DYNAMICS OF BANKRUPT STOCKS

BY

XIAO LI

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2011

Urbana, Illinois

Doctoral Committee:

Professor Renming Song, Chair
Professor Richard Sowers, Director of Research
Assistant Professor Robert Edward Lee DeVille
Assistant Professor Liming Feng

Abstract

In this thesis, we study the behavior of bankrupt stocks. Bankrupt stock is a special case of the Hard-to-Borrow stocks. Besides the general nice feature of the Hard-to-borrow feedback for the buy-in demand, the bankrupt stocks could exclude the diffusive effects. This nice property would modify the Marco Avellaneda and Mike Lipkin's jump-diffusion model for the Hard-to-Borrow stocks into the pure jump systems with stochastic intensity. Under this main assumption, our model is a two-dimensional integrate-and-fire model which is recursively tractable. By investigating the dynamics of the model, we could capture the self-reinforcing aspect of the buy-ins and subsequent crashes of the stock price.

Having the recursively explicit dynamics of the stock prices and buy-in rate on hand, we can calibrate the model under the physical measure by error minimization. One way to justify the fit of the calibration results is to compare the sample path and the real prices. On the other hand, we match the option pricing theory against observed behavior of the options to see how the periodic buy-ins would act to cover the the cost of the short position, which gives the mechanism of the essential feedback of the Hard-to-Borrowness.

Keywords: Bankrupt Stock; Hard-to-Borrow; Dynamics; Calibration; Option

To my beloved parents, for their love and support.

Acknowledgments

I wish to thank my advisor and my friend Professor Richard B. Sowers. Within my five years' study in the university of Illinois, he patiently advised my research work, guided me on the way for the industry job. Also his help on English and social skill sets also benefits me a lot to adapt to the life here as a foreign student. Frankly speaking, I could not have my achievement so far without his unlimited support.

I also thank Professors Renming Song, Lee DeVille, and Liming Feng for serving on my thesis committee. Also I wish to express special thanks to Professor Mike Lipkin for his co-work on the dynamics of bankrupt stocks. I gained plenty of market intuitions from him.

Furthermore, I am really glad that I met a lot great friends here. Many thanks to the guys from Math department, like Qiang Zeng, and my soccer teammates, like Di Li, you all make my life colorful for those five years.

Finally, I would like to send my sincere gratitude to my whole family. Meanwhile, I thank my girlfriend, Evelyn for her steady support and endless love. No matter how difficult problems I encounter, she is always on my side to encourage me.

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Chapter 1

Introduction

1.1 Motivation

The intuition for this research is from Marco Avellaneda and Mike Lipkin's introduction to the hard-to-borrow stocks [1]. The Hard-to-Borrow Stock category is an interesting case in the real world now, which could illustrate plenty of market phenomena.

In general, hard-to-borrow stocks refers to those, which are either subject to the short-selling restrictions or having limited floats available. For those stocks, a big concern will be the short interest, which is the percentage of the float currently held short in the market. As the short interest increases, the stock becomes even harder to borrow, and at some extreme occasion, the clearing firm will require the trader with the short position to forcibly repurchase the stocks to the short holdings. These buy-ins will artificially drive up the stock price, and after the buy-ins complete, there's no price support, then the stocks will drop down immediately. Sometimes, it will lead to illusion of the stock dynamics. For example, at the end of 2002, United Airlines field for Chapter 11 bankruptcy protection and its assets were far exceeded by its debts. It will be reasonable to expect that the stock price should go to zero within a short period. Nevertheless, it was traded as high as $4 and even above $1 for more than one year. As Lipkin said this illusion could be illustrated by the Hard-to-Borrowness feedback mechanism. Indeed, in his paper [1] with Avellaneda, they had constructed the model for the Hard-to-Borrow stocks to capture most of the properties. Though they have also used that model to explain the effective dividends and option pricing, the main weakness of the model is that it is too complicate to investigate the full dynamics. Then we take a look into the Bankrupt Stocks, which provides important insights into markets but could be easier for dynamics checking.

One feature of bankrupt stocks which we are focusing on is that their price will always decrease in the long run. The market naturally takes advantage of this fact by shorting the stock or buying puts. Prohibitions on naked shorting require access to the inventory of available stocks, and it is this feature we wish to model and study. Since the available inventory is finite, increasing demand for the stock leads to *buy-ins*, which temporarily drive up the price of the stock. Therefore, when the price support for the stock vanishes, the price crashes downward, and the cycle repeats. All those reveals that bankrupt stocks are the special case of the Hard-to-Borrow Stocks.

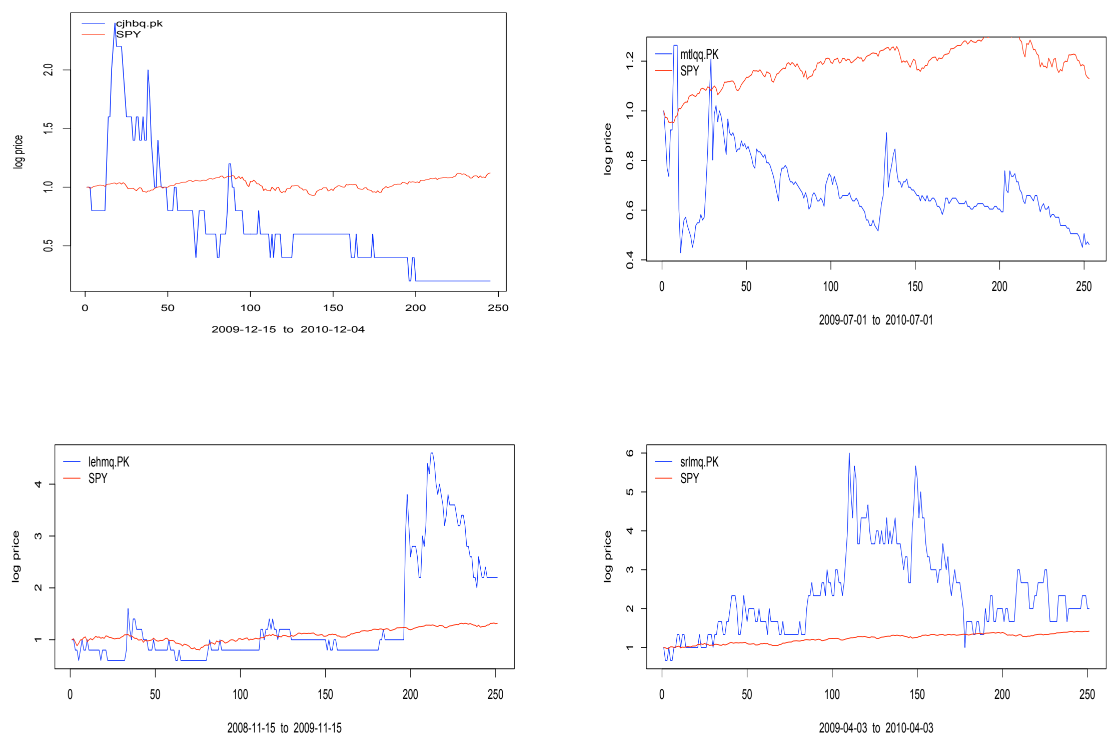
Moreover, the great advantage we could take for investigating the bankrupt stocks is that for the bankruptcy model we can remove the diffusive effects, and we claim that the dominant dynamics are indeed the buy-ins. Diffusivity in valuation can be interpreted as an expression of a wide range of beliefs about future behavior. One of the interesting consequences of this is that we can explicitly convert from the physical measure to the risk-neutral measure, since there is only one source of randomness.

Our goal here is to re-model for hard-to-borrow stocks [1] to understand the dynamics of bankrupt stocks. We claim that in many cases, the dominant behavior of bankrupt issues is the hard-to-borrowness and corresponding buy-ins needed to cover positions with negative delta exposure. Our model posits a type of *integrate-and-fire* dynamics for the dynamics of a coupled system which consists of *price* and *short interest*; we use the short interest as a proxy for the hard-to-borrowness. The stochasticity in this model is the time of crashes, when price support vanishes after a run-up in the short interest. Since our model allows fairly explicit calculations, calibration against real data is reasonably straightforward. Furthermore, we study pricing of *options* and compare our results to empirical options prices.

1.2 Several Examples for the Bankrupt Stocks

To start, let's consider several examples which single out the dynamics we wish to study. We look at prices for CJHBQ (Champion Enterprises, bankruptcy file November 15, 2009) MTLQQ (General Motors Liquidation, bankruptcy file June 1, 2009), LEHMQ (Lehman Brothers, bankruptcy filing September 15, 2008), and SRLMQ (Sterling Mining, bankruptcy filing March 3, 2009). In Figures 1.1 we plot the log prices of these issues (normalized to S_0) against time for a year's worth of time starting a month after bankruptcy filing (e.g., for CJHBQ, December 15, 2009–December 15, 2010).

Figure 1.1: Prices of CJHBQ, MTLQQ, LEHMQ, and SRLMQ



For both CJHBQ and MTLQQ, there are recurring periods where the price trends downward, followed by sharp spikes up, and then crashes downward. Volume increases at these spikes and crashes. In our model, these sharp spikes upward are modeled by an unstable feedback loop in the buy-in-rate. The dynamics of LEHMQ and SRLMQ are a bit more challenging to model via (2.3), but we claim that the proposed dynamics are still valid.

1.3 Structure of the Thesis

We start with a quick review of the Avellaneda and Lipkin's work [1] for the hard to borrow stocks in Section 2.1, and explained how they set up the model parameters. Before our new bankrupt model, We talk a bit with the asymptotic property of the single jump model as a preparation for the Chapter 3, in which We will fully discussed the asymptotic dynamics of the bankrupt stocks. In Section 2.3, We elaborate our mathematics model for the bankrupt stocks and also give some parameter assumptions to support our model.

The main work We've done in the Chapter 3 is to explicitly solve the dynamics of the stock price and short interest in a recursive way. Those details are really important for further calibration and option pricing. And in Section 3.2, we carefully discuss the asymptotic behavior for the bankrupt stocks evolutions. Those fundamental work justify our significant assumption for the bankrupt stocks—in the long term view, both the stock price and short interest will vanish for sure.

In Chapter 4, we show how we could use the dynamics of the model we get from Chapter 3 to calibrate the parameter. We use the data source around 1 trading year for Lehman Brother's stocks as an example to illustrate how the stock prices and short interest interact each other. In the Section 4.2.4, we attach a comparison figure between the real data and the calibration results to reveal how our calibration methods from Section 4.1 to Section 4.2 fit the real world.

The last main research work is to study the option pricing for the bankruptcy in Chapter 5. We started from Section 5.1.1 to give the Black-Scholes like equations for the European style options by replication methods. Also the conversion of the physical model to the risk neutral model is discussed here. Later on in Section 5.2, we discuss the least-squares method introduced by Longstaff and Schwats [5] to simulate the American put options.

Finally, we list the future work we will continuous on the research, and also list all the numerical codes for this dissertation in the Appendix.

Chapter 2

Mathematics of the Model

In this Chapter, we will elaborate the main idea to set up our model for the Bankrupt Stocks from the idea of the Hard-to-Borrow Stocks. First of all, we will briefly review how Avellaneda and Lipkin [1] present the model for Hard-to-Borrow Stocks in 2009. Secondly, we will simply take a look into the constant intensity version to see that we should expect the stock price will tend to zero for the compensate poisson model with proper parameter, which will guide us to get the asymptotic property of our generalized model for the bankrupt stocks. In the last of this chapter, we will discuss the mathematics conjecture of our bankrupt stock model.

2.1 Review of the Hard-to-Borrow Stocks Model

As we talked before in the introduction, Avellaneda and Lipkin [1] believe that for the hard to borrow stocks, the demand of the buy-ins (or forcibly purchasing requirement) will give positive feedback to the stock price. And once the buy-ins have finished, the stock price will crash down immediately. Hence, if we denote S_t and λ_t be the stock price and buy-in rate respectively, they have the following system of coupled equations [1]:

$$\begin{aligned}\frac{dS_t}{S_{t-}} &= \sigma dW_t + \gamma\lambda_t dt - \gamma dN_{\lambda_t}(t) \\ \frac{d\lambda_t}{\lambda_{t-}} &= \kappa dZ_t + \alpha(\bar{\lambda} - \lambda_t)dt + \beta\frac{dS_t}{S_{t-}}\end{aligned} \tag{2.1}$$

2.1.1 Parameter Illustration for the HTB Model

Observe the above coupled system, we have the equations follows the jump-diffusion processes with correlated stochastic intensity:

Lemma 2.1.1. *W_t,Z_t are independent Brownian motion for the diffusion parts*

Since W_t drives the stock price while Z_t drives the buy-in rate fluctuation with totally different future behavior believes, they should be independent of each other. Moreover σ and κ are the volatilities respectively.

Lemma 2.1.2. *$dN_{\lambda_t}(t)$ denotes the increment of a standard Poisson Process with stochastic intensity λ_t over the interval $(t, t+dt)$, and γ is the price elasticity of demand due to buy-ins.*

As we discussed before, without the support of the buy-in demand, the stock price will have a sharp drop, so it is reasonable here to capture this crash by a jump process. Also the frequency of the jump occurs depends on the buy-in rate, so we should explain this rate by setting up the intensity to follow the buy-in rate process.

Lemma 2.1.3. *β is positive.*

The more frequent the buy-ins, the higher the stock price will be driven by the market effect. And right after the disappear of the buy-in demand, the stock price will drop down. This conjecture shows that the stock price and buy-in rate are positively correlated. (β denotes the correlation)

Lemma 2.1.4. *$\bar{\lambda}$ is the stationary (long term) value for the buy-in rate*

2.1.2 Further Discussion of the HTB Model

This model gives us the features of the Hard-to-Borrow stocks, which shows the feedback mechanism of the buy-in rate to the stock price.And Avellaneda and Lipkin [1] using this

Model in their paper show how to calculate the effective dividends to cover the short position. But the main weakness of this model is that for the jump-diffusion system [3] with stochastic intensity, the process evolution is difficult to track, which also made the parameter calibration really complicated. In order to reduce the complexity and make the process tractable, we investigate the bankrupt stocks, which are definitely hard-to-borrow. But since bankrupt stocks will certainly go to zero, there's no wide range of beliefs about future behavior. In this sense, we could exclude diffusive effects for the bankrupt stocks, which will yield the complicate Hard-to-Borrow Stocks Model by Avellaneda and Lipkin [1] to a solvable system. (We will discuss this new model in Chapter 2.3).

2.2 Asymptotic behavior of the simple jump model

Before we go to have our mathematics model of bankrupt stocks, let's take a look at the asymptotic behavior of the simple jump model(Geometric Poisson Process [8]). Since as a main assumption for the bankrupt stocks, the price will eventually vanish. And also as we talked above, the main force to drive the stock price down is the downward jump. To guarantee this important assumption, we should check the asymptotic behavior of our stock models. And as a preparation for the later check for our model, let's investigate the simple jump model:

$$\begin{aligned} \frac{dS_t}{S_{t-}} &= \gamma\lambda dt - \gamma dN_{\lambda t} \\ S_0 &= S_o \end{aligned} \tag{2.2}$$

Theorem 2.2.1. *if $0 < \gamma < 1$, then we could have $S_t \to 0$, when $t \to +\infty$.*

By Itô-Doeblin Formula [8] for One Jump Process:

We could solve: $S_t = S_o \exp[\gamma\lambda t + \ln(1-\gamma)N_{\lambda t}]$, for all $t \geq 0$.

Lemma 2.2.2. *The strong law of large numbers for Poisson Process [2]: $\frac{N_{\lambda t}}{t} \to \lambda$, as $t \to +\infty$.*

Meanwhile: $S_t = S_o \exp[(\gamma\lambda + \ln(1-\gamma)\frac{N_{\lambda t}}{t})t]$, then:

$$\begin{aligned}\lim_{t\to+\infty} \gamma\lambda + \ln(1-\gamma)\frac{N_{\lambda t}}{t} &= \gamma\lambda + \ln(1-\gamma)\lambda \\ &= (\gamma + \ln(1-\gamma))\lambda\end{aligned}$$

Lemma 2.2.3. *if* $0 < \gamma < 1$, $\gamma + \ln(1-\gamma) < 0$

denote $f(x) = x + \ln(1-x)$, then for $0 < x < 1$:

$$\begin{aligned}f'(x) &= 1 + \frac{1}{x-1} \\ &= \frac{x}{x-1} \\ &< 0\end{aligned}$$

Hence, $f(x) < f(0) = 0$ for all $0 < x < 1$. Finally Lemma 2.2.3 together with the equation above shows that:

$$\begin{aligned}\lim_{t\to+\infty} S_t &= \lim_{t\to+\infty} S_o \exp[(\gamma + \ln(1-\gamma)\lambda)t] \\ &= 0\end{aligned}$$

We will later guarantee that our stock price has the similar asymptotic behavior and satisfies the important feature that the bankrupt stock will go to zero eventually.

2.3 Mathematical Model for the Bankrupt Stocks

In this section, we will use the non-diffusive feature of the bankrupt stocks to reset the equations system for Hard-to-Borrow stocks in (2.1) to our pure jump system. Then After removing the diffusion parts and adding the buy-in rate feedback, our basic model is two-dimensional system:

$$\begin{aligned}
\frac{dS_t}{S_{t-}} &= (\alpha_q + \alpha_h \lambda_t)dt - \gamma dN_t^{\kappa_q + \kappa_h \lambda_t} \\
\frac{d\lambda_t}{\lambda_{t-}} &= (\beta_q + \beta_h \lambda_t)dt + \rho \frac{dS_t}{S_{t-}} \\
S_0 &= S_o \\
\lambda_0 &= \lambda_o
\end{aligned} \tag{2.3}$$

2.3.1 Parameter Explanation

In this modified model:

The Process S_t still represents the stock prices process, which is driven by the time drift and the instantaneous jump caused by the disappearance of the buy-in demand.

The λ_t is the short interest rate(the number of shares sold short divided by the daily volume),which represents the strength of Hard-to-Borrowness. Though this is a different item from the buy-in rate, they do vary in the exactly the same direction: it is obviously that the greater the short interest, the more difficult to borrow, then the more frequent the buy-ins will occur. While there's no need for the buy-in demand, the desired the short positions has been covered, this also lower the short interest. Hence we may assume that those two things could be proportional to each other. And at the modeling level, we can replace these two without losing the essential feedback.

α_q,β_q,κ_q are the quiescent rates of growth, while α_h,β_h,κ_h capture the exposure to the Hard-to-Borrowness which reflects the feedback mechanism.

$-\gamma dN_t^{\kappa_q + \kappa_h \lambda_t}$ denotes the downward jumps with jump size γ corresponding to crashes.

ρ is the correlation between the stock price and the short interest, which is supposed to be positive.

2.3.2 Model Reformulation and Discussion

Replacing the $\frac{dS_t}{S_{t-}}$ from the first equation in (2.3) to the second part, we could rewrite our system as:

$$\begin{aligned} dS_t &= (\alpha_q + \alpha_h \lambda_t) S_t dt - \gamma S_{t-} dN_t^{\kappa_q + \kappa_h \lambda_t} \\ d\lambda_t &= (\tilde{\beta}_1 \lambda_t + \tilde{\beta}_2 \lambda_t^2) dt - \tilde{\gamma} \lambda_{t-} dN_t^{\kappa_q + \kappa_h \lambda_t} \end{aligned} \tag{2.4}$$

Remark 2.3.3. *For the drift part, the $t-$ is replaced by t, since this part is continuous, with new coefficients:*

$$\tilde{\beta}_1 \overset{def}{=} \beta_q + \rho \alpha_q \qquad \tilde{\beta}_2 \overset{def}{=} \beta_h + \rho \alpha_h \qquad and \qquad \tilde{\gamma} \overset{def}{=} \rho \gamma.$$

A quick observation tells that the system (2.4) is triangular, which means the stock prices depend on the short interest but not vice versa. So we can first solve for λ and then solve for S. Also, before the jumps, we have the system is only ODE dynamics, which makes the model is explicitly solvable. This nice feature will yield our main approach for this problem later on that we could first determine the dynamics of the model before the crash, and after the crash we just restart the system, and solve the dynamics iteratively.

Hence, it is straightforward for us to check the dynamics of the coupled system within the first crash cycle before we generalized to the full dynamics model.

Let's take a look into the ODE for the short interest before the crash, and define:

$$f(\ell) \overset{\text{def}}{=} \tilde{\beta}_1 \ell + \tilde{\beta}_2 \ell^2 \tag{2.5}$$

for all $\ell \in \mathbb{R}$, we can solve

$$\begin{aligned} \dot{\lambda}_t &= f(\lambda_t) \qquad t > 0 \\ \lambda_0 &= \lambda_\circ \end{aligned} \tag{2.6}$$

To find the first random crash time τ, let $\mathfrak{e}$ be an exponential(1) random variable, and define

$$\tau \stackrel{\text{def}}{=} \inf\left\{t \geq 0 : \int_{s=0}^{t} (\kappa_q + \kappa_h \lambda_s)\, ds \geq \mathfrak{e}\right\}. \tag{2.7}$$

Define $\lambda_\tau \stackrel{\text{def}}{=} \lambda_{\tau-}\{1-\tilde{\gamma}\}$, which is the short interest right after the crash decay.

Similarly, the dynamics of S can be solved as:

$$S_t = S_\circ \exp\left[\alpha_q t + \alpha_h \int_{s=0}^{t} \lambda_s ds\right] \qquad 0 \leq t < \tau \tag{2.8}$$

and define $S_\tau = S_{\tau-}\{1-\gamma\}$ for the stock price immediately after the steep drop .

At time τ we restart the system.

Furthermore, let's give two assumptions here to validate the model:

Assumption 2.3.4. *We require that $\gamma \in (0,1)$ and $\tilde{\gamma} \in (0,1)$.*

Then both λ and S are positive at τ. Thus both λ and S remain positive for a whole cycle, and thus for all time.

Assumption 2.3.5. *We assume that $\alpha_q \leq 0$ and that β_q, κ_q, α_h, β_h, and κ_h are all in $\mathbb{R}_+$.*

We want the stock to have a downward trend (i.e., we allow $\alpha_q < 0$). Similarly, if the short rate is very small, we want to have nonnegative short interest and baseline jump frequency (i.e., $\beta_q \geq 0$ and $\kappa_q \geq 0$). We also naturally want an increase in the short interest to increase the price ($\alpha_h \geq 0$), cause more buy-ins (to cover shorts; i.e., $\beta_h \geq 0$), and to cause more frequent crashes ($\kappa_h \geq 0$).

Since f is smooth and $f(0) = 0$, the dynamics of (2.6) and Assumption 2.3.5 ensure that $\lambda_t > 0$ for all $t \in [0, \tau)$.

Finally, we should notice that this model has an important aspect we will use later on to calculate the dynamics and solve the calibration problem is that our model is Markovian [7].

The generator [7] of (S, λ) is

$$
\begin{aligned}
(\mathcal{L}\varphi)(S,\lambda) \stackrel{\text{def}}{=} \{\alpha_q + \alpha_h \lambda\} S \frac{\partial \varphi}{\partial S}(S,\lambda) + \left\{\beta_1 \lambda + \beta_2 \lambda^2\right\} \frac{\partial \varphi}{\partial \lambda}(\lambda) \\
+ \{\kappa_q + \kappa_2 \lambda\} \{\varphi\left((1-\gamma)S, (1-\tilde{\gamma})\lambda\right) - \varphi(S,\lambda)\}
\end{aligned}
\tag{2.9}
$$

for all $f \in C^2(\mathbb{R})$. For any $c > 0$, we then have $(\hat{S}_t, \hat{\lambda}_t) \stackrel{\text{def}}{=} (S_{ct}, \lambda_{ct})$ is also Markovian, and has generator

$$
\begin{aligned}
c(\mathcal{L}\varphi)(S,\lambda) = \{c\alpha_q + c\alpha_h \lambda\} S \frac{\partial \varphi}{\partial S}(S,\lambda) + \left\{c\beta_1 \lambda + c\beta_2 \lambda^2\right\} \frac{\partial \varphi}{\partial \lambda}(\lambda) \\
+ \{c\kappa_q + c\kappa_2 \lambda\} \{\varphi\left((1-\gamma)S, (1-\tilde{\gamma})\lambda\right) - \varphi(S,\lambda)\}
\end{aligned}
$$

Thus the dynamics are closed under rescaling with the constant c.

Chapter 3

Dynamics of the Bankruptcy Model

3.1 General System Evolution

First notice that (2.6) is a Bernoulli's Equation [10], which could be explicitly solved. But from Assumption 2.3.5, we know that $\tilde{\beta}_2 > 0$ will make $f(\ell) \approx \tilde{\beta}_1 \ell^2$ for $\ell \gg 1$. This quadratic dominant part will lead the ODE to explosion in finite time. Then based on this observation, if we denote $\phi_\ell(t)$ to be the solution to the equation (2.6) with the initial condition $\phi_\ell(0) = \ell$ and also t_ℓ^* is the finite explosion time constraint, we will have the explicit form in the following lemma:

Lemma 3.1.1. $t_\ell^* = \frac{1}{\tilde{\beta}_1} \ln \frac{\tilde{\beta}_1 + \tilde{\beta}_2 \ell}{\tilde{\beta}_2 \ell}$, *and for* $0 \le t < t_\ell^*$, *we have* $\phi_\ell(t) = \frac{\tilde{\beta}_1 \ell e^{\tilde{\beta}_1 t}}{\tilde{\beta}_1 - \tilde{\beta}_2 \ell \left(e^{\tilde{\beta}_1 t} - 1\right)}$

Proof. from (2.6) we have:

$$\begin{aligned} \dot{\phi}_\ell(t) &= \tilde{\beta}_1 \phi_\ell(t) + \tilde{\beta}_2 \phi_\ell(t)^2 \\ \phi_\ell(0) &= \ell. \end{aligned} \tag{3.1}$$

Divided $\phi_\ell(t)^2$ from the both sides of the first equation in (3.1):

$$\frac{\dot{\phi}_\ell(t)}{\phi_\ell(t)^2} = \frac{\tilde{\beta}_1}{\phi_\ell(t)} + \tilde{\beta}_2$$

This yields:

$$-\left(\frac{1}{\phi_\ell(t)}\right)' = \tilde{\beta}_1 \frac{1}{\phi_\ell(t)} + \tilde{\beta}_2$$

Multiplies the Integrate Factor $e^{\tilde{\beta}_1 t}$ for both side to the above equation:

$$(\frac{e^{\tilde{\beta}_1 t}}{\phi_\ell(t)})' = -e^{\tilde{\beta}_1 t}\tilde{\beta}_2$$

then

$$\frac{e^{\tilde{\beta}_1 t}}{\phi_\ell(t)} = -e^{\tilde{\beta}_1 t}\frac{\tilde{\beta}_2}{\tilde{\beta}_1} + C$$

together with the initial condition $\phi_\ell(0) = \ell$, we can solve:

$$\phi_\ell(t) = \frac{\tilde{\beta}_1 \ell e^{\tilde{\beta}_1 t}}{\tilde{\beta}_1 - \tilde{\beta}_2 \ell \left(e^{\tilde{\beta}_1 t} - 1\right)} \tag{3.2}$$

To get the explosion constraint, we could get this easily by setting the denominator of the solution above to be zero:

$$\tilde{\beta}_1 - \tilde{\beta}_2 \ell \left(e^{\tilde{\beta}_1 t} - 1\right) = 0$$

This gives:

$$t^*_\ell = \frac{1}{\tilde{\beta}_1} \ln \frac{\tilde{\beta}_1 + \tilde{\beta}_2 \ell}{\tilde{\beta}_2 \ell}$$

□

Using Lemma 3.1.1, we can also derive the stock price explicitly before the crash, just define:

$$\Phi_\ell(t) \stackrel{\text{def}}{=} \int_{s=0}^{t} \phi_\ell(s) ds = \frac{1}{\tilde{\beta}_2} \ln \frac{\tilde{\beta}_1}{\tilde{\beta}_1 - \tilde{\beta}_2 \ell \left(e^{\tilde{\beta}_1 t} - 1\right)} \qquad 0 < t < t^*_\ell. \tag{3.3}$$

And similar to what we defined the first crash time as in the (2.7), Let $\{\mathfrak{e}_n\}_{n \in \mathbb{N}}$ be an i.i.d. collection of exponential(1) random variables. And denotes J_n to be the interarrival times, and τ_n to be the crash times

$$\begin{aligned}
\tau_1 = J_1 &\stackrel{\text{def}}{=} \inf\left\{t \geq 0 :\ \kappa_q t + \kappa_h \Phi_{\lambda_0}(t) \geq \mathfrak{e}_1\right\} \\
\lambda_{\tau_1 -} &= \frac{\tilde{\beta}_1 \ell e^{\tilde{\beta}_1 \tau_1}}{\tilde{\beta}_1 - \tilde{\beta}_2 \ell \left(e^{\tilde{\beta}_1 \tau_1} - 1\right)} \\
\tau_{n+1} &\stackrel{\text{def}}{=} \sum_{i=1}^{n+1} J_i \qquad n \geq 1 \\
J_{n+1} &\stackrel{\text{def}}{=} \inf\left\{t > \tau_n :\ \kappa_q t + \kappa_h \Phi_{(1-\tilde{\gamma})\lambda_{\tau_n -}}(t) \geq \mathfrak{e}_n\right\} \qquad n \geq 1 \\
\lambda_{t-} &= \phi_{(1-\tilde{\gamma})\lambda_{\tau_n -}}(t - \tau_n) \qquad n \geq 1 \qquad \tau_n < t \leq \tau_{n+1}
\end{aligned} \tag{3.4}$$

Iterative in the system (3.4), we will have the full dynamics of the short interest.

And once we know the evolution of the λ_t, we can construct S in the mean time:

$$S_{t-} = \begin{cases} S_0 \exp\left[\alpha_q t + \alpha_h \Phi_{\lambda_0}(t)\right], & \text{if } 0 \leq t \leq \tau_1; \\ (1-\gamma) S_{\tau_n -} \exp\left[\alpha_q (t - \tau_n) + \alpha_h \Phi_{(1-\gamma)\lambda_{\tau_n -}}(t - \tau_n)\right], & \text{if } \tau_n < t \leq \tau_{n+1} \qquad n \geq 1; \end{cases} \tag{3.5}$$

The equations (3.5) track all the evolutions of the stock price, and by using (3.4) and (3.5), the asymptotic of the stock price is approachable.

3.2 Asymptotic Behavior of the System

With the explicit calculation for the crash cycle in Section 3.1, we can find the long-term dynamics of Stock price S and λ.

As we discussed before, for bankrupt stocks, we should expect that the stock price will go to zero eventually, so in the long term, few trading will incur in those stocks. Under this idea, no perpetual buy-in demands are needed. Hence, we need to adjust the parameter coefficients in our model to have:

$$
\begin{aligned}
&\lim_{t\to\infty} \lambda_t = 0 \\
&\lim_{t\to\infty} S_t = 0
\end{aligned} \tag{3.6}
$$

And since the asymptotic of the stock price is determined by the short interest by the triangular property of the system, let's first check the long term effect of the λ.

Furthermore, observing the equation (3.1), we know that during the crash cycle the λ should be monotonic, and decay right after the crash, this property tell us that investigating on the asymptotic of the λ at the crash time is enough to have the full picture of the whole short interest process.

Theorem 3.2.1. *If* $\lim_{n\to\infty} \lambda_{\tau_n} = 0$*, then* $\kappa_q > 0$*.*

This Lemma shows that the quiescent rate support for the jump is significant to vanish the short interest rate. Intuitively, you will see, if $\kappa_q = 0$, then together with $\lim_{n\to\infty} \lambda_{\tau_n -} = 0$, in the wide long range, the short interest rate will be dominated by the positive drift speed, this will yield a positive long term short interest, which is a contradiction.

Proof. Let's assume $\kappa_q = 0$, then apply this to evolutions (3.4), for $n > 1$:

$$
\begin{aligned}
\lambda_{\tau_n} &= (1-\gamma)\lambda_{\tau_{n-}} \\
&= (1-\gamma)\phi_{\lambda_{\tau_{n-1}}}(J_n) \\
&= (1-\gamma)\phi_{\lambda_{\tau_{n-1}}}(\Phi^{-1}_{\lambda_{\tau_{n-1}}}(e_n/\kappa_h))
\end{aligned}
$$

From equation (3.3), we could solve:

$$
\Phi_\ell^{-1}(x) = \frac{1}{\tilde{\beta}_1} \ln \frac{\tilde{\beta}_1 + \tilde{\beta}_2 \ell - \tilde{\beta}_1 e^{-\tilde{\beta}_2 x}}{\tilde{\beta}_2 \ell} \tag{3.7}
$$

Then replacing (3.2) and (3.7) into the above evolution we can get:

$$
\begin{aligned}
\lambda_{\tau_n} &= (1-\tilde{\gamma})\frac{\tilde{\beta}_1\lambda_{\tau_{n-1}}\frac{\tilde{\beta}_1+\tilde{\beta}_2\lambda_{\tau_{n-1}}-\tilde{\beta}_1 e^{-\tilde{\beta}_2(e_n/\kappa_h)}}{\tilde{\beta}_2\lambda_{\tau_{n-1}}}}{\tilde{\beta}_1+\tilde{\beta}_2\lambda_{\tau_{n-1}}-\tilde{\beta}_2\lambda_{\tau_{n-1}}\frac{\tilde{\beta}_1+\tilde{\beta}_2\lambda_{\tau_{n-1}}-\tilde{\beta}_1 e^{-\tilde{\beta}_2(e_n/\kappa_h)}}{\tilde{\beta}_2\lambda_{\tau_{n-1}}}} \\
&= (1-\tilde{\gamma})\frac{\tilde{\beta}_1+\tilde{\beta}_2\lambda_{\tau_{n-1}}-\tilde{\beta}_1 e^{-\tilde{\beta}_2(e_n/\kappa_h)}}{\tilde{\beta}_2 e^{-\tilde{\beta}_2(e_n/\kappa_h)}} \\
&= \frac{(1-\tilde{\gamma})}{\tilde{\beta}_2}(\tilde{\beta}_1 e^{\tilde{\beta}_2(e_n/\kappa_h)}+\tilde{\beta}_2\lambda_{\tau_{n-1}}e^{\tilde{\beta}_2(e_n/\kappa_h)}-\tilde{\beta}_1)
\end{aligned}
$$

Apply Recursion algorithm, we could figure out that:

$$
\lambda_{\tau_n} = (1-\tilde{\gamma})^n(\frac{\tilde{\beta}_1}{\tilde{\beta}_2}+\lambda_0)e^{\frac{\tilde{\beta}_2}{\kappa_h}\sum\limits_{k=1}^{n}e_k}+\frac{\tilde{\beta}_1\tilde{\gamma}}{\tilde{\beta}_2}\sum_{k=1}^{n-1}(1-\tilde{\gamma})^k e^{\frac{\tilde{\beta}_2}{\kappa_h}\sum\limits_{j=1}^{k}e_{n+1-j}}-\frac{\tilde{\beta}_1}{\tilde{\beta}_2}(1-\tilde{\gamma}) \tag{3.8}
$$

denote:

$$
\begin{aligned}
I_1 &= (1-\tilde{\gamma})^n(\frac{\tilde{\beta}_1}{\tilde{\beta}_2}+\lambda_0)e^{\frac{\tilde{\beta}_2}{\kappa_h}\sum\limits_{k=1}^{n}e_k} \\
I_2 &= \frac{\tilde{\beta}_1\tilde{\gamma}}{\tilde{\beta}_2}\sum_{k=1}^{n-1}(1-\tilde{\gamma})^k e^{\frac{\tilde{\beta}_2}{\kappa_h}\sum\limits_{j=1}^{k}e_{n+1-j}} \\
I_3 &= \frac{\tilde{\beta}_1}{\tilde{\beta}_2}(1-\tilde{\gamma})
\end{aligned} \tag{3.9}
$$

Lemma 3.2.2. *If* $\frac{\tilde{\beta}_2}{\kappa_h} < -\ln(1-\tilde{\gamma})$, *then* $I_1 \to 0$, *as* $n \to \infty$

Actually $I_1 = \exp[-n(-\ln(1-\tilde{\gamma}) - \frac{\tilde{\beta}_2}{\kappa_h}\frac{\sum_{k=1}^{n}e_k}{n})]$

Since e_n are i.i.d exponential(1) random variables, strong law of large numbers tells us: $\frac{\sum_{k=1}^{n}e_k}{n} \to 1$ as $n \to \infty$.

So when $\frac{\tilde{\beta}_2}{\kappa_h} < -\ln(1-\tilde{\gamma})$, we have $-\ln(1-\tilde{\gamma}) - \frac{\tilde{\beta}_2}{\kappa_h}\frac{\sum_{k=1}^{n}e_k}{n} > 0$ as as $n \to \infty$.

Then the exponential decay will drive $I_1 \to 0$, for $n \to \infty$.

Lemma 3.2.3. $\lim_{n\to\infty} I_2 - I_3 > 0, \quad a.s$

It is trivial that the continuity and nonnegativity of the exponential random variables yield that $\forall k > 0$, we always have $e_k > 0 \;\; a.s$, and also use the I.I.D property so:

$$\begin{aligned}\lim_{n\to\infty} I_2 &\geq e^{\frac{\tilde{\beta}_2}{\kappa_h}e_1}\frac{\tilde{\beta}_1\tilde{\gamma}}{\tilde{\beta}_2}\sum_{k=1}^{\infty}(1-\tilde{\gamma})^k \\ &= e^{\frac{\tilde{\beta}_2}{\kappa_h}e_1}\frac{\tilde{\beta}_1}{\tilde{\beta}_2}(1-\tilde{\gamma}) \\ &> I_3 \qquad a.s\end{aligned} \tag{3.10}$$

Lemma 3.2.2 shows us the one cumulative part of the short interest might approach to zero in the long term, while Lemma 3.2.3 indicates that another part will make the long term short interest greater than zero almost surely. This contradicts our assumption that $\lim_{n\to\infty}\lambda_{\tau_n} = 0$. Hence we should have $\kappa_q > 0$ to avoid this contradiction. $\square$

Remark 3.2.1. *Under the naive case $\kappa_q = 0$, if $1 > \tilde{\gamma} > \frac{\tilde{\beta}_2}{\kappa_h} > 0$ we will have:*

$$\lim_{n\to+\infty}\mathbb{E}[\lambda_{\tau_n}] = \frac{\tilde{\beta}_1(1-\gamma)}{\gamma\kappa_h - \tilde{\beta}_2} > 0$$

This give us the stationary sense of the short interest.

Proof. Using the facts e_i are i.i.d exponential-1:

$$\begin{aligned}\mathbb{E}[\lambda_{\tau_n}] &= \mathbb{E}[(1-\tilde{\gamma})^n(\frac{\tilde{\beta}_1}{\tilde{\beta}_2}+\lambda_0)e^{\frac{\tilde{\beta}_2}{\kappa_h}\sum_{k=1}^{n}e_k} + \frac{\tilde{\beta}_1\tilde{\gamma}}{\tilde{\beta}_2}\sum_{k=1}^{n-1}(1-\tilde{\gamma})^k e^{\frac{\tilde{\beta}_2}{\kappa_h}\sum_{j=1}^{k}e_{n+1-j}} - \frac{\tilde{\beta}_1}{\tilde{\beta}_2}(1-\tilde{\gamma})] \\ &= (1-\tilde{\gamma})^n(\frac{\tilde{\beta}_1}{\tilde{\beta}_2}+\lambda_0)\mathbb{E}[e^{\frac{\tilde{\beta}_2}{\kappa_h}e_1}]^n + \frac{\tilde{\beta}_1\tilde{\gamma}}{\tilde{\beta}_2}\sum_{k=1}^{n-1}(1-\tilde{\gamma})^k\mathbb{E}[e^{\frac{\tilde{\beta}_2}{\kappa_h}e_1}]^k - \frac{\tilde{\beta}_1}{\tilde{\beta}_2}(1-\tilde{\gamma}) \\ &= (1-\tilde{\gamma})^n(\frac{\tilde{\beta}_1}{\tilde{\beta}_2}+\lambda_0)(\frac{\kappa_h}{\kappa_h-\tilde{\beta}_2})^n + \frac{\tilde{\beta}_1\tilde{\gamma}}{\tilde{\beta}_2}(\frac{\frac{(1-\tilde{\gamma})\kappa_h}{\kappa_h-\tilde{\beta}_2}[1-(\frac{(1-\tilde{\gamma})\kappa_h}{\kappa_h-\tilde{\beta}_2})^n]}{1-\frac{(1-\tilde{\gamma})\kappa_h}{\kappa_h-\tilde{\beta}_2}}) - \frac{\tilde{\beta}_1}{\tilde{\beta}_2}(1-\tilde{\gamma})\end{aligned}$$

Let $n \to +\infty$, we have:

$$\lim_{n\to+\infty}\mathbb{E}[\lambda_{\tau_n}] = \frac{\tilde{\beta}_1(1-\gamma)}{\gamma\kappa_h - \tilde{\beta}_2} > 0$$

$\square$

Remark 3.2.2. *After we obtain the important feature that the quiescent rate $\kappa_q > 0$, then if we consider the discussion above for the simple Poisson Jump Model 2.2.1. We could understand that this quiescent rate will guarantee that the jump always occurs at the positive rate, then just control the relationship between the drift rate and the quiescent rate as we did before for the simple jump model, we could also get $\lim_{n\to\infty} \lambda_{\tau_n-} = 0$. Hence without proving the existence of this asymptotic, we just use this as an assumption to find the necessary conditions we need to validate this.*

Theorem 3.2.4. *When $\kappa_q > 0$, then $\lim_{n\to\infty} \lambda_{\tau_n-} = 0$, only if:*

$$\frac{\tilde{\beta}_1}{\kappa_q} < \ln \frac{1}{1-\tilde{\gamma}}.$$

Proof. Let's proceed similarly as we did before with the system evolutions (3.4).

For each ℓ and E in $(0, \infty)$, let $\bar{\tau}_E(\ell)$ solve

$$\kappa_q \bar{\tau}_E(\ell) + \kappa_h \Phi_{(1-\tilde{\gamma})\lambda_{\tau_n-}}(\bar{\tau}_E(\ell)) = E \tag{3.11}$$

(this is uniquely possible since $t \mapsto \Phi_\ell(t)$ is nondecreasing). Then

$$\lambda_{\tau_{n+1}-} = \phi_{(1-\tilde{\gamma})\lambda_{\tau_n-}}(\tau_{\mathfrak{e}_{n+1}}((1-\tilde{\gamma})\lambda_{\tau_n-})). \tag{3.12}$$

We want to see what happens to this recursion when $\lambda_{\tau_n-} \ll 1$.

When ℓ is really small and E is of order 1 (E represents the exponential(1) variable). Since Φ is nonnegative, (3.11) implies that $\bar{\tau}_E(\ell) \leq E/\kappa_q$. Thus for small ℓ, we have that

$$\tilde{\beta}_2 \ell (e^{\tilde{\beta}_1 \bar{\tau}_E(\ell)} - 1) \approx 0;$$

This shows that:

$$
\begin{aligned}
\Phi_\ell(\bar{\tau}_E(\ell)) &= \frac{1}{\tilde{\beta}_2} \ln \frac{\tilde{\beta}_1}{\tilde{\beta}_1 - \tilde{\beta}_2 \ell (e^{\tilde{\beta}_1 \bar{\tau}_E(\ell)} - 1)} \\
&\approx \frac{1}{\tilde{\beta}_2} \ln \frac{\tilde{\beta}_1}{\tilde{\beta}_1} \\
&= 0
\end{aligned}
\tag{3.13}
$$

in other words,

$$
\bar{\tau}_E(\ell) \approx \frac{E}{\kappa_q} \tag{3.14}
$$

for $\ell \ll 1$ This then implies that:

$$
\begin{aligned}
\phi_{(1-\tilde{\gamma})\ell}(\bar{\tau}_E((1-\tilde{\gamma})\ell)) &= \frac{\tilde{\beta}_1 (1-\tilde{\gamma}) \ell e^{\tilde{\beta}_1 \bar{\tau}_E((1-\tilde{\gamma})\ell)}}{\tilde{\beta}_1 - \tilde{\beta}_2 (1-\tilde{\gamma}) \ell (e^{\tilde{\beta}_1 \bar{\tau}_E((1-\tilde{\gamma})\ell)} - 1)} \\
&\approx (1-\tilde{\gamma}) \ell e^{\frac{\tilde{\beta}_1 E}{\kappa_q}}
\end{aligned}
\tag{3.15}
$$

Thus if $\lambda_{\tau_n -} \ll 1$, together with equations (3.12) and (3.15), we should approximately have

$$
\ln \lambda_{\tau_{n+1} -} \approx \ln \lambda_{\tau_n -} + \ln(1-\tilde{\gamma}) + \frac{\tilde{\beta}_1}{\kappa_q} \mathfrak{e}_{n+1}
$$

Recursively, we will have

$$
\frac{\ln \lambda_{\tau_n -}}{n} \approx \ln(1-\tilde{\gamma}) + \frac{\tilde{\beta}_1}{\kappa_q} \frac{\sum e_n}{n}
$$

The strong law of large numbers shows that $\frac{\sum e_n}{n} \to 1$ almost surely. Thus if $\lim_{n\to\infty} \lambda_{\tau_n -} = 0$, we get:

$$
\lim_{n\to\infty} \frac{\ln \lambda_{\tau_n -}}{n} = \ln(1-\tilde{\gamma}) + \frac{\tilde{\beta}_1}{\kappa_q}
$$

And by the fact $\lambda_{\tau_n -} \ll 1$, we know that $\ln \lambda_{\tau_n -} < 0$, this yields:

$$
\ln(1-\tilde{\gamma}) + \frac{\tilde{\beta}_1}{\kappa_q} < 0 \tag{3.16}
$$

Inequality (3.16) justify our main condition that: $\frac{\tilde{\beta}_1}{\kappa_q} < \ln \frac{1}{1-\tilde{\gamma}}$ $\square$

Also because the dynamics of the stock price is fully determined by the evolutions of the short interest, based on our discussion for the asymptotic property for λ, it's straightforward to get:

Theorem 3.2.5. *if* $\lim_{n\to\infty} \lambda_{\tau_n -} = 0$*, then* $\lim_{n\to\infty} S_{\tau_n -} = 0$*, and:*

$$\lim_{n\to\infty} \frac{\ln S_{\tau_n -}}{n} = \ln(1-\gamma) + \frac{\alpha_q}{\kappa_q}$$

Proof. Using results (3.5) and set-up (3.11), we could derive the approximation for the stock price:

$$\ln S_{\tau_{n+1}-} = \ln S_{\tau_n -} + \ln(1-\gamma) + \alpha_q \bar{\tau}_E(\ell) + \alpha_h \Phi_{(1-\tilde{\gamma})\lambda_{\tau_n -}}(\bar{\tau}_E(\ell)) \tag{3.17}$$

Plug in the approximation (3.13) and (3.14) back to the above equation, we would have:

$$\ln S_{\tau_{n+1}-} \approx \ln S_{\tau_n -} + \ln(1-\gamma) + \frac{\alpha_q}{\kappa_q} \mathfrak{e}_{n+1}$$

This approximation is guaranteed by the fact that $\lim_{n\to\infty} \lambda_{\tau_n -} = 0$. Recursively we could obtain:

$$\frac{\ln S_{\tau_{n+1}-}}{n} \approx \ln(1-\gamma) + \frac{\alpha_q}{\kappa_q} \frac{\sum \mathfrak{e}_n}{n}$$

Apply the strong law of large numbers for exponential(1) variables again, we could have:

$$\lim_{n\to\infty} \frac{\ln S_{\tau_n -}}{n} = \ln(1-\gamma) + \frac{\alpha_q}{\kappa_q}$$

Furthermore by Assumption 2.3.4 and Assumption 2.3.5, we know that $0 < \gamma < 1$ and $\alpha_q \leqslant 0$. Together with Theorem 3.2.1, we have $\kappa_q > 0$. These conditions make $\ln(1-\gamma) + \frac{\alpha_q}{\kappa_q} < 0$, therefore:

$$\begin{aligned} \lim_{n\to\infty} S_{\tau_n -} &= \lim_{n\to\infty} \exp[n(\ln(1-\gamma) + \frac{\alpha_q}{\kappa_q})] \\ &= 0 \end{aligned}$$

□

Theorem 3.2.5 tells us that the subsets of the stock price and short interest on the crash time will vanish eventually. And the nice feature of the evolutions will guide us to our main requirements (3.6) that for bankrupt stocks both the stock prices and short interest will finally turn down to zero in the long term. And with this on mind, to remove the diffusive parts from the original Hard-to-Borrow Model does make sense.

Chapter 4

Parameter Calibration and Real Data Analysis

In this Chapter, we will focus on using the dynamics of the bankrupt stocks we established in Chapter 3 to calibrate the parameters in the model.

The naive approach would be to start with the Markov property and use a discrete-time approximation of (2.9). Instead, we would like to find a calibration procedure which better reflects what the model attempts to capture. Namely, the evolution of S and λ between crashes is deterministic, and the crashes occur with a clock which depends on λ.

4.1 Calibration Set Up

Taking stock of what we know, we have a sequence $\{t_j\}_{j=1}^J$ of times at which we have observations. We assume of course that $0 = t_0 < t_1 \dots t_J$. At each t_j, we can observe $S^{\text{ob}}_{t_j-}$ and the short interest $\lambda^{\text{ob}}_{t_j-}$. We are careful in that we observe the left-continuous limits of these processes at the observation times since the processes in the model are right-continuous, but the model is designed to only approximate the behavior right after the crashes. By identifying times when the price hits a maximum when volume is high, we can observe the times of the crashes; i.e., a sequence $\{\tau_n\}_{n=1}^N$. Since these are of course observation times, we have that $\{\tau_n\}_{n=1}^N \subset \{t_j\}_{j=1}^J$.

For the calibration, our goal is to find a vector $\tilde{v} = (\alpha_q, \alpha_h, \tilde{\beta}_0, \tilde{\beta}_1, \kappa_q, \kappa_h, \gamma, \tilde{\gamma}) \in S \stackrel{\text{def}}{=} \mathbb{R}_- \times \mathbb{R}_+^5 \times [0,1]^2$ which best fits the model (2.4) to data. We can then recover the original model (2.3) by setting

$$\beta_q = \tilde{\beta}_1 - \alpha_q \frac{\tilde{\gamma}}{\gamma} \qquad \beta_h = \tilde{\beta}_2 - \alpha_h \frac{\tilde{\gamma}}{\gamma} \qquad \rho \overset{\text{def}}{=} \frac{\tilde{\gamma}}{\gamma}$$

Between the crashes, the evolutions are deterministic, then we could use least square minimization to find $\tilde{v}$ which gives the smallest square error. This approach could apply to both the stock price and short interest. In order to have a better sense for the calibration, let's expand the notations of Section 3.1 here. For $\tilde{v} = (\alpha_q, \alpha_h, \tilde{\beta}_0, \tilde{\beta}_1, \kappa_q, \kappa_q, \kappa_h, \gamma, \tilde{\gamma}) \in S$, define:

$$\begin{aligned}
t_\ell^{\tilde{v},*} &\overset{\text{def}}{=} \frac{1}{\tilde{\beta}_1} \ln \frac{\tilde{\beta}_1 + \tilde{\beta}_2 \ell}{\tilde{\beta}_2 \ell} \\
\phi_\ell^{\tilde{v}}(t) &= \frac{\tilde{\beta}_1 \ell e^{\tilde{\beta}_1 t}}{\tilde{\beta}_1 - \tilde{\beta}_2 \ell \left(e^{\tilde{\beta}_1 t} - 1\right)} \qquad 0 \le t < t_\ell^{\tilde{v},*} \\
\Phi_\ell^{\tilde{v}}(t) &\overset{\text{def}}{=} \frac{1}{\tilde{\beta}_2} \ln \frac{\tilde{\beta}_1}{\tilde{\beta}_1 - \tilde{\beta}_2 \ell \left(e^{\tilde{\beta}_1 t} - 1\right)} \qquad 0 < t < t_\ell^{\tilde{v},*}
\end{aligned}$$

Hence by using the results of (3.4) and (3.5), The best values of $\tilde{v}$ should be given by minimizing

$$\begin{aligned}
I_1(\tilde{v}) &\overset{\text{def}}{=} \frac{1}{2}\Big(\sum_{0 \le t_j \le \tau_1} \left| \phi^{\tilde{v}}_{\lambda_0^{ob}}(t_j) - \lambda^{\text{ob}}_{t_j-} \right|^2 + \sum_{1 \le n \le N} \sum_{\substack{1 \le j \le J \\ \tau_n < t_j \le \tau_{n+1}}} \left| \phi^{\tilde{v}}_{(1-\tilde{\gamma})\lambda^{\text{ob}}_{\tau_n -}}(t_j - \tau_n) - \lambda^{\text{ob}}_{t_j-} \right|^2 \Big) \\
I_{21}(\tilde{v}) &\overset{\text{def}}{=} \frac{1}{2} \sum_{0 \le t_j \le \tau_1} \left| \alpha_q t_j + \alpha_h \Phi^{\tilde{v}}_{\lambda_0^{ob}} t_j - \ln(S^{\text{ob}}_{t_j-}/S^{\text{ob}}_0) \right|^2 \\
I_{22}(\tilde{v}) &\overset{\text{def}}{=} \frac{1}{2} \sum_{1 \le n \le N} \sum_{\substack{1 \le j \le J \\ \tau_n < t_j \le \tau_{n+1}}} \left| \ln(1-\gamma) + \alpha_q (t_j - \tau_n) + \alpha_h \Phi^{\tilde{v}}_{(1-\tilde{\gamma})\lambda^{\text{ob}}_{\tau_n -}}(t_j - \tau_n) - \ln(S^{\text{ob}}_{t_j-}/S^{\text{ob}}_{\tau_n -}) \right|^2 \\
I_2(\tilde{v}) &\overset{\text{def}}{=} I_{21}(\tilde{v}) + I_{22}(\tilde{v})
\end{aligned}$$

(where for convenience we set $\tau_{J+1} \overset{\text{def}}{=} \infty$) We want to minimize these functions over the set

$$S^* \stackrel{\text{def}}{=} \left\{ \tilde{v} \in S : \tau_{n+1} - \tau_n \le t^{\tilde{v},*}_{\lambda^{\text{ob}}_{\tau_n -}} \text{ for all } 1 \le n \le N \right\}. \tag{4.1}$$

Since we want to avoid the system explosive within the finite time.

Finally, let's minimize over the crash times; these are the only source of randomness in the model. We note here that if $\{\mathscr{F}_t\}_{t\ge 0}$ is the filtration generated by the system (2.3), then if the parameters are indeed given by $\tilde{v}$, we have that by using the Markov Property:

$$\mathbb{P}\left\{\tau_{n+1} - \tau_n \ge t \middle| \mathscr{F}_{\tau_n}\right\} = \exp\left[- \int_{s=0}^{t} \left\{ \kappa_q + \kappa_h \phi^{\tilde{v}}_{(1-\tilde{\gamma})\lambda^{\text{ob}}_{\tau_n -}}(s) \right\} ds \right]$$

so the $\tau_{n+1} - \tau_n$ has conditional (on $\mathscr{F}_{\tau_n}$) density with respect to Lebesgue measure of the form

$$f(t) = \left\{ \kappa_q + \kappa_h \phi^{\tilde{v}}_{(1-\tilde{\gamma})\lambda^{\text{ob}}_{\tau_n -}}(t) \right\} \exp\left[-\kappa_q t - \kappa_h \Phi^{\tilde{v}}_{(1-\tilde{\gamma})\lambda^{\text{ob}}_{\tau_n -}}(t) \right]$$

Then take the log form to be more convenient for the calculation

$$f(t) = \ln\left\{ \kappa_q + \kappa_h \phi^{\tilde{v}}_{(1-\tilde{\gamma})\lambda^{\text{ob}}_{\tau_n -}}(t) \right\} + \left[-\kappa_q t - \kappa_h \Phi^{\tilde{v}}_{(1-\tilde{\gamma})\lambda^{\text{ob}}_{\tau_n -}}(t) \right]$$

We can then find maximum-likelihood estimates of $\tilde{v}$ by minimizing over

$$I_3(\tilde{v}) \stackrel{\text{def}}{=} \sum_{\tau_n \le T} (\kappa_q (\tau_{n+1} - \tau_n) - \kappa_h \Phi^{\tilde{v}}_{(1-\tilde{\gamma})\lambda^{\text{ob}}_{\tau_n -}}(\tau_{n+1} - \tau_n) - \ln\left\{ \kappa_q + \kappa_h \phi^{\tilde{v}}_{(1-\tilde{\gamma})\lambda^{\text{ob}}_{\tau_n -}}(\tau_{n+1} - \tau_n) \right\}).$$

We now combine things. We calibrate by minimizing

$$\bar{I}(\tilde{v}) \stackrel{\text{def}}{=} I_1(\tilde{v}) + I_2(\tilde{v}) + I_3(\tilde{v}) \tag{4.2}$$

over all $\tilde{v} \in S^*$.

4.2 Real Data Analysis and Numerical Problems

4.2.1 Data Choices

In this section We will pick LEHMQ (Lehman Brothers Holdings Inc.) as an example to illustrate how we calibrate the model parameters numerically.

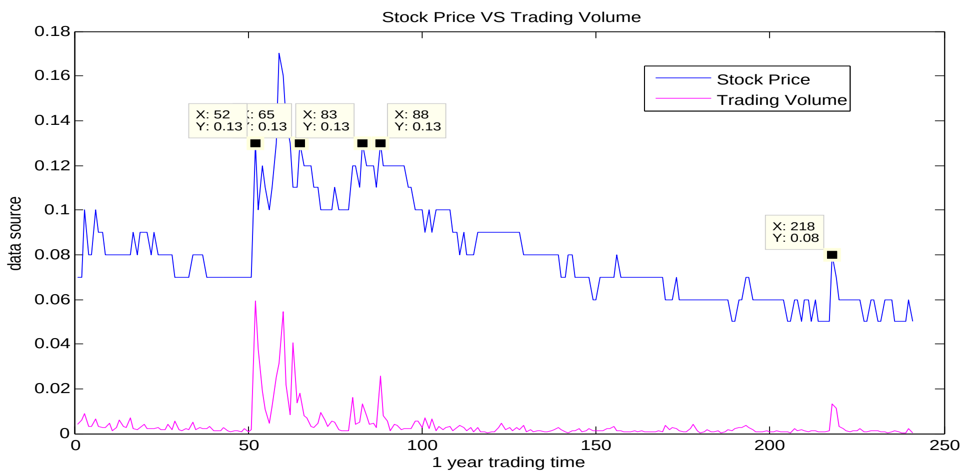
We choose the market data for LEHMQ from 12/10/2009 to 11/23/2010 [11] containing 241 trading days as a whole trading year cycle. LEHMQ files Chapter 11 Bankruptcy protection in Sep.2008, after several months trading, in the period we pick, it is extremely hard-to-borrow, which means this is a good example to study.

So let's set $0 = t_0 < \frac{1}{241} = t_1 < ... < t_{240} = 1$, with time increments $\Delta_t = \frac{1}{241}$

Before we use the Calibration Set-Up in Section 4.1, we need to have the crash time observation first. The idea to get this is from the feedback mechanism of the buy-ins: the greater the short interest, the even harder to short, then more frequent buy-ins occurring, which will finally drive up the stock price. And immediately without this buy-in support, the stock price will crash down. With this in mind we could denote the crash time to be the time, when both the stock price and the trading volume hits the local maximal, and also the volume should reach a LARGE amount. (Let's say state here "LARGE" as 1 standard deviation larger than the mean)

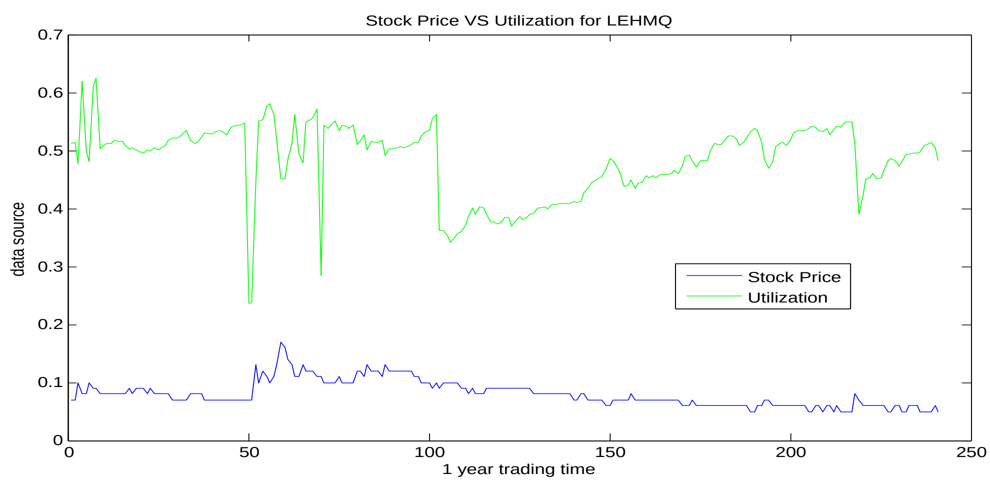
The Matlab codes in Appendix A Section 6 will gives us the crash time for the LEHMQ data source above. The results shows within these 241 trading dates, only 5 significant crash observed, at position: 52,65,83,88,218. Let's take a look into the figure 4.1 for the comparison of the trading volume (which is rescale by $\frac{1}{10^9}$ to have the same size order of the stock price) and stock price. This figure clearly shows that the stock price and trading volume has almost exactly the on-going trends. Especially they attain their local peaks at the close time range. And marking the 5 crash times on the picture, you will see, before those times, the stock price continuously goes up, and right after it, there's a huge drop, which does capture the main feature of the Hard-to-Borrowness.

Figure 4.1: Stock Price VS Trading Volume



Before we go to the general Calibration Methods, just put the comparison graph of the stock price and utilization(the placement for the short interest) here to see how they interact each other.

Figure 4.2: Stock Price VS Utilization



4.2.2 Calibration Methods

The main Calibration Procedure is to find the minimizer for the objective function (4.2). And to simplify the problem, let's first neglect the nonlinear explosive time constraint (4.1). Then our numerical work reduced to a unconstraint minimization problem. And in Matlab, we do this by using function: **fminsearch**, which is used to finds the minimizer of several variables for a scalar value function, and given the initial start estimation. The way to use it is to put $calibrationresults = fminsearch(@Objective, initial)$; in the main script, which means that the computer will use the Nelder-Mead algorithm [6] finding a local minimum calibration results for the self-defined desired Objective functions.

The Objective Functions we constructed for this calibration problem is totally determined by the structure of $\bar{I}(\tilde{v})$, which could be found in Section 4.1. And the details for this Objective Function could be found in Section 6 of Appendix A.

4.2.3 Minimization Initialization

Besides the Objective Function mentioned in Section 4.2.2, another important issue for the minimization search is the initial start. A bad initial condition will result in imprecise result, hence we need to carefully choose the initialization to have the problem converge into a reasonable range.

As a reasonable place to start searching for values of $\tilde{v}$, let's consider the case where $\kappa_h = 0$, which means no interaction between jump time and short interest. This should give us an idea of the order of magnitude of things. If $\kappa_h = 0$, then the shocks occur at exponential times. In other words,

$$\tau = \inf \{t\kappa_q \geq \mathfrak{e}\} = \frac{\mathfrak{e}}{\kappa_q}.$$

In other words, τ is exponential(κ_q). Thus $\mathbb{E}[\tau] = \frac{1}{\kappa_q}$. One might thus take an an initial guess of κ_q the reciprocal of the average empirical time between crashes.

Then for β_q and β_h should be the order of the inverse or the regular trading dates and several weeks separately. We could just set $\beta_q = 252$ and $\beta_h = 252/20$.

Furthermore, also neglect the short interest impact to the stock evolution drift speed, so we could set $\alpha_h = 0$. (but in the codes, in order to avoid some unexpected numerical problems, we will use some small number instead of zero),then:

$$S_{\tau_1 -} = S_0 e^{\alpha_q \tau_1} \qquad \text{and} \qquad \ln \frac{S_{\tau_{n+1}-}}{S_{\tau_n -}} = \ln(1-\gamma) + \alpha_q \tau$$

so

$$S_{\tau_1 -} = S_0 e^{\alpha_q \tau_1} \qquad \text{and} \qquad \mathbb{E}\left[\ln \frac{S_{\tau_{n+1}-}}{S_{\tau_n -}}\right] = \ln(1-\gamma) + \frac{\alpha_q}{\kappa_q} \tag{4.3}$$

By solving (4.3), the initialization for α_q and γ are ready.

Finally for $\tilde{\gamma}$, just try some value from $(0,1)$ as the set-up for short.

4.2.4 Results Analysis

Section 4.2.1,4.2.2 and 4.2.3 together will give us the calibration results in the following tables:

Table 1

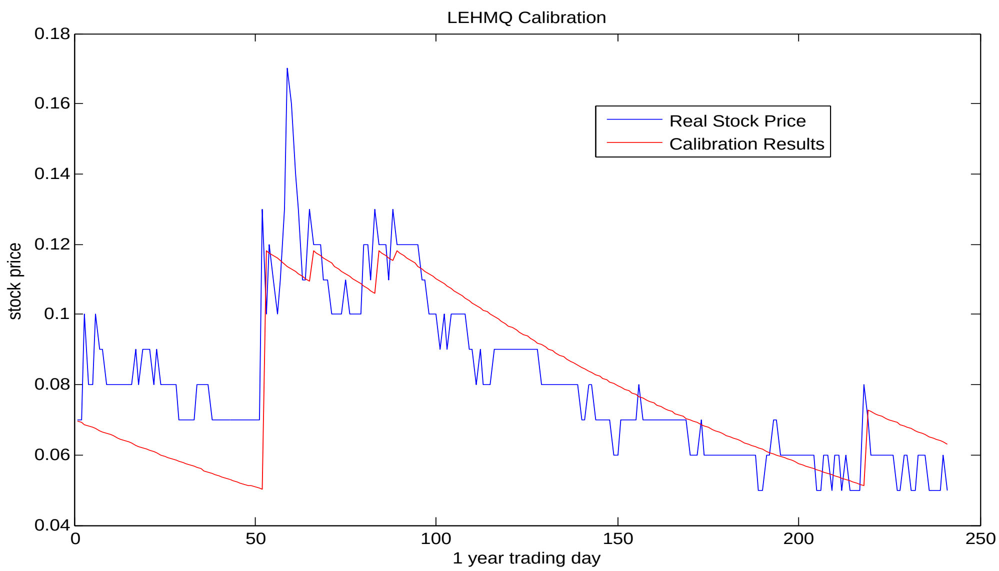
α_q	α_h	$\tilde{\beta}_1$	$\tilde{\beta}_2$	κ_q	κ_h	γ	$\tilde{\gamma}$
-1.1759	0.005	0.0934	0.0002	5.5191	0.0108	0.2028	0.0472

First of all, this gives us $\frac{\tilde{\beta}_1}{\kappa_q} \approx 0.016 < 0.02 \approx \ln \frac{1}{1-\tilde{\gamma}}$, which corresponds with Theorem 3.2.4

Also by using this result, we could plot the piecewise calibrated stock prices compared to the real values in figure 4.3. The View of the calibration figure tells us plenty of things.

Though not that exactly matching all the real stock price, the calibration does capture all the main trend of the evolution: like when to jump and how the jump follows, which really reflects the hard-to-borrow feedback mechanism. Also on the other hand, this is a way shows the fit of our model to the real market.

Figure 4.3: LEHMQ Calibration



Chapter 5

Option Pricing for Calibrating Validation

In the last part of Chapter 4, we have already used the comparison between the real stock data and the calibration results to show that our model fits really well. In this chapter we will convince us again the validation of our calibration by another way, which is to match the theory against the behavior of the option pricing things.

5.1 European Option Pricing for the Bankruptcy Model

Let's first understand the options pricing four European style. Since we can only trade S, we would normally not have a unique risk-neutral probability measure for the two-dimensional pair (S, λ). However, since there is in fact only one source of noise; the $\mathfrak{e}_n$'s, we can in fact proceed. Since one of the important aspects of the model of (2.1) is that the hard-to-borrowness leads one to an effective dividend rate which quantifies the correct risk-neutral cost of the crashes for a trader, we shall initially allow the stock to have a time-varying dividend rate δ_t.

5.1.1 Deriving the PIDE by Replication Strategy

Let V be the option price process. Since we have already excluded the diffusions, and the only uncertainty here is the jump time, so it's enough for us to hedge it with a combination of stock and a bond (which we assume to have interest rate r). We thus assume that:

$$V_t = w_t^{(1)} S_t + w_t^{(2)} B_t. \tag{5.1}$$

And also by the feature of the Hard-to-Borrowness, we should have this portfolio is self-financing [9] with an effective dividend yield to cover the short position cost. Then we get:

$$dV_t = w_t^{(1)}(dS_t + \delta_t S_t dt) + w_t^{(2)} dB_t. \tag{5.2}$$

On the other hand Itô-Doeblin Theorem for Jump Process together with system (2.4) shows that:

$$\begin{aligned} dV_t =& \frac{\partial \tilde{V}}{\partial t}(t, S_t, \lambda_t)dt + \{\alpha_q + \alpha_h \lambda_t\} S_t \frac{\partial \tilde{V}}{\partial S}(t, S_t, \lambda_t)dt + \left\{\tilde{\beta}_1 \lambda_t + \tilde{\beta}_2 \lambda_t^2\right\} \frac{\partial \tilde{V}}{\partial \lambda}(t, S_t, \lambda_t)dt \\ & + \left\{\tilde{V}(t, S_{t-}(1-\gamma), \lambda_{t-}(1-\tilde{\gamma})) - \tilde{V}(t, S_{t-}, \lambda_{t-})\right\} dN_t^{\kappa_q + \kappa_h \lambda_t} \end{aligned} \tag{5.3}$$

Matching equation (5.2) and (5.3), we have:

$$\begin{aligned} & \frac{\partial \tilde{V}}{\partial t}(t, S_t, \lambda_t)dt + \{\alpha_q + \alpha_h \lambda_t\} S_t \frac{\partial \tilde{V}}{\partial S}(t, S_t, \lambda_t)dt + \left\{\tilde{\beta}_1 \lambda_t + \tilde{\beta}_2 \lambda_t^2\right\} \frac{\partial \tilde{V}}{\partial \lambda}(t, S_t, \lambda_t)dt \\ & \quad + \left\{\tilde{V}(t, S_{t-}(1-\gamma), \lambda_{t-}(1-\tilde{\gamma})) - \tilde{V}(t, S_{t-}, \lambda_{t-})\right\} dN_t^{\kappa_q + \kappa_h \lambda_t} \\ & = w_t^{(1)} \left\{\{\alpha_q + \alpha_h \lambda_t\} S_t dt - \gamma S_{t-} dN_t^{\kappa_q + \kappa_h \lambda_t} + \delta_t S_t dt\right\} + r w_t^{(2)} B_t dt. \end{aligned} \tag{5.4}$$

Equating the coefficients of $dN_t^{\kappa_q + \kappa_h \lambda_t}$, we have that

$$w_t^{(1)} = -\frac{\tilde{V}(t, S_{t-}(1-\gamma), \lambda_{t-}(1-\tilde{\gamma})) - \tilde{V}(t, S_{t-}, \lambda_{t-})}{\gamma S_{t-}}. \tag{5.5}$$

Equating the dt terms and using the fact that $w_t^{(2)} B_t = \tilde{V}(t, S_t, \lambda_t) - w_t^{(1)} S_t$, we have that

$$\begin{aligned}\frac{\partial \tilde{V}}{\partial t}(t, S_t, \lambda_t) + \{\alpha_q + \alpha_h \lambda_t\} S_t \frac{\partial \tilde{V}}{\partial S}(t, S_t, \lambda_t) + \left\{\tilde{\beta}_1 \lambda_t + \tilde{\beta}_2 \lambda_t^2\right\} \frac{\partial \tilde{V}}{\partial \lambda}(t, S_t, \lambda_t) \\ = w_t^{(1)} \{\alpha_q + \alpha_h \lambda_t + \delta_t\} S_t + r \left\{\tilde{V}(t, S_t, \lambda_t) - w_t^{(1)} S_t\right\}. \end{aligned} \tag{5.6}$$

Replacing $w_t^{(1)}$ by (5.5) back to (5.6), we have the Black-Scholes like equation as follows:

$$\begin{aligned}\frac{\partial \tilde{V}}{\partial t}(t, S, \lambda) + \{\alpha_q + \alpha_h \lambda\} S \frac{\partial \tilde{V}}{\partial S}(t, S, \lambda) + \left\{\tilde{\beta}_1 \lambda + \tilde{\beta}_2 \lambda^2\right\} \frac{\partial \tilde{V}}{\partial \lambda}(t, S, \lambda) \\ + \frac{1}{\gamma} \{\alpha_q + \alpha_h \lambda_t + \delta_t - r\} \left\{\tilde{V}(t, S(1-\gamma), \lambda(1-\tilde{\gamma})) - \tilde{V}(t, S, \lambda)\right\} = r\tilde{V}(t, S, \lambda). \end{aligned} \tag{5.7}$$

5.1.2 Risk Neutral Model and Effective Dividend

In reality, when we want to calculate the option price, we only care about the risk neutral process, since under the risk neutral measure, the discount process is a martingale, which is comfortable for the calculation. Let's now get the risk-neutral model by setting $\delta \equiv 0$. Then the Black-Scholes equation is

$$\begin{aligned}\frac{\partial \tilde{V}}{\partial t}(t, S, \lambda) + \{\alpha_q + \alpha_h \lambda\} S \frac{\partial \tilde{V}}{\partial S}(t, S, \lambda) + \left\{\tilde{\beta}_1 \lambda + \tilde{\beta}_2 \lambda^2\right\} \frac{\partial \tilde{V}}{\partial \lambda}(t, S, \lambda) \\ + \frac{1}{\gamma} \{\alpha_q + \alpha_h \lambda_t - r\} \left\{\tilde{V}(t, S(1-\gamma), \lambda(1-\tilde{\gamma})) - \tilde{V}(t, S, \lambda)\right\} = r\tilde{V}(t, S, \lambda). \end{aligned}$$

Unwinding this, let's define

$$\tilde{\kappa}_q \stackrel{\text{def}}{=} \frac{\alpha_q - r}{\gamma} \qquad \text{and} \qquad \tilde{\kappa}_h \stackrel{\text{def}}{=} \frac{\alpha_h}{\gamma}.$$

Then the risk-neutral dynamics are given by

$$
\begin{aligned}
dS_t &= \{\alpha_q + \alpha_h \lambda_t\} S_t dt - \gamma S_{t-} dN_t^{\tilde{\kappa}_q + \tilde{\kappa}_h \lambda_t} \\
&= r S_t dt + \gamma \{\tilde{\kappa}_q + \tilde{\kappa}_h \lambda_t\} S_t dt - \gamma S_{t-} dN_t^{\tilde{\kappa}_q + \tilde{\kappa}_h \lambda_t} \\
d\lambda_t &= \left\{\tilde{\beta}_1 \lambda_t + \tilde{\beta}_2 \lambda_t^2\right\} dt - \tilde{\gamma} \lambda_{t-} dN_t^{\tilde{\kappa}_q + \tilde{\kappa}_h \lambda_t}
\end{aligned}
\tag{5.8}
$$

The second representation of the dynamics of S characterizes the stock price in the usual way; a combination of interest rate and a martingale. Note that we can again write the dynamics of λ as the second equation of (2.3). One of the key insights of (5.8) is that the risk-neutral cost of holding a short delta position at time t is the product of the crash size and the instantaneous likelihood of a crash. In particular, if a trader is short a share at time t, the value of the crash is

$$
\gamma \times \{\tilde{\kappa}_q + \tilde{\kappa}_h \lambda_t\} = \alpha_q - r + \alpha_h \lambda_t.
$$

Thus the clearing firm can demand a payment of $\alpha_q - r + \alpha_h \lambda_t$ for a short share at that time. Thus, let's set the dividend rate to

$$
\delta_t \stackrel{\text{def}}{=} \alpha_q - r + \alpha_h \lambda_t = \gamma \{\tilde{\kappa}_q + \tilde{\kappa}_h \lambda_t\}. \tag{5.9}
$$

Note that if our position is short delta, then $w_t^{(1)} < 0$ so a positive dividend rate implies that $w_t^{(1)} \delta_t S_t < 0$, corresponding to a payment made by the trader. Note also that then

$$
\frac{1}{\gamma} \{\alpha_q - r + \alpha_h \lambda_t + \delta_t\} = 2\tilde{\kappa}_q + 2\tilde{\kappa}_h \lambda_t.
$$

Thus the Black-Scholes equation for a European option is given by (5.7) with δ_t given by

(5.9). Note that the Feynman-Kac formula for this uses the process

$$\begin{aligned} d\hat{S}_t &= \{\alpha_q + \alpha_h \lambda_t\} \hat{S}_t dt - \gamma \hat{S}_{t-} dN_t^{2\tilde{\kappa}_q + 2\tilde{\kappa}_h \lambda_t} \\ &= \{r - \delta_t\} \hat{S}_t dt + \gamma \{2\tilde{\kappa}_q + 2\tilde{\kappa}_h \lambda_t\} S_t dt - \gamma \hat{S}_{t-} dN_t^{2\tilde{\kappa}_q + 2\tilde{\kappa}_h \hat{\lambda}_t} \\ d\hat{\lambda}_t &= \left\{\tilde{\beta}_1 \lambda_t + \tilde{\beta}_2 \hat{\lambda}_t^2\right\} dt - \tilde{\gamma} \hat{\lambda}_{t-} dN_t^{2\tilde{\kappa}_q + 2\tilde{\kappa}_h \hat{\lambda}_t}. \end{aligned} \tag{5.10}$$

Again, the second equation of (2.3) gives the dynamics of λ.

Note that since the only randomness occurs in the jumps (as opposed to the diffusion), the dividend-adjusted dynamics of S is *not* equivalent to the risk-neutral dynamics with a dividend discount; i.e., the dynamics of (5.10) is not the same as

$$S_t \exp\left[-\int_{s=0}^{t} \delta_s ds\right]$$

with S given by (5.8).

5.2 Numerical Approach for American Option Pricing

Let's next consider pricing American puts (which represent a significant portion of trading activity in bankrupt stocks). Since the only source of randomness in our model is the crash times, several well-known methods are likely to have numerical challenges. Fourier methods, for example, depend in various ways on smoothing properties (like that of the heat kernel), which are not available here. In the same vein, a numerical implementation of a something like a differential inequality would suffer from non-local effects. We will develop an algorithm which is suited to the fact that the only source of randomness in our model is the crash times. As is usual in pricing American options, the challenge is to functionally approximate the risk-neutral expectation operator.

Back to our model here, from the dynamics we know that, the evolutions for both the stock price is determined and the uncertainty is due to the jump time. Hence, with the calibration results on hand, it will be really easy to generate the sample path (See the Code in Appendix 6 for details). Then the efficient way came into our mind is the Monte Carlo simulation, which is used to generate the sample path first and then apply the least square estimation terminology introduced by Longstaff and Schwartz [5] to numerically calculate the desired American Put price.

However due to the limitation of the short interest data, the empirical results for this option validation will be done in the future work, which will be discussed later in the next Chapter.

Chapter 6

Future Research

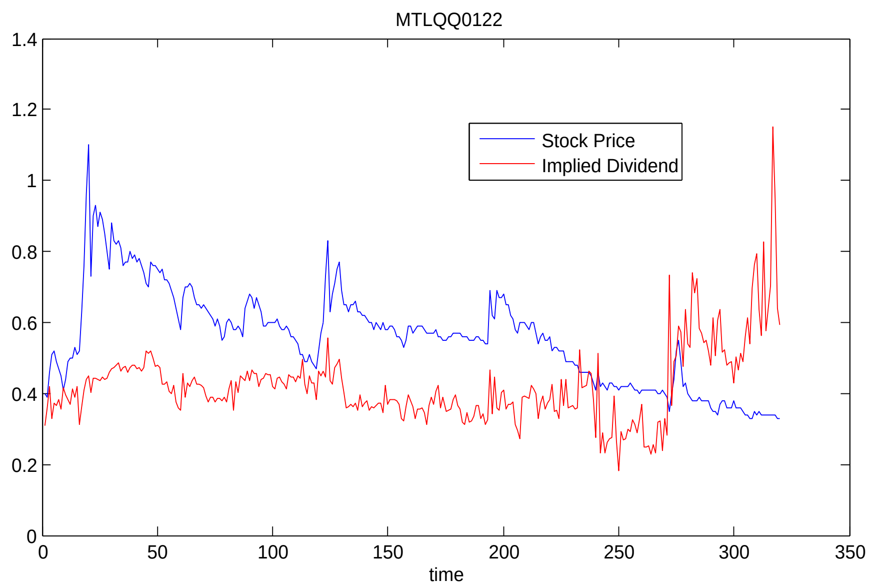
As we showed above in the Chapter 4, the calibration results we obtained by using the utilization data is in some extent catch the hard to borrowness mechanism. However those stock evolution is generated upon each crash period piecewisely, which would result in non-fit when generated in the whole period randomly. Hence in Chapter 5 we mentioned that this would yield the difficulty to calculate the option price. The main reason We observed is that the utilization data within that range has some conflicts with our main assumption and also doesn't match the stock evolution well enough, which could be seen from Figure 4.2.

Therefore, we are thinking of looking into other source of data to work out the option pricing issue. One way we try to investigate is to use the implied dividend yield from the put-call conversion [1], such that $d_{imp}(K,T) \equiv \frac{P_{pop}(K,T)-C_{pop}(K,T)+KRT}{ST}$. For here, d_{imp} is the implied dividend yield from the option market, and C_{pop},P_{pop} represents the premium over parity for call/put separately. As Mike Lipkin suggests, this dividend is the key index when he trades the option on Hard-to-Borrow Stocks, which would replace the role for the short interest. In order to see how this works, we plot the historical stock prices VS the implied dividend yield for the 01/22/11 options on equity MTLQQ(general motors) from period July.2009-Jan.2011 for example in Figure 6.1. We shall see just within the 1 year's bankrupt(from 06.2009), the dividend yield might be used as a good reference to catch the short interest. And as the time elapse, when the value of the stocks approach zero, the dividend yield looks like irrelated. Hence, it is reasonable for us to drop the tail range of the stocks and investigate the inverse effect of the dividend yield to justify the validation of the stock price. We will also try to find some ways nice enough to implement the put price.

After we finish the above research work, our next step would be naturally get into some more complicate model. Since as we talked in the beginning, bankrupt stocks are only the special case of the Hard-to-Borrow stocks. We choose them as our interested topic is based on their tractability, but good enough to show the Hard-to-Borrowness feedback. However, in the real market there are lots of limitations on the bankrupt stocks. For example, few people will have interest to trade the bankrupt stocks, and then after the stock died, the historical data will also hard to extract from the database. So, based on the mechanism we got from the bankrupt stocks, it might be more interesting to investigate the general Hard-to-Borrow stocks, which means we need to add the diffusivity part back into our model. This should be a challenging but attractive problem in the future.

Finally,the macro picture of this research work is built on a feedback mechanism, which might be a good guide for us to have the insight into other market phenomena.

Figure 6.1: MTLQQ0122



Appendix A Codes for finding the crash time

The data source for the codes is stored in a txt files, which contains trading volume, stock price and utilization (short interest).

```
%% This sub-function is used to find the crash time
global time tau lambda S lambda_0 Stock_0 delta_t crash_position;
load databankruptstock.txt;
datastock=databankruptstock;
J=max(size(datastock));
delta_t=1/J;
time=1/J:delta_t:1;
volume=datastock(:,1);
S=datastock(:,2);
lambda=datastock(:,3);
lambda_0=lambda(1);
Stock_0=S(1);
[pks_stock,locs_stock]=findpeaks(S);
[pks_volume,locs_volume]=findpeaks(volume);
mean_volume=mean(volume);
std_volume=std(volume);
j=1;
k=1;
temp=0;
```

```
n=max(size(locs_stock));
m=max(size(locs_volume));
tau=zeros(min(m,n),1);
for i=1:n
    while (j<=m)&&(locs_stock(i)~=locs_volume(j))
        j=j+1;
    end
    if j<=m
        if volume(locs_volume(j))>(mean_volume+std_volume)
        tau(k)=locs_volume(j);
        k=k+1;
        end
        temp=j;
      elseif temp~=0
         j=temp;
    else j=1;
    end
end
N=k-1;
tau=tau(1:N,1).*delta_t;
crash_position=tau/delta_t;
```

Appendix B Codes for Objective Function

```
%x(1)=\alpha_q: quiescent rates of growth for stocks
%x(2)=\alpha_h: exposure to HTB for stocks
%x(3)=\tilde{\beta}_1
%x(4)=\tilde{\beta}_2
%x(5)=\kappa_q: rate of growth for intensity
%x(6)=\kappa_h: exposure to HTB for intensity
%x(7)=\gamma: jump size;
%x(8)=\tilde{\gamma}=\gamma*\rho
%\rho: correlation of buy-in rate and stock change rate.
%I_i: objective function i, i=1,2,3
%y=I_1+I_2+I_3
%intensity: deterministic intensity during the crash
function y=Objective(x)
global time tau lambda S lambda_0 Stock_0 delta_t crash_position I_2
global I2 I1 I3
N=max(size(tau));
J=max(size(time));
I_1=zeros(J,1);
I_2=zeros(J,1);
initial_lambda=zeros(N+1,1);
initial_stock=zeros(N+1,1);
```

```
%construct cycle between crashes
increaments=zeros(N+2,1);
increaments(1)=0;
increaments(2:(N+1),1)=crash_position;
increaments(N+2,1)=J;
%generate initial lambda condition for each cycle
jump_lambda=lambda(crash_position);
initial_lambda(1)=lambda_0;
initial_lambda(2:(N+1),1)=(1-x(8))*jump_lambda;
%generate initial stock price condition for each cycle
jump_stock=S(crash_position);
initial_stock(1)=Stock_0;
initial_stock(2:(N+1),1)=(1-x(7))*jump_stock;
%time within the cycle
time_difference=zeros(J,1);
for n=1:(N+1)
    time_difference((increaments(n)+1):increaments(n+1),1)=
    time((increaments(n)+1):increaments(n+1),1)-increaments(n)*delta_t;
    I_1((increaments(n)+1):increaments(n+1),1)=
    x(3)*initial_lambda(n)
    *exp(x(3)*(time_difference((increaments(n)+1):increaments(n+1),1)))
    ./(x(3)-x(4)*(exp(x(3)
    *(time_difference((increaments(n)+1):increaments(n+1),1)))-1));
    I_2((increaments(n)+1):increaments(n+1),1)=log(initial_stock(n))
    +x(1)*time_difference((increaments(n)+1):increaments(n+1),1)+
    x(2)*log(x(3)./(x(3)-x(4)*initial_lambda(n)*
    (exp(x(3)
    *time_difference((increaments(n)+1):increaments(n+1),1))-1)))/x(4);
```

```
end
    I1=sum((I_1-lambda).^2);
    I2=sum((I_2-log(S)).^2);
%Maximum-likelihood Estimation:
crash_cycle=(increaments(2:(N+1),1)-increaments(1:N,1))*delta_t;
I_3=x(5)*crash_cycle-x(6)*log(x(3)./(x(3)-x(4)*initial_lambda(1:N,1).*
(exp(x(3)*crash_cycle)-1)))/x(4)-log(x(5)
+x(6)*x(3)*initial_lambda(1:N,1).*exp(x(3)*crash_cycle)
./(x(3)-x(4)*initial_lambda(1:N,1)
.*(exp(x(3)*crash_cycle)-1)));
I3=sum(I_3);
y=I1+I2+I3;
end
```

Appendix C Codes for Simulating Sample Path

```
global time tau lambda exponential calibration_result previous_start
N=max(size(time));
L=max(size(tau));
Sample_Number=1;
y=zeros(Sample_Number,N);
time_difference=zeros(1,N);
time=time';
%calibration_result=calibration;
crash_time=-log(rand(Sample_Number,10));
for i=1:Sample_Number
    sample_crash=0;
    previous_start=lambda(1);
    S_initial=S(1);
    count=1;
    start=1;
    while count<11
        guess=tau(min(count,L));
        exponential=crash_time(i,count);
        crash_time(i,count)=fsolve('crash_moment',guess);
        sample_crash=sample_crash+floor(crash_time(i,count)/time(1,1));
        if sample_crash>N
```

```
        break
    end
    time_difference(i,start:sample_crash)=
    time(1,start:sample_crash)-time(1,start);
    y(i,start:sample_crash)=S_initial*exp(calibration_result(1)
    *time_difference(i,start:sample_crash)
    +calibration_result(2)*(calibration_result(3)
    *previous_start*exp(calibration_result(3)
    *(time_difference(i,start:sample_crash)))
    ./(calibration_result(3)-calibration_result(4)*previous_start
    *(exp(calibration_result(3)
    *(time_difference(i,start:sample_crash)))-1))));
    S_initial=(1-calibration_result(7))*y(i,sample_crash);
    previous_start=(1-calibration_result(8))*phi(crash_time(i,count));
    start=sample_crash+1;
    count=count+1;
  end
end
```

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