...x_{k-1}1,x_{k+1}...x_{k+n-1}0}$ for some $x_i \in \{0,1\}$ (in fact, $x_1x_2...x_{k-1}1(x_{k+1}...x_{k+n-1}0)$ is the kneading sequence of f in the sense of [BS02]).

By Proposition 2.6.3, there exists $\alpha \in P\Sigma O_{k+n}$ such that $[\alpha] \otimes \mathfrak{M}_{\mathfrak{K}(f)} = \mathfrak{M}_{1...1,0...0}$. Let $\mathfrak{M}_0 = \mathfrak{M}_{1...1,0...0} = [\alpha] \otimes \mathfrak{M}_{\mathfrak{K}(f)}$.

Define $\beta \in P\Sigma O_{k+n}$ by $\beta = \phi^{-1}\gamma_{n-1}\gamma_{n-2}...\gamma_1$ where the γ_i are as in Lemma 2.6.6. Notice that $[\alpha] \otimes \mathfrak{M}_f \otimes [\beta] = \mathfrak{M}_0 \otimes [\gamma_{n-1}\gamma_{n-2}...\gamma_1]$.

Recall $\gamma = \gamma_n\gamma_{n-1}...\gamma_1$. By Lemma 2.6.6, we have that $[\gamma] \otimes \mathfrak{M}_0 = \mathfrak{M}_0 \otimes [\gamma_{n-1}\gamma_{n-2}...\gamma_1\gamma_n]$.

Notice that

$$[\gamma] \otimes ([\alpha] \otimes \mathfrak{M}_f \otimes [\beta]) = [\gamma] \otimes \mathfrak{M}_0 \otimes [\gamma_{n-1}\gamma_{n-2}...\gamma_1] =$$

$$= \mathfrak{M}_0 \otimes [\gamma_{n-1}\gamma_{n-2}...\gamma_1\gamma_n] \otimes [\gamma_{n-1}\gamma_{n-2}...\gamma_1] =$$

$$= \mathfrak{M}_0 \otimes [\gamma_{n-1}\gamma_{n-2}...\gamma_1] \otimes [\gamma] =$$

$$= ([\alpha] \otimes \mathfrak{M}_f \otimes [\beta]) \otimes [\gamma].$$

In $\mathfrak{M}_0$, we have that $(s_1....s_n)^2 \cdot 0 = s_1...s_n \cdot 1 \cdot s_{k+n}s_{k+1}...s_{k+n-1} = 0 \cdot s_1...s_k s_{k+n}s_{k+1}...s_{k+n-1}$. Let $\mathfrak{M}_g$ be the bimodule $[\alpha] \otimes \mathfrak{M}_f \otimes [\beta] = \mathfrak{M}_0 \otimes [\gamma_{n-1}\gamma_{n-2}...\gamma_1]$. In $\mathfrak{M}_g$, we have the same induced cyclic ordering except that it now has the appropriate conjugations for the braid action that first takes s_{k+1} to fall between s_{k+n} and s_1 (this is the action of γ_1), then takes s_{k+2} to fall between s_{k+1} and s_1 (action of γ_2), and so on through s_{k+n-1}.

So $\mathfrak{M}_g$ is a twisted kneading bimodule, and by Proposition 2.5.2 there exists a topological polynomial g whose associated bimodule is $\mathfrak{M}_g$. Notice that $P_g = P_f$ and that the mapping schemes of these two topological polynomials are the same. While the labels of the period are all 0 and of the preperiod are all 1 in the Moore diagram of $\mathfrak{K}(g)$, they might not be in the Moore diagram of $\mathfrak{K}(f)$.

So we have that $[\gamma] \otimes \mathfrak{M}_g = \mathfrak{M}_g \otimes [\gamma]$, where γ acts on $S_2 \setminus P_g$ by the Dehn twist T_Γ about a non-peripheral simple closed curve Γ that separates the period of P_g from the rest of P_g. By Lemma 2.6.5, g is not equivalent to any complex polynomial.

$\square$

This yields the following corollary, which is also a consequence of Koch's result:

Corollary 2.6.8. *Every quadratic mapping scheme with preperiod length $k \geq 1$ and period length $n \geq 2$ is realized by a topological polynomial that is not equivalent to any complex polynomial.*

Recall that by Berstein-Levy (and an easy additional observation), every quadratic mapping scheme not meeting the hypotheses of Corollary 2.6.8 can only be realized by topological polynomials that *are* equivalent to complex polynomials.

Chapter 3

Sub-hyperbolicity of the mega-bimodule

3.1 The quadratic periodic case

Let A denote the standard generating set for $P\Sigma O_n$; $A = \{a_{i,j} \mid 1 \leq i, j \leq n, i \neq j\}$. We will use a superset $S \supseteq A$ and take it as our generating set. Gutiérrez and Krstić used this generating set S to show that $P\Sigma O_n$ has a regular language of normal forms [GK98]. Their proof relies on the combinatorial analysis of the automorphisms in S found in McCullough and Miller's monograph on symmetric automorphisms of free products [MM96]. We will exploit this combinatorial description of the generating set S to describe computations in forms easily implemented by a computer.

For $1 \leq j \leq n$, let $S_j \subset [n] \setminus \{j\}$ (we use $[n]$ to denote the set $\{1, 2, ..., n\}$). Define $A_{S_j,j} = \prod_{i \in S_j} a_{i,j}$. For $\chi \subset [n]$ such that $1, n \in \chi$, define $w_\chi = x_1 x_2 ... x_n \in \{0,1\}^n$ with $x_{i-1} = 1$ if and only if $i \in \chi$ for all $1 < i \leq n$ ($x_n = 1$ if and only if $1 \in \chi$, which always holds by definition). We denote the complement of χ by $\bar{\chi}$.

Lemma 3.1.1. *For $j \neq 1$, then $[A_{S_j,j}] \otimes \mathfrak{M}_{w_\chi} = \mathfrak{M}_{w_\chi} \otimes [A_{S'_{j-1},j-1}]$, where:*

- *if $1 \notin S_j, j \in \chi$, then $S'_{j-1} = (S_j \cap \chi) - 1$,*

- *if $1 \notin S_j, j \notin \chi$, then $S'_{j-1} = (S_j \cap \bar{\chi}) - 1$,*

- *if $1 \in S_j, j \in \chi$, then $S'_{j-1} = (S_j \cup \bar{\chi}) - 1$, and*

- *if $1 \in S_j, j \notin \chi$, then $S'_{j-1} = (S_j \cup \chi) - 1$.*

  For $j = 1$, then $[A_{S_1,1}] \otimes \mathfrak{M}_{w_\chi} = \mathfrak{M}_{w_{\chi'}} \otimes [A_{S'_n,n}]$, where:

- *if $n \notin S_1$, then $\chi' = \chi \triangle S_1$ and $S'_n = (S_1 \cap \bar{\chi}) - 1$, and*

- *if $n \in S_1$, then $\chi' = \overline{(\chi \triangle S_1)} \cup \{1\}$ and $S'_n = (S_1 \cup \bar{\chi}) - 1$*

Proof. This follows directly from Proposition 2.6.1.

For example, if $j \neq 1$, $1 \notin S_j$, and $j - 1 \in \chi$, then $[A_{S_j,j}] \otimes \mathfrak{M}_{w_\chi}$ consists of $[a_{i,j}] \otimes \mathfrak{M}_{x_1 \ldots x_{n-2} 11}$ where $x_{j-1} = 1$ and $i \neq 1$. So by Proposition 2.6.1, if $x_{i-1} = 0$, then $[a_{i,j}] \otimes \mathfrak{M}_{x_1 \ldots x_{n-2} 11} = \mathfrak{M}_{x_1 \ldots x_{n-2} 11}$. If $x_{i-1} = 1$, then $[a_{i,j}] \otimes \mathfrak{M}_{x_1 \ldots x_{n-2} 11} = \mathfrak{M}_{x_1 \ldots x_{n-2} 11} \otimes [a_{i-1,j-1}]$. So for $i \in S_j$, we only have $i - 1 \in S'_{j-1}$ if $x_{i-1} = 1$; that is, if $i \in \chi$. Since there is no $a_{1,j}$, we cannot have $i - 1 \in S'_{j-1}$ if $i \notin S_j$. Hence, $[A_{S_j,j}] \otimes \mathfrak{M}_{w_\chi} = \mathfrak{M}_{w_\chi} \otimes [A_{S'_{j-1},j-1}]$ with $S'_{j-1} = (S_j \cap \chi) - 1$.

$\square$

Define $A_{S_1,S_2,\ldots,S_n} = A_{S_1,1} \cdot A_{S_2,2} \cdot \ldots \cdot A_{S_n,n}$ with $A_{S_j,j} = \prod_{i \in S_j} a_{i,j}$ and $S_j \subset [n] \setminus \{j\}$ as before.

Definition 3.1.2. We will call $A_{S_1,S_2,\ldots,S_n}$ *simple* if for all $1 \leq j_1 \neq j_2 \leq n$ we have that $[A_{S_{j_1},j_1}, A_{S_{j_2},j_2}] = \mathtt{I}$ (i.e. if all the $A_{S_j,j}$ pairwise commute).

Fortunately, we have a nice combinatorial characterization of simplicity:

Lemma 3.1.3. *[MM96]*

$A_{S_1,S_2,\ldots,S_n}$ *is simple if and only if for all $i \neq j \in [n]$ we have that:*

- $i \in S_j, j \in S_i$ *implies* $S_j \cup S_i = [n]$

- $i \in S_j, j \notin S_i$ *implies* $S_j \cup \bar{S}_i = [n]$ *(i.e. $S_i \subset S_j$)*

- $i \notin S_j, j \in S_i$ *implies* $\bar{S}_j \cup S_i = [n]$ *(i.e. $S_j \subset S_i$)*

- $i \notin S_j, j \notin S_i$ *implies* $\bar{S}_j \cup \bar{S}_i = [n]$ *(i.e. $S_i \cap S_j = \emptyset$)*

So when $n = 4$ for example, the simple automorphisms are of the following forms:

$$\mathtt{I} \qquad a_{i,j} \qquad a_{i,j} a_{k,j} \qquad a_{i,j} a_{k,l} \qquad a_{i,j} a_{i,k} a_{j,k} \qquad a_{i,j} a_{k,j} a_{l,k} a_{j,k}$$

where $1 \leq i, j, k, l \leq n$ are all distinct. There are 73 (including $\mathtt{I}$) such automorphisms.

Let S be the set of all simple pure symmetric outer automorphisms $A_{S_1,S_2,\ldots,S_n}$. Notice that S generates $P\Sigma O_n$ since it contains A. By Lemma 3.1.1 and the fact that restrictions of commuting elements will commute, we see that the restriction of an element of S will also be an element of S. Thus, the restriction map will be non-increasing in wordlength with respect to S. We compute these restrictions below.

Proposition 3.1.4. *For $A_{S_1,S_2,\ldots,S_n}$ a simple automorphism and $T_A = \{j \mid 1 \in S_j \text{ for } j > 1, n \in S_j \text{ for } j = 1\}$, then $[A_{S_1,S_2,\ldots,S_n}] \otimes \mathfrak{M}_{w_\chi} = \mathfrak{M}_{w_{\chi'}} \otimes [A_{S'_1,S'_2,\ldots,S'_n}]$ where:*

for $j > 1$:

- *if $j \in \chi \cap \bar{T}_A$, then $S'_{j-1} = (S_j \cap \chi) - 1$,*

- *if $j \in \chi \cap T_A$, then $S'_{j-1} = (S_j \cup \bar{\chi}) - 1$,*

- *if $j \in \bar{\chi} \cap \bar{T}_A$, then $S'_{j-1} = (S_j \cap \bar{\chi}) - 1$,*

- *if $j \in \bar{\chi} \cap T_A$, then $S'_{j-1} = (S_j \cup \chi) - 1$,*

and for $j = 1$:

- *if $1 \notin T_A$ (i.e. $n \notin S_1$) then $\chi' = \chi \triangle S_1$ and $S'_n = (S_1 \cap \bar{\chi}) - 1$, else*

- *if $1 \in T_A$ (i.e. $n \in S_1$) then $\chi' = \overline{(\chi \triangle S_1)} \cup \{1\}$ and $S'_n = (S_1 \cup \bar{\chi}) - 1$.*

Proof. This result follows directly from Lemma 3.1.1.

$\square$

Definition 3.1.5. For $A_{S_1,S_2,\ldots,S_n} \in S$ and $\chi \subset [n]$ such that $1, n \in \chi$, define $A_{S_1,S_2,\ldots,S_n}|_{w_\chi} = A_{S'_1,S'_2,\ldots,S'_n}$, where $[A_{S_1,S_2,\ldots,S_n}] \otimes \mathfrak{M}_{w_\chi} = \mathfrak{M}_{w_{\chi'}} \otimes [A_{S'_1,S'_2,\ldots,S'_n}]$ and the S'_j are as given in Proposition 3.1.4.

Notice that if $j \notin T_A$, then $\#S'_{j-1} \leq \#S_j$ (this still holds for S'_n and S_1). So if $S_j = \varnothing$, then $S'_{j-1} = \varnothing$ as well. Thus, if $A_{S_1,S_2,\ldots,S_n}$ is a simple automorphism with exactly k of its S_j being non-empty, then any of its restrictions will be simple automorphisms with at most k of its S'_{j-1} non-empty. So we augment our length function to distinguish between elements of equal wordlength in S by also considering the number of non-empty S_j.

Definition 3.1.6. The *augmented length function* $\ell : S^* \to \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ is given by $\ell(a_1 a_2 \ldots a_m) = (m, \#$ of non-empty S_j in $a_1 a_2 \ldots a_m)$

Proposition 3.1.7. *If there exists an $m > 0$ such that for all $a_1 a_2 \ldots a_m \in S^*$ a representative of some $g \in P\Sigma O_n$ with $m = |g|_S$ and for every $\chi \subset [n]$ such that $1, n \in \chi$ we have that $\ell(a_1 a_2 \ldots a_m|_{w_\chi}) < \ell(a_1 a_2 \ldots a_m)$ in the lexicographic ordering on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, then the $P\Sigma O_n$-megabimodule $\mathfrak{G}_n$ is sub-hyperbolic.*

Proof. An element g of a self-similar group is the nucleus of the action on the tree T if and only if there exists some vertex $v \in T$ such that $g|_v = g$ or g is the restriction of such an element. Since ℓ is non-increasing under restrictions, if an m satisfying the hypotheses exists, any elements of $P\Sigma O_n$ of S-wordlength m cannot self-restrict at any vertex. So these elements can only be in the nucleus if there is an element h of length greater than m that does self-restrict at some vertex v. But if such an h existed, a length m subword of a representative of h would also self-restrict at v, which contradicts the above. Hence, all the elements of $P\Sigma O_n$ that are in the nucleus of the self-similar action of the faithful quotient have S-wordlength less than m, and the nucleus is finite.

$\square$

By writing a C++ program to implement the computations from Proposition 3.1.4 and applying Proposition 3.1.7, we answer Question 1 in the simplest non-trivial case:

Theorem 1.3.3. *If f is a quadratic topological polynomial that is periodic with period length $n = 4$, then $\mathfrak{G}_f$ is sub-hyperbolic.*

3.2 Description of the program

We encode a simple automorphism $A_{S_1, S_2, \ldots, S_n} \in S$ as an $n \times n$ binary matrix with entry i, j being 1 if and only if $i \in S_j$. We can easily determine which of these matrices correspond to simple automorphisms by checking McCullough and Miller's conditions (Lemma 3.1.3).

We also parametrize the basis of kneading bimodules with length n binary strings that begin and end with the letter 1. These strings will play the role of χ in Proposition 3.1.4.

In this context, the program can perform the computations of Proposition 3.1.4 as Boolean operations between the n string representing the kneading bimodule and the columns of the matrix representing the simple automorphism. Reducing all the indices by 1 becomes a (cyclical) shift in the column.

To deal with minimal length representatives of group elements, we next take each word of length 2 in S and find a unique word that represents the same group element. To do this, first we eliminate any inner automorphisms created by the product. For example, in the $n = 4$ case $a_{3,1}a_{4,1}a_{3,4} \cdot a_{2,1}$ will use representative $a_{3,4} \cdot \mathtt{I}$. In the simple automorphism notation, this is $A_{\{3,4\}, \varnothing, \varnothing, \{3\}} \cdot A_{\{2\}, \varnothing, \varnothing, \varnothing} = A_{\varnothing, \varnothing, \varnothing, \{3\}} \cdot A_{\varnothing, \varnothing, \varnothing, \varnothing}$. Then we move as many $a_{i,j}$ as possible to the left automorphism. So $a_{2,1} \cdot a_{3,1}$ will use representative $a_{2,1}a_{3,1} \cdot \mathtt{I}$. Again in simple automorphism notation: $A_{\{2\}, \varnothing, \varnothing, \varnothing} \cdot A_{\{3\}, \varnothing, \varnothing, \varnothing} = A_{\{2,3\}, \varnothing, \varnothing, \varnothing} \cdot A_{\varnothing, \varnothing, \varnothing, \varnothing}$. To perform these operations, the program deletes or moves subsets within a S_j (starting as large as possible and working its way down to single $a_{i,j}$s) and then checks the simplicity conditions.

After assigning these representatives, the program checks uniqueness by confirming that these reductions induce a partial order on the set of $S \times S$.

The program then takes all reduced representatives of words on length 3 in S that do not involve $A_{\varnothing, \varnothing, \varnothing, \varnothing}$ and checks if they restrict to themselves after any n restrictions. If so, the program stores the word as a representative of an element in the nucleus. If not, the program checks if ℓ decreases under all of these restrictions. If so, the representative does not ever self-restrict. If not, the program tries another n restrictions and repeats. In the $n = 4$ case, all length 3 representatives after $2n$ restrictions either yield a decrease in ℓ under all such restrictions or self-restrict under at least one such restriction.

So now these representatives give us length 3 subwords of any representatives of longer length that

self-restrict. The program computes all possible representatives of length 4 whose 3-prefix and 3-suffix self-restrict on matching vertices. Note that matching does not mean equal; if the 3-prefix self-restricts on vertex v, then the 3-suffix must self-restrict on vertex $a(v)$, where a is the first letter of the 3-prefix.

The program then checks if all of these length 4 representatives actually correspond to group elements of S-wordlength 4. To do so, the program reduces the words by moving as many $a_{i,j}$s as possible to the *right*. For example, $a_{1,4}a_{3,4} \cdot a_{4,3} \cdot a_{2,3}a_{4,3} \cdot a_{2,1}$ reduces to $\mathtt{I} \cdot a_{1,4}a_{3,4} \cdot a_{2,3}a_{4,3} \cdot a_{2,1}a_{4,3}$. That is, $A_{\varnothing,\varnothing,\varnothing,\{1,3\}} \cdot A_{\varnothing,\varnothing,\{4\},\varnothing} \cdot A_{\varnothing,\varnothing,\{2,4\},\varnothing} \cdot A_{\{2\},\varnothing,\varnothing,\varnothing} = A_{\varnothing,\varnothing,\varnothing,\varnothing} \cdot A_{\varnothing,\varnothing,\varnothing,\{1,3\}} \cdot A_{\varnothing,\varnothing,\{2,4\},\varnothing} \cdot A_{\{2\},\varnothing,\{4\},\varnothing}$. In the $n = 4$ case, all the length 4 representatives reduce. Hence, no self-restricting elements of S-wordlength greater than 3 exist, and so the nucleus is finite.

3.3 Remaining cases

Our program for the quadratic periodic with period length $n = 5$ case is similar to the above, however it involves more refined methods to reduce the words in S^*. Also, the size of the generating set S is over a thousand in this case (it is under a hundred for $n = 4$), and the size of the basis of kneading bimodules grows exponentially in n. So while the $n = 4$ program runs in about half a minute, the $n = 5$ program takes weeks to finish. For these reasons, we do not yet have a positive answer to Question 1 in this case, but we are confident that this will change soon.

Clearly, this method is not tenable for periods $n > 5$. Periodic cubic polynomials with small period length should have relations similar to those computed in Section 3.1, so we anticipate being able to extend Theorem 1.3.3 to these cases as well.

We hope to compare the nucleii of these simple cases and produce a conjecture for the nucleii for the computationally unrealistic cases. We already have some sense for how those cases should work out.

Conjecture 3.3.1. *For $n > 2$, the value $m = n$ with Proposition 3.1.7 implies the sub-hyperbolicity of $\mathfrak{G}_n$.*

We have not attempted here to prove sub-hyperbolicity in any preperiodic cases. Unlike the periodic case, there exist obstructed topological polynomials with stable Levy cycles realizing these mapping schemes. As a result, to prove sub-hyperbolicity, we need to prove that the elements of $P\Sigma O_{k+n}$ representing the Levy cycle have finite order in the faithful quotient of the action (see the next chapter for a more thorough discussion). With few generators, finding the order of these elements is straightforward. It is less clear how to program a computer to do a similar analysis when the size of the generating set is large. We hope to use some of the properties of the kneading bimodules $\mathfrak{M}_{x_1...x_{k-1}1, x_{k+1}...x_{k+n-1}0}$ proved in [BN08] to develop a method to determine the order of these elements that is implementable by a computer.

Chapter 4

Applications to complex dynamics

4.1 Realizability of obstructed topological polynomials

The proof of Theorem 2.6.7 that there exist obstructed quadratic preperiodic topological polynomials only uses the condition on the degree to produce the basis of kneading bimodules $\mathfrak{M}_{x_1 \ldots x_{k-1} 1, x_{k+1} \ldots x_{k+n-1} 0}$. For arbitrary degree $d \geq 3$, we will not be able to produce an explicit basis in this way. However, we establish a few conditions on the mapping scheme when we can replicate the argument from Section 2.6.

We will show the existence of a kneading bimodule that acts for our mapping scheme as $\mathfrak{M}_0$ did in the proof of Theorem 2.6.7. In order for the bimodule to play this role, the kneading automaton that generates it needs to have:

1. the states associated with the post-critical points inside the Levy cycle must restrict to each other all in the same coordinate,

2. any pair of these same states must not share any other coordinates with non-trivial restrictions,

3. the permutations of these states must not interact with the non-trivial restrictions, and

4. the induced cyclic ordering on the generating set matches the twists from the γ_i.

We need 1 so that we can cycle the $a_{i,j}$ and γ_i as in Lemma 2.6.5. We need 2 so that we do not pick up any extra generators while cycling through. We need 3 to guarantee that the presence of active states does not disturb the two previous properties. Finally, we need 4 to guarantee that our resulting bimodule is twisted kneading.

In the case of a period of length at least two which has no critical values (in some sense, a period which is *strongly* not an attractor), the proof follows very similar lines to the quadratic case. For a period containing critical values, there need to be sufficiently many critical values outside the period so that their associated states can act on the non-trivial restriction coordinates, so that the states associated with the critical values in the period can act trivially on these coordinates. The following proposition precisely defines "sufficiently many" for a given period.

Proposition 4.1.1. *For $S = S(C, P, \alpha, \nu)$ a polynomial mapping scheme of degree d with $\Omega = \{\omega_1, ..., \omega_n\} \subset Z \setminus C$ such that $\alpha(\Omega) = \Omega$ and*

$$\#(\alpha^{-1}(\Omega) \setminus \Omega) \leq \sum_{z \in C \setminus \{\infty\}, \alpha(z) \notin \Omega} (\nu(z) - 1),$$

then there exist kneading automata for S such that the states $s_\Omega = \{s_1, ..., s_n\}$ associated with Ω have all of the following properties:

1. *the arrows from states in s_Ω to other states in s_Ω are labeled by 0,*

2. *the arrows from states in s_Ω to states not in s_Ω have pairwise disjoint labels from $\{1, ..., d-1\}$,*

3. *the sets of labels on which the states in s_Ω act non-trivially are pairwise disjoint, and*

4. *for all $1 \leq i, j \leq n, i \neq j$, state s_i acts trivially on any letter labeling an arrow leaving s_j.*

Proof. Choose a planar generating set for $\pi_1(S^2 \setminus P)$. Since S is a polynomial mapping scheme, there exists some topological polynomial f realizing it. Let $\mathfrak{K}(f)$ be the kneading automaton of f. In its Moore diagram, we will relabel some of the arrows leaving the states $s_\Omega = \{s_1, ..., s_n\}$ associated with Ω and redefine the actions of possibly all of the active states to produce the Moore diagram of a kneading automaton $\mathfrak{K}$ with the desired properties.

First, relabel all arrows within s_Ω by 0. Notice there are no arrows entering s_Ω from outside it (since $\alpha(\Omega) = \Omega$).

Second, since

$$\#(\alpha^{-1}(\Omega) \setminus \Omega) \leq \sum_{z \in C \setminus \{\infty\}, \alpha(z) \notin \Omega} (\nu(z) - 1),$$

we have that

$$\sum_{z \notin \Omega, \alpha(z) \in \Omega} (\nu(z) - 1) + \#(\alpha^{-1}(\Omega) \setminus \Omega) \leq \sum_{z \in C \setminus \{\infty\}} (\nu(z) - 1),$$

which by the Riemann-Hurwitz Formula gives us

$$\sum_{z \notin \Omega, \alpha(z) \in \Omega} \nu(z) \leq d - 1.$$

Thus, the number of arrows entering Ω in S is at most $d - 1$. Redefine the actions of the states in s_Ω so that the sets on which they act non-trivially are pairwise disjoint (note we can do this by the above calculation). As required by Lemma 2.5.4, label any arrows from s_Ω to states associated with critical points

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by an appropriate letter on which the originating state acts non-trivially. As for the remaining arrows leaving s_Ω, notice by the calculation above there are enough letters remaining in $\{1, ..., d-1\}$ so that these may be labeled from this set so that these labels are pairwise disjoint with each other and with the non-trivial actions.

Notice that states s_Ω in our new automaton now fit the requirements listed in the statement of the proposition. However, since we redefined some of the actions, we may need to redefine the actions of the states not in s_Ω in order to ensure that our automaton is dendroid.

Let k be the number of critical values of S not in Ω. Let $\mathcal{H}_0$ be the hypergraph with vertices $\{0, 1, ..., d-1\}$ and hyperedges defined by the actions of the states in s_Ω. We will redefine the actions of the k active states outside of s_Ω one-by-one by considering their resulting hypergraphs $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_k$. We will think of $\mathcal{H}_{i-1}$ as a sub-hypergraph of $\mathcal{H}_i$.

First, define the actions of these active states so as to connect the partial hypergraph induced by the hyperedges of $\mathcal{H}_0$, without adding cycles. By our assumption that

$$\#(\alpha^{-1}(\Omega) \setminus \Omega) \le \sum_{z \in C \setminus \{\infty\}, \alpha(z) \notin \Omega} (\nu(z) - 1),$$

we will add more edges than we already have in $\mathcal{H}_0$, so this will eventually yield a hypergraph $\mathcal{H}_{i_0}$ where the partial hypergraph induced by the hyperedges is connected. Continue redefining the actions of the active states so as to maintain this property and not create cycles.

By a standard result in combinatorics (see e.g. Proposition 4 in Chapter 17 of [Ber73]), a connected hypergraph with no cycles on d vertices with m hyperedges containing the vertices $\{E_i\}_{i=1}^m$, obeys the formula:

$$\sum_{i=1}^m (\#E_i - 1) = d - 1.$$

This, of course, is exactly the Riemann-Hurwitz Formula in our setting. Therefore, we may redefine the actions of all k active states so that the final hypergraph H_k is connected (notice that every vertex lies in some hyperedge) and contains no cycles. In other words, the sequence of permutations defined by our automaton is dendroid. Note that we may need to redefine the restrictions of these active states to preserve property 2 of Lemma 2.5.4.

Let $\mathfrak{K}$ be this automaton. Notice that $\mathfrak{K}$ is dendroid by construction. Further, since its input and output sets are equal (as they were in $\mathfrak{K}(f)$), it is kneading.

$\square$

Lemma 4.1.2. *For $S = S(C, P, \alpha, \nu)$ a polynomial mapping scheme of degree d meeting the hypotheses of Proposition 4.1.1, g a topological polynomial realizing S with the kneading automaton guaranteed by Proposition 4.1.1, and $\gamma \in P\Sigma O_{\#(P \setminus \{\infty\})}$ acting by a Dehn twist T_Γ about a curve Γ separating Ω (as in the Proposition) from the rest of P, if $[\gamma] \otimes \mathfrak{M}_g \simeq \mathfrak{M}_g \otimes [\gamma]$, then Γ is a g-stable Levy cycle of length 1 and g is not equivalent to any complex polynomial.*

Proof. As in Lemma 2.6.5, note that the since γ acts by a Dehn twist, it preserves the induced ordering of $\mathfrak{M}_g$ and we have that $T_\Gamma \circ g$ homotopic to $g \circ T_\Gamma$. So $g(\Gamma)$ must be homotopic to Γ, and g must map it by degree 1. By property (2) of the kneading automaton $\mathfrak{K}(g)$ from Proposition 4.1.1, every component of $g^{-1}(\Gamma)$ except Γ itself is peripheral. Hence, Γ is a stable Levy cycle of length 1 for g.

$\square$

Our main result follows from the above proposition using the argument of the Section 2.6.

Theorem 1.2.1. *Suppose that a polynomial mapping scheme satisfies one of the following conditions:*

1. *at least one (non-attractor) period of length at least two and not containing critical values,*

2. *at least two (non-attractor) periods not containing critical values,*

3. *at least two non-attractor periods both of length at least two, or*

4. *at least four non-attractor periods.*

Then this scheme is realized by a topological polynomial that is not equivalent to any complex polynomial.

Proof. Let S be the mapping scheme satisfying one of the cases.

1. S has one period of length at least two not containing critical values.

   Let $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ be the period given (with $\alpha(\omega_i) = \omega_{i+1}$ and $\alpha(\omega_n) = \omega_1$) and let $P = \{\infty, \omega_1, ..., \omega_m\}$. Choose a planar generating set $\{s_1, ..., s_m\}$ such that s_i loops around ω_i. By the Riemann-Hurwitz Formula, $C \setminus \{\infty\}$ has at most $d - 1$ elements. So Ω has at most $d - 1$ arrows incoming (and none outgoing) in the mapping scheme S. Thus, we may apply Proposition 4.1.1; let $\mathfrak{M}_0$ be the bimodule generated by the kneading automaton $\mathfrak{K}$ given by this proposition.

   For $1 \leq i \leq n$, define

   $$\gamma_i = a_{i,i-1}a_{i,i-2}...a_{i,1}a_{i,n}a_{i,n-1}...a_{i,i+1}$$

   and $\gamma = \gamma_n\gamma_{n-1}...\gamma_2\gamma_1$ (similar to the γ_i in Lemma 2.6.6 but reducing all the indices by k).

Consider $[\gamma_i] \otimes \mathfrak{M}_0$. First, we have $[a_{i,i+1}] \otimes \mathfrak{M}_0$. Now, we do not know the full wreath recursions for the states s_i and s_{i+1}, however we do know they are inactive and that $s_i|_0 = s_{i-1}$ and $s_{i+1}|_0 = s_i$. So $s_i^{s_{i+1}}|_0 = s_{i-1}^{s_i}$. Further, since none of the other arrows from these two states share labels with each other, the wreath recursions do not share non-trivial coordinates besides the one we have already considered. Thus, the only twist produced is $a_{i-1,i}$, and the kneading bimodule remains unchanged.

We may repeat the above argument for the rest of γ_i, and we find that

$$[\gamma_i] \otimes \mathfrak{M}_0 = [a_{i,i-1}a_{i,i-2}...a_{i,1}a_{i,n}a_{i,n-1}...a_{i,i+1}] \otimes \mathfrak{M}_0 =$$
$$= \mathfrak{M}_0 \otimes [a_{i-1,i-2}a_{i-1,i-3}...a_{i-1,n}a_{i-1,n-1}a_{i-1,n-2}...a_{i-1,i}] =$$
$$= \mathfrak{M}_0 \otimes [\gamma_{i-1}].$$

So $[\gamma_i] \otimes \mathfrak{M}_0 = \mathfrak{M}_0 \otimes [\gamma_{i-1}]$ and $[\gamma_1] \otimes \mathfrak{M}_0 = \mathfrak{M}_0 \otimes [\gamma_n]$ just as in the lemma and for the exact same reasons.

Consider the induced ordering by $\mathfrak{M}_0$ on the generating set as follows: $(s_m...s_{n+1}s_1...s_n)^d \cdot 0 = (s_m...s_{n+1}s_1...s_n)^{d-1}s_m...s_{n+1}\cdot 0 \cdot s_n s_1...s_{n-1}$. Just as in Theorem 2.6.7, this rearrangement of $\{s_1,..,s_n\}$ is consistent with the action of $\gamma_{n-1}...\gamma_2\gamma_1$. We will get the action of some other $\beta \in P\Sigma O_m$ on the rest of the cyclic order. Notice that $\gamma_{n-1}...\gamma_2\gamma_1$ and β commute since they act on disjoint coordinates (in fact, γ and β commute).

Set $\mathfrak{M}_g = \mathfrak{M}_0 \otimes [\gamma_{n-1}\gamma_{n-2}...\gamma_1\beta]$. This is a twisted kneading bimodule, so let g be the topological polynomial uniquely determined by this twisted kneading bimodule and this planar generating set. Notice that g has mapping scheme S.

As before in the proof of Theorem 2.6.7, we have that

$$[\gamma] \otimes \mathfrak{M}_g = [\gamma_n\gamma_{n-1}...\gamma_1] \otimes \mathfrak{M}_0 \otimes [\gamma_{n-1}...\gamma_1\beta] =$$
$$= \mathfrak{M}_0 \otimes [\gamma_{n-1}...\gamma_1\gamma_n] \otimes [\gamma_{n-1}...\gamma_1\beta] =$$
$$= \mathfrak{M}_0 \otimes [\gamma_{n-1}...\gamma_1] \otimes [\gamma\beta] =$$
$$= \mathfrak{M}_0 \otimes [\gamma_{n-1}...\gamma_1\beta] \otimes [\gamma] =$$
$$= \mathfrak{M}_g \otimes [\gamma].$$

So again we see that for Γ a non-peripheral simple closed curve separating $\{\omega_1, ..., \omega_n\}$ from the rest of P, we have $g(\Gamma)$ homotopic to Γ with g mapping it by degree 1.

So one component of $g^{-1}(\Gamma)$ is homotopic to Γ (this is the preimage labeled by 0 in our automaton). The other preimages will be peripheral since each of them separate at most 1 post-critical point from the rest of P (these are the preimages labeled by 1 through $d-1$). Thus, Γ is a g-stable Levy cycle, and g is obstructed.

2. S has two periods not containing critical values.

   By 1, we need only consider when we have $\omega_1, \omega_2 \in P$ such that α fixes both and neither are critical values. Let $P = \{\infty, \omega_1, \omega_2, \omega_3, ..., \omega_m\}$. Choose a planar generating set $\{s_1, ..., s_m\}$ with s_i looping around ω_i.

   By the Riemann-Hurwitz Formula, in S there are at most $d-1$ arrows incoming to $\Omega = \{\omega_1, \omega_2\}$ from outside the set (there are no outgoing arrows). So by Proposition 4.1.1, there exists a keading automaton $\mathfrak{K}$ consistent with S with the properties outlined in the statement of the proposition.

   We consider the induced ordering of automaton $\mathfrak{K}$ by $(s_m...s_3s_2s_1)^d \cdot 0 = (s_m...s_3s_2s_1)^{d-1} s_m...s_3 \cdot 0 \cdot s_2 s_1$. So this portion of the cyclic order will remain unchanged. Let $\beta \in P\Sigma O_m$ act consistent with the remainder of the above calculation on $\{s_3, ..., s_m\}$. Notice that β commutes with γ.

   Let Γ be a simple closed curve separating Ω from the rest of P. Note that Γ is non-peripheral. Let $\gamma, \gamma_1, \gamma_2$ be defined by $\gamma_1 = a_{1,2}, \gamma_2 = a_{2,1}, \gamma = \gamma_2 \gamma_1$. Note that γ acts by a Dehn twist about the curve Γ, and that β and γ commute.

   Let the bimodule generated by the kneading automaton $\mathfrak{K}$ with twist β be $\mathfrak{M}_g$. Let g be the unique topological polynomial determined by the twisted kneading bimodule $\mathfrak{M}_g$ and the planar generating set $\{s_1, ..., s_m\}$. Notice that g has mapping scheme S.

   Since $s_1|_0 = s_1$ and $s_2|_0 = s_2$ in $\mathfrak{M}_g$, and s_1 and s_2 share no other non-trivial restrictions, we have that $[a_{1,2}] \otimes \mathfrak{M}_g = \mathfrak{M}_g \otimes [a_{1,2}]$ and $[a_{2,1}] \otimes \mathfrak{M}_g = \mathfrak{M}_g \otimes [a_{2,1}]$. Thus, $[\gamma] \otimes \mathfrak{M}_g = \mathfrak{M}_g \otimes [\gamma]$.

   Finishing the argument as in 1, we see that Γ is a g-stable Levy cycle obstructing g.

3. S has two non-attractor periods both of length at least two.

   Let Ω_1 and Ω_2 be the two periods in question. For $i = 1, 2$, let $p_i = \#(\alpha^{-1}(\Omega_i) \setminus \Omega_i)$. We may assume $p_1 \le p_2$. Notice that

   $$\#(\alpha^{-1}(\Omega_1) \setminus \Omega_1) = p_1 \le p_2 \le \sum_{z \in C \setminus \{\infty\}, \alpha(z) \notin \Omega_1} (\nu(z) - 1).$$

   So Proposition 4.1.1 applies to $\Omega = \Omega_1$.

Now proceed as in 1, with the additional observation that the actions of the states associated with critical values do not affect the conjugations, since all the non-trivial actions of a single state are disjoint from the non-trivial actions and restrictions of all the other states associated with Ω.

4. S has four non-attractor periods.

   Let Ω_1 be the union of two of the non-attractor periods, and Ω_2 be the union of the other two. By the same argument as above in 3, we may assume that $\Omega = \Omega_1$ satisfies the hypotheses of Proposition 4.1.1.

   If Ω is the union of two periods both of length one, then proceed as in 2, with the additional observation that the actions of the states associated with critical values do not affect the conjugations, since all the non-trivial actions of a single state are disjoint from the non-trivial actions and restrictions of all the other states associated with Ω.

   If not, consider a period of length greater than one in Ω and proceed as in 3.

   □

4.2 Examples of obstructed topological polynomials

While our proofs explicitly construct a single obstructed polynomial realizing a particular mapping scheme, we can easily extend the ideas to produce many more obstructed polynomials realizing the same set of mapping schemes.

The simplest way to do so is by pre-composing with Dehn twists about the obstruction Γ (as noted in [BN06]). That is, changing the twisted kneading bimodule from $\mathfrak{M}_g$ to $\mathfrak{M}_g \otimes [\gamma^l]$ for any $l \in \mathbb{Z}$. The proof that Γ is a Levy cycle of length 1 still holds.

See Figure 4.1 for a topological polynomial produced by our proof, Figure 4.2 for this example twisted as in the previous paragraph with $l = 1$, and Figure 4.3 for $l = -1$.

This method can also produce topological polynomials obstructed by Levy cycles of length greater than 1.

Theorem 4.2.1. *For S a polynomial mapping scheme of degree d with a single finite critical point, whose post-critical set contains a period of length $n \geq 2$ which does not contain any critical values, and $1 < l < n$, $l \leq d$ such that l divides n, then there exists a topological polynomial g which realizes S and admits a Levy cycle of length l, but does not admit a Levy cycle consisting only of a single curve surrounding the period in question.*
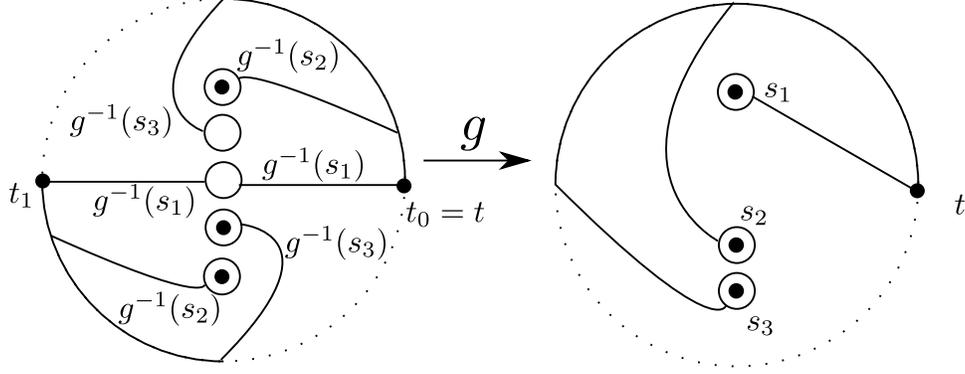
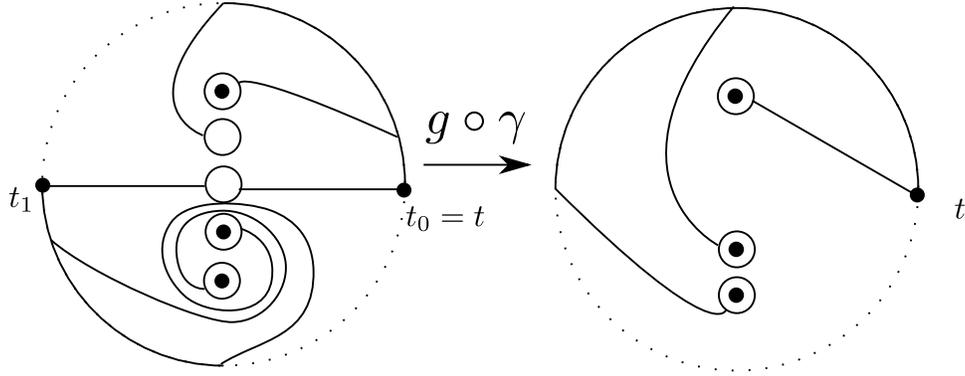Figure 4.1: The obstructed topological polynomial g

Figure 4.2: The obstructed topological polynomial $g \circ \gamma$

Proof. Let $k = \frac{n}{l}$. Let $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ be the period given with $\alpha(\omega_i) = \omega_{i+k}$ for $1 \le i \le n - k$ and $\alpha(\omega_{n-k+i}) = \omega_{i+1}$ for $1 \le i < k$, and $\alpha(\omega_n) = \omega_1$. Let $P = \{\infty, \omega_1, ..., \omega_m\}$ such that for $n < i \le m$, $\alpha(\omega_i) = \omega_{i-1}$ (note this is not our usual convention). Choose a planar generating set $\{s_1, ..., s_m\}$ so that s_i loops around ω_i in the positive direction.

Let f be a topological polynomial realizing S. As in the proof of Proposition 4.1.1, we create a kneading automaton by copying $\mathfrak{K}(f)$ and changing some of the labels on arrows leaving the period Ω. Instead of labeling all the arrows within the period by 0, we cycle through the labels $\{0, 1, ..., l - 1\}$ in the following way:

The arrows leaving $\Omega_0 = \{\omega_1, \omega_2, ...\omega_k\}$ (but staying within Ω) are labeled by 0. The arrows leaving $\Omega_1 = \{\omega_{l+1}, ..., \omega_{2k}\}$ are labeled by 1. We continue in this way, labeling the arrows leaving $\Omega_i = \{\omega_{ik+1}, ..., \omega_{(i+1)k}\}$ (but staying within the period Ω) by i for $0 \le i \le l - 1$.

Additionally, we label all the arrows in the pre-period by any label i_0 such that $l - 1 \le i_0 \le d - 1$, and define the action of the active state to be $(0 i_0 1 ... d - 2)$. Define the bimodule generated by this kneading automaton to be $\mathfrak{M}_0$.
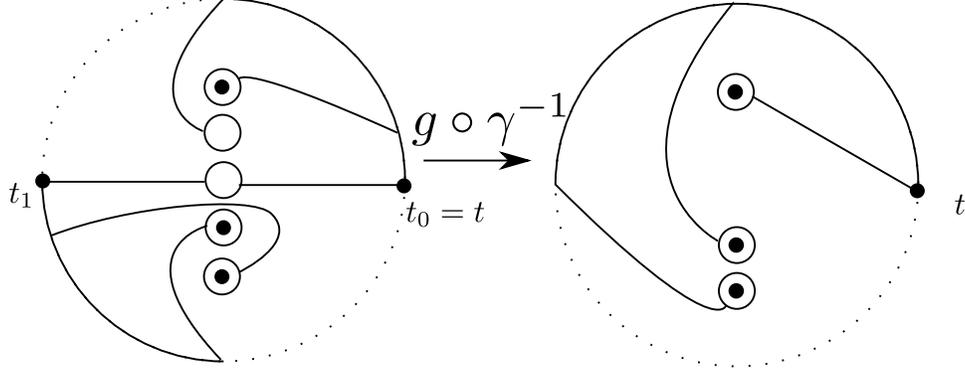
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Figure 4.3: The obstructed topological polynomial $g \circ \gamma^{-1}$

Define $\gamma_{i,j} \in P\Sigma O_m$ for $0 \leq i \leq l-1, 1 \leq j \leq k$ by

$$\gamma_{i,j} = a_{ik+j,ik+j-1}a_{ik+j,ik+j-2}...a_{ik+j,ik+1}a_{ik+j,(i+1)k}a_{ik+j,(i+1)k-1}...a_{ik+j,ik+j+1}.$$

Let $\gamma_i = \gamma_{i,k}\gamma_{i,k-1}...\gamma_{i,1}$, and $\gamma = \gamma_{l-1}\gamma_{l-2}...\gamma_1\gamma_0$.

So, for example, if $n = 4$ and $l = 2$, we would have:

$$\gamma_{0,1} = a_{1,2}$$

$$\gamma_{0,2} = a_{2,1}$$

$$\gamma_{1,1} = a_{3,4}$$

$$\gamma_{1,2} = a_{4,3}$$

$$\gamma_0 = a_{2,1}a_{1,2}$$

$$\gamma_1 = a_{4,3}a_{3,4}.$$

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Another example: if $n = 9$ and $l = 3$, we would have:

$$\gamma_{0,1} = a_{1,3}a_{1,2}$$

$$\gamma_{0,2} = a_{2,1}a_{2,3}$$

$$\gamma_{0,3} = a_{3,2}a_{3,1}$$

$$\gamma_{1,1} = a_{4,6}a_{4,5}$$

$$\gamma_{1,2} = a_{5,4}a_{5,6}$$

$$\gamma_{1,3} = a_{6,5}a_{6,4}$$

$$\gamma_{2,1} = a_{7,9}a_{7,8}$$

$$\gamma_{2,2} = a_{8,7}a_{8,9}$$

$$\gamma_{2,3} = a_{9,8}a_{9,7},$$

which gives

$$\gamma_0 = a_{3,2}a_{3,1}a_{2,1}a_{2,3}a_{1,3}a_{1,2}$$

$$\gamma_1 = a_{6,5}a_{6,4}a_{5,4}a_{5,6}a_{4,6}a_{4,5}$$

$$\gamma_2 = a_{9,8}a_{9,7}a_{8,7}a_{8,9}a_{7,9}a_{7,8}.$$

Notice that for $i_1 \neq i_2$, we have $[\gamma_{i_1,j_1}, \gamma_{i_2,j_2}]$ trivial.

Define Γ to be the multicurve $\{\Gamma_0, ..., \Gamma_{l-1}\}$, where Γ_i is a nonperipheral simple closed curve separating Ω_i from the rest of P_f. Notice that γ_i acts by a Dehn twist about Γ_i.

Consider $[\gamma_{i,j}] \otimes \mathfrak{M}_0$. By construction, all the states associated with post-critical points in Ω_i pairwise share exactly one non-trivial restriction coordinate: i. So we have that $[\gamma_{i,j}] \otimes \mathfrak{M}_0 = \mathfrak{M}_0 \otimes [\gamma_{i-1,j}]$ for $1 \leq i < l-1$, $[\gamma_{0,j}] \otimes \mathfrak{M}_0 = \mathfrak{M}_0 \otimes [\gamma_{l-1,j-1}]$, and $[\gamma_{0,k}] \otimes \mathfrak{M}_0 = \mathfrak{M}_0 \otimes [\gamma_{l-1,1}]$. Thus, $[\gamma_i] \otimes \mathfrak{M}_0 = \mathfrak{M}_0 \otimes [\gamma_{i-1}]$ for $1 \leq i \leq l-1$, and

$$[\gamma_0] \otimes \mathfrak{M}_0 = \mathfrak{M}_0 \otimes [\gamma_{l-1,k-1}\gamma_{l-1,k-2}\cdots\gamma_{l-1,1}\gamma_{l-1,k}].$$

Define

$$\mathfrak{M}_g = \mathfrak{M}_0 \otimes [\gamma_{l-1,k-1}\gamma_{l-1,k-2}\cdots\gamma_{l-1,1}].$$

We consider the induced ordering of $\mathfrak{M}_0$ in a more complicated way than usual:

$$(s_m...s_{n+1}s_{n-k-1}...s_n...s_{3k}s_{k+1}...s_{2k}s_1...s_{k-1}s_k)^d \cdot 0 =$$

$$= (s_m...s_{n+1}s_{n-k-1}...s_n...s_{3k}s_{k+1}...s_{2k}s_1...s_{k-1}s_k)^{d-1} \cdot i_0 \cdot s_n s_{n-k+1}...s_{n-2}s_{n-1} =$$

$$= (s_m...s_{n+1}s_{n-k-1}...s_n...s_{3k}s_{k+1}...s_{2k}s_1...s_{k-1}s_k)^{d-2} \cdot 1 \cdot s_m...s_{n+1}s_n s_{n-k+1}...s_{n-2}s_{n-1} =$$

$$= (s_m...s_{n+1}s_{n-k-1}...s_n...s_{3k}s_{k+1}...s_{2k}s_1...s_{k-1}s_k)^{d-3} \cdot 2 \cdot s_1 s_2...s_k s_m...s_{n+1}s_n s_{n-k+1}...s_{n-2}s_{n-1} =$$

$$...$$

$$= 0 \cdot s_{n-2k}...s_{n-k}...s_1 s_2...s_k s_m...s_{n+1}s_n s_{n-k+1}...s_{n-2}s_{n-1}.$$

So the changes in the induced ordering of $\mathfrak{M}_0$ match the action of $\gamma_{l-1,k-1}\gamma_{l-1,k-2}...\gamma_{l-1,1}$ on $\{s_{n-k+1}, ..., s_n\}$, and $\mathfrak{M}_g$ is twisted kneading.

Notice that for $1 \leq i \leq l-1$, we have $[\gamma_i] \otimes \mathfrak{M}_g = \mathfrak{M}_g \otimes [\gamma_{i-1}]$. Further, for $i = 0$,

$$[\gamma_0] \otimes \mathfrak{M}_g = \mathfrak{M}_0 \otimes [\gamma_{l-1,k-1}\gamma_{l-1,k-2}...\gamma_{l-1,1}\gamma_{l-1,k}] \otimes [\gamma_{l-1,k-1}...\gamma_{l-1,1}] =$$

$$= \mathfrak{M}_0 \otimes [\gamma_{l-1,k-1}..\gamma_{l-1,1}] \otimes [\gamma_{l-1}] =$$

$$= \mathfrak{M}_g \otimes [\gamma_{l-1}].$$

Since $\mathfrak{M}_g$ is twisted kneading, let g be the unique topological polynomial determined by the bimodule $\mathfrak{M}_g$ and the planar generating set $\{s_1, ..., s_m\}$, notice that g has mapping scheme S. By the above computation, g takes each curve Γ_i to Γ_{i+1} (adding mod l) by a degree 1 map. Further, we see by the labeling of our kneading automaton that the components of $g^{-1}(\Gamma_i)$ not labeled by i are all peripheral. So the multicurve Γ is a g-stable Levy cycle of length l.

We leave to the reader the verification that the simple closed curve surrounding the entire period is not an obstruction.

$\square$

For example, both the topological polynomials g and f in Figures 4.4 and 4.5 are quadratic with a prepeTiodic mapping scheme with preperiod length one and period length four. However, while g admits a Levy cycle of length one (not pictured), f only admits a Levy cycle of length two, shown in Figure 4.5 with its inverse images.
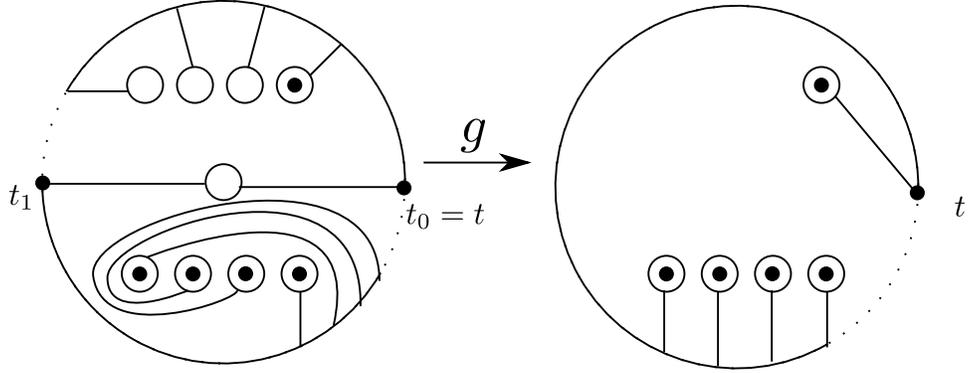
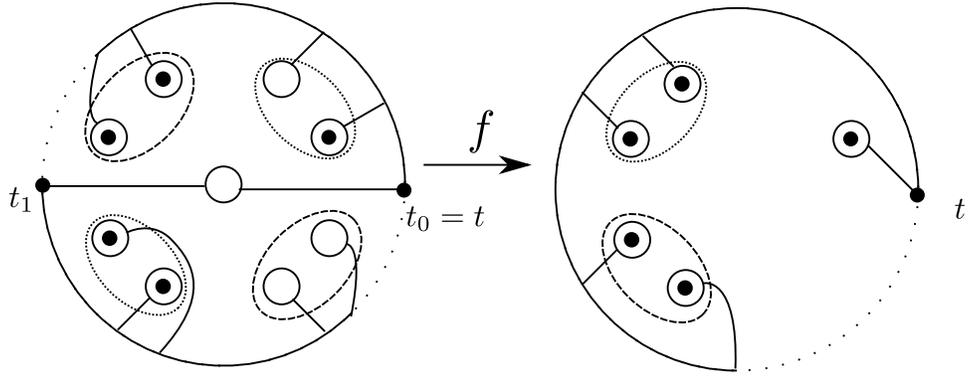Figure 4.4: An obstructed topological polynomial with Levy cycle of length one

Figure 4.5: An obstructed topological polynomial with Levy cycle of length two

4.3 Non-classified mapping schemes

While there exist many polynomial mapping schemes that fall outside the purview of both Theorem 1.2.1 and the Berstein-Levy Theorem, in many of these cases we may still apply Proposition 4.1.1 to show the existence of obstructed topological polynomials realizing the mapping scheme. For example, relatively few mapping schemes with exactly three non-attractor periods do not meet the hypotheses of Proposition 4.1.1, even if they do not meet the conditions of Theorem 1.2.1.

However, there do exist non-hyperbolic mapping schemes for which Proposition 4.1.1 does not apply. For instance, mapping schemes which have only a single period (say of length at least two), and that period contains all the critical values.

Open Problem 1. What can we say about mapping schemes for which no $\Omega \subset Z \setminus C$ satisfies the conditions of Proposition 4.1.1?

Even the single period example mentioned above seems more subtle than the cases we have addressed in this paper. For instance, consider the two schemes in Figure 4.6.
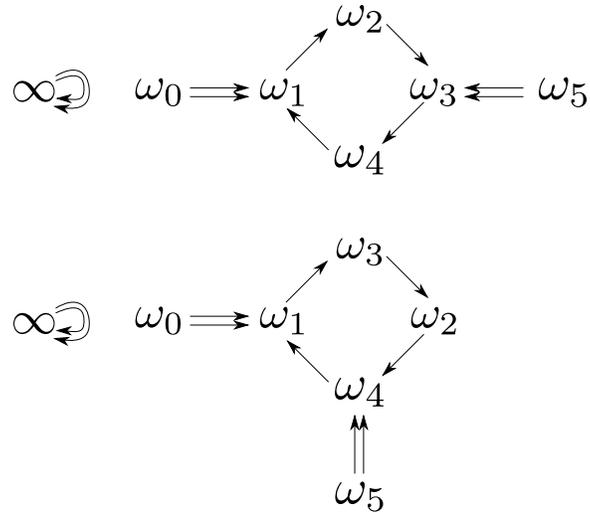
Figure 4.6: Two mapping schemes outside the scope of our results

These mapping schemes are identical in every sense that we have used to distinguish those realizable by obstructed polynomials from those not realizable. However, notice in Figure 4.7 that for the second scheme we may avoid the issues that arise with having all the critical values in the only period by considering a Levy cycle of length greater than 1.
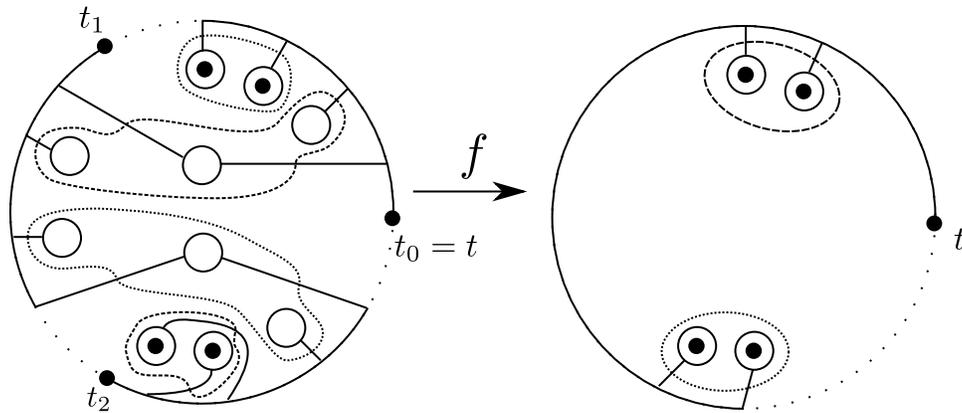

Figure 4.7: An obstructed topological polynomial outside the scope of our results

Somehow, the relative position of the critical values within this period affects the mapping scheme's realizability by obstructed topological polynomials. It seems that one could combine the ideas of Theorem 4.2.1 with the treatment of critical values in Proposition 4.1.1 to gain some ground within this class of mapping schemes.

Open Problem 2. How does the relative position of critical values within a non-attractor period of a mapping scheme affect the scheme's realizability by obstructed topological polynomials?

However, it remains unclear how to prove the (probable) non-realizability of the remaining mapping schemes. Nekrashevych proved a criterion for the equivalence to a complex polynomial for topological polynomials whose associated bimodule is kneading (i.e. has trivial twist) [Nek05]. It would be interesting to extend this criteria and connect it with the work here.

Open Problem 3. Can we employ the $P\Sigma O_n$-bimodule theory of Nekrashevych to prove the non-realizability of mapping schemes by obstructed topological polynomials?

We have only produced stable Levy cycles for polynomials. Perhaps we could use the machinery of this paper to find non-stable Levy cycles for some of these unclassified schemes. Instead of showing that the bimodule $[\gamma]$ restricts to itself under $\mathfrak{M}_g$, we would need to show that a particular product of powers of Dehn twists restricts to a different particular multitwist (these extra powers and twists coming from the other non-peripheral preimages of the Levy cycle). While the bimodules certainly still make these computations straightforward, one would need a systematic way to determine which multitwists to consider.

Open Problem 4. Can we extend these results if we consider non-stable Levy cycles?

Also, while we have demonstrated that certain mapping schemes are realizable by obstructed topological polynomials, for most of these mapping schemes we have only produced obstructed topological polynomials with Levy cycles of length 1.

Open Problem 5. Extend this method to produce more examples of topological polynomials admitting Levy cycles of length greater than 1.

Even for obstructed topological polynomials with Levy cycles of length 1, it is not clear if our method constructs all such polynomials.

Open Problem 6. Do there exist topological polynomials realizing these mapping schemes and admitting Levy cycles of length 1 other than those given in this paper?

Further, we have not attempted to determine when these various examples of obstructed topological polynomials are equivalent. A result like that for the quadratic polynomials with preperiod length 1 and period length 2 in [BN06] would be interesting, even if only for preperiodic quadratic polynomials.

Open Problem 7. What are the Thurston equivalence classes of these obstructed topological polynomials?

If the answer to Question 1 is positive in all cases, then the various elements represented by γ in our proofs have finite order in the faithful quotient of the action.

Open Problem 8. What are the orders of the obstructing elements in the faithful quotient of $P\Sigma O_n$ by the self-similar action?

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