CENTRAL EXTENSIONS OF DIVISIBLE GROUPS

BY

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DISSERTATION

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Abstract

This thesis contributes to the classification of central extensions of divisible groups with finite abelian quotient, so called "d-ab extensions." We give a matrix classification of equivalence classes of d-ab extensions and explicitly provide a family of group presentations. We provide a criterion for determining when two d-ab extensions are isomorophic in the case when the quotient is homocyclic. When the kernel has rank 1, we parametrize isomorphism classes of d-ab extensions with homocyclic quotient by constructing a family of group presentations. We also give a general reduction of d-ab extensions to the case when the kernel and center of the extensions coincide. For this case we give a classification of isomorphism classes when the kernel has rank 1. We highlight the applications of central extensions of divisible groups to nilpotent groups.

For Jojo

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Notation

Notation	Meaning
$R^\times$	The multiplicative group of units in the ring R with identity
R^+	The additive group of the ring R
$\mathbb{N}$	The set of natural numbers (including 0)
$\mathrm{Ker}(\varphi)$	The kernel of the homomorphism φ
$\mathrm{Im}(\varphi)$	The image of the function φ
$\|G\|$	The order of the group G, or the cardinality of the set G
$\mathrm{Hom}(G, H)$	The set of homomorphisms $G \to H$
$\mathrm{Aut}\,(G)$	The group of automorphisms of G
$\mathrm{Aut}_Q(M)$	The automorphisms of M as a Q-module
$\varphi\|_S$	The map φ restricted to the subset S
$\langle \sigma_1, \ldots, \sigma_n \rangle$	The subgroup generated by $\sigma_1, \ldots, \sigma_n$
$A \cong B$	A is isomorphic to B
$G \leq H$	G is a subgroup of H
$Z\,(G)$	The center of the group G
$\mathbb{Z}/n$	The integers modulo n
$\mathbb{Z}_p$	The p-adic integers
$[x, y]$	The commutator $xyx^{-1}y^{-1}$
$[G, H]$	The subgroup generated by the set $\{[g, h] \colon g \in G, h \in H\}$
G'	The group $[G, G]$
x^y	The conjugate $y^{-1}xy$
${}^y x$	The conjugate yxy^{-1}
$[x]_\sim$	The equivalence class of x under the equivalence relation $\sim$
$\otimes$	The (nonabelian) tensor product
$\wedge$	The (nonabelian) exterior product
$G^{\wedge 2}$	The exterior square $G \wedge G$
$\varphi^{\wedge 2}$	The map $x \wedge y \mapsto \varphi(x) \wedge \varphi(y)$
α_*	The map of extensions induced (covariantly) by α
β^*	The map of extensions induced (contravariantly) by β
$\cong$	Isomorphism of extensions
$\equiv$	Equivalence of extensions
$\sim$	Isomorphism of extensions fixing the kernel elementwise
$\boxplus$	The Baer sum

$H_2(\beta)$	The map in homology induced by β
$M(Q)$	The Schur multiplier of Q
$\xi_{\mathbf{e}}$	The map corresponding to the extension $\mathbf{e}$
$\twoheadrightarrow$	Surjection
$\rightarrowtail$	Injection
$\mathscr{E}$	The category of equivalence classes of group extensions
$\mathscr{E}(Q, K)$	Equivalence classes of extensions of K by Q
$\mathscr{E}_\varphi(Q, K)$	Equivalence classes of extensions of K by Q inducing the action $\varphi\colon Q \to \mathrm{Aut}(K)$
$\mathscr{E}_0(Q, K)$	Equivalence classes of central extensions of K by Q
$\cdot_{\mathrm{tgt}}$	The tightening functor $\mathscr{E}_{\text{d-ab}} \to [\mathscr{E}_{\mathrm{tgt}}]_\sim$
$\mathscr{E}_{\mathrm{tgt}}$	Equivalence classes of tight extensions
$\mathscr{E}_{\text{d-ab}}$	Equivalence classes of d-ab extensions
$\mathbf{e}_{\mathrm{tgt}}$	A tightening of $\mathbf{e}$
K_ξ	The set of degeneracies for ξ
$K_{\mathbf{e}}$	The set of degeneracies for $\xi_{\mathbf{e}}$
A_ξ	The domain of ξ_{tgt}
$\mathbb{Z}G$	The integral group ring of G

0 Introduction

The main goal of this thesis is to contribute to the classification of groups G that fit into a central extension of groups

$$D \rightarrowtail G \twoheadrightarrow Q$$

such that D is divisible and torsion and Q is finite abelian. One reason for our interest in these groups is that every finite nilpotent group of class 2 embeds into such a group in a nice way (Proposition 2.1.3). There is a similar embedding property for a class-c nilpotent group (namely, Q should be nilpotent of class $c-1$), and some of our foundational results, particularly in Chapter 2, apply to this general situation. However, our chief contributions are to the case when $c = 2$.

Another reason for studying these groups comes from the fact that abelian extensions of divisible (abelian) groups are always split. Thus the Universal Coefficient Theorem provides us with an isomorphism that makes d-extensions easier to understand (Theorem 2.1.2).

In Chapter 1 we give a general introduction to the theory of group extensions. We develop some key results, particularly starting with Section 1.3 on induced maps, which will be integral to our considerations in later chapters.

In Chapter 2 we specialize our considerations to central extensions of divisible groups, i.e., "d-extensions," and present some key properties of d-extensions. In Section 2.1 we discuss the relevance of induced maps to d-extensions and we develop a criterion for determining when an embedding of kernels produces an embedding of extensions (Corollary 2.1.9). Sections 2.2 and 2.3 are devoted to structural properties of divisible abelian groups and their automorphisms. In Section 2.4 we show how our situation reduces to considering p-groups.

In Chapter 3 we specialize further to d-extensions with abelian quotient, i.e., "d-ab extensions." We define "d-matrices" for a given finite abelian p-group Q in Section 3.1 and use d-matrices to parametrize equivalence classes of d-ab extensions: if $\mathscr{A}$ is an ordered m-set of d-matrices for a finite abelian p-group Q, then there is a group $G(\mathscr{A})$ with a presentation that depends only on $\mathscr{A}$, and there is a d-ab extension

$$\mathbf{e}(\mathscr{A})\colon \qquad \mathbb{Z}(p^\infty)^m \rightarrowtail G(\mathscr{A}) \twoheadrightarrow Q.$$

We show that each d-ab extension is equivalent to $\mathbf{e}(\mathscr{A})$ for some such $\mathscr{A}$ (Theorem 3.1.3) and that if $\mathscr{Z}$ is another ordered m-set of d-matrices for Q, then $\mathbf{e}(\mathscr{A}) \equiv \mathbf{e}(\mathscr{Z})$ if and only if $\mathscr{A} = \mathscr{Z}$ (Theorem 3.1.4). In other words, the function $\mathscr{A} \to \mathbf{e}(\mathscr{A})$ is a parametrization. In the case when Q is homocyclic of exponent p^e, we prove that $\mathbf{e}(\mathscr{A}) \cong \mathbf{e}(\mathscr{Z})$ if and only if there exist $R \in \mathrm{GL}_m(\mathbb{Z}_p)$ and $S \in \mathrm{GL}_r(\mathbb{Z}/p^e)$ such that

$$[Z_1|\cdots|Z_m] = S\big(\,[A_1|\cdots|A_m]\,(R \otimes 1_r)\,\big) * S^{\mathrm{t}},$$

where S^{t} denotes the transpose of S and $*$ is a product of partitioned matrices (Theorem 3.1.7).

In Sections 3.2 and 3.3 we specialize to d-ab extensions of the prüfer p-group. This is the rank-1 torsion case and, modulo Baer sums, actually accounts for all d-ab extensions with kernel having no torsion-free component (Corollary 3.2.8). We give some structural results and provide a family of group presentations (Definition 3.2.4) which, we show (Theorem 3.2.7) accounts for all such extensions up to isomorphism. In the case when the quotient is homocyclic, we are able to parametrize the isomorphism classes with a family of group presentations (Theorem 3.3.8).

In Chapter 4 we consider the special case when the kernel and center of a central extension coincide exactly. We refer to these extensions as "tight." We show that there is a functor $\cdot_{\mathrm{tgt}}$ that assigns a tight extension to a general d-ab extension (Theorem 4.3.11). In Sections 4.1 - 4.4 we investigate the properties of this functor and use these properties to give a parametrization of d-ab extensions (Theorem 4.3.16). Tight extensions are the extensions that apply to nilpotent groups directly, so it is satisfying that we are able to give a family of group presentations that parametrize tight d-ab extensions of the prüfer p-group (up to isomorphism), which is the content of Section 4.5.

1 Group extensions

We begin by giving some basic definitions and results from extension theory. Our primary concern is with extensions of abelian groups, so for the most part our kernels will be abelian, but we shall explicitly state when this is needed.

If K and Q are groups, then an *extension of K by Q* is a short exact sequence

$$\mathbf{e}\colon \qquad K \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} Q$$

of groups and homomorphisms. The groups K and Q are called, respectively, the *kernel* and *quotient* of $\mathbf{e}$. We will sometimes abuse terminology and refer to the group E as the extension. Note that $K \cong \operatorname{Im}(\iota)$ and $Q \cong E/\operatorname{Im}(\iota)$. When the kernel is abelian, it will generally be written additively, though the quotient and extension will be written multiplicatively.

Any extension comes with functions $\tau\colon Q \to E$ such that $\pi\tau = 1$. Such a function is called a *transversal* for $\mathbf{e}$; transversals are not generally homomorphisms. If an extension has a transversal that is a homomorphism, then it *splits* as an exact sequence.

Definition 1.0.1. A *map of extensions* is a commutative diagram

$$\begin{array}{lccccc} \mathbf{e}\colon & K & \rightarrowtail & G & \twoheadrightarrow & Q \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ \mathbf{e}'\colon & L & \rightarrowtail & H & \twoheadrightarrow & R \end{array}$$

of groups and homomorphisms in which the rows are short exact sequences (extensions). Such a map is really a triple (α, γ, β) where

$$\alpha\colon K \to L$$
$$\gamma\colon G \to H$$
$$\beta\colon Q \to R$$

satisfy the obvious commutativity properties with respect to $\mathbf{e}$ and $\mathbf{e}'$. We shall write $(\alpha, \gamma, \beta)\colon \mathbf{e} \to \mathbf{e}'$ or simply (α, γ, β) to denote the above map.

Let $\varphi = (\alpha, \gamma, \beta)$ be a map of extensions. We say φ is *surjective* if $\alpha = 1$ and β is surjective; note that this implies that γ surjective. We say φ is *injective* if $\beta = 1$ and α is injective; note that this implies that γ injective.

Definition 1.0.2. An *equivalence of extensions* is a map $(1, \gamma, 1)$, i.e., a commutative diagram

$$\begin{array}{lccccc} \mathbf{e}\colon & K & \rightarrowtail & G & \twoheadrightarrow & Q \\ & \| & & \downarrow{\scriptstyle \gamma} & & \| \\ \mathbf{e}'\colon & K & \rightarrowtail & H & \twoheadrightarrow & Q \end{array}$$

in which the rows $\mathbf{e}$ and $\mathbf{e}'$ are short exact sequences (extensions). We call the extensions $\mathbf{e}$ and $\mathbf{e}'$ *equivalent* and write $\mathbf{e} \equiv \mathbf{e}'$. We write $[\mathbf{e}]$ for $[\mathbf{e}]_{\equiv}$, i.e., the equivalence class of $\mathbf{e}$.

By the 5-lemma the nonidentity vertical map in an equivalence must be an isomorphism. Equivalence clearly forms an equivalence relation on the class of all extensions that restricts to the class of extensions of K by Q.

The category whose objects are equivalence classes of group extensions and whose morphisms are the resulting classes of maps of extensions will be denoted by $\mathscr{E}$. We will mostly be interested in understanding extensions that can arise with given kernel and given quotient. We let

$$\mathscr{E}(Q, K)$$

denote the full subcategory of $\mathscr{E}$ whose objects are (equivalence classes of) extensions of K by Q.

Definition 1.0.3. An *isomorphism of extensions* is an isomorphism in $\mathscr{E}$, i.e., a map with an inverse. Specifically, this is a commutative diagram

$$\begin{array}{ccccc} K & \rightarrowtail & G & \twoheadrightarrow & Q \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ L & \rightarrowtail & H & \twoheadrightarrow & R \end{array}$$

with isomorphisms for vertical arrows. If two extensions $\mathbf{e}$ and $\mathbf{e}'$ are isomorphic, we write $\mathbf{e} \cong \mathbf{e}'$.

Our main concern in this thesis is the so-called "isomorphism problem" for extensions. That is, we are interested in determining when two extensions are isomorphic (at least in certain specific situations), and in providing a classification of extensions up to isomorphism (in certain situations).

1.1 Group modules and cohomology

When K is an abelian group, there is a well-known correspondence between extensions of K by Q and the second cohomology group $H^2(Q, K)$ with a suitable Q-module structure on K. We summarize this correspondence here.

Definition 1.1.1. Let Q be any group. The integral group ring $\mathbb{Z}Q$ is the ring of finite (formal) sums $\sum n_q q$ with each $n_q \in \mathbb{Z}$ and $q \in Q$. Addition in $\mathbb{Z}Q$ is defined as

$$\big(\sum_{q\in Q} m_q q\big) + \big(\sum_{q\in Q} n_q q\big) = \sum_{q\in Q}(m_q + n_q)q.$$

Multiplication in $\mathbb{Z}Q$ is defined by putting $(n_g g)(n_h h) = n_g n_h gh$ and extending linearly. Thus, we define

$$\big(\sum_{q\in Q} m_q q\big)\big(\sum_{q\in Q} n_q q\big) = \sum_{q\in Q} s_q q,$$

where

$$s_k = \sum_{gh=k} m_g n_h$$

for each $k \in Q$. By a *Q-module* we mean a $\mathbb{Z}Q$-module, i.e., an abelian group on which Q acts linearly. We refer to a Q-module homomorphism as a *Q-map*.

If K is abelian and

$$\mathbf{e}\colon \qquad K \rightarrowtail^{\iota} E \xrightarrow{\pi} \!\!\!\!\twoheadrightarrow Q$$

is an extension with transversal τ, then K becomes a left Q-module via the action

$$^{q}k = \iota^{-1}\big(\tau(q)\iota(k)\tau(q)^{-1}\big).$$

This definition is independent of τ, and equivalent extensions yield the same Q-action. Thus, we obtain an action $\varphi\colon Q \to \operatorname{Aut} K$, which makes K into a Q-module. This action depends only on the equivalence class of $\mathbf{e}$ and we say φ is *realized* by $\mathbf{e}$. If two extensions are equivalent then they realize the same action, so there is a full subcategory $\mathscr{E}_\varphi(Q,K)$ of $\mathscr{E}(Q,K)$ whose objects are equivalence classes of extensions realizing φ.

It is worth remarking that if φ is any Q-action on K, then φ can be realized by a split extension of K by Q, namely the extension

$$\mathbf{s}_\varphi\colon \qquad K \rightarrowtail K \rtimes_\varphi Q \twoheadrightarrow Q$$

with the obvious maps. In fact, if $\mathbf{e}$ is any split extension of K by Q that realizes φ, then $\mathbf{e} \equiv \mathbf{s}$, and conversely.

If K is an abelian group, there is a binary operation $\boxplus$ on $\mathscr{E}(Q,K)$ that makes this set into an abelian group. This operation was first introduced by R. Baer ([Bae34]) and thus carries the name *Baer sum.*

Given two extensions

$$\mathbf{e}_1\colon \qquad K \overset{\iota_1}{\rightarrowtail} E_1 \overset{\pi_1}{\twoheadrightarrow} Q$$

and

$$\mathbf{e}_2\colon \qquad K \overset{\iota_2}{\rightarrowtail} E_2 \overset{\pi_2}{\twoheadrightarrow} Q$$

that both induce the module action φ we define

$$E_1 \boxplus E_2 = \frac{\{(g_1, g_2) \in E_1 \times E_2 \colon \pi_1(g_1) = \pi_2(g_2)\}}{\{(\iota_1(k), -\iota_2(k)) \colon k \in K\}}.$$

We have maps

$$\begin{aligned}(\iota_1 \boxplus \iota_2) \colon K &\rightarrowtail (E_1 \boxplus E_2)\\ \colon k &\mapsto (\iota_1(k), 1)\end{aligned}$$

and

$$\begin{aligned}(\pi_1 \boxplus \pi_2) \colon (E_1 \boxplus E_2) &\twoheadrightarrow Q\\ \colon (g_1, g_2) &\mapsto \pi_1(g_1).\end{aligned}$$

It is easy to verify that $\mathrm{Ker}(\pi_1 \boxplus \pi_2) = \mathrm{Im}(\iota_1 \boxplus \iota_2)$, so we have the extension

$$\mathbf{e}_1 \boxplus \mathbf{e}_2 \colon \qquad K \xrightarrow{\iota_1 \boxplus \iota_2} E_1 \boxplus E_2 \xrightarrow{\pi_1 \boxplus \pi_2} Q.$$

Verification of the following proposition is straightforward.

Proposition 1.1.2. *If* $\mathbf{e}$ *and* $\mathbf{e}'$ *are two extensions realizing a given module action* φ*, then* $\mathbf{e} \boxplus \mathbf{e}'$ *also realizes* φ*.*

It is well-known that $\boxplus$ makes $\mathscr{E}_\varphi(Q, K)$ into an abelian group, and that the zero element is the class of split extensions.

Definition 1.1.3. Let Q be any group and M be any Q-module. The *cohomology of Q with coefficients in M*, denoted $H^*(Q, M)$, is defined to be $\mathrm{Ext}^*_{\mathbb{Z}Q}(\mathbb{Z}, M)$, where the Q-action on $\mathbb{Z}$ is trivial ($qn = n$ for $q \in Q, n \in \mathbb{Z}$). Dually, the *homology of Q with coefficients in M*, denoted $H_*(Q, M)$, is defined to be $\mathrm{Tor}^{\mathbb{Z}Q}_*(\mathbb{Z}, M)$.

If the coefficients are taken in $\mathbb{Z}$ with trivial Q-action, it is customary to omit them from the notation and write $H_n(Q)$ and $H^n(Q)$.

The groups $H^*(Q, M)$ (respectively $H_*(Q, M)$) are computed by constructing a Q-projective resolution of $\mathbb{Z}$ and applying the functor $\mathrm{Hom}_{\mathbb{Z}Q}(\_, M)$ (respectively $\_ \otimes_{\mathbb{Z}Q} M$). There is a well-known standard resolution defined by taking P_n to be the free $\mathbb{Z}Q$-module generated by n-tuples $[q_1|\cdots|q_n]$ of elements of Q. We take the differential map in degree $n > 0$ to be

$$\partial = \sum_{i=1}^{n} (-1)^i \partial_i,$$

where

$$\begin{aligned}
\partial_0([q_1|\cdots|q_n]) &= q_1[q_2|\cdots|q_n] \\
\partial_i([q_1|\cdots|q_n]) &= [q_1|\cdots|q_{i-1}|q_iq_{i+1}|q_{i+2}|\cdots|q_n] \qquad \text{for } 0 < i < n; \\
\partial_n([q_1|\cdots|q_n]) &= [q_1|\cdots|q_{n-1}].
\end{aligned}$$

We take $\varepsilon\colon P_0 \to \mathbb{Z}$ to be the augmentation $q \mapsto 1$ for all $q \in Q$. This can easily be shown to be a Q-projective (in fact, free) resolution of $\mathbb{Z}$. This resolution is known as the *bar resolution*. It is useful in obtaining group-theoretic interpretations of the cohomology groups, especially in low degree. We shall be interested in degree 2.

We see that 2-cocycles are identified with functions $f\colon G\times G \to M$ satisfying

$$xf(y,z) - f(xy,z) + f(x,yz) - f(x,y) = 0.$$

Such functions are called *factor sets*. The 2-coboundaries are of the form

$$g(x,y) = xg(y) - g(xy) + g(x).$$

Let

$$\mathbf{e}\colon \qquad M \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q.$$

be an extension with M abelian. Assume for ease of notation that ι is simply embedding M as a subgroup, i.e., identify M with $\iota(M)$.

Let τ be a transversal for $\mathbf{e}$. For $x, y \in Q$ we can write

$$\tau(x)\tau(y) = f(x,y)\tau(xy)$$

for some element $f(x,y) \in M$, because $\tau(x)\tau(y)\tau(xy)^{-1} \in \operatorname{Ker}\pi = M$. Applying the associative law to the product $\tau(x)\tau(y)\tau(z)$ and simplifying shows that f is a factor set.

If another transversal τ' is chosen, then the resulting factor set f' has the property that $f - f'$ is a 2-coboundary. Thus, we obtain a unique cohomology class corresponding to the extension.

Conversely, if $f\colon Q\times Q \to M$ is a factor set, then we can define the extension G which consists, as a set, of elements of $M \times Q$. The group operation is

$$(m,q)(m',q') = (m + {}^{q}m' + f(q,q'), q * q'),$$

where $*$ is the group operation in Q, $+$ is the operation in M, and ${}^{q}m$ specifies the Q-action on M.

This correspondence actually provides a bijection between $\mathscr{E}_\varphi(Q,M)$ and $H^2(Q,M)$, where M is regarded as a Q-module via φ. We remark that the class of split extensions corresponds with the zero factor set. This gives a general classification of extensions of M by Q in terms of pairs (ξ, φ) where $\varphi\colon Q \to \operatorname{Aut} M$ and ξ is a factor set.

1.2 Central extensions

An extension

$$\mathbf{e}: \qquad C \rightarrowtail^{\iota} E \twoheadrightarrow Q$$

is said to be *central* if $\operatorname{Im}(\iota) \le Z(G)$. Of course, in this case C is abelian. Since the induced Q-action on C arises from conjugation in G, central extensions are precisely the extensions that induce the trivial action. Therefore, we use the notation $\mathscr{E}_0(Q, A)$ to denote central extensions of A by Q. These extensions are parametrized by $H^2(Q, C)$ with C given the trivial Q-action.

A group N is *nilpotent* if there is a central series

$$1 \le Z_1 \le Z_2 \le \cdots \le Z_c = N$$

in which $Z_i/Z_{i-1} = Z(N/Z_{i-1})$ (this is referred to as the *upper central series*). The integer $c = c(N)$ is called the *nilpotency class* or simply the *class* of N. Nilpotent groups can be built up by successive central extensions with nilpotent kernels. In particular, we have the extension

$$Z \overset{\iota}{\rightarrowtail} N \overset{\pi}{\twoheadrightarrow} N/Z$$

and we note that $c(N/Z) = c(N) - 1$.

It is well-known that a finite group is nilpotent if and only if it is a direct product of its Sylow subgroups (cf. [Rob91, p. 130]). Thus, in the finite case, the study of nilpotent groups reduces to the study of groups of prime-power order.

1.3 Induced maps of the kernel

Assume M and $\overline{M}$ are (left) Q-modules and $M \xrightarrow{\alpha} \overline{M}$ is a Q-map. Then there is an induced map $H^2(Q, M) \to H^2(Q, \overline{M})$, and hence an induced map $\alpha_*\colon \mathscr{E}(Q, M) \to \mathscr{E}(Q, \overline{M})$, which we will construct in this section. These induced maps will figure prominently in our investigations. In particular, the maps induced by isomorphisms provide insight into isomorphism classes of extensions. Later in the thesis we shall make use of the maps induced by injections.

We identify M and $\overline{M}$ with their underlying abelian groups and write them additively. Suppose we have an extension

$$\mathbf{e}: \qquad M \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$$

of groups that realizes the Q-action on M. The map π induces an action of

G on $\overline{M}$ (by defining ${}^{g}\overline{m} = {}^{\pi(g)}\overline{m}$) and we may form the semidirect product $S = \overline{M} \rtimes G$.

Lemma 1.3.1. *The subset $N = \{\big(-\alpha(m), \iota(m)\big) \colon m \in M\}$ is a normal subgroup of S.*

Proof. Recall that inverses in S are given by $(m,g)^{-1} = \big(-{}^{(g^{-1})}m, g^{-1}\big)$ and products by $(m_1,g_1)(m_2,g_2) = (m_1 + {}^{g_1}m_2, g_1g_2)$.

If (m_1,g_1), $(m_2,g_2) \in S$, then

$$\big(m_1,g_1\big)\big(m_2,g_2\big)^{-1} = \big(m_1,g_1\big)\big(-{}^{(g_2^{-1})}m_2, g_2^{-1}\big) = \big(m_1 - {}^{(g_1g_2^{-1})}m_2, g_1g_2^{-1}\big).$$

If g_1 and g_2 are in $\operatorname{Im}\iota = \operatorname{Ker}\pi$, then $g_1g_2^{-1}$ acts trivially, and it follows that N is a subgroup.

For normality, assume $g_1 \in G$, $m \in M_1$ and $m_2 \in \overline{M}$. Then

$$\begin{aligned}&(m_2,g_1)\big(-\alpha(m_1),\iota(m_1)\big)(m_2,g_1)^{-1} = \\ &= \Big(m_2 - {}^{g_1}\big(\alpha(m_1)\big) - {}^{(g_1\iota(m_1)g_1^{-1})}m_2, g_1\iota(m_1)g_1^{-1}\Big).\end{aligned}$$

Since $\iota(m_1)$ acts trivially on $\overline{M}$ and $\overline{M}$ is abelian this simplifies to

$$= \Big(-{}^{g_1}\big(\alpha(m_1)\big), {}^{g_1}\big(\iota(m_1)\big)\Big).$$

By definition, G acts on $\overline{M}$ by ${}^{g}m = {}^{\pi(g)}m$ and α is a Q-map, so

$${}^{g_1}\big(\alpha(m_1)\big) = \alpha\big({}^{\pi(g_1)}m_1\big).$$

Also, since the Q-action on M is realized by the extension $\mathbf{e}$, we have

$$\iota^{-1}\Big({}^{g_1}\big(\iota(m_1)\big)\Big) = {}^{\pi(g_1)}m_1,$$

so that ${}^{g_1}\big(\iota(m_1)\big) = \iota\big({}^{\pi(g_1)}m_1\big)$, which completes the proof. □

We put $H = S/N$. There are obvious maps $\varphi \colon G \to H$ and $\varepsilon \colon \overline{M} \to H$, namely $\varphi \colon g \mapsto (0,g)N$ and $\varepsilon \colon m \mapsto (m,1_G)N$. It is easy to see that ε is injective: if $m_2 \in \overline{M}$ with $(m_2,1_G) \in N$ then $m_2 = -\alpha(m_1)$ with $m_1 \in \operatorname{Ker}\iota = 0$. Thus, $m_2 = 0$.

We also have a map $\rho \colon H \twoheadrightarrow Q$ given by $\rho \colon (m,g)N \mapsto \pi(g)$, which is well defined because if $(m,g) \in N$ then $g \in \operatorname{Im}\iota = \operatorname{Ker}\pi$. Obviously $\operatorname{Im}\varepsilon \leq \operatorname{Ker}\rho$. If $(m,g)N \in \operatorname{Ker}\rho$, then $g \in \operatorname{Ker}\pi = \operatorname{Im}\iota$ and, writing $g = \iota(m_1)$ with $m_1 \in M$, we see that $(m,g) = (m,\iota(m_1)) \equiv \big(m + \alpha(m_1), 1_G\big) \pmod N$. Thus, $(m,g) \in \operatorname{Im}\varepsilon$. We therefore have an extension

$$\alpha_*(\mathbf{e})\colon \qquad \overline{M} \overset{\varepsilon}{\rightarrowtail} H \overset{\rho}{\twoheadrightarrow} Q.$$

It is quite easy to see that the following diagram commutes:

$$\begin{array}{lccccc} \mathbf{e}\colon & M & \overset{\iota}{\rightarrowtail} & G & \overset{\pi}{\twoheadrightarrow} & Q \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\varphi} & & \| \\ \alpha_*(\mathbf{e})\colon & \overline{M} & \overset{\varepsilon}{\rightarrowtail} & H & \overset{\rho}{\twoheadrightarrow} & Q. \end{array} \tag{1.1}$$

We thus have a map of extensions $(\alpha, \varphi, 1)\colon \mathbf{e} \to \alpha_*(\mathbf{e})$. We remark that, conversely, if $(\alpha, \varphi, 1)$ is any map of extensions, say with quotient Q, then α is necessarily a Q-module homomorphism. It is a direct consequence of the 5-lemma that if α is injective (respectively surjective) then φ is injective (surjective). Thus, if α is injective, so is $(\alpha, \varphi, 1)$.

Following the literature in extension theory, we call α_* the *pushout* of $\mathbf{e}$ by α. It should be noted, however, that H is *not* the pushout of G and $\overline{M}$, which involves the free product instead of the semidirect product.

Proposition 1.3.2. *Let Q be any group and let M and $\overline{M}$ be left Q-modules with Q-actions φ_1 and φ_2, respectively. If $M \xrightarrow{\alpha} \overline{M}$ is a Q-map and $\mathbf{e}$ is an extension of M by Q inducing φ_1, then the extension $\alpha_*(\mathbf{e})$ realizes φ_2.*

Proof. For concreteness we use the notation from the diagram (1.1) above. Write ${}^{q}m$ for the given Q-action on $\overline{M}$ (i.e., for $\varphi_2(q)(m)$) and $q \cdot m$ for the Q-action induced by $\alpha_*(\mathbf{e})$.

If $h = (m, g)N \in H$ and $x \in \overline{M}$ then

$$\begin{aligned} h\varepsilon(x)h^{-1} &= (m, g)\big((x, 1)N\big)\big({}^{(g^{-1})}m^{-1}, g^{-1}\big) \\ &= \big(m \cdot {}^{g}xm^{-1}, 1\big)N \\ &= ({}^{g}x, 1)N \\ &= ({}^{\pi(g)}x, 1)N. \end{aligned}$$

Note that $\rho(h) = \pi(g)$ and that if τ is a transversal for $\mathbf{e}$ then $\sigma\colon q \mapsto \big(1, \tau(q)\big)N$ is a transversal for the extension $\alpha_*(\mathbf{e})$. Thus, taking $h = \sigma(q) = \big(1, \tau(q)\big)$ with $q \in Q$, we have

$$\begin{aligned} q \cdot x &= \varepsilon^{-1}\big({}^{\sigma(q)}\varepsilon(x)\big) \\ &= \varepsilon^{-1}\Big(({}^{q}x, 1)N\Big) \\ &= {}^{q}x. \end{aligned}$$

□

In the context above, the extension $\alpha_*(\mathbf{e})$ is in fact uniquely distinguished up to equivalence by the fact that it fits into a commutative diagram like (1.1), as the next proposition asserts.

Proposition 1.3.3. *Assume M is a Q-module,*

$$\mathbf{e}\colon \qquad M \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$$

an extension realizing the Q-action on M, and $\alpha\colon M \to \overline{M}$ is a map of Q-modules. If

$$\begin{array}{cccccc} \mathbf{e}\colon & M & \overset{\iota}{\rightarrowtail} & G & \overset{\pi}{\twoheadrightarrow} & Q \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\psi} & & \| \\ \mathbf{e}'\colon & \overline{M} & \overset{\varepsilon_2}{\rightarrowtail} & \overline{G} & \overset{\rho_2}{\twoheadrightarrow} & Q. \end{array}$$

is a map of extensions with $\mathbf{e}'$ realizing the Q-action on $\overline{M}$, then $\mathbf{e}' \equiv \alpha_*(\mathbf{e})$.

Proof. Using the notation in diagram (1.1), we define a map $\theta\colon H \to G_2$ by $(m, g)N \mapsto \varepsilon_2(m)\psi(g)$. We now show that θ is an isomorphism.

First, for $x \in M$ we have

$$\theta\big(\alpha(x^{-1}), \iota(x)\big) = \varepsilon_2\alpha(x^{-1}) \cdot \psi\iota(x) = \psi\iota(x^{-1})\psi\iota(x) = 1,$$

so θ is well-defined. Obviously θ is a homomorphism. To see that θ is injective, assume $\theta(m, g) = 1$; then $\varepsilon_2(m^{-1}) = \psi(g)$. Now, $\pi(g) = \rho_2\psi(g) = \rho_2\varepsilon_2(m^{-1}) = 1$, so $g = \iota(x)$ for some $x \in M$. Then $\varepsilon_2(m^{-1}) = \psi\iota(x) = \varepsilon_2\alpha(x)$, and since ε_2 is injective, $m = \alpha(x^{-1})$. Thus, $(m, g) \in N$ and θ is injective. For surjectivity, assume $y \in \overline{G}$ and let $g \in G$ satisfy $\pi(g) = \rho_2(y)$. Then $\rho_2\big(y\psi(g^{-1})\big) = 1$, so there is an $m \in \overline{M}$ such that $y = \varepsilon_2(m)\psi(g)$, and θ is surjective.

Finally, it is obvious that $(1, \theta, 1)\colon \alpha_*(\mathbf{e}) \to \mathbf{e}'$ is an equivalence. □

Definition 1.3.4. If $\alpha\colon M \to \overline{M}$ is a map of Q-modules, then the *map induced by α* is the map $\alpha_*\colon \mathscr{E}(Q, M) \to \mathscr{E}(Q, \overline{M})$ that assigns to each class $[\mathbf{e}]$ of extensions the class $[\alpha_*(\mathbf{e})]$.

That induced maps are well-defined is obvious by Proposition 1.3.3. We also have the following covariance property.

Corollary 1.3.5. *Let Q be any group. If $\alpha\colon M_1 \to M_2$ and $\alpha'\colon M_2 \to M_3$ are maps of Q-modules, then $(\alpha\alpha')_* = \alpha_*\alpha'_*$.*

Proof. Let $\mathbf{e}$ be any extension of M_1 by Q. If $\mathbf{e} \mapsto \alpha_*(\mathbf{e})$ is given by the triple of maps $(\alpha, \gamma, 1)$ and $\alpha_*(\mathbf{e}) \mapsto \alpha'_*(\alpha_*(\mathbf{e}))$ is given by $(\alpha', \gamma', 1)$, then $\mathbf{e} \mapsto \alpha_*\alpha'_*(\mathbf{e})$ is given by $(\alpha\alpha', \gamma\gamma', 1)$. By Proposition 1.3.3 we must have $\alpha_*\alpha'_*(\mathbf{e}) \equiv (\alpha\alpha')_*(\mathbf{e})$, as desired. □

Corollary 1.3.6. *If Q is any group and α is an invertible Q-map, then α_* is also invertible and $(\alpha^{-1})_* = (\alpha_*)^{-1}$.*

A priori the map α_* is merely a function. The following proposition guarantees that α_* is in fact a map of abelian groups.

Proposition 1.3.7. *For any Q-map $\alpha\colon M \to \overline{M}$, the function α_* is a homomorphism of abelian groups $\mathscr{E}(Q,M) \to \mathscr{E}(Q,\overline{M})$.*

Proof. Assume we have the extensions

$$\mathbf{e}_1\colon \qquad M \xrightarrow{\iota_1} G_1 \xrightarrow{\pi_1} Q$$

$$\mathbf{e}_2\colon \qquad M \xrightarrow{\iota_2} G_2 \xrightarrow{\pi_2} Q,$$

and for concreteness take $\alpha_*(\mathbf{e}_1)$ and $\alpha_*(\mathbf{e}_2)$ to be the extensions

$$\alpha_*(\mathbf{e}_1)\colon \qquad \overline{M} \xrightarrow{\varepsilon_1} H_1 \xrightarrow{\rho_1} Q$$

$$\alpha_*(\mathbf{e}_2)\colon \qquad \overline{M} \xrightarrow{\varepsilon_2} H_2 \xrightarrow{\rho_2} Q,$$

where H_i, ε_i, and ρ_i are given as in diagram 1.1 ($i \in \{1,2\}$); also, put $N_i = \Big\{\big(-\alpha(m), \iota_i(m)\big)\colon m \in M\Big\}$ (cf. Lemma 1.3.1). Thus, $H_i = (M \rtimes G_i)/N_i$.

Recall that

$$G_1 \boxplus G_2 = \frac{\{(g_1,g_2) \in G_1 \times G_2 \colon \pi_1(g_1) = \pi_2(g_2)\}}{\{(\iota_1(m), -\iota_2(m))\colon m \in M\}}$$

and note that

$$H_1 \boxplus H_2 = \frac{\{(h_1,h_2) \in H_1 \times H_2 \colon \rho_1(h_1) = \rho_2(h_2)\}}{\{(\varepsilon_1(m), -\varepsilon_2(m))\colon m \in \overline{M}\}}.$$

There is a map $\theta\colon G_1 \boxplus G_2 \to H_1 \boxplus H_2$ given by taking the coset of (g_1, g_2) to the coset of $\big((0,g_1)N_1, (0,g_2)N_2\big)$. Indeed, we have $\rho_1\big((0,g_1)N_1\big) = \pi_1(g_1) = \pi_2(g_2) = \rho_2\big((0,g_2)N_2\big)$, and

$$\Big(\big(0,\iota_1(m)\big)N_1, \big(0,-\iota_2(m)\big)N_2\Big) = \Big(\big(\alpha(m),1\big)N_1, \big(-\alpha(m),1\big)N_2\Big),$$

which is the trivial coset (recall ε_1 is the map $m \mapsto (m,1)N_1$; likewise for ε_2). Thus, θ is well-defined.

It is straight-forward to verify that the diagram

$$\begin{array}{lccccc}
(\mathbf{e}_1 \boxplus \mathbf{e}_2)\colon & M & \xrightarrow{\iota_1 \boxplus \iota_2} & G_1 \boxplus G_2 & \xrightarrow{\pi_1 \boxplus \pi_2} & Q \\
& \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\theta} & & \| \\
\big(\alpha_*(\mathbf{e}_1) \boxplus \alpha_*(\mathbf{e}_2)\big)\colon & \overline{M} & \xrightarrow{\varepsilon_1 \boxplus \varepsilon_2} & H_1 \boxplus H_2 & \xrightarrow{\rho_1 \boxplus \rho_2} & Q
\end{array}$$

commutes. By Proposition 1.3.3, we have

$$\alpha_*(\mathbf{e}_1 \boxplus \mathbf{e}_2) \equiv \big(\alpha_*(\mathbf{e}_1) \boxplus \alpha_*(\mathbf{e}_2)\big),$$

as desired. $\square$

We have the following direct corollary to Proposition 1.3.2.

Corollary 1.3.8. *For any Q-map $\alpha\colon M_1 \to M_2$, the map α_* restricts to a homomorphism $\mathscr{E}_{\varphi_1}(Q, M_1) \to \mathscr{E}_{\varphi_2}(Q, M_2)$. In particular, α_* restricts to a homomorphism $\mathscr{E}_0(Q, M_1) \to \mathscr{E}_0(Q, M_2)$.*

Also of interest is the case when $\varphi_1 = \varphi_2$ (and $M_1 = M_2$), so that α_* restricts to a map $\mathscr{E}_\varphi(Q, M) \to \mathscr{E}_\varphi(Q, M)$.

1.4 Induced maps of the quotient

Given a homomorphism of groups $\beta\colon \overline{Q} \to Q$ and a Q-module M, we obtain a $\overline{Q}$-module structure on M by defining ${}^q m = {}^{\beta(q)} m$. There is an induced map $H^2(Q, M) \to H^2(\overline{Q}, M)$; thus, there is an induced map $\beta^*\colon \mathscr{E}(Q, M) \to \mathscr{E}(\overline{Q}, M)$, which we shall construct in this section. Dual to the maps induced by homomorphisms of the kernels above, these induced maps of the quotient also figure prominently in our investigations. Isomorphism classes of extensions can be fully understood using maps induced by isomorphisms of the kernels and quotients of extensions, the subject of the sequel (Section 1.5). Later we shall also make use of the maps β^* when β is a surjection.

Suppose we have an extension

$$\mathbf{e}\colon \qquad M \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$$

of groups that realizes the Q-action on M. Let H be the pullback of the maps π and β; that is, put

$$H = G \times_Q \overline{Q} = \{(g, q) \in G \times \overline{Q} \colon \pi(g) = \beta(q)\}.$$

The pullback comes with maps $\rho\colon H \to \overline{Q}$ and $\varphi\colon H \to G$, which are restrictions of the respective projection maps from $G \times \overline{Q}$. We also have the map $\varepsilon\colon M \to H$ defined by $\varepsilon(m) = (\iota(m), 1)$. Clearly $\operatorname{Ker} \rho = \operatorname{Im}(\varepsilon)$, and we have the extension

$$\beta^*(\mathbf{e})\colon \qquad M \overset{\varepsilon}{\rightarrowtail} H \overset{\rho}{\twoheadrightarrow} \overline{Q},$$

which fits into the commutative diagram

$$\begin{array}{llcccccc} \beta^*(\mathbf{e})\colon & M & \overset{\varepsilon}{\rightarrowtail} & H & \overset{\rho}{\twoheadrightarrow} & \overline{Q} \\ & \| & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\beta} \\ \mathbf{e}\colon & M & \overset{\iota}{\rightarrowtail} & G & \overset{\pi}{\twoheadrightarrow} & Q \end{array} \tag{1.2}$$

Thus, we have the map $(1, \varphi, \beta)\colon \beta^*(\mathbf{e}) \to \mathbf{e}$ of extensions. As a consequence of the 5-lemma, if β is injective (respectively surjective) then φ is injective (surjective); hence if β is surjective, then so is $(1, \varphi, \beta)$.

Proposition 1.4.1. *Assume that Q and $\overline{Q}$ are groups, M is a Q-module, and $\beta\colon \overline{Q} \to Q$ is a group homomorphism. Then the extension $\beta^*(\mathbf{e})$ induces the action of $\overline{Q}$ on M given by ${}^{q}m = {}^{\beta(q)}m$. Thus, if $\mathbf{e}$ induces the trivial Q-action on M, then so does $\beta^*(\mathbf{e})$.*

Proof. We use the notation in diagram 1.2 above. Let τ be a transversal for $\mathbf{e}$. The function $\overline{Q} \to H$ given by $q \mapsto \big(\tau\beta(q), q\big)$ is a transversal for the extension $\beta^*(\mathbf{e})$. If $q \cdot m$ denotes the action of $\overline{Q}$ on M induced by the extension $\beta^*(\mathbf{e})$, then

$$\begin{aligned} q \cdot m &= \varepsilon^{-1}\Big(\big(\tau\beta(q), q\big)\big(\iota(m), 1\big)\big(\tau\beta(q)^{-1}, q^{-1}\big)\Big) \\ &= \varepsilon^{-1}\big(\tau\beta(q)\iota(m)\tau\beta(q)^{-1}, 1\big) \\ &= \iota^{-1}\big(\tau(\beta(q))\iota(m)\tau(\beta(q))^{-1}\big) \\ &= {}^{\beta(q)}m. \end{aligned}$$

□

As in the previous section, $\beta^*(\mathbf{e})$ is uniquely distinguished up to equivalence by fitting into a diagram like (1.2).

Proposition 1.4.2. *Assume that Q and $\overline{Q}$ are groups, M is a Q-module, and $\beta\colon \overline{Q} \to Q$ is a group homomorphism. Let*

$$\mathbf{e}\colon \qquad M \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$$

be an extension realizing the Q-action on M, and $\beta\colon \overline{Q} \to Q$ a group homomorphism. If

$$\begin{array}{lccccc} \widetilde{\mathbf{e}}\colon & M & \rightarrowtail & \widetilde{H} & \overset{\widetilde{\rho}}{\twoheadrightarrow} & \overline{Q} \\ & \| & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\beta} \\ \mathbf{e}\colon & M & \overset{\iota}{\rightarrowtail} & G & \overset{\pi}{\twoheadrightarrow} & Q. \end{array}$$

is a map of extensions, then $\widetilde{\mathbf{e}} \equiv (\beta^*\mathbf{e})$.

Proof. Using the notation in diagram 1.2, we define a map $\theta\colon \widetilde{H} \to H$ by $\theta(h) = \big(\psi(h), \widetilde{\rho}(h)\big)$ for $h \in \overline{H}$. Obviously θ is a homomorphism into H and commutativity of the diagram

$$\begin{array}{lccccc} \widetilde{\mathbf{e}}\colon & M & \overset{\varepsilon}{\rightarrowtail} & \widetilde{H} & \overset{\widetilde{\rho}}{\twoheadrightarrow} & \overline{Q} \\ & \| & & \downarrow{\scriptstyle\theta} & & \| \\ \beta^*(\mathbf{e})\colon & M & \overset{\varepsilon}{\rightarrowtail} & H & \overset{\rho}{\twoheadrightarrow} & \overline{Q} \end{array}$$

is easy to check. By the 5-lemma it follows that θ is an isomorphism, so that $(1, \theta, 1)\colon \widetilde{\mathbf{e}} \to \beta^*(\mathbf{e})$ is an equivalence. □

Definition 1.4.3. If M is a Q-module and $\beta\colon \overline{Q} \to Q$ is a group homomorphism, then the *map induced by* β is the map $\beta^*\colon \mathscr{E}(Q,M) \to \mathscr{E}(\overline{Q},M)$ that assigns to each class $[\mathbf{e}]$ of extensions the class $[\beta^*(\mathbf{e})]$.

These induced maps are well-defined, as is obvious by Proposition 1.4.2. We also have the following contravariance property.

Corollary 1.4.4. *Let* $\beta\colon Q_2 \to Q_1$ *and* $\beta'\colon Q_3 \to Q_2$ *be group homomorphisms. Then* $(\beta\beta')^* = \beta'^*\beta^*$.

Proof. Let $\mathbf{e}$ be any extension by Q_1. If $\beta^*(\mathbf{e}) \to \mathbf{e}$ is the triple $(1,\gamma,\beta)$ and $\beta'^*\beta^*(\mathbf{e}) \to \beta^*(\mathbf{e})$ is the triple $(1,\gamma',\beta')$, then $\beta'^*\beta^*(\mathbf{e}) \to \mathbf{e}$ is the triple $(1,\gamma\gamma',\beta\beta')$. By Proposition 1.4.2 we have $\beta'^*\beta^*(\mathbf{e}) \equiv (\beta\beta')^*(\mathbf{e})$, as desired. □

Corollary 1.4.5. *If* $\beta\colon Q_2 \to Q_1$ *is an invertible group homomorphism, then* β^* *is also invertible and* $(\beta^{-1})^* = (\beta^*)^{-1}$.

As with the induced maps α_* in the previous section, the functions β^* are maps of abelian groups.

Proposition 1.4.6. *For any* Q*-map* $\beta\colon \overline{Q} \to Q$*, the function* β^* *is a homomorphism of abelian groups* $\mathscr{E}(\overline{Q},M) \to \mathscr{E}(Q,M)$.

Proof. Assume we have the extensions

$$\mathbf{e}_1\colon \qquad M \overset{\iota_1}{\rightarrowtail} G_1 \overset{\pi_1}{\twoheadrightarrow} Q$$

$$\mathbf{e}_2\colon \qquad M \overset{\iota_2}{\rightarrowtail} G_2 \overset{\pi_2}{\twoheadrightarrow} Q,$$

and for concreteness take $\beta^*(\mathbf{e}_1)$ and $\beta^*(\mathbf{e}_2)$ to be the extensions

$$\beta^*(\mathbf{e}_1)\colon \qquad M \overset{\varepsilon_1}{\rightarrowtail} H_1 \overset{\rho_1}{\twoheadrightarrow} \overline{Q}$$

$$\beta^*(\mathbf{e}_2)\colon \qquad M \overset{\varepsilon_2}{\rightarrowtail} H_2 \overset{\rho_2}{\twoheadrightarrow} \overline{Q},$$

where H_i, ε_i, and ρ_i are given as in diagram 1.2 ($i \in \{1,2\}$), i.e., $H_i = G_i \times_Q \overline{Q}$.

Recall that

$$G_1 \boxplus G_2 = \frac{\{(g_1,g_2) \in G_1 \times G_2\colon \pi_1(g_1) = \pi_2(g_2)\}}{\{(\iota_1(m), -\iota_2(m))\colon m \in M\}}$$

and note that

$$H_1 \boxplus H_2 = \frac{\{(h_1,h_2) \in H_1 \times H_2\colon \rho_1(h_1) = \rho_2(h_2)\}}{\{(\varepsilon_1(m), -\varepsilon_2(m))\colon m \in M\}}.$$

There is a map $\theta\colon H_1 \boxplus H_2 \to G_1 \boxplus G_2$ defined by taking the coset of $\big((g_1,q_1),(g_2,q_2)\big)$ to the coset of (g_1,g_2).

It is straight-forward to verify that the diagram

$$\begin{array}{lccccc} \beta^*(\mathbf{e}_1) \boxplus \beta^*(\mathbf{e}_2)\colon & M & \xrightarrow{\varepsilon_1 \boxplus \varepsilon_2} & H_1 \boxplus H_2 & \xrightarrow{\rho_1 \boxplus \rho_2} & \overline{Q} \\ & \| & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \beta} \\ \mathbf{e}_1 \boxplus \mathbf{e}_2\colon & M & \xrightarrow{\iota_1 \boxplus \iota_2} & G_1 \boxplus G_2 & \xrightarrow{\pi_1 \boxplus \pi_2} & Q \end{array}$$

commutes. By Proposition 1.3.3, we have

$$\beta^*(\mathbf{e}_1 \boxplus \mathbf{e}_2) \equiv \beta^*(\mathbf{e}_1) \boxplus \beta^*(\mathbf{e}_2),$$

as desired. □

Corollary 1.4.7. *For any homomorphism $\alpha\colon \overline{Q} \to Q$, the map β^* restricts to a homomorphism $\mathscr{E}_\varphi(Q, M) \to \mathscr{E}_{\varphi\beta}(\overline{Q}, M)$. In particular, β^* restricts to a homomorphism $\mathscr{E}_0(Q, M) \to \mathscr{E}_0(\overline{Q}, M)$.*

1.5 Isomorphism classes of extensions

Lemma 1.5.1. *Let $(\alpha, \gamma, \beta)\colon \mathbf{e} \to \mathbf{e}'$ be an isomorphism of extensions. Then $\mathbf{e}' \equiv (\beta^{-1})^*\alpha_*(\mathbf{e}) \equiv \alpha_*(\beta^{-1})^*(\mathbf{e})$.*

Proof. For definiteness, assume $\mathbf{e}$ and $\mathbf{e}'$ are the extensions

$$\mathbf{e}\colon \qquad M_1 \xrightarrow{\iota_1} G_1 \xrightarrow{\pi_1} Q_1$$

$$\mathbf{e}'\colon \qquad M_2 \xrightarrow{\iota_2} G_2 \xrightarrow{\pi_2} Q_2.$$

Consider the commutative diagram

$$\begin{array}{lccccc} \mathbf{e}\colon & M_1 & \xrightarrow{\iota_1} & G_1 & \xrightarrow{\pi_1} & Q_1 \\ & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \varphi_1} & & \| \\ \alpha_*(\mathbf{e})\colon & M_1 & \xrightarrow{\varepsilon_1} & H_1 & \xrightarrow{\rho_1} & Q_1 \\ & \| & & \uparrow{\scriptstyle \varphi_2} & & \uparrow{\scriptstyle \beta^{-1}} \\ (\beta^{-1})^*\big(\alpha_*(\mathbf{e})\big)\colon & M_2 & \xrightarrow{\varepsilon_2} & H_2 & \xrightarrow{\rho_2} & Q_2. \end{array}$$

Note that since α and β are isomorphisms, so are φ_1 and φ_2. We define an equivalence by functional composition

$$\begin{array}{lccccc} (\beta^{-1})^*\big(\alpha_*(\mathbf{e})\big)\colon & M_2 & \xrightarrow{\varepsilon_2} & H_2 & \xrightarrow{\rho_2} & Q_2 \\ & \| & & {\scriptstyle \psi}\downarrow & & \| \\ \mathbf{e}'\colon & M_2 & \xrightarrow{\iota_2} & G_2 & \xrightarrow{\pi_2} & Q_2, \end{array}$$

where $\psi = \gamma\varphi_1^{-1}\varphi_2$. Commutativity of the diagram is easy to check.

We see similarly that $\mathbf{e}' \equiv \alpha_*(\beta^{-1})^*(\mathbf{e})$. □

Corollary 1.5.2. *If* $\mathbf{e}$ *is a central extension and* $\mathbf{e} \cong \mathbf{e}'$ *then* $\mathbf{e}'$ *is also a central extension.*

Proof. This is a direct result of Lemma 1.5.1 together with Corollaries 1.4.7 and 1.3.8. □

It is elementary that isomorphisms between groups G and H are parametrized by $\operatorname{Aut} G$ by choosing a fixed isomorphism $\varphi\colon G \xrightarrow{\sim} H$ and composing with φ. If $\gamma\colon G \xrightarrow{\sim} H$ is another isomorphism, then $\gamma = \varphi\alpha$ for some $\alpha \in \operatorname{Aut} G$ (namely $\varphi^{-1}\gamma$). Thus, in studying isomorphisms and their classes, it suffices to consider automorphisms.

We fix the groups Q and M with M abelian. It is clear by Corollary 1.3.8 that there is a left action of $\operatorname{Aut}(M)$ on $\mathscr{E}(Q, M)$ given by $[\alpha \cdot \mathbf{e}] = [\alpha_*(\mathbf{e})]$ for $\alpha \in \operatorname{Aut}(M)$ and $[\mathbf{e}] \in \mathscr{E}(Q, M)$. We also obtain a left action of $\operatorname{Aut} Q$ on $\mathscr{E}(Q, M)$ by defining $[\beta \cdot \mathbf{e}] = [(\beta^{-1})^*(\mathbf{e})]$. By Lemma 1.5.1 these actions commute.

We therefore can define a left action of $\operatorname{Aut} M \times \operatorname{Aut} Q$ on $\mathscr{E}(Q, M)$ by

$$[(\alpha, \beta) \cdot \mathbf{e}] = [\alpha_*(\beta^{-1})^*(\mathbf{e})].$$

What interests us is that, as a result of Corollaries 1.4.7 and 1.3.8, this action restricts to an action on $\mathscr{E}_0(Q, M)$.

Corollary 1.5.3. *Two extensions of* M *by* Q *are isomorphic if and only if their equivalence classes are in the same* $(\operatorname{Aut} M \times \operatorname{Aut} Q)$*-orbit.*

Proof. Suppose $\mathbf{e}$ and $\mathbf{e}'$ are two extensions of M by Q and assume we have the isomorphism $(\alpha, \gamma, \beta)\colon \mathbf{e} \to \mathbf{e}'$ of extensions. Then by Lemma 1.5.1 $\mathbf{e}' \equiv \alpha_*(\beta^{-1})^*(\mathbf{e})$. Of necessity $\alpha \in \operatorname{Aut}(M)$ and $\beta \in \operatorname{Aut}(Q)$, so that $[\mathbf{e}'] = [(\alpha, \beta) \cdot \mathbf{e}]$. The converse being trivial, this completes the proof. □

1.6 The Schur multiplier

Let C be an abelian group and Q any group. In the next section we will associate each central extension of C by Q with a homomorphism from the second homology group $H_2(Q)$ of Q into C. We therefore devote this section to understanding the group $H_2(Q)$.

Schur first studied the groups H_2 in his investigation of projective representations, which predates homology theory. The notation $M(Q)$ for $H_2(Q)$ has become standard. We shall use $M(Q)$ to denote a specific group (isomorphic with $H_2(Q)$) to be defined momentarily. The group $M(Q)$ (or $H_2(Q)$) is called the *Schür multiplier* of Q.

There is a formula for $M(Q)$ in terms of a presentation for Q.

Proposition 1.6.1 (Hopf's formula)**.** *If $R \rightarrowtail F \twoheadrightarrow Q$ is a free presentation of a group Q, then*

$$M(Q) \cong \frac{F' \cap R}{[F, R]}.$$

A proof of Hopf's formula can be found in [Rob91, p. 347]. An interesting corollary of this Proposition is that the group $F' \cap R/[F, R]$ does not depend the presentation of Q.

Though we do make use of Hopf's formula, another formula for $M(Q)$, which involves the nonabelian tensor square, is often more convenient. We summarize the relevant definitions and results here. For more details, see [BL87] and [BJR87].

Definition 1.6.2. For any group G, the *(nonabelian) tensor square* $G \otimes G$ is the group generated by symbols $g \otimes h$ for elements g and h of G, with relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h)$$
$$g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h').$$

The *exterior square* $G \wedge G$ is the quotient of $G \otimes G$ obtained by imposing the additional relation $g \otimes g = 1$ for all $g \in G$. The image of the element $g \otimes h$ in $G \wedge G$ is written $g \wedge h$.

It is obvious that in the abelian case, both the tensor and the exterior squares are just the classical ones. There is a well-defined map $\kappa' \colon Q \wedge Q \to Q'$ defined by $\kappa(x \wedge y) = [x, y]$ on the generators. We have the following alternative formula for $M(Q)$, the proof of which can be found in [Mil52].

Proposition 1.6.3. *Let $R \rightarrowtail F \xrightarrow{\pi} Q$ be a free presentation of Q. There is a homomorphism $\omega \colon Q \wedge Q \to F/[F, R]$ such that $\omega \colon x \wedge y \mapsto [\bar{x}, \bar{y}][F, R]$, where $\pi(\bar{x}) = x$ and $\pi(\bar{y}) = y$, and such that ω restricts to an isomorphism*

$$\mathrm{Ker}(\kappa') \xrightarrow{\sim} \frac{[F, F] \cap R}{[F, R]}.$$

By $M(Q)$ we shall mean the group $\mathrm{Ker}(\kappa')$. One feature of this formula that is especially useful is that it is functorial. If $\beta \colon Q_1 \to Q_2$ is any homomorphism then there is an induced map $(\beta \wedge \beta) \colon Q_1 \wedge Q_1 \to Q_2 \wedge Q_2$ defined by $(\beta \wedge \beta)(x \wedge y) = \beta(x) \wedge \beta(y)$ on the generators. From this we obtain a formulation of the induced map in homology, namely $H_2(\beta) \colon M(Q_1) \to M(Q_2)$ is the restriction of $\beta \wedge \beta$ to $M(Q_1)$.

Corollary 1.6.4. *If Q is abelian, then $M(Q) = Q \wedge Q$.*

As a direct result of Proposition 5 in [BJR87], we obtain the following.

Proposition 1.6.5. *If Q is a finite group, then $M(Q)$ is finite. If in addition Q is a p-group for a prime p, then so is $M(Q)$.*

Lemma 1.6.6. *If G and H are finite groups with coprime orders, then $M(G \times H) \cong M(G) \times M(H)$ (naturally).*

Proof. Assume $\varphi\colon Q \to G \times H$ is an isomorphism and write $\varphi = \varphi_G \times \varphi_H$, where $\varphi_G\colon Q \to G$ is the composition of φ with the natural projection $G \times H \to G$ and likewise for φ_H. By Proposition 11 of [BJR87], we have

$$Q \otimes Q \cong (G \otimes G) \times (G \otimes H) \times (H \otimes G) \times (H \otimes H).$$

All of the relevant actions are given by conjugation in $G \times H$. Thus, G acts on H trivially, and vice versa. Thus, by Proposition 2.4 of [BL87] we have $G \otimes H = G_{\mathrm{ab}} \otimes_{\mathbb{Z}} H_{\mathrm{ab}}$, and likewise for $H \otimes G$, both of which are trivial since $|G|$ and $|H|$ are coprime. Hence, $Q \otimes Q \cong (G \otimes G) \times (H \otimes H)$. From the proof of Proposition 11 of [BJR87] we see that the isomorphism is given by

$$x \otimes y \mapsto (\varphi_G(x) \otimes \varphi_G(y)) \times (\varphi_H(x) \otimes \varphi_H(y)).$$

It follows easily that $(\varphi_G \wedge \varphi_G) \times (\varphi_H \wedge \varphi_H)$ is an isomorphism restricting to the isomorphism

$$\big(H_2(\varphi_G) \times H_2(\varphi_H)\big)\colon M(Q) \xrightarrow{\sim} M(G) \times M(H).$$

□

Our concern in this thesis will mainly be with nilpotent quotients. We recall that a finite group is nilpotent if and only if it is a direct product of its Sylow p-subgroups. We therefore obtain the following decomposition for the Schür multiplier of a finite nilpotent group.

Proposition 1.6.7. *Assume N is a finite nilpotent group. Let $p_1, \ldots, p_k$ be the distinct primes dividing $|N|$ and for $i = 1, \ldots, k$ let P_i be the (unique) Sylow p_i-subgroup of N. Thus $N = P_1 \times \cdots \times P_k$. Let $\pi_i\colon N \to P_i$ be the natural projection for each $i = 1, \ldots, k$. Then the map $H_2(\pi_1) \times \cdots \times H(\pi_k)$ is an isomorphism*

$$M(N) \xrightarrow{\sim} M(P_1) \times \cdots \times M(P_k).$$

1.7 The Universal Coefficient Theorem

For abelian groups A and B, let $\mathscr{E}_{\mathrm{ab}}(A, B)$ denote the set of equivalence classes of abelian extensions of B by A, that is, the extensions $B \rightarrowtail G \twoheadrightarrow A$ such that G is abelian. It is easy to check that $\mathscr{E}_{\mathrm{ab}}(A, B)$ is a subgroup of $\mathscr{E}_0(A, B)$, and it is well known that $\mathscr{E}_{\mathrm{ab}}(A, B) \cong \mathrm{Ext}(A, B)$.

Assume throughout this section that C is an abelian group and Q is any group. Regarding C as a trivial Q-module, the well-known Universal Coefficient

Theorem (cf. [Bro82]) guarantees the existence of a short exact sequence

$$\mathscr{E}_{\mathrm{ab}}(Q_{\mathrm{ab}}, C) \overset{\widetilde{\nu}}{\rightarrowtail} \mathscr{E}_0(Q, C) \overset{\xi_\bullet}{\twoheadrightarrow} \mathrm{Hom}(M(Q), C) \tag{1.3}$$

of groups.The maps $\xi_{\mathbf{e}}$, for extensions $\mathbf{e}$, will play a crucial role in our classifications later in this thesis. In this section we construct the sequence (1.3) above.

To obtain the injection, let $\nu\colon Q \twoheadrightarrow Q_{\mathrm{ab}}$ denote the natural map. Then there is an induced map

$$\nu^*\colon\ \mathscr{E}_0(Q_{\mathrm{ab}}, C) \to \mathscr{E}_0(Q, C),$$

and by restriction we have the map

$$\widetilde{\nu}\colon\ \mathscr{E}_{\mathrm{ab}}(Q_{\mathrm{ab}}, C) \to \mathscr{E}(Q, C).$$

Specifically, if $\mathbf{a} \in \mathscr{E}_{\mathrm{ab}}(Q_{\mathrm{ab}}, C)$, then we use the pullback construction to construct the commutative diagram

$$\begin{array}{lccccc} \mathbf{e}\colon & C & \rightarrowtail & E & \twoheadrightarrow & Q \\ & \| & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\nu} \\ \mathbf{a}\colon & C & \rightarrowtail & A & \overset{\pi}{\twoheadrightarrow} & Q_{\mathrm{ab}}. \end{array}$$

Since $\widetilde{\nu}$ is the restriction of the induced map ν of quotients, $\widetilde{\nu}$ is a group homomorphism. Also, if $\mathbf{e}$ is split, then so is $\mathbf{a}$: for suppose there is a splitting map $\sigma\colon Q \to E$. Since $\gamma\sigma([Q, Q]) \leq [A, A] = 1$, the map $Q_{\mathrm{ab}} \to A$ defined by

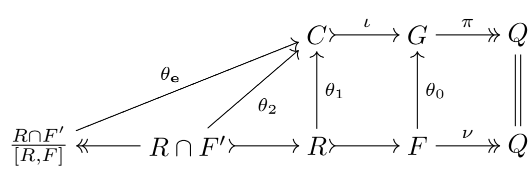
$$q[Q, Q] \mapsto \gamma\sigma(q)$$

for $q \in Q$ is a well defined splitting homomorphism for $\mathbf{a}$. Thus, $\widetilde{\nu}$ is injective.

To construct the surjection $\xi_\bullet$, assume we have the extension

$$\mathbf{e}\colon \qquad C \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$$

of groups. Let $R \rightarrowtail F \twoheadrightarrow Q$ be a free presentation for Q. Consider the following diagram:



The existence of the map $\theta_0\colon F \to G$ follows from the projective property of free groups. We see that $\theta_0(R) \leq \operatorname{Ker}\pi = \iota(C)$, so we obtain the map $\theta_1 = (\iota^{-1}\theta_0)\colon R \to C$. Let θ_2 denote the restriction of θ_1 to $R \cap F'$. Finally, $\theta_2([R, F]) \leq \iota^{-1}([\theta_0(R), \theta_0(F)]) \leq \iota^{-1}([\iota(C), G]) = 1$. Thus, θ_2 factors through

$R \cap F'/[R,F]$, as shown in the diagram. Let $\theta_{\mathbf{e}}$ denote the corresponding map

$$\theta_{\mathbf{e}} \colon \frac{R \cap F'}{[R,F]} \to C.$$

Lemma 1.7.1. *The map $\theta_{\mathbf{e}}$ does not depend on the choice of θ_0.*

Proof. Suppose θ_0' is another choice. We see that for $f \in F$, $\theta_0(f)\theta_0'(f)^{-1} \in \operatorname{Ker}\pi = \iota(C)$. Thus, there is $m_f \in \iota(C)$ such that $\theta_0'(f) = \theta_0(f)m_f$. Hence, if $g \in F$, then

$$\begin{aligned} \theta_0'([f,g]) &= [\theta_0'(f), \theta_0'(g)] \\ &= [\theta_0(f)m_f, \theta_0(g)m_g] \\ &= [\theta_0(f), \theta_0(g)], \end{aligned}$$

because $\iota(C) \leq Z(G)$. Since $[\theta_0(f), \theta_0(g)] = \theta_0([f,g])$, we see that θ_0 and θ_0' agree on F' and the result follows. □

We now describe the map $\theta_{\mathbf{e}}$. Let τ be any transversal for $\mathbf{e}$. We assume without loss of generality that F is the free group with basis $\{\bar{x} \colon x \in Q\}$. By the preceding Lemma we may also assume that θ_0 is the map

$$\theta_0 \colon \prod_i \bar{x}_i \mapsto \prod_i \tau(x_i).$$

Then

$$\theta_1 \colon \prod_i \bar{x}_i \mapsto \iota^{-1}\Big(\prod_i \tau(x_i)\Big)$$

for products $\prod_i \bar{x}_i$ that are in R, and

$$\theta_2 \colon \prod_i [\bar{x}_i, \bar{y}_i] \mapsto \iota^{-1}\Big(\prod_i [\tau(x_i), \tau(y_i)]\Big), \tag{1.4}$$

again when such products are in R; we note, however, that not every element of $R \cap F'$ is of the form in 1.4. Rather we have products of commutators of the form $[\prod_j \bar{x}_j, \prod_k \bar{z}_k]$. Note however that for $x, y \in Q$ we have $\bar{x}\bar{y} = r\overline{xy}$ with $r \in R$, so that $[\bar{x}\bar{y}, z] = {}^r[\overline{xy}, z][r,z]$, which is congruent to $[\overline{xy}, z]$ modulo $[F,R]$.

Thus we see that $\theta_{\mathbf{e}}$ is defined as

$$\theta_{\mathbf{e}} \colon \prod_i [\bar{x}_i, \bar{y}_i][F,R] \mapsto \iota^{-1}\Big(\prod_i [\tau(x_i), \tau(y_i)]\Big)$$

for all such products as $\prod_i [x_i, y_i] = 1$ in Q.

Definition 1.7.2. We define $\xi_{\mathbf{e}} \colon M(Q) \to C$ to be the map obtained by composing $\theta_{\mathbf{e}}$ with the isomorphism in Proposition 1.6.3, i.e.,

$$\xi_{\mathbf{e}} \colon \sum_i x_i \wedge y_i \mapsto \iota^{-1}\Big(\prod_i [\tau(x_i), \tau(y_i)]\Big)$$

whenever $\prod_i [x_i, y_i] = 1$ in Q. We refer to $\xi_{\mathbf{e}}$ as the *map corresponding to* $\mathbf{e}$.

Proposition 1.7.3. *The map* $\mathbf{e} \mapsto \xi_{\mathbf{e}}$ *is a group homomorphism, i.e.,* $\xi_{\mathbf{e} \boxplus \mathbf{e}'} = \xi_{\mathbf{e}} + \xi_{\mathbf{e}'}$.

Proof. Suppose we have two extensions

$$\mathbf{e}_1\colon \qquad C \overset{\iota_1}{\rightarrowtail} E_1 \overset{\pi_1}{\twoheadrightarrow} Q$$

and

$$\mathbf{e}_2\colon \qquad C \overset{\iota_2}{\rightarrowtail} E_2 \overset{\pi_2}{\twoheadrightarrow} Q.$$

and consider the extension

$$\mathbf{e}_1 \boxplus \mathbf{e}_2\colon \qquad K \overset{\iota_1 \boxplus \iota_2}{\rightarrowtail} E_1 \boxplus E_2 \overset{\pi_1 \boxplus \pi_2}{\twoheadrightarrow} Q.$$

For $(u, v) \in E_1 \times E_2$ we use the notation $(u, v)_{\boxplus}$ to denote a coset in $E_1 \boxplus E_2$, that is, modulo the relation induced by setting $\iota_1(c) \sim \iota_2(c)$ for $c \in C$.

If τ_1 and τ_2 are transversals for $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively, then we see that $(\tau_1 \boxplus \tau_2)\colon q \mapsto (\tau_1(q), \tau_2(q))$ is a transversal for $\mathbf{e}_1 \boxplus \mathbf{e}_2$. Thus, for sets $\{x_i, y_i\}_i \subset Q$ such that $\prod_i [x_i, y_i] = 1$ we have

$$\begin{aligned}
\xi_{\mathbf{e} \boxplus \mathbf{e}'}\big(\sum_i x_i \wedge y_i\big) &= (\iota_1 \boxplus \iota_2)^{-1}\Big(\prod_i \big[(\tau_1 \boxplus \tau_2)(x_i), (\tau_1 \boxplus \tau_2)(y_i)\big]_{\boxplus}\Big) \\
&= (\iota_1 \boxplus \iota_2)^{-1}\Big(\prod_i \big[(\tau_1(x_i), \tau_2(x_i)), (\tau_1(y_i), \tau_2(y_i))_{\boxplus}\big]\Big) \\
&= (\iota_1 \boxplus \iota_2)^{-1}\Big(\big(\prod_i \big[(\tau_1(x_i), \tau_1(y_i)\big], \prod_i \big[(\tau_2(x_i), \tau_2(y_i)\big]\big)_{\boxplus}\Big).
\end{aligned}$$

We note that $\prod_i \big[(\tau_2(x_i), \tau_2(y_i)\big] \in \iota_2(C)$ since applying π_2 yields $\prod_i [x_i, y_i]$, which is trivial by assumption. The element

$$\Big(\prod_i \big[(\tau_1(x_i), \tau_1(y_i)\big], \prod_i \big[(\tau_2(x_i), \tau_2(y_i)\big]\Big)$$

of $E_1 \times E_2$ represents the same coset as

$$\Big(\prod_i \big[(\tau_1(x_i), \tau_1(y_i)\big] \cdot \iota_1 \iota_2^{-1}\big(\prod_i \big[(\tau_2(x_i), \tau_2(y_i)\big]\big), 1\Big)$$

in $E_1 \boxplus E_2$, and we see that its preimage under $\iota_1 \boxplus \iota_2$ in C is

$$\iota_1^{-1}\Big(\prod_i \big[(\tau_1(x_i), \tau_1(y_i)\big]\Big) + \iota_2^{-1}\Big(\prod_i \big[(\tau_2(x_i), \tau_2(y_i)\big]\Big),$$

which is equal to $(\xi_{\mathbf{e}_1} + \xi_{\mathbf{e}_2})\Big(\sum_i x_i \wedge y_i\Big)$, as desired. □

Lemma 1.7.4. *The map* $\mathbf{e} \mapsto \xi_{\mathbf{e}}$ *is surjective onto* $\operatorname{Hom}(M(Q), C)$.

Proof. Assume that $R \rightarrowtail F \twoheadrightarrow Q$ is a free presentation, and let $\theta \colon F' \cap R/[F,R] \to C$ be a homomorphism. Consider the extension

$$\frac{F' \cap R}{[F,R]} \overset{\varepsilon}{\rightarrowtail} \frac{R}{[F,R]} \twoheadrightarrow \frac{R}{F' \cap R}. \tag{1.5}$$

of abelian groups. Since $R/R \cap F' \cong RF'/F' \leq F/F'$ is free abelian, the extension splits. Hence, there is a map

$$\sigma \colon \frac{R}{[F,R]} \to \frac{F' \cap R}{[F,R]}$$

such that $\sigma\varepsilon$ is the identity. Put $\bar{\theta} = \theta\sigma$, which is a homomorphism $R/[F,R] \to C$.

We now consider the induced map $\bar{\theta}_*$ as given by the push-out diagram

$$\begin{array}{rccccc} & \frac{R}{[F,R]} & \overset{\varepsilon}{\rightarrowtail} & \frac{F}{[F,R]} & \twoheadrightarrow & Q \\ & \downarrow{\scriptstyle \bar{\theta}} & & \downarrow & & \| \\ \mathbf{e}\colon & C & \overset{\iota}{\rightarrowtail} & E & \overset{\pi}{\twoheadrightarrow} & Q. \end{array}$$

Our claim is that $\theta_{\mathbf{e}} = \theta$, which would complete the proof. Put $F^0 = F/[R,F]$ and $R^0 = R/[R,F]$. Write μ for the map $F \twoheadrightarrow Q$. Now $E = C \rtimes F^0/N$, where $N = \{\big(\bar{\theta}(r), -\varepsilon(r)\big) \colon r \in R^0\}$ (cf. 1.3.1). The maps in $\mathbf{e}$ are given by

$$\iota \colon c \mapsto (c,1)N$$
$$\pi \colon (c, x[R,F])N \mapsto \mu(x).$$

Choose elements $\bar{x} \in F$ such that $\mu(\bar{x}) = x$. A transversal for $\mathbf{e}$ is then given by $\tau \colon x \mapsto (0, \bar{x}[R,F])$. Thus, for collections of pairs $\{x_i, y_i\}_i \subset Q$ such that $\prod_i [x_i, y_i] = 1$ we have

$$\theta_{\mathbf{e}}\Big(\prod_i [x_i,y_i][R,F]\Big) = \iota^{-1}\Big(\prod_i \big[\tau(x_i), \tau(y_i)\big]N\Big) \tag{1.6}$$

$$= \iota^{-1}\Big(\prod_i \big[(0, x_i[R,F]), (0, y_i[R,F])\big]N\Big) \tag{1.7}$$

$$= \iota^{-1}\Big(\big(0, \prod_i [x_i,y_i][R,F]\big)N\Big). \tag{1.8}$$

Now, $\prod_i [x_i,y_i] \in R \cap F' \leq R$ by assumption, so that

$$\Big(0, \prod_i [x_i,y_i][R,F]\Big)N = \Big(\bar{\theta}(\prod_i [x_i,y_i][R,F]), 1\Big)N. \tag{1.9}$$

Also, since $\prod_i [x_i,y_i] \in R \cap F'$, we have

$$\sigma\big(\prod_i [x_i,y_i][R,F]\big) = \prod_i [x_i,y_i][R,F],$$

which implies that

$$\bar{\theta}\Big(\prod_i [x_i, y_i][R, F]\Big) = \theta\Big(\prod_i [x_i, y_i][R, F]\Big). \tag{1.10}$$

Putting together 1.8, 1.9, and 1.10, we obtain

$$\begin{aligned}\theta_{\mathbf{e}}\Big(\prod_i [x_i, y_i][R, F]\Big) &= \iota^{-1}\Big(\big(\theta(\prod_i [x_i, y_i][R, F]), 1\big)N\Big)\\ &= \theta\Big(\prod_i [x_i, y_i][R, F]\Big),\end{aligned}$$

verifying the claim and completing the proof. □

We remark that in establishing surjectivity we have in fact constructed a splitting map for the sequence. This depended on the splitting of the sequence (1.5), which was not natural. Thus we see the group theoretic interpretation of the well-known result that the Universal Coefficient Sequence splits unnaturally.

Proposition 1.7.5. *The sequence*

$$\mathscr{E}_{ab}(Q_{ab}, C) \overset{\widetilde{\nu}}{\rightarrowtail} \mathscr{E}_0(Q, C) \overset{\xi_\bullet}{\twoheadrightarrow} \mathrm{Hom}(M(Q), C)$$

is exact.

Proof. What is left to show is exactness at the term $\mathscr{E}_0(Q, C)$. Suppose $\mathbf{a}$ is an abelian extension of C by Q_{ab}. Consider the diagram

$$\begin{array}{lccccc} \widetilde{\nu}(\mathbf{a})\colon & C & \overset{\iota}{\rightarrowtail} & E & \overset{\rho}{\twoheadrightarrow} & Q \\ & \| & & \downarrow & & \downarrow \nu \\ \mathbf{a}\colon & C & \overset{\varepsilon}{\rightarrowtail} & G & \overset{\pi}{\twoheadrightarrow} & Q_{\mathrm{ab}}. \end{array}$$

We recall that $E = G \times_{Q_{\mathrm{ab}}} Q$. If τ is a transversal for $\mathbf{a}$ then $\sigma\colon q \mapsto (\tau(qQ', q)$ is a transversal for $\widetilde{\nu}(\mathbf{a})$. Thus, for collections of pairs $\{x_i, y_i\}_i \subseteq Q$ such that $\prod_i [x_i, y_i] = 1$, we have

$$\begin{aligned}\xi_{\widetilde{\nu}(\mathbf{a})}\Big(\sum_i x_i \wedge y_i\Big) &= \iota^{-1}\Big(\prod_i \big[\sigma(x_i), \sigma(y_i)\big]\Big)\\ &= \iota^{-1}\Big(\prod_i \big[(\tau(x_iQ'), x_i), (\tau(y_iQ'), y_i)\big]\Big)\\ &= \iota^{-1}\Big(\prod_i \big[\tau(x_iQ'), \tau(y_iQ')\big], \prod_i \big[x_i, y_i\big]\Big).\end{aligned}$$

Since G is abelian, $\big[\tau(x_iQ'), \tau(y_iQ')\big] = 1$ for each i. Also, by assumption we have $\prod_i \big[x_i, y_i\big] = 1$. Hence

$$\xi_{\widetilde{\nu}(\mathbf{a})}\Big(\sum_i x_i \wedge y_i\Big) = 0,$$

verifying that $\mathrm{Im}(\widetilde{\nu}) \leq \mathrm{Ker}(\xi_\bullet)$.

Conversely, suppose $\mathbf{e}$ is the extension

$$\mathbf{e}\colon \qquad C \rightarrowtail^{\iota} E \overset{\rho}{\twoheadrightarrow} Q$$

and that $\xi_{\mathbf{e}} = 0$. Consider the extension

$$\mathbf{e}_0\colon \qquad C \overset{\iota}{\rightarrowtail} \rho^{-1}(Q') \overset{\rho}{\twoheadrightarrow} Q', \tag{1.11}$$

which we claim is split. Indeed, let τ be a transversal for $\mathbf{e}$ and consider the map $\sigma\colon Q' \to G$ defined by

$$\sigma\colon \prod [x_i, y_i] \mapsto \prod [\tau(x_i), \tau(y_i)].$$

for collections of pairs $\{x_i, y_i\}_i \subseteq Q$. Since $\xi_{\mathbf{e}} = 0$, we have the condition

$$\prod [x_i, y_i] = 1 \Longrightarrow \prod [\tau(x_i), \tau(y_i)] = 1$$

for all collections of pairs $\{x_i, y_i\}_i \subseteq Q$, and it follows that σ is well-defined. By design σ is multiplicative, and evidently $\mathrm{Im}(\sigma) \leq \rho^{-1}(Q')$. Moreover, it is obvious that $\rho\sigma$ is the identity map on Q'. Thus, the sequence (1.11) above is split. Specifically, $\rho^{-1}(Q') = \iota(C) \times \mathrm{Im}(\sigma)$.

In fact, $\mathrm{Im}(\sigma) \triangleleft E$, which we now verify. Let $g \in G$ and write $q = \rho(g)$. Then for $x, y \in Q$ we have ${}^g[\tau(x), \tau(y)] = [{}^g\tau(x), {}^g\tau(y)] = [\tau({}^qx), \tau({}^qy)]$ since $\tau({}^qx)$ and ${}^g\tau(x)$ differ by an element of $Z(G)$ (and likewise for y). Normality of $\mathrm{Im}(\sigma)$ easily follows.

Put $K = \mathrm{Im}(\sigma)$, let $\mu\colon E \to E/K$ be the natural map, and consider the diagram

$$\begin{array}{llcccc}
\mathbf{e}\colon & C & \overset{\iota}{\rightarrowtail} & E & \overset{\rho}{\twoheadrightarrow} & Q \\
& \| & & \downarrow \mu & & \downarrow \nu \\
\mathbf{a}\colon & C & \overset{\mu\iota}{\rightarrowtail} & E/K & \overset{\pi}{\twoheadrightarrow} & Q_{\mathrm{ab}}.
\end{array}$$

Since $K \cap \iota(C) = 1$ the map $\mu\iota$ is injective. The map π is defined as $gK \mapsto \rho(g)Q'$ and is clearly surjective with kernel $\iota(C)K/K = \mathrm{Im}(\mu\iota)$. The diagram evidently commutes.

Finally, to see that E/K is abelian we remark that for $x, y \in E$ we have $[x, y] = [\tau(x), \tau(y)]$ because x and $\tau(x)$ differ by an element of $Z(G)$ (and likewise for y). Hence, in fact, $\mathrm{Im}(\sigma) = E'$ and E/K is abelian.

Thus, $\mathrm{Ker}(\xi_\bullet) = \mathrm{Im}(\widetilde{\nu})$ and the sequence is exact. □

We end this chapter with a description of how the map $\xi_\bullet$ behaves with respect to maps of extensions.

Proposition 1.7.6. *If* $(\alpha, \gamma, \beta)\colon \mathbf{e}_1 \mapsto \mathbf{e}_2$ *is a map of extensions with* $\mathbf{e}_1$ *and*

$\mathbf{e}_2$ *central, then*

$$\xi_{\mathbf{e}_2} H_2(\beta) = \alpha \xi_{\mathbf{e}_1}.$$

Proof. For definiteness, assume $(\alpha, \gamma, \beta)\colon \mathbf{e}_1 \mapsto \mathbf{e}_2$ is given by the commutative diagram

$$\begin{array}{lccccc}
\mathbf{e}_1\colon & C_1 & \xrightarrow{\iota_1} & G_1 & \xrightarrow{\pi_1} & Q_1 \\
 & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\
\mathbf{e}_2\colon & C_2 & \xrightarrow{\iota_2} & G_2 & \xrightarrow{\pi_2} & Q_2.
\end{array}$$

If τ_1 is a transversal for $\mathbf{e}_1$, we define a transversal τ_2 for $\mathbf{e}_2$ as follows. For $q \in \operatorname{Im}(\beta)$ let $\sigma(q)$ be any element of Q_1 such that $\beta\bigl(\sigma(q)\bigr) = q$ and define $\tau_2(q) = \gamma\tau_1\sigma(q)$. For $q \notin \operatorname{Im}(\beta)$ choose $\tau_2(q)$ arbitrarily with the property $\pi_2\bigl(\tau_2(q)\bigr) = q$. Clearly τ_2 is a transversal for $\mathbf{e}_2$. We note that then

$$\tau_2\beta \equiv \gamma\tau_1 \pmod{Z(G_2)}, \tag{1.12}$$

for if $q \in Q_1$ then $\pi_2\bigl(\tau_2\beta(q)\bigr) = \beta(q) = \beta\pi_1\tau_1(q) = \pi_2\bigl(\gamma\tau_1(q)\bigr)$.

We note also that for $x \in \operatorname{Im}\iota_1$ we have

$$\alpha\iota_1^{-1}(x) = \iota_2^{-1}\gamma(x). \tag{1.13}$$

To see this, write $x = \iota_1(c)$ with $c \in C_1$. Then $\iota_2^{-1}\gamma(x) = \iota_2^{-1}\gamma\iota_1(c) = \iota_2^{-1}\iota_2\alpha(c) = \alpha(c) = \alpha\iota_1^{-1}(x)$.

Finally, for collections of pairs $\{x_i, y_i\} \subseteq Q$ such that $\prod_i [x_i, y_i] = 1$ we have

$$\begin{aligned}
\alpha\xi_{\mathbf{e}_1}\Bigl(\sum_i x_i \wedge y_i\Bigr) &= \alpha\iota_1^{-1}\Bigl(\prod_i [\tau_1(x_i), \tau(y_i)]\Bigr) && (1.14)\\
&= \iota_2^{-1}\gamma\Bigl(\prod_i \bigl[\tau_1(x_i), \tau_1(y_i)\bigr]\Bigr) && (1.15)\\
&= \iota_2^{-1}\Bigl(\prod_i \bigl[\gamma\tau_1(x_i), \gamma\tau_1(y_i)\bigr]\Bigr) && (1.16)\\
&= \iota_2^{-1}\Bigl(\prod_i \bigl[\tau_2\beta(x_i), \tau_2\beta(y_i)\bigr]\Bigr) && (1.17)\\
&= \xi_{\mathbf{e}_2}\Bigl(\sum_i \beta(x_i) \wedge \beta(y_i)\Bigr) && (1.18)\\
&= \xi_{\mathbf{e}_2} H_2(\beta). && (1.19)
\end{aligned}$$

In the above (1.15) comes from (1.13), (1.17) from (1.12), and (1.18) from the definition of $\xi_{\mathbf{e}_2}$. We note that whenever $\prod_i [x_i, y_i] = 1$ we have $\prod_i [\beta(x_i), \beta(y_i)] = 1$, so that the expression in (1.18) is defined. $\square$

2 D-Extensions

2.1 Central extensions with divisible kernel

In this chapter we introduce the notion of a *d-extension*, which is a certain type of central extension that turns out to have a universal property, namely, every central extension embeds into a d-extension. D-extensions play a role analogous to that played by divisible groups for abelian groups.

Definition 2.1.1. An abelian group D is said to be *divisible* provided that for every $x \in D$ and nonzero $n \in \mathbb{Z}$ there is $y \in D$ such that $ny = x$.

Divisible groups have been studied extensively and there is a thorough analysis of them in [Fuc73]. An equivalent condition for divisibility is that every subgroup is a direct summand. Divisible groups are precisely those satisfying Baer's criterion for injectivity [Bae40]; thus, divisible groups are precisely the injective objects in the category of abelian groups. As a consequence, if D is divisible then $\operatorname{Ext}(A, D) = 0$ for all abelian groups A. The Universal Coefficient Theorem (cf. Section 1.7) thus gives the following.

Theorem 2.1.2. *If Q is any group and D is a divisible abelian group, then the map $\mathbf{e} \mapsto \xi_{\mathbf{e}}$ is an isomorphism*

$$\mathscr{E}_0(Q, D) \xrightarrow{\sim} \operatorname{Hom}(M(Q), D).$$

It follows that a central extension with divisible kernel is uniquely determined up to equivalence by a homomorphism from $M(Q)$ into the kernel, where Q is the quotient. We will refer to a central extension with divisible kernel as a *d-extension*.

It is well-known that the category of abelian groups has enough injectives; that is, if A is an abelian group, then there is a divisible group D and an embedding $A \rightarrowtail D$ (see, for example, [Fuc73, p. 106]). By considering induced maps of kernels (cf. Section 1.3) we obtain the following fundamental embedding property of d-extensions.

Proposition 2.1.3. *Let*

$$\mathbf{e}: \qquad C \rightarrowtail E \twoheadrightarrow Q$$

be a central extension. Then there exists a d-extension

$$\mathbf{e}_d\colon \qquad D \rightarrowtail G \twoheadrightarrow Q$$

and an injective map $\mathbf{e} \rightarrowtail \mathbf{e}_d$ such that the map $Q \to Q$ is the identity.

Proof. Let $\varepsilon\colon C \rightarrowtail D$ be an embedding of C into a divisible abelian group D. Then the induced map $\varepsilon_*\colon \mathbf{e} \mapsto \varepsilon_*(\mathbf{e})$ of extensions has the desired properties. □

For any group Q, let $\mathscr{E}_0(Q, \_)$ be the subcategory of $\mathscr{E}$ whose objects consist of equivalence classes of extensions with quotient Q and morphisms are of the form $(\alpha, \gamma, 1)$, i.e., morphisms fixing the quotient pointwise. In [Gru70, p. 195] Gruenberg shows that the injective objects in $\mathscr{E}_0(Q, \_)$ are precisely those extensions with divisible kernel. Proposition 2.1.3 can then be restated as "The category $\mathscr{E}_0(Q, \_)$ has enough injectives."

Nilpotent groups are one reason for our interest in d-extensions.

Proposition 2.1.4. *Let N be a nilpotent group of class c. Then there is a nilpotent group G of class c and an embedding $\varepsilon\colon N \rightarrowtail G$ such that $Z(G)$ is divisible, $\varepsilon\big(Z(N)\big) \le Z(G)$, and the induced map $\bar{\varepsilon}\colon N/Z(N) \to G/Z(G)$ is the identity.*

First we prove the following lemma.

Lemma 2.1.5. *If*

$$\begin{array}{ccccc} C_1 & \stackrel{\varepsilon}{\rightarrowtail} & E & \stackrel{\rho}{\twoheadrightarrow} & Q \\ \downarrow{\scriptstyle \omega} & & \downarrow & & \| \\ C_2 & \stackrel{\iota}{\rightarrowtail} & G & \stackrel{\pi}{\twoheadrightarrow} & Q \end{array}$$

is an injection of extensions with $Z(E) = \mathrm{Im}(\varepsilon)$ then $Z(G) = \mathrm{Im}(\iota)$.

Proof. We can identify the lower extension with the push-out (see Section 1.3). By Corollary 1.3.8 we have $\mathrm{Im}(\iota) \le Z(G)$. Recall that

$$G = (E \times C_2)/N,$$

where

$$N = \Big\{ \big(\varepsilon(z), -\omega(z)\big)\colon z \in Z(E) \Big\}$$

(here the action is trivial, so the semi-direct product is direct). Now, for $e, e' \in E$ and $d, d' \in C_2$ we have $\big[(e,d),(e',d')\big]N = \big([e,e'],0\big)N$. Also $\big([e,e'],0\big) \in N$ if and only if there exists $z \in C_1$ such that $\varepsilon(z) = [e,e']$ and $\omega(z) = 0$, i.e. $z = 0$ and $[e,e'] = 1$. Thus $(e,d)N \in Z(G)$ if and only if $e \in Z(E) = \mathrm{Im}(\varepsilon)$. Now, for $e \in Z(E)$ and $d \in C_2$ we have $\pi\big((e,d)N\big) = \rho(e) = 1$, so that $Z(G) \le \mathrm{Ker}(\pi) = \mathrm{Im}(\iota)$, thus completing the proof. □

Proof of Proposition 2.1.4. Apply Proposition 2.1.3 to obtain the injection

$$\begin{array}{ccccc} Z(N) & \rightarrowtail & N & \twoheadrightarrow & N/Z(N) \\ \downarrow & & \downarrow & & \| \\ D & \rightarrowtail & G & \twoheadrightarrow & N/Z(N). \end{array}$$

By Lemma 2.1.5 we see that $Z(G) = D$, from which it follows that G is nilpotent of class c. ☐

Thus, d-extensions contain all the nilpotent groups in a nice way, that is, such that containment occurs centrally and the central quotients are identical.

The next proposition justifies to some extent our abuse of terminology in referring to the middle group of an extension as the extension. It also justifies our interest in isomorphism classes of extensions. The problem of classifying middle groups (up to isomorphism) that can appear in d-extensions translates directly into classifying the d-extensions up to isomorphism.

Proposition 2.1.6. *Let N be a finitely generated nilpotent group and D a divisible abelian group. If*

$$\mathbf{e}_1\colon \qquad D \overset{\iota}{\rightarrowtail} G_1 \twoheadrightarrow N$$

and

$$\mathbf{e}_2\colon \qquad D \overset{\varepsilon}{\rightarrowtail} G_2 \twoheadrightarrow N$$

are two extensions, then $\mathbf{e}_1 \cong \mathbf{e}_2 \iff G_1 \cong G_2$ as groups.

Proof. Necessity is clear.

For sufficiency, we show that D embeds as the (unique) maximal divisible abelian subgroup in G_1 and G_2. Let $G = G_1$ or G_2 and suppose $\widetilde{D} \leq G$ is a divisible abelian subgroup. Note that every subgroup of N is finitely generated. Thus, by the classification of divisible abelian groups, N contains no nontrivial divisible subgroups. Now, $\widetilde{D}/D \cap \widetilde{D} \cong \widetilde{D}D/D$, which is a divisible subgroup of N and thus is trivial. Hence, $\widetilde{D} \leq D$, which shows that D embeds as the maximal divisible subgroup of G.

Now let $\varphi\colon G_1 \to G_2$ be an isomorphism. Since $\varphi(\iota D)$ is divisible, necessarily $\varphi(\iota D) \leq \varepsilon D$, and likewise $\varphi^{-1}(\varepsilon D) \leq \iota D$. Hence, in fact, $\varphi(\iota D) = \varepsilon D$. Let $\alpha = \varepsilon^{-1}\varphi\iota$, which is an isomorphism $D \to D$.

Assume τ is a transversal for $\mathbf{e}_1$ and let $\beta = \rho\varphi\tau$. We verify that β is an isomorphism. Since τ is multiplicative modulo ιD, and $\rho\varphi\iota = 0$, it follows that β is a homomorphism. Now, $\rho\varphi$ is surjective, and $\iota D \leq \operatorname{Ker}(\rho\varphi)$, so since $\{\tau n\colon n \in N\}$ is a complete set of coset representatives modulo ιD, β is surjective. If $n \in \operatorname{Ker}\beta$ then $\tau(n) \in \iota D$ and $n = 0$, so β is also injective.

Finally, commutativity of the diagram making $(\alpha, \varphi, \beta)\colon \mathbf{e}_1 \to \mathbf{e}_2$ an isomorphism is easy to check. ☐

Another motivation for studying d-extensions comes from an application to capable groups. By definition a group is *capable* if and only if it is isomorphic with the central quotient of some group. Obviously these groups play crucial roles in any study of central extensions since they are the groups that can appear as quotients when the kernel is the center. It turns out that it is sufficient to consider d-extensions.

Corollary 2.1.7. *A group Q is capable if and only if there is a d-extension*

$$D \overset{\iota}{\rightarrowtail} E \twoheadrightarrow Q$$

with $\mathrm{Im}(\iota) = Z(E)$*.*

Proof. Sufficiency is clear. For necessity, suppose that Q is capable and let E be a group such that $Q \cong E/Z(E)$. Let $\omega\colon Z(E) \to D$ be an embedding into a divisible abelian group D. Apply Proposition 2.1.3 to the extension $Z(E) \rightarrowtail E \twoheadrightarrow Q$ to obtain the commutative diagram

$$\begin{array}{ccccc} Z(E) & \overset{\varepsilon}{\rightarrowtail} & E & \overset{\rho}{\twoheadrightarrow} & Q \\ \omega\downarrow & & \downarrow & & \| \\ D & \overset{\iota}{\rightarrowtail} & G & \overset{\pi}{\twoheadrightarrow} & Q \end{array}$$

By Lemma 2.1.5 we have $Z(G) = \mathrm{Im}(\iota)$. $\square$

By Theorem 2.1.2 a d-extension $D \overset{\iota}{\rightarrowtail} G \twoheadrightarrow Q$ with transversal τ is uniquely classified up to equivalence by the values

$$\iota^{-1}\Big(\prod_i [\tau x_i, \tau y_i]\Big)$$

for which $\prod_i [x_i, y_i] = 1$ (cf. Section 1.7). We shall be more explicit when further restrictions are placed on Q. For now, we make the following observation that for d-extensions, the converse of Proposition 1.7.6 holds.

Proposition 2.1.8. *If $\mathbf{e}_1$ and $\mathbf{e}_2$ are d-extensions such that there exist group homomorphims α (of the kernels) and β (of the quotients) for which $\xi_{\mathbf{e}_2} H_2(\beta) = \alpha\xi_{\mathbf{e}_1}$, then there is a group homomorphism γ such that $(\alpha, \gamma, \beta)\colon \mathbf{e}_1 \mapsto \mathbf{e}_2$ is a map of extensions.*

Proof. By Proposition 1.7.6 we have $\xi_{\alpha_* \mathbf{e}_1} = \alpha\xi_{\mathbf{e}_1}$ and $\xi_{\beta^* \mathbf{e}_2} = \xi_{\mathbf{e}_2} H_2(\beta)$. Thus, assuming $\xi_{\mathbf{e}_2} H_2(\beta) = \alpha\xi_{\mathbf{e}_1}$, we have $\xi_{\alpha_* \mathbf{e}_1} = \xi_{\beta^* \mathbf{e}_2}$. By Theorem 2.1.2 there is an equivalence $\varphi\colon \alpha_* \mathbf{e}_1 \mapsto \beta^* \mathbf{e}_2$. Composing the maps $\alpha_*\colon \mathbf{e}_1 \mapsto \alpha_* \mathbf{e}_1$, φ, and $\beta^*\colon \beta^* \mathbf{e}_2 \mapsto \mathbf{e}_2$ we obtain the desired map $\mathbf{e}_1 \mapsto \mathbf{e}_2$. $\square$

Suppose C is any abelian group. Then we may understand extensions of C by looking at extensions of a divisible group D such that C embeds into D. Indeed, any extension $\mathbf{e}$ of C must embed into an extension $\mathbf{d}$ of D. The following Corollary gives us some insight into when such an embedding exists if the extensions $\mathbf{e}$ and $\mathbf{d}$ are given.

Corollary 2.1.9. *Let Q be any group and let C and D be a abelian groups with D divisible. Given a homomorphism $\alpha\colon C \to D$ and extensions $\mathbf{e} \in \mathscr{E}_0(Q, C)$ and $\mathbf{d} \in \mathscr{E}_0(Q, D)$, we have an induced map $\alpha_*\colon \mathbf{e} \mapsto \mathbf{d}$ of extensions if and only if $\xi_{\mathbf{d}} = \alpha\xi_{\mathbf{e}}$.*

Proof. If $\alpha_*\colon \mathbf{e} \mapsto \mathbf{d}$ is a map of extensions, then by Proposition 1.7.6 we have $\xi_{\mathbf{d}} = \alpha\xi_{\mathbf{e}}$. The converse is a direct result of Proposition 2.1.8. □

2.2 Divisible abelian groups

In the next few sections we consider the classification problem for d-extensions (with certain restrictions on the quotient). In doing so we use some properties of divisible abelian groups and their automorphisms. In this section we summarize these properties and develop some useful terminology.

The question arises as to which abelian groups are divisible and what are their automorphisms. It is easy to see that the additive group of rationals $\mathbb{Q}$ is divisible. An important class of examples is the so-called "prüfer p-groups."

Let p be any prime integer and consider the groups $\mathbb{Z}/p^i$ for $i \in \mathbb{Z}_{>0}$. For each i assume we have a chosen generator x_i for $\mathbb{Z}/p^i$. If $i \leq j$ then there is a map $\mathbb{Z}/p^i \hookrightarrow \mathbb{Z}/p^j$ given by $x_i \mapsto p^{j-i}x_j$. This collection of maps forms an inductive system, whose limit is denoted $\mathbb{Z}(p^\infty)$ and called the *prüfer group of type p^∞* (or *prüfer p-group*). A presentation for $\mathbb{Z}(p^\infty)$ is

$$\mathbb{Z}(p^\infty) = \langle d_1, d_2, \cdots : pd_1 = 1, pd_{i+1} = d_i, i > 0\rangle.$$

It is easy to see that the prüfer groups are divisible (cf. [Rob91, p. 94]). In fact, there is a well known classification, which shows that, modulo direct sums, this exhausts all examples.

Theorem 2.2.1 (Classification of Divisible Abelian Groups)**.** *Let D be a divisible abelian group. Then there is a multiset $\mathscr{P}$ of prime integers and an index set $\mathscr{I}$ such that*

$$D \cong \Big(\bigoplus_{p\in\mathscr{P}} \mathbb{Z}(p^\infty)\Big) \oplus \Big(\bigoplus_{i\in\mathscr{I}} \mathbb{Q}\Big).$$

This is proved in [Rob91, p. 97], for example.

Definition 2.2.2. Let A be an abelian group. The *torsion-free rank* of A, denoted $\mathrm{r}_0(A)$, is the dimension of $A \otimes \mathbb{Q}$ as a vector space over $\mathbb{Q}$. For primes p we let $\mathrm{r}_p(A)$ denote the dimension of $\mathrm{Hom}(\mathbb{Z}/p, A)$ as a vector space over $\mathbb{F}_p$. The *total rank* of A is $\mathrm{r}(A) = \sum_p \mathrm{r}_p(A)$.

If D is a divisible group, then $\mathrm{r}_p(D)$ is simply the cardinal number of copies of $\mathbb{Z}(p^\infty)$ and $r_0(D)$ the cardinal number of copies of $\mathbb{Q}$ appearing in a direct sum decomposition.

The following notion is useful when considering the Prüfer groups.

Definition 2.2.3. Let G be any group, which we write additively. We say that a set $\mathscr{X}$ of elements of G is *p-inductive* if

1. $0 \in \mathscr{X}$,
2. for each $x \in \mathscr{X}$ there is $y \in \mathscr{X}$ such that $py = x$ (p-divisibility), and
3. for each pair $x, y \in \mathscr{X}$ there is an $a \in \mathbb{N}$ such that either $p^a x = y$ or $p^a y = x$.

Proposition 2.2.4. *If $\mathscr{X}$ is a p-inductive set in any group G, then for each $i \in \mathbb{N}$ there is a unique $x_i \in \mathscr{X}$ such that $|x_i| = p^i$. Consequently, the elements of $\mathscr{X}$ can be indexed as $\{x_i\}_{i=0}^{\infty}$ where $x_0 = 0$ and $px_i = x_{i-1}$ for $i > 0$.*

Proof. Since $0 \in \mathscr{X}$, the proposition is trivial for $i = 0$. Assume that $i > 0$ and the proposition holds for $i - 1$. Using property 2, choose x_i so that $px_i = x_{i-1}$. Then $|x_i| = p^i$. Now suppose that $y \in \mathscr{X}$ and $|y| = p^i$. Using property 3, there exists $a \in \mathbb{N}$ such that $p^a y = x_i$ (without loss of generality). Then $|y| = |x_i| \cdot p^a \Rightarrow a = 0$, so that $y = x_i$. □

We will therefore use the convention that if $\{0, d_1, d_2, \ldots\}$ is a p-inductive set, then $pd_1 = 0$ and $pd_i = d_{i-1}$ for $i > 1$. In particular, p-inductive sets are ordered and their elements are distinct.

Proposition 2.2.5. *There is a p-inductive set in $\mathbb{Z}(p^\infty)$, and any p-inductive set in $\mathbb{Z}(p^\infty)$ generates $\mathbb{Z}(p^\infty)$. In particular, if $\{0, d_1, d_2, \ldots\}$ is a p-inductive set in $\mathbb{Z}(p^\infty)$, then each element of $\mathbb{Z}(p^\infty)$ can be written uniquely in the form λd_i with $\lambda \in \mathbb{Z}/p^i$.*

Proof. The existence of a p-inductive set in $\mathbb{Z}(p^\infty)$ follows from the fact that $\mathbb{Z}(p^\infty)$ is the inductive limit of the system

$$\mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 \hookrightarrow \cdots .$$

Choose any generator x_1 of $\mathbb{Z}/p$. Since $\mathbb{Z}/p^i$ is cyclic of order p^i and x_1 has order p, it follows that there is an $x_i \in \mathbb{Z}/p^i$ such that $x_1 \mapsto p^{i-1}x_i$. Moreover, x_i is unique, for if $p^{i-1}y = p^{i-1}x_i$, then $x_i - y$ has order dividing p^{i-1}, which contradicts that $|x_i - y| = \operatorname{lcm}(|x_i|, |y|) = p^i$. The set $\{0, x_1, x_2, \ldots\}$ is p-inductive.

Assume $\{0, d_1, d_2, \ldots\}$ is a p-inductive set in D with $|d_i| = p^i$ for $i > 1$. By identifying $\mathbb{Z}/p^i$ with its image in $\mathbb{Z}/p^{i+1}$ for each i, we may take the underlying set for the inductive limit to be the union $\bigcup_i \mathbb{Z}/p^i$. Now let $d \in D$. There is a unique $i \in \mathbb{N}$ such that $d \in \mathbb{Z}/p^i$ but $d \notin \mathbb{Z}/p^{i+1}$. Likewise, $d_i \in \mathbb{Z}/p^i$, and it is easily seen that d_i generates $\mathbb{Z}/p^i$. The result now follows. □

One of our motivations for studying d-extensions is the embedding property (Proposition 2.1.3). Naturally, for a given abelian group A there is a "best" choice for divisible group into which to embed A. This is the content of the following proposition, which is proved, for example, in [Fuc73, p. 106].

Proposition 2.2.6. *Let A be an abelian group. There is a divisible group A^{div} and an embedding $\delta_A\colon A \rightarrowtail A^{\mathrm{div}}$ such that if $\iota\colon A \rightarrowtail D$ is any embedding with D divisible, then ι factors through δ_A. The group A^{div} is unique up to isomorphism.*

It is obvious that if $A \rightarrowtail D$ is any embedding of A into a divisible group D then A^{div} may be taken as a subgroup of D. The group A^{div} is called the *divisible hull* of A.

Recall that a d-extension is uniquely determined by its corresponding homomorphism $\xi_{\mathbf{e}}\colon M(Q) \to D$, where D is the kernel and Q is the quotient. The next proposition limits our scope to d-extensions where $D \cong \mathrm{Im}(\xi_{\mathbf{e}})^{\mathrm{div}}$.

Proposition 2.2.7. *Let $\mathbf{e}$ be the d-extension*

$$\mathbf{e}\colon \qquad D \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$$

and let $\widetilde{D} \leq D$ be any divisible subgroup containing $\mathrm{Im}(\xi_{\mathbf{e}})$. Let $\hat{D}$ be a complement to $\widetilde{D}$ in D. There is an extension

$$\widetilde{D} \overset{\widetilde{\iota}}{\rightarrowtail} \widetilde{G} \overset{\widetilde{\pi}}{\twoheadrightarrow} Q$$

such that $\mathbf{e}$ is equivalent to the extension

$$\widetilde{D} \oplus \hat{D} \overset{\widetilde{\iota}\oplus 1}{\rightarrowtail} \widetilde{G} \times \hat{D} \overset{\widetilde{\pi}\times 0}{\twoheadrightarrow} Q.$$

Proof. Let $\widetilde{\varepsilon}$ be the inclusion map $\widetilde{D} \rightarrowtail D$. Then $\xi_{\mathbf{e}}$ factors through $\widetilde{\varepsilon}$, say $\xi_{\mathbf{e}} = \widetilde{\varepsilon}\widetilde{\xi}$, with $\widetilde{\xi} \in \mathrm{Hom}(M(Q), \widetilde{D})$. Now $\widetilde{\xi}$ corresponds to an extension of $\widetilde{D}$ by Q, say the extension

$$\widetilde{\mathbf{e}}\colon \qquad \widetilde{D} \overset{\widetilde{\iota}}{\rightarrowtail} \widetilde{G} \overset{\widetilde{\pi}}{\twoheadrightarrow} Q.$$

Let φ be the inclusion map $\widetilde{G} \rightarrowtail \widetilde{G} \times \hat{D}$. It is easy to see that the diagram

$$\begin{array}{llcccccc}
\widetilde{\mathbf{e}}\colon & & \widetilde{D} & \overset{\widetilde{\iota}}{\rightarrowtail} & \widetilde{G} & \overset{\widetilde{\pi}}{\twoheadrightarrow} & Q \\
& & \downarrow{\scriptstyle \widetilde{\varepsilon}} & & \downarrow{\scriptstyle \varphi} & & \| \\
\hat{\mathbf{e}}\colon & & \widetilde{D} \oplus \hat{D} & \overset{\widetilde{\iota}\oplus 1}{\rightarrowtail} & \widetilde{G} \times \hat{D} & \overset{\widetilde{\pi}\times 0}{\twoheadrightarrow} & Q.
\end{array}$$

commutes. By Proposition 1.7.6 we have $\xi_{\hat{\mathbf{e}}} = \widetilde{\varepsilon}\widetilde{\xi} = \xi$ and by Theorem 2.1.2 we have $\mathbf{e} \equiv \hat{\mathbf{e}}$. $\square$

This result means that effectively a complement to $\widetilde{D}$ splits off from G as a direct factor. We remark that we may take $\widetilde{D}$ to be the divisible hull of $\mathrm{Im}(\xi_{\mathbf{e}})$.

Proposition 2.2.8. *$r_p(A) = r_p(A^{\mathrm{div}})$ for $p = 0$ or p prime.*

This is proved in [Fuc73, p. 107], for example. Thus, in considering d-extensions $\mathbf{e}$ of D by Q we may as well assume that $\mathrm{r}_p(D) \leq \mathrm{r}_p(M(Q))$ for all p. Otherwise $\mathrm{Im}(\xi_{\mathbf{e}})^{\mathrm{div}}$ has a nontrivial complement in D, which splits off per Proposition 2.2.7.

2.3 Automorphisms of prüfer p-groups

In considering the isomorphism problem for d-extensions, we will need to make use of some basic facts about automorphisms of divisible abelian groups. The automorphism group of $\mathbb{Q}^+$ is well-known to be $\mathbb{Q}^\times$. Because of the classification of divisible abelian groups, the study of automorphisms reduces (by way of matrices) to studying the endomorphisms of $\mathbb{Z}(p^\infty)$. Here we study the automorphisms of $\mathbb{Z}(p^\infty)$.

Heuristically, p-inductive sets behave much like bases of vector spaces. The following proposition further confirms this intuition.

Proposition 2.3.1. *$\mathrm{Aut}\,\mathbb{Z}(p^\infty)$ is in bijective correspondence with p-inductive sets in $\mathbb{Z}(p^\infty)$. In particular, the automorphisms of $\mathbb{Z}(p^\infty)$ are precisely the changes of p-inductive sets.*

Proof. Let $\varphi \in \mathrm{Aut}\,\mathbb{Z}(p^\infty)$ and let $\mathscr{D} = \{1, d_1, d_2, \ldots\}$ be a p-inductive set in $\mathbb{Z}(p^\infty)$. It is easy to see that $\varphi(\mathscr{D}) = \{0, \varphi(d_1), \varphi(d_2) \ldots\}$ is also a p-inductive set. Conversely, if $\mathscr{D}' = \{0, d_1', d_2', \ldots\}$ is a p-inductive set in $\mathbb{Z}(p^\infty)$, then assigning $d_i \mapsto d_i'$ and extending linearly clearly defines an automorphism of $\mathbb{Z}(p^\infty)$. $\square$

Assume again that for each $i \in \mathbb{Z}_{>0}$ we have a chosen generator x_i for the additive group of $\mathbb{Z}/p^i$, which we regard as a ring. Dual to $\mathbb{Z}(p^\infty)$, the ring $\mathbb{Z}_p$ of p-adic integers is the projective limit of the system of maps $\mathbb{Z}/p^i \twoheadrightarrow \mathbb{Z}/p^j$ given by $x_j \mapsto x_i$ for $i \leq j$. As such, $\mathbb{Z}_p$ consists of sequences $a = (a_i)_{i=1}^\infty$ such that $a_i \in \mathbb{Z}/p^i$ and if $i \leq j$ then $a_i \equiv a_j \pmod{p^i}$. Hence each p-adic integer uniquely determines a sequence $(\alpha_i)_{i=0}^\infty$ of elements of $\mathbb{Z}/p$, where

$$a_i = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \cdots + \alpha_{i-1} p^{i-1}$$

for each $i \in \mathbb{Z}_{>0}$. Conversely, any such sequence determines a unique p-adic integer, and we may regard a p-adic integer as a formal power series $\sum_{i=0}^\infty \alpha_i p^i$.

Now suppose $\psi \colon \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)$ is a homomorphism. For each i we may write $\psi(d_i) = a_i d_i$ with $a_i \in \mathbb{Z}/p^i$. Since $p^{i-j} d_i = d_j$ for $i > 0$ and $j < i$, we have $a_j d_j = \psi(d_j) = \psi(p^{i-j} d_i) = p^{i-j} a_i d_i = a_i d_j$, so that $a_i \equiv a_j \pmod{p^j}$).

Hence the sequence $(a_i)_i$ is a p-adic integer, and

$$\mathrm{End}(\mathbb{Z}(p^\infty)) \cong \mathbb{Z}_p$$

as rings. Now, the list $\{a_i d_i\}_{i\geq 0}$ is a p-inductive set if and only if $p \nmid a_i$ for all i (otherwise 0 is repeated), that is, if and only if $a_1 \neq 0$.

Definition 2.3.2. Let $a \in \mathbb{Z}_p$. The *p-adic valuation* of a, denoted $\upsilon_p(a)$, is the largest i such that $p^i|a$.

Using the notation above, $\upsilon_p(a)$ is the smallest integer i such that $\alpha_i \neq 0$. It is easy to see that the units in $\mathbb{Z}_p$ are the p-adic integers a with $\upsilon_p(a) = 0$. Thus, $\psi \in \mathrm{Aut}(\mathbb{Z}(p^\infty))$ if and only if $(a_i)_i \in \mathbb{Z}_p^\times$, i.e.,

$$\mathrm{Aut}\,\mathbb{Z}(p^\infty) \cong (\mathbb{Z}_p)^\times.$$

2.4 D-extensions with finite nilpotent quotients

Throughout this section let N be a finite nilpotent group and let D be a divisible abelian group. For each prime p let D_p be the primary p-component of D. We wish to consider d-extensions of D by N. Since N is finite, so is $M(N)$, and moreover only primes dividing $|N|$ can divide $|M(N)|$ (cf. Proposition 1.6.5 and Lemma 1.6.6). Thus, we may assume that $r_p(D) = 0$ for $p = 0$ and all primes $p \nmid |N|$, since other D_p split off (cf. Proposition 2.2.7). By the Primary Decomposition Theorem for abelian groups we have

$$D \cong \bigoplus_{p||N|} D_p.$$

Next, for $p \mid |N|$, let N_p be the (unique) Sylow p-subgroup of N. Proposition 1.6.7 states that we have a decomposition

$$M(N) \cong \bigoplus_{p||N|} M(N_p). \tag{2.1}$$

Thus,

$$\mathrm{Hom}(M(N), D) \cong \bigoplus_{p,q||N|} \mathrm{Hom}(M(N_p), D_q) \cong \bigoplus_{p||N|} \mathrm{Hom}(M(N_p), D_p).$$

Therefore, with Theorem 2.1.2, we have

$$\mathscr{E}_0(N, D) \cong \bigoplus_{p||N|} \mathscr{E}_0(N_p, D_p).$$

To understand this decomposition of extensions, suppose $\mathbf{e}$ is the d-extension

$$\mathbf{e}: \qquad D \rightarrowtail G \twoheadrightarrow N$$

with corresponding map $\xi \in \mathrm{Hom}(M(N), D)$. For the decomposition of D, let $\iota_p \colon D_p \to D$ be the natural embedding for each prime $p \mid |N|$. There is a collection of maps $\{\theta_p \colon M(N) \to D_p \colon p \mid |N|\}$ such that

$$\xi = \sum_{p||N|} \iota_p \theta_p.$$

Now for each p, $\mathrm{Ker}\,\theta_p$ contains the complement of $M(N_p)$ in $M(N)$, so that each θ_p factors through the natural projection $\omega_p \colon M(N) \twoheadrightarrow M(N_p)$. Of course, $\omega_p = H_2(\nu_p)$, where $\nu_p \colon N \twoheadrightarrow N_p$ is the natural projection. Thus, for each p we have

$$\theta_p = \xi_p H_2(\nu_p),$$

where $\xi_p \in \mathrm{Hom}(M(N_p), D_p)$. This yields

$$\xi = \sum_{p||N|} \iota_p \xi_p H_2(\nu_p).$$

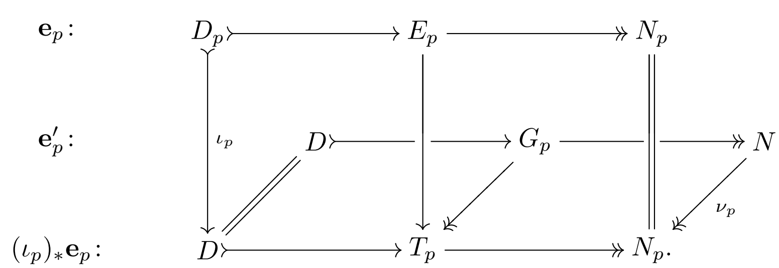
Corresponding to each map ξ_p there is an extension

$$\mathbf{e}_p\colon \qquad D_p \overset{\varepsilon_p}{\rightarrowtail} E_p \overset{\pi_p}{\twoheadrightarrow} N_p.$$

If the extension

$$\mathbf{e}'_p\colon \qquad D \rightarrowtail G_p \twoheadrightarrow N$$

corresponds to the map $\iota_p \xi_p H_2(\nu)$, then by Propositions 1.7.6 we see that $\mathbf{e}'_p$ is given by the induced maps of extensions



We remark that if $\hat{D}_p$ is the complement to D_p in D, then by Proposition 2.2.7 we have $T_p \cong E_p \times \hat{D}_p$.

The extension $\mathbf{e}$ is necessarily equivalent to the Baer sum $\bigoplus_{p||N|} (\iota_p)_* \mathbf{e}_p$. The following proposition provides a simpler expression for $\mathbf{e}$.

Proposition 2.4.1. *Let* $\mathbf{e}$ *be a central extension of* D *by the finite nilpotent group* N*. Assume that* $\mathrm{r}_p(D) \le \mathrm{r}_p(M(N))$ *for all* p*. Then, with the notation above,* $\mathbf{e}$ *is equivalent to the extension*

$$\mathbf{s}\colon \qquad \left(\bigoplus\nolimits_p D_p\right) \overset{\oplus_p \varepsilon_p}{\rightarrowtail} \times_p E_p \overset{\times_p \pi_p}{\twoheadrightarrow} \times_p N_p.$$

Proof. Let $p_1, \dots, p_k$ be the primes dividing $|N|$ and $\xi_i = \xi_{p_i}$. Write ε_i for ε_{p_i} and let τ_i be a transversal for $\mathbf{e}_{p_i}$. Then a transversal for $\mathbf{s}$ is given by $\times_i \tau_i$. Thus,

$$\begin{aligned}
\xi_{\mathbf{s}}\big((x_1, \dots x_k) \wedge (y_1, \dots y_k)\big) &= (\oplus_p \varepsilon_p)^{-1}\Big(\big[(\tau_1 x_1, \dots \tau_k x_k), (\tau_1 y_1, \dots \tau_k y_k)\big]\Big) \\
&= (\oplus_p \varepsilon_p)^{-1}\Big(\big([\tau_1 x_1, \tau_1 y_1], \dots, [\tau_k x_k, \tau_k y_k]\big)\Big) \\
&= \Big(\varepsilon_1^{-1}\big([\tau_1 x_1, \tau_1 y_1]\big), \dots, \varepsilon_k^{-1}\big([\tau_k x_k, \tau_k y_k]\big)\Big) \\
&= (\xi_1(x_1 \wedge y_1), \dots, \xi_k(x_k \wedge y_k)) \\
&= \xi_{\mathbf{e}}\big((x_1, \dots x_k) \wedge (y_1, \dots y_k)\big).
\end{aligned}$$

Thus $\mathbf{s} \equiv \mathbf{e}$. □

What interests us is that this decomposition respects isomorphism classes, as the following proposition states.

Proposition 2.4.2. *Using the notation and assumptions from Proposition 2.4.1, for each prime $p \mid |N|$ let*

$$\widetilde{\mathbf{e}}_p\colon \qquad D_p \overset{\widetilde{\varepsilon}_p}{\rightarrowtail} \widetilde{E}_p \overset{\widetilde{\pi}_p}{\twoheadrightarrow} N_p$$

be any central extension and let $\widetilde{\mathbf{e}}$ be the extension

$$\widetilde{\mathbf{e}}\colon \qquad \Big(\bigoplus_p D_p\Big) \overset{\oplus_p \widetilde{\varepsilon}_p}{\rightarrowtail} \times_p \widetilde{E}_p \overset{\times_p \widetilde{\pi}_p}{\twoheadrightarrow} \times_p N_p.$$

Then $\mathbf{e} \cong \widetilde{\mathbf{e}}$ if and only if $\mathbf{e}_p \cong \widetilde{\mathbf{e}}_p$ for each prime $p \mid |N|$.

Proof. Sufficiency is obvious. For necessity, let

$$\begin{array}{llcccccc}
\mathbf{e}\colon & D & \overset{\varepsilon}{\rightarrowtail} & E & \overset{\pi}{\twoheadrightarrow} & N \\
& \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow{\scriptstyle \beta} \\
\widetilde{\mathbf{e}}\colon & D & \overset{\widetilde{\varepsilon}}{\rightarrowtail} & \widetilde{E} & \overset{\widetilde{\pi}}{\twoheadrightarrow} & N
\end{array}$$

be an isomorphism. Let $\widetilde{\xi} \in \operatorname{Hom}(M(N), D)$ be the map corresponding to $\widetilde{\mathbf{e}}$. Then $\widetilde{\xi} = \alpha \circ \xi_{\mathbf{e}} \circ H_2(\beta^{-1})$. Now, the action of $\operatorname{Aut} D$ on D restricts to actions on the primary components. Thus, we may decompose α as a sum

$$\sum_p \iota_p \alpha_p \mu_p,$$

where $\alpha_p \in \operatorname{Aut}(D_p)$ and $\mu_p\colon D \twoheadrightarrow D_p$ is the natural projection. Similarly,

$$\beta^{-1} = \sum_p \delta_p \beta_p^{-1} \nu_p$$

with $\beta_p \in \operatorname{Aut}(N_p)$, $\delta_p\colon N_p \rightarrowtail N$ the inclusion map and $\nu_p\colon N \twoheadrightarrow N_p$ the natural projection. Moreover this sum is respected by H_2 so that

$$H_2(\beta^{-1}) = \sum_p H_2(\delta_p \beta_p^{-1} \nu_p) = \sum_p H_2(\delta_p) H_2(\beta_p^{-1}) H_2(\nu_p).$$

Thus,

$$\widetilde{\xi} = \sum_{\text{primes } p,q,r} \Big(\iota_p \alpha_p \mu_p\Big)\Big(\iota_q \xi_q H_2(\nu_q)\Big)\Big(H_2(\delta_r) H_2(\beta_r^{-1}) H_2(\nu_r)\Big),$$

where $\xi_p \in \operatorname{Hom}(M(N_p), D_p)$ is as defined on page 36. Now, $\mu_p \iota_q$ is nonzero only if $p = q$, in which case it is the identity. Likewise for $\nu_q \delta_r$ and hence (by functoriality) for $H_2(\nu_q) H_2(\delta_r)$. Hence,

$$\xi' = \sum_p \iota_p \alpha_p \xi_p H_2(\beta_p^{-1}) H_2(\nu_p).$$

Also,

$$\xi' = \sum_p \iota_p \xi'_p H_2(\nu_p),$$

where $\xi'_p \in \operatorname{Hom}(H_2(N_p), D_p)$ corresponds to the extension $\widetilde{\mathbf{e}}_p$. Since these sums are both direct decompositions, we conclude that the summands are equal:

$$\iota_p \alpha_p \xi_p H_2(\beta_p^{-1}) H_2(\nu_p) = \iota_p \xi'_p H_2(\nu_p)$$

for each prime p.

Finally, composing with μ_p on the left and $H_2(\delta_p)$ on the right, we obtain

$$\alpha_p \xi_p H_2(\beta_p^{-1}) = \xi'_p$$

for each prime p. We conclude that $\mathbf{e}_p \cong \widetilde{\mathbf{e}}_p$ for each prime p. □

For d-extensions by a finite nilpotent group N, we have reduced our considerations to the case where N is in fact a p-group (p a prime) and the kernel is

$$D \cong \bigoplus_{i=1}^{s} \mathbb{Z}(p^\infty).$$

We remark that under these circumstances

$$\operatorname{Hom}(M(N), D) \cong \bigoplus_{i=1}^{s} \operatorname{Hom}(M(N), \mathbb{Z}(p^\infty)),$$

so

$$\mathscr{E}(N, D) \cong \bigoplus_{i=1}^{r} \mathscr{E}(N, \mathbb{Z}(p^\infty)).$$

This decomposition does not, however, respect isomorphism classes, since an automorphism of D does not necessarily restrict to the components. In other words, there are isomorphisms unaccounted for. Consequently, while any classification of extensions of $\mathbb{Z}(p^\infty)$ by N would account for all extensions of D by N, the isomorphism problem would remain, i.e., some of the resulting extensions can be isomorphic.

In the next chapter we give such a classification when the class of N is 1, that is, when N is abelian.

3 D-extensions with abelian quotient

In this chapter we investigate the structure of d-extensions with finite abelian quotient Q, which we assume to be a p-group (p a prime). In this case $M(Q) = Q \wedge Q$, so for divisible abelian groups D we have

$$\mathscr{E}_0(Q, D) \cong \mathrm{Hom}(Q \wedge Q, D).$$

We can be explicit about the structure of d-extensions by Q. We may take D to be a p-group of finite rank. The homomorphisms of $M(Q) \to D$ can be viewed as matrices with certain properties. We shall use these matrices to give a parametrization of equivalence classes of d-extensions, which we follow with a solution to the isomorphism problem when Q is homocyclic.

When $D \cong \mathbb{Z}(p^\infty)$ we can understand d-extensions in terms of so-called "pairing maps," which allow us to give a relatively nice presentation of the extension. If Q is homocyclic we are able to give presentations which parametrize the isomorphism classes of d-extensions.

Definition 3.0.3. We shall refer to any d-extension with finite abelian quotient as a *d-ab extension*.

3.1 D-matrices and d-extensions

Assume Q is a finite abelian p-group of exponent p^e and rank r. Let $\mathscr{B} = \{b_1, \ldots, b_r\}$ be a basis for Q. Then Q is the internal direct sum $\langle b_1 \rangle \oplus \cdots \oplus \langle b_r \rangle$. We can obviously write $Q = Q_1 \oplus \cdots \oplus Q_n$ with $Q_i \cong (\mathbb{Z}/p^{e_i})^{r_i}$, $r_i > 0$, such that $e_1 < e_2 < \cdots < e_n$. The groups Q_i are called the *homocyclic components* of Q. If $n = 1$, then Q is said to be *homocyclic*.

The homocyclic components of Q are clearly unique up to isomorphism, though they are not unique as subgroups of Q. Indeed, the basis $\mathscr{B}$ determines the subgroups Q_i. It will be convenient to partition $\mathscr{B}$ into subsets

$$\mathscr{B}[p^{e_i}] = \{x \in \mathscr{B} \colon |x| = p^{e_i}\}.$$

The set $\mathscr{B}[p^{e_i}]$ is a basis for the homocyclic component Q_i. Write $\mathscr{B}[p^{e_j}] = \{b_1^{(j)}, \ldots, b_{r_j}^{(j)}\}$. It will be convenient to let $R_i = r_1 + r_2 + \cdots + r_i$ for $1 \leq i \leq n$ and $R_0 = 0$. Assume for definiteness that $e_1 < e_2 < \cdots < e_n = e$.

We define the p-adic valuation for elements of the group $\mathbb{Z}/p^e$ in a similar way to the discussion in Section 2.3: for $a \in \mathbb{Z}/p^e$ we define $\upsilon_p(a)$ to be the largest integer i such that $i \le e$ and $p^i | a$, that is, such that there exists $b \in \mathbb{Z}/p^e$ such that $p^i b = a$.

Definition 3.1.1. Suppose $A = [\alpha_{i,j}]$ is an $r \times r$ matrix over $\mathbb{Z}/p^e$. Then we refer to A as a *d-matrix for* Q if A satisfies

1. $-A = A^{\mathrm{T}}$ (skew-symmetry), and
2. $\upsilon_p(\alpha_{i,j}) \ge e - \min(e_k, e_\ell)$, whenever $R_{k-1} < i \le R_k$ and $R_{\ell-1} < j \le R_\ell$.

Let $\mathscr{A} = (A_1, \ldots, A_m)$ be an ordered set of d-matrices for Q, with $A_k = [\alpha_{ij}^{(k)}]$. Let $D = D_1 \oplus \cdots \oplus D_m$ with each $D_i \cong \mathbb{Z}(p^\infty)$. We define a group $G(\mathscr{A})$ and a d-extension

$$\mathbf{e}(\mathscr{A})\colon \qquad D \rightarrowtail G(\mathscr{A}) \twoheadrightarrow Q$$

as follows. To increase readability, we write $G(\mathscr{A})$ additively. Let $\mathscr{D}_i$ be a p-inductive set in D_i ($i = 1, \ldots m$). Thus $\mathscr{D} = \mathscr{D}_1 \cup \cdots \cup \mathscr{D}_m$ generates D. For each j write $\mathscr{D}_j = \{d_i^{(j)} : i \ge 0\}$ so that $pd_i^{(j)} = d_{i-1}^{(j)}$ and $d_0^{(j)} = 0$. We point out that $\{d_k^{(1)}, \ldots, d_k^{(m)}\}$ is a basis for $D[p^k]$, the subgroup of elements of D with order dividing p^k.

Our construction will depend on the sets $\mathscr{D}$ and $\mathscr{B}$, which we regard as fixed. Let $G(\mathscr{A})$ be defined by generators $\mathscr{S} = \mathscr{C} \cup \mathscr{X}$, where

$$\mathscr{C} = \{c_i^{(j)} : i \in \mathbb{Z}_{>0}; j = 1, \ldots m\}$$
$$\mathscr{X} = \{x_i^{(j)} : i = 1, \ldots r_j; j = 1, \ldots n\},$$

and relations

$$pc_1^{(j)} = 0 \text{ for } j = 1, \ldots m; \tag{3.1}$$
$$pc_i^{(j)} = c_{i-1}^{(j)} \text{ for } i > 1 \text{ and } j = 1, \ldots m; \tag{3.2}$$
$$p^{e_j} x_i^{(j)} = 0 \text{ for } i = 1, \ldots, r_j; \tag{3.3}$$
$$[x_i^{(j)}, x_k^{(\ell)}] = \sum_t \alpha_{g,h}^{(t)} \cdot c_e^{(t)}, \text{ where } g = R_{j-1} + i \text{ and } h = R_{\ell-1} + k; \tag{3.4}$$
$$[c_i^{(j)}, y] = 0 \text{ for all } y \in \mathscr{S}. \tag{3.5}$$

Now put $G = G(\mathscr{A})$ and let $C = \langle c_i^{(j)} : i \in \mathbb{Z}_{>0} \ \&\ 1 \le j \le m \rangle$. Obviously $C \le Z(G)$, and we have $G/C \cong Q$ by the isomorphism $\bar{\pi}\colon x_i^{(j)} \mapsto b_i^{(j)}$. Thus, $\pi\colon G \twoheadrightarrow Q$, which is defined by composing the natural map $G \twoheadrightarrow G/C$ with $\bar{\pi}$, has kernel C.

Next, the function $\mathscr{D} \to C$ defined by $d_i^{(j)} \mapsto c_i^{(j)}$ extends to a surjective homomorphism $D \twoheadrightarrow C$ by Von Dyck's Theorem, and by composing with the inclusion map $C \hookrightarrow G$ we obtain the homomorphism $\iota\colon D \to G$. Now, it is obvious that the $\mathrm{Ker}(\iota)$ is finite, so that we obtain an embedding of $D \rightarrowtail G$

(noting $D \cong D/\operatorname{Ker}(\iota)$); the question as to whether this embedding is the obvious one is answered by the following lemma.

Lemma 3.1.2. *The homomorphism $\iota\colon D \to G$ is injective.*

Proof. Let $j, \ell \in \{1\dots,r\}$, $i \in \{1,\dots,r_j\}$ and $k \in \{1,\dots,r_\ell\}$. Put $g = R_{j-1}+i$ and $h = R_{\ell-1}+k$ so that $R_{j-1} < g \le R_j$ and $R_{\ell-1} < h \le R_\ell$, and $\upsilon_p(\alpha_{g,h}) \ge e - \min(e_j, e_\ell)$. Thus, $e - \upsilon_p(\alpha^{(t)}_{gh}) \le \min(e_j, e_\ell)$, and

$$\begin{aligned}\big|\alpha^{(1)}_{g,h}\cdot c^{(1)}_e + \cdots + \alpha^{(m)}_{g,h}\cdot c^{(m)}_e\big| &\le \max_t\{|\alpha^{(t)}_{g,h}\cdot c^{(t)}_e|\} = \max_t\{p^{e-\upsilon_p(\alpha^{(t)}_{gh})}\}\\ &\le \min\{p^{e_j}, p^{e_\ell}\} = \big|b^{(j)}_i \wedge b^{(\ell)}_k\big|.\end{aligned}$$

It follows that there is a (unique) well-defined homomorphism $\theta\colon Q\wedge Q \to D$ such that

$$\theta\big(b^{(j)}_i \wedge b^{(\ell)}_k\big) = \alpha^{(1)}_{g,h}\cdot c^{(1)}_e + \cdots + \alpha^{(m)}_{g,h}\cdot c^{(m)}_e$$

for each such i, j, k, l. Let $\mathbf{e}$ be an extension (unique up to equivalence)

$$\mathbf{e}\colon \qquad D \overset{\varepsilon}{\rightarrowtail} G \overset{\rho}{\twoheadrightarrow} Q$$

such that $\theta = \xi_{\mathbf{e}}$ (note that we are not assuming $G = G(\mathscr{A})$). We recall that if τ is any transversal for $\mathbf{e}$, then

$$\xi_{\mathbf{e}}\big(b^{(j)}_i \wedge b^{(\ell)}_k\big) = \varepsilon^{-1}\big([\tau(b^{(j)}_i), \tau(b^{(\ell)}_k)]\big).$$

Now G is generated by the set

$$\{\tau(b^{(j)}_i)\colon j = 1,\dots r; i = 1,\dots,r_j\} \cup \{\varepsilon(d^{(v)}_u)\colon u \in \mathbb{Z}_{>0}; v = 1,\dots m\},$$

which is clearly in bijective correspondence with $\mathscr{X}\cup\mathscr{C}$, with $x^{(j)}_i \leftrightarrow \tau(b^{(j)}_i)$ and $c^{(v)}_u \leftrightarrow \varepsilon(d^{(v)}_u)$. Hence we may take $\mathscr{X}\cup\mathscr{C}$ as generating set for G. For simplicity, we identify the elements of $\mathscr{X}\cup\mathscr{C}$ with their corresponding images in G.

Now, the relations 3.1, 3.2, 3.4, and 3.5 defining $G(\mathscr{A})$ all clearly hold in G, with 3.4 coming from $\theta = \xi_{\mathbf{e}}$. Regarding the relation 3.3, we must certainly have $p^{e_j}\tau(b^{(j)}_i) \in D$. For each pair i and j put $p^{e_j}\tau(b^{(j)}_i) = \delta_{i,j}$. Since D is divisible, there exists $\widetilde{\delta}_{i,j} \in D$ such that $p^{e_j}\widetilde{\delta}_{i,j} = \delta^{-1}_{i,j}$, and we put $\widetilde{x^{(j)}_i} = \tau(b^{(j)}_i)\widetilde{\delta}_{i,j}$ for each i and j. Then $p^{e_j}\widetilde{x^{(j)}_i} = 0$ for each i and j. In fact, $b^{(j)}_i \mapsto \widetilde{x^{(j)}_i}$ is simply another choice of transversal for $\mathbf{e}$, and we may replace our original choice for τ with this one (recall $\xi_{\mathbf{e}}$ is independent of τ). This gives us the relation 3.3 as well.

Now, since $\mathscr{C}\cup\mathscr{X}$ generates both $G(\mathscr{A})$ and G and all the defining relations for $G(\mathscr{A})$ hold in G, we may apply Von Dyck's Theorem to obtain a surjective homomorphism $\varphi\colon G(\mathscr{A}) \twoheadrightarrow G$ extending the identity function on $\mathscr{C}\cup\mathscr{X}$. Evidently $\varphi\iota = \varepsilon$, and since ε is injective, we see that ι is also injective. $\square$

We therefore have the extension

$$\mathbf{e}(\mathscr{A})\colon \qquad D \overset{\iota}{\rightarrowtail} G(\mathscr{A}) \overset{\pi}{\twoheadrightarrow} Q.$$

We remark that in the notation of the proof of Lemma 3.1.2, it easily follows that φ is in fact an isomorphism, and $(1, \varphi, 1)$ is an equivalence of extensions. An obvious choice for transversal here is obtained by defining $\tau(b_i^{(j)}) = x_i^{(j)}$ and extending linearly. We note that τ is a homomorphism if and only if all the elements of $\mathscr{X}$ commute, which holds if and only if $\mathscr{A} = (0, \dots, 0)$.

Write $\xi_{\mathscr{A}}$ for $\xi_{\mathbf{e}(\mathscr{A})}$. Thus,

$$\begin{aligned}\xi_{\mathscr{A}}(b_i^{(j)}, b_k^{(\ell)}) &= \iota^{-1}\big([\tau b_i^{(j)}, \tau b_k^{(\ell)}]\big) = \iota^{-1}\big([x_i^{(j)}, x_k^{(\ell)}]\big) \\ &= \alpha_{g,h}^{(1)} \cdot d_e^{(1)} + \dots + \alpha_{g,h}^{(m)} \cdot d_e^{(m)},\end{aligned}$$

where $g = R_{j-1} + i$ and $h = R_{\ell-1} + k$.

The following two theorems provide us with a classification of d-extensions by finite abelian groups (up to equivalence) in terms of d-matrices.

Theorem 3.1.3. *If $\mathbf{e}$ is any d-extension of D by Q, then $\mathbf{e} \equiv \mathbf{e}(\mathscr{A})$ for some ordered m-set $\mathscr{A}$ of d-matrices for Q.*

Proof. Let $j, \ell \in \{1 \dots, r\}$, $i \in \{1, \dots, r_j\}$ and $k \in \{1, \dots, r_\ell\}$ be arbitrary. Put $g = R_{j-1} + i$ and $h = R_{\ell-1} + k$ so that $R_{j-1} < g \le R_j$ and $R_{\ell-1} < h \le R_\ell$. Since $|b_i^{(j)} \wedge b_k^{(\ell)}|$ divides both $|b_i^{(j)}|$ and $|b_k^{(\ell)}|$, we must have $\xi_{\mathbf{e}}(b_i^{(j)} \wedge b_k^{(\ell)}) \in D[p^e]$. For $t \in \{1, \dots, m\}$ define $\alpha_{g,h}^{(t)}$ by the relation

$$\xi_{\mathbf{e}}(b_i^{(j)} \wedge b_k^{(\ell)}) = \alpha_{g,h}^{(1)} \cdot d_e^{(1)} + \dots + \alpha_{g,h}^{(m)} \cdot d_e^{(m)}$$

We thus obtain matrices $A^{(t)} = \big(\alpha_{g,h}^{(t)}\big)$ for $t = 1, \dots, m$. We note that by order considerations we have

$$\upsilon_p(\alpha_{g,h}) \ge e - \min(e_j, e_\ell).$$

Put $\mathscr{A} = (A^{(1)}, \dots, A^{(m)})$. Evidently $\xi_{\mathbf{e}} = \xi_{\mathscr{A}}$, so $\mathbf{e} \equiv \mathbf{e}(\mathscr{A})$. □

Theorem 3.1.4. *If $\mathscr{A}$ and $\mathscr{Z}$ are two ordered m-sets of d-matrices for Q, then $\mathbf{e}(\mathscr{A}) \equiv \mathbf{e}(\mathscr{Z})$ if and only if $\mathscr{A} = \mathscr{Z}$. Hence, there is a bijective correspondence between ordered m-sets of d-matrices for Q and extensions of D by Q (up to equivalence).*

Proof. Sufficiency is obvious. For necessity, assume $\mathbf{e}(\mathscr{A}) \equiv \mathbf{e}(\mathscr{Z})$. Then $\xi_{\mathscr{A}} = \xi_{\mathscr{Z}}$. Write $[\zeta_{i,j}^{(t)}]$ for the t-th matrix in $\mathscr{Z}$. Consider arbitrary $i \in \mathbb{Z}_{>0}$ and $j \in \{1, \dots, m\}$ and set $g = R_{j-1} + i$ and $h = R_{\ell-1} + k$. Then

$$\xi_{\mathscr{A}}(b_i^{(j)}, b_k^{(\ell)}) = \alpha_{g,h}^{(1)} \cdot d_e^{(1)} + \dots + \alpha_{g,h}^{(m)} \cdot d_e^{(m)}$$

and

$$\xi_{\mathscr{Z}}(b_i^{(j)}, b_k^{(\ell)}) = \zeta_{g,h}^{(1)} \cdot d_e^{(1)} + \cdots + \zeta_{g,h}^{(m)} \cdot d_e^{(m)}$$

Since $\{c_e^{(1)}, \ldots, c_e^{(m)}\}$ is a basis for $D[p^e]$, we must have $\alpha_{g,h}^{(t)} = \zeta_{g,h}^{(t)}$ for $t = 1, \ldots, m$, that is, $\mathscr{A} = \mathscr{Z}$. □

Next we obtain the criterion for two ordered m-sets of d-matrices for Q to yield isomorphic extensions. We shall need the following notions for our next result.

Definition 3.1.5. Let $M_1, \ldots, M_s$ and N be $t \times t$ matrices. The *partitioned matrix* $\left[M_1|\ldots|M_s\right]$ is the $t\times ts$ matrix whose $\{i,j\}$th entry is the $\{i,k\}$th entry of M_ℓ, where $j = (\ell-1)t + k$. The product $\left[M_1|\ldots|M_s\right] * N$ is the partitioned matrix $\left[M_1 N|\ldots|M_s N\right]$.

Definition 3.1.6. Let $A = (\alpha_{i,j})$ and $B = (\beta_{i,j})$ be square matrices of dimensions $m \times m$ and $n \times n$ respectively; then the *tensor product* $A \otimes B$ of A and B is defined to be the $mn \times mn$ matrix

$$\begin{bmatrix} \alpha_{11}\beta_{11} & \cdots & \alpha_{11}\beta_{1n} & \cdots & \alpha_{1m}\beta_{11} & \cdots & \alpha_{1m}\beta_{1n} \\ & \vdots & & & & \vdots & \\ \alpha_{11}\beta_{n1} & \cdots & \alpha_{11}\beta_{nn} & \cdots & \alpha_{1m}\beta_{n1} & \cdots & \alpha_{1m}\beta_{nn} \\ & \vdots & & & & \vdots & \\ \alpha_{m1}\beta_{11} & \cdots & \alpha_{m1}\beta_{1n} & \cdots & \alpha_{mm}\beta_{11} & \cdots & \alpha_{mm}\beta_{1n} \\ & \vdots & & & & \vdots & \\ \alpha_{m1}\beta_{n1} & \cdots & \alpha_{m1}\beta_{nn} & \cdots & \alpha_{mm}\beta_{n1} & \cdots & \alpha_{mm}\beta_{nn} \end{bmatrix}$$

Although difficult to use in practice, the following theorem solves the isomorphism problem for the classification of d-extensions given above in the case when Q is homocyclic.

Theorem 3.1.7. *Assume that Q is homocyclic of exponent p^e and rank r. Let $\mathscr{A} = (A_1, \ldots, A_m)$ and $\mathscr{Z} = (Z_1, \ldots, Z_m)$ be ordered m-sets of d-matrices for Q. Then $G(\mathscr{A}) \cong G(\mathscr{Z})$ if and only if there exist $R \in \mathrm{GL}_m(\mathbb{Z}_p)$ and $S \in \mathrm{GL}_r(\mathbb{Z}/p^e)$ such that*

$$[Z_1|\cdots|Z_m] = S\big(\,[A_1|\cdots|A_m]\,(R \otimes 1_r)\,\big) * S^{\mathrm{t}},$$

where S^{t} denotes the transpose of S, and $[A_1|\cdots|A_m]\,(R \otimes 1_r)$ is the ordinary matrix product, resulting in an $(r \times mr)$ matrix, which is regarded as being partitioned into $r \times r$ matrices.

Proof. Since Q is homocyclic, $\mathscr{B} = \{b_1^{(1)}, \ldots, b_r^{(1)}\}$; we write b_i for $b_i^{(1)}$. We also write $d^{(i)}$ for $d_e^{(i)}$. Also we mention that $\mathrm{Aut}(Q) \cong \mathrm{GL}_r(\mathbb{Z}/p^e)$. To ease notation, we identify D with its images in $G(\mathscr{A})$ and $G(\mathscr{Z})$, and we identify $G(\mathscr{A})/D$ and $G(\mathscr{Z})/D$ with Q. To distinguish elements of $G(\mathscr{A})$ and $G(\mathscr{Z})$, we regard $G(\mathscr{A})$ as defined by the presentation above with generating set $\mathscr{S}$ and $G(\mathscr{Z})$ as given by a similar presentation with generating set $\mathscr{S}' = \{s' \colon s \in \mathscr{S}\}$.

Assume that $\varphi\colon G(\mathscr{A}) \xrightarrow{\sim} G(\mathscr{Z})$ is an isomorphism. Then $\varphi|_{D[p^e]}$ is an automorphism of $D[p^e]$ since $D[p^e]$ is a characteristic subgroup. This is describable by an invertible matrix

$$R = \Big[\rho_{ij}\Big]$$

with entries in $\mathbb{Z}/p^e$, so that

$$\varphi(d^{(i)}) = \rho_{i1}d^{(1)} + \cdots + \rho_{im}d^{(m)}.$$

For $v \in D[p^e]$ there is a unique coefficient vector $[v] = (v_1, \ldots, v_m)$ defined by

$$v = v_1 d^{(1)} + \cdots + v_m d^{(m)}.$$

Then, in terms of coefficients, we have $[\varphi(v)] = [v]R$ for $v \in D[p^e]$.

We also have the isomorphism $\bar{\varphi}\colon G(\mathscr{A})/D \to G(\mathscr{Z})/D$ determined by $\bar{\varphi}(gD) = \varphi(g)D$ for $g \in G(\mathscr{A})$. Thus, $\bar{\varphi} \in \operatorname{Aut} Q$, which is describable by an invertible matrix

$$T = \Big[\tau_{ij}\Big]$$

with entries in $\mathbb{Z}/p^e$, so that, in terms of coefficients, $[\bar{\varphi}(w)] = [w]T$ for $w \in Q$.

For fixed i and $j \in \{1, \ldots, r\}$ we have

$$\begin{aligned}
\varphi\big([x_i, x_j]\big) &= \big[\varphi(x_i), \varphi(x_j)\big] \\
&= [\tau_{i1}x'_1 + \cdots + \tau_{ir}x'_r, \tau_{j1}x'_1 + \cdots + \tau_{jr}x'_r] \\
&= \sum_{\substack{1 \le k \le r \\ 1 \le \ell \le r}} \tau_{ik}\tau_{jl}[x'_k, x'_\ell] \\
&= \sum_{\substack{1 \le k \le r \\ 1 \le \ell \le r}} \tau_{ik}\tau_{jl}\Big(\sum_{t=1}^{m} \zeta_{kl}^{(t)} d^{(t)}\Big).
\end{aligned}$$

On the other hand, $[x_i, x_j] = \alpha_{ij}^{(1)} d^{(1)} + \cdots + \alpha_{ij}^{(m)} d^{(m)}$, so, in terms of coefficient vectors, we have

$$\begin{aligned}
\Big[\varphi\big([x_i, x_j]\big)\Big] &= \Big[[x_i, x_j]\Big]R \\
&= \Big[\alpha_{ij}^{(1)} \cdots \alpha_{ij}^{(m)}\Big]R \\
&= \Big[\sum_{1 \le k \le m} \alpha_{ij}^{(k)}\rho_{k1} \cdots \sum_{1 \le k \le m} \alpha_{ij}^{(k)}\rho_{km}\Big]
\end{aligned}$$

Thus, for $1 \le t \le m$, we have

$$\sum_{1 \le k \le m} \alpha_{ij}^{(k)}\rho_{kt} = \sum_{\substack{1 \le k \le r \\ 1 \le \ell \le r}} \tau_{ik}\tau_{jl}\zeta_{kl}^{(t)}. \tag{3.6}$$

The right hand side of (3.6) is precisely the ij-th entry of the product

$TZ^{(t)}T^{\mathrm{t}}$, that is, the $\{i,(t-1)r+j\}$th entry of the product $T[Z_1|\cdots|Z_m]*T^{\mathrm{t}}$. By inspection we see also that the left hand side is the $\{i,(t-1)r+j\}$th entry of the product $[A_1|\cdots|A_m]\,(R\otimes 1_r)$. Hence

$$T[Z_1|\cdots|Z_m]*T^{\mathrm{t}} = [A_1|\cdots|A_m]\,(R\otimes 1_r)\,.$$

Finally, writing $S=T^{-1}$, we complete the proof of necessity.

For sufficiency, assume the matrix relation holds and put $T=S^{-1}$. Then we obtain again (3.6). Define $\Phi\colon D\to D$ by

$$\Phi\colon d_i^{(j)}\mapsto \rho_{j1}d_i^{(1)}+\cdots+\rho_{jm}d_i^{(m)}$$

and $\Omega\colon Q\to Q$ by

$$\Omega\colon b_i\mapsto \tau_{i1}b_1+\cdots+\tau_{ir}b_r.$$

Then

$$\begin{aligned}\Phi\circ\xi_{\mathscr{A}}(b_i\wedge b_j) &= \Phi(\alpha_{ij}^{(1)}d^{(1)}+\cdots+\alpha_{ij}^{(m)}d^{(m)})\\ &= \alpha_{ij}(\rho_{11}d^{(1)}+\cdots+\rho_{1m}d^{(m)})+\cdots+\alpha_{ij}^{(m)}(\rho_{m1}d^{(1)}+\cdots+\rho_{mm}d^{(m)})\\ &= \Big(\sum_{1\le k\le m}\alpha_{ij}^{(k)}\rho_{k1}\Big)d^{(1)}+\cdots+\Big(\sum_{1\le k\le m}\alpha_{ij}^{(k)}\rho_{km}\Big)d^{(m)}.\end{aligned}$$

Also,

$$\begin{aligned}\xi_{\mathscr{Z}}\circ\Omega^{\wedge 2}(b_i\wedge b_j) &= \xi_{\mathscr{Z}}(\tau_{i1}b_1+\cdots+\tau_{ir}b_r\wedge\tau_{j1}b_1+\cdots+\tau_{jr}b_r)\\ &= \sum_{\substack{1\le k\le r\\ 1\le \ell\le r}}\tau_{ik}\tau_{jl}\xi_{\mathscr{Z}}(b_k\wedge b_\ell)\\ &= \sum_{\substack{1\le k\le r\\ 1\le \ell\le r}}\tau_{ik}\tau_{jl}\Big(\sum_{t=1}^{m}\zeta_{kl}^{(t)}d^{(t)}\Big).\end{aligned}$$

Since (3.6) holds we have $\Phi\circ\xi_{\mathscr{A}}=\xi_{\mathscr{Z}}\circ\Omega^{\wedge 2}$, so that $\mathbf{e}(\mathscr{A})\cong\mathbf{e}(\mathscr{Z})$ by Proposition 2.1.8; thus, $G(\mathscr{A})\cong G(\mathscr{Z})$. □

3.2 Central extensions of prüfer groups

Our solution to the isomorphism problem in Theorem 3.1.7 could be difficult to apply in practice. In the case when D has rank 1, i.e., $D\cong\mathbb{Z}(p^\infty)$, the situation simplifies greatly. Even when Q is not homocyclic we can say a bit more than the theorems of the previous section.

To investigate the isomorphism problem of d-by-a extensions of $\mathbb{Z}(p^\infty)$ we find the following notion useful.

Definition 3.2.1. Let Q and D be two groups. Let θ be a homomorphism

$Q \wedge Q \to D$, $\mathscr{B}$ a subset of Q, and $\mathscr{D}$ a subset of D. We say that θ *pairs* $\mathscr{B}$ on $\mathscr{D}$ if we have the following two properties:

1. for each $x \in \mathscr{B}$ there is at most one $y \in \mathscr{B}$ such that $\theta(x \wedge y) \neq 0$;

2. for each pair $x, y \in \mathscr{B}$ we have $\theta(x \wedge y) = \pm d$ for some $d \in \mathscr{D}$.

First we observe that when Q is a finite abelian p-group, pairings exist between generating sets for Q and p-inductive sets in $\mathbb{Z}(p^\infty)$.

Lemma 3.2.2. *Let $\theta \in \operatorname{Hom}\big(Q \wedge Q, \mathbb{Z}(p^\infty)\big)$ with Q a finite abelian p-group, and let $\mathscr{D}$ be a p-inductive set in $\mathbb{Z}(p^\infty)$. Then there is a generating set $\mathscr{G}$ for Q such that $|\mathscr{G}| = \operatorname{rank}(Q)$ and such that θ pairs $\mathscr{G}$ on $\mathscr{D}$.*

Proof. The proof is by induction on $\operatorname{rank}(Q)$. If $\operatorname{rank}(Q) \leq 1$ then there is nothing to prove. Assume $\operatorname{rank}(Q) \geq 2$ and that our assertion holds for abelian groups of rank $< \operatorname{rank}(Q)$.

Let $\mathscr{B}'$ be a basis for Q and choose x_1' and $y_1 \in \mathscr{B}'$ such that $\theta(x_1' \wedge y_1)$ has maximum order, say $\theta(x_1' \wedge y_1) = d$ with $|d| = p^\alpha$. Write $\mathscr{D} = \{0, d_1, d_2, \ldots\}$, so that $|d_i| = p^i$ for each i. Then $d = \lambda d_\alpha$ for a unique $\lambda \in \mathbb{Z}$ with $1 \leq \lambda < p^\alpha$ and $p \nmid \lambda$. Thus, λ is invertible modulo p^α and there is (a unique) $\lambda' \in \mathbb{Z}$ such that $1 \leq \lambda' < p^\alpha$ and $\lambda\lambda' \equiv 1 \pmod{p^\alpha}$. Put $x_1 = {x_1'}^{\lambda'}$. Then $\theta(x_1 \wedge y_1) = d_\alpha$. We put $\mathscr{B} = (\mathscr{B}' - \{x_1'\}) \cup \{x_1\}$.

We remark also that by maximality of $|\theta(x_1' \wedge y_1)|$ we have $\operatorname{Im}(\theta) = \langle d_\alpha \rangle$. For $x \in \mathscr{B} - \{y_1\}$ define λ_x by $\theta(x_1 \wedge x) = \lambda_x d$ and put $\mathscr{B}_{x_1} = \big\{x - \lambda_x y_1 \colon x \in \mathscr{B} - \{y_1\}\big\}$. Then $\theta(x_1 \wedge x') = 0$ for $x' \in \mathscr{B}_{x_1}$. Next, for $x \in \mathscr{B}_{x_1} - \{x_1\}$ define μ_x by $\theta(x \wedge y) = \mu_x d$ and put $\mathscr{B}_1 = \big\{x - \mu_x x_1 \colon x \in \mathscr{B}_{x_1} - \{x_1\}\big\}$. Then for $x \in \mathscr{B}_1$ we have $\theta(x \wedge y_1) = 0 = \theta(x_1 \wedge x)$.

Note that $Q = \langle x_1, y_1, \mathscr{B}_1 \rangle$. Put $Q_1 = \langle \mathscr{B}_1 \rangle$. Then $\operatorname{rank}(Q_1) = \operatorname{rank}(Q) - 2 = |\mathscr{B}_1|$ since $Q = \langle x_1, y_1, Q_1 \rangle$ implies that $\operatorname{rank}(Q_1) \geq \operatorname{rank}(Q) - 2$ and the opposite inequality is obvious. Hence, our induction hypothesis applies, and we obtain a generating set $\mathscr{G}_1$ for Q_1 such that $|\mathscr{G}_1| = \operatorname{rank}(Q) - 2$ and $\theta|_{Q_1 \wedge Q_1}$ pairs $\mathscr{G}_1$ on $\mathscr{D}$. Put $\mathscr{G} = \mathscr{G}_1 \cup \{x_1, y_1\}$. Clearly $\mathscr{G}$ generates Q, and it is easy to see that θ pairs $\mathscr{G}$ on $\mathscr{D}$. ∎

The next result provides important structural information about d-ab extensions of $\mathbb{Z}(p^\infty)$.

Theorem 3.2.3. *Let* $\mathbf{e}$ *be the d-extension*

$$\mathbf{e}\colon \qquad \mathbb{Z}(p^\infty) \overset{\iota}{\rightarrowtail} G \twoheadrightarrow Q$$

with Q a finite abelian p-group. Put $r = \operatorname{rank}(G/Z(G))$. Then

1. *r is even;*

2. *there exists a subgroup C and normal 2-generator subgroups G_i for $i = 1, \ldots, r/2$ such that*

$$G = G_1 \cdot G_2 \cdots G_{r/2} \cdot C$$

with $D \leq C \leq Z(G)$, and $[G_i, G_j] = 1$ if $i \neq j$;

3. *there exist generators x_i, y_i of G_i for each i such that $[x_i, y_i] \neq 1$, so in particular x_i and y_i are independent mod $Z(G)$ (hence mod D);*

4. *$G/Z(G) = G_1Z(G)/Z(G) \times \cdots \times G_kZ(G)/Z(G)$.*

Proof. Let $\mathscr{D}$ be any p-inductive set in $\mathbb{Z}(p^\infty)$ and let $\mathscr{G}$ be a generating set for Q such that $\xi_{\mathbf{e}}$ pairs $\mathscr{G}$ on $\mathscr{D}$. Put $\xi = \xi_{\mathbf{e}}$ and write

$$\mathscr{G} = \{x_1, y_1, \ldots, x_k, y_k, z_1, \ldots, z_\ell\},$$

where $\xi(x_i \wedge y_i) \neq 0$ for each i, $\xi(x_i \wedge y_j) = 0 = \xi(x_i \wedge x_j) = \xi(y_i \wedge y_j)$ for $i \neq j$, and $\xi(z_j \wedge x) = 0$ for all $x \in Q$ and all j. Let τ be a transversal for $\mathbf{e}$ and put $G_i = \langle \tau x_i, \tau y_i \rangle$ for $i = 1, \ldots k$. We recall that $\xi(x \wedge y) = \iota^{-1}([\tau x, \tau y])$ for each $x, y \in Q$. It follows that $z_i \in Z(G)$ for $i = 1, \ldots \ell$ and that for $i \neq j$ we have $[G_i, G_j] = 1$. Put $C = \langle z_1, \ldots, z_\ell, \mathrm{Im}(\iota) \rangle$. Then $G = G_1 \cdot G_2 \cdots G_k \cdot C$, with C and the G_i having the properties required in 2.

We now prove 4, which will show that $\mathrm{rank}(G/Z(G)) = 2k$ and complete the proof. Clearly $G/Z(G) = \prod_{i=1}^k G_iZ(G)/Z(G)$. Also, since $i \neq j$ implies that $[G_i, G_j] = 1$, we have $G_i \cap G_j \leq Z(G)$ whenever $i \neq j$: for if $z \in G_i \cap G_j$ $(i \neq j)$, then necessarily $[z, G_\alpha] = 1$ for each α. The result now follows. □

Next we define a collection of d-a extensions of $\mathbb{Z}(p^\infty)$ which, we show, accounts for all such extensions up to isomorphism. Here we will write our groups multiplicatively.

Definition 3.2.4. Let F be a finite abelian p-group with presentation $F = \langle \mathscr{G} | \mathscr{R} \rangle$. Let $k \in \mathbb{N}$ and suppose

$$\mathscr{F} = (f_1, g_1, \ldots, f_k, g_k)$$

is an ordered set of elements in F. Let

$$\mathscr{X} = (e_1, \ldots, e_k)$$

be an ordered set of natural numbers. We define a group $G_{F,\mathscr{F},\mathscr{X}}$ with generators

$$\mathscr{Y} = \{x_1, y_1, \ldots, x_k, y_k\} \cup \mathscr{C} \cup \mathscr{G},$$

where $\mathscr{C} = \{c_i \colon i \in \mathbb{Z}_{<0}\}$, subject to the relations

$$\mathscr{R}; \tag{3.7}$$

$$[z, w] = 1 \text{ for all } z \in \mathscr{C} \cup \mathscr{G} \text{ and } w \in \mathscr{Y}; \tag{3.8}$$

$$c_1^p = 1; \tag{3.9}$$

$$c_i^p = c_{i-1} \text{ for } i > 1; \tag{3.10}$$

$$[x_i, y_i] = c_{e_i} \text{ for } i = 1, \ldots, k; \tag{3.11}$$

$$[x_i, y_j] = 1 \text{ whenever } i \neq j; \tag{3.12}$$

$$[x_i, x_j] = [y_i, y_j] = 1 \text{ for all } i, j; \tag{3.13}$$

$$x_i^{p^{e_i}} = f_i; \text{ and} \tag{3.14}$$

$$y_i^{p^{e_i}} = g_i. \tag{3.15}$$

We note that since $\mathscr{G}$ appears in the generating set and $\mathscr{R}$ in the relations, $G_{F,\mathscr{F},\mathscr{X}}$ is independent (up to group isomorphism) of the presentation for F. We remark that, in fact, we are abusing notation by using the f_i and g_i in the relations 3.14 and 3.15 above. More properly we should choose words in $F(\mathscr{G})$ which are preimages of the f_i and g_i under the presentation. We simply omit the choice function (i.e., transversal) from the notation as it does not affect $G_{F,\mathscr{F},\mathscr{X}}$. For definiteness, we take

$$\mathscr{G} = \{u_1, \ldots, u_\ell\} \tag{3.16}$$

and

$$\mathscr{R} = \{u_1^{p^{\alpha_1}}, \ldots, u_\ell^{p^{\alpha_\ell}}\} \cup \{[u_i, u_j] \colon 1 \le i < j \le \ell\}, \tag{3.17}$$

Assume F, $\mathscr{F}$, and $\mathscr{X}$ are fixed and put $G = G_{F,\mathscr{F},\mathscr{X}}$. Let $C = \langle \mathscr{C} \rangle$ and note that $G' \le C$, so the group $Q = G/C$ is abelian and clearly finite.

Lemma 3.2.5. *Put $D = \mathbb{Z}(p^\infty)$ and let $\mathscr{D} = \{0, d_1, d_2, \ldots\}$ be a p-inductive set in D. Then, with $F, \mathscr{F}$, and $\mathscr{X}$ as in Definition 3.2.4, the homomorphism $\iota \colon D \to G_{F,\mathscr{F},\mathscr{X}}$ given by $\lambda d_i \mapsto c_i^\lambda$ is injective.*

Proof. Let $Q = G/C$ as above. A presentation for Q can be obtained by adding the relations

$$c = 1 \text{ for all } c \in \mathscr{C}$$

to those defining G. Let $\overline{\mathscr{G}}$ be the set of symbols $\{\overline{g} \colon g \in \mathscr{G}\}$ and let $\overline{f}$ denote the word corresponding to f via the obvious bijection $\overline{\mathscr{G}} \cong \mathscr{G}$ given by $g \mapsto \overline{g}$ for $g \in \mathscr{G}$ (i.e., f and $\overline{f}$ represent the same element in F). Then Q can be given by generators

$$\overline{\mathscr{Y}} = \{\overline{x}_1, \overline{y}_1, \ldots, \overline{x}_k, \overline{y}_k\} \cup \overline{\mathscr{G}}$$

subject to the relations

$$\overline{u}_1^{p^{\alpha_1}} = \cdots = \overline{u}_\ell^{p^{\alpha_\ell}} = 1;$$
$$[\overline{y}, \overline{z}] = 1 \text{ for all } \overline{y}, \overline{z} \in \overline{\mathscr{Y}};$$
$$\overline{x}_i^{p^{e_i}} = \overline{f}_i; \text{ and}$$
$$\overline{y}_i^{p^{e_i}} = \overline{g}_i.$$

Since $|d_{e_i}| \leq p^{e_i} \leq \min\{|\overline{x}_i|, |\overline{y}_i|\} = |\overline{x}_i \wedge \overline{y}_i|$ for each $i, j \in \{1, \ldots, k\}$, we may define a homomorphism

$$\theta \colon Q \wedge Q \to D$$

by $\theta(\overline{x}_i \wedge \overline{y}_i) = d_{e_i}$ for $i = 1, \ldots k$ and $\theta(\overline{x} \wedge \overline{y}) = 0$ for all other pairs $\overline{x}, \overline{y} \in \overline{\mathscr{Y}}$. Let

$$\mathbf{e}\colon \qquad D \overset{\varepsilon}{\rightarrowtail} E \overset{\rho}{\twoheadrightarrow} Q$$

be an extension (unique up to equivalence) such that $\theta = \xi_{\mathbf{e}}$.

We recall that if τ is any transversal for $\mathbf{e}$, then

$$\xi_{\mathbf{e}}(\overline{y} \wedge \overline{z}) = \varepsilon^{-1}\big([\tau(\overline{y}), \tau(\overline{z})]\big).$$

Now E is generated by the set

$$\{\tau(\overline{y}) \colon \overline{y} \in \overline{\mathscr{Y}}\} \cup \{\varepsilon(d_u) \colon u \in \mathbb{Z}_{>0}\}$$

which is clearly in bijective correspondence with $\mathscr{Y}$, with $x_i \leftrightarrow \tau(\overline{x}_i)$, $y_i \leftrightarrow \tau(\overline{y}_i)$, $u_i \leftrightarrow \tau(\overline{u}_i)$, and $c_u \leftrightarrow \varepsilon(d_u)$. Hence we may take $\mathscr{Y}$ as generating set for E. For simplicity, we identify the elements of $\mathscr{Y}$ with their corresponding images in G.

The relations 3.8, 3.11, 3.12, and 3.13 then hold in G because of our definition of θ. Also, that the relations 3.9 and 3.10 hold in E follow from the relations in D. Moreover, the relations 3.7, 3.14, and 3.15 must hold modulo $\varepsilon(D)$ in E. Thus for each $i = 1, \ldots, \ell$ there is $\delta_i \in D$ such that $\big(\tau(\overline{u}_i)\big)^{p^{\alpha_i}} = \delta_i$. Since D is divisible there exists $\widetilde{\delta}_i \in D$ such that $p^{\alpha_i}\widetilde{\delta}_i = \delta_i^{-1}$, and we put $\widetilde{u}_i = \tau(\overline{u}_i)\widetilde{\delta}_i$ for each i. Then $p^{e_j}\widetilde{x}_i = 0$ for each i. Moreover, $\rho(\widetilde{u}_i) = \overline{u}_i$ for each i and, since the validity of the relations 3.8-3.12 does not depend on our choice of the transversal τ, we may assume that $\tau(\overline{u}_i) = \widetilde{u}_i$. With this modification the relation 3.7 holds in E as well.

In a similar fashion we may modify the choices $\tau(\overline{x}_i)$ and $\tau(\overline{y}_i)$ to obtain the relations 3.14 and 3.15. Specifically, if $\tau(\overline{x}_i)^{p^{e_i}} = f_i\gamma_i$ with $\gamma_i \in D$ then we let $\widetilde{\gamma}_i \in D$ satisfy $p^{e_i}\widetilde{\gamma}_i = \gamma_i^{-1}$ and let $\widetilde{x}_i = \tau(\overline{x}_i)\widetilde{\gamma}_i$. We then may modify τ by replacing $\tau(\overline{x}_i)$ with $\widetilde{x}_i$. The process for the choices $\tau(\overline{y}_i)$ is identical.

Now, since $\mathscr{Y}$ generates both G and E and all the defining relations for G hold in E, we may apply Von Dyck's Theorem to obtain a surjective homomorphism $\varphi \colon G \twoheadrightarrow E$ extending the identity function on $\mathscr{Y}$. Evidently $\varphi\iota = \varepsilon$, and since ε is injective, we see that ι is also injective. $\square$

Definition 3.2.6. Define $\mathbf{e}_{F,\mathscr{F},\mathscr{X}}$ to be the extension

$$\mathbf{e}_{F,\mathscr{F},\mathscr{X}}: \qquad D \overset{\iota}{\rightarrowtail} G_{F,\mathscr{F},\mathscr{X}} \overset{\pi}{\twoheadrightarrow} G_{F,\mathscr{F},\mathscr{X}}/\langle \mathscr{C} \rangle,$$

where π is the quotient map. Evidently $\mathbf{e}_{F,\mathscr{F},\mathscr{X}}$ is a central extension (by relation 3.8).

The following theorem asserts that the extensions $\mathbf{e}_{F,\mathscr{F},\mathscr{X}}$ account, up to isomorphism, for every d-ab extension of $\mathbb{Z}(p^\infty)$ with finite quotient.

Theorem 3.2.7. *Let* $\mathbf{e}$ *be the d-extension*

$$\mathbf{e}: \qquad \mathbb{Z}(p^\infty) \overset{\varepsilon}{\rightarrowtail} E \overset{\rho}{\twoheadrightarrow} Q$$

with Q a finite abelian p-group. Then there is a finite subgroup $F \leq Z(E)$, and sets $\mathscr{F} \subset F$ and $\mathscr{X}$ such that $E \cong G_{F,\mathscr{F},\mathscr{X}}$. Moreover, $\mathbf{e} \cong \mathbf{e}_{F,\mathscr{F},\mathscr{X}}$.

Proof. Let $Z = Z(E)$. We first show that there is a set of elements of G that we may identify with the generators of $G_{F,\mathscr{F},\mathscr{X}}$, and which, after this identification, satisfy the relations defining $G_{F,\mathscr{F},\mathscr{X}}$. We use the notation of Theorem 3.2.3 and let $k = r/2$. Then E is generated by Z and the set $\{x_1, y_1, \ldots x_k, y_k\}$, with $[x_i, y_i] \neq 1$ and if $\langle x_i, y_i \rangle = G_i$ for each $i = 1, \ldots k$; then for $i \neq j$ we have $[G_i, G_j] = 1$. Now, for each i we have $|[x_i, y_i]| = |x_i Z| = |y_i Z|$; for if $x_i^n \in Z$, then $[x_i, y_i]^n = [x_i^n, y_i] = 1$, and conversely if $[x_i^n, y_i] = [x_i, y_i]^n = 1$, then, since $[x_i, y_j] = [x_i, x_j] = 1$ for $i \neq j$, we see that $x_i^n \in Z$. For each i put $|x_i Z| = p^{e_i}$.

Now, since Q is abelian, the commutators $[x_i, y_i]$ are in $\iota(D)$ for each i. Let $\{0, \widetilde{c}_1, \widetilde{c}_2, \ldots\}$ be a p-inductive set in $\iota(D)$. Then for each i there exists $\lambda_i \in \mathbb{Z}/p^{e_i}$, $p \nmid \lambda_i$, such that $[x_i, y_i] = \lambda_i \widetilde{c}_{e_i}$. Then $[x_i^{1/\lambda_i}, y_i] = \widetilde{c}_{e_i}$. Since x_i^{1/λ_i} has the same properties as x_i in Theorem 3.2.3, we may assume $\lambda_i = 1$ for each $i = 1, \ldots k$.

Since D is divisible and $\iota(D) \leq Z(E)$, we see that $Z(E) = \iota(D) \times F$ for some subgroup $F \leq Z(E)$. Thus, for each i we have $x_i^{p^{e_i}} = \delta_i f_i$ with $f_i \in F$ and $\delta_i \in \iota(D)$. Since $\iota(D)$ is divisible, we may find $\widetilde{\delta}_i \in \iota(D)$ such that $\widetilde{\delta}_i^{p^{e_i}} = \delta_i^{-1}$.

For each i put $\widetilde{x}_i = x_i \widetilde{\delta}_i$, so that $\widetilde{x}_i^{p^{e_i}} = f_i$. We note that $\widetilde{x}_i x_i^{-1} \in Z$, which implies $[\widetilde{x}_i, g] = [x_i, g]$ for any $g \in E$. In an identical fashion we obtain $\widetilde{y}_i$ so that $\widetilde{y}_i^{p^{e_i}} = g_i \in F$ and $\widetilde{y}_i y_i^{-1} \in Z$. Also, since $\widetilde{x}_i x_i^{-1} \in Z$ and $\widetilde{y}_i y_i^{-1} \in Z$, we see that E is generated by Z and $\{\widetilde{x}_1, \widetilde{y}_1, \ldots, \widetilde{x}_k, \widetilde{y}_k\}$.

Finally, let $\{0, \widetilde{c}_1, \widetilde{c}_2, \ldots\}$ be a p-inductive set in $\iota(D)$ and $\langle \widetilde{\mathscr{G}} | \widetilde{\mathscr{R}} \rangle$ be a presentation for F. For definiteness, we take

$$\widetilde{\mathscr{G}} = \{\widetilde{u}_1, \ldots, \widetilde{u}_\ell\}$$

and

$$\widetilde{\mathscr{R}} = \{\widetilde{u}_1^{p^{\alpha_1}}, \ldots, \widetilde{u}_\ell^{p^{\alpha_\ell}}\} \cup \{[\widetilde{u}_i, \widetilde{u}_j] \colon 1 \leq i < j \leq \ell\}.$$

Let $\mathscr{F} = (f_1, g_1, \ldots, f_k, g_k)$ and $\mathscr{X} = (e_1, \ldots e_k)$. Since $Z = \iota(D) \times F$, we see that the relations defining $G_{F,\mathscr{F},\mathscr{X}}$ hold in E, with $\widetilde{c}_i$, $\widetilde{x}_i$, $\widetilde{y}_i$, and $\widetilde{u}_i$ replacing c_i, x_i, y_i, and u_i respectively for each i. Thus, there is a surjection $\varphi\colon G_{F,\mathscr{F},\mathscr{X}} \twoheadrightarrow E$ taking

$$\begin{aligned} c_i &\mapsto \widetilde{c}_i \text{ for } i = 1, 2, \ldots; \\ x_i &\mapsto \widetilde{x}_i \text{ for } i = 1, \ldots, k; \\ y_i &\mapsto \widetilde{y}_i \text{ for } i = 1, \ldots, k; \\ u_i &\mapsto \widetilde{u}_i \text{ for } i = 1, \ldots, \ell. \end{aligned}$$

We construct the following commutative diagram:

$$\begin{array}{llcccccc} \mathbf{e}_{F,\mathscr{F},\mathscr{X}}: & & D & \stackrel{\iota}{\rightarrowtail} & G_{F,\mathscr{F},\mathscr{X}} & \stackrel{\pi}{\twoheadrightarrow} & G_{F,\mathscr{F},\mathscr{X}}/\langle\mathscr{C}\rangle \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\gamma} \\ \mathbf{e}: & & D & \stackrel{\varepsilon}{\rightarrowtail} & E & \stackrel{\rho}{\twoheadrightarrow} & Q. \end{array}$$

Let $\mathscr{D} = \{0, d_1, d_2, \ldots\}$ be the p-inductive set in D such that $\iota(d_i) = c_i$ in the extension $\mathbf{e}_{F,\mathscr{F},\mathscr{X}}$. Of course, the set $\{0, \varepsilon^{-1}(\widetilde{c}_1), \varepsilon^{-1}(\widetilde{c}_2), \ldots\}$ is also a p-inductive set in D. Obviously the map $d_i \mapsto \varepsilon^{-1}(\widetilde{c}_i)$ uniquely defines an automorphism of β of D. Moreover, we plainly have $\varphi\iota = \varepsilon\beta$. This also gives us a homomorphism $\gamma\colon G_{F,\mathscr{F},\mathscr{X}}/\langle\mathscr{C}\rangle \to Q$ by passing to the quotients, i.e., if τ is a transversal for $\mathbf{e}_{F,\mathscr{F},\mathscr{X}}$, then $\gamma = \rho\varphi\tau$ is a homomorphism $G_{F,\mathscr{F},\mathscr{X}}/\langle\mathscr{C}\rangle \to Q$ because $\varphi(\mathrm{Ker}(\pi)) = \mathrm{Ker}\,\rho$.

Finally, γ is easily seen to be surjective. Since Q is finite, γ must in fact be bijective, i.e., an isomorphism. By the 5-lemma it then follows that φ is an isomorphism, which verifies that $\mathbf{e} \cong \mathbf{e}_{F,\mathscr{F},\mathscr{X}}$. □

Corollary 3.2.8. *Any d-ab extension with finite quotient is a Baer sum of extensions of the form* $\mathbf{e}_{F,\mathscr{F},\mathscr{X}}$.

Proof. Let Q and D be abelian p-groups with D divisible and Q finite. Recall that by Theorem 2.2.1 we have

$$D \cong \bigoplus_{i \in I} D_i$$

for some index set I, with each $D_i \cong \mathbb{Z}(p^\infty)$. Thus, if $\xi \in \mathrm{Hom}(Q \wedge Q, D)$ then ξ is a sum

$$\xi = \sum_{i \in I} \xi_i$$

where $\mathrm{Im}(\xi_i) \leq D_i$. Now take $\xi = \xi_{\mathbf{e}}$ with $\mathbf{e}$ an arbitrary central extension of D by Q. For each $i \in I$ let $\mathbf{e}_i$ be a d-extension (unique up to equivalence) corresponding to ξ_i. Then each $\mathbf{e}_i$ is an extension of $\mathbb{Z}(p^\infty)$ by Q, and by Theorem 2.1.2 we have that $\mathbf{e}$ is equivalent to the Baer sum of the extensions

$\mathbf{e}_i$. $\square$

We have a precise description of the center and central quotient for the group $G_{F,\mathscr{F},\mathscr{X}}$.

Proposition 3.2.9. *Let $G = G_{F,\mathscr{F},\mathscr{X}}$. Then $Z(G) \cong D \times F$, and*

$$G/Z(G) \cong (\mathbb{Z}/p^{e_1})^2 \times \cdots \times (\mathbb{Z}/p^{e_k})^2.$$

Proof. To ease notation, put $Z = Z(G)$; recall that $C = \langle \mathscr{C} \rangle = \iota(D)$. We claim that $Z = \langle \mathscr{C} \cup \mathscr{G} \rangle$. Note that from relation 3.8 we have $\langle \mathscr{C} \cup \mathscr{G} \rangle \leq Z$. For the reverse containment, suppose $g \in Z$. Modulo $\langle \mathscr{C} \cup \mathscr{G} \rangle$ we may write

$$g = \prod_{i=1}^{k} x_i^{\ell_i} y_i^{m_i}$$

with $\ell_i, m_i \in \mathbb{Z}$. Then for each i we have $[x_i, g] = [x_i, y_i]^{m_i} = c_{e_i}^{m_i} = 1$, which implies that $p^{e_i} | m_i$ and $y_i^{m_i}$ is a power of g_i, which is in $\langle \mathscr{G} \rangle$. Likewise, for each i we have $[g, y_i] = c_{e_i}^{\ell_i}$, which implies $p^{e_i} | \ell_i$ for each i and $x_i^{\ell_i} \in \langle \mathscr{G} \rangle$. Hence, $g \in \langle \mathscr{C} \cup \mathscr{G} \rangle$, which verifies our claim.

Set $\hat{x}_i = x_i Z$ and $\hat{y}_i = y_i Z$. We have shown that

$$G/Z = \langle \hat{x}_1, \hat{y}_1, \ldots \hat{x}_k, \hat{y}_k \rangle$$

and $|\hat{x}_i| = |\hat{y}_i| = p^{e_i}$. Clearly G/Z is abelian. Also, as above, if $\prod_i \hat{x}_i^{\ell_i} \hat{y}_i^{m_i} = Z$, then $\hat{x}_i^{\ell_i} = \hat{y}_i^{m_i} = Z$ for each i. This implies the description of G/Z.

To prove that $Z \cong D \times F$, we observe that since C is divisible and $C \cong D$, it suffices to prove $Z/C \cong F$. Since $\langle \mathscr{G} \rangle$ and C commute, we have $Z = \langle \mathscr{C} \cup \mathscr{G} \rangle = \langle \mathscr{G} \rangle C$, so

$$Z/C \cong \langle \mathscr{G} \rangle / \langle \mathscr{G} \rangle \cap C.$$

Thus, $\langle \overline{\mathscr{G}} \rangle = \pi(\langle \mathscr{G} \rangle) = \langle \mathscr{G} \rangle / \langle \mathscr{G} \rangle \cap C \cong Z/C$. For definiteness we continue to assume that $\mathscr{G}$ and $\mathscr{R}$ are the sets given in 3.16 and 3.17. Since the relations $\mathscr{R}$ hold in G, we have a homomorphism $\omega \colon F \to G$ that extends the identity map on $\mathscr{G}$ (i.e., takes $u_i \mapsto u_i$). Let $\overline{\omega} \colon F \to Q$ be the composition $\pi\omega$, which takes $u_i \mapsto \overline{u}_i$ for $i = 1, \ldots \ell$. Evidently $\operatorname{Im}(\overline{\omega}) = \langle \overline{\mathscr{G}} \rangle$, so now it suffices to show that $\overline{\omega}$ is injective.

To accomplish this, we follow a procedure similar to our proof of Lemma 3.2.5. First, put $A = G/Z$; we construct an abelian extension of F by A. Note that $\{\hat{x}_1, \hat{y}_1, \ldots, \hat{x}_k, \hat{y}_k\}$ is a basis for A. For $i = 1, \ldots k$ let $\varphi_{x_i} \colon \langle \hat{x}_i \rangle \times \langle \hat{x}_i \rangle \to F$ be the function

$$\varphi_{x_i}(\hat{x}_i^{\gamma}, \hat{x}_i^{\delta}) = \begin{cases} f_i & \text{if } \gamma + \delta \geq p^{e_i} \\ 0 & \text{otherwise,} \end{cases}$$

where γ and δ are assumed in the set $\{0, 1, \ldots, p^{e_i} - 1\}$. Likewise, define

$$\varphi_{y_i}(\hat{y}_i^\gamma, \hat{y}_i^\delta) = \begin{cases} g_i & \text{if } \gamma + \delta \geq p^{e_i} \\ 0 & \text{otherwise.} \end{cases}$$

Let $f\colon A \times A \to F$ be the function

$$f = \sum_{i=1}^{k} f_i,$$

that is,

$$(\hat{x}_1^{\gamma_1}, \hat{y}_1^{\mu_1}, \ldots, \hat{x}_k^{\gamma_k}, \hat{y}_k^{\mu_1}, \hat{x}_1^{\delta_1}, \hat{y}_1^{\nu_1}, \ldots, \hat{x}_k^{\delta_k}, \hat{y}_k^{\nu_k}) \mapsto \sum_{i=1}^{k} \left(\varphi_{x_i}(\hat{x}_i^{\gamma_i}, \hat{x}_i^{\delta_i}) + \varphi_{y_i}(\hat{y}_i^{\mu_i}, \hat{y}_i^{\nu_i})\right)$$

We claim that f is a factor set. Before we verify this, we note that f is symmetric, and assuming that the A-action on F is trivial, we obtain an abelian extension

$$F \rightarrowtail E \twoheadrightarrow A$$

corresponding to f.

We must verify the 2-cocycle condition

$$f(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0.$$

Since f behaves independently on the basis elements, we may assume without loss of generality that x, y, and z are in $\langle \hat{x}_i \rangle$ for some i. Write $\hat{x}_i = u$ and $e_i = e$ and assume $x = u^\lambda$, $y = u^\mu$, and $z = u^\nu$. Note that $xy = u^{\overline{\lambda+\mu}}$, where

$$\overline{\lambda + \mu} = \begin{cases} \lambda + \mu & \text{if } \lambda + \mu < p^e \\ \lambda + \mu - p^{e_i} & \text{if } \lambda + \mu \geq p^e. \end{cases}$$

Similarly, of course, $yz = u^{\overline{\mu+\nu}}$.

We now consider cases. First, suppose $\lambda + \mu \geq p^e$. Then

$$f(x, y) = f_i$$

and $\overline{\lambda + \mu} = \lambda + \mu - p^e$. In case $\overline{\lambda + \mu} + \nu < p^e$ we have

$$f(xy, z) = 0.$$

If further $\mu + \nu \geq p^e$ then $\lambda + \overline{\mu + \nu} = \lambda + \mu + \nu - p^e = \overline{\lambda + \mu} + \nu < p^e$. Thus,

$$f(y, z) = f_i \Longrightarrow f(x, yz) = 0.$$

If instead $\mu + \nu < p^e$ then $\lambda + \overline{\mu + \nu} = \lambda + \mu + \nu \geq p^e$ since $\lambda + \mu \geq p^e$ by supposition. Thus,

$$f(y,z) = 0 \Longrightarrow f(x,yz) = f_i.$$

In either event, for the case when $\overline{\lambda+\mu}+\nu < p^e$ the 2-cocycle condition is satisfied.

Next, suppose $\overline{\lambda+\mu}+\nu \geq p^e$ (still with $\lambda+\mu \geq p^e$). Then

$$f(xy,z) = f_i.$$

Note that we must have $\mu+\nu \geq p^e$, for $p^e \leq \overline{\lambda+\mu}+\nu = \lambda+\mu+\nu-p^e$ together with $\lambda < p^e$ implies $\mu+\nu \geq p^e$. Now $\lambda+\overline{\mu+\nu} = \overline{\lambda+\mu}+\nu$ as before, but now this is assumed $\geq p^e$. Hence we have

$$f(y,z) = f_i \text{ and } f(x,yz) = f_i,$$

which verifies the 2-cocycle in this case.

Now suppose $\lambda+\mu < p^e$. Then

$$f(x,y) = 0$$

and $\overline{\lambda+\mu} = \lambda+\mu$. If $\lambda+\mu+\nu < p^e$ then each term in the 2-cocycle condition is 0. Assume $\lambda+\mu+\nu \geq p^e$. Then $\overline{\lambda+\mu}+\nu = \lambda+\mu+\nu \geq p^e$, so

$$f(xy,z) = f_i.$$

If $\mu+\nu < p^e$. Then $f(y,z) = 0$ and $\lambda+\overline{\mu+\nu} = \lambda+\mu+\nu \geq p^e$ and so $f(x,yz) = f_i$. In this case the 2-cocycle condition is satisfied.

If instead $\mu+\nu \geq p^e$, then $f(y,z) = f_i$, and and $\lambda+\overline{\mu+\nu} = \lambda+\mu+\nu-p^e < \lambda+\mu+p^e-p^e = \lambda+\mu < p^e$. Thus, $f(x,yz) = 0$ and we have verified that in each case the 2-cocycle condition is satisfied.

Now let

$$F \overset{\zeta}{\rightarrowtail} E \twoheadrightarrow A$$

be an extension (unique up to equivalence) corresponding with the cohomology class of f. Recall that we may take E to be the group that is $F \times A$ as a set, with group operation

$$(h,a)(h',a') = (h+h'+f(a,a'), aa').$$

for $h,h' \in F$ and $a,a' \in A$. Doing so, ζ is the map $h \mapsto (h,1)$ for $h \in F$.

Note that unless $p^{e_i} = 2$ we have $(0,\hat{x}_i)^2 = (0,\hat{x}_i^2)$, and an easy induction shows that for $\lambda < p^{e_i}$ we have

$$(0,\hat{x}_i)^\lambda = (0,\hat{x}_i^\lambda).$$

It then follows that

$$(0,\hat{x}_i)^{p^{e_i}} = (f_i, 1).$$

Similarly, we have

$$(0,\hat{y}_i)^{p^{e_i}} = (g_i, 1).$$

Now, E is generated by $(\mathscr{G}, 1) \cup \{(0,\hat{x}_1), (0,\hat{y}_1), \ldots, (0,\hat{x}_k), (0,\hat{y}_k)\}$, which we may identify with the generating set $\overline{\mathscr{Y}}$ for Q in the obvious way: $\mathscr{G} \leftrightarrow (\mathscr{G}, 1)$, $(0,\hat{x}_i) \leftrightarrow \overline{x}_i$, and $(0,\hat{y}_1) \leftrightarrow \overline{y}_i$ for each $i = 1, \ldots, k$. With this identification, the only relations we see that the relations defining Q also hold in E. Thus, by Von Dyck's Theorem there is a surjective homomorphism $\psi\colon Q \twoheadrightarrow E$ taking $\overline{u}_i \mapsto (u_i, 1)$ for $i = 1, \ldots, \ell$. Obviously $\psi\overline{\omega} = \zeta$, and since ζ is injective, so is $\overline{\omega}$. This completes the proof. □

3.3 D-extensions with finite homocyclic quotients

In this section our aim is to give a class of group presentations which parametrize central extensions of prüfer groups with finite homocyclic quotients. We begin with a lemma that is similar to Lemma 3.2.2. We require our generating set to be a basis, so we must be content with pairing only on the homocyclic components.

Lemma 3.3.1. *Let $\theta \in \mathrm{Hom}(Q \wedge Q, \mathbb{Z}(p^\infty))$ with Q a finite abelian p-group, and let $\mathscr{D}$ be a p-inductive set in $\mathbb{Z}(p^\infty)$. Then there is a basis $\mathscr{B}$ for Q such that θ pairs $\mathscr{B}[p^i]$ on $\mathscr{D}$ for each $i \in \mathbb{N}$.*

Proof. The proof is by induction on the rank of Q. If the homocyclic components of Q all have rank ≤ 1, then there is nothing to prove. Thus we can assume that at least one of the homocyclic components H of Q has rank > 1. Let p^a be the exponent of H. Assume that our lemma holds for groups of rank less than rank(Q).

Write $\mathscr{D} = \{0, d_1, d_2, \ldots\}$, so that $|d_i| = p^i$ for each i. Let $\mathscr{B}_0$ be a basis for Q. Choose $x, y \in \mathscr{B}_0[p^a]$ such that $|\theta(x \wedge y)|$ is maximal over the elements of $\mathscr{B}_0[p^a]$. Write $\theta(x \wedge y) = \lambda d_i$ with $i \geq 0$ and λ a unit in $\mathbb{Z}/p^i$. If $\lambda \neq 1$, then we may replace x with $x = (1/\lambda)x$ without violating the maximality property for x and still retaining a basis. Hence we assume $\theta(x \wedge y) = d_i$.

Next, for $w \in \mathscr{B}_0[p^a] - \{x, y\}$ write $\theta(x \wedge w) = \lambda_w d_{i_w}$ with $\lambda_w \in (\mathbb{Z}/p^{i_w})^\times$ and $0 \leq i_w \leq i$ (by maximality). Put

$$w' = w - p^{i-i_w}\lambda_w y.$$

Then $|p^{i-i_w}\lambda_w y| \leq p^a$, so $|w'| = |w|$. Also, for $u \in \mathscr{B}_0[p^a]$ we have

$$|\theta(u \wedge w')| \leq \max\{|\theta(u \wedge w)|, |p^{i-i_w}\lambda_w \theta(u \wedge y)|\} \leq |\theta(x \wedge y)| \qquad (3.18)$$

by the maximality of our choice. We put $w' = w$ for $w \in \mathscr{B}_0 - \mathscr{B}_0[p^a]$ and let

$$\mathscr{B}_1 = \big\{w' \colon w \in \mathscr{B}_0 - \{x, y\}\big\} \cup \{x, y\}.$$

Since $|w'| = |w|$ for each w and $\mathscr{B}_1$ generates Q, we see that $\mathscr{B}_1$ is a basis for Q. By 3.18 we have that $|\theta(x \wedge y)|$ is still maximal over the elements of $\mathscr{B}_1[p^a]$. Moreover, for each $w' \in \mathscr{B}_1[p^a] - \{y\}$ we have $\theta(x \wedge w') = 0$.

Similarly, for $v \in \mathscr{B}_1[p^a] - \{x, y\}$ we write $\theta(v \wedge y) = \mu_v d_{j_v}$. Set

$$v'' = v + p^{i-j_v}\mu_v x$$

and for $v \in \mathscr{B}_1 - \mathscr{B}_1[p^a]$ simply set $v'' = v$. Define

$$\mathscr{B}' = \big\{v'' \colon v \in \mathscr{B}_1 - \{x, y\}\big\} \cup \{x, y\},$$

which is a basis for Q and has the property that $\theta(x \wedge w) = 0 = \theta(w \wedge y)$ for all $w \in \mathscr{B}'[p^a] - \{x, y\}$.

Let $\widetilde{Q} = \langle \mathscr{B}' - \{x, y\}\rangle$. Now, θ can be decomposed as a sum

$$\theta = \sum_{\{u,v\} \subset \mathscr{B}'} \theta_{u \wedge v} \tag{3.19}$$

where $\theta_{u \wedge v}$ is the restriction of θ to the one-dimensional subspace $\langle u \wedge v\rangle$. Then for $v \in \mathscr{B}'[p^a]$ we have $\theta_{x \wedge v} = 0 = \theta_{v \wedge y}$. We remark that for $w \in \mathscr{B}' - \mathscr{B}'[p^a]$ we could have $\theta(x \wedge w) \neq 0$ (and likewise for y). Collecting terms in 3.19, we write $\theta = \theta_{x \wedge y} + \theta_{x,y} + \widetilde{\theta}$, where

$$\theta_{x,y} = \sum_{w \in \mathscr{B}' - \mathscr{B}'[p^a]} \big(\theta_{x \wedge w} + \theta_{w \wedge y}\big)$$

and

$$\widetilde{\theta} = \sum_{\{u,v\} \subset \mathscr{B}' - \{x,y\}} \theta_{u \wedge v}.$$

We remark that $\widetilde{\theta}$ is a map $\widetilde{Q} \wedge \widetilde{Q} \to D$. Since $\widetilde{Q}$ has rank $r - 2$, we may apply our induction hypothesis to $\widetilde{\theta}$. Thus there is a basis $\widetilde{\mathscr{B}}$ for $\widetilde{Q}$ such that $\widetilde{\theta}$ pairs $\widetilde{\mathscr{B}}[p^i]$ on $\mathscr{D}$ for all i. Finally, we let $\mathscr{B} = \{x, y\} \cup \widetilde{\mathscr{B}}$, which is evidently a basis for Q with the desired properties. □

Corollary 3.3.2. *Fix a basis $\mathscr{B}$ for the finite abelian p-group Q. Let $D \cong \mathbb{Z}(p^\infty)$, and fix a p-inductive set $\mathscr{D}$ in D. Then each isomorphism class of d-extensions of D by Q contains an extension* **pr** *such that for each $i \in \mathbb{N}$, $\xi_{\mathbf{e}}$ pairs $\mathscr{B}[p^i]$ on $\mathscr{D}$.*

Proof. Let $\mathbf{e}$ be an extension of D by Q By Lemma 3.3.1 there is a basis $\mathscr{B}'$ for Q such that $\xi_{\mathbf{e}}$ pairs $\mathscr{B}'[p^i]$ on $\mathscr{D}$ for each i. Let $\alpha \in \operatorname{Aut} Q$ be the automorphsim defined by the change of basis $\mathscr{B} \to \mathscr{B}'$. Then the map $\xi_{\mathbf{e}} \circ \alpha^{\wedge 2}$ clearly pairs

$\mathscr{B}[p^i]$ on $\mathscr{D}$ for each $i \in \mathbb{N}$. Define **pr** to be the extension corresponding to the map $\xi_{\mathbf{e}} \circ \alpha^{\wedge 2}$. By Proposition 2.1.8 there is an isomorphism $\mathbf{e} \cong \mathbf{pr}$. □

For the remainder of this section we tacitly assume that

$$Q = (\mathbb{Z}/p^a)^r$$

with a and $r > 0$.

We fix a basis $\mathscr{B}$ for Q and a p-inductive set $\mathscr{D}$ in $\mathbb{Z}(p^\infty)$ and we write $\mathscr{D} = \{0, d_1, d_2 \ldots\}$, so that $|d_i| = p^i$ for each i. By Corollary 3.3.2, each isomorphism class of d-extensions of D by Q corresponds to a map $Q \wedge Q \to D$ that pairs $\mathscr{B}$ on $\mathscr{D}$. Our aim is to use these pairing maps to obtain a parametrization of isomorphism classes of d-extensions of $\mathbb{Z}(p^\infty)$ with finite homocyclic quotient Q.

Definition 3.3.3. Let $\theta\colon Q \wedge Q \to \mathbb{Z}(p^\infty)$ be a map that pairs $\mathscr{B}$ on $\mathscr{D}$. For $1 \le i \le a$, if s_i is the number of pairs $(x, x') \in \mathscr{B} \times \mathscr{B}$ such that $\theta(x \wedge x') = \pm d_i$, then we define

$$\mathbf{T}(\theta) = (s_1, \ldots, s_a).$$

Proposition 3.3.4. *If $\theta\colon Q \wedge Q \to D$ is a homomorphism that pairs $\mathscr{B}$ on $\mathscr{D}$ and $\mathbf{T}(\theta) = (s_1, \ldots, s_a)$, then:*

1. *s_i is even and nonnegative for each $i = 1, \ldots, a$, and*

2. *$\sum_i s_i \le r$.*

Moreover, if $\mathbf{T}$ is any such tuple, then $\mathbf{T} = \mathbf{T}(\xi)$ for some $\xi \in \mathrm{Hom}(Q \wedge Q, D)$ that pairs $\mathscr{B}$ on $\mathscr{D}$.

Proof. That $\mathbf{T}(\theta)$ satisfies the given properties is obvious. Suppose that $\mathbf{T} = (t_1, \ldots, t_a)$ satisfies 1 and 2. Write

$$t_0 = r - \sum_i t_i$$

and define ξ by setting $\xi(u \wedge v) = 0$ for all $u, v \in \mathscr{B}$ except for the following:

$$\begin{aligned}
&\xi(x_{t_0+1} \wedge x_{t_0+2}) = \cdots = \xi(x_{t_0+t_1-1} \wedge x_{t_0+t_1}) = d_1;\\
&\xi(x_{t_0+t_1+1} \wedge x_{t_0+t_1+2}) = \cdots = \xi(x_{t_0+t_1+t_2-1} \wedge x_{t_0+t_1+t_2}) = d_2;\\
&\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots\\
&\xi(x_{t_0+\cdots+t_{a-1}+1} \wedge x_{t_0+\cdots+t_{a-1}+2}) = \cdots = \xi(x_{t_0+\cdots+t_a-1} \wedge x_{t_0+\cdots+t_a}) = d_a.
\end{aligned}$$

Clearly ξ pairs $\mathscr{B}$ on $\mathscr{D}$ and $\mathbf{T} = \mathbf{T}(\xi)$. □

Definition 3.3.5. We refer to a tuple satisfying properties 1 and 2 of Proposition 3.3.4 as a *pairing tuple*.

Corollary 3.3.6. *Homomorphisms from $Q \wedge Q$ to D that pair $\mathscr{B}$ on $\mathscr{D}$ are parametrized by pairing tuples $\theta \leftrightarrow \mathbf{T}(\theta)$.*

Thus, d-extensions of $\mathbb{Z}(p^\infty)$ by finite homocyclic groups can be understood in terms of pairing tuples. We now proceed to show that, in fact, pairing tuples provide a parametrization for such extensions.

Definition 3.3.7. Assume we have a pairing tuple $\mathbf{T} = (s_1, \ldots, s_a)$ and write $s_0 = r - \sum_i s_i$. We define the group $G(\mathbf{T})$ to be the group with generators

$$\{x_1, \ldots, x_r, d_1, d_2, \ldots\},$$

and relations

$d_1^p = 1$,

$d_i^p = d_{i-1}$ for $i \geq 2$;

$x_i^{p^a} = 1$ for $1 \leq i \leq a$;

$[x_i, d_j] = 1$ for all i and j;

$[x_i, x_j] = 1$ for all i and j with the following exceptions:

(S1) $[x_{s_0+1}, x_{s_0+2}] = \cdots = [x_{s_0+s_1-1}, x_{s_0+s_1}] = d_1$;

(S2) $[x_{s_0+s_1+1}, x_{s_0+s_1+2}] = \cdots = [x_{s_0+s_1+s_2-1}, x_{s_0+s_1+s_2}] = d_2$;

$\vdots \qquad \vdots \qquad \vdots$

(Sb) $[x_{s_0+\cdots+s_{b-1}+1}, x_{s_0+\cdots+s_{b-1}+2}] = \cdots = [x_{s_0+\cdots+s_b-1}, x_{s_0+\cdots+s_a}] = d_a$.

We note that the group $G(\mathbf{T})$ is actually just the group $G(\mathscr{A})$ defined in Section 3.1 in the case when D has rank 1, so there is only one matrix, and the matrix is block diagonal with blocks

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, in the same fashion here we note that by Lemma 3.1.2 we have $\mathbb{Z}(p^\infty) \cong \langle d_i \colon i \geq 1\rangle$, which is a central subgroup, and $G(\mathbf{T})/\langle d_1, d_2, \ldots\rangle \cong (\mathbb{Z}/p^a)^r$. We thus obtain a d-extension

$$\mathbf{e}(\mathbf{T})\colon \qquad \mathbb{Z}(p^\infty) \rightarrowtail G(\mathbf{T}) \twoheadrightarrow Q$$

with obvious maps.

Our aim is to prove the following theorem, which gives an explicit solution to the isomorphism problem for d-ab extensions with rank 1 kernel and homocyclic quotient.

Theorem 3.3.8. *1. If $\mathbf{e}$ is a central extension of $\mathbb{Z}(p^\infty)$ by a homocyclic p-group Q, then $\mathbf{e} \cong \mathbf{e}(\mathbf{T})$ for some pairing tuple $\mathbf{T}$.*

2. *If* $\mathbf{T}$ *and* $\widetilde{\mathbf{T}}$ *are two pairing tuples, then* $\mathbf{e}(\mathbf{T}) \cong \mathbf{e}(\widetilde{\mathbf{T}}) \Leftrightarrow \mathbf{T} = \widetilde{\mathbf{T}}$.

Before proving Theorem 3.3.8 we introduce the following terminology.

Definition 3.3.9. Let A be any finite abelian p-group and j a positive integer. We refer to the number of times p^j occurs as an invariant factor for A as the p^j***-invariant*** of A.

We remark that the p^j-invariant for an abelian group A is the maximal integer m such that $(\mathbb{Z}/p^j\mathbb{Z})^m$ is a direct summand of A. Consequently, if $A \cong \widetilde{A}$ then for each j the p^j-invariants of A and $\widetilde{A}$ are identical.

Proof of Theorem 3.3.8. For 1, sufficiency is trivial, while necessity follows by Corollary 3.3.2. Thus what is left to show is that if $\mathbf{T}$ and $\widetilde{\mathbf{T}}$ are two pairing tuples and $G(\mathbf{T}) \cong G(\widetilde{\mathbf{T}})$, then $\mathbf{T} = \widetilde{\mathbf{T}}$. Put $G = G(\mathbf{T})$ and $\widetilde{G} = G(\widetilde{\mathbf{T}})$. We identify the subgroups $\langle d_i \colon i \geq 1 \rangle$ in both groups and write $D = \langle d_i \colon i \geq 1 \rangle$.

For definiteness, write $\mathbf{T} = (s_1, \ldots, s_a)$ and $\widetilde{\mathbf{T}} = (\widetilde{s}_1, \ldots, \widetilde{s}_{\widetilde{a}})$. We remark that $G/D \cong Q \cong \widetilde{G}/D \cong Q$, so that $\widetilde{a} = a$.

Write Z for $Z(G)$, which is of course characteristic in G. Now $G/Z \cong (G/D)/(Z/D)$ is a homomorphic image of Q, and hence a finite abelian p-group. Let

$$\mathscr{B}_{G/Z} = \{x_{s_0+1}, \ldots, x_r\}$$

and note that for every $x \in \mathscr{B}_{G/Z}$ there is a unique $x' \in \mathscr{B}_{G/Z}$ such that $[x, x'] \neq 1$. The collection $\mathscr{B}_{G/Z}Z = \{xZ \colon x \in \mathscr{B}_{G/Z}\}$ of cosets clearly generates G/Z.

We claim that in fact $\mathscr{B}_{G/Z}Z$ is a basis for G/Z. To verify this, let $a_{s_0+1}, \ldots, a_r \in \mathbb{Z}$, put

$$y = x_{s_0+1}^{a_{s_0+1}} \cdots x_r^{a_r},$$

and assume $y \in Z$. We show that $x_i^{a_i} \in Z$ for each $i \in \{s_0+1, \ldots, r\}$. Choose $i \in \{s_0+1, \ldots, r\}$ and put

$$j = \begin{cases} i-1 & \text{if } i - s_0 \text{ is even;} \\ i+1 & \text{if } i - s_0 \text{ is odd.} \end{cases}$$

In other words, j is the unique integer such that $[x_i, x_j] \neq 1$. Using bilinearity of the commutator map (since $G' \leq Z$) we have $1 = [y, x_j] = [x_i^{a_i}, x_j]^{\pm 1}$. Also, for $\ell \neq j$ we have $[x_i^{a_i}, x_\ell] = 1$, so we conclude that $x_i^{a_i} \in Z$, as desired.

Now, by observing the orders of the elements in $\mathscr{B}_{G/Z}$, we see that the p^i-invariant of G/Z is s_i (for $i = 1, \ldots, a$). Similarly the p^i-invariant of $\widetilde{G}/Z(\widetilde{G})$ is $\widetilde{s}_i$. Now, the isomorphism $G \cong \widetilde{G}$ induces an isomorphism $G/Z \cong \widetilde{G}/\widetilde{Z}$. Hence, we have $s_i = \widetilde{s}_i$ for each i; that is., $\mathbf{T} = \widetilde{\mathbf{T}}$. □

4 Tight extensions

4.1 Extensions of the center

Given a central extension $C \rightarrowtail E \twoheadrightarrow G$, Theorem 3.2.3 suggests that it might be easier to think of E as a central extension of $Z(E)$ by $E/Z(E)$. Motivated by this we make the following definition.

Definition 4.1.1. A (central) extension

$$C \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} G$$

is said to be *tight* provided that $\operatorname{Im}\iota = Z(G)$.

Beyond interest in their own right, tight extensions can be used to shed light on general d-ab extensions via a functor that assigns to each d-ab extension a tight d-ab extension. In this chapter we investigate the properties of this functor and how it can contribute to an understanding of the general situation. In particular, we are able to give a parametrization of d-ab extensions up to isomorphism using our functor. We also give a classification of tight extensions of prüfer groups with finite abelian quotient.

We recall Lemma 2.1.5, which can be restated using the language of tight extensions.

Lemma 4.1.2. *If* $\mathbf{e} \mapsto \widetilde{\mathbf{e}}$ *is an injective map of extensions and* $\mathbf{e}$ *is tight, then* $\widetilde{\mathbf{e}}$ *is also tight.*

A particularly interesting application of this is regarding the extensions

$$Z(N) \rightarrowtail N \twoheadrightarrow N/Z(N)$$

with N nilpotent. Using Proposition 2.1.3 we see that class-c nilpotent groups embed into tight d-extensions with class-$(c-1)$ quotients. In particular, class-2 nilpotent groups embed into tight d-ab extensions.

Conveniently, tight extensions behave well with respect to isomorphisms, as the next proposition asserts.

Proposition 4.1.3. *If* $\mathbf{t}$ *is a tight extension and* $\mathbf{e}$ *is an extension such that* $\mathbf{e} \cong \mathbf{t}$*, then* $\mathbf{e}$ *is tight.*

Proof. Assume we have the isomorphism

$$\begin{array}{lccccc} \mathbf{t}) & C & \overset{\iota}{\rightarrowtail} & E & \twoheadrightarrow & G \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\varphi} & & \downarrow \\ \mathbf{e}) & \widetilde{C} & \overset{\varepsilon}{\rightarrowtail} & \widetilde{E} & \twoheadrightarrow & \widetilde{G} \end{array}$$

of extensions with $\mathbf{t}$ is tight. Then $Z(E) = \iota(C)$. Also note that $\alpha(C) = \widetilde{C}$. Now since φ is an isomorphism of groups, we have $Z(\widetilde{E}) = \varphi(Z(E)) = \varphi(\iota(C)) = \varepsilon(\alpha(C)) = \varepsilon(\widetilde{C})$. Hence, $\mathbf{e}$ is tight. $\square$

Our main concern will be with tight d-extensions that have abelian quotients, i.e., tight d-ab extensions. In this case there is a straight-forward criterion distinguishing tight extensions based on their corresponding maps.

Let A be any abelian group and assume $\theta \in \mathrm{Hom}(Q \wedge Q, A)$. For $x \in Q$ we may define a map $\theta_x \colon Q \to A$ by $\theta_x(y) = \theta(x \wedge y)$. We have a homomorphism

$$\begin{aligned} Q &\to \mathrm{Hom}(Q, A) \\ x &\mapsto \theta_x \end{aligned}$$

depending on θ, which we denote by $\theta_\square$.

Definition 4.1.4. A map $\theta \in \mathrm{Hom}(Q \wedge Q, A)$ is called *non-degenerate* provided that $\theta_\square$ is injective. We denote the set of all non-degenerate maps $Q \wedge Q \to A$ by $\mathrm{Hom}_{\mathrm{nd}}(Q \wedge Q, A)$.

Proposition 4.1.5. *A central extension* $\mathbf{e}$ *with abelian quotient is tight if and only if* $\xi_{\mathbf{e}}$ *is non-degenerate.*

Proof. Assume $\mathbf{e}$ is the central extension

$$\mathbf{e}) \qquad C \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} A$$

with transversal τ. Put $\xi = \xi_{\mathbf{e}}$. Then ξ is defined by $\xi(x \wedge y) = \iota^{-1}\big([\tau x, \tau y]\big)$. Clearly $\xi_x = 0 \iff \tau x \in Z(G)$. Thus, if $\mathbf{e}$ is tight and $x \in A$ satisfies $\xi_x = 0$, then $\tau x \in Z(G) = C$, so that $x = 1$. Conversely, if $\mathbf{e}$ is not tight, then there is $z \in Z(G) - C$; then $\pi z \neq 0$ but $\xi_{\pi z} = 0$. $\square$

The following proposition greatly reduces the scope of possibilities for tight extensions.

Proposition 4.1.6. *Assume that* A *is a finitely generated abelian group and that*

$$\mathbf{e}\colon \qquad C \rightarrowtail T \twoheadrightarrow A.$$

is a tight extension. Let p *be any prime.*

1. *If* C *is torsion, then* A *is finite.*

2. *If A has a nontrivial p-primary component, then so does C.*

3. *If C is a p-group, then so is A.*

4. *If C is torsion free, then so is A.*

Proof. Put $\xi = \xi_{\mathbf{e}}$.

(1) Suppose A has an element, say x, of infinite order and assume C is torsion. Let $\mathscr{B}$ be a basis for A and set $\ell = \operatorname{lcm}\{|\xi(x \wedge y)| \colon y \in \mathscr{B}\}$. Then $\xi_{x^\ell} = 0$, but $x^\ell \neq 1$. This contradicts the fact that ξ is non-degenerate.

(2) Assume $x \in A$ has order p. Let A_q be the q-primary component of A for each prime q and let $\mathscr{B}_q$ be a basis for A_q. Let $\mathscr{B} = \cup_q \mathscr{B}_q$. Then $\mathscr{B}$ generates A. Since $\mathbf{e}$ is tight, there is a $y \in \mathscr{B}$ such that $\xi(x \wedge y) \neq 0$. Now, $|\xi(x \wedge y)|$ divides $\gcd\{|x|, |y|\}$, which must divide p. Since $\xi(x \wedge y) \neq 0$, we have $|\xi(x \wedge y)| = p$. The result now follows.

(3) If A is not a p-group, then either A has a nontrivial q-primary component for some prime $q \neq p$ or A is torsion free. In the first case, C also has a nontrivial q-primary component (by 2), and hence is not a p-group. If A is torsion free, then A is infinite, and by 1, the group C has an element of infinite order and thus is not a p-group.

(4) is the contrapositive of (2) applied to all primes. □

4.2 Tightening d-ab extensions

We now describe how a tight d-ab extension $\mathbf{e}_{\text{tgt}}$ can be obtained in a natural way from an arbitrary d-a extension $\mathbf{e}$.

Assume we have the d-ab extension

$$\mathbf{e}\colon \qquad D \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$$

with Q a finite abelian p-group. Put $Z = Z(G)$. Then ι factors through the inclusion $Z \hookrightarrow G$, so we may regard ι as a map $D \rightarrowtail Z$. Put $F = Z/\iota(D)$. Since D is divisible, the extension

$$\mathbf{e}\colon \qquad D \overset{\iota}{\rightarrowtail} Z \overset{\pi}{\twoheadrightarrow} F$$

(of abelian groups) splits, and there is a map $\sigma\colon Z \twoheadrightarrow D$ such that $\sigma\iota$ is the identity map on D. Put $K = \operatorname{Ker}\sigma$. Thus $Z \cong \operatorname{Im}(\iota) \times K$.

Let $G_T = G/K$. The embedding $\iota_T\colon D \rightarrowtail G_T$ given by $d \mapsto \iota(d)K$ has image Z/K, and we note that $\iota_T(D) \leq Z(G_T)$. If $xK \in Z(G_T)$, then $[G, x] \leq K \cap [G, G] \leq K \cap D = 1$, so $x \in Z$. Thus $Z(G_T) = Z/K = \iota_T(D)$. Also, $G_T/\iota_T(D) = (G/K)/(Z/K) \cong G/Z$, so, writing $A = G/Z$, we have the tight d-extension

$$\mathbf{e}_{\text{tgt}}\colon \qquad D \overset{\iota_T}{\rightarrowtail} G_T \twoheadrightarrow A.$$

Let τ be a transversal for $\mathbf{e}$. Then since $\operatorname{Ker}\pi = \operatorname{Im}\iota \leq Z$, the coset $\tau(q)Z$ is independent of τ, and we have a map $\nu\colon Q \to A$ defined by $\nu(q) = \tau(q)Z$ that depends only on $\mathbf{e}$. Evidently ν is a surjective homomorphism.

Commutativity of the following diagram is obvious:

$$\begin{array}{lccccc} \mathbf{e}\colon & D & \overset{\iota}{\rightarrowtail} & G & \overset{\pi}{\twoheadrightarrow} & Q \\ & \Vert & & \big\downarrow & & \big\downarrow \nu \\ \mathbf{e}_{\mathrm{tgt}}\colon & D & \overset{\iota_T}{\rightarrowtail} & G_T & \twoheadrightarrow & A. \end{array}$$

4.3 The tightening functor for d-ab extensions

In the appropriate setting the process described in Section 4.2 is functorial. In this section we describe the setting required for this to be true. We will see that as a result the process described in Section 4.2 respects isomorphism classes of extensions. We conclude this section by using our functor to parametrize d-a extensions up to isomorphism.

We continue to assume tacitly that Q is finite abelian, and without loss of generality a p-group.

Definition 4.3.1. For a homomorphism $\xi\colon Q \wedge Q \to D$ into any group D, we define the *set of degeneracies* of ξ to be the set

$$K_\xi = \{a \in Q\colon (\forall x \in Q)\xi(a \wedge x) = 0\}.$$

If $\xi = \xi_{\mathbf{e}}$, then we write $K_{\mathbf{e}}$ for K_ξ. We remark that K_ξ is plainly a subgroup of Q.

Proposition 4.3.2. *If* $\mathbf{e}$ *is the d-extension*

$$\mathbf{e}\colon \qquad D \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q,$$

then $\pi^{-1}(K_{\mathbf{e}}) = Z(G)$.

Proof. Put $K = K_{\mathbf{e}}$ and let g and h be in G with $\pi(g) \in K$. Let τ be a transversal for $\mathbf{e}$. Then, since $\iota(D) \leq Z(G)$ we have $[g,h] = [\tau\pi(g), \tau\pi(h)] = [\tau(k), \tau\pi(h)]$ with $k \in K$. Now, $\iota^{-1}\big([\tau(k), \tau\pi(h)]\big) = \xi(k \wedge \pi(h)) = 0$, so $[g,h] = 1$ and $g \in Z(G)$, which shows $\pi^{-1}(K_{\mathbf{e}}) \leq Z(G)$.

Conversely, let $z \in Z(G)$. Then for $y \in Q$ we have

$$\xi(\pi(z) \wedge y) = \iota^{-1}\big([\tau\pi(z), \tau(y)]\big) = \iota^{-1}\big([z, \tau(y)]\big) = 0,$$

providing the reverse containment and completing the proof. ∎

It is worth noting that, in the notation of Proposition 4.3.2, there is an extension

$$D \rightarrowtail Z(G) \overset{\pi}{\twoheadrightarrow} K, \tag{4.1}$$

which is necessarily split because $Z(G)$ is abelian and D is divisible.

Proposition 4.3.3. *Let $\xi \in \mathrm{Hom}(Q \wedge Q, D)$ with D any group. There is a group A_ξ, a surjective homomorphism $\nu_\xi\colon Q \twoheadrightarrow A_\xi$, and non-degenerate map $\xi_{tgt}\colon A_\xi \wedge A_\xi \to D$ such that $\xi = \xi_{tgt} \circ \nu_\xi^{\wedge 2}$. Moreover, if ξ factors as $\xi = \theta \circ \mu^{\wedge 2}$ with $\theta\colon \widetilde{A} \wedge \widetilde{A} \to D$ non-degenerate and μ surjective, then then there is a group isomorphism $\alpha\colon \widetilde{A} \to A_\xi$ such that $\theta = \xi_{tgt} \circ \alpha^{\wedge 2}$.*

Proof. Put $K = K_\xi$ and let $\nu\colon Q \twoheadrightarrow A$ be any surjective homomorphism with kernel K; then $A \cong Q/K$. Define A_ξ to be A. For each $a \in A$ choose $\overline{a} \in Q$ such that $\nu(\overline{a}) = a$. For $q \in Q$ define $k_q \in K$ by the condition $q = k_q\overline{\nu(q)}$.

Define $\xi_{\mathrm{tgt}}\colon A \wedge A \to D$ by $x \wedge y \mapsto \xi(\overline{x} \wedge \overline{y})$ for x and y in A and extending linearly. Then for u and v in Q we have

$$\begin{aligned}\xi(u \wedge v) &= \xi\big(k_u\overline{\nu(u)} \wedge k_v\overline{\nu(v)}\big)\\ &= \xi\big(\overline{\nu(u)} \wedge \overline{\nu(v)}\big)\\ &= \xi_{\mathrm{tgt}} \circ \nu^{\wedge 2}(u \wedge v).\end{aligned}$$

Thus, ξ_{tgt} and ν have the desired properties.

Next assume $\xi = \theta \circ \mu^{\wedge 2}$ with θ nondegenerate and $\mu\colon Q \twoheadrightarrow \widetilde{A}$. Let $k \in K$. Then $\theta\big(\mu(k) \wedge \mu(x)\big) = \xi(k \wedge x) = 0$ for all $x \in Q$, so since μ is surjective, it follows that $\mu(k) = 1$ and $K \le \mathrm{Ker}\,\mu$. Conversely, if $\widetilde{k} \in \mathrm{Ker}\,\mu$, then $\xi(\widetilde{k} \wedge x) = \theta\big(1 \wedge \mu(x)\big) = 0$ for all $x \in Q$. Thus $\mathrm{Ker}\,\mu = K = \mathrm{Ker}\,\nu$ and it follows that there is a group isomorphism $\alpha\colon \widetilde{A} \to A$ such that $\nu = \alpha\mu$. Then

$$\theta \circ \mu^{\wedge 2} = \xi_{\mathrm{tgt}} \circ \nu^{\wedge 2} = \xi_{\mathrm{tgt}} \circ \alpha^{\wedge 2} \circ \mu^{\wedge 2}$$

and since $\mu^{\wedge 2}$ is surjective, it follows that $\theta = \xi_{\mathrm{tgt}} \circ \alpha^{\wedge 2}$. □

We remark that A_ξ is necessarily a finite abelian p-group since it is a quotient of Q. In particular, $A_\xi \cong Q/K_\xi$.

Corollary 4.3.4. *If $\mathbf{e}$ is a d-ab extension with kernel D, then there is a tight d-ab extension $\mathbf{e}_{tgt}$ and a surjection $\cdot_{tgt}\colon \mathbf{e} \twoheadrightarrow \mathbf{e}_{tgt}$. Moreover, $\mathbf{e}_{tgt}$ is unique up to isomorphisms fixing D elementwise.*

Proof. Apply Proposition 4.3.3 to $\xi_{\mathbf{e}}$ to obtain a surjective homomorphism ν and non-degenerate map ξ_{tgt} such that $\xi_{\mathbf{e}} = \xi_{\mathrm{tgt}} \circ \nu^{\wedge 2}$. Let $\mathbf{e}_{\mathrm{tgt}}$ be a d-extension corresponding to ξ_{tgt}. By Proposition 2.1.8 there is a surjection $\mathbf{e} \twoheadrightarrow \mathbf{e}_{\mathrm{tgt}}$. Since ξ_{tgt} is non-degenerate, $\mathbf{e}_{\mathrm{tgt}}$ is tight.

Next, assume $(1, \gamma, \mu)\colon \mathbf{e} \twoheadrightarrow \mathbf{t}$ is a surjective map of extensions with $\mathbf{t}$ tight. Then $\xi_{\mathbf{t}}$ is non-degenerate and by Proposition 1.7.6 we have the factorization $\xi_{\mathbf{e}} = \xi_{\mathbf{t}} \circ \mu^{\wedge 2}$. Thus, Proposition 4.3.3 guarantees the existence of a group isomorphism α such that $\xi_{\mathbf{t}} = \xi_{\mathrm{tgt}} \circ \alpha^{\wedge 2}$. By Proposition 2.1.8 there is a group homomorphism φ such that $(1, \varphi, \alpha)\colon \mathbf{t} \to \mathbf{e}_{\mathrm{tgt}}$ is a map of extensions. By the 5-lemma φ is necessarily an isomorphism, which completes the proof. □

Corollary 4.3.5. *Every d-ab extension of the group D can be described as a pullback of a tight d-a extension of D along a surjective map of extensions (fixing D elementwise).*

It appears that the surjective map of Proposition 4.3.4 might be best understood in terms of an equivalence relation involving isomorphisms that fix the kernel pointwise. With this in mind we make the following definition.

Definition 4.3.6. Define the equivalence relation $\sim$ on extensions by $\mathbf{e} \sim \mathbf{e}'$ if and only if there is an isomorphism $(1, \varphi, \theta) \colon \mathbf{e} \to \mathbf{e}'$.

Corollary 4.3.7. *For a given d-ab extension $\mathbf{e}$ there is a unique class $[\mathbf{e}_{tgt}]_\sim$ of tight d-a extensions such that $\mathbf{t} \in [\mathbf{e}_{tgt}]_\sim$ if and only if there is a surjection $\mathbf{e} \twoheadrightarrow \mathbf{t}$.*

Proof. This follows immediately from Corollary 4.3.4 upon noting that, if $\mathbf{t} \cong \mathbf{e}_{\mathrm{tgt}}$ via an isomorphism fixing D, then there is a surjection $\mathbf{e} \twoheadrightarrow \mathbf{t}$ (namely the composition $\mathbf{e} \twoheadrightarrow \mathbf{e}_{\mathrm{tgt}} \xrightarrow{\cong} \mathbf{t}$). ☐

Definition 4.3.8. We refer to any extension in the class $[\mathbf{e}_{\mathrm{tgt}}]_\sim$ of extensions in Corollary 4.3.7 as a *tightening* of $\mathbf{e}$.

We now give the setting in which we shall define our tightening functor.

Definition 4.3.9. The *category of d-by-abelian extensions*, denoted $\mathscr{E}_{\text{d-ab}}$, is the category whose objects are equivalence classes of d-extensions with abelian quotient, and whose morphisms are (equivalence classes of) commutative diagrams

$$\begin{array}{ccccc} C_1 & \rightarrowtail & G_1 & \twoheadrightarrow & Q_1 \\ \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow \\ C_2 & \rightarrowtail & G_2 & \twoheadrightarrow & Q_2. \end{array}$$

for which $\varphi(Z(G_1)) \leq Z(G_2)$.

The *category of tight extensions*, denoted $\mathscr{E}_{\mathrm{tgt}}$, is the full subcategory of $\mathscr{E}_{\text{d-ab}}$ whose objects are tight extensions.

Recall that the group $K_{\mathbf{e}}$ is the set of degeneracies of $\xi_{\mathbf{e}}$. It turns out that these groups are the key to understanding the category $\mathscr{E}_{\text{d-ab}}$.

Proposition 4.3.10. *A map $(\alpha, \gamma, \beta) \colon \mathbf{e} \to \widetilde{\mathbf{e}}$ is a morphism in $\mathscr{E}_{\text{d-ab}}$ if and only if $\beta(K_\xi) \leq K_{\widetilde{\xi}}$, where $\xi = \xi_{\mathbf{e}}$ and $\widetilde{\xi} = \xi_{\widetilde{\mathbf{e}}}$.*

Proof. Consider the commutative diagram:

$$\begin{array}{cccccc} \mathbf{e}: & D & \overset{\iota}{\rightarrowtail} & G & \overset{\pi}{\twoheadrightarrow} & Q \\ & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ \widetilde{\mathbf{e}}: & \widetilde{D} & \overset{\widetilde{\iota}}{\rightarrowtail} & \widetilde{G} & \overset{\widetilde{\pi}}{\twoheadrightarrow} & \widetilde{Q}. \end{array}$$

Suppose $\beta(K_\xi) \leq K_{\widetilde{\xi}}$. We must show $\gamma\pi^{-1}(K_\xi) \leq (\widetilde{\pi})^{-1}(K_{\widetilde{\xi}})$. Take $x \in \gamma\pi^{-1}(K_\xi)$, so $x = \gamma(g)$ with $\pi(g) \in K_\xi$. Then $\widetilde{\pi}(x) = \widetilde{\pi}(\gamma(g)) = \beta(\pi(g)) \in \beta(K_\xi) \leq K_{\widetilde{\xi}}$, so that $x \in (\widetilde{\pi})^{-1}(K_{\widetilde{\xi}})$.

Conversely, assume (α, γ, β) is a morphism in $\mathscr{E}_{\text{d-ab}}$. Let $k \in K_\xi$. Then by Proposition 4.3.2 we have $k = \pi(z)$ with $z \in Z(G)$ and $\gamma(z) \in \gamma(Z(G)) \leq Z(\widetilde{G})$. Thus $\beta(k) = \beta\pi(z) = \widetilde{\pi}\gamma(z) \in \widetilde{\pi}(Z(\widetilde{G})) = K_{\widetilde{\xi}}$ and $\beta(K_\xi) \leq K_{\widetilde{\xi}}$. □

We remark that if ξ is non-degenerate then $K_\xi = 1$. Hence, for tight extensions the condition in Proposition 4.3.10 is trivial, and $\mathscr{E}_{\text{tgt}}$ is in fact a full subcategory of $\mathscr{E}$.

Theorem 4.3.11. *There is a functor*

$$\cdot_{tgt}\colon \mathscr{E}_{\text{d-ab}} \to [\mathscr{E}_{tgt}]_\sim$$

that assigns to each d-extension $\mathbf{e}$ *its class of tightenings* $[\mathbf{e}_{tgt}]_\sim$. *Moreover,* $\cdot_{tgt}$ *induces a functor*

$$[\cdot_{tgt}]_\cong\colon [\mathscr{E}_{\text{d-ab}}]_\cong \to [\mathscr{E}_{tgt}]_\cong.$$

Proof. That $\cdot_{\text{tgt}}$ is well-defined on objects follows from Corollary 4.3.7.

Let $(\delta, \gamma, \beta)\colon \mathbf{e} \to \widetilde{\mathbf{e}}$ be a morphism in $\mathscr{E}_{\text{d-ab}}$, that is, a map of extensions such that $\beta(K_\xi) \leq K_{\widetilde{\xi}}$, where $\xi = \xi_{\mathbf{e}}$ and $\widetilde{\xi} = \xi_{\widetilde{\mathbf{e}}}$. By Proposition 1.7.6 we have $\widetilde{\xi}\beta^{\wedge 2} = \delta\xi$.

For definiteness, assume $[\mathbf{e}] \in \mathscr{E}(Q, D)$ and $[\widetilde{\mathbf{e}}] \in \mathscr{E}(\widetilde{Q}, \widetilde{D})$. As in Proposition 4.3.3, write $\xi = \xi_{\text{tgt}}\nu^{\wedge 2}$ and $\widetilde{\xi} = \widetilde{\xi}_{\text{tgt}}\widetilde{\nu}^{\wedge 2}$, with $\nu\colon Q \twoheadrightarrow A$ having kernel K_ξ, and likewise $\widetilde{\nu}\colon \widetilde{Q} \twoheadrightarrow \widetilde{A}$ with kernel $K_{\widetilde{\xi}}$. Then

$$\widetilde{\xi}_{\text{tgt}}\widetilde{\nu}^{\wedge 2}\beta^{\wedge 2} = \widetilde{\xi}\beta^{\wedge 2} = \delta\xi = \delta\xi_{\text{tgt}}\nu^{\wedge 2}. \tag{4.2}$$

Now since $\beta(\mathrm{Ker}(\nu)) \leq \mathrm{Ker}(\widetilde{\nu})$, there is a unique group homomorphism $\alpha\colon A \to \widetilde{A}$ such that $\alpha\nu = \widetilde{\nu}\beta$. Thus,

$$\widetilde{\xi}_{\text{tgt}}\widetilde{\nu}^{\wedge 2}\beta^{\wedge 2} = \widetilde{\xi}_{\text{tgt}}\alpha^{\wedge 2}\nu^{\wedge 2}. \tag{4.3}$$

Putting (4.2) and (4.3) together, we have

$$\widetilde{\xi}_{\text{tgt}}\alpha^{\wedge 2}\nu^{\wedge 2} = \delta\xi_{\text{tgt}}\nu^{\wedge 2}.$$

Since $\nu^{\wedge 2}$ is surjective, we obtain

$$\widetilde{\xi}_{\text{tgt}}\alpha^{\wedge 2} = \delta\xi_{\text{tgt}}.$$

Now, if $\mathbf{e}_{\text{tgt}}$ corresponds to ξ_{tgt} and $\widetilde{\mathbf{e}}_{\text{tgt}}$ corresponds to $\widetilde{\xi}_{\text{tgt}}$, then by Proposition 2.1.8, there is a group homomorphism φ such that $(\delta, \varphi, \alpha)\colon \mathbf{e}_{\text{tgt}} \to \widetilde{\mathbf{e}}_{\text{tgt}}$ is a map of extensions. Also, any extension corresponding to ξ_{tgt} represents the tightening of $\mathbf{e}$, i.e, in this case $[\mathbf{e}_{\text{tgt}}]_\sim$ is the tightening of $\mathbf{e}$; likewise, $[\widetilde{\mathbf{e}}_{\text{tgt}}]_\sim$ is the tightening of $\widetilde{\mathbf{e}}$. Thus, passing to classes modulo $\sim$, we obtain the morphism $[(\delta, \varphi, \alpha)]_\sim$ in the category $\mathscr{E}_{\mathrm{tgt}}$. Functoriality now follows from the fact that the definition of α respects composition of functions.

Finally, the existence of the induced functor $[\cdot_{\mathrm{tgt}}]_\cong$ follows from the fact that if (δ, γ, β) is an isomorphism, then so is $(\delta, \varphi, \alpha)$. □

We now make the following important observation.

Proposition 4.3.12. *If* $\mathbf{e}$ *and* $\widetilde{\mathbf{e}}$ *are two d-extensions with tightenings* $\mathbf{e}_{tgt}$ *and* $\widetilde{\mathbf{e}}_{tgt}$ *respectively, and if* $\mathbf{e} \cong \widetilde{\mathbf{e}}$*, then* $\mathbf{e}_{tgt} \cong \widetilde{\mathbf{e}}_{tgt}$*.*

Proposition 4.3.12 follows immediately from the following lemma.

Lemma 4.3.13. *If* $(\alpha, \gamma, \beta)\colon \mathbf{e} \to \mathbf{e}'$ *is a map with* β *surjective and* α *injective, then* $\beta(K_{\mathbf{e}}) = K_{\mathbf{e}'}$*.*

Proof. Put $\xi = \xi_{\mathbf{e}}$ and $\xi' = \xi_{\mathbf{e}'}$. Choose $k \in K_\xi$. Then $\xi(x \wedge k) = \xi_{\mathrm{tgt}}(\nu_\xi(x) \wedge 0) = 0$ for all x. Then

$$\xi'_{\mathrm{tgt}}\big(\nu_{\xi'}\beta(x) \wedge \nu_{\xi'}\beta(k)\big) = \xi'(\beta(x) \wedge \beta(k)) = \alpha\xi(x \wedge k) = 0$$

for all x. Since ξ'_{tgt} is non-degenerate, we have $\nu_{\xi'}\beta(k) = 0$ and $\beta(k) \in K_{\xi'}$. Hence, $\beta(K_\xi) \leq K_{\xi'}$.

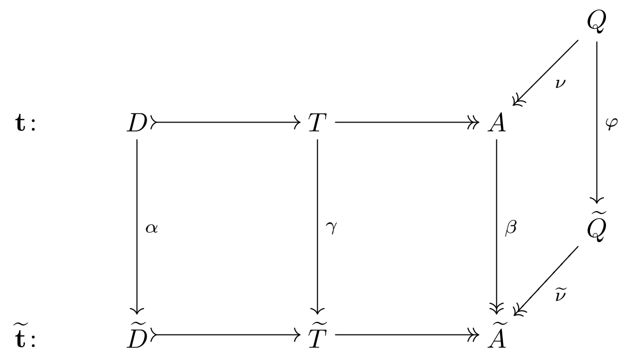
Conversely, if $k' \in K_{\xi'}$, then we may write $k' = \beta(k)$ with $k \in Q$. Then

$$\alpha\xi_{\mathrm{tgt}}\big(\nu_\xi(x), \nu_\xi(k)\big) = \alpha\xi(x, k) = \xi'\big(\beta(x), \beta(k)\big) = 0$$

for all x. Since α is injective, we conclude that $\xi_{\mathrm{tgt}}\big(\nu_\xi(x), \nu(k)\big) = 0$ for all x, and thus $\nu_\xi(k) = 0$. Hence, $k' \in \beta(K_\xi)$. □

Corollary 4.3.14. *If* Φ *is a map of d-by-a extensions and* Φ *is either an injection or a surjection, then* Φ *is a morphism in the category* $\mathscr{E}_{\text{d-ab}}$*. In particular, isomorphisms of extensions are morphisms in the category* $\mathscr{E}_{\text{d-ab}}$*. Also, the maps of Propositions 2.1.3 and 4.3.4 are morphisms in the category* $\mathscr{E}_{\text{d-ab}}$*.*

Definition 4.3.15. Let $\mathbf{t}$ and $\widetilde{\mathbf{t}}$ be tight extensions by A and $\widetilde{A}$, respectively. Let Q and $\widetilde{Q}$ be groups and let $\nu\colon Q \twoheadrightarrow A$ and $\widetilde{\nu}\colon \widetilde{Q} \twoheadrightarrow \widetilde{A}$. We say the pairs $(\mathbf{t}, \nu)$ and $(\widetilde{\mathbf{t}}, \widetilde{\nu})$ are *isomorphic* and write $(\mathbf{t}, \nu) \cong (\widetilde{\mathbf{t}}, \widetilde{\nu})$ provided that there is an isomorphism $(\alpha, \gamma, \beta)\colon \mathbf{t} \to \widetilde{\mathbf{t}}$ and an isomorphism $\varphi\colon Q \to \widetilde{Q}$ such that the following diagram commutes:

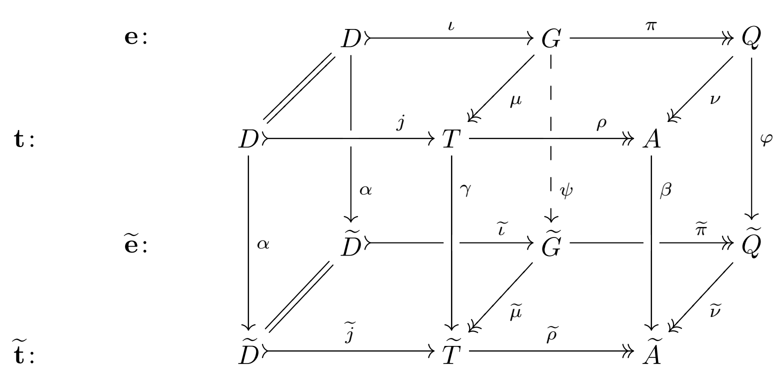


We now state our main theorem of this section.

Theorem 4.3.16. *Let $\mathscr{T}$ be the set of isomorphism classes $[(\mathbf{t},\nu)]_{\cong}$ such that $\mathbf{t}$ is a tight d-ab extension and ν is a surjective homomorphism with codomain equal to the quotient of $\mathbf{t}$. Then the function $f\colon \mathscr{T} \to \mathrm{obj}\big([\mathscr{E}_{\text{d-a}}]_{\cong}\big)$ defined by $f\colon [(\mathbf{t},\nu)]_{\cong} \mapsto \nu^*(\mathbf{t})$ gives a parametrization of d-ab extensions up to isomorphism.*

Proof. To see that f is well-defined, assume we have an isomorphism $(\mathbf{t},\nu) \cong (\widetilde{\mathbf{t}},\widetilde{\nu})$ with notation as in Definition 4.3.15 above. Put $\mathbf{e} = \nu^*(\mathbf{t})$ and $\widetilde{\mathbf{e}} = \widetilde{\nu}^*(\widetilde{\mathbf{t}})$.

Consider the diagram



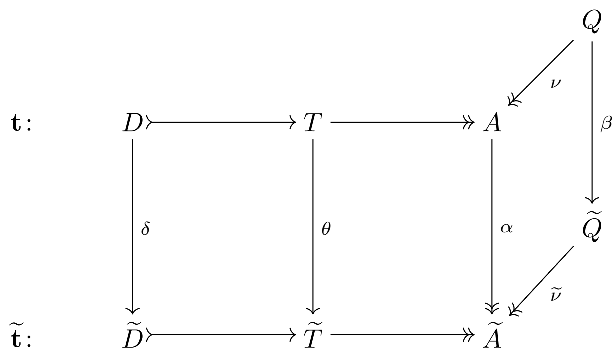
where ψ is defined using the universal property of pullbacks in the category **Grp**. That is, we recall that $\widetilde{G}$ is the pullback in the category **Grp** of the maps $\widetilde{\rho}$ and $\widetilde{\nu}$. What is required is that the maps $\varphi\pi\colon G \to Q$ and $\gamma\mu\colon G \to \widetilde{T}$ have the property that $\widetilde{\nu}\varphi\pi = \widetilde{\rho}\gamma\mu$. Indeed, we have $\widetilde{\rho}\gamma\mu = \beta\rho\mu = \beta\nu\pi = \widetilde{\nu}\varphi\pi$. Commutativity of the right cube follows immediately.

To see commutativity of the left cube, note that by the properties of pullbacks, $\mathrm{Ker}(\widetilde{\mu}) \cap \mathrm{Ker}(\widetilde{\pi}) = 1$. Thus, it is enough to verify $\widetilde{\mu}\psi\iota = \widetilde{\mu}\widetilde{\iota}\alpha$ and $\widetilde{\pi}\psi\iota = \widetilde{\pi}\widetilde{\iota}\alpha$. Indeed, $\widetilde{\pi}\psi\iota = \varphi\pi\iota = 0 = \widetilde{\pi}\widetilde{\iota}\alpha$, and

$$\widetilde{\mu}\psi\iota = \gamma\mu\iota = \gamma j = \widetilde{j}\alpha = \widetilde{\mu}\widetilde{\iota}\alpha.$$

Finally, since α and φ are isomorphisms, the Five Lemma guarantees that ψ is an isomorphism. Hence, $\mathbf{e} \cong \widetilde{\mathbf{e}}$ and f is well defined.

Now suppose $(\delta, \gamma, \beta)\colon \mathbf{e} \to \widetilde{\mathbf{e}}$ is an isomorphism. Following the proof of Theorem 4.3.11 we obtain a commutative diagram



Thus, $(\mathbf{t}, \nu) \cong (\widetilde{\mathbf{t}}, \widetilde{\nu})$, so $[(\mathbf{t}, \nu)]_{\cong}$ is an isomorphic invariant for $\mathbf{e}$, and f is injective.

Finally, by Corollary 4.3.4 every d-ab extension is of the form $\nu^*(\mathbf{t})$ with $\mathbf{t}$ tight and ν a surjection. Hence, f is a parametrization. □

4.4 The fibres of $\cdot_{\mathbf{tgt}}$

We now wish to consider what properties an extension $\mathbf{e}$ has if its tightening is given. That is, we consider the fibres of $\cdot_{\mathrm{tgt}}$ over a fixed class $[\mathbf{t}]_\sim$ of tight extensions. We therefore fix the tight extension

$$\mathbf{t}\colon \qquad D \rightarrowtail T \xrightarrow{\pi_{\mathbf{t}}} A.$$

Here we may assume that $D \leq T$ and $A = T/D$ up to isomorphism. Let $\nu\colon Q \twoheadrightarrow A$, where Q is an abelian group, and consider the pullback extension $\mathbf{e} = \nu^*(\mathbf{t})$. In other words, we wish to consider the surjection of extensions given by the diagram

$$\begin{array}{lccccc}
\mathbf{e}\colon & D & \overset{\iota}{\rightarrowtail} & G & \overset{\pi}{\twoheadrightarrow} & Q \\
& \| & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\nu} \\
\mathbf{t}\colon & D & \overset{\iota_{\mathbf{t}}}{\rightarrowtail} & T & \overset{\pi_{\mathbf{t}}}{\twoheadrightarrow} & A.
\end{array}$$

The extension $\mathbf{e}$ is uniquely determined (up to equivalence) by $\mathbf{t}$ and the surjective homomorphism ν, namely,

$$G \cong \{(x, q) \in T \times Q\colon \pi_{\mathbf{t}}(x) = \nu(q)\},$$

with the obvious maps in $\mathbf{e}$ (cf. Section 1.4 on pullbacks).

Let $S = T \times Q$. The isomorphism above is made explicit with the embedding $G \to S$ given by $g \mapsto \big(\varphi(g), \pi(g)\big)$. To see this embedding another way, put

$Z = Z(G)$ and recall that the extension

$$\mathbf{e}\colon \qquad D \overset{\iota}{\rightarrowtail} Z \twoheadrightarrow Z/\iota(D)$$

splits, so there is a homomorphism $\sigma\colon Z \twoheadrightarrow D$ such that $\sigma\iota$ is the identity on D. If $K = \mathrm{Ker}(\sigma)$, then $Z = \iota(D) \times K$. In particular, $\iota(D) \cap K = 1$. Thus, there is an embedding $G \rightarrowtail G/K \times G/\iota(D)$ given by $g \mapsto \big(gK, g\iota(D)\big)$. Recall from Section 4.2 that $T \cong G/K$, and since $Q \cong G/\iota(D)$, we have $S \cong G/K \times G/\iota(D)$, giving the embedding $G \rightarrowtail S$ (which is the same map as above).

For a subgroup $H \leq G$, we let $\overline{H}$ denote its embedded image in S. With this notation we have:

Proposition 4.4.1. *For $q \in Q$ choose $t_q \in T$ such that $\pi_{\mathbf{t}}(t_q) = \nu(q)$. Then*

$$\overline{G} = (D \times 1) \cdot \{(t_q, q) \colon q \in Q\}.$$

Proof. For $d \in D$ and $q \in Q$ we have $\pi_{\mathbf{t}}(dt_q) = \pi_{\mathbf{t}}(t_q) = \nu(q)$, so $(dt_q, q) \in \overline{G}$ and $(D \times 1) \cdot \{(t_q, q) \colon q \in Q\} \leq \overline{G}$.

Conversely, let $g \in G$. Put $q = \pi(g)$. Then $\pi_{\mathbf{t}}(t_q) = \nu\pi(g) = \pi_{\mathbf{t}}\varphi(g)$, so there exists $d \in D$ such that $\varphi(g) = dt_q$. Thus, $\big(\varphi(g), \pi(g)\big) = \big(d, 1\big)\big(t_q, q\big)$. $\square$

Next, let $F = \mathrm{Ker}(\nu)$ and note that $\pi^{-1}(F) = Z$ by Proposition 4.3.2. Thus, $K \cong \pi(Z) \cong F$.

The next result gives information about the image $\overline{G}$ of G in S. To simplify notation, if $H \leq T$ we also write H for the corresponding subgroup $H \times 1 \leq S$, and likewise for subgroups of Q.

Proposition 4.4.2. *The group $\overline{G}$ has the following properties:*

1. *$\overline{G} \cap Q = F$,*
2. *$\overline{G} \cap T = D$,*
3. *$\overline{G} \trianglelefteq S$, and*
4. *$\overline{G}T = S = \overline{G}Q$.*

Proof. 1. For $g \in G$, $\big(\varphi(g), \pi(g)\big) \in Q \iff \varphi(g) = 1 \Rightarrow \nu(\pi(g)) = \pi_{\mathbf{t}}(\varphi(g)) = 1$ and $\pi(g) \in \mathrm{Ker}(\nu) = F$. Conversely, for $k \in F$ we obviously have $(1, k) \in \overline{G} \cap Q$.

2. If $\big(\varphi(g), \pi(g)\big) \in T$, then $\pi(g) = 1$, so $g \in \iota(D)$ and $\big(\varphi(g), \pi(g)\big) \in D \times 1$. The containment $D \times 1 \leq \overline{G} \cap T$ is obvious.

3. Since Q is abelian we have $S' = T' \times 1 = \varphi(G') \leq \overline{G}$, which implies that $\overline{G} \trianglelefteq S$.

4. Let $(t, q) \in S$ and choose $g \in G$ such that $\pi(g) = q$. Then

$$\big(t, q\big) = \big(\varphi(g), \pi(g)\big)\big(\varphi(g^{-1})t, 1\big) \in \overline{G}T.$$

Also, if $h \in G$ is such that $\varphi(h) = t$, then

$$\big(t, q\big) = \big(\varphi(h), \pi(h)\big)\big(1, \pi(h^{-1})q\big) \in \overline{G}Q.$$

□

4.5 Tight d-ab extensions of prüfer groups

In this section we give a classification of tight d-ab extensions up to isomorphism. By Proposition 4.1.6, there are no such extensions unless the quotient is a finite p-group. Fix $D = \mathbb{Z}(p^\infty)$ and assume Q is a finite abelian p-group. We will establish the following:

Theorem 4.5.1. *Assume Q is a finite abelian p-group such that each homocyclic component of Q has even rank. Then there is a unique isomorphism class of tight extensions of $\mathbb{Z}(p^\infty)$ by Q. If some homocyclic component has odd rank, then there are no tight extensions of $\mathbb{Z}(p^\infty)$ by Q.*

We give a nice presentation for a tight extension of $\mathbb{Z}(p^\infty)$ by Q in the case when a tight extension exists. For definiteness assume that

$$Q \cong (\mathbb{Z}/p^{e_1})^{r_1} \times \cdots \times (\mathbb{Z}/p^{e_a})^{r_a}$$

with $0 < e_1 < \cdots < e_a$ and r_i even for each i. Put $r = \operatorname{rank}(Q) = \sum_i r_i$. Recall that if $\mathscr{B}$ is a basis for Q we write

$$\mathscr{B}[p^{e_i}] = \{x \in \mathscr{B} \colon |x| = p^{e_i}\}.$$

For non-degenerate maps we can improve Lemmas 3.2.2 and 3.3.1.

Lemma 4.5.2. *Let $\xi \in \operatorname{Hom}_{\mathrm{nd}}(Q \wedge Q, D)$ and let $\mathscr{D}$ be a p-inductive set in D. There is a basis $\mathscr{B}$ of Q such that*

1. *ξ pairs $\mathscr{B}$ on $\mathscr{D}$;*
2. *if $x, y \in \mathscr{B}$ and $\xi(x \wedge y) \neq 0$ then $|x| = |y| = |\xi(x \wedge y)|$.*

Proof. The proof is by induction on r. If $r = 0$, then the result is trivial. Assume that $r \geq 1$ and that our result holds for groups of rank $< r$.

Choose a basis $\widetilde{\mathscr{B}}$ for Q. Let $x \in \widetilde{\mathscr{B}}[p^{e_a}]$. If $m = \max\{|\xi(x \wedge y)| \colon y \in \widetilde{\mathscr{B}}\}$, then $\xi(x^m, y) = 0$ for all $y \in Q$. Since ξ is non-degenerate, $m = |x| = p^{e_a}$, i.e., there is $x' \in \widetilde{\mathscr{B}}$ such that $\xi(x \wedge x')$ has order p^{e_a}. We remark that necessarily $r \geq 2$ and that $x' \in \widetilde{\mathscr{B}}[p^{e_a}]$. Write $\xi(x \wedge x') = \lambda d_a$ with $\lambda \in (\mathbb{Z}/p^a)^\times$. If $\lambda \neq 1$ then we can replace x by the element $(1/\lambda)x$ and retain all the properties of x mentioned so far. Hence, we assume without loss of generality that $\xi(x \wedge x') = d_a$.

Next, for $y \in \widetilde{\mathscr{B}} - \{x, x'\}$, write $\xi(x \wedge y) = \lambda_y d_{i_y}$ with $\lambda_y \in (\mathbb{Z}/p^{i_y})^\times$. Put

$$y' = y - \lambda_y p^{a - i_y} x'.$$

It is easy to see that the set

$$\mathscr{B}' = \{y' : y \in \widetilde{\mathscr{B}} - \{x, x'\}\} \cup \{x, x'\}$$

is a basis for Q since $|y'| = \operatorname{lcm}\{|y|, \lambda_y p^{a-i_y}|x'|\} = p^{i_y} = |y|$ and $\langle \mathscr{B}' \rangle = Q$. Also, $\xi(x \wedge y') = 0$ for each $y' \in \mathscr{B}' - \{x, x'\}$.

Similarly, we consider the terms $w \wedge x'$ with $w \in \mathscr{B}' - \{x, x'\}$. We set $\xi(w \wedge x') = \mu_w d_{j_w}$, put

$$w'' = w - \mu_w p^{a-j_w} x$$

and note

$$\mathscr{B}'' = \{w'' : w \in \mathscr{B}' - \{x, x'\}\} \cup \{x, x'\}$$

is a basis for Q. Moreover, for all $w'' \in \mathscr{B}'' - \{x, x'\}$ we have $\xi(x \wedge w'') = 0 = \xi(w'', x')$.

Let $\widetilde{Q} = \langle \mathscr{B}'' - \{x, x'\} \rangle$. Now ξ can be decomposed as a sum

$$\xi = \sum_{\{u,v\} \subset \mathscr{B}} \xi_{u \wedge v},$$

where $\xi_{u \wedge v}$ is the restriction of ξ to the one-dimensional subspace $\langle u \wedge v \rangle$.

For $u \in \mathscr{B}'' - \{x, x'\}$ we have $\xi|_{x \wedge u} = 0 = \xi|_{u \wedge x'}$. Consequently,

$$\xi = \xi_{x \wedge x'} + \xi'$$

where

$$\xi' = \sum_{\{u,v\} \subset \mathscr{B} - \{x, x'\}} \xi_{u \wedge v},$$

Note that $\xi' : \widetilde{Q}^{\wedge 2} \to D$. Since $\widetilde{Q}$ has rank $r - 2$, we may apply our induction hypothesis to ξ' to obtain a basis $\mathscr{B}_2$ for $\widetilde{Q}$ with the properties of the lemma with respect to ξ'. The basis $\mathscr{B} = \mathscr{B}_2 \cup \{x, x'\}$ for Q then has the desired properties with respect to ξ. □

Proof of Theorem 4.5.1. First, we prove that if a tight extension of D by Q exists, then all homocyclic components of Q have even rank. Assume $\mathbf{t}$ is a tight extension of D by Q. Let $\mathscr{B}$ be a basis with the properties described in Lemma 4.5.2 with for the map $\xi_{\mathbf{t}} \in \operatorname{Hom}(Q \wedge Q, D)$. Consider one of the sets $\mathscr{B}[p^{e_i}]$ which forms a basis for the homocyclic component Q_i of Q; thus $\operatorname{rank}(Q_i) = r_i$. Now, for each $x \in \mathscr{B}[p^{e_i}]$ there is a unique $x' \in \mathscr{B}[p^{e_i}]$ such that $\xi_{\mathbf{t}}(x \wedge x') \neq 0$. Of course, $x' \neq x$, and since $x'' = x$ we see that $\mathscr{B}[p^{e_i}]$ is partitioned into pairs, and hence has an even number of elements.

Now assume that the homocyclic components of Q all have even ranks. We show that there is a tight extension of D by Q with a nice presentation.

Let T be the group given by the set of generators

$$\{x_1^{(1)}, \ldots, x_{r_1}^{(1)}\} \cup \{x_1^{(2)}, \ldots, x_{r_2}^{(2)}\} \cup \cdots \cup \{x_1^{(a)}, \ldots, x_{r_a}^{(a)}\} \cup \{d_1, d_2, \ldots\},$$

subject to the defining relations

$$\begin{aligned}
&d_1^p = 1 \text{ and } d_i^p = d_{i-1} \text{ for } i \geq 2;\\
&(x_i^{(j)})^{p^{e_j}} = 1 \text{ for } 1 \leq i \leq r_j;\\
&[x_i^{(k)}, d_j] = 1 \text{ for all } i,\ j, \text{ and } k;\\
&[x_i^{(k)}, x_j^{(\ell)}] = 1 \text{ for all } i,\ j,\ k, \text{ and } \ell, \text{ with the following exceptions:}\\
&[x_1^{(1)}, x_2^{(1)}] = [x_3^{(1)}, x_4^{(1)}] = \cdots = [x_{r_1-1}^{(1)}, x_{r_1}^{(1)}] = d_{e_1};\\
&\quad\vdots \qquad\qquad\qquad \vdots\\
&[x_1^{(a)}, x_2^{(a)}] = [x_3^{(a)}, x_4^{(a)}] = \cdots = [x_{r_a-1}^{(a)}, x_{r_a}^{(a)}] = d_{e_a}.
\end{aligned}$$

Let $\mathscr{D} = \{0, c_1, c_2, \ldots\}$ be a p-inductive set in D and $\widetilde{\mathscr{B}}$ a basis for Q. Then by Lemma 3.1.2, the map $\iota\colon D \to T$ given by $\iota\colon \lambda c_i \mapsto \lambda d_i$ is an injective homomorphims. Obviously $T/\langle d_1, d_2, \ldots\rangle \cong Q$. Explicitly, if $\widetilde{\mathscr{B}}[p^{e_i}] = \{\widetilde{b}_1^{(i)}, \ldots, \widetilde{b}_{r_i}^{(i)}\}$, then we have the map $\pi\colon T \twoheadrightarrow Q$ defined by $\pi\colon x_i^{(j)} \mapsto \widetilde{b}_i^{(j)}$ for each relevant i and j and $\pi\colon d_k \mapsto 1$ for each k. Hence, we obtain the tight extension

$$\mathbf{e}\colon \qquad D \overset{\iota}{\rightarrowtail} T \overset{\pi}{\twoheadrightarrow} Q.$$

Now let $\mathbf{t}$ be any tight extension of D by Q. Let $\mathscr{B}$ be a basis with the properties of Lemma 4.5.2 with respect to $\xi_{\mathbf{t}}$ and the p-inductive set $\mathscr{D}$. For each i we can write $\mathscr{B}[p^{e_i}] = \{b_1^{(i)}, \ldots, b_{r_i}^{(i)}\}$ so that $\xi_{\mathbf{t}}(b_j^{(i)} \wedge b_{j+1}^{(i)}) = c_i$ for odd j. Let $\beta \in \mathrm{Aut}(Q)$ be the automorphism defined by the change of basis $\mathscr{B} \to \widetilde{\mathscr{B}}$.

We note that for odd j and each i we have

$$\xi_{\mathbf{e}}(\widetilde{b}_j^{(i)} \wedge \widetilde{b}_{j+1}^{(i)}) = c_i$$

(with all other pairs of elements from $\widetilde{\mathscr{B}}$ mapping to 0). Thus it is apparent that $\xi_{\mathbf{e}} = \xi_{\mathbf{e}} \circ \beta^{\wedge 2}$, so that $\mathbf{t} \cong \mathbf{e}$. □

We note that the group T constructed in the proof of Theorem 4.5.1 is a special case of the groups $G(\mathscr{A})$ defined in Section 3.1. In the case of T, $\mathscr{A}$ consists of the single block diagonal matrix

$$\begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & M_a \end{bmatrix},$$

where M_i is the $r_i \times r_i$ block diagonal matrix with 2×2 blocks

$$\begin{bmatrix} 0 & p^{e_a - e_i} \\ -p^{e_a - e_i} & 0 \end{bmatrix}$$

for each $i = 1, \ldots, a$.

Also, the group T is a special case of the family of groups $G_{F,\mathscr{F},\mathscr{X}}$ constructed in Definition 3.2.4, where $F = 1$.

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