CHARACTERISTIC MODES FOR IMPEDANCE MATCHING AND BROADBANDING OF ELECTRICALLY SMALL ANTENNAS

BY

JACOB J. ADAMS

DISSERTATION

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Doctoral Committee:

Professor Jennifer T. Bernhard, Chair
Professor Andreas C. Cangellaris
Professor Steven J. Franke
Professor Jianming Jin
ABSTRACT

Antennas smaller than a quarter wavelength are fundamentally constrained in a variety of ways. One of the more problematic limitations is that the antenna’s bandwidth declines sharply as the size of the antenna decreases. Myriad studies have sought antennas that perform close to the fundamental limits, and they use a patchwork of good and bad design approaches. Our primary goal is to describe a new, complete framework to model the fundamental behavior of small antennas. We base our analysis in characteristic mode theory which allows us to decompose the antenna behavior into the behavior of a few well-defined modes. Using this decomposition we can better understand, design, and analyze small antennas. First, we explain a unified approach to model the antenna input impedance, rather than the haphazard array of approaches that are currently used. Using our model for the input impedance, we are then able to establish the conditions under which a small antenna can be effectively impedance matched, and analyze some simple methods for matching an antenna without using an external matching network. Through this study, we find that near-optimum modes actually exist in nearly every geometry but are often masked by higher order modes. From this result, a new design paradigm is proposed in which designs seek to couple into these existing modes and match using the simple methods described herein, rather than creating ever more complex and impractical structures.

We also design and fabricate two novel, spherical, electrically small antennas, the TM$_{10}$ antenna and the spherical meanderline antenna. Both of these antennas exhibit quality factor close to the lower limit, and hence, a near-optimum bandwidth. The spherical meanderline antenna is particularly well-suited for automated fabrication and can achieve bandwidth comparable to the best known values. In collaboration with materials scientists, we demonstrate the spherical meanderline antenna, which is one of the first
microwave structures printed on a curved surface using a direct-ink write process.

Finally, to circumvent some of the bandwidth limitations imposed on small antennas, we propose an approach to design multimode antennas. Estimates are derived for the bandwidth increases that can be achieved with this approach to antenna broadbanding, and a simple figure of merit is suggested. A case study in broadbanding the TM$_{10}$ antenna provides some idea of what types of modal combinations are practical. Finally, a multimode spherical meanderline antenna matched with the simple techniques described herein is designed and fabricated.
To Mom, Dad, and Theresa
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CHAPTER 1

INTRODUCTION

Emerging communication, sensing, and tracking applications continue to require smaller and smaller antennas, driven by the final form factor of wireless devices. However, while many electronic components benefit from rapidly decreasing size according to Moore’s law, antennas face miniaturization limitations. When their sizes are below a quarter-wavelength, they are considered “electrically small,” and they will suffer from reduced gain, efficiency, system range, and bandwidth [1].

These limitations were rigorously derived in 1948 by Chu [2] although Wheeler had developed some approximate relations a year earlier [3]. We will discuss the electrically small antenna limitations in further detail in Chapter 2. Since the late 1940s, engineers have sought an antenna that approaches or exceeds the size-bandwidth performance limit described by Chu [2], but performance has always fallen short. Because applications of electrically small antennas (ESAs) are so numerous, the field of electrically small antenna design is filled with questionable design approaches. Too many of the designs reveal a lack of understanding of, or total disregard for, electromagnetic theory in an attempt to circumvent physics. These are what Hansen refers to as “pathological antennas” [1], a term which can also be extended to include design methods. Common offenders in this regard are genetically optimized antennas, fractal antennas, metamaterial structures, and antennas with resistive components for increased bandwidth.

This is not to say that these approaches have no merit. In cases where the physics of the problem is too complex to model, numerical optimization plays an important role, but a numerically optimized design gives little physical insight that can be carried over to similar problems. Using fractals is an interesting approach but ultimately offers no benefit for small antennas. The term metamaterial can refer to a range of infinite periodic structures with unusual electromagnetic properties. However, in the case of
small antennas, quasi-infinite periodic structures are not realizable, but the term “metamaterial-inspired” is instead used to describe traditional parasitic coupling. While there are potential applications for this parasitic coupling approach, we believe that framing it as a “metamaterial” approach is detrimental to understanding the real phenomena at work and to the underlying research. Finally, the approach of using resistive components to increase bandwidth is simply poor design. The bandwidth can always be lowered by dissipating power in resistors to reduce the radiation efficiency of the antenna, but in almost no cases is this a desirable tradeoff.

Our goal in this work is to take a more fundamental approach to small antenna design. The research described here provides a framework for understanding ESAs both in terms of their matching behavior and their impedance bandwidth performance. As we will discuss in Chapter 3, characteristic mode theory tells us that the total response of an antenna can be decomposed into individual characteristic modes. Using this decomposition method, it is easier to study the behavior of the individual modes and understand the apparent antenna response. From the characteristic mode analysis, we can develop a general approach to understanding small antenna physics, including how to model input impedance, perform simple impedance matching without external circuitry, and extend the bandwidth of small antennas by combining modes. Furthermore, our study of the modes leads to some interesting conclusions about the quality factor of the fundamental mode, and suggests a new paradigm for designing small antennas.

In addition to these theoretical studies, we also fabricate a variety of small antennas whose characteristics are described from a fundamental viewpoint. In particular, we fabricate several three-dimensional antennas, which can be shown to achieve greater bandwidth than planar or linear wire antennas. Some of the antennas in this work demonstrate the application of a new process for depositing conductive ink on curved surfaces, which represents a significant practical advance for conformal antenna fabrication.

1.1 Organization of the Document

In Chapter 2 we review the fundamental limitations of small antennas in terms of the quality factor as developed by several authors over the past
60 years. We then discuss some common electrically small antenna design techniques and the state of the art in small antenna design. In Chapter 3, we introduce the theory of characteristic modes, which is the primary analysis method used in this work. This method has been significantly underutilized since it was introduced in the 1970s, and we demonstrate its benefits through the analysis of several antennas. However, we first review the theory and some of the practical aspects behind the calculation of characteristic modes. Applying this rigorous modal approach to analysis of both single and multiresonant antennas provides an enhanced understanding of small antenna behavior and properties.

To elucidate how the modes govern the antenna’s behavior, we first consider the intrinsic relationship between the antenna’s modes and its driving point impedance in Chapter 4. This relationship then suggests a unified approach to modeling small antenna impedance. Also, since small antenna modes are naturally poorly matched to typical system impedances, we analyze a commonly used matching technique, shunt stub matching, and gain new insight using the modal framework. Finally, we discover an interesting property of the fundamental characteristic mode, and suggest a new approach to designing small antennas.

In Chapter 5 we begin from basic principles to design a near-optimal single mode antenna by identifying a low quality factor ($Q$) current distribution and developing a structure to support that distribution. The antenna is fabricated and tested, confirming that the design does have a $Q$ that approaches the lower limit. We then apply the theory of characteristic modes to reveal the behavior of individual antenna modes underlying the aggregate response. The study of the antenna modes reveals another matching technique that is applicable for a variety of structures. Next, we re-design the structure so that it can be fabricated using a conformal printing method, rather than by hand. The ability to modify the design without any measurable change in the antenna’s behavior underscores the benefits of designing the antenna from fundamental principles. This also demonstrates the first application of this three-dimensional direct-write printing technology to microwave structures.

Finally, in Chapter 6 we discuss an approach for extending ESA bandwidth beyond Chu’s limit using multiple modes. Using this approach and our new understanding of the modal behavior, we provide estimates of the achievable bandwidth when multiple radiating modes are combined. We find that by
using two characteristic modes to bring a resonance and antiresonance together with similar impedances, we can create wideband performance that is significantly better than the traditional limitations suggest. The design approach is applied to the TM$_{10}$ antenna designed in previous chapters for single mode operation. When updated to take advantage of multiple radiating modes, the antenna’s bandwidth can be significantly increased. Finally, we conclude with a summary of the contributions of this work to date and a roadmap for future research directions.
CHAPTER 2

FUNDAMENTAL LIMITATIONS OF ELECTRICALLY SMALL ANTENNAS

Electrically small antennas (ESAs) have long been studied in the antenna community because of the obvious advantages of miniaturization and the challenges of small antenna design. An antenna is typically considered to be electrically small when its maximum dimension is less than \( \frac{1}{\pi} \lambda_0 \) (where \( \lambda_0 \) is the free space wavelength at the center of the frequency band). Antennas operating below this size are plagued with some combination of low efficiency, small bandwidth, low input impedance, and high input reactance [1]. ESAs have been studied since the late 1940s when Wheeler published one of the first papers in the area [3]. Comparing small antennas to capacitors and inductors, Wheeler suggested that small radiation resistance and large reactances were fundamental properties of ESAs and that they would result in low efficiency and bandwidth. In this chapter, these limitations of small antennas will be discussed and some small antenna geometries introduced.

2.1 Quality Factor for Antennas

One of the key parameters for electrically small antennas is the radiation quality factor \((Q)\). The use of \(Q\) comes from resonant circuits and is based on the idea that antenna input impedance can be represented by a single parallel or series RLC circuit. Like the \(Q\) of resonant circuits, the \(Q\) of an antenna represents the ratio of the energy stored per cycle \((W_{\text{max}})\) to the dissipated power \((P_d)\). If we assume the antenna is lossless, then the dissipated power is equal to the radiated power \((P_{\text{rad}})\) and \(Q\) is defined as

\[
Q_{\text{rad}} = \frac{2 \pi_0 |W_{\text{max}}|}{P_{\text{rad}}}. \tag{2.1}
\]

When the antenna is modeled as a single resonator with quality factor \(Q \gg 1\), the antenna’s fractional impedance bandwidth is inversely proportional to
Q. For $Q \gg 1$, the fractional bandwidth defined at the s:1 VSWR level is [4]

$$\text{FBW} = \frac{s-1}{Q\sqrt{s}}. \quad (2.2)$$

Thus, a low $Q$ antenna is desirable for many applications.

Recently Yaghjian and Best showed that $Q$ can be calculated from the impedance $Z = R + jX$ and its frequency derivative as [4]

$$Q(\omega_0) = \frac{\omega_0}{2R(\omega_0)} \left[ \left( \frac{\partial R(\omega_0)}{\partial \omega} \right)^2 + \left( \frac{\partial X(\omega_0)}{\partial \omega} + \frac{|X(\omega_0)|}{\omega_0} \right)^2 \right] \quad (2.3)$$

and it can be dually calculated from the admittance $Y = G + jB$ as

$$Q(\omega_0) = \frac{\omega_0}{2G(\omega_0)} \left[ \left( \frac{\partial G(\omega_0)}{\partial \omega} \right)^2 + \left( \frac{\partial B(\omega_0)}{\partial \omega} + \frac{|B(\omega_0)|}{\omega_0} \right)^2 \right]. \quad (2.4)$$

These equations can be used interchangeably, but in resonant regions where $\frac{\partial G}{\partial \omega_0}$ and $\frac{\partial B}{\partial \omega_0}$ are changing quickly and difficult to estimate for sampled data, Equation 2.3 provides more accurate results. The opposite is true in antiresonant regions where $\frac{\partial Z}{\partial \omega_0}$ is rapidly varying.

### 2.1.1 Chu’s limit for small antennas

An early seminal work by Chu [2] showed that the minimum radiation quality factor ($Q$) of an electrically small antenna is constrained by the size of the antenna. The size of the antenna is defined by the product, $ka$, where $k$ is the wavenumber at the center of the band and $a$ is the radius of the minimum sphere that circumscribes the antenna. Figure 2.1 illustrates the dimension $a$. By considering circuit models of the spherical modes that could exist external to such a sphere, Chu determined the minimum $Q$ to be [2, 5, 4]

$$Q_{\text{Chu}} = \eta \left( \frac{1}{ka} + \frac{1}{(ka)^3} \right) \quad (2.5)$$

where $\eta$ is the antenna efficiency. We will refer to this bound as the Chu limit.

The minimum $Q$ then increases cubically as the electrical size of the antenna goes to zero. Thus, the achievable bandwidth of an ESA is small and drops very rapidly as the size decreases. The efficiency of the antenna also
has a significant effect on the attainable $Q$. Antennas can have very low $Q$ if they are very inefficient. Obviously, low efficiency is not desirable, so this is not generally an acceptable approach.

2.1.2 Thal’s limit for currents on a spherical surface

In 2006, Thal revisited the small antenna $Q$ limit [6] and showed that the minimum achievable $Q$ is actually higher than suggested by Chu. Chu’s model did not consider energy stored within the circumscribing sphere, as Chu himself observed. The additional stored energy increases the $Q$, so the actual bound must be higher than Chu calculated. Thal addressed this shortcoming by deriving a modal circuit model which also includes internal stored energy [6]. As a simplification, Thal considered only currents on a spherical surface, and expanded the fields inside the circumscribing sphere, resulting in a higher limit on the minimum $Q$. Figure 2.2 shows Chu’s original circuit for the TM$_{10}$ spherical mode and the internally stored energy that was included by Thal.

For the lowest order spherical modes, TM$_{1n}$ and TE$_{1n}$, Thal calculated ratios

$$ Q_{TM_{1n}} = 1.5 \times Q_{Chu} \quad \text{ka} \to 0 \quad (2.6) $$
$$ Q_{TE_{1n}} = 3 \times Q_{Chu} \quad \text{ka} \to 0 \quad (2.7) $$

The higher order TM$_{mn}$ and TE$_{mn}$ modes have higher $Q$ than these low order
Figure 2.2: Circuit model of the TM_{10} spherical mode. Chu’s circuit model [2] is shown to the left of the generator. Thal’s extended circuit model which includes energy stored inside the radius-$a$ sphere [6] is shown to the right of the generator.

2.1.3 Gustafsson’s limit for arbitrary antennas

In two recent papers [7, 8], Gustafsson, Sohl, and Kristensson developed a method of calculating the lower bound on $Q$ for arbitrary antenna geometries. Prior to their work, lower bounds on $Q$ had used a sphere as the circumscribing geometry. Since the standard measure of the size of the antenna has been viewed in terms of a sphere, it is no surprise that we will find in the next section that the optimal antenna geometries fill a spherical volume. Gustafsson’s has the flexibility to determine the optimal $Q$ for an antenna constrained to a given volume that is not necessarily spherical.

The main result of Gustafsson’s work is an upper bound on the ratio of directivity, $D$, to $Q$ of a single resonance, linearly polarized antenna using non-magnetic materials, expressed as

$$\frac{D}{Q} \leq \eta_a \frac{k^3}{2\pi} \gamma_1$$

(2.8)

where $\gamma_1$ is the largest eigenvalue of the high contrast polarizability dyadic $\gamma_\infty$ and $\eta_a$ is the absorption efficiency of the antenna defined by
\[
\eta_a(\hat{k}, \hat{\varepsilon}) = \frac{\int_0^\infty \sigma_a(k; \hat{k}, \hat{\varepsilon}) \, dk}{\int_0^\infty \sigma_a(k; \hat{k}, \hat{\varepsilon}) + \sigma_s(k; \hat{k}, \hat{\varepsilon}) \, dk}
\]

where \(\sigma_a\) is the absorption cross-section and \(\sigma_s\) is the scattering cross-section of the antenna.

For several antenna types, Gustafsson finds that \(\eta_a \approx 1/2\) [8]. Theoretically, \(0 \leq \eta_a \leq 1\), but antennas with \(\eta_a > 1/2\) tend to be electrically large or intentionally mismatched [9].

The high-contrast polarizability dyadic \(\gamma_\infty\) is a property of the antenna shape and can be calculated analytically for several canonical shapes or numerically for any shape. For details on this calculation, see [7, 8] and references therein.

For example, in the case of a sphere of radius \(a\), \(\gamma_1 = 4\pi a^3\) [8]. Substituting this into Equation 2.8 leads to a bound for the sphere

\[
\frac{D}{Q} \leq 2\eta_a (ka)^3.
\]

When \(\eta_a = 1/2\) and \(D = 1.5\) (Gustafsson shows that the monopole and dipole cases are equivalent and we should use the dipole directivity), this leads to the Thal bound on \(Q\) [6]. In [8], limits on \(D/Q\) are given for rectangular parallelepipeds, cylinders, toroids, cones, and 2D rectangular surfaces.

2.2 Electrically Small Antenna Designs

Linear antennas such as monopoles are among the simplest and oldest antennas, and can also be operated when electrically small. However, the standard monopole is below resonance when it is electrically small and presents a small resistance and large capacitive reactance at the feed point, which must be modified with a matching network. For example, in the case of a small, straight monopole with height \(h\) and wire radius \(b \ll h\), the resistance and reactance are approximately [1]

\[
R_{\text{monopole}} = 10(kh)^2
\]

\[
X_{\text{monopole}} = 60(1 - \ln(h/b)) \cot(kh)
\]
If connected directly to a typical system impedance, such as 50 Ω, a monopole in the electrically small region would have a large impedance mismatch loss, rendering such a design useless. However, we can determine what the bandwidth of a dipole would be if it were matched (with a single lumped element and ideal transformer) by calculating the $Q$ and then applying Equation 2.2. This value can be thought of as the “intrinsic” bandwidth of the antenna, and is more fundamental than the impedance at the operating frequency, which can be matched with an external network (or an internal network as we will discuss in following sections). Equations 2.3 and 2.4 allow us to calculate the $Q$ at every frequency, and reveal this “intrinsic” bandwidth.

Returning to the small, thin monopole, we can use Equations 2.11 and 2.12 in Equation 2.3 to calculate the $Q$ of the monopole. Figure 2.3 shows the ratio of the monopole $Q$ to the Chu limit of Equation 2.5 across electrical size and for several wire radii. The $Q$ of the monopole is significantly above the lower bound, although it decreases as the wire thickness increases. The $Q$ is fairly high relative to the Chu limit because a quasi-linear structure like a monopole does not effectively utilize the space inside the radius-$a$ sphere.

To impedance match the small monopole, we first make it self-resonant (that is, $X_m = 0$) and then a transformer adjusts the resonant resistance to the desired value. The self-resonant frequency of the monopole can be modified by increasing the overall wire length (i.e., increasing the inductance). Typical methods of lowering resonance frequency include meandering the wire or coiling it into a helix. In recent years, fractals have garnered interest in the antenna community as a way of increasing wire length [10, 11]. Genetic algorithms have also been used to design electrically small monopole antennas [12]. However, Best showed that regardless of the method used to reduce the self-resonance frequency, the key parameter, $Q$, is largely set by the height of the antenna and the amount of the radius-$a$ sphere that contains radiating currents [13].

Clearly, the small monopole is far from optimal in terms of its impedance bandwidth. However, by expanding the volume of the antenna, the performance can be significantly improved. For example, two simple cylindrical designs, the cylindrical helix and top-loaded monopole, have $Q : Q_{Chu}$ ratios of 2.5 and 2.0, respectively [14].
Figure 2.3: Ratio of the monopole \( Q \) to \( Q_{Chu} \) of a thin monopole of height \( h \) and radius \( b \), shown for \( b = \lambda/10^3 \), \( b = \lambda/10^4 \), and \( b = \lambda/10^5 \). The value of \( a \) used to calculate the electrical size is \( a = \sqrt{h^2 + b^2} \).

2.2.1 Spherical antennas

Foltz and McLean showed mathematically that the minimum \( Q \) is achievable with an antenna occupying a spherical volume rather than a dipole or planar structure \[15\]. Gustafsson’s limit for arbitrary geometries also leads to a similar conclusion \[8\]. As an experimental confirmation of the result, Best introduced a spherical wire antenna with among the lowest \( Q \) to date at approximately \( 1.5 \times Q_{Chu} \) \[16, 17\]. However in \[17\], the antenna was not self-resonant at 50 \( \Omega \) except for a few particular electrical sizes, and no clear guidelines were provided as to how impedance matching could be achieved.

Best compared several electrically small antenna designs in \[14\]. He found that the spherical helix offered the best \( Q/Q_{Chu} \) ratio (1.5) at most electrical sizes, but was difficult to match to 50 \( \Omega \) as mentioned. The spherical cap dipole, essentially a spherical metallic cap fed by a helical wire in the center, also showed a \( Q/Q_{Chu} \) ratio between 1.5 and 2 at several electrical sizes. Other designs considered included a cylindrical helix, a disk-loaded dipole, and the spherical resonator. Since these designs are three-dimensional, they have low \( Q \)—though not as low as the spherical dipoles. These other designs exhibit \( Q/Q_{Chu} \) ratios between 2 and 3.

We developed the first antenna structure to specifically excite the TM\(_{10}\) spherical mode in \[18, 19, 20\], which we will discuss in Chapter 5. The antenna offers similar \( Q \) performance to Best’s spherical helix with the added
advantage of tunability. In addition, our design approach is more firmly grounded in theory than many earlier designs that rely on genetic algorithms or other optimization techniques. In Chapter 5, we first describe a low $Q$ current distribution mathematically, and then manipulate the structure of the antenna to achieve the desired distribution. We also demonstrate that the antenna approaches a fundamental limit on $Q$ recently described by Thal [6] and compare it to the $Q$ performance of the aforementioned designs.

First we will introduce the theory of characteristic modes, so that the modal theory can be applied to our antenna design in Chapter 5.
CHAPTER 3

CHARACTERISTIC MODE THEORY

The theory of characteristic modes originated as a result of work by Garbacz and Turpin [21] and was refined soon after by Harrington and Mautz [22, 23]. The theory proposes a set of characteristic current modes that form an orthonormal basis for the current on a metal structure such as an antenna. A recent review of characteristic mode theory (CMT) can be found in [24].

In the four decades since its inception, CMT has been applied to a handful of problems in antenna analysis and synthesis [25, 26, 27], packaging [28, 29] and MIMO antenna design [30, 31, 32]. After a long hiatus, there has been some renewed interest in CMT. However, the method is broadly applicable and insightful, and we believe its full potential remains untapped. The characteristic mode approach is valuable because it reveals information (namely, the individual modes and their eigenvalues) that is otherwise inaccessible using standard electromagnetic simulation or measurement techniques [33]. From observations of characteristic modes, we can draw some general conclusions about common antenna behaviors.

3.1 The Operator Equation

We begin with the boundary condition at a perfect electric conductor (PEC) surface \( S \) with normal \( \hat{n} \)

\[
\hat{n} \times \left( \mathbf{E}^i + \mathbf{E}^s \right) = 0 \quad (3.1)
\]

where \( \mathbf{E}^i \) and \( \mathbf{E}^s \) are the incident and scattered fields at the interface. We can express \( \mathbf{E}^s \) in terms of the induced current \( \mathbf{J} \) and define the operator \( L \) as

\[
L(\mathbf{J}) \big|_{tan} = -\mathbf{E}^s = j\omega \mathbf{A}(\mathbf{J}) + \nabla \Phi(\mathbf{J}) \quad (3.2)
\]
with the electric vector potential defined as

\[
A(J) = \mu \iint_S J(r')G(r, r')\,ds'
\] (3.3)

electric scalar potential as

\[
\Phi(J) = \frac{-1}{j\omega \varepsilon} \iint_S \nabla' \cdot J(r')G(r, r')\,ds'
\] (3.4)

and the free space Green’s function as

\[
G(r, r') = \frac{e^{-jk|r-r'|}}{4\pi |r-r'|}.
\] (3.5)

The resulting equation

\[
[L(J) - E^i]_{tan} = 0
\] (3.6)

is typically solved with the method of moments [34] which results in a large, dense matrix equation.

### 3.2 Characteristic Modes

We first define the inner product for the Hilbert space of all square integrable vector functions on the surface \(S\) as

\[
\langle B^*, C \rangle = \iint_S B^* \cdot C \, ds.
\] (3.7)

Then, we note that the operator \(L\) has dimensions of impedance and can be written as

\[
L(J) \mid_{tan} = Z(J) = R(J) + jX(J)
\] (3.8)

where \(R(J)\) and \(X(J)\) are real. Because of reciprocity, the operator \(Z\) is symmetric, \(\langle B, ZC \rangle = \langle ZB, C \rangle\), so \(R(J)\) and \(X(J)\) are also symmetric.

Harrington proposes the following eigenvalue problem [22]:

\[
Z(J) = \nu W(J).
\] (3.9)
Any symmetric choice of the weighting operator $W$ will diagonalize $Z$, but the choice of $W = R$ will lead to orthogonal radiated fields and real eigenfunctions.

The continuous problem can be discretized into a matrix form, and the resulting generalized eigenvalue problem with $[W] = [R]$ is

$$[X] J_n = \lambda_n [R] J_n$$  \hspace{1cm} (3.10)

where $\nu = 1 + j\lambda$.

The eigenvectors are typically normalized such that $\langle J_n, R(J_n) \rangle = 1$. The normalized eigenvectors are orthogonal. That is, [22]

$$\langle J_m, [R] J_n \rangle = \delta_{mn}$$  \hspace{1cm} (3.11)

where

$$\delta_{mn} = \begin{cases} 
1 & m = n \\
0 & m \neq n 
\end{cases}$$  \hspace{1cm} (3.12)

The $J_n$ diagonalize $[Z]$ and form a basis for the total current on the structure, such that

$$J_{tot} = \sum_n \frac{V^n_i}{1 + j\lambda_n} J_n$$  \hspace{1cm} (3.13)

where $V^n_i$ is the modal excitation coefficient defined as

$$V^n_i = \langle J_n, E^i \rangle = \iint_S J_n \cdot E^{iT} \, ds.$$  \hspace{1cm} (3.14)

Similarly, the total electric and magnetic fields can be expressed as a summation of the fields due to each mode, and the fields are orthogonal over the sphere at infinity [22].

15
3.3 Calculation of the Characteristic Modes

To solve the eigenvalue problem, we first used FEKO [35] to model the antenna and generate the impedance matrix. FEKO was chosen because it is based on the method of moments, allows access to the raw impedance matrix, and allows externally generated current vectors to be imported back into the program. At every frequency of interest, a large set of eigenvalues and eigenvectors is calculated in Matlab. Since the eigenvectors are real, they represent the co-phasal modal current distribution at each moment method node, and the real eigenvalues determine how capacitive or inductive the mode is and how well it radiates. The eigenvalues and eigenvectors both vary with frequency, but the eigenvectors tend to vary slowly, retaining the same general modal pattern.

Having obtained a large set of vectors and scalars associated with each mode at each frequency, Matlab algorithms are then used to identify modes across frequencies since both eigenvalues and eigenvectors are frequency varying. Then the organized set of modes is exported back to FEKO where current distributions and fields can be plotted. A discussion of the FEKO settings used and the Matlab user interface for the program can be found in Appendix A.

Accurately finding the eigenvalues of the impedance matrix is not trivial. After experimenting with a variety of solution approaches, we applied the approach Harrington and Mautz originally used to calculate the modes [23]. More discussion about the solution approach is found in Appendix B.

Characteristic mode theory can be applied to both radiation and scattering problems. In this work, we will deal with the case of radiation problems. Until now, we have not specified an excitation for the problem, and thus do not have the particular solution for a real radiation problem. In fact, the entire eigenvalue problem is solved without knowledge of any excitation, but to find the final solution for an excited problem, we must calculate the modal excitation coefficient of each mode. For the simple case of a gap voltage source on a wire located at the $m^{th}$ node, the modal excitation coefficient is

$$\langle J_n, E^i \rangle = \iint_S J_n \cdot E^i \, ds = J_n [m].$$

(3.15)
As we see from Equation 3.13, the total current on the antenna can be represented as a weighted sum of the eigencurrents. Similarly, the input admittance can be expressed as a summation of the admittances of each mode as [36]

\[ Y_{in}(\vec{r}_m) = \sum_n \frac{\langle J_n, E^i \rangle}{1 + j\lambda_n} J_n(\vec{r}_m) \]

\[ Y_{in}[m] = \sum_n \frac{J_n[m]^2}{1 + \lambda_n^2}(1 - j\lambda_n). \] (3.16)

From this equation, we can see that a mode is resonant when \( \lambda_n = 0 \), and that the sign of the eigenvalue determines whether the input admittance is capacitive or inductive. In general, characteristic modes are generalized into two categories: resonant and non-resonant [37]. Resonant modes are capacitive at low frequencies, resonate, and become inductive beyond their resonance. These modes contribute most of the radiated energy when their eigenvalues are small. Non-resonant modes (inductive modes) begin as inductive modes with large positive eigenvalues and never resonate, only contributing inductive susceptance to the antenna at all frequencies.

The modal eigenvalues are often presented in terms of their characteristic angle to better show their behavior near resonance. The characteristic angle is defined as

\[ \alpha_n = 180^\circ - \tan^{-1}(\lambda_n) \] (3.17)

Modes are capacitive when \( 180^\circ < \alpha_n < 270^\circ \), inductive when \( 90^\circ < \alpha_n < 180^\circ \), and resonant when \( \alpha_n = 180^\circ \).

3.4 Characteristic Modes of Simple Antennas

To verify our calculations, we compare the characteristic modes of some simple antennas to those computed in the literature. In a dissertation by M. Cabedo-Fabres, detailed characteristic modes are given for a 0.5 m long dipole with wire radius of 1 mm [37]. The modes are calculated using an internally developed moment method code. The eigenvalues of this antenna given by Cabedo-Fabres show excellent agreement with those we have calcu-
lated using FEKO as shown in Figure 3.1. As an additional verification of our calculations, some of the dipole’s eigenvectors from [37] are compared to our calculations in Figure 3.2.

We have verified that our program can accurately calculate the eigen-modes of a wire antenna in both the electrically small and electrically large regimes. To verify that we can also produce accurate eigenvalues for an antenna with metallic surfaces, we use data from a recent dissertation by C. Tamgue Famdie [38]. In this work, the author calculates several of the smallest eigenvalues of a 100 mm × 40 mm rectangular plate from 820 MHz to 4820 MHz. Figure 3.3 shows that our calculations again agree extremely well with those from [38]. The first six modes of the plate are shown in Figure 3.4. Insight into where to feed the antenna and the radiation patterns of various modes can be gained from observing these types of plots.
Figure 3.1: Eigenvalues of 0.5 m dipole with wire radius = 1 mm (a) from [37] and (b) calculated using our program.
Figure 3.2: Eigenvectors 1 and 2 of 0.5 m dipole with wire radius = 1 mm (a) from [37] and (b),(c) calculated using our program.
Figure 3.3: Eigenvalues of a 100 mm $\times$ 40 mm rectangular plate (a) from [38] and (b) calculated using our program.
Figure 3.4: Modal currents of the first six characteristic modes of a 100 mm $\times$ 40 mm PEC plate.
CHAPTER 4
MODELS FOR SMALL ANTENNAS

The first step in analyzing small antennas is to devise some abstractions to model their behavior. Since small antenna radiation patterns tend to be omnidirectional TM$_{10}$- or TE$_{10}$-like patterns, we primarily need to model the antenna’s input impedance. Antenna impedance is commonly modeled via lumped electrical circuit elements and transmission lines. Though the literature is replete with circuit models for antennas, there is no commonly accepted approach to developing these models and little insight can be transferred from modeling one antenna to the next.

In the electrically small region, simple antenna models like series and parallel RLC circuits are often quite accurate. However, multiple resonances can exist when the antenna is electrically large in part of the band or if it is intentionally designed to be multiresonant in the electrically small region as we will see in Chapter 6. When such multiresonant antennas are modeled, a simple resonant circuit will not suffice. Thus, higher-order RLC circuits are used to approximate the impedance behavior of these types of antennas.

As a simple example, consider modeling the dipole antenna over a broad frequency band where it ranges from electrically small to several wavelengths in size. A variety of models have been proposed to solve this problem [39, 40, 41, 42], yet each has limitations [43], and there is no clear way of extending the analysis to other antennas.

Thus, we seek a fundamental understanding of antennas that can provide some guidelines for impedance modeling. Because characteristic mode theory allows us to break down the component parts of the total antenna response, it can provide us with a set of guidelines for each individual mode and a method for combining the responses to find the total response.
4.1 Circuit Model from Characteristic Modes

Characteristic mode theory suggests natural circuit models for the modes. As we saw in Chapter 3, the input impedance of a gap voltage source on a wire is given by

\[ Y_{in}[m] = \sum_{n} \frac{J_{n}(\vec{r}_{feed})^2}{1 + \lambda_n^2} (1 - j\lambda_n) \]  

(4.1)

where \( J_{n}(\vec{r}_{feed}) \) is the normalized current of the \( n \)th mode at the feed point and \( \lambda_n \) is the \( n \)th eigenvalue. Because the modal admittances sum, they represent admittances connected in parallel.

Let us consider resonant characteristic modes first. As we observed earlier, the modal eigenvalues are typically negative (capacitive) at low frequencies, cross through zero, and then become positive (inductive). This behavior is reminiscent of a series resonant circuit since \( \frac{dX}{df} > 0 \). The idea of representing the modes of the antenna as series resonators connected in parallel was pointed out by Cabedo-Fabres [37] and elaborated somewhat by Obeidat [44]. Consider the admittance of a simple series RLC resonator:

\[ Y = \frac{1}{R(1 + Q^2\Omega^2)}(1 - jQ\Omega) \]  

(4.2)

where \( Q \) is the quality factor of the resonator and \( \Omega = \frac{\omega}{\omega_r} - \frac{\omega_r}{\omega} \) with \( \omega_r \) being the resonant frequency of the circuit.

Comparing the series RLC admittance in 4.2 to the expression for a single mode admittance from 4.1, we find some interesting parallels:

\[ R = \frac{1}{J_{n}(\vec{r}_{feed})^2} \]  

(4.3)

\[ Q\Omega = \lambda \]  

(4.4)

Thus, if \( \lambda \) has similar frequency variation to \( Q\Omega \), then this circuit model is valid. To verify this, we can fit a curve of the form

\[ \lambda_n = Q_n(f/f_n - f_n/f) \]  

(4.5)

to the eigenvalues of the monopole antenna discussed in Section 3.4.

Figure 4.1a shows the eigenvalues, and it appears that the fit to this model
Figure 4.1: Mode 1 eigenvalues of the dipole from Section 3.4 fit to a \( Q(f/f_n - f_n/f) \) model.

is not very good, particularly at the low frequencies. The characteristic angle of the mode better displays the behavior of the eigenvalues near their resonance and is defined as \( \alpha_{c,n} = 180 - \tan^{-1}(\lambda_n) \). Figure 4.1b shows the characteristic angle to illuminate the behavior near resonance. Again, it seems that the fit is not particularly good even near resonance and it does not find the correct resonance frequency.

From Equation 3.16 we know that the conductance and susceptance of the mode are proportional to \( \frac{1}{\lambda_n^2} \) and \( \frac{1}{\lambda_n} \), respectively, so it is reasonable to strongly weight the fit near resonance and disregard the values taken when the eigenvalue is large. Thus, Figure 4.2 shows the fit to the same function, but with a strong weighting applied when the eigenvalues are small, proportional to \( \frac{1}{\lambda_n^2} \). The fit is still inaccurate for many frequencies, but it aligns well near the resonance point as shown in Figure 4.2b. When far away from the resonance point, the eigenvalue is irrelevant because the mode looks like a very large reactance in parallel with smaller reactances of other modes.

If we apply this model of the resonant modes as series RLC circuits, finding the best fit to the eigenvalues is an unnecessary step. The series resonator is completely defined by \( f_n, R_n, \) and \( Q_n \). The value of \( f_n \) can be found quite easily by choosing the frequency at which the eigenvalues cross the x-axis. The resistance, \( R_n \), is readily found by taking

\[
R_n = \frac{1}{J_n^2 (\vec{r}_{feed})} \tag{4.6}
\]

where \( J_n^2 (\vec{r}_{feed}) \) is current of the nth mode at the feed point.

Finally we can determine the \( Q \) of the mode at resonance by taking the
derivative with respect to $f$ of Equation 4.4 and evaluating at $f = f_n$ to find

$$Q_n = \frac{f_n}{2} \frac{d\lambda_n}{df} \bigg|_{f=f_n}.$$  \hspace{1cm} (4.7)

Since the characteristic modes are essentially solutions to the source-free Maxwell’s equations, their quality factors will be bounded by the same limits as discussed in Chapter 2.

Other models for the eigenvalues are possible, and in our studies we have found that a third order polynomial in $\frac{1}{f}$ can model the eigenvalues very accurately over a wide bandwidth. However, these models offer no guarantee of physical realizability of the circuit and can significantly complicate our model of the mode. Thus, we will continue with the assumption that a resonant modal admittance can be sufficiently modeled by a series RLC circuit.

Up to this point, we have assumed that the modal feed point current is constant (resulting in a constant modal resistance). However in most cases, the current varies with frequency. The overall modal current pattern changes slowly with the frequency and this causes some shift in the magnitude of the current at the feed. Figure 4.3 shows the square of the modal feed current for the dipole simulated in Section 3.4. Only results for Modes 1 and 3 are shown since the feed point is at the center of the dipole. For a center-fed dipole, the even modes are not excited and thus their feed current is zero. For both odd modes, the feed current decreases as the frequency increases, though at different rates. For increased accuracy, the circuit model should incorporate this into the model by using a frequency dependent resistance for each mode.
4.2 Stub Matching of Small Antennas

Decades ago, antenna engineers realized that they could excite monopole antennas by shorting the base and feeding a shunt stub without significantly affecting the radiation pattern of the antenna [45, 46]. While this was originally done to reduce construction costs, Baudoux [46] did note that the input resistance of the antenna differed substantially from a standard monopole. In the following decades, folded dipoles became a significant research topic [47] for their impedance transformation properties. Related to both of these designs is the stub match or gamma match, which consists of a shorted wire parallel to the main antenna wire, but not necessarily shorted to the top of the antenna as in a folded dipole. Some recent work has applied this technique to a variety of small resonant antennas [13, 48]. Altshuler qualitatively describes the effect as the addition of a parallel inductance, but this interpretation is not entirely satisfying, especially when compared to the Smith charts provided [48]. In Chapter 6 we wish to apply this technique to the matching of a multimode antenna, so we need to understand the matching mechanism in more detail.

Analysis of these antennas typically uses transmission line theory, calculating differential and common mode impedances to find the total impedance. The transmission line approach applies to simple geometries like straight, parallel wires, but would fail to explain the behavior of complex geometries like the genetic antenna of Reference [48]. A comprehensive review of
Figure 4.4: Illustration of the characteristic modes of the stub matched straight monopole.

the transmission line approach to these types of antennas can be found in [49]. This work does not seek to replace the existing theories, but instead demonstrates an alternate approach to the problem using modes and lumped circuits. The result is new physical insight into small antenna behavior that cannot be gained from the transmission line approach.

4.2.1 Circuit model for the stub matched geometry

Consider the basic stub-matched straight monopole shown in Figure 4.4. Solving for the characteristic modes reveals that there are indeed two modes excited by the feed point, as envisioned by the transmission line model. As discussed earlier in this chapter, characteristic mode theory gives us a natural way of modeling the impedance of these two modes. In the case of the small monopole antenna, the circuit model is quite simple, and mirrors the one suggested by Altshuler [48]. The simple model consists of a radiating mode (series RLC resonator) in parallel with an inductor as shown in Figure 4.5. However, we will see shortly that the shunt stub’s effect encompasses much more than just adjusting the shunt inductance.
Let us consider the limitations of the matching technique that are implied by this simple circuit model. First, we note that the shunt inductor in parallel acts as an impedance transformer from the natural resonant resistance ($R_1$) to a higher resistance. For example, the conductance of a series RLC circuit, representing the radiating mode, is shown in Figure 4.6. The radiating mode’s self-resonance occurs at the peak, where the input resistance is $R_1$ (5 Ω in this example). Below the resonant frequency, the modal admittance is capacitive and can be resonated with the shunt inductor. The resonant resistance is then the inverse of the admittance at that frequency. Because the conductance peak occurs at the modal self-resonant frequency, it is apparent that the resistance can only be transformed to a higher value. When the antenna is electrically small, the modal resistance tends to be small as well, so this should not present a problem. However, we observe that the modal conductance falls very quickly away from the self-resonant frequency. This suggests that the mode must be self-resonant just above the desired match frequency, or the modal conductance will be very small, resulting in an input resistance orders of magnitude above most standard system impedances.

Second, we observe that the impedance of the series RLC branch in Figure 4.5 is given as

$$Z_1 = R_1(1 + jQ_1\Omega_1(f))$$  \hspace{1cm} (4.8)

where $\Omega_1(f) = (f/f_1 - f_1/f)$ . Applying the definition of $Q$ from [4], we find

$$Q(f) = \begin{cases} Q_1\frac{f_1}{f} & f > f_1 \\ Q_1\frac{f}{f_1} & f < f_1 \end{cases} .$$  \hspace{1cm} (4.9)
Figure 4.6: Example conductance variation of series resonator ($R_1 = 5 \, \Omega$).

As we move away from the self-resonance of the mode, the apparent $Q$ of the mode becomes larger. For example, if we try to match a straight monopole (self-resonant at $\lambda/4$) at a height of $\lambda/20$, the $Q$ at that size will be a factor of 5 larger than the $Q$ at resonance.

Considering these two problems, we expect this matching technique to be most useful when the desired match frequency is near the self-resonant frequency of the radiating mode. For an electrically small straight monopole, this is not the case and the stub matching technique is not well-suited for the task. Self-resonance can be induced at a lower frequency using a variety of techniques to increase the mode’s inductance or capacitance [50]. For instance, Altshuler’s antenna was self-resonant before matching due to randomly oriented wires chosen based on a genetic algorithm [48]. However, because the simpler geometry allows some analytical calculations, we will briefly examine the impedances of the straight monopole. We then will investigate a helical monopole to force the radiating mode to be self-resonant in the electrically small region.

4.2.2 Stub matched straight monopole

Although we have predicted that the stub matching technique will not work well for the small, straight monopole, it is instructive to study this simple structure initially. Consider a stub-matched straight monopole with dimensions as given in Figure 4.4a. As previously discussed, there are two
Table 4.1: Circuit parameters at 300 MHz (see Figure 4.5) for straight monopole with $h = 10$ cm, wire radius = 1 mm, $l_s = 1$ cm.

<table>
<thead>
<tr>
<th>$h_s$ (cm)</th>
<th>$L_0$ (nH) from loop inductance</th>
<th>$L_0$ (nH) from CMT</th>
<th>$R_1$ (Ω)</th>
<th>$Q_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>--</td>
<td>--</td>
<td>--</td>
<td>7.5</td>
<td>67.4</td>
</tr>
<tr>
<td>1.0</td>
<td>10</td>
<td>12</td>
<td>21.0</td>
<td>67.2</td>
</tr>
<tr>
<td>2.5</td>
<td>23</td>
<td>26</td>
<td>24.2</td>
<td>66.5</td>
</tr>
<tr>
<td>5.0</td>
<td>45</td>
<td>48</td>
<td>25.1</td>
<td>63.6</td>
</tr>
<tr>
<td>10.0</td>
<td>89</td>
<td>88</td>
<td>32.3</td>
<td>44.0</td>
</tr>
</tbody>
</table>

modes excited when the stub is grounded, Mode 0 and Mode 1. The Mode 0 currents circulate from one grounded point to the other, making it a non-resonant mode that always has an inductive admittance. It is also a very poor radiator so its impedance can be approximated by a single inductor. The inductance of Mode 0 is well approximated by the inductance of a rectangular loop over a ground plane [51]. Tables 4.1 and 4.2 show the Mode 0 inductance compared to theoretical values of the inductance as $h_s$ and $l_s$ are changed.

Since the inductance of Mode 0 is easily calculated, it appears that we could easily match the Mode 1 susceptance at the required frequency by changing $h_s$ and $l_s$. However, from Tables 4.1 and 4.2, we also can see that Mode 1 is not independent of the stub parameters.

In fact, the Mode 1 resonant parameters are strongly linked to the stub geometry. For this electrically short antenna, the resonance is much higher in frequency than the operating band, so we will ignore it for now. Let us examine the other two parameters as we change the stub geometry as shown in Tables 4.1 and 4.2. The Mode 1 resistance varies significantly as the stub is changed and also differs greatly from the resistance of the monopole without the stub (7.5 Ω). However, $Q_1$ remains mostly constant and very close to the stub-less $Q$ (67.4). $Q_1$ only begins to differ from this value when the stub approaches the same height as the monopole. In this case, the volume of the radiating portion of the antenna is significantly increased. $Q_1$ seems to be fundamentally related to the size of the radiating structure.
Table 4.2: Circuit parameters at 300 MHz (see Figure 4.5) for straight monopole with $h = 10$ cm, wire radius = 1 mm, $h_s = 1$ cm.

<table>
<thead>
<tr>
<th>$l_s$ (cm)</th>
<th>$L_0$ (nH) from loop inductance</th>
<th>$L_0$ (nH) from CMT</th>
<th>$R_1$ (Ω)</th>
<th>$Q_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>2</td>
<td>17</td>
<td>18</td>
<td>16.4</td>
<td>67.4</td>
</tr>
<tr>
<td>4</td>
<td>29</td>
<td>31</td>
<td>12.6</td>
<td>67.2</td>
</tr>
<tr>
<td>10</td>
<td>65</td>
<td>64</td>
<td>9.4</td>
<td>66.8</td>
</tr>
<tr>
<td>20</td>
<td>124</td>
<td>106</td>
<td>7.3</td>
<td>68.6</td>
</tr>
</tbody>
</table>

Figure 4.7: Illustration of the characteristic modes of the stub matched helical monopole.

4.2.3 Stub matched helical monopole

As discussed in Section 4.2.1, this matching technique cannot produce a useful conductance for the short straight monopole. To apply the technique, the radiating mode must be resonant near the desired operating frequency. In order to force the resonance to occur in the electrically small region, we wind the wire into a narrow, electrically short helix as shown in Figure 4.7. With a rectangular stub attached, this structure now fulfills the self-resonant condition discussed in Section 4.2.1.

Under this configuration, Mode 0 does not change except that its inductance is slightly increased because the helix now makes up one of the sides of the rectangle instead of a straight wire. $L_0$ can be approximately calculated the same way, with a small adjustment for the increased inductance.

Before the stub is added, the helical monopole has $Q_1 = 38.1$, $R_1 = 7.7$
Ω, and the resonant frequency is 300 MHz, as designed. Table 4.3 presents the circuit parameters of Mode 1 for a feed point at the base of the helix (1h subscripts) or at the base of the stub (1s subscripts). \( R_1 \) takes on a variety of values as the stub is modified, but is never less than the stub-less helical monopole. The resonant frequency also changes significantly. However, once again, the modal \( Q \) does not change much despite the large geometric modification, so the stub tuning method preserves the \( Q \) of the original radiating mode. It is lower than the \( Q \) of the straight monopole of the same height, due to the larger volume of the helical coil. This is more evidence that the modal \( Q \) is fundamentally related to the size of the radiating portion of the antenna.

Even using the helix, it again seems difficult to match the antenna to a desired impedance since not only is \( L_0 \) related to the stub parameters, but so are \( R_1 \) and \( f_1 \). It may be instructive to understand the dependence of \( R_1 \) and \( f_1 \) on the stub geometry. First, let us study the dependence of the resistance. Because the characteristic modes are normalized to radiate unit power, we found in Section 4.1 that

\[
R_1 = \frac{1}{J_1 \tau_{feed}^2}.
\]

Therefore, studying the behavior of the modal current will lead to an understanding of the input resistance.

When a perturbation such as the stub is added but the radiating portion of the antenna is not changed significantly, we expect the total current required to radiate unit power should not change appreciably. If the helical monopole has no stub, the entirety of the required current can only be drawn from the wire connected to the ground plane. In the example given in Tables 4.3 and 4.4, the Mode 1 current required for the stub-less antenna to radiate unit power is 361 mA. When the stub is added, approximately the same amount of current should be needed to radiate unit power. However, the current at the center feed point drops significantly to 50-200 mA. The remaining current comes from the shorted end of the stub. Unlike Mode 0, where one shorted end draws current from the ground plane and the other deposits an equal amount of current onto the ground plane, the Mode 1 currents are both drawn from the ground plane at the two shorted points. Thus, the sum of these currents \((J_{1h} + J_{1s})\) should be approximately equal to the total current for the stub-less helix \((J_1)\). Observing the last two columns of Tables 4.3 and 4.4, one sees that the total current tends to be equal to or slightly less than the current of the stub-less monopole. For some perspective on how similar these total current values are, we observe that for the same geometric
Table 4.3: Circuit parameters at 300 MHz for helical monopole of height = 10 cm, helix pitch = 6.37 mm, helix radius = 5 mm, wire radius = 1 mm, \( l_s = 1 \) cm.

<table>
<thead>
<tr>
<th>( h_s ) (cm)</th>
<th>( f_1 ) (MHz)</th>
<th>( R_{1h} ) (Ω)</th>
<th>( Q_{1h} )</th>
<th>( J_{1h} ) mA</th>
<th>( R_{1s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>---</td>
<td>300</td>
<td>7.7</td>
<td>38.1</td>
<td>361</td>
<td>---</td>
</tr>
<tr>
<td>0.5</td>
<td>305</td>
<td>64</td>
<td>38.0</td>
<td>125</td>
<td>19.2</td>
</tr>
<tr>
<td>1.0</td>
<td>314</td>
<td>130</td>
<td>38.1</td>
<td>88</td>
<td>14.9</td>
</tr>
<tr>
<td>2.0</td>
<td>338</td>
<td>197</td>
<td>36.5</td>
<td>71</td>
<td>14.0</td>
</tr>
<tr>
<td>4.0</td>
<td>409</td>
<td>193</td>
<td>34.0</td>
<td>72</td>
<td>14.0</td>
</tr>
<tr>
<td>8.0</td>
<td>458</td>
<td>84</td>
<td>39.0</td>
<td>109</td>
<td>15.7</td>
</tr>
</tbody>
</table>

Table 4.4: Circuit parameters at 300 MHz for helical monopole of height = 10 cm, helix pitch = 6.37 mm, helix radius = 5 mm, wire radius = 1 mm, \( h_s = 1 \) cm.

<table>
<thead>
<tr>
<th>( l_s ) (cm)</th>
<th>( f_1 ) (MHz)</th>
<th>( R_{1h} ) (Ω)</th>
<th>( Q_{1h} )</th>
<th>( J_{1h} ) mA</th>
<th>( R_{1s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>---</td>
<td>300</td>
<td>7.7</td>
<td>38.1</td>
<td>361</td>
<td>---</td>
</tr>
<tr>
<td>-0.5</td>
<td>316</td>
<td>1362</td>
<td>37.3</td>
<td>27</td>
<td>9.8</td>
</tr>
<tr>
<td>1.0</td>
<td>314</td>
<td>130</td>
<td>38.1</td>
<td>88</td>
<td>14.9</td>
</tr>
<tr>
<td>2.0</td>
<td>312</td>
<td>58</td>
<td>38.3</td>
<td>131</td>
<td>21.3</td>
</tr>
<tr>
<td>4.0</td>
<td>310</td>
<td>29</td>
<td>38.6</td>
<td>186</td>
<td>36.5</td>
</tr>
</tbody>
</table>

modifications, the Mode 0 current ranges from 4,000 to 77,000 mA. It is important to realize that this is not the current on the actual antenna, but the amount of current needed to radiate 1 W if Mode 0 could be excited by itself with no mismatch loss. If these values seem large, it is because Mode 0 is a very poor radiator, and behaves primarily as an inductor.

The stub reduces the Mode 1 current at both feed points below the current of the single feed. Therefore, in this shorted helix configuration, the stub acts like a transformer for the Mode 1 resistance. The stub also adjusts the division of the Mode 1 current between the potential feed points, and the impedance transformation ratio is a function of the stub geometry and the choice of feed point. Letting \( J_{1h} = \alpha J_1 \), then \( J_{1s} \approx (1-\alpha)J_1 \). Because CMT does not require that we choose a feed point until we calculate the input impedance, there are two possible resistances that can be seen at the input: \( R_{1h} = \frac{R_1}{\alpha^2} \) and \( R_{1s} = \frac{R_1}{(1-\alpha)^2} \).

In a practical scenario, the stub is close to the helix (\( l_s < h_s \)), and the current path through the stub is less inductive than the bottom portion of
the helix. Therefore, \( \alpha \) is usually less than 0.5 (i.e., most of the current flows through the stub) so that \( R_{1h} > R_{1s} > R_1 \). In many cases, only a small impedance step up is needed (from 7.7 \( \Omega \) to 50 \( \Omega \) in this case), and the stub feed point provides this small increase while the helix-center feed sees a very large and undesirable impedance.

Interestingly, in the case of equal current division (\( \alpha = 0.5 \)), this analysis predicts a 4:1 impedance step up, just as with a folded dipole. Furthermore, it predicts a \( N^2:1 \) step up in the case of \( N \) stubs that each draw equal current, the same as an \( N \)-element folded dipole [52].

Kraus notes that feeding a shunt tap increases the impedance of the electrically small helix to match to common transmission lines [53]. However, no design intuition is given. Calculating precise formulas for the current division coefficient \( \alpha \) is beyond the scope of this work, but it is useful to develop some general intuition about how \( \alpha \) changes so that this approach can be applied to a variety of complex antenna shapes in addition to the short helix. From Table 4.1, we see that changing the height of the stub has little effect on the splitting of the current, so the height may be neglected when approximating the current division. However, it is clear from Table 4.2 that the distance between the stub and the helix has a large influence on the way the current splits. As the stub is moved farther away from the helix, the percentage of current drawn through the stub decreases. In general, when the inductance of one path is increased, the current flowing on that path decreases. For example, if the wire radius of the stub is decreased, more of the current flows through the helix. Alternatively, if the antenna is made electrically smaller (i.e., the helix pitch is decreased), less current will flow through the helix for the same \( h_s, l_s \) because the helix path is now even more inductive.

Under most configurations, because of the stub’s lower resistance, among other reasons, feeding the stub is preferable to feeding the helix center. Most of the Mode 1 current usually flows through the stub, so Mode 1 is well excited by the stub feed. Because the Mode 0 current circulates without much radiation, Mode 0 is excited equally at either feed. It is better to excite the radiating mode strongly while weakly exciting the matching mode, or the total response may be very inductive, high \( Q \), and inefficient.

If only \( R_1 \) and \( L_0 \) depended on the stub geometry, the antenna could be readily matched to the desired impedance by making the antenna resonant slightly above the desired frequency, and selecting \( R_1 \) and \( L_0 \) as necessary to
resonate the antenna at the desired frequency and resistance. However, there is one additional complication: the Mode 1 resonant frequency also changes with modifications to the stub. The frequency, $f_1$, is primarily affected by $h_s$. If the stub taps the helix very close to the bottom, $f_1$ is only slightly higher than the original frequency. As the stub moves towards the top, $f_1$ becomes close to the resonant frequency of a short, straight monopole. For this particular geometry, the new $f_1$ can be approximated by

$$f_1' \approx \frac{f_1}{1 + \frac{h_s}{\lambda}(s - 1)}$$

where $s$ is the shortening ratio of the helix (i.e., the helix height divided by a quarter wavelength at resonance). Now using the physical intuition of the geometry and modes, we can design a match for an antenna of any height or geometry.

### 4.3 On the Existence of Low $Q$ Modes

As we saw in our analysis of the stub matching technique, the modal $Q$ seems to be fundamentally determined by the size of the antenna, but factors like the feed location and small geometric features are extraneous. For monopole-like electrically small antennas, the lowest order mode exists with a particular $Q$ regardless of these factors. However, they do play a significant role in the antenna’s behavior. The feed point will affect how the low order and higher order modes are excited. The boundary conditions set by certain geometric features (e.g., helices, meanderlines) will determine the resonant frequency of the low order mode and the modal excitations. Thus, these factors will control the total input impedance and total $Q$.

Consider the thin monopole discussed in Section 4.2.2. The monopole is 10 cm tall with a wire radius of 1 mm. From Tables 4.1 and 4.2, the fundamental radiating mode of this antenna has $Q_1 \approx 67$ at 300 MHz regardless of the geometry. The antenna is very poorly matched, but using characteristic modes, we can observe its $Q$. Gustafsson predicts that the minimum achievable $Q$ in that thin cylindrical space is 64.4 [8], which is strikingly close to the $Q$ calculated for the fundamental mode. Furthermore, consider the $Q$ of the stub matched helical monopole in Section 4.2.3. From Tables 4.3 and 4.4,
we see that the fundamental mode has $Q_1 \approx 38$ for a variety of feed locations and geometries. For a cylindrical volume of radius = 5 mm and height = 10 cm, Gustafsson calculates that the minimum $Q$ is 35.2 \cite{8}. Again, the lowest order mode has a $Q$ almost precisely equal to the value calculated by Gustafsson regardless of the feed point or geometry within the volume.

The constancy of the $Q$ also extends to three-dimensional antennas. As an example, several cylindrical antennas were simulated and the $Q$ of their lowest order mode and total $Q$ were calculated. Some representative examples of the simulated structures are shown in Figure 4.8 along with their total $Q$ and the $Q$ of their fundamental mode. First, we observe that the low order modal $Q$ is remarkably consistent regardless of geometry or feed point. Furthermore, a $Q$ of 122.4 corresponds to Gustafsson’s calculated minimum $Q$ for a cylindrical volume with height = 5 cm and radius = 1 cm \cite{8}. Once again, the fundamental mode $Q$ is almost exactly at the Gustafsson limit despite significant changes in the geometry and feed location.

Secondly, we note that the total $Q$ ranges over several orders of magnitude and never approaches the low order modal $Q$. Higher order modes excited by the feed configurations contribute to the increased $Q$. Although the structures support an effective radiating mode, this fact is masked by the higher order modes that are excited. The characteristic mode approach allows us to observe the low order mode alone, unlike measurements or traditional numerical analysis where the existence of a low $Q$ mode would be obfuscated.

Traditionally, engineers have expended great effort designing low $Q$ structures. However, since a space-filling, low $Q$ mode is excited by a structure as simple as a conductive cylinder shorted to ground, a paradigm shift is needed in small antenna design. Rather than struggling to design complex, low $Q$ structures, the focus should be on coupling into modes that exist in basic structures and minimizing higher order modes. A reasonable approach is to first fill the given space as fully as possible, which should excite a low $Q$ mode in addition to several other modes. Next, the structure should be made resonant near the desired operating frequency by introducing boundary conditions to force the current to follow a longer electrical path (e.g., helical coils, meanderlines). Once this is done, a matching technique, perhaps as simple as the stub technique discussed in Section 4.2, can be applied to excite the low $Q$ mode with a useful impedance.
Figure 4.8: Several configurations of cylindrical antennas (height = 5 cm, radius = 1 cm) with their respective $Q$ values at 300 MHz for the lowest order mode and the total $Q$ at one of the feeds. (a) Shorted cylinder with feed wire along centerline. (b) Raised cylinder with three wires shorted to ground. (c) Shorted cylinder with feed wire along centerline and shorted stub. (d) Cylinder with several shorted arms and feed on centerline.
In Section 2.1.2, we discussed a modified limit on the $Q$ of electrically small antennas derived by Thal [6]. In [6], Thal also showed that the $\text{TM}_{10}$ spherical mode has the lowest $Q$ possible. The $\text{TM}_{10}$ mode can be excited by a surface current distribution of the form

$$J_{\theta} = \sin(\theta).$$

(5.1)

Based on Thal’s analysis, we developed the first antenna structure to specifically excite the $\text{TM}_{10}$ mode in [18], followed by initial experimental results in [19]. The structure is designed to approximately support the current distribution of Equation 5.1. This distribution can exist in either a monopole or a dipole form. In this work, we will consider the monopole version of the antenna because it is more practical to fabricate a hemisphere on a ground plane than full sphere. The monopole $\text{TM}_{10}$ antenna is shown in Figure 5.1. The structure consists of helical wires coiled along the constant-$\phi$ lines of a sphere which support the current distribution of Equation 5.1.

The wires are coiled around the arms to add electrical length for the desired sinusoidal current distribution of Equation 5.1. Without the coils, the arms are electrically short and the current distribution is simply triangular in $\theta$ as shown in Figure 5.2. This produces a significantly higher $Q$ as higher order (and higher $Q$) modes are excited instead of the $\text{TM}_{10}$. More than two arms are desirable, as the increasing number of arms better approximates the full $\text{TM}_{10}$ distribution and results in a lower $Q$. However, with more arms, more wire length is required to create the same electrical length.

In the monopole configuration, the sphere is bisected by a ground plane along $\theta = \pi/2$. The ground plane provides image currents allowing the dipole nature of the $\text{TM}_{10}$ mode to be realized with a monopole antenna. A copper feed trace suspended above the ground plane contacts all of the arms at their
Figure 5.1: A four-arm TM$_{10}$ monopole. The wire pitch, $p$, is measured on the outer radius of the arms. The helical coils have radius $r_h$ and their centerlines follow a circle of radius $b$ from the center of the structure. The width of the trace is $w$ and the probe feed makes contact from underneath the ground plane at the center of the feed trace.

Figure 5.2: The normalized magnetic field, proportional to the current, along the centerline of the arms versus $\theta$. Simulated results from HFSS are plotted when the arms are straight wires and helical coils.
bases. The trace is fed in the center by a probe from behind the ground plane.

5.1 Properties of the Antenna

The complete antenna structure, including wire coils, was simulated in HFSS [54] using two symmetry planes to reduce computation time. The antenna is observed to have some notable properties in the electrically small region. A typical plot of the input impedance of the antenna is shown in Figure 5.3a. The first resonance of the antenna occurs when the physical length of wire on each arm is on the order of a half wavelength. Thus, changing the electrical length of the arms via the pitch of the coils acts as the primary mechanism to shift location of these resonances. Decreasing the pitch lowers the resonant frequency. An antiresonance also appears in the electrically small region, slightly higher in frequency than the resonance.

Typical of most electrically small antennas, the radiation pattern resembles that of an electric monopole. On an infinite ground plane, the pattern is nearly omnidirectional in $\phi$ (variation of less than $\sim 0.4$ dB for the four-arm design), and it features a null at $\theta = 0$ and peak directivity of 4.9 dBi along the direction of the ground plane ($\theta = 90^\circ$). The maximum cross polarization ratio in any direction is less than -40 dB.

5.1.1 Behavior of the quality factor

The antenna’s bandwidth performance will be discussed in terms of the ratio $Q/Q_{Chu}$. This figure of merit clearly compares the TM$_{10}$ antenna’s performance to the optimum, and performance can be fairly compared across different $ka$ values. In all $Q$ calculations, the input impedance is used to find $Q$ using Equation 2.3.

Figure 5.3b shows the $Q/Q_{Chu}$ ratio for the same antenna whose impedance characteristics are shown. After the first resonance, the steep $Q/Q_{Chu}$ curve flattens and reaches a minimum value. Then just before the subsequent antiresonance, the $Q/Q_{Chu}$ value begins rising again. We refer to this flat region where the $Q/Q_{Chu}$ minimum occurs as the “low $Q$ region,” indicating that the $Q$ is low relative to the electrical size. Operation anywhere in this
Figure 5.3: Typical properties of the TM$_{10}$ antenna. (a) The input impedance behavior of the antenna in the electrically small region. If the antenna is tuned properly, it exhibits both a resonance and antiresonance. (b) The $Q/Q_{Chu}$ for the same antenna. The operating region or “low $Q$ region” exists between the first resonance and the first antiresonance.

region is desirable.

5.1.2 Comparison to the lower bound

Our goal is to approach the lower bound on $Q$ with a realizable electrically small antenna. As discussed earlier, Thal has shown that the real achievable $Q$ is higher than Chu’s limit when the currents are constrained to a spherical surface. Because of the prevalence of the Chu limit in the literature, we will use it as the primary basis for comparison but also present Thal’s limit for a more realistic assessment of antenna performance. For antennas with efficiency less than 100%, the correspondingly lower $Q$ limit will be reflected in the comparison as per Equation 2.5.
Figure 5.4: $Q/Q_{Chu}$ ratio of several TM$_{10}$ antennas compared to existing designs from [14]. The Thal limit for currents on a spherical surface [6] is also shown.

Illustrating the excellent performance of our design, Figure 5.4 shows the $Q/Q_{Chu}$ ratio for the TM$_{10}$ antenna with four arms and eight arms and with two different pitches. Also shown in the plot are the Thal limit and several other designs from [14].

With eight arms, the TM$_{10}$ antenna closely approaches the Thal limit, achieving nearly optimal $Q$ and maximizing bandwidth. In addition to approaching the lower bound, the antenna exhibits $Q$ lower than or equal to that of the comparison designs.

Finally, it can also been seen in Figure 5.4 that the operating frequency of the TM$_{10}$ antenna can be moved by increasing or decreasing the pitch of the wires. The small pitch and large pitch designs shown here are given only as examples. An even smaller or larger pitch could be used to tune the operating frequency lower or higher, respectively.
5.1.3 Impedance matching

Even if the antenna is operating in the low $Q$ region, it still must be matched to the system impedance to achieve maximum bandwidth (the $Q$ discussed here represents the $Q$ when resonated with a lossless reactance). As seen in Figure 5.3, the resistance at resonance is very small ($\sim 4 \Omega$) and changes little when the geometric parameters are changed. However, at the antiresonance, the resistance can be varied greatly.

The primary factor in determining the antiresonant resistance ($R_{ar}$) is the electrical size of the structure. Figure 5.5 shows how $R_{ar}$ varies as the antenna is tuned via pitch changes for a four-arm antenna. The total size of the structure stays constant but the resonant frequency changes, so $ka$ at antiresonance changes. When the antenna is tuned for very small electrical sizes, $R_{ar}$ becomes very large.

How we might match this structure without an external network is unclear. To gain a more fundamental understanding of the origin of the resonances and antiresonances, we will study its characteristic modes. The modes will allow us to separate effects from different parts of the structure, ultimately allowing us to control its impedance match.

Figure 5.5: Antiresonant resistance ($R_{ar}$) versus electrical size (at antiresonance) of a four-arm TM$_{10}$ antenna with a 0.050 inch substrate ($\epsilon_r = 1$) and a trace width of 2 mm. As the antenna is tuned to smaller electrical sizes, $R_{ar}$ increases rapidly.
5.2 Characteristic Modes of the TM$_{10}$ Antenna

Since we are interested in the behavior of the TM$_{10}$ antenna in the electrically small region, the number of significant characteristic modes is small when excited at the feed location indicated. As we will show, the superposition of just two modes results in a tunable, low $Q$ antiresonance and allows a multiresonant, wideband response if properly combined.

The characteristic modes of the structure were calculated using the approach outlined in Chapter 3. A large set of modes results from this computation, but the vast majority of these modes are not well-excited and contribute little to the overall response. Figure 5.6 illustrates the currents of the two most significant characteristic modes. Mode 1 looks very similar to the current distribution of the TM$_{10}$ spherical mode that the antenna was designed to excite, further verifying our initial design methodology. Mode 2 has a larger current along the feed trace than in the arms and a null near the base of each arm. Both characteristic modes are normal resonant modes as discussed in Chapter 3 (\textit{i.e.}, they are not special inductive modes).

The characteristic angle of the two modes over frequency is shown in Figure 5.7. Mode 1 has the smallest eigenvalue magnitude over the operating band and provides most of the antenna’s radiation. The Mode 1 resonance falls at the same frequency as the resonance of the entire structure. As with all resonant characteristic modes, the mode is capacitive before its resonance.
and inductive after its resonance.

The mode 2 resonance is significantly higher in frequency than either the first resonance or antiresonance of the entire antenna. Thus it presents a capacitive admittance in the operating band of the antenna.

Figure 5.8 shows the $\theta$ polarized radiation pattern of each mode at 1050 MHz. Cross-polarization is insignificant for both modes.

### 5.3 Modal Interactions and Antenna Input Impedance

As discussed in Section 3.3, the input admittance of an antenna can be expressed as a sum of the admittances of its characteristic modes. Individual characteristic modes behave as series RLC circuits in that they have declining capacitive susceptance before their single resonance and increasing inductive susceptance after their resonance (except in the case of special inductive modes which are always inductive). When multiple modes are excited in the same structure, it is similar to connecting these series RLC resonators in parallel since the admittances add directly from Equation 3.16. This circuit model suggests that resonances ($\frac{\partial B}{\partial f} < 0$) and antiresonances ($\frac{\partial B}{\partial f} > 0$) of the entire structure are caused by fundamentally different phenomena. Antenna
Figure 5.8: Normalized elevation and azimuth patterns of the first two characteristic modes of TM$_{10}$ antenna on a substrate with dielectric constant $\epsilon_r = 2.94$ at 1050 MHz. Only the $\theta$ component is shown because cross-polarization is insignificant.
resonances are generally caused by the resonance of a single dominant characteristic mode while antiresonances must be the result of the interaction of two or more characteristic modes, some that are capacitive and some that are inductive (either non-resonant or at a frequency above their resonance).

Because resonances are the result of a single characteristic mode, resonant properties depend almost entirely on the mode itself. Resonant conductance is very high for electrically small antennas due to large currents at the feed, resulting in the small resonant resistance that is often observed. Reducing the feed current and controlling the antenna’s resonant conductance is difficult.

5.3.1 Tuning the TM\textsubscript{10} antenna with characteristic modes

The TM\textsubscript{10} antenna exhibits both a resonance and an antiresonance in the electrically small region. According to Figure 5.7, the antenna’s resonance occurs at nearly the same frequency as the mode 1 resonance, and only mode 1 contributes significantly to the admittance at that frequency. However, consider the antenna’s antiresonance, which falls between the mode 1 resonance and the mode 2 resonance. We can postulate that the antiresonance occurs when the capacitive susceptance of mode 2 cancels, then exceeds, the inductive susceptance of mode 1.

To further investigate the source of the antenna’s antiresonance, the modal admittances of the antenna are calculated using Equation 3.16. The conductance and susceptance of the two significant modes are shown in Figure 5.9 for several different values of $\epsilon_r$, the substrate dielectric constant. From the figure, it is clear mode 2 resonates at a significantly lower frequency when a higher dielectric constant material is used, while mode 1’s resonant frequency changes little. The mode 1, mode 2, and antenna resonances and the antenna antiresonance are given by Table 5.1.

Since mode 2 has a larger current along the feed trace than in the arms, its resonant frequency is more strongly influenced by changes in the feed lines and substrate. On the other hand, mode 1’s resonance is moved primarily by changing the wire pitch since the majority of the current is in the arms. For an impedance match to 50 $\Omega$, the conductance at resonance or antiresonance should be 20 mS. In all cases, it is clear that the conductance at the modal resonances is orders of magnitude above this value, but the antiresonant
Figure 5.9: Admittance of the significant characteristic modes for substrates with varying dielectric constant. The total antenna admittance is shown by the black line.
Table 5.1: Resonant frequencies (GHz) of the antenna and its significant characteristic modes (1 and 2).

<table>
<thead>
<tr>
<th>εr</th>
<th>f1</th>
<th>f2</th>
<th>fR</th>
<th>fAR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.87</td>
<td>&gt;1.6</td>
<td>0.87</td>
<td>1.25</td>
</tr>
<tr>
<td>2.94</td>
<td>0.84</td>
<td>1.56</td>
<td>0.84</td>
<td>1.02</td>
</tr>
<tr>
<td>6</td>
<td>0.83</td>
<td>1.26</td>
<td>0.83</td>
<td>0.98</td>
</tr>
<tr>
<td>8</td>
<td>0.82</td>
<td>1.17</td>
<td>0.82</td>
<td>0.98</td>
</tr>
<tr>
<td>10.2</td>
<td>0.78</td>
<td>1.10</td>
<td>0.79</td>
<td>0.97</td>
</tr>
</tbody>
</table>

conductance is closer.

When the air substrate is used, the conductance at the antiresonance is \( \sim 1 \) mS. As the modes are moved closer together, the antenna antiresonance occurs at a frequency where the conductance is higher, and when \( ε_r = 6 \), the antiresonant conductance is \( \sim 20 \) mS. Thus, by modifying the spacing between the modes, the antenna’s antiresonant resistance can be controlled with very small changes in the operating frequency.

To illustrate this effect, Figure 5.10 shows the impedance of the antenna for the three different substrates. As \( ε_r \) increases, \( R_{ar} \) decreases while the resonant resistance hardly changes. This emphasizes that the resonant resistance is due to the characteristics of the first mode, while the antiresonant resistance is due to the interaction of the first and second mode and can be more easily modified.

The impedance loop closes noticeably as the permittivity is increased and the antiresonant frequency is shifted slightly lower. Other than the frequency shift, the \( Q/Q_{Chu} \) ratio is not significantly affected by the changing permittivity.

The substrate permittivity provides a coarse tuning option while the width of the feed trace allows finer control of the impedance. Figure 5.11 shows the effect of changing the trace width. The impedance change achieved with the trace width is much finer than when changing the substrate permittivity. The change in antiresonant frequency is also smaller.

5.3.2 Circuit model for the TM\(_{10}\) antenna

Following the framework developed in Section 4.1 for modeling the resonant characteristic modes as series RLC circuits, we can quickly develop a simple
Figure 5.10: Impedance of a four-arm $TM_{10}$ antenna ($ka = 0.47 - 0.62$) on a 0.050 inch substrate with varying permittivity. Increasing the permittivity decreases $R_{ar}$. Trace width is 2 mm, $p = 3$ mm, $b = 21$ mm, $r_h = 2.5$ mm.

Figure 5.11: Impedance of a four-arm $TM_{10}$ antenna ($ka = 0.47 - 0.60$) with a 0.050 inch substrate ($\varepsilon_r = 2.94$). Decreasing the trace width increases $R_{ar}$. $p = 2.85$ mm, $b = 21$ mm, $r_h = 2.5$ mm.
Figure 5.12: Comparison between simulated TM$_{10}$ admittance and admittance calculated using a simple two-resonator circuit model.

Figure 5.13: Comparison between simulated TM$_{10}$ impedance and impedance calculated using a simple two-resonator circuit model.

circuit model for the antenna (the case given above with $\epsilon_r = 2.94$). First we find the frequencies where $\lambda_1$ and $\lambda_2$ cross the x-axis. These are approximately $f_1 = 853.7$ MHz and $f_2 = 1.573$ GHz, which are the resonant frequencies of the two characteristic modes. Then, applying Equation 4.6, we find that $R_1 = 1.44\, \Omega$ and $R_2 = 0.15\, \Omega$. Finally, we calculate the $Q$ of each mode with Equation 4.7 to find $Q_1 = 40.3$ and $Q_2 = 167$. Figures 5.12 and 5.13 compare the admittance and impedance of the simulation to our simple model. The model is most accurate near the resonances of the antenna where we know precisely how the modes behave. As we move away from either resonance into the antiresonant region, the conductance and resistance become less accurate. Still, the antiresonant frequency is predicted fairly accurately.

This simple model is clearly missing some aspect of the modal behavior.
that occurs away from resonance. An obvious omission from our model is the frequency dependence of the modal feed current as discussed at the end of Section 4.1. Incorporating a simple linear frequency dependence to the modal resistance significantly improves the model as seen in Figures 5.14 and 5.15. The model is nearly indistinguishable from the simulation except for a small discrepancy in the antiresonant frequency. The procedure demonstrated here is extremely general and can be readily applied to nearly any antenna depending on the amount of complexity allowed in the circuit model.
Figure 5.16: Photographs of the TM\textsubscript{10} antenna fabricated for (a) \(ka = 0.49\) and (b) \(ka = 0.29\).

Table 5.2: Physical characteristics of the TM\textsubscript{10} antennas.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>(r_h)</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22.5</td>
<td>21</td>
<td>2.5</td>
<td>2.82</td>
</tr>
<tr>
<td>2</td>
<td>25.5</td>
<td>21</td>
<td>3.1</td>
<td>1.84</td>
</tr>
</tbody>
</table>

5.4 Measurements of the Single Mode Antenna

To verify the simulation results discussed in the previous sections, two antennas were constructed which were approximately the same physical size, but designed to operate at two different frequencies, 1.04 GHz and 0.541 GHz. Since their physical sizes were the same, their electrical sizes, \(ka = 0.49\) and \(ka = 0.29\), were different at their respective frequencies of operation. The antennas are pictured in Figure 5.16 and their dimensions are found in Table 5.2. The arms were constructed by winding 22 AWG wire around a plastic ring \((\epsilon_r \approx 4)\). The feed traces were laid out on a 50 mil Duroid 6002 \((\epsilon_r = 2.94)\) board which measured 30.7 x 23.1 cm. At 1.04 GHz this is 1.1 x 0.8 \(\lambda_0\) and at 0.541 GHz it is 0.55 x 0.42 \(\lambda_0\). Efficiency was measured using a Wheeler cap [55].

As discussed in the previous sections, the antennas are matched to 50 \(\Omega\) by modifying the feed trace and moving the second characteristic mode. In the case of the first antenna, the appropriate matching structure is a cross-shaped feed trace with 3 mm width. Since the second antenna is electrically smaller, it can be seen from Figure 5.5 that the antiresonant resistance will tend to be larger than that of the first antenna. Thus, mode 2 will need to be moved even lower in frequency to match to 50 \(\Omega\). A disk-shaped trace was
Table 5.3: Measured electrical characteristics of the TM\textsubscript{10} antennas.

<table>
<thead>
<tr>
<th>( \kappa a )</th>
<th>( f_0 ) (GHz)</th>
<th>( BW_{HP} )</th>
<th>( \eta_f )</th>
<th>( Q )</th>
<th>( Q_{Q_{Chu}} )</th>
<th>( Q_{Q_{Thal}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.49</td>
<td>1.04</td>
<td>9.1%</td>
<td>88%</td>
<td>22</td>
<td>2.3</td>
</tr>
<tr>
<td>2</td>
<td>0.29</td>
<td>0.541</td>
<td>2.1%</td>
<td>71%</td>
<td>95</td>
<td>3.0</td>
</tr>
<tr>
<td>2*</td>
<td>0.29</td>
<td>0.541</td>
<td>2.5%</td>
<td>71%</td>
<td>82</td>
<td>2.5</td>
</tr>
</tbody>
</table>

used instead to reduce the resonant frequency of mode 2. The radius of the disk was 19 mm, almost the size of the antenna.

Table 5.3 shows the measured values of center frequency, bandwidth, \( Q \), efficiency, and the \( Q \) ratios (including losses). The second antenna has a smaller electrical size and, as predicted by the Chu limit, a higher \( Q \) and a smaller fractional bandwidth. A better measure of the antennas’ relative performance is the \( Q \) ratios. The first antenna has smaller Chu and Thal ratios, so its bandwidth is closer to the optimum for its electrical size.

The difference in the two antennas’ \( Q \) ratios is attributed to bandwidth reduction from the electrically small ground plane on which the second antenna is mounted. Appendix C contains a detailed discussion of the effects of finite ground planes on small antennas. At the center frequency of the first antenna the ground plane is 1.1 x 0.8 \( \lambda_0 \), but for the second antenna it is only 0.55 x 0.42 \( \lambda_0 \). At ground plane sizes below a wavelength, the achievable bandwidth tends to fall precipitously for electrically small monopole-type antennas. Therefore, the second antenna was placed on a larger ground plane measuring 111 x 111 cm (2.0 x 2.0 \( \lambda_0 \)) to measure its bandwidth in a situation comparable to that of the first antenna. Since the matching feed trace was designed with the antenna on the smaller ground plane, the much larger ground plane changes the impedance match. After accounting for this impedance mismatch, the bandwidth was found to increase and the \( Q \) ratio decreased. The measurements are shown in Table 5.3 as antenna 2*. Though the Chu ratio is still slightly larger than that of the first antenna, the Thal ratios are now equal, suggesting that the two antennas have equivalent performance relative to their electrical sizes, but different absolute performance.

As one would expect, a lower efficiency was measured for the electrically smaller antenna, which was also observed in simulations. We attribute this to increased ohmic loss in the more tightly coiled wires, and expect the trend to continue with further reductions in electrical size.

The measurements largely agree with simulations of the antennas on a
finite ground plane with lossy materials, although the measured $Q$ is slightly higher than expected. The measured input impedance is compared with simulation in Figure 5.17, and the measured radiation patterns are shown in Figure 5.18. The antennas were constructed by hand and this led to some imperfections in the construction, particularly inconsistent wire pitch and a slightly prolate shape to the sphere. These deviations from the ideal shape reduce coupling to the $TM_{10}$ mode and increase the $Q$.

5.5 Direct Ink Writing of Spherical Meanderline Antennas

Because of the challenges of fabricating the $TM_{10}$ antenna described in previous sections, we considered other geometries that would be more suitable for automated fabrication. In collaboration with Prof. Jennifer Lewis’s research group in Materials Science and Engineering at the University of Illinois, we developed a structure which can be fabricated automatically and precisely using a direct ink writing process [56, 57, 58].

The conformal antenna is shown in Figure 5.19; we will refer to this antenna as the spherical meanderline (SM). The new antenna consists of metallic meanderlines printed on the surface of a Pyrex hemisphere. The metallic lines

Figure 5.17: Measured vs. simulated impedance of the $ka = 0.49$ $TM_{10}$ antenna operating at 1.04 GHz.
are fabricated using a printed silver ink with a conductivity of approximately $2 \times 10^7$ S/m. The conductivity of the ink is determined by the annealing temperature and time as seen in Figure 5.20a [56]. The stated value was estimated from the DC conductivity of ink when heat treated at 550 °C for 3 hours, and confirmed when measured efficiency was similar to the simulated value. The ink was cured at this high temperature to maximize the conductivity, as the efficiency is dominated by conductor loss in the meanderlines. Figure 5.20b shows the relationship between the ink resistivity, $\rho$, and the antenna efficiency for an antenna operating at $ka = 0.5$. As expected, the efficiency is reduced as the antenna is operated at smaller electrical sizes. The ink can be printed using nozzles with inner diameters from 2 μm to 200 μm.

Ideally, the hemisphere would have a low permittivity to reduce the energy stored in the material and the radiation $Q$. However, since the antenna needs to be annealed at temperatures around 500 °C, common circuit materials such as teflon and other plastics cannot be used. The maximum working temperature of Pyrex 7740 glass is over 500 °C, and it has a relatively low permittivity ($\epsilon_r \approx 4.6$), low loss, and low cost [59]. The Pyrex hemisphere was chosen to be hollow with walls as thin as possible. The approximate thickness of the hemisphere is 1.2 mm. The hemisphere is epoxied to the planar substrate using a non-conductive epoxy and then the arms are con-

Figure 5.18: Radiation patterns of a four-arm TM$_{10}$ antenna.
Figure 5.19: An illustration showing the geometry of the spherical meanderline antenna.

Figure 5.20: (a) Resistivity of the silver ink vs. annealing time for a variety of annealing temperatures [56]. (b) Radiation efficiency vs. ink resistivity of an antenna at $ka = 0.5$. 

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connected to the feed lines with a conductive silver epoxy. The theory of operation of this antenna is the same as for the original single mode TM$_{10}$ antenna. The structure of the wires excites the TM$_{10}$ mode and a higher order mode as described in Section 5.2. The higher order mode is conditioned by the feed lines to tune the input impedance as described in Section 5.3.1. To control the radiating TM$_{10}$ mode, the self-inductance of the meanderlines is adjusted by modifying the spacing between them. The ink cross-section is also important as this will contribute to the per-unit-length inductance of the meanderlines. Based on profilometry measurements of a meanderline printed with a 100 $\mu$m nozzle, shown in Figure 5.21, the simulations approximate the wire cross-section as a $W_n \times W_n/5$ rectangle where $W_n$ is the inner nozzle diameter.

5.5.1 Measurements of the spherical meanderline antennas

To demonstrate this approach, we created four electrically small antennas with varying $ka$ value, operating frequency, and surface type (concave or convex) [60]. The structures were again simulated in HFSS to determine the precise geometric parameters, summarized in Table 5.4, to achieve the desired operating frequency and impedance.
Table 5.4: Geometry of the fabricated spherical meanderline antennas (refer to Figure 5.19).

<table>
<thead>
<tr>
<th>Design</th>
<th>Hemisphere radius, $a$, mm</th>
<th>Number of arms</th>
<th>$\alpha$, $^\circ$</th>
<th>$w$, mm</th>
<th>$p$, $\mu$m</th>
<th>Duroid height, mm</th>
<th>Copper trace width, mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>SM1</td>
<td>12.7</td>
<td>8</td>
<td>33.2</td>
<td>100</td>
<td>662</td>
<td>1.6</td>
<td>0.6</td>
</tr>
<tr>
<td>SM2</td>
<td>12.8</td>
<td>4</td>
<td>80.0</td>
<td>100</td>
<td>635</td>
<td>0.8</td>
<td>5.5</td>
</tr>
<tr>
<td>SM3</td>
<td>6.55</td>
<td>8</td>
<td>34.6</td>
<td>30</td>
<td>620</td>
<td>0.8</td>
<td>0.1</td>
</tr>
<tr>
<td>SM4</td>
<td>12.7</td>
<td>8</td>
<td>37.3</td>
<td>100</td>
<td>560</td>
<td>1.6</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Table 5.5: Radiation characteristics of the spherical meanderline antennas.

<table>
<thead>
<tr>
<th>Design</th>
<th>$ka$</th>
<th>$f_0$ (GHz)</th>
<th>$BW_{HP}$</th>
<th>$\eta_r$</th>
<th>$Q$</th>
<th>$Q_{Q_{\text{typ}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SM1</td>
<td>0.459</td>
<td>1.73</td>
<td>15.2%</td>
<td>71%</td>
<td>13.1</td>
<td>1.5</td>
</tr>
<tr>
<td>SM2</td>
<td>0.211</td>
<td>0.79</td>
<td>6.2%</td>
<td>14%</td>
<td>31.9</td>
<td>2.0</td>
</tr>
<tr>
<td>SM3</td>
<td>0.490</td>
<td>3.57</td>
<td>12.1%</td>
<td>63%</td>
<td>16.5</td>
<td>2.5</td>
</tr>
<tr>
<td>SM4</td>
<td>0.450</td>
<td>1.70</td>
<td>12.6%</td>
<td>66%</td>
<td>15.9</td>
<td>1.8</td>
</tr>
</tbody>
</table>

Initially, the antenna measurements did not match well with HFSS simulations. The measured antennas had lower radiation resistance, lower efficiency, and many small impedance loops. The antennas were also very susceptible to flexing of the substrate and feed cable. We determined that the cause of this was the asymmetry in the fabricated structure. In simulation when everything was symmetric, the apex of the antenna (where all of the arms meet) could be connected or disconnected with no difference in the antenna’s performance. However, once the symmetry was broken, if the arms remained apex-connected, the antenna performance changed dramatically as the current at the apex was no longer forced to be zero. Small asymmetries allowed additional modes to exist, which destroyed the antenna’s desired impedance behavior. Once the apex connection was removed, even large asymmetries had little effect. All antennas discussed below were fabricated with a 1.6 mm diameter circular gap at the apex and align well with simulated results in a repeatable fashion. Their radiation characteristics are summarized in Table 5.5.

The first of the four antenna designs (SM1) was optimized to operate at 1.7 GHz. The particular frequency was chosen to result in a $ka = 0.5$ antenna using the hemispheres that were available (25.4 mm diameter). An eight-arm design was fabricated using a 100 $\mu$m ink nozzle. The spacing between
the meanderlines was 660 μm, and the meanderlines subtended an azimuthal angle of 37.7°. A photograph of one of the fabricated antennas is shown in Figure 5.22a. Figure 5.22b shows a representative antenna being patterned on the outer surface of the glass hemisphere.

Four copies of this antenna were fabricated to evaluate the repeatability of the fabrication process. Figure 5.23 shows the simulated and measured VSWR for four fabricated antennas, and Figure 5.23b shows the impedance of one of these antennas. Their center frequencies range from 1.69 to 1.74 GHz and have similar bandwidths, indicating that this fabrication process provides excellent control and reproducibility. The efficiencies of these antennas fall within the range of 68% to 72%, very close to the predicted value of 66%.

One of the four samples had a bandwidth of 15.2% and an efficiency of 71%. Applying the relation between $Q$ and bandwidth in Equation 2.2, the antenna’s $Q$ is 13.1. This $Q$ value is 1.5× the Chu limit and 1.1× the Thal limit, representing a greater than 50% improvement in performance relative to the helical TM$_{10}$ antenna reported in 5.4 and nearly an order of magnitude bandwidth increase compared to a conventional monopole design. Given that this antenna closely approaches the Thal limit, its design is essentially optimum for its size.

Since the antenna is electrically small, it radiates in an omnidirectional monopole pattern with a slight tilt and small back-plane radiation due to the finite ground plane used during measurements (Figure 5.24). Importantly, the SM1 exhibits excellent performance relative to the fundamental limits as well as good agreement with simulations.

In a similar fashion to Section 5.4, we next demonstrate the facile tuning of the design to a different frequency while maintaining the same physical size. This is again equivalent to showing that the antenna can be made to operate at different electrical sizes. The antenna (SM2) is designed with a smaller electrical size ($ka \approx 0.2$) by conformal printing of four conductive meander-line arms on the convex surface of an identical glass hemisphere ($a = 12.7$ mm), targeting an operating frequency in the 700 MHz band. This band was made available by the transition from analog to digital broadcast television in the United States and is highly valued for mobile communications due to its propagation characteristics.

A photograph of the antenna is shown in Figure 5.25a, and its measured VSWR is shown in Figure 5.25b. The measured center frequency of SM2
Figure 5.22: Photographs of (a) the completed antenna (SM1) and (b) the antenna during printing.
Figure 5.23: Comparison between simulation and (a) measured VSWR of four copies of SM1 and (b) the measured impedance of one of these copies.

Figure 5.24: Normalized azimuth and elevation patterns of SM1.
is 790 MHz with a half power bandwidth of 6.2% and measured efficiency of 14%. This reduced efficiency is expected in self-resonant antennas at smaller electrical sizes [61]. While the bandwidth is widened by the lower efficiency, no net effect is observed in the ratio of the antenna’s $Q$ to the Chu and Thal limits because these limits are also lowered proportionally to the efficiency. These ratios are slightly higher than for SM1, because SM2 only has four arms. As the number of arms increases, the antenna more closely approximates the $TM_{10}$ distribution and exhibits a lower $Q$.

Next, we create an antenna (SM3) of nominally the same $ka (\approx 0.5)$ as SM1 on the convex surface of a smaller glass hemisphere ($a = 6.55$ mm), raising the targeted frequency to 3.6 GHz. This demonstrates that the antenna can be easily redesigned for a new form factor, and that the printing process is capable of producing very small features, enabling electrically small antennas operating up to several GHz. In this case, the eight meanderline arms are patterned by conformally printing the silver nanoparticle ink through a 30 $\mu$m nozzle which is more appropriate for the smaller physical size of the hemisphere. A photograph of the antenna is provided in Figure 5.26 along with its measured VSWR values. The measured center frequency of SM3 is 3.57 GHz with a half power bandwidth of 12.1%, while its measured efficiency is 63%, similar to SM1. SM3 possesses $Q$ ratios that are slightly higher than the other designs, because the wall thickness of the smaller glass hemisphere
is the same as the larger hemisphere. More of the internal volume of the antenna is occupied by this higher permittivity material, and thus more energy is stored internally.

The radiating structure on the outside of the hemisphere is easily damaged by incidental contact with objects. Thus, as an additional variation to the SM designs, we fabricated a fourth antenna (SM4) on the concave (interior) surface of the glass hemisphere. Printing on the concave surface of the hemisphere is more challenging than on the exterior surface because the nozzle begins to contact the hemisphere, particularly in the tight space near the apex of the radiating structure. The glass hemisphere is embedded in a polydimethylsiloxane (PDMS) mold to hold it in place during printing (Figure 5.27a). After printing is completed, the patterned structure is removed from the mold. The hollow glass hemisphere now serves as a support structure and protective barrier, allowing the device to be easily handled and mounted on a low-loss laminate substrate (Figure 5.27b). Designed to perform nearly identically to SM1, this antenna also exhibits a $Q$ close to the lower bound. Its center frequency is at 1.70 GHz with bandwidth and efficiency of 12.6% and 66%, respectively. The Chu and Thal ratios are 1.8 and 1.4, respectively. These are slightly higher than those of SM1 because the radiating component of the antenna (i.e., the meanderline arms) is electrically smaller than SM1 by the thickness of the glass hemisphere. However,
when calculating $ka$, we include the entire thickness of the hemisphere in the size.

### 5.5.2 Direct ink writing of a spherical helix

As a test of the fabrication technique and to directly compare the spherical meanderline antenna to another effective design, we also fabricated a spherical helix antenna similar to those in [16, 17]. A slight modification was made to the spherical helix to allow more flexibility in terms of the electrical sizes and impedances for which it could be designed. The original spherical helix was only matched to 50 Ω for particular electrical sizes. Recalling the
modal theory discussed in earlier chapters, the original spherical helix can be thought of as a combination of a radiating TM_{10}-like mode, and a non-resonant, inductive mode with currents flowing in the grounded feed point and flowing out one of the other grounded points. Both the radiating mode and the inductive mode are conditioned by the same parameter, namely, the number of turns on the helix. When the number of turns is chosen to resonate the radiating mode at a particular frequency, the non-resonant mode is set to a particular inductance that cannot be changed without also affecting the radiating mode. Thus, the spherical helix has two modes with only one parameter to effectively control them, and the choices of antiresonant impedance are limited.

Rather than matching with a non-resonant inductive mode as done in [16, 17], we can instead match the helix with a higher order resonant mode that contributes capacitive susceptance to the total response. As with the TM_{10} and spherical meanderline antennas, the higher order resonant mode is excited by putting a thin substrate between the helix and the ground plane, feeding the center of a copper trace, and designing the trace appropriately. Now, the primary mechanisms controlling the two modes are distinct and a match can be achieved at a variety of impedances and electrical sizes. To demonstrate this approach, we fabricated a spherical helix designed for a match to 50 Ω at 3.6 GHz shown in Figure 5.28. The antenna is printed on a (nominally) half inch diameter hollow Pyrex hemisphere like the ones used for the SM antennas, so the electrical size of the antenna is nominally ka = 0.5. The fabricated antenna differed somewhat from the simulations, and the center frequency was measured to be 3.2 GHz as shown in Figure 5.29. The shift is most likely due to a thinner ink cross-section than assumed in simulation and the asphericity of the hemisphere. Due to the frequency shift, the match deteriorated somewhat, but still shows that a clear radiating mode exists and that the capacitive matching approach will also work with the spherical helix. In the future, we plan to make a spherical helix printed with the direct ink writing process that is matched using Best’s approach [16, 17]. That is, all arms except one would be connected shorted to ground and the unshorted arm would be fed. The arms would be connected at the apex to allow an inductive mode to flow from feed to short and provide impedance matching. We believe a comparison between the two matching approaches would show that they are essentially equivalent.
Figure 5.28: Photograph of the printed spherical helix.

Figure 5.29: Measured reflection coefficient of the printed spherical helix from 3 to 4 GHz.
CHAPTER 6

MULTIMODE ELECTRICALLY SMALL ANTENNAS

The previous sections discussed historical ESA design and our new design based on the excitation of the lowest $Q$ spherical mode. We believe that, with a $Q$ at the Thal limit, the spherical helix and the TM$_{10}$ antenna have the lowest $Q$ possible for an antenna whose currents are constrained to a spherical surface. Therefore, new research questions must be posed in the area of electrically small antenna design.

One research direction with a great deal of promise is the design of multimode (or multiresonant) small antennas. Multimode antennas offer the promise of bandwidths that are not constrained by the Chu limit, as we will see in the following section. While we have demonstrated an optimal single mode design, the study of multimode ESAs has just begun.

The Goubau antenna is probably the most famous multimode design [62]. Goubau achieved a very large impedance bandwidth (2:1 VSWR bandwidth $\approx 68\%$), but the antenna was not truly electrically small ($ka$ was slightly larger than 1) and the design methodology was not clear. A body of work on wideband antennas just outside of the electrically small region followed Goubau’s result and Best summarized the most significant contributions in [63].

Few multiresonant electrically small antennas ($ka \leq 0.5$) have been reported. Recently Stuart and Tran designed a multiresonant antenna with $ka = 0.54$ [64]. With a matching network, the antenna achieved 2:1 VSWR bandwidth approximately equal to that expected for a singly resonant antenna at the Chu limit. Stuart and Best also reported a wideband antenna with $ka = 0.53$ and 1.7 times the impedance bandwidth predicted by Chu’s limit [65].
6.1 Multimode Bandwidth Estimates

It is of interest to predict the maximum achievable bandwidth when two resonant modes are combined. Like Chu’s limit, this analysis would serve as an important design consideration and figure of merit with which actual multiresonant antennas could be compared. Other authors have approached this problem by considering the case of two coupled resonators that are not electrically small, such as patches [66, 67]. Both approaches result in smaller “maximum” bandwidths than found in this work, and only modes with the same resonant frequency are considered. However, as we will see in Section 6.3, creating modes with the same resonant frequency is difficult in strongly coupled structures such as electrically small antennas.

As we have shown in Chapter 4, radiating small antenna modes can be reasonably equated to series resonant circuits described by a resonant frequency, radiation resistance, and quality factor. Having this simple model for the circuits, we can readily calculate the bandwidth when two such resonators are combined in parallel. The general approach to enlarging the bandwidth in this manner is to separate the modes in terms of resonant frequency, but not so much as to violate the VSWR specifications between the resonant frequencies. In addition, we want to consider cases in which no external matching network is required, and the antenna is matched using an internal network such as the inductive or capacitive matches described in Sections 4.2 and 5.3, respectively.

To begin with, consider a series RLC circuit defined by \( R_1, f_1, Q_1 \) such that \( Z_1 = R_1 \left(1 + jQ_1 \left(\frac{f}{f_1} - \frac{f}{f_1}\right)\right) \). Assuming that \( Q_1 \gg 1 \), it is well known that such a circuit has a fractional bandwidth (FBW) centered at \( f_1 \) and equal to [4]

\[
FBW = \frac{s - 1}{Q_1 \sqrt{s}}
\]  

(6.1)

where \( s \) is the maximum allowable VSWR, \( Z_0 \) is the system impedance, and we have assumed that the circuit is matched at the resonant frequency (i.e., \( R_1 = Z_0 \)).

Adding a second series resonator (defined by \( R_2, f_2, Q_2 \)) in parallel we find the total admittance of the system
Figure 6.1: An illustration of two different types of VSWR response possible with two arbitrary resonant circuits.

\[
Y = \frac{1}{R_1 \left( 1 + jQ_1 \left( \frac{f}{f_1} - \frac{\Delta f}{f_1} \right) \right)} + \frac{1}{R_2 \left( 1 + jQ_2 \left( \frac{f}{f_2} - \frac{\Delta f}{f_2} \right) \right)}.
\]  

(6.2)

From this expression, we can find the reflection coefficient, \( \Gamma = \frac{Y_0 - Y}{Y_0 + Y} \), as a function of \( R_1, f_1, Q_1, R_2, f_2, Q_2, f, \) and \( Y_0 \). The resulting expression is extremely complicated and offers little insight into the problem at hand.

Instead, we take a heuristic approach to the problem. We know that if the resonances are spaced very far apart, there will be two separate matched bands with each mode operating as if the other were not present (see Figure 6.1a). Then, if we move the modes closer together we will eventually observe a “double-tuned” type of response in the VSWR such as that of Figure 6.1b. Each VSWR dip is still near the resonant frequency of the individual mode, but the region between the resonances differs significantly. However, outside the resonant frequencies of the modes, the total response tracks almost exactly with the response of the nearest mode. Thus, we can postulate that the problem of maximizing the bandwidth of this two-resonator system is actually a matter of separating the two modes by as large a frequency range as possible while still maintaining the specified match in the region between them. From circuit simulations in Agilent ADS, we observe that the maximum mismatch between the two modes is a monotonic function of the frequency difference between the two modes. The only time this does not hold is in the trivial case when the modes are very close and there is no middle VSWR peak. The farther apart the modes are, the larger the VSWR peak between them, regardless of the other circuit parameters.
With this observation, we can develop a procedure to find the maximum bandwidth attainable for the two modes. First, we assume that Mode 1 is at a fixed frequency and we can control the resonant frequency of Mode 2. Then, we incrementally move Mode 2 away from Mode 1 and numerically compute the VSWR from the admittance of Equation 6.2. For each incremental frequency spacing, we search for the peak value of VSWR between the two resonances. When the VSWR peak exceeds a set specification, we stop moving Mode 2 away and estimate the total bandwidth given the chosen circuit parameters. Since the outer edges of the band behave approximately as a single isolated mode, the total bandwidth is estimated by applying Equation 6.1

\[
BW = f_{\text{upper}} - f_{\text{lower}}
\approx f_2 \left(1 + \frac{s - 1}{2Q_2 \sqrt{s}}\right) - f_1 \left(1 - \frac{s - 1}{2Q_1 \sqrt{s}}\right)
\approx (f_2 - f_1) + \frac{s - 1}{2\sqrt{s}} \left(\frac{f_1}{Q_1} + \frac{f_2}{Q_2}\right)
\] (6.3)

and we estimate the center frequency as

\[
f_c = \frac{f_{\text{upper}} + f_{\text{lower}}}{2}
\approx \frac{f_2}{2} \left(1 + \frac{s - 1}{2Q_2 \sqrt{s}}\right) + \frac{f_1}{2} \left(1 - \frac{s - 1}{2Q_1 \sqrt{s}}\right)
\approx \frac{1}{2} \left(f_2 + f_1 + \frac{s - 1}{2\sqrt{s}} \left(\frac{f_2}{Q_2} - \frac{f_1}{Q_1}\right)\right).
\] (6.4)

For electrically small antennas, our primary concern is the bandwidth attainable at a particular electrical size. Happily, we also have models for the resonant circuit parameters of the antenna for a variety of electrical sizes. In particular, we know that the \(Q\) of each mode is bounded from below by \(Q_{Chu} = \frac{1}{ka} + \frac{1}{(kaj)^2}\) [2], which will give the largest possible bandwidth. We also know that the resistance of each mode tends to vary linearly with frequency, but this complicates our simple model so we will assume a constant resistance. Since the resistance can be selected to some extent (by varying the feed current with stubs or other design modifications), we will assume that
we can choose the resistances of the modes. The modal resonant frequency is easily changed as we have seen, so we will also allow it to be chosen.

First let us examine the case where the resistances of the modes are set to be perfectly matched at their resonant frequencies, \( R_1 = R_2 = Z_0 \). A Matlab program was written to perform the procedure discussed above in order to find the maximum frequency spacing that meets the VSWR specification. Figure 6.2a shows the fractional bandwidth (FBW) resulting from this calculation for different VSWR values and across a range of electrical sizes. The maximum FBW for the two mode case has a shape similar to that of the limits calculated by Chu, and does not diverge toward infinite bandwidth at larger electrical sizes like the limits calculated by Hujanen et al. [68] and Villalobos et al. [69]. Figure 6.2b shows the ratio of the maximum FBW for the two mode case to the maximum FBW for a single mode at the same \( ka \) value. The ratio is nearly constant over electrical size, although there is clearly more improvement when the VSWR specification on the antenna is tighter. The bandwidth increase is approximately 3.4 and 2.6, for the \( s = 2 \) and \( s = 5.828 \) cases, respectively. These bandwidth improvements are significant and greater than the simple two-fold increase which might be expected.

As an additional test, the same procedure was repeated for a case when the two modes have an increased \( Q \) to represent antennas with \( Q \) approximately equal to Thal’s limit \( (Q_1 = Q_2 = 1.5 \times Q_{Chu}) \). Figure 6.3a shows the maximum FBW versus electrical size for this case, and it appears that the only change is a small decrease in bandwidth at a given \( ka \). Figure 6.3b reveals that the bandwidth reduction from the first case is inversely proportional to the increased modal \( Q \). The ratios of maximum FBW to the FBW of a single mode at the Chu limit are now 2.3 and 1.7, respectively, representing a factor of 1.5 reduction from the case where both modes had \( Q \) at the Chu limit. Thus, we expect that two modes with \( Q \) much above the Chu limit can still outperform a single ideal mode \( (i.e., \text{with } Q \text{ at the Chu limit}) \). For example, referring back to Figure 6.2, two modes with \( Q_{1,2} \leq 3.4 \times Q_{Chu} \) could perform as well as a single mode with \( Q = Q_{Chu} \) in terms of the 2:1 VSWR bandwidth. This suggests that a multimode antenna does not need to be optimal to outperform Chu’s limit, so simple, non-spherical shapes are feasible.
Figure 6.2: (a) Maximum attainable fractional bandwidth (FBW) of two matched modes with $Q_{1,2}$ at the Chu limit for different VSWR ($s$). (b) The ratio of the attainable FBW using the two modes to the FBW of a single mode at exactly the Chu limit for different VSWR ($s$).
Figure 6.3: (a) Maximum attainable fractional bandwidth (FBW) of two matched modes with $Q_{1,2}$ at $1.5 \times$ the Chu limit for different VSWR ($s$).
(b) The ratio of the attainable FBW using the two modes to the FBW of a single mode at exactly the Chu limit for different VSWR ($s$).
6.1.1 Multimode bandwidth estimates for optimally mismatched antennas

The preceding analysis assumes that the modes are perfectly matched, i.e., at resonance $s = 1$. For maximum bandwidth, this is not the optimal matching. In fact, the antenna should be intentionally mismatched for a wider bandwidth. Consider the equation for VSWR, $s$, in terms of the antenna impedance $Z_a$ and the system impedance $Z_0$:

$$s = \frac{|Z_a + Z_0| + |Z_a - Z_0|}{|Z_a + Z_0| - |Z_a - Z_0|} \quad (6.5)$$

Near resonance, the series resonator model, $Z_a = R \left(1 + jQ \left(\frac{f}{f_0} - \frac{1}{f_0}\right)\right)$, can be approximated by expanding $\text{Im}\{Z_a\}$ in a Taylor series about $f_0$. Taking the first term of the series gives $Z_a \approx R \left(1 + j2Q \left(\frac{f}{f_0} - 1\right)\right)$. Using this expression in Equation 6.5 we find

$$s \approx \frac{\sqrt{(R + Z_0)^2 + 4R^2Q^2 \left(\frac{f}{f_0} - 1\right)^2} + \sqrt{(R - Z_0)^2 + 4R^2Q^2 \left(\frac{f}{f_0} - 1\right)^2}}{\sqrt{(R + Z_0)^2 + 4R^2Q^2 \left(\frac{f}{f_0} - 1\right)^2} - \sqrt{(R - Z_0)^2 + 4R^2Q^2 \left(\frac{f}{f_0} - 1\right)^2}} \quad (6.6)$$

Solving for $f$, we find

$$f = f_0 \left(1 \pm \sqrt{\frac{(s - Z_0 R)(s Z_0 R - 1)}{2Q\sqrt{s}}} \right). \quad (6.7)$$

It follows that we can write the fractional bandwidth as

$$FBW = \frac{\sqrt{(s - Z_0 R)(s Z_0 R - 1)}}{Q\sqrt{s}}. \quad (6.8)$$

When

$$Z_0 = Z_{0,\text{opt}} = R \frac{s^2 + 1}{2s} \quad (6.9)$$

the maximum fractional bandwidth is obtained,
\[ FBW_{\text{opt}} = \frac{s^2 - 1}{2Qs}. \]  
\[ (6.10) \]

Compared to the FBW of the perfectly matched resonator given in Equation 6.1, this optimal mismatch provides a bandwidth increase of

\[ \frac{FBW_{\text{opt}}}{FBW_{R=Z_0}} = \frac{s + 1}{2\sqrt{s}}. \]  
\[ (6.11) \]

It is not immediately clear how mismatching the modes will affect the bandwidth when two modes are combined in parallel. Thus, we modify Equations 6.3 and 6.4 using Equation 6.8 to find

\[ BW \approx (f_2 - f_1) + \sqrt{(s - \frac{Z_0}{R})(s \frac{Z_0}{R} - 1)} \left( \frac{f_1}{Q_1} + \frac{f_2}{Q_2} \right) \]  
\[ (6.12) \]

\[ f_c \approx \frac{1}{2} \left( f_2 + f_1 + \sqrt{(s - \frac{Z_0}{R})(s \frac{Z_0}{R} - 1)} \left( \frac{f_2}{Q_2} - \frac{f_1}{Q_1} \right) \right) \]  
\[ (6.13) \]

Then, applying the same procedure outlined earlier, we find the maximum spacing of the modes and the bandwidth for a variety of electrical sizes. There is no guarantee that the optimal mismatch for a single mode is the same as for two modes, so the mismatch is varied. The two modes are assumed to have the same resonant resistance \( R_1 = R_2 = 50 \Omega \), but the system impedance \( Z_0 \) is allowed to take a variety of values. Figure 6.4 shows the improvement above a single mode with \( Q \) at the Chu limit for \( s = 2 \) and \( s = 5.828 \). The blue line shows the perfectly matched case where \( R_1 = R_2 = Z_0 \). The green line shows the result when the match is chosen using the Equation 6.9 for the optimal single mode match. Finally, the red line shows the match that gives the largest bandwidth increase as determined by running the program for several values of \( Z_0 \). Clearly, Equation 6.9 does not give the optimal match for the two mode case, although it still gives a significant bandwidth improvement.

From Equation 6.11, a single mode match can be improved by factors of 6.1% and 41%, for \( s = 2 \) and 5.828, respectively. For two modes, however, the bandwidth gained by mismatching the antenna is even greater. When \( s = 2 \), a 23% increase in bandwidth is realized, while an 82% increase is
Figure 6.4: The ratio of the attainable FBW using two modes to the FBW of a single mode at exactly the Chu limit for VSWR ($s$) of (a) $s = 2$ and (b) $s = 5.828$. Both modes are assumed to have $R_1 = R_2 = 50\,\Omega$, but the system impedance takes several values.
found when \( s = 5.828 \).

Adding a second mode benefits the bandwidth most when the VSWR requirement is tight (e.g., \( s = 2 \)) if the modes are perfectly matched. However, we have shown that if they are mismatched even greater bandwidth can be obtained by selecting a system impedance of \( Z_0 \approx R \frac{3s^2+1}{4s} \). Based on these results, we expect that an antenna with two ideal modes (i.e., \( Q \) at the Chu limit), could have a bandwidth 4 to 5 times that of a single, perfectly matched mode. Note that this improvement is greater than that seen by adding an infinite order matching network to the antenna (3.8 and 3.2 for the \( s = 2 \) and \( s = 5.828 \) cases, respectively [70]). This can be explained by the fact that the additional “matching” elements in the antenna are the radiating modes which have desirable “loss.”

### 6.2 Conductance Ratio As a Figure of Merit

For singularly resonant small antennas, \( Q \) is the salient factor in evaluating an antenna’s bandwidth potential. One of the implicit assumptions required to relate bandwidth to \( Q \) is that the antenna’s impedance response can be modeled as a single series or parallel RLC circuit over the frequency range of interest. However, two or more resonances created near enough to each other cannot be modeled by a single RLC circuit [71]. While Chu’s limit still applies to the \( Q \) of each mode, the inverse relationship between \( Q \) and bandwidth no longer holds [72]. As we have seen, several additional factors contribute to the attainable bandwidth, including the frequency spacing of the modes and their conductance maxima and minima. The \( Q \) of an individual mode no longer has critical importance, although it will affect how far apart in frequency modes can be before VSWR specifications are violated. To reduce the complications that these additional factors bring, we wish to develop a simple figure of merit to evaluate the multimode potential of a design.

Consider matching an antenna to an arbitrary system impedance \( G_0 \). In order for the VSWR to remain below a desired value at both a resonance and an antiresonance, the ratio of the resonant conductance to the antiresonant conductance must be less than or equal to the square of the desired VSWR. This observation is easily derived from the equations for VSWR. At a resonant point (\( Y_{ant} = G_{ant} \)), then, the VSWR can be written
We then assume that at a resonance, $G_R > G_0$, and at an antiresonance, $G_{AR} < G_0$, since we desire that the antenna’s impedance encircle the center of the Smith chart for minimum VSWR over the entire bandwidth. If we assume that $VSWR = VSWR_{max}$ at both resonances, we find that $G_R/G_{AR} = VSWR^2_{max}$ and $G_0$ should be chosen as $G_0 = \sqrt{G_R G_{AR}}$.

For example, if the half-power bandwidth ($VSWR = 5.828$) is to extend from a resonance to the following antiresonance, $G_R/G_{AR}$ must be less than 34.0, assuming that we use a matching network to move the system impedance to the geometric mean of the resonant and antiresonant resistances and match out any reactance at the minima and maxima.

Of course, this approach neglects the behavior of the susceptance and $Q$ of the modes and is not a guarantee of multiresonant matchability but rather a minimum requirement. While approximate, this resonance-antiresonance conductance ratio (CR) is useful because it allows us to quickly evaluate an antenna’s multiresonant potential even when the conductance values are not centered around the desired system conductance. If the conductance has a swing greater than the square of the maximum VSWR between resonances and antiresonances, it is clear that it cannot have a VSWR below the desired value at both the resonant and antiresonant frequencies. In the next section, we use the CR to evaluate the multiresonant potential of the TM$_{10}$ antenna.

### 6.3 Multiresonant TM$_{10}$ Antenna Using Low Order Characteristic Modes

The goal is to create an electrically small antenna with bandwidth beyond that predicted by the Chu limit by matching the antenna at several resonances caused by the interaction of multiple modes. We will focus on designing an antenna using the TM$_{10}$ structure with $ka \sim 0.5$ so that the antenna is electrically small.

The most obvious way to create an additional mode in the TM$_{10}$ antenna is to offset the pitches of half of the helices so that two low order, TM$_{10}$-
like modes appear and resonate at different frequencies. The initial design consists of two arms with pitch $p_a$ and two arms with pitch $p_b$, where the arms with the same pitch are placed opposite each other. As anticipated, this approach leads to two characteristic modes with slightly offset resonant frequencies. In the low frequency mode (Mode 1a), the current is highest in the arms with the smaller pitch, and in the high frequency mode (Mode 1b), the current exists in the arms with the larger pitch. Figure 6.5 contains a schematic showing these modes.

However, while both modes are generally well excited by the feed location, an anomalous drop in the conductance of Mode 1a occurs with this configuration. A typical example of this response can be seen in Figure 6.6. When the conductance drops to nearly zero in the band of interest, the CR becomes very large and multiresonant behavior is not possible.

According to (3.16), such a drop in the conductance can either come from a large increase in the modal eigenvalue at that frequency or a sharp drop in the feed point current of the mode. In this case, the mode is near resonance and the eigenvalue is actually decreasing. Thus, there must be a drop in the feed point current. Figure 6.5a shows the currents of Mode 1a. There are two notable current nulls on the feed traces associated with the inactive arm. As frequency increases, the nulls are observed to move along the trace...
Figure 6.6: Modal input conductance of a two mode antenna with offset pitches showing anomalous falloff of Mode 1a conductance.

toward the center until they reach the feed point and cause the conductance null.

The current nulls approach the feed symmetrically along the inactive arms. Observing the characteristic modes, it seems that the symmetry of the structure supports the current null anomaly. To eliminate the nulls, the symmetry of the structure is broken by placing the arms with the same pitch adjacent to each other, rather than across from each other. This asymmetric configuration eliminates the current null and allows distinct modes 1a and 1b to be excited as seen in Figure 6.7.

However, while the CR is lower than symmetric configuration, it is still much too high for multiresonant operation (Figure 6.7a). While the conductance peaks are fundamental to the mode and difficult to change, the antiresonance can often be adjusted to some extent as discussed in Section 5.3. If the modes were closer together, their conductance curves might cross at a higher value and result in a higher antiresonance conductance minimum.

To test this, the pitches $p_a$ and $p_b$ are moved closer together. Figure 6.7b shows the conductance response under this scenario. However, the effect is not entirely as intended; the modes move closer together but Mode 1a begins to deteriorate as it moves close to 1b. As the pitches move even closer, Mode 1a continues to deteriorate, eventually collapsing into the single TM$_{10}$-like
Figure 6.7: Modal input conductance of two mode asymmetric, offset pitch antenna with (a) $p_a = 3.0$ mm, $p_b = 9.5$ mm and (b) $p_a = 3.5$ mm, $p_b = 4.5$ mm. Both designs are on a 0.050 inch Duroid 6002 substrate and have $b = 21$ mm, $r_H = 2.5$ mm, $w = 2$ mm, and wire diameter of 0.2 mm.
mode that appears when the pitches are the same. In order to operate the antenna at multiple resonances, a different set of modes, which have either a lower resonant conductance or a higher antiresonant conductance, must be found.

6.4 Multiresonant TM$_{10}$ Antenna Using Higher Order Characteristic Modes

Combining the two first order modes to yield a multiresonant structure seems impossible because of the very high CR that was found. However, higher order modes can also be excited in the structure in the electrically small region. As shown in Section 5.3, when the pitches are equal, a higher order mode (Mode 2) is excited. In Chapter 5, we used this mode to tune the impedance of a single antiresonance. Now we will attempt to use it to reduce the CR so that several resonances can be combined.

For example, Figure 5.9 shows Modes 1 and 2 when the substrate dielectric constant ($\epsilon_r$) is changed. Mode 2’s resonance decreases greatly when $\epsilon_r$ increases while the Mode 1 resonance hardly moves. As this happens, the minimum conductance increases and the CR decreases. As these modes get closer together, they do not degenerate into one as was observed with the low order modes. Thus, the combination of Mode 1 and Mode 2 can be made to have a much smaller CR than the two low order modes.

To further reduce the CR, variations of the original design were simulated. Geometric parameters such as the pitch of the arms, feed trace width, substrate dielectric constant, and number of helical arms were varied and their effects cataloged.

Since increasing the number of arms allows the currents to better approximate the TM$_{10}$ mode, the $Q$ is closer to the lower bound when there are many arms. However, the coupling between the more closely spaced arms requires a smaller wire pitch to achieve the same operating frequency.

When the feed arms are very wide and the substrate dielectric constant is high, an interesting effect is observed. The currents of Mode 2 differ somewhat from those shown in Figure 5.6. Because the electrical length of the feed trace is now quite large, the null in the Mode 2 distribution, which was previously in the arms, is shifted to the feed trace. As a result, Mode
Figure 6.8: Modal input conductance of two mode, eight-arm antenna on 0.050 inch Duroid 6010 with $b = 21$ mm, $p = 2.0$ mm, $w = 6$ mm, $r_H = 2.5$ mm, and wire diameter = 0.6 mm.

2 now has a TM$_{10}$-like current distribution in the arms similar to Mode 1. Both modes’ feed trace currents remain in opposite directions. Under this configuration, they both have a very low $Q$, and either could be used as an effective radiating mode. However, since Mode 2 resonates at a frequency approximately 50% higher than Mode 1, it has a significantly lower conductance. This makes it a much better candidate for multimode combination with the antiresonance between Modes 1 and 2. Instead of combining the low-order TM$_{10}$ mode (Mode 1) with the following antiresonance, we combine the higher-order TM$_{10}$ mode (Mode 2) with its preceding antiresonance and find that a much lower CR can be achieved. Furthermore, these two modes have very similar radiation patterns, resembling that of a monopole, so the total pattern remains consistent across the band and cross-polarization is very low.

6.4.1 Multiresonant antenna with matching network

To demonstrate this concept, an eight-arm antenna on 0.050 inch Duroid 6010 was designed with the following parameters: $b = 21$ mm, $p = 2.0$ mm, $w = 6$ mm, $r_H = 2.5$ mm, and wire diameter = 0.6 mm. Simulations of the lossless antenna were performed in FEKO. Given that the antenna structure has not changed significantly from the antenna in Chapter 5, we expect the efficiency of this antenna to be 80-90%.

The input conductance of the two modes is shown in Figure 6.8. In Fig-
ure 5.9, the TM_{10}-like mode was resonant around 850 MHz. Now, the higher order mode is resonant at 1050 MHz, and we expect that at this larger electrical size, the conductance of the resonance will be lower. This is precisely what is observed: the resonant conductance of mode 2 in Figure 6.8 is an order of magnitude lower than mode 1 in Figure 5.9. Meanwhile the antiresonant conductance is approximately the same as it was in Figure 5.9c. Therefore, the CR is reduced by an order of magnitude to approximately 12; however, there is a large inductive susceptance. Agilent ADS was used to simulate connecting the antenna to a 21 Ω system impedance and using a 43 pF shunt capacitor to resonate it.

Figure 6.9 shows a VSWR plot of the simulated antenna using this matching network. Centered at approximately 1050 MHz, the antenna has 120 MHz of 2:1 VSWR bandwidth, which is about 11.4%. The radiation pattern of the antenna is consistent across the band and is shown in Figure 6.10. The antenna’s electrical size, from the bottom of the substrate to the outer radius of the furthest helix, is \( ka = 0.54 \). Using Chu’s limit to estimate the bandwidth of a singly resonant antenna at this electrical size, we find that this antenna’s fractional bandwidth is about 31% larger than that achievable with a single resonance in the ideal case. Compared to the more practical limit derived by Thal [6], the antenna has 96% greater bandwidth than an ideal singly resonant antenna. It is important to emphasize that this result does not violate Chu’s fundamental principle, which is a constraint only on the \( Q \) of an antenna. Instead we use the fact that the \( Q \)-bandwidth relationship no longer holds for closely spaced resonances [72] to increase the bandwidth.

If we consider the non-radiating mode to be an additional “matching circuit” for the radiating mode and apply Fano’s matching limitations, the bandwidth of the mode can be improved by at most a factor of 2.31 [70]. The ideal \( Q \) at \( ka = 0.54 \) is approximately 11 [6], so the optimal 2:1 VSWR bandwidth for such an antenna is 15%, close to the value we have attained. On the other hand, if we consider an ideal matching network for a TM_{10} mode, the attainable bandwidth is over 60% at this electrical size [68], suggesting significant room for improvement through further research.

The applicability of this method to electrically small structures around \( ka = 0.5 \) is clear as bandwidth has been increased by a significant amount. However, very electrically small structures (\( ka < 0.4 \)) may require new ap-
Figure 6.9: VSWR of the multimode antenna with shunt capacitor connected to a 21 Ω system impedance.

Figure 6.10: Normalized radiation patterns for the multimode TM_{10} antenna from HFSS simulation.
proaches because the necessarily large conductances associated with the resonances result in very large conductance ratios that are difficult to adjust. Wideband operation may still be possible in terms of half power bandwidth (VSWR = 5.828), but the CR needed to achieve wideband operation in terms of VSWR = 2 bandwidth may be difficult.

6.5 Design and Measurement of an Electrically Small Multiresonant Antenna

Following this analysis, we wish to fabricate a multimode antenna that achieves the wide bandwidth demonstrated in the previous section, but that also avoids the use of an external matching network. Achieving both of these goals is very challenging, but we will use the framework developed in previous sections to guide the design.

To begin with, we have already developed a basic structure for an antenna that radiates two modes with a low conductance ratio in the previous section. This structure uses the TM$_{10}$ topology that was originally developed for optimal single-mode $Q$ in Section 5.1-5.3. This structure is effective, but difficult to fabricate precisely or at small physical sizes. Therefore, we elected to use the spherical meanderline (SM) topology as the basis for a multimode design. In Section 5.5 we have shown that the behavior of the SM antenna is well-understood and similar to that of the TM$_{10}$ antenna. The translation from the TM$_{10}$ design to a SM design is straightforward; the only change is how we create increased electrical length of the conductor on the hemispherical surface. Thus, we expect that multimode behavior can be created in a similar way (i.e., by creating a first order mode and a second order mode resonant at similar frequencies using a thin, high permittivity substrate).

Simulating the structure in HFSS, we found that a low conductance ratio ($\approx 14$) could be obtained using this type of geometry. These HFSS simulations are not presented here since the design process is similar to the previous section. However, we found that the attainable conductance ratio was slightly higher than that of the TM$_{10}$ antennas simulated in Section 6.4. We suspect that this is due to the additional stored energy in the hemisphere. In Section 6.4, we assumed that there were no dielectrics in the space except for the planar dielectric substrate. However, for a practical design, the conduc-
tive ink must be printed on Pyrex. Since the Pyrex stores additional energy relative to the air, it becomes more difficult to combine the two modes. A similar effect would be observed if we designed the antennas at a smaller electrical size.

The next challenge was matching the antenna to 50 Ω without using a matching network. A matching network could be used as in the previous section, but the additional size of a network increases the size of the antenna and is counterproductive to our goal of miniaturizing the antenna. Thus, we seek to integrate the matching network into the antenna. For a single mode, this is a fairly simple undertaking. As we have shown, in Sections 4.2 and 5.3.1 respectively, a single mode can be easily matched using an inductive mode excited via a stub or a higher-order capacitive mode below its resonance. Once we have two radiating modes, independently matching one without affecting the other is extremely difficult. Just as we need at least two independent geometric parameters to match a single mode (one for the radiating mode and one for the matching mode), we need at least four independent geometric parameters to match two modes.

The simplest technique to add an additional matching mode is to make a short path to ground. This approach is similar to the stub matching approach discussed in Section 4.2. The feed trace can be shorted to ground with little difficulty. Ideally, this shorting pin would only affect one mode at a time and could be moved around to find an appropriate match. However, as we saw with the shorted monopole antennas in Section 4.2, the shorting pin tends to change the response of the radiating mode as well. To some extent, this effect is necessary and beneficial. For example, adding the shorting pins tends to decrease the modal feed current and thus increases the driving point impedance above its natural value of a few ohms. The problem is that the shorting pin changes the resonant frequencies and impedance of both radiating modes simultaneously, and the effects are difficult to model. Ultimately, we found that an optimization was necessary to find the appropriate geometry.

The main change to the antenna design is the pattern on the planar substrate. The new board design is shown in Figure 6.11. The main modification to the simple cross pattern used for the single mode antenna is a stub in the azimuth direction that ends in a shorting pin. The radial position of the stub center is designated $r_s$, its width is $w_s$, its length is $l_s$ and the pin diameter
Figure 6.11: Schematic of the board for the multimode antenna showing key dimensions.

is \( d \). We set a pin diameter of \( d = 1.628 \) mm to represent the diameter of the 14 AWG wire. The board is also chosen to be thin with a high dielectric constant so we used a piece of 25 mil thick Duroid 6010 (\( \epsilon_r = 10.2 \)).

We optimized the board for use with a hemisphere of radius 12.7 mm, a 100 \( \mu \)m nozzle, and center-to-center spacing \( s = 4.1 \) mm. The optimization was restricted such that the gap between the end of the stub and the adjacent arm was more than a stub width \( w_s \). The stub was also required to be wholly within the inner diameter of the hemisphere (\( \approx 1.2 \) mm less than the outer diameter) to avoid misalignment of the hemisphere due to the solder bump at the shorting pin. The dimensions were restricted to the following ranges: \( r_s = [3, 12] \) mm, \( w_s = [0.5, 3] \) mm, \( l_s = [0, 12] \) mm, \( w = [0.5, 6] \) mm. The cost function to be maximized was

\[
f_c = N_{2:1}^2 + N_{3:1}^2
\]

(6.15)

where \( N_{2:1} \) and \( N_{3:1} \) represent the number of frequency points where VSWR < 2 and VSWR < 3, respectively.
The optimization resulted in a board with the following parameters: $r_s = 10.3\, \text{mm}$, $w_s = 2.3\, \text{mm}$, $l_s = 6.6\, \text{mm}$, $w = 3.35\, \text{mm}$, $l = 13.0\, \text{mm}$. The four-arm radiating hemispherical structure is fabricated in a similar manner to the antennas discussed in Section 5.5. Its geometric parameters, with reference to Figure 5.19 are: $a = 12.7\, \text{mm}$, $\alpha = 68^\circ$, $w = 100\, \mu\text{m}$, and $p = 4.1\, \text{mm}$. The fabricated antenna is shown in Figure 6.12.

The antenna was measured attached to a 50 $\Omega$ system with no matching network. The measured VSWR, shown in Figure 6.13, clearly shows two radiation modes with similar impedance. The measured efficiencies at the VSWR minima of the low and high frequency modes are 75% and 60%, respectively. However, the match is not as good as the simulated case due to some additional inductance in the measured antenna. This illustrates that the antenna is operating with both modes as we expect, but it is also very sensitive to small fabrication details. The measured mismatch is likely due in part to the soldering of the shorting pins and the large amount of silver epoxy used to make the electrical connection at the base. The overall behavior is also very sensitive to the ink cross-section and hemisphere sphericity, both of which can vary.
Figure 6.13: Simulated and measured VSWR of the multimode antenna attached to a 50 Ω system impedance.
CHAPTER 7

CONCLUSION

The topic of small antennas has always garnered interest in the microwave community because of the wide variety of applications in which miniaturized antennas would be useful. Because they are very limited in geometry, one might expect that small antennas are the simplest antennas to understand. If the designer has a fundamental understanding of their behavior, then they can be understood in terms of several principles. However, a proliferation of design approaches, some of them misguided, has led to a wide body of research that is difficult to understand as a whole.

Essentially, small antennas are able to radiate in one of two modes with any reasonable bandwidth. First is the TM$_{10}$ mode, which has the prototypical dipole pattern with a null along $\theta = 0, \pi$ and $E_\theta$ polarization. As shown by Thal [6], an antenna coupling entirely into this mode would have the minimum $Q$. Small antennas also occasionally operate in the TE$_{10}$ mode, which has a similar pattern shape and nulls along $\theta = 0$, but $E_\phi$ polarization [73]. This mode tends to have a $Q$ approximately twice that of the TM$_{10}$ mode and so is not commonly used [6].

Because a large amount of current is required to radiate a unit of power, the input resistance of a wire-fed electrically small antenna is typically on the order of a few ohms, and without special care to create appropriate electrical length, the antenna will have a large capacitive or inductive reactance. As we discussed in this work, there are two simple approaches to impedance matching a small antenna using internal distributed elements, but both require that the radiating mode be self-resonant near the match frequency for the best results. Self-resonance can be achieved using a variety of approaches to create more electrical length. We also saw that most conducting structures support a characteristic mode that approaches the theoretical bound on $Q$ for the enclosing volume. Ideally, we would like to couple into this low $Q$ mode, and use some an additional mode to match to the necessary
impedance. Again, this leads to the conclusion that we must design the structure and feed so that this fundamental mode is self-resonant near the design frequency. Then, we can excite an additional mode using the stub (inductive) matching technique or the higher order (capacitive) matching technique to provide the impedance matching.

Finally, it appears that the Chu limit is, in fact, an accurate lower bound for $Q$. However, there are ways to circumvent the strict inverse relationship between $Q$ and bandwidth. In particular, multiple radiating modes can be combined to create a response with significantly enhanced bandwidth, but the design complexity is significantly increased.

7.1 Contributions

This body of research has made several important contributions to the field of small antenna theory and design.

- Introduced a new type of electrically small antenna that excites the TM$_{10}$ mode for minimum $Q$.
  This antenna exhibits excellent $Q$ performance, approaching the lower bound derived by Thal [6] as the number of arms is increased. Our antenna’s performance also verifies Thal’s result by showing that the antenna’s $Q$ approaches Thal’s limit but does not exceed it. The antenna also exhibits $Q$ comparable to some of the best existing designs from [14]. As an added advantage, the design can be tuned to operate in a quasi-TM$_{10}$ mode at any frequency and for any electrical size in the electrically small region via changes in coil pitch. It can also be matched to 50 $\Omega$ without an external matching network using several tuning mechanisms introduced in this work. Prototypes of the original design were constructed, verifying simulated results.

- Used a unique direct ink writing process to fabricate smaller, more stable, and more precise TM$_{10}$ antennas.
  Because of our understanding of the physics of the antenna’s operation, the changes to the antenna’s geometry resulted in little change in its behavior. Thus, our fundamental design approach allows modifications to be made to
the antenna with ease. This spherical meanderline exhibits even lower $Q$ than the TM$_{10}$ design, and it shares the TM$_{10}$ antenna’s versatility in terms of possible electrical sizes, frequencies, and matched impedances. Our success in fabricating this small structure has implications for increasing frequency and decreasing the physical size at which these structures can be practically built. Up to now, most three-dimensional electrically small antennas have been built for the hundreds of MHz or below. The new conformal structure can now be fabricated to make three-dimensional, electrically small antennas perhaps as high as 10 GHz. Such antennas would have numerous applications in commercial mobile communications and wireless data links, and the printing technique could enable many interesting new electromagnetic devices. The ability to deposit conductors on curved surfaces could have an impact on future antenna packaging approaches.

- Used characteristic mode theory to formulate a unified approach to model the input impedance of electrically small antennas.

  Characteristic mode theory allowed us to decompose the antenna response into the individual current modes that can be more easily analyzed. The natural behavior of the modes motivated a simple framework for developing circuit models for small antennas. The small antenna prototype circuit consists of series RLC resonators connected in parallel, and this topology illuminates the sharp differences between antenna resonances and antiresonances. Antenna resonances are the natural resonances of the characteristic modes while antiresonances are caused by modal interactions.

- Described two simple matching techniques for small antennas using characteristic mode concepts.

  We described both inductive and capacitive matching techniques for small antennas. Both methods require no external components and are enabled by exciting an additional characteristic mode. However, we explained that they do require that the antenna be self-resonant near the desired match frequency. We showed that an inductive mode can be easily excited in a monopole antenna via a shorted stub. Through an analysis of this shorted stub technique, we learned why the stub acts as both an impedance transformer and a shunt reactance. We specifically demonstrated how the capacitive approach can be applied to the TM$_{10}$ antenna. Observations of how the modes respond to
geometric changes resulted in an understanding of how the antenna’s antiresonance can be tuned easily to a large range of resistances without affecting other performance parameters.

- Discovered that a minimum $Q$ mode exists in a variety of structures but is often masked by excitation of higher order modes.

  Again using characteristic mode theory, we found that even as a structure’s feed point or arrangement of conductors was changed, a minimum $Q$ mode exists. The $Q$ of this fundamental mode only changed when the shape of the radiating portion of the antenna changed appreciably, and using Gustafsson’s limit, we found that this mode is near the minimum bound for a variety of cylindrical and spherical shapes [8]. Typically, this mode is hidden when the total antenna response is measured or simulated because higher order modes are significantly excited. Only through a modal decomposition does it become apparent that a low $Q$ mode is persistently excited. This analysis suggests that small antenna designers should strive to couple into these existing modes and minimize higher order mode excitation rather than develop complex new structures in an attempt to excite modes that are already present.

- Estimated the maximum achievable bandwidth when combining two modes to form a multimode response.

  To achieve bandwidth larger than the Chu limit, we concluded that we needed to break the inverse $Q$-bandwidth relationship by exciting modes at closely spaced frequencies. Using the standard circuit models developed in this dissertation, we calculated approximate values for the bandwidth improvement that can be obtained by combining two ideal modes. If the modes are perfectly matched we found an improvement over Chu’s limit by a factor of 2.5 to 3.5. However, if the modes are optimally mismatched, bandwidth increases by factors of 4 to 5 are possible.

- Developed a simple figure of merit to enable a multiresonant response using characteristic modes and applied the method to achieve a bandwidth greater than the Chu limit with the TM$_{10}$ antenna.

  We determined that a reasonable design approach is to keep the resonance-antiresonance conductance ratio to a minimum. We applied characteristic
mode theory to identify modes which are candidates for multiresonant operation and eliminate those which are not. The first design attempted with a low order TM$_{10}$-like mode was found to exhibit an interesting modal null, but was ultimately not useful for multiresonant operation. By using a higher-order TM$_{10}$-like characteristic mode to achieve a small conductance ratio, an electrically small multiresonant antenna ($ka = 0.54$) was designed with half-power bandwidth greater than Chu’s limit by 31% and than Thal’s limit by 96% for an ideal singly resonant antenna.

- Applied the stub matching technique to match the impedance of a multimode antenna and fabricated a multimode antenna.

  Given the analysis of the stub matching antenna as an impedance transformer, we developed a distributed circuit matching approach for a multimode spherical meanderline antenna. Using the stub matching technique, we designed a practical, internally matched antenna with bandwidth equal to the Chu limit. The fabricated prototype exhibited two radiating modes, and correct modal frequencies with a slight impedance mismatch.

  We believe this body of research leads to a new way of thinking about electrically small antenna design in which aggregate antenna physics is viewed in terms of individual modal behavior. This view of the antenna’s operation is a more insightful approach to designing antennas. Recent work by Best [13] has shown that brute force optimizations like genetic algorithms provide no advantage in electrically small design. Ultimately, these optimization techniques are used when the physics of the problem are too complex to be understood. The application of characteristic mode theory reduces the complexity of the problem to a degree, resulting in simpler and more versatile designs than brute force calculation.

7.2 Future Work

There are a variety of future research avenues stemming from this work. In particular, we see a few main areas of further research.
• Single mode small antenna design

In the past few years, researchers, including the present author, have demonstrated antennas approaching the revised physical limit developed by Thal [6]. Therefore, it would seem that there is little potential for continued research into broadband small antennas. In some respects, this is true. Traditional, single resonance antennas have reached the performance limit ascribed to them. However, there is still some space for continued research in this area. The most significant advances are likely to come from new materials. As one example, Thal predicted that a material with very high permeability could create an antenna with \( Q \) exactly at the Chu limit [6]. This would represent a 50% increase in bandwidth from current designs, but the lower bound would prevent further increases. A variation of this approach using magnetic materials was recently shown in [74].

Hujanen et al. [68] and Villalobos et al. [69] have recently shown that when the actual, low pass circuit model of the \( \text{TM}_{10} \) mode is considered instead of the resonant approximation assumed by most authors, a significantly larger bandwidth can be achieved. At larger electrical sizes, this bandwidth rises rapidly towards infinity. The reason is that the low pass model is not well-approximated by a resonant circuit, which has been traditionally applied to calculate the bandwidth. Why then have no authors reported bandwidths beyond the traditional limits? We postulate that the reason is because no antennas have yet been designed that can adequately excite the \( \text{TM}_{10} \) mode over a broad range of frequencies. Instead, the resonant approximation is actually more accurate for practical antennas because they only excite the \( \text{TM}_{10} \) mode in a narrow bandwidth. Consider the spherical meanderline antenna designed to excite the \( \text{TM}_{10} \) mode at a particular frequency. If the antenna were filled with a material whose permittivity profile varied as \( \epsilon_r \propto f^{-2} \), the distances within the structure would be the same in terms of wavelengths, and a \( \text{TM}_{10} \) mode that was previously narrowband would now be excited over a broad bandwidth. Of course, many other aspects of this design would need to be studied such as dispersion, loss, and the availability of such a material, but it is an interesting avenue for future investigation.

We are also interested in comparing fabricated spherical meanderline and spherical helix antennas that use both inductive and capacitive matching. This will illuminate the advantages and disadvantages of the two designs and of the matching techniques.
Multimode small antenna design

Without some exotic new materials, significant improvements in single mode small antennas are likely to be incremental. However, there is a vast design space available to develop multimode antennas. These antennas have exciting potential; as we have shown in this work, four-fold bandwidth improvements are possible by adding just one additional mode. We have demonstrated simulated results of an antenna with bandwidth just at the Chu limit, so significant space remains for further improvement. Of course, we intend to continue the fabrication of the current multimode design presented in this work but are also considering other geometries. One idea is to print different radiators on the interior and exterior of the hemisphere and possibly even nest hemispheres of different sizes to create several radiating modes that can be independently matched using a redesigned feed structure.

Characteristic modes

Since characteristic modes are extremely general, we can apply them to any radiation or scattering problem, and there are many avenues for further research. One area that needs continued investigation is the models for eigenvector and eigenvalue behavior. I believe these models can be further refined to apply over a broader range of frequencies while also imposing physical realizability constraints on the resulting circuit models. If the variation of the eigenvalues could be predicted accurately from just a few frequency points, they could then be interpolated over a wide frequency range. The normal solution of the method of moments (MoM) equations involves the inversion of the $[Z]$ matrix. It has been shown that the characteristic modes can express a spectral form of this matrix inversion [33]. Since only a few modes are significantly excited unless the object is electrically large, $[Z]^{-1}$ can be represented by the summation of just a few of these modes. If the modes can be estimated in some fundamental way over frequency, MoM calculations could potentially be sped up.

Finally, we have observed many interesting phenomena in our study of the characteristic modes of small antennas. One is the occurrence of modal current nulls at the feed point. The nulls are sometimes very narrowband and an antenna could be designed to null out a particular mode under certain conditions. This phenomenon could be used to sense material properties or
as a reconfiguration mechanism.
APPENDIX A

COMPUTATION OF MODES WITH MATLAB AND FEKO

The procedure for calculating the characteristic modes involves several steps carried out using both FEKO and Matlab. We will discuss the key steps here, although not every feature of the code can be sufficiently explained in this document.

A.1 Solving for the Characteristic Modes

A.1.1 FEKO procedure

The user begins the simulation in FEKO Suite 5.5 [35] using the component called CADFEKO. This component allows the user to create the necessary geometry, boundary conditions, and material properties for the problem. Note that the characteristic mode program only supports lossless materials at this time.

In CADFEKO, the user designs the geometry as they would for the solution of a normal antenna problem with a few small differences. First, no ports should be defined. At this time, the characteristic mode program only supports excitation by a wire gap feed. However, rather than define the feed point, the user should short the wire where they expect the port to be, or if the port location is yet to be determined, place a short wire at any potential port location. For example, in the case of a characteristic mode analysis of a helical monopole with a shorted stub, both the stub and the helical coil are shorted to ground. The input impedance at any wire node can be determined after the eigenvalue problem is solved.

The other primary difference is that the user must edit certain solution settings that cannot be modified using CADFEKO, so the component EDIT-FEKO must be called after the model is completed. When the model is com-
plete and has been meshed appropriately, the user selects Run→EDITFEKO and a window will ask if the CADFEKO solution configuration should be disabled. The CADFEKO solution should be disabled so that the EDITFEKO solution can supersede it. Several settings will need to be edited in EDITFEKO.

- **EG card**: The EG card must be edited in two ways. First, the box “Write geometrical output to FEKO output file (*.out)” must be checked and the solution accuracy must be set to double precision.

- **FR card**: This card should be edited to create the moment method matrix at the desired frequency points. The continuous data option should not be used.

- **A0 card**: This card must be used to trick FEKO into thinking there is a plane wave excitation for the problem since we have not defined a port. To do this, set the number of $\theta$ and $\phi$ angles to 1 and set the magnitude of the excitation to zero.

- **PS card**: The PS card allows us to save the impedance matrix and the currents. Make sure that “Save matrix elements to *.mat file” and “Save currents to *.str file” are selected.

- **OS card**: The OS card should be set so that all currents are written to the output file.

Additional cards for far field calculations, ground planes, and other features should be set as defined in the FEKO documentation. Once these settings are made, the user selects Run→FEKO to calculate the impedance matrices.

### A.1.2 Matlab procedure

Several files are generated during the FEKO solution process. Most important are the *.mat, *.str, and *.out files. These contain the impedance matrix, output currents (which should all be zero at this point since the excitation was zero), and geometric data in ASCII format. A Matlab user interface has been created to parse these files, solve the eigenvalue problem, and then write the characteristic currents to the *.str file. The Matlab graphical user interface (GUI) is shown in Figure A.1.

To use the GUI, the user first selects “Browse” and looks for the *.mat file associated with the problem they are interested in solving. Then when the
Figure A.1: Matlab user interface for the solution of characteristic modes.
file is loaded, the user should set the values of $M$ and the maximum number of modes to be calculated. The $[R]$ matrix is approximated as a positive definite matrix by retaining only the singular values which are larger than $M$ times the maximum singular value of $[R]$ (see Appendix B for further details on $M$). $M$ must range between 0 and 1. The larger $M$ is, the fewer modes will be kept, and the more approximation is involved in the calculation. However, as $M$ becomes very small, a huge number of insignificant and numerically noisy modes are retained, complicating the problem unnecessarily. The best value of $M$ greatly depends on the size of the problem and whether it involves wires only or wires and surfaces. Typically $M$ is somewhere between $10^{-6}$ and $10^{-3}$. When the problem is saved, only the first $N$ modes with the lowest eigenvalues will be retained, where $N$ is the maximum number of modes that are set.

Once these settings are made, the user can select “Calculate CM” to solve the eigenvalue problem and sort solutions. Then the eigenvalues can be plotted using “Plot eigenvalues.” To find modal admittances at a feed location, its node must be specified in the “node number” box. Then the user can use “Plot Modal Admittances” to show the individual modal admittances, the total impedance, the modal currents at that node, and the $Q$ of each individual mode. The node number can be found by inspecting the geometric data in the *.out file.

Once a desirable set of data has been calculated, the user selects “Save CM data” to save the entire set of modal eigenvalues and eigenvectors to a file. This data can then be loaded at a later time using “Load CM data” rather than resolving the problem.

There are several other features of the Matlab interface that are not discussed here.

A.2 Visualization of the Characteristic Modes

The currents, near fields, and far fields of each mode can be displayed in FEKO after being solved with the Matlab GUI. First, the EDITFEKO file must be modified again. The A0 card should be changed so that the number of $\theta$ angles is equal to the number of modes. This makes FEKO behave as if the currents from each mode are actually caused by plane waves arriving
at different angles. Then, the PS card must be modified so that “Read matrix elements from *.mat file” and “Reads currents from *.str file” are set. Finally any desired far or near field cards should be added so those fields are computed. The user then runs FEKO once to generate a new Checksum value with the updated settings. Note that this should generate a FEKO error which is ignored.

Next, the user deletes the original *.str file which should contain all zero currents. The user then renames the *.modes files to *.str and opens the file. The user should overwrite the second line of the new *.str file with the Checksum value from the most recent *.out file. Now all settings should be complete and the user can run FEKO. FEKO will compute the modal near and far fields from the provided modal currents and the post-processing component POSTFEKO can be used to visualize these quantities. Each mode can be displayed individually by choosing the appropriate number from the “Solution” column in POSTFEKO.
APPENDIX B
CHARACTERISTIC MODE SOLUTION
APPROACH

The approach taken to solve the eigenvalue problem has some bearing on the accuracy and meaningfulness of the results. In theory, the eigenvalues and eigenvectors found by solving (3.10) should be real because \([R]\) is a symmetric positive semi-definite matrix. However, due to the numerical inaccuracies in calculating the matrix entries, \([R]\) usually has small negative eigenvalues. Solving (3.10) directly using \([R]\) will result in complex eigenvalues and eigenvectors that have unclear physical meaning.

To resolve these complex eigenvalues, we investigated several other methods of computing eigenvalues.

B.1 Harrington’s Method

In his early work on characteristic modes, Harrington also encountered this problem. He addressed the problem by diagonalizing \([R]\) and removing the small eigenvalues \([23]\).

Let \([U]\) be the matrix which diagonalizes \([R]\) such that

\[
\mu = [U^T R U] = diag(\mu_i)
\]  

(B.1)

where \(diag(\mu_i)\) indicates a diagonal matrix with the eigenvalues of \([R]\), \(\mu_i\), along the diagonal.

Then we remove any negative and very small eigenvalues by setting the eigenvalues \(\mu_i < M \mu_1\) to zero, where \(\mu_1\) is the largest eigenvalue and \(M\) is a parameter set by the user.
The matrix $[\mu]$ can then be partitioned as

$$[\mu] = \begin{bmatrix} \mu_{11} & 0 \\ 0 & 0 \end{bmatrix}$$  \hspace{1cm} (B.2)$$

where $\mu_{11}$ is the diagonal matrix of non-zero eigenvalues of $[R]$.

Then define

$$[x] = [U^TJ] = \begin{bmatrix} [x_1] \\ [x_2] \end{bmatrix}$$  \hspace{1cm} (B.3)$$

and

$$[A] = [U^TXU] = \begin{bmatrix} [A_{11}] & [A_{12}] \\ [A_{12}^T] & [A_{22}] \end{bmatrix}.$$

Premultiplying (3.10) by $[U^T]$ and using the definitions in (B.2), (B.3), and (B.4) we find

$$[\mu_{11}^{-1/2}][A_{11} - A_{12}A_{22}^{-1}A_{12}^T][\mu_{11}^{-1/2}][\mu_{11}^{1/2}x_1] = \lambda[\mu_{11}^{1/2}x_1]$$ \hspace{1cm} (B.5)$$

which is a standard eigenvalue problem of the form $[B]y = \lambda y$. The eigenvalues $\lambda$ are the same as those in (3.10) while the eigenvectors $J$ can be found as

$$J = [U] \begin{bmatrix} [I] \\ [-A_{22}^{-1}A_{12}^T] \end{bmatrix} [\mu_{11}^{-1/2}]y.$$ \hspace{1cm} (B.6)$$

The number of eigenmodes that can be found from this calculation is the same as the number of non-zero eigenvalues in $[\mu]$. This is, in turn, determined by $M$, the ratio of the smallest eigenvalue kept to the largest. $M$ is typically set between $10^{-4}$ and $10^{-5}$, depending on the problem being solved and the number of eigenmodes desired. The Matlab interface allows the user to easily modify $M$ to capture all desired eigenvalues at all frequencies.

### B.2 Higham-Cheng Algorithm

Another approach that has been used to solve the characteristic modes problem is the Higham-Cheng algorithm [75]. The algorithm was applied in [76]
to solve a 2-D eigenproblem with greater accuracy than direct matrix inversion. Based on this result, we applied the algorithm to our 3-D problem. The algorithm searches for a rotation \((e^{-j\theta})\) and perturbation \((\Delta R + j \Delta X)\) to the original matrices \([R]\) and \([X]\) that results in the nearest definite pair that can be simultaneously diagonalized. Ultimately, the new pair is expressed as

\[
\hat{R} + j \hat{X} = e^{-j\theta} (R + \Delta R + j (X + \Delta X)).
\]  

(B.7)

Finding the minimizing point requires many solutions of an eigenproblem of a matrix of the same size as \([R]\) and \([X]\). In most cases, we found that while this method gave accurate results, it took a significantly longer time than Harrington’s method, so Harrington’s method was used to calculate the characteristic modes in this work.
The monopole-type antennas that are discussed in Chapters 4, 5, and 6 all require the use of a ground plane. An infinite ground plane simplifies the simulation process, but the finite ground planes used in practice can be a variety of sizes. Therefore, it is important to get a sense of the behavior of the antennas over finite ground planes.

It is well-known that an antenna on a finite ground plane behaves differently than on an infinite ground plane. The most commonly observed effect is that the radiation pattern of a monopole-like antenna tilts upwards, away from the azimuthal plane, and backlobes appear in the pattern. This effect can be seen even when the maximum dimension of the ground plane is several wavelengths [77]. When the ground plane is very small, the input impedance of the antenna begins to change relative to the value on an infinite ground plane. When the maximum dimension of the ground plane is on the order of a wavelength, significant perturbations are seen in the monopole’s input impedance [77, 78]. As the ground plane increases in size, the impedance exhibits some ripple that oscillates around and eventually converges to the infinite ground value. As expected, this manifests itself in the VSWR of the antenna at both the center frequency and band edges. Figure C.1 shows the variation of the VSWR vs. maximum ground plane dimension for a simple monopole antenna and a TM_{10} antenna of the same height. Even though the TM_{10} antenna has a significantly wider footprint than the monopole, it is interesting to note that they both experience similar degradation of VSWR as the ground plane size shrinks.

A common concern about the TM_{10} antenna is that the relative complexity of the antenna might cause its performance to be strongly affected by a finite ground plane. However, as we have already seen, the VSWR degrades with ground plane size at the same rate as a simple monopole. Even more importantly, the bandwidth of the TM_{10} antenna also degrades similarly to
the monopole. Consider Figure C.2, which plots the maximum achievable half power bandwidth of each antenna. The maximum bandwidth is calculated from $Q$ using Equation 2.2. The ratio of achievable bandwidth of the TM$_{10}$ antenna over the monopole is about 5.8 on an infinite ground plane. When the ground plane shrinks below a wavelength, both antennas rapidly lose bandwidth. The ratio of their bandwidth varies but oscillates around 5.8 until the ground plane becomes smaller than a half wavelength and the TM$_{10}$ antenna loses bandwidth more rapidly. At this size, the square ground plane has sides of about 1/3 of a wavelength. The TM$_{10}$ antenna simulated has electrical size $ka = 0.5$, so the ground plane extends only 1/10 of a wavelength beyond the edge of the antenna. Thus, we can see that the ground plane does affect the TM$_{10}$ antenna more strongly than the monopole, but only if it is around the size of the antenna itself. Still, even in this extreme case, the TM$_{10}$ antenna offers a significant (approximately three-fold) improvement in bandwidth over a simple monopole.
Figure C.2: Maximum achievable half power bandwidth (assuming first order matching network and ideal transformer) of antennas mounted on a finite ground plane.
REFERENCES


AUTHOR’S BIOGRAPHY

Jacob Adams was born in Plain City, OH. He received the B.S. (summa cum laude with distinction) and M.S. degrees in Electrical and Computer Engineering from the Ohio State University in Columbus, OH, in 2005 and 2007, respectively. In December 2010, he completed his Ph.D. degree in Electrical and Computer Engineering at the University of Illinois at Urbana-Champaign. He was awarded a National Merit Scholarship in 2001 to pursue his undergraduate studies. In 2005, he was awarded a Graduate Research Fellowship from the National Science Foundation and the Dean’s Distinguished University Fellowship from the Ohio State University. From 2002-2005, he was an undergraduate research assistant at the Ohio State University ElectroScience Laboratory, and in 2004 he worked as an intern in RF Apertures at Northrop Grumman Electronic Systems in Linthicum, MD. In January 2011, he will begin work at the University of Illinois as a postdoctoral research associate. The position is funded through the Intelligence Community Postdoctoral Research Fellowship Program under a proposal based on some of the work in this dissertation.