HAMILTONIAN CYCLES THROUGH SPECIFIED EDGES IN BIPARTITE GRAPHS, DOMINATION GAME, AND THE GAME OF REVOLUTIONARIES AND SPIES

BY

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DISSERTATION

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ABSTRACT

This thesis deals with the following three independent problems.

Pósa proved that if G is an n-vertex graph in which any two nonadjacent vertices have degree sum at least $n+k$, then G has a spanning cycle containing any specified family of disjoint paths with a total of k edges. We consider the analogous problem for a bipartite graph G with n vertices and parts of equal size. Let F be a subgraph of G whose components are nontrivial paths. Let k be the number of edges in F, and let t_1 and t_2 be the numbers of components of F having odd and even length, respectively. We prove that G has a spanning cycle containing F if any two nonadjacent vertices in opposite partite sets have degree-sum at least $n/2+\tau(F)$, where $\tau(F)=\lceil k/2\rceil+\epsilon$ (here $\epsilon=1$ if $t_1=0$ or if $(t_1,t_2)\in\{(1,0),(2,0)\}$, and $\epsilon=0$ otherwise). We show also that this threshold on the degree-sum is sharp when $n>3k$.

Bostjan Brešar, Sandi Klavžar and Douglas F. Rall proposed a game involving the notion of graph domination number. Two players, Dominator and Staller, occupy vertices of a graph G, playing alternatingly. Dominator starts first. A vertex is valid is to be occupied if adding it to the occupied set enlarges the set of vertices dominated by the occupied set. The game ends when the occupied set becomes a dominating set (A *dominating set* is a set of vertices U such that every vertex is in U or has a neighbor in U; the minimum size of a dominating set is the *domination number*, written $\gamma(G)$). Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. The size of the dominating set obtained when both players play optimally is the *game domination number* of G, written as $\gamma_g(G)$. The *Staller-first game domination number*, written as $\gamma_g'(G)$, is defined similarly; the only difference is that Staller starts the game. Brešar et al. showed that $\gamma(G)\le\gamma_g(G)\le 2\gamma(G)-1$ and that for any k and k' such that $k\le k'\le 2k-1$, there exists a graph G with $\gamma(G)=k$ and $\gamma_g(G)=k'$. Their constructions use graphs with many vertices of degree 1. We present an n-vertex graph G with domination number, minimum degree and connectivity of order $\theta(\sqrt{n})$ that satisfies $\gamma_g(G)=2\gamma(G)-1$. Building on the work of Brešar et al., Kinnersley proved that $|\gamma_g(G)-\gamma_g'(G)|\le 1$. Brešar et al. defined a

pair (k, k') to be *realizable* if $\gamma_g(G) = k$ and $\gamma'_g(G) = k'$ for some graph G. They showed that the pairs (k, k), $(k, k+1)$ and $(2k+1, 2k)$ are realizable for $k \geq 1$. Their constructions for $(k, k+1)$ and $(2k+1, 2k)$ are not connected. We show that for $k \geq 1$, the pairs $(k, k+1)$, $(2k+1, 2k)$ and $(2k+2, 2k+1)$ are realizable by connected graphs.

Józef Beck invented the following game, the game of *revolutionaries and spies*. It is a two-player game $\mathcal{RS}(G, m, r, s)$ played on a graph G by two players $\mathcal{R}$ and $\mathcal{S}$. Player $\mathcal{R}$ controls r pieces called *revolutionaries* and player $\mathcal{S}$ controls s pieces called *spies*. At the start, $\mathcal{R}$ places his pieces on vertices of G, and then $\mathcal{S}$ does so also. At each subsequent round, $\mathcal{R}$ moves some of his pieces from their current vertex to a neighboring vertex, and then $\mathcal{S}$ does so also. If at the end of a round there is a meeting of at least m revolutionaries on some vertex without a spy, then $\mathcal{R}$ wins. Player $\mathcal{S}$ wins if he can prevent such a meeting forever. We show that $s \geq \gamma(G)\lfloor r/m \rfloor$ suffices for $\mathcal{S}$ to win $\mathcal{RS}(G, m, r, s)$. Given r and s, let H be a complete bipartite graph with at least $r+s$ vertices in each partite set. We will show that $7r/10 + O(1)$ is the minimum number of spies needed to win $\mathcal{RS}(H, 2, r, s)$. We also show $r/2 + O(1)$ is the minimum number of spies needed to win $\mathcal{RS}(H, 3, r, s)$. For $m \geq 4$, we show that the minimum number of required spies to win $\mathcal{RS}(H, m, r, s)$ is at least $\lfloor \lfloor r/2 \rfloor / \lceil m/3 \rceil \rfloor - 1$ and at most $(1 + 1/\sqrt{3})r/m + 1$.

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CHAPTER 1

INTRODUCTION

We present results on three combinatorial problems.

1.1 Hamiltonian Cycles through Specified Edges in Bipartite Graphs

The first problem is about Hamiltonian cycles on bipartite graphs. In 1859, the Irish mathematician William Rowan Hamilton invented a game that he sold to a toy manufacturer in Dublin. The game was a wooden regular dodecahedron with its 20 corners labeled with the names of big cities (see Figure 1.1). The objective was to find a closed path along the edges of the dodecahedron such that every city is visited exactly once.

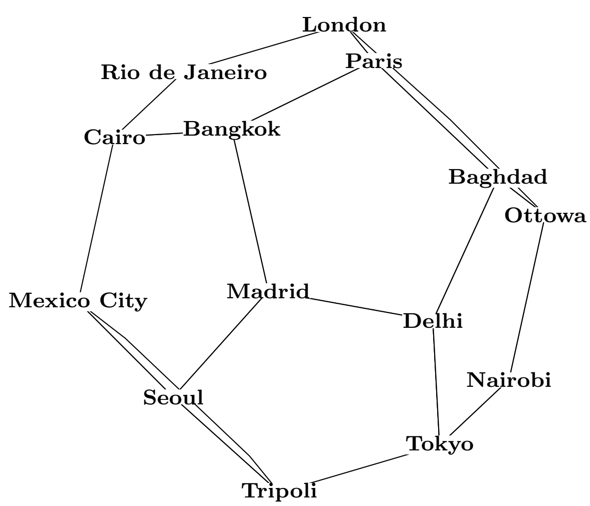


Figure 1.1: Hamilton's game.

The problem turns out to be extremely hard when considered for an arbitrary graph. In 1972, Richard Karp [18] proved that finding such a path in a directed or undirected graph is NP-complete. Later, Garey and Johnson [12] proved that the directed version restricted to planar graphs is NP-complete, and the undirected version remains NP-complete even for cubic planar graphs. In 1980, Akiyama, Nishizeki, and Saito [1] showed that the problem is NP-complete even when restricted to bipartite graphs.

Consistent with the computational aspects, no nice characterization of Hamiltonian graphs is known. Probably the most famous necessary condition is 1-toughness[1] introduced by Chvátal [6]. In 1952, Dirac [9] observed a sufficient condition: every n-vertex graph with $n \geq 3$ and minimum degree at least $n/2$ is Hamiltonian. Ore [20] obtained a stronger version: every n-vertex graph with $n \geq 3$ whose pairs of nonadjacent vertices have degree-sum at least n is Hamiltonian. Pósa [21] realized that if half of the vertices have degree at least $n/2$, then we may allow smaller degrees on the other vertices: if d_i is the i-th smallest degree in an n-vertex graph with $d_i > i$ for all $i < (n-1)/2$, and $d_{\lceil \frac{n}{2} \rceil} \geq \lceil \frac{n}{2} \rceil$ if n is odd, then the graph is Hamiltonian. Chvátal [5] extended this by showing that $d_i > i$ or $d_{n-i} \geq n-i$ for all $i < n/2$ suffices. Moon and Moser [19] proved a similar result for bipartite graphs: every bipartite graph with exactly $n/2$ vertices in each partite set is Hamiltonian when any two nonadjacent vertices taken from different partite sets have degree-sum at least $n/2+1$. All the mentioned sufficient conditions are sharp in the sense that one cannot make their inequalities weaker and still obtain the conclusion.

Mathematicians have also studied similar degree conditions conditions under which a Hamiltonian cycle satisfying some additional restrictions exists [11]. Pósa [22] proved that given integers t and k where $0 \leq t \leq k$, an n-vertex graph G with $n \geq 3$ whose pairs of nonadjacent vertices have a degree-sum at least $n+k$, and a subgraph F of G consisting of t nontrivial paths with a total of k edges, one can always find a spanning cycle of G that contains F. He showed that the threshold $n+k$ is also sharp. In Chapter 2 we generalize this result to bipartite graphs. Let G be a bipartite graph with exactly $n/2$ vertices in each of its partite sets, and let F be a subgraph of G consisting of t_1 odd-length paths and t_2 even-length nontrivial paths. If any two nonadjacent vertices belonging to different partite sets have degree-sum at least $n/2+\lceil k/2 \rceil+\epsilon$ (where $\epsilon = 1$ if $t_1 = 0$ or $(t_1, t_2) \in \{(1,0),(2,0)\}$, and otherwise $\epsilon = 0$), then one can find a Hamiltonian cycle in G that contains F. We will show also that the result is sharp when $n > 3k$. This work is joint with Douglas B. West and will appear in Journal of Graph Theory.

1.2 Domination Game

The second problem extends a classical optimization problem. A set S of vertices of a graph is a *dominating set* if every vertex not in S is adjacent to some vertex of S. The size of

[1] A graph is 1-tough if removing any separating set of size k leaves at most k components.

a smallest dominating set in a graph G is the *domination number* of G and is denoted by $\gamma(G)$. A vast amount of research has been dedicated to studying the properties of $\gamma(G)$. The extension that we will consider here is obtained via adding a flavor of competition to the initial scenario in which one is selecting the vertices of some optimum dominating set one by one. The competition arises when another independent will, an opponent or the nature, gets a share in the selection process.

There is more than one classical approach to formalize the notion of competition. What we are considering here is the kind of competitions that are studied in the combinatorial game theory. Many of the classical games are examples of the combinatorial games. Probably the most famous among them is chess, and it is guessed that the most ancient one is Senet[2]. There are two main differences between combinatorial game theory and the notion of game theory studied in economics. First, the state of the game and the actions that the players can perform are known to all of them. The second difference is that in combinatorial games players necessarily play in sequence. Therefore probabilistic strategies are unnecessary. This makes analysis adhoc and less generalizable. Despite this, a few frameworks have been developed that capture the essence of some common categories of these games, for example "surreal numbers" by John Horton Conway [7].

The game that we study is played by two players, Dominator and Staller. Dominator starts first and they alternate occupying available vertices. Vertices whose addition to the occupied set increases the set of dominated vertices are available to be occupied. The game finishes when the occupied set is a dominating set. Dominator's goal is to minimize the size of this dominating set, and Staller's goal is to maximize it. The size of the final set when both players play optimally in a graph G is its *game domination number*, written as $\gamma_g(G)$. The same quantity when Staller starts the game is the *Staller-first game domination number*, written as $\gamma'_g(G)$. Brešar et. al. [4] introduced this game and proved $\gamma_g(G) - 1 \leq \gamma'_g(G) \leq \gamma_g(G) + 2$ for every graph G. A natural question is whether every possible pair of values allowed by this inequality will be observed on some graph, or one might be able to improve the inequality in some other way. They call a pair (k, k') *realizable* if there exists a graph G with $\gamma_g(G) = k$ and $\gamma'_g(G) = k'$. In [4], they built a connected graph that realizes (k, k) for $k \geq 1$. They also presented disconnected graphs that realize $(k, k+1)$ and $(2k+1, 2k)$ for $k \geq 1$. Bill Kinnersley proved that $\gamma'_g(G) \leq \gamma_g(G) + 1$, so pairs $(k, k+2)$ are not realizable. Also, $(2, 1)$ is trivially not realizable. We will identify pairs realizable by connected graphs. We present connected graphs that realize $(k, k+1)$, $(2k+1, 2k)$ and $(2k+2, 2k+1)$ for $k \geq 1$. Therefore all the pairs allowed by the inequality $\gamma_g(G) - 1 \leq \gamma'_g(G) \leq \gamma_g(G) + 1$

[2] or Senat; meaning "the game of passing"

are realizable by connected graphs excepting $(2,1)$. Our results on this topic, together with additional results, will appear in a joint paper with Bill Kinnersley and Douglas B. West.

1.3 The Game of Revolutionaries and Spies

The third problem is about another combinatorial game. This work is joint with Jane V. Butterfield, Daniel W. Cranston, Gregory J. Puleo and Douglas B. West. This game is not impartial, that is the set of moves available to players not only depends on the state of the game (the state of the board) but also depends on which player is playing. Two players $\mathcal{R}$ and $\mathcal{S}$ play this game on a graph G. Integer parameters r, s and m are fixed. Player $\mathcal{R}$ controls r identical pieces called *revolutionaries*, and $\mathcal{S}$ controls s identical pieces called *spies*. At the beginning of the game $\mathcal{R}$ puts his pieces on vertices of G, then $\mathcal{S}$ also does so. Players alternate playing. At his turn, a player can move each of his pieces along one of its incident edges. During the game if immediately after $\mathcal{S}$ moves, there are m revolutionaries on some vertex of G without any spy on that vertex, then $\mathcal{R}$ wins. Player $\mathcal{S}$ wins if he prevents this forever. The primary question of interest is: given G, m, r, and s, who wins? The game was invented by Józef Beck in 1994 [17]. Howard et al. [17] proved that if G is acyclic, then spies win if their population is at least $\lfloor r/m \rfloor$. Let $\mathbb{Z}^{\boxtimes d}$ be the graph with vertex set $\mathbb{Z}^d$ and edge set $\{(a_1,\ldots,a_d)(b_1,\ldots,b_d)\colon\ |a_i - b_i| \le 1 \text{ for all } i\}$. Let $s_d(r)$ be the minimum number of spies needed to beat r revolutionaries on $\mathbb{Z}^{\boxtimes d}$ with meeting size 2. Howard et al. [17] proved that $\liminf_{r\to\infty} s_d(r)/r \ge 3/4$ for $d \ge 2$.

An *m-large complete bipartite graph* is a bipartite graph with at least m vertices in each of its partite sets. We will show that if G is an $r+s$-large complete bipartite graph and $m=2$, then s should be at least $7r/10+O(1)$ for the spies to win. If $m=3$, then s should be at least $r/2+O(1)$. For larger m, we show that the winning threshold is between $3r/(2m)+O(1)$ and $(1+1/\sqrt{3})r/m+O(1)$. We also know that when a graph has a dominating vertex or has a single cycle or is an interval graph [17], the threshold is $\lfloor r/m \rfloor + O(1)$.

1.4 Basic Concepts and Notations

A *graph* G is a pair consisting of a vertex set $V(G)$ and an edge set $E(G)$, where $E(G)$ is a set of unordered pairs of vertices. We represent the edge $\{u,v\}$ consisting of vertices u and v simply as uv. When $\{u,v\} \in E(G)$, we say u is *adjacent* to v, and $\{u,v\}$ is *incident* to u. The *neighborhood* of a vertex u is the set of all the vertices adjacent to u, denoted by $N_G(u)$. The *closed neighborhood* of a vertex u is $N_G(u) \cup \{u\}$, denoted by $N_G[u]$. The *degree* of a vertex u in a graph G is $|N_G(u)|$, denoted by $\deg_G(u)$. The minimum degree of a graph G is the smallest vertex degree in that graph, denoted by $\delta(G)$; the maximum degree is the largest one, denoted by $\Delta(G)$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, denoted as $H \subseteq G$. A *spanning subgraph* of G is a subgraph of G whose vertex set is the same as G. An *induced subgraph* H of G is a subgraph of G that contains all the edges of G whose endpoints lie in H. For a subset A of vertices of G, we use $G[A]$ to denote the induced subgraph of G with vertex set A. The notation $G - A$ is a shorthand for $G[V - A]$. A graph H is *isomorphic* to G, written as $H \cong G$, if there exists a bijection $f : V(G) \to V(H)$ such that for any two vertices u and v, we have u adjacent to v if and only if $f(u)$ is adjacent to $f(v)$. The *union* of graphs G and H, written as $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *disjoint union* of graphs G and H, written as $G + H$, is the union of G and H under the condition that the vertex sets of G and H are specified to be disjoint. The *Cartesian product* of two graphs G and H, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H) = \big\{(u, v)\colon\ u \in V(G), v \in V(H)\big\}$ and edge set $\big\{(u, v)(u, v')\colon\ u \in V(G), vv' \in E(H)\big\} \bigcup \big\{(u, v)(u', v)\colon\ uu' \in E(G), v \in V(H)\big\}$.

A *path* P is a graph isomorphic to the graph with vertices $v_1, \ldots, v_n$ and edges $\{v_i v_{i+1}\colon\ 1 \leq i \leq n - 1\}$. Sometimes we represent a path by listing its vertices in order in angle brackets: $\langle v_1, \ldots, v_n \rangle$. We use the notation $P(v_i, v_j)$ to represent the list of vertices from v_i to v_j along a path P that contains them both. We also use the term *v_1, v_n-path* for a path that starts at vertex v_1 and ends at vertex v_n. A *cycle* is a graph obtained from a path by adding one more edge joining its endpoints. We may represent a cycle by listing its vertices in order in brackets, that is as $[v_1, \ldots, v_n]$. Given a cycle C and an edge uv on C, we write $C(u, v)$ for the list of vertices along the path $C - uv$ from u to v. The *length* of a cycle or path is the number of edges in it. The *distance* between two vertices u and v in a graph G, written as $d_G(u, v)$, is the length of a shortest u, v-path contained in G. A graph is *Hamiltonian* if it contains a spanning cycle. We use P_n and C_n to indicate the isomorphism class of n-vertex paths and cycles, respectively. We write $H \subseteq G$ to mean that G contains a subgraph isomorphic to H or in the isomorphism class designated by H. Similarly, when ρ is a graph invariant that depends only on the isomorphism class, we write $\rho(H)$ to mean its value on any graph isomorphic to H or in the isomorphic class designated by H.

A graph is *complete* if its vertices are pairwise adjacent; the isomorphism class of n-vertex complete graphs is denoted by K_n. A set of vertices in a graph is *independent* if the subgraph induced by it has no edges. A graph is *k-partite* (*bipartite* when $k = 2$) if its vertex set can be covered by k independent sets. A *complete bipartite graph* is a bipartite graph where all pairs of vertices from distinct partite sets are edges; the isomorphism class is denoted by

$K_{m,n}$, where m and n are the sizes of the partite sets.

Two vertices u and v in a graph G are *connected* if G contains a u, v-path. A graph G is *connected* if every two vertices of G are connected. A *component* of G is a maximal connected subgraph of G. A vertex is a *cut-vertex* in a graph, if removing it increases the number of components of that graph. A *block* in a graph is an induced subgraph with no cut vertices. A subset S of vertices of a graph G is a *cutset* if $G - S$ is not connected. The size of a smallest cutset of a graph G is the *connectivity* of G.

A set D of vertices of a graph G is a *dominating set* if every vertex of G not in D is adjacent to some vertex of D. The size of a smallest dominating set in G is denoted by $\gamma(G)$.

For a predicate $\mathcal{P}$, the notation $[\mathcal{P}]$ equals 1 if $\mathcal{P}$ is true; otherwise $[\mathcal{P}] = 0$.

CHAPTER 2

HAMILTONIAN CYCLES THROUGH SPECIFIED EDGES IN BIPARTITE GRAPHS

The study of sufficient conditions for Hamiltonian cycles is a classical topic in graph theory. Dirac's Theorem [9] states that every n-vertex graph with minimum degree at least $n/2$ is Hamiltonian. Ore [20] strengthened this: it suffices to have $\sigma_2(G) \geq n$, where $\sigma_2(G) = \min\{\deg_G(x) + \deg_G(y)\colon\ xy \notin E(G)\}$. Further refinements have studied sufficient conditions on degrees for spanning cycles through specified edges (loops and multiple edges are forbidden).

We consider analogues of these results for bipartite graphs. An X, Y*-bigraph* is a bipartite graph with partite sets X and Y. It is *balanced* if $|X| = |Y|$. For an X, Y-bigraph G, let $\sigma(G) = \min\{\deg_G(x) + \deg_G(y)\colon\ x \in X, y \in Y, xy \notin E(G)\}$. Gould [13] used $\sigma_{1,1}(G)$ for this quantity to distinguish it from $\sigma_2(G)$. Since we study only balanced bipartite graphs in this chapter, we use the simplified notation $\sigma(G)$. Always n denotes $|V(G)|$.

The analogue of Ore's Theorem for balanced bipartite graphs was proved by Moon and Moser [19]: $\sigma(G) \geq n/2+1$ implies that G is Hamiltonian. The disjoint union of the complete bipartite graphs $K_{a,a}$ and $K_{n/2-a,n/2-a}$ shows that the result is sharp (see Figure 2.1(a)).

Researchers also studied degree thresholds for the existence of spanning cycles through a specified set F of edges, calling a graph F*-Hamiltonian* when such a cycle exists. Of course, F must be a *linear forest*, meaning that every component of F is a path. We require all the paths to be nontrivial (positive length). When F is a perfect matching in a graph G, Häggkvist [14] proved that $\sigma_2(G) \geq n + 1$ is sufficient for G to be F-Hamiltonian. Las Vergnas [23] proved the bipartite analogue, showing that $\sigma(G) \geq n/2 + 2$ suffices when F is a perfect matching. Again the threshold is sharp.

More generally, we seek a spanning cycle through a linear forest with k edges. For general graphs, $\sigma_2(G) \geq n + k$ suffices (Pósa [22]). Faudree, Gould, and Jacobson [11] proved that when F has t components and k edges, with $2 \leq k + t \leq n$, the condition $\sigma_2(G) \geq n + k$ guarantees that G has a cycle of length r containing F for all r such that $2t + k \leq r \leq n$.

We seek the threshold on $\sigma(G)$ to guarantee that G is F-Hamiltonian whenever G is an n-vertex balanced bipartite graph and F is a linear forest in G having k edges. When F is a matching, the requirement on $\sigma(G)$ as a function only of n was studied by Amar, Flandrin, Gancarzewicz, and Wojda [8]. They proved that if $\sigma(G) > 2n/3$, then every matching in G lies in some Hamiltonian cycle, and this threshold on $\sigma(G)$ is sharp. Our problem adds the parameter k, and we seek the sufficiency threshold for $\sigma(G)$ in terms of n and k.

Usually the answer is $\sigma(G) \geq n/2 + \lceil k/2 \rceil$, but the threshold is larger by 1 for some arrangements of k edges. Suppose that the k edges of F form t_1 components of odd length and t_2 components of positive even length. Let

$$\epsilon(t_1, t_2) = \begin{cases} 1 & t_1 = 0 \\ 1 & (t_1, t_2) \in \{(1,0), (2,0)\} \\ 0 & \text{otherwise} \end{cases},$$

and let $\tau(F) = \lceil k/2 \rceil + \epsilon(t_1, t_2)$. Our main result is that if $\sigma(G) \geq n/2 + \tau(F)$, then G is F-Hamiltonian. Furthermore, this threshold on $\sigma(G)$ is sharp when $n > 3k$. Note that when $n = 2k$, the result of Las Vergnas yields $n/2 + 2$ as the threshold. When $n < 3k$ and F is a matching, the result of Amar et al. [8] yields $2n/3$ as the threshold, but the sharpness example for their result requires $n > 3k$, like ours.

Pósa's result for linear forests in general graphs does not depend on the number of components in the forest. His general result follows easily from the case of matchings. In the bipartite analogue, the general case reduces analogously to the case where each specified path has length 1 or 2. Paths of odd and even lengths behave differently in the bipartite setting because traversing them does or does not switch partite sets.

In Section 2.1 we present sharpness constructions for all cases with $n > 3k$. In Section 2.2 we reduce the sufficiency argument to the case where all components of the linear forest have length at most 2, and we outline the steps needed to complete the proof. The remainder of this chapter proves the remaining needed structural statement that if $\sigma(G) \geq n/2 + \tau(F)$ and G has a spanning path through F (where paths in F have length at most 2), then G also has a spanning cycle through F.

2.1 Sharpness Constructions

In this section we introduce needed terminology and provide constructions showing that the results are sharp. We begin with sharpness constructions when all paths in the linear forest F have length 1 or 2. This will be the main case in the sufficiency proof, so we introduce special terminology.

Definition 2.1.1. A *short forest* is a linear forest whose components have length 1 or 2.

When there are t_1 components of length 1 and t_2 of length 2, we also call this a (t_1, t_2)-*short forest*.

We will abuse notation slightly by often viewing F as a specified set of edges rather than a subgraph, but the usage will be clear from context. For example, when P is a path (or a cycle) in G, we say that P *passes through* F if $F \subseteq E(P)$.

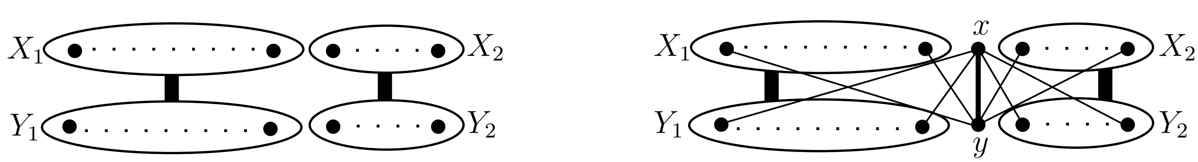
Since k always denotes the number of edges in F, we have $k = t_1 + 2t_2$ when F is a short forest, which includes all cases with $k \leq 2$. We first consider the special case $\epsilon(t_1, t_2) = 1$. The construction in Figure 2.1(a) for $k = 0$ proves sharpness for the Moon–Moser result [19]. Since the graph in Figure 2.1(b) has a perfect matching containing xy, that construction also proves sharpness of Las Vergnas's result. Note that the short forests for which $\epsilon(t_1, t_2) = 1$ are those with $(t_1, t_2) \in \{(0, t), (1, 0), (2, 0)\}$, where t is any nonnegative integer.

Lemma 2.1.2. Let n be even and greater than $2(t_1 + 2t_2 + 1)$. If $\epsilon(t_1, t_2) = 1$, then there is an n-vertex balanced X, Y-bigraph G and a (t_1, t_2)-short forest F in G such that $\sigma(G) = n/2 + \tau(F) - 1$ and G has no spanning cycle through F.

Proof. Since $\epsilon(t_1, t_2) = 1$ and $k = t_1 + 2t_2$, we have $\tau(F) - 1 = \lceil t_1/2 \rceil + t_2$.

For $t_1 = t_2 = 0$, the graph G in Figure 2.1(a) is $K_{a,a} + K_{n/2-a,n/2-a}$. It is disconnected and hence has no spanning cycle, but $\sigma(G) = n/2$.

For $t_2 = 0$ and $t_1 \in \{1, 2\}$, where $\tau(F) - 1 = 1$, we construct G in Figure 2.1(b) from $K_{a-1,a-1} + K_{n/2-a,n/2-a}$ by adding x to X and y to Y with $N(x) = Y \cup \{y\}$, $N(y) = X \cup \{x\}$, and $xy \in F$. Although $\sigma(G) = n/2 + 1$, there is no spanning cycle through xy. If $t_1 = 2$, then F has another edge not incident to x or y, but still there is no cycle through xy.



(a) construction for $k = 0$. (b) construction for $k \in \{1, 2\}$.

Figure 2.1: Sharpness constructions for (t_1, t_2)-short forests with $t_2 = 0$ and $t_1 \leq 2$.

The remaining case is $t_1 = 0$ and $t_2 > 0$. Let G have partite sets $X_1 \cup X_0 \cup X_2$ and $Y_1 \cup Y_0 \cup Y_2$, with $|X_1| = |Y_1| = |X_2| = |Y_2| = m$ and $|X_0| = |Y_0| = t_2$, where $m \geq t_2$. Let $E(G)$ consist of all edges joining the partite sets except those from X_1 to Y_2 and from X_2 to Y_1; see Figure 2.2. Let F consist of a perfect matching in $G[X_0 \cup Y_0]$ plus a matching of size t_2 in $G[X_2 \cup Y_0]$; note that F is $(0, t_2)$-short. If G has a spanning cycle C through F, then deleting $V(F)$ cuts C into t_2 paths. Since covering $G - V(F)$ requires at least $t_2 + 1$ paths (one for $G[X_1 \cup Y_1]$ and at least t_2 for $G[X_2 \cup Y_2]$), no such cycle exists. □

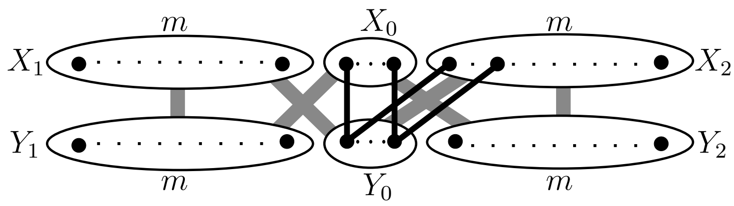


Figure 2.2: Sharpness when $t_1 = 0$.

The pairs (t_1, t_2) with $\epsilon(t_1, t_2) = 0$ are those such that $t_1 \geq 3$ or $t_1 t_2 > 0$. The next construction differs from those above because $|X_1| \neq |Y_1|$. Note that $n > 3k$ is required.

Lemma 2.1.3. Fix t_1 and t_2 with $\epsilon(t_1, t_2) = 0$ and let $k = t_1 + 2t_2$. For $n \in \mathbb{N}$ with $n \geq 2\lceil \frac{k+1}{2} \rceil + 2k$ and $n \equiv 2\lceil \frac{k}{2} \rceil - 2 \pmod 4$, there is an n-vertex balanced bipartite graph G and a (t_1, t_2)-short forest F in G such that $\sigma(G) = n/2 + \tau(F) - 1$ and G has no spanning cycle through F.

Proof. Since $\epsilon(t_1, t_2) = 0$, we have $\tau(F) = \lceil k/2 \rceil$. Fix $m \in \mathbb{N}$ with $m \geq \lfloor t_1/2 \rfloor + t_2 + 1$. Let G have partite sets $X_0 \cup X_1 \cup X_2$ and $Y_0 \cup Y_1 \cup Y_2$ with $|X_0| = |Y_0| = t_1 + t_2$, $|X_2| = |Y_1| = m - \lfloor t_1/2 \rfloor - 1$, and $|X_1| = |Y_2| = m$. Let $E(G)$ consist of all edges joining the partite sets except those from X_1 to Y_2 and from X_2 to Y_1; see Figure 2.3. Let F consist of a perfect matching in $G[X_0 \cup Y_0]$ plus a matching of size t_2 in $G[X_2 \cup Y_0]$; note that F is (t_1, t_2)-short. For $x \in X_1$ and $y \in Y_2$, we have $d_G(x) + d_G(y) = 2(m - \lfloor t_1/2 \rfloor - 1 + t_1 + t_2) = n/2 + \lceil k/2 \rceil - 1$, and such a pair has the smallest degree-sum. The construction exists for $m \geq \lfloor t_1/2 \rfloor + t_2 + 1$, yielding all values of n specified in the hypothesis.

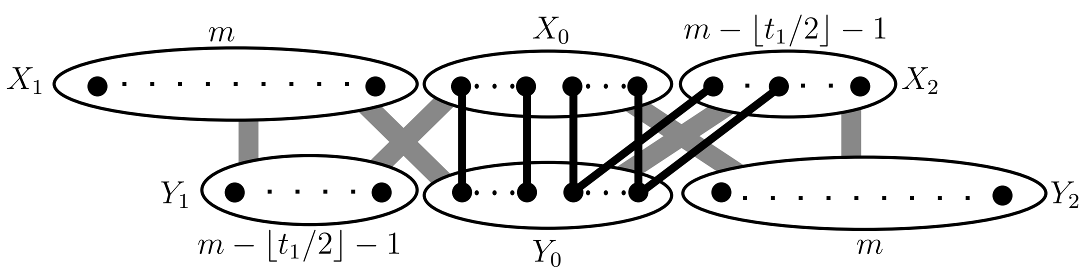


Figure 2.3: Sharpness when $\epsilon(t_1, t_2) = 0$.

Assume a spanning cycle C through F. Since F consists of $t_1 + t_2$ paths, deleting $V(F)$ cuts C into at most $t_1 + t_2$ paths. Since $|X_1| - |Y_1| = \lfloor t_1/2 \rfloor + 1$, covering $G[X_1 \cup Y_1]$ needs at least $\lfloor t_1/2 \rfloor + 1$ paths; similarly, covering $G[(X_2 - V(F)) \cup Y_2]$ needs at least $\lfloor t_1/2 \rfloor + t_2 + 1$ paths. Since covering $G - V(F)$ needs more than $t_1 + t_2$ paths, no such cycle exists. □

Lemmas 2.1.2 and 2.1.3 provide sharpness constructions whenever $k = t_1 + 2t_2$. From the sharpness constructions for (t_1, t_2)-short forests, we obtain sharpness for linear forests with longer paths.

Lemma 2.1.4. Let F be a k-edge linear forest in an n-vertex bipartite graph G with $\sigma(G) = n/2 + \tau(F) - 1$. If G is not F-Hamiltonian, then there is an $(n+2)$-vertex bipartite graph G' containing a $(k+2)$-edge linear forest F' with the same number of components of each parity as F, such that $\sigma(G') = (n+2)/2 + \tau(F') - 1$ and G' is not F'-Hamiltonian.

Proof. Let xy be an edge in F with $x \in X$. Form G' from G by adding two new vertices x' and y' and setting $N(y') = X$ and $N(x') = Y$. Note that $\sigma(G') = \sigma(G) + 2$. Form F' by adding to $F - \{xy\}$ the edges $\{xy', y'x', x'y\}$. This does not change the parity of the length of any path, so $\tau(F') = \tau(F) + 1$. Hence $\sigma(G') = |V(G')|/2 + \tau(F') - 1$.

Any spanning cycle through F' in G' can be converted to a spanning cycle through F in G by replacing the path through x, y', x', y with the edge xy. Thus G' is not F'-Hamiltonian. □

Repeating this construction yields examples for any desired list of path-lengths showing that $\sigma(G) = n/2 + \tau(F) - 1$ is not sufficient, given that such an example exists with the same number of odd and even components when the lengths of the paths are at most 2. We have exhibited such examples when $n > 3k$.

2.2 Outline of the Sufficiency Proof

Our first step is to reduce proving sufficiency to the case of short forests, by in essence reversing the construction in Lemma 2.1.4.

Lemma 2.2.1. Let G be an n-vertex balanced X,Y-bigraph. If $\sigma(G) \geq n/2 + \tau(F)$ guarantees a spanning cycle through F whenever F is a short linear forest in G, then it also suffices without the length restriction.

Proof. Let F consist of k edges forming t_1 paths of odd length and t_2 paths of even (positive) length. When $k = t_1 + 2t_2$, the forest F is short and there is nothing to prove. We proceed by induction on k with t_1 and t_2 fixed. For $k > t_1 + 2t_2$, some path in F has length at least 3; let x, y', x', y be consecutive vertices along it. Form G' from $G - \{x', y'\}$ by adding the edge xy (if not already present). Let F' in G' be the same as F except for replacing the specified path through x, y', x', y with the edge xy.

Since t_1 and t_2 do not change, $\tau(F') = \tau(F) - 1$. Since each vertex of G' loses at most one neighbor in $\{x', y'\}$, we have $\sigma(G') \geq \sigma(G) - 2 = |V(G')|/2 + \tau(F')$. Hence the induction hypothesis yields a spanning cycle C' through F' in G'. Obtain the desired cycle C in G by replacing xy in C' with the path through x, y', x', y. □

Our main task, which takes the bulk of this chapter, will be to prove that G is F-Hamiltonian when the following conditions hold: F is a short forest, $\sigma(G) \geq n/2 + \tau(F)$, and G has a spanning path through F. A relatively easy induction on k then completes the sufficiency proof. To clarify the structure of the proof, we present this induction first. The basis step, for $k = 0$, is the Moon-Moser result. We prove it here to make our result self-contained and to motivate some notions that we will use frequently later. When $k = 0$, we only need a spanning cycle. Note also that $\epsilon(t_1, t_2) = 1$ when $k = 0$.

Proposition 2.2.2. [19] If G is an n-vertex balanced X, Y-bigraph and $\sigma(G) \geq n/2 + 1$, then G has a spanning cycle.

Proof. Adding edges preserves the condition $\sigma(G) \geq n/2 + 1$, so a maximal counterexample has a spanning path P with nonadjacent endpoints x and y. Since $\deg_G(x) + \deg_G(y) \geq \sigma(G) \geq n/2 + 1$ and there are $n/2$ odd-indexed edges along P, some odd-indexed edge $x'y'$ contains neighbors of both x and y. Now $(P - x'y') \cup \{xy', x'y\}$ is a spanning cycle (Figure 2.4). □

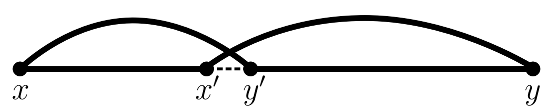


Figure 2.4: Substituting xy' and $x'y$ for $x'y'$.

The cycle produced in this proof is a concatenation of subpaths of P with adjacent endpoints. To express it in this way, we need appropriate notation for paths and subpaths.

Now we can write the cycle in Figure 2.4 as $[P(x, x'), P(y, y')]$. Next we present the overall induction argument that uses the structural claim.

Lemma 2.2.3. If $\sigma(G) \geq n/2 + \tau(F)$ implies that G is F-Hamiltonian whenever F is a short forest and G has a spanning path through F, then $\sigma(G) \geq n/2 + \tau(F)$ implies that G is F-Hamiltonian for every linear forest F in G.

Proof. By Lemma 2.2.1, we may restrict our attention to short forests. For these we use induction on k, the number of edges. The case $k = 0$ is the Moon–Moser result proved in Proposition 2.2.2, since $\epsilon(t_1, t_2) = 1$ when $k = 0$.

For $k > 0$, let uv be an edge of F, and let $F' = F - uv$ and $k' = k - 1$. Note that F' is a (t_1', t_2')-short forest in G for some t_1' and t_2' with $k' = t_1' + 2t_2'$. Since $\tau(F) = \lceil k/2 \rceil + \epsilon(t_1, t_2)$

and $\tau(F') = \lceil k'/2 \rceil + \epsilon(t'_1, t'_2)$, we have $\tau(F) \geq \tau(F')$ unless k is even, $\epsilon(t_1, t_2) = 0$, and $\epsilon(t'_1, t'_2) = 1$. This requires $t_1 = 2$, and then no choice for t_2 is possible.

We conclude that $\sigma(G) \geq n/2 + \tau(F')$. Now the induction hypothesis implies that G has a spanning cycle C through $F - uv$. If $uv \in E(C)$, then C is a spanning cycle through F, as desired. Otherwise, let u' and u'' be the neighbors of u on C, and let v' and v'' be the neighbors of v on C, with v' and u' on different sides of the chord uv as in Figure 2.5.

Since paths in F have length at most 2, at most one edge in $\{uu', uu'', vv', vv''\}$ is in F. If such an edge exists, then by symmetry we may assume it is uu''. Let Q be the path $C - uu'$. The path $\langle Q(u', v), Q(u, v') \rangle$, is a spanning path through F in G. Now the structural hypothesis guarantees that G has a spanning cycle through F. □

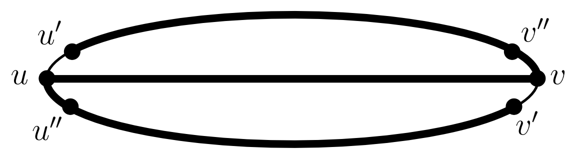


Figure 2.5: Cycle C.

Thus our task is to prove that the hypothesis in Lemma 2.2.3 is a true statement. We begin by formalizing two important concepts from Proposition 2.2.2: parity of edges along a spanning path and having both endpoints of the path as neighbors.

Definition 2.2.4. Let G be an X, Y-bigraph containing an x, y-path P of odd length. An edge of P is an *odd edge* or *even edge* (with respect to P) when it has *odd position* or *even position* in a listing of the edges in order from one end of P. We write $E_{\mathsf{odd}}(P)$ for the set of all odd edges on P and $E_{\mathsf{even}}(P)$ for the set of all even edges on P. An edge on an x, y-path P is *full* (with respect to P) if one endpoint is adjacent to x and the other is adjacent to y. The edge is *half-full* (with respect to P) if exactly one of these edges exists.

In this language, we generalize the idea used in the Moon-Moser result; we will use this remark frequently. We write $u \leftrightarrow v$ when u and v are adjacent in G; otherwise, $u \not\leftrightarrow v$.

Remark 2.2.5. Let G be an n-vertex balanced X, Y-bigraph, and let P be a spanning x, y-path in G. Since each endpoint of an edge along P has at most one neighbor in $\{x, y\}$, the pigeonhole principle implies that if $x \not\leftrightarrow y$ and $\deg_G(x) + \deg_G(y) \geq n/2 + p$, then there are at least p full odd edges and at least $p + 1$ full even edges along P. Moreover, if $\deg_G(x) + \deg_G(y) = n/2 + p$ and there are exactly p full odd edges on P, then all other odd edges on P are half-full.

Having reduced our task to proving the hypothesis of Lemma 2.2.3, we henceforth adopt the setting of that statement as a uniform restriction on G and F. We will not continue to repeat these hypotheses, so we gather them here as a definition.

Definition 2.2.6. *The Scenario.* Throughout the rest of this chapter, G denotes a fixed n-vertex balanced X,Y-bigraph, F is a short forest in G consisting of k edges, with t_1 single-edge components and t_2 double-edge components, and $\sigma(G) \geq n/2 + \tau(F)$. All uses of x, x', x_i indicate vertices in X, and all uses of y, y', y_i indicate vertices in Y. We call the edges of F the *selected* edges. Let F_1 denote the set of isolated edges in F, and let F_2 denote the set of edges of F in paths of length 2. Always P denotes a given spanning path through F with nonadjacent endpoints $x \in X$ and $y \in Y$; hence $\deg_G(x) + \deg_G(y) \geq \sigma(G)$ and the end-edges of P are not full.

Our task, given the scenario of Definition 2.2.6, is to produce a spanning cycle through F. We show successively that various conditions suffice to ensure such a cycle. We already observed in proving the Moon-Moser result that having an unselected full odd edge suffices. The subsequent sufficient conditions are:

On P there are fewer than $\tau(F)$ selected odd edges (Lemma 2.2.7).
Some full even edge on P is in F_1 (Lemma 2.2.8).
Along P, half of the selected edges are odd and half are even (Section 2.3).
Both end-edges of P are unselected (Section 2.4).
One end-edge of P is unselected (Section 2.5).
Both end-edges of P are selected (Section 2.6).

The last three steps together include all cases for P and hence imply that the specified conditions guarantee a spanning cycle through F. This will complete the proof. We do not start with those cases because their proofs use the earlier, easier cases. The first two conditions are easy to show sufficient, and we close this section with that.

Lemma 2.2.7. If fewer than $\tau(F)$ odd edges of P are selected, then G is F-Hamiltonian.

Proof. Since $\sigma(G) \geq n/2 + \tau(F)$, at least $\tau(F)$ odd edges are full. Since fewer than $\tau(F)$ are selected, some full odd edge is unselected, which we have observed is sufficient. □

Lemma 2.2.8. If F_1 contains a full even edge of P, then G is F-Hamiltonian.

Proof. Let $y'x'$ be such an edge. Consecutive vertices x'', y', x', y'' exist along P. Let $Q = \langle P(x'', x), y', x', P(y, y'') \rangle$ (see Figure 2.6). Since $y'x' \in F_1$, we have $x''y', x'y'' \notin F$, and hence Q passes through F. We may therefore assume $x'' \not\leftrightarrow y''$, which yields $\deg_G(x'') + \deg_G(y'') \geq$

$n/2+\tau(F)$. Since every edge other than $y'x'$ has different parity on P and Q, one of P and Q has fewer than $\lceil k/2 \rceil$ selected odd edges. Since $\tau(F) \geq \lceil k/2 \rceil$, Lemma 2.2.7 applies. $\square$

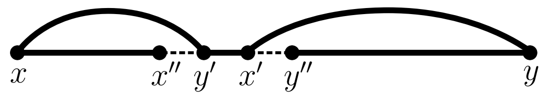


Figure 2.6: The path Q.

Henceforth, the phrase "Lemma A.B applies" means the hypotheses of that lemma (often Lemma 2.2.7) have been satisfied and hence its conclusion (always existence of a spanning cycle through F) holds, thereby completing the proof of that case.

2.3 Paths Splitting F by Parity

Given spanning paths P and Q through F such that every selected edge has opposite parity in P and Q, one of $\{P, Q\}$ has at most $\lfloor k/2 \rfloor$ selected odd edges. Since $\tau(F) \geq \lceil k/2 \rceil$, Lemma 2.2.7 thus suffices when k is odd (or $\epsilon(t_1, t_2) = 1$) and such P and Q exist. When k is even, this observation is not sufficient, and we need an additional structural lemma.

Definition 2.3.1. The spanning path P through F *splits* F if $|F \cap E_{\mathsf{odd}}(P)| = |F \cap E_{\mathsf{even}}(P)|$. When $x'y'$ is a full odd edge on P (hence not an end edge), preceded by y'' and followed by x'' on P, we define $P^{x'y'}$ to be the path $\langle P(x'', y), x', y', P(x, y'') \rangle$ (see Figure 2.7).

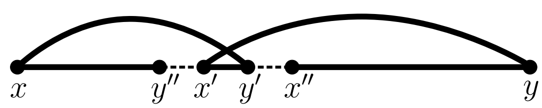


Figure 2.7: The path $P^{x'y'}$

Every edge in both P and $P^{x'y'}$ has the same parity on both paths, because movement from the "X-end" to the "Y-end" of $P^{x'y'}$ traverses common edges of P and $P^{x'y'}$ in the same direction (contrast this with Figure 2.6, where all edges except $y'x'$ change parity).

Lemma 2.3.2. If k is even and P splits F, then G is F-Hamiltonian.

Proof. Suppose G is not F-Hamiltonian. Lemma 2.2.7 applies unless at least $\tau(F)$ full odd edges are selected. Since P splits F, there are exactly $k/2$ selected odd edges. Hence $\tau(F) = k/2$, which requires $\epsilon(t_1, t_2) = 0$ and hence $t_1 \geq 2$ (t_1 is even when k is even). Every selected odd edge is full, and the other odd edges are half-full. Since $x \not\leftrightarrow y$, the end-edges of P are not full. Thus the end-edges of any path through F that splits F are unselected.

Since every path through F splits F_2 by parity, P also must split F_1. Let $x'y'$ be an odd edge in F_1. Since all selected odd edges are full, $P^{x'y'}$ exists. Since $x'y' \in F_1$, this path also contains F, and it splits F since all edges of F have the same parity in P and $P^{x'y'}$. If the odd edge nearest to $x'y'$ in either direction is selected, then an end-edge of $P^{x'y'}$ is selected. The preceding paragraph forbids this when G is not F-Hamiltonian. Hence we may assume that any two selected odd edges of P incident to a common even edge are both in F_2.

Let $r = t_1/2$. Let x_1y_1 be the odd edge in F_1 closest to x on P; similarly choose x_2y_2 closest to y. These edges are distinct when $r \geq 2$. We consider three cases for r.

Case 1: $r \geq 3$. Let x_3y_3 be a third selected odd edge in F_1. Let $\langle u_1, v_1, x_3, y_3, u_2, v_2\rangle$ be the 6-vertex portion of P centered at x_3y_3 (see Figure 2.8). Since $x_3y_3 \in F_1$ and successive selected odd edges lie in F_2, none of $u_1v_1, v_1x_3, y_3u_2, u_2v_2$ is selected.

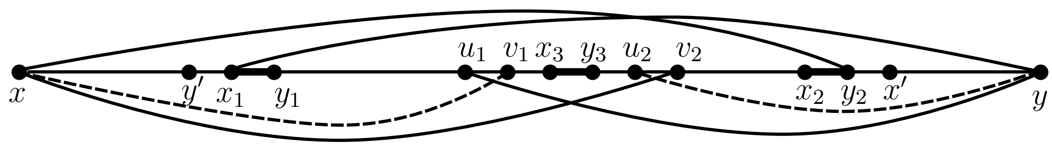


Figure 2.8: Three selected odd edges.

If $u_2 \leftrightarrow y$, then let $Q = \langle P(x', y), P(u_2, y_2), P(x, y_3)\rangle$, where x' follows y_2 on P. Edges in $E(Q) \cap E(P)$ have the same parity in both paths, so Q splits F but has a selected end-edge. This is forbidden, so $u_2 \not\leftrightarrow y$. By symmetry, $v_1 \not\leftrightarrow x$. Since all unselected odd edges on P are half-full, $v_2 \leftrightarrow x$ and $u_1 \leftrightarrow y$.

Consider paths $\langle P(x, u_1), P(y, v_1)\rangle$ and $\langle P(u_2, x), P(v_2, y)\rangle$ through F. Every edge of F_1 except x_3y_3 has different parity on these paths. Since x_3y_3 is even on both and $|F| = 2r+2t_2$, one of the two has fewer than $r + t_2$ selected odd edges, and Lemma 2.2.7 applies.

Case 2: $r = 2$. Let $\langle y', x_1, y_1, u_1, v_1\rangle$ and $\langle u_2, v_2, x_2, y_2, x'\rangle$ be the 5-vertex portions of P centered at y_1 and x_2 (see Figure 2.9). Again $x_1y_1, x_2y_2 \in F_1$ implies that the other edges of these two subpaths are unselected. If $u_1 \leftrightarrow y$, then $\langle P(x', y), P(u_1, y_2), P(x, y_1)\rangle$ has a selected end-edge, so $u_1 \not\leftrightarrow y$. Similarly, $x \not\leftrightarrow v_2$. Now, since unselected odd edges are half-full, $x \leftrightarrow v_1$ and $y \leftrightarrow u_2$. Hence $u_1v_1 \neq u_2v_2$.

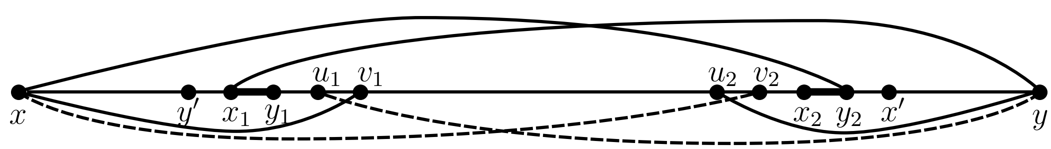


Figure 2.9: Exactly two selected odd edges.

The two edges of $F_1 - \{x_1y_1, x_2y_2\}$ are both even edges of P. If either lies in $\langle P(v_1, u_2)\rangle$, then $\langle P(u_1, x), P(v_1, u_2), P(y, v_2)\rangle$ has at most $1 + t_2$ selected edges in odd position, and

Lemma 2.2.7 applies. If each even edge of F_1 lies in $\langle P(x,u_1)\rangle$ or $\langle P(v_2,y)\rangle$, then by symmetry we may assume that $\langle P(x,u_1)\rangle$ contains such an edge e. Now the edges y_1x_1, y_2x_2, and e all have even position in $\langle P(u_1,x_1), P(y,v_1), P(x,y')\rangle$. Hence this path through F has at most $1+t_2$ selected odd edges, and Lemma 2.2.7 applies.

Case 3: $r=1$. Let x_1y_1 be the odd edge of F_1, and let e be the even edge. By symmetry in X and Y, we may assume $e \in \langle P(x,x_1)\rangle$. Let $\langle x_0,y_0,x_1,y_1\rangle$ be the 4-vertex portion of P ending with x_1y_1. Since $x_1y_1 \in F_1$, both y_0x_1 and x_0y_0 are unselected.

If $x_0 \leftrightarrow y$, then $\langle P(x,x_0), P(y,y_0)\rangle$ has only t_2 selected odd edges, and Lemma 2.2.7 applies. Hence $x_0 \not\leftrightarrow y$. Since unselected odd edges are half-full, $x \leftrightarrow y_0$. Since $\epsilon(t_1,t_2)=0$, we have $t_2>0$, and F_2 is nonempty. Let b be the center of a component $\langle P(a,b,c)\rangle$ of F_2.

Compare P with the path obtained from $P^{x_1y_1}$ by interchanging X and Y; in both the end-edges are unselected, F is split, and x_1y_1 is full. Both paths have b and e on the same side of x_1y_1, or both have them on opposite sides. Hence we may assume $b \in Y$ in the first case (Figure 2.10) and $b \in X$ in the second case (Figure 2.11). Let d be the vertex before a on P, and let x' be the vertex before y.

In the first case, with $a \in X$, the edge ab is selected and odd, hence full, hence $a \leftrightarrow y$. Now $\langle P(x',x_1), y, P(a,y_0), P(x,d)\rangle$ has only t_2 selected odd edges, and Lemma 2.2.7 applies.

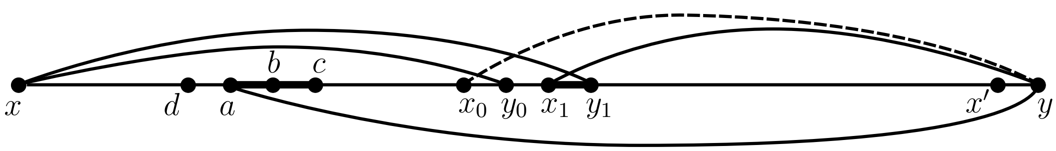


Figure 2.10: The even edge of F_1 and $\langle d,a,b,c\rangle$ on the same side of x_1y_1.

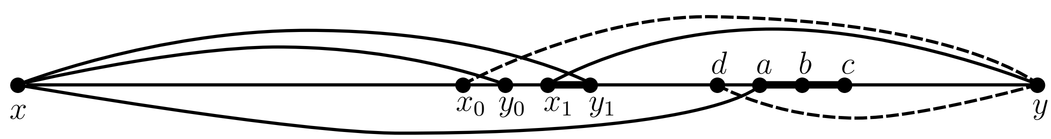


Figure 2.11: The even edge of F_1 and $\langle d,a,b,c\rangle$ on opposite sides of x_1y_1.

In the second case, if $d \leftrightarrow y$, then $\langle P(a,y), P(d,x)\rangle$ splits F and has a selected end-edge, so we may assume $d \not\leftrightarrow y$. Unselected odd edges are half-full, so $x \leftrightarrow a$. Now $\langle P(d,x_1), P(y,a), P(x,y_0)\rangle$ has only t_2 selected odd edges, and Lemma 2.2.7 applies. □

2.4 Paths with Both End-edges Unselected

In this section we complete the proof for the case of a spanning path whose end-edges are unselected. In the previous section our focus was on such a path, with the additional hypothesis that it splits F. Having eliminated that case, we may now assume that the numbers of even and odd selected edges along P differ. The first two lemmas are tools.

Lemma 2.4.1. If there are at most $\lfloor k/2 \rfloor$ selected odd edges along P, then G is F-Hamiltonian.

Proof. Lemma 2.2.7 applies when fewer than $\lceil k/2 \rceil$ odd edges are selected, and Lemma 2.3.2 applies when equality holds. □

Lemma 2.4.2. Given that P has unselected full even edges $y_i x_i$ and $y_j x_j$, let Q be the portion of P between them. If the end-edges of P are unselected, and the inequality below holds, then G is F-Hamiltonian.

$$|E_{\mathsf{odd}}(Q) \cap F_1| - |E_{\mathsf{even}}(Q) \cap F_1| \leq 2\lfloor t_1/2 \rfloor - t_1 + 1$$

Proof. Name the vertices so that $y_i x_i$ is later than $y_j x_j$ along P, so $Q = \langle P(x_j, y_i) \rangle$. With y' and x' neighboring x and y, let $R = \langle P(x', x_i), y, P(x_j, y_i), x, P(y_j, y') \rangle$ (see Figure 2.12). Edges in both P and R have different parity in R and P, except for those in Q. Thus

$$|E_{\mathsf{odd}}(R) \cap F_1| + |E_{\mathsf{odd}}(P) \cap F_1| = t_1 + |E_{\mathsf{odd}}(Q) \cap F_1| - |E_{\mathsf{even}}(Q) \cap F_1| \leq 2\lfloor t_1/2 \rfloor + 1.$$

We conclude that P or R has at most $\lfloor t_1/2 \rfloor$ edges of F_1 in odd position. Since $y_i x_i$, $y_j x_j$, and the end-edges of P are unselected, R and P both pass through F. One of them has at most $\lfloor t_1/2 \rfloor + t_2$ selected odd edges, so Lemma 2.4.1 applies. □

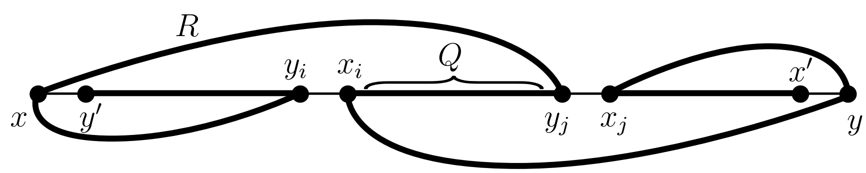


Figure 2.12: Horizontal path P and modified path R.

Lemma 2.4.3. If $t_1 \leq 2$, or if both end-edges of P are unselected, then G is F-Hamiltonian.

Proof. There are at least $\tau(F) + 1$ full even edges along P. Let S be the set of full even edges outside F_2; note that $|S| \geq \lceil t_1/2 \rceil + \epsilon(t_1, t_2) + 1$. If any edge of S is in F_1, then G is F-Hamiltonian by Lemma 2.2.8, so we may assume $S \cap F = \varnothing$. Index S as $y_1 x_1, \ldots, y_s x_s$ in order along P from x to y. Let $Q_j = \langle P(x_j, y_{j+1}) \rangle$ for $1 \leq j < s$. Note that always $s \geq 2$.

Case 1: $t_1 \leq 2$. Note first that if $t_1 = 0$, then there are exactly $k/2$ selected odd edges along P, and Lemma 2.4.1 applies. If $t_1 \in \{1, 2\}$, then $\lfloor t_1/2 \rfloor = 2\lfloor t_1/2 \rfloor - t_1 + 1$. If $s \geq 3$, then paths Q_1 and Q_2 exist; one of them contains at most $\lfloor t_1/2 \rfloor$ edges of F_1, so in this case Lemma 2.4.2 applies. If $t_2 = 0$, then $\epsilon(t_1, t_2) = 1$ and $s \geq 3$, as desired. If $t_2 > 0$, then $\epsilon(t_1, t_2) = 0$, but still $s \geq 3$ if some even edge in F_2 is not full. Hence we may assume that $t_2 > 0$ and that all even edges of F_2 are full.

Since $|F_1| \leq 2$, Lemma 2.4.1 applies unless every edge of F_1 is odd. Since $\tau(F) = 1 + t_2$, there are at least $t_2 + 1$ full odd edges, with at most t_2 in F_2. Since an unselected full odd edge yields a spanning cycle through F, we may assume that some odd edge $\hat{x}\hat{y}$ in F_1 is full.

Since $t_2 > 0$, by symmetry we may assume F_2 has an edge in $\langle P(\hat{y}, y) \rangle$. Let d, a, b, c be four vertices in order along $\langle P(\hat{y}, y) \rangle$ such that $ab, bc \in F_2$ (see Figure 2.11, with $\hat{x}\hat{y}$ replacing x_1, y_1 in the figure). If $a \notin Y$, then consider $P^{\hat{x}\hat{y}}$ instead of P and interchange X and Y; hence we may assume $a \in Y$. Now ab is a full even edge in F_2, so $x \leftrightarrow a$ (as in Figure 2.11). The path $\langle P(d, x), P(a, y) \rangle$ has $\hat{y}\hat{x}$ in even position, so it has at most $\lfloor k/2 \rfloor$ selected odd edges, and Lemma 2.4.1 applies.

Case 2: $t_1 \geq 3$. In this case $\epsilon(t_1, t_2) = 0$. Let y' and x' be the neighbors of x and y on P. For $1 \leq j < s$, let $R_j = \langle P(x', x_{j+1}), y, P(x_j, y_{j+1}), x, P(y_j, y') \rangle$; this is just the path R in Figure 2.12 with $i = j + 1$. If $|E_{\mathsf{odd}}(Q_j) \cap F_1| - |E_{\mathsf{even}}(Q_j) \cap F_1| \leq 2\lfloor t_1/2 \rfloor - t_1 + 1$ for some j, then Lemma 2.4.2 applies. For t_1 even, we may thus assume that $|E_{\mathsf{odd}}(Q_j) \cap F_1| \geq 2$ for all j. Hence $|E_{\mathsf{odd}}(P) \cap F_1| \geq 2s - 2 \geq t_1 = |F_1|$. We conclude that all edges of F_1 have odd position in P, and that every Q_j contains exactly two of them. Hence exactly two members of F_1 (those in Q_j) have odd position in R_j; since $t_1 \geq 4$, Lemma 2.4.1 applies.

The remaining case is t_1 odd. Let $p = \lceil t_1/2 \rceil$; note that $s > p$. Lemma 2.4.2 applies unless

$$|E_{\mathsf{odd}}(Q_j) \cap F_1| - |E_{\mathsf{even}}(Q_j) \cap F_1| \geq 1 \qquad \text{for } 1 \leq j \leq p. \tag{$*$}$$

Since $|F_1| = t_1 < 2p$, we have $|E_{\mathsf{odd}}(Q_j) \cap F_1| = 1$ and $|E_{\mathsf{even}}(Q_j) \cap F_1| = 0$ for some j. Since selected edges outside Q_j have opposite parity in P and R_j, for this j we have $|E_{\mathsf{odd}}(R_j) \cap F_1| + |E_{\mathsf{odd}}(P) \cap F_1| = t_1 + 1$. If R_j or P has at most $\lfloor t_1/2 \rfloor$ odd edges in F_1, then Lemma 2.4.1 applies. Hence each has exactly p, meaning that F_1 has exactly p odd edges and $p-1$ even edges on P. Now $(*)$ requires $|E_{\mathsf{odd}}(Q_j) \cap F_1| = 1$ and $|E_{\mathsf{even}}(Q_j) \cap F_1| = 0$ for $1 \leq j \leq p$. Hence each even edge of F_1 is in $\langle P(x, y_1) \rangle$ or $\langle P(x_{p+1}, y) \rangle$, and all odd edges are in $\langle P(x_1, y_{p+1}) \rangle$. By symmetry, we may assume that $P(x_{p+1}, y)$ has at most $\lfloor (p-1)/2 \rfloor$ even edges of F_1. Now $\langle P(x', x_2), y, P(x_1, y_2), P(x, y_1) \rangle$ has at most $\lfloor (p-1)/2 \rfloor + 1 + t_2$ selected odd edges (counting one in Q_1). Since $p + 1 \leq t_1$ for $t_1 \geq 3$, Lemma 2.4.1 applies. □

2.5 Paths with One End-edge Selected

Several types of alternate paths will be useful in this section. We assume throughout this section that on P the initial edge xy' is selected and the final edge $x'y$ is unselected.

Lemma 2.5.1. If P has a full unselected even edge $\bar{y}\bar{x}$ preceded somewhere by an unselected odd edge $\hat{x}\hat{y}$ whose X-endpoint is adjacent to y, then G is F-Hamiltonian.

Proof. Let $Q = \langle P(x', \bar{x}), y, P(\hat{x}, x), P(\bar{y}, \hat{y})\rangle$ (see Figure 2.13); note that Q passes through F. Since Q travels backward along P from x', every edge of F has opposite parity on P and Q, so Lemma 2.4.1 applies. □

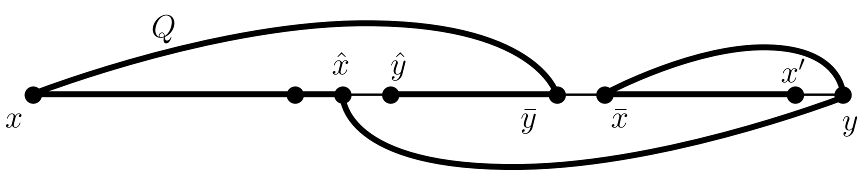


Figure 2.13: The path Q, toggling parity.

Lemma 2.5.2. If P has a full unselected even edge $\bar{y}\bar{x}$ followed somewhere by an unselected odd edge $\hat{x}\hat{y}$ whose Y-endpoint is adjacent to x, and $\langle P(\bar{x}, y)\rangle$ contains at least $\lceil t_1/2\rceil$ odd edges of F_1, then G is F-Hamiltonian.

Proof. Let $Q' = \langle P(\hat{x}, \bar{x}), P(y, \hat{y}), P(x, \bar{y})\rangle$ (see Figure 2.14). All selected edges of $\langle P(\bar{x}, y)\rangle$ appear with opposite parity on P and Q', including at least $\lceil t_1/2\rceil$ edges of F_1 in odd position on P. Hence $|E_{\mathsf{odd}}(Q') \cap F| \leq t_1 - \lceil t_1/2\rceil + t_2 \leq \lfloor k/2\rfloor$, and Lemma 2.4.1 applies. □

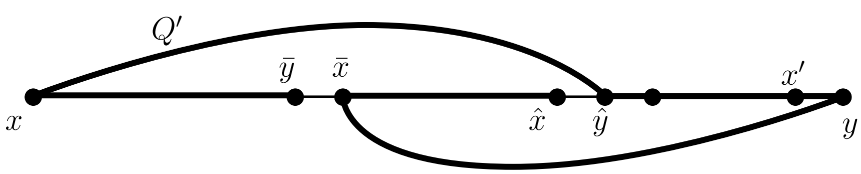


Figure 2.14: The path Q', mostly toggling parity.

Lemma 2.5.3. If one end-edge of P is unselected, then G is F-Hamiltonian.

Proof. By Lemma 2.4.3, we may assume $t_1 \geq 3$ and $\epsilon(t_1, t_2) = 0$. Since $\sigma(G) \geq n/2 + \lceil t_1/2\rceil + t_2$, there are at least $\lceil t_1/2\rceil + t_2 + 1$ full even edges along P.

The components of F_2 are paths of length 2. Let S be the set of full even edges on P that do not lie in F_2 and are not incident to a component of F_2 whose even edge is not full. Index the edges of S as $y_1x_1, \ldots, y_sx_s$ along P from x to y. Since S is obtained by discarding from the set of all full even edges at most one for each component of F_2, we have $s > \lceil t_1/2 \rceil \geq 2$. If any edge of S is in F_1, then Lemma 2.2.8 applies; hence we may assume $S \cap F = \varnothing$.

For $1 \leq j < s$, let y'_j be the neighbor of x_j on P other than y_j. If $x_jy'_j \notin F$ for some such j, then Lemma 2.5.1 applies, using $x_jy'_j$ as $\hat{x}\hat{y}$ and y_sx_s as $\bar{y}\bar{x}$. Hence we may assume $x_jy'_j \in F$. Introducing the next two vertices, let $y_j, x_j, y'_j, x'_j, y''_j$ be consecutive along P. If $x_jy'_j \in F_2$, then $\langle x_j, y'_j, x'_j \rangle$ is a component of F_2, since $y_jx_j \notin F$. Since $y_jx_j \in S$, the next even edge $y'_jx'_j$ must also be full (by the definition of S). We conclude that $x'_j \leftrightarrow y$ and $x'_jy''_j \notin F$. Again Lemma 2.5.1 applies, with $x'_jy''_j$ as $\hat{x}\hat{y}$ and y_sx_s as $\bar{y}\bar{x}$.

Therefore, we may assume $x_jy'_j \in F_1$ for $1 \leq j < s$. For such j, let x^-_{j+1} be the vertex before y_{j+1} on P. If $x^-_{j+1}y_{j+1} \notin F$, then Lemma 2.5.2 applies with $x^-_{j+1}y_{j+1}$ as $\hat{x}\hat{y}$ and y_1x_1 as $\bar{y}\bar{x}$, since $s - 1 \geq \lceil t_1/2 \rceil$. Hence we may assume $x^-_{j+1}y_{j+1} \in F$. Introducing the two preceding vertices, let $x^=_{j+1}, y^-_{j+1}, x^-_{j+1}, y_{j+1}, x_{j+1}$ be consecutive along P. If $x^-_{j+1}y_{j+1} \in F_2$, then $\langle y^-_{j+1}, x^-_{j+1}, y_{j+1} \rangle$ is a component of F_2, since $y_{j+1}x_{j+1} \notin F$. Since $y_{j+1}x_{j+1} \in S$, the preceding even edge $y^-_{j+1}x^-_{j+1}$ must also be full (by the definition of S), and hence $y^-_{j+1} \leftrightarrow x$. Again Lemma 2.5.2 applies, with $x^=_{j+1}y^-_{j+1}$ as $\hat{x}\hat{y}$ and y_1x_1 as $\bar{y}\bar{x}$.

Therefore, we may assume for $1 \leq j < s$ that $x^-_{j+1}y_{j+1} \in F_1$, along with our previous conclusion that $x_jy'_j \in F_1$. Let $Q'' = \langle P(x', x_2), y, P(x_1, y_2), P(x, y_1) \rangle$ (see Figure 2.15). All edges of $\langle P(x_2, x') \rangle$ have opposite parity in P and Q'', including $x_jy'_j$ and $x^-_{j+1}y_{j+1}$ for $2 \leq j < s$. If for any such j the edges $x_jy'_j$ and $x^-_{j+1}y_{j+1}$ are not the same, then $|E_{\mathsf{odd}}(Q'') \cap F| \leq t_1 + t_2 - (s-1) \leq \lfloor k/2 \rfloor$ (again using $s - 1 \geq \lceil t_1/2 \rceil$), and Lemma 2.4.1 applies.

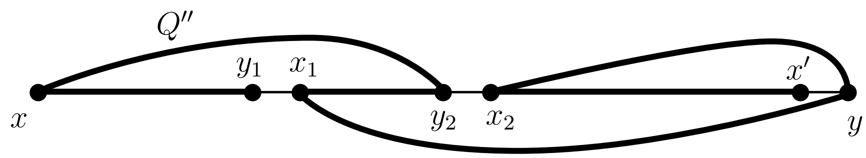


Figure 2.15: The path Q''.

Hence we may assume that $x^-_2, y_2, x_2, \ldots, x_{s-1}, y_s, x_s$ are consecutive on P, forming edges that alternate between F_1 and S. Let T denote the set of these edges in F_1. (Note $s \geq 3$.)

Now let $R = \langle P(x_s, y), P(x_1, y_s), P(x, y_1) \rangle$ (see Figure 2.16). Since R passes through F, we may assume its endpoints are non-adjacent, so $d_G(x_s) + d_G(y_1) \geq \sigma(G) \geq n/2 + \lceil k/2 \rceil$, and at least $\lceil k/2 \rceil$ odd edges of R are full. Lemma 2.2.7 applies unless at least $\lceil k/2 \rceil$ of them are in F. Exactly t_2 are in F_2, so at least $\lceil t_1/2 \rceil$ full odd edges of R are in F_1; call this set D.

All edges of F have the same parity on P and R, so the edges of D are in odd position

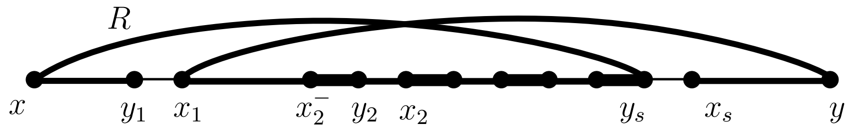


Figure 2.16: The path R.

also on P. We claim that $D \cap T = \varnothing$. If $x_j^- y_j \in D$ for some j with $2 \le j < s$, then fullness on R yields $y_j \leftrightarrow x_s$, and $[P(x_s, y), P(x_j, y_s), P(x, y_j)]$ is a spanning cycle through F. Also, if $x_{s-1}y_s \in D$, then fullness on R yields $x_{s-1} \leftrightarrow y_1$, and $[P(x_{s-1}, y), P(x_1, y_{s-1}), P(x, y_1)]$ is a spanning cycle through F.

Hence $D \cap T = \varnothing$. We have therefore found $\lceil t_1/2 \rceil + s - 1$ edges of F_1 in odd position on P. Since $s > \lceil t_1/2 \rceil$ and $|F_1| = t_1$, we conclude that $F_1 \subseteq E_{\mathsf{odd}}(P)$ and $s = t_1/2 + 1$.

Consider three consecutive vertices x_0^-, y_0, x_0 in $P(x, x_1)$, with y_0x_0 in even position (possibly $y_0x_0 = y_1x_1$). We prove Claim $(*)$: *If y_0x_0 is full and $x_0^- y_0 \notin F$, then G is F-Hamiltonian.* Since we may assume by Lemma 2.4.3 that the edge of P incident to x is selected, we may assume $x_0^- \ne x$. Introducing the two preceding vertices, we have $x_0^=, y_0^-, x_0^-, y_0, x_0$ consecutive on P. If $y_0^- x_0^- \notin F$, then $\langle P(x_0^-, x), P(y_0, y)\rangle$ is a spanning path through F with both end-edges unselected, and Lemma 2.4.3 applies.

Hence we may assume $y_0^- x_0^- \in F$. Since $F_1 \subseteq E_{\mathsf{odd}}(P)$, we have $y_0^- x_0^- \in F_2$. Since $x_0^- y_0 \notin F$, the component C of F_2 containing $y_0^- x_0^-$ is $\langle x_0^=, y_0^-, x_0^-\rangle$. Since P has at least $t_1/2 + t_2 + 1$ full even edges and $s = t_1/2 + 1$, every component of F_2 has a full even edge or is incident to a full even edge (by the definition of S). If the even edge of C is full, then $x_0^- \leftrightarrow y$, and $[P(x, x_0^-), P(y, y_0)]$ is a spanning cycle through F. Otherwise, $y_0^= x_0^=$ exists before $x_0^=$ and is full. Now $\langle x_0^-, y_0^-, x_0^=, P(y, y_0), P(x, y_0^=)\rangle$ is a spanning path through F; call it R' (see Figure 2.17). All selected edges after $y_0^=$ on P have opposite parity in P and R', including all $t_1/2$ edges of T. Hence at most $k/2$ selected edges have odd position on R', and Lemma 2.4.1 applies. This proves Claim $(*)$.

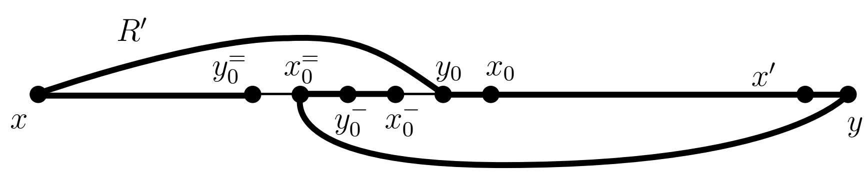


Figure 2.17: The path R' through F.

Now consider y_1x_1, the first edge of S. Let x_1^- be the other neighbor of y_1. By $(*)$, we may assume $x_1^- y_1 \in F$. If $x_1^- y_1 \in F_2$, then $x_1^- \ne x$, and we have $x_1^=, y_1^-, x_1^-, y_1, x_1$ consecutive on P with $y_1^- x_1^- \in F_2$ and $x_1^= y_1^- \notin F$. Since $y_1x_1 \in S$, we conclude that $x_1^- y_1^-$ is full (by the definition of S). Now $(*)$ applies with $x_1^{=}, y_1^-, x_1^-$ playing the role of x_0^-, y_0, x_0.

Hence we may assume that $x_1^- y_1 \in F_1$. Since the $t_1/2$ edges of T are in $\langle P(x_1, y_s)\rangle$, we have $x_1^- y_1 \in D$. Hence $x_1^- y_1$ is full with respect to R, so $y_1 \leftrightarrow x_s$, which contradicts the assumption that the endpoints of R are not adjacent. This completes the proof. □

2.6 Paths with Both End-edges Selected

The final case is when both end-edges of P lie in F. First we prove that this is sufficient under a threshold on n.

Lemma 2.6.1. If both end-edges of P are selected and $n > 2t_1 + 3t_2$, then G is F-Hamiltonian.

Proof. Since $n > 2t_1 + 3t_2 = |V(F)|$, some vertex of G is not incident to F. By symmetry in X and Y, we may assume it is in X and name it x_1. By Lemma 2.4.3, we may assume $t_1 \geq 3$ and $\epsilon(t_1, t_2) = 0$. Again let $p = \lceil t_1/2 \rceil$. Since $\sigma(G) \geq n/2 + p + t_2$, at least $p + t_2 + 1$ even edges of P are full. At least $p + 1$ are in $E_{\mathsf{even}}(P) - F_2$; let $y_0 x_0$ be one of them. If $y_0 x_0 \in F_1$, then Lemma 2.2.8 applies, so we may assume $y_0 x_0 \notin F$.

Since $y_0 x_0$ is full, $[P(x, y_0)]$ and $[P(x_0, y)]$ are disjoint cycles that together cover $V(G)$ and all edges of F. Among these two cycles, let C be the one containing x_1 and C' be the other. Let y_1 and y_2 be the neighbors of x_1 on C; the choice of x_1 yields $x_1 y_1, x_1 y_2 \notin F$. Let $P_1 = \langle C(x_1, y_1)\rangle$ and $P_2 = \langle C(x_1, y_2)\rangle$. Let $m = |V(C)|$ and $m' = |V(C')|$, so $m + m' = n$. Let $s = |F_2 \cap E(C)|$ and $s' = |F_2 \cap E(C')|$, so $s + s' = 2t_2$.

If y_1 has a neighbor v on C' such that an edge uv of C' is not in F, then $\langle C(x_1, y_1), C'(v, u)\rangle$ is a spanning path through F with $x_1 y_2$ as an unselected end-edge, and Lemma 2.5.3 applies. Hence we may assume that both edges on C' incident to any neighbor of y_1 on C' are in F_2. Thus the only neighbors of y_1 in $V(C')$ are centers of components of F_2 contained in C', which yields $d_C(y_1) \geq d_G(y_1) - s'/2$.

Since F is a forest, we can choose an edge $x'y'$ of C' not in F (see Figure 2.18). Since $x' \not\leftrightarrow y_1$, we have $d_G(x') + d_G(y_1) \geq n/2 + p + t_2$. Since $d_C(y_1) \geq d_G(y_1) - s'/2$ and $d_C(x') \geq d_G(x') - m'/2$, we have $d_C(x') + d_C(y_1) \geq m/2 + p + s/2$. We conclude that among the $m/2$ edges in odd position on P_1, at least $p + s/2$ have neighbors of both y_1 and x'. Let $x''y''$ be one such edge. If $x''y'' \notin F$, then $\langle P_1(x_1, x''), P_1(y_1, y''), C'(x', y')\rangle$ is a spanning path through F having $x_1 y_2$ as an unselected end-edge, and Lemma 2.5.3 applies.

Hence we may assume that $|E_{\mathsf{odd}}(P_1) \cap F| \geq p + s/2$. By applying these arguments using y_2 and P_2 in place of y_1 and P_1, also $|E_{\mathsf{odd}}(P_2) \cap F| \geq p + s/2$. Edges have opposite parity

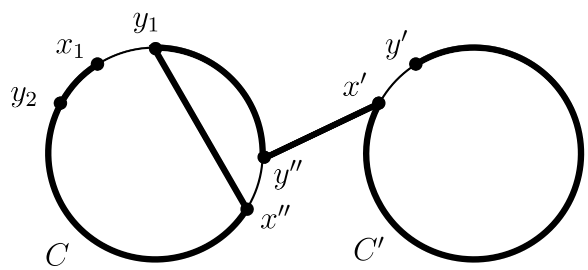


Figure 2.18: Two cycles.

on P_1 and P_2, so $2p + s \leq |E(C) \cap F| \leq t_1 + s$. Since $p = \lceil t_1/2 \rceil$, equality must hold, and $F_1 \subseteq E(C)$, with half of F_1 in each of $E_{\mathsf{odd}}(P_1)$ and $E_{\mathsf{odd}}(P_2)$. Since P_1 and P_2 move in opposite directions from x_1 on $\langle P(x, y_0) \rangle$, the edges of $E_{\mathsf{odd}}(P_1)$ and $E_{\mathsf{odd}}(P_2)$ appear with opposite parity on the original path P. Therefore P splits F, and Lemma 2.3.2 applies. □

Lemma 2.6.2. Under the scenario of Definition 2.2.6, G is F-Hamiltonian.

Proof. We are left with the case where F is a spanning forest, both end-edges of P are selected, and $t_1 \geq 3$ (hence $\tau(F) = \lceil k/2 \rceil \geq 2$). If $t_2 = 0$, then F is a perfect matching in G; since $\sigma(G) \geq n/2 + 2$, the result of Las Vergnas [23] applies. Hence we may assume $t_2 \geq 1$.

We may name X and Y so that F_2 has a component with center in X; call it $\langle y_1, x_1, y_2 \rangle$. Let $G' = G - x_1 - y_1$, $F' = F - \{x_1y_1, x_1y_2\}$, and $n' = n - 2$. Now F' is a short forest in G'. We have $\sigma(G') \geq \sigma(G) - 2$ and $\tau(F') = \tau(F) - 1$; hence $\sigma(G') \geq n'/2 + \tau(F')$.

Since y_2 is not incident to any edge of F', we have $n' > 2t'_1 + 3t'_2$, and Lemma 2.6.1 yields a spanning cycle C through F' in G'. Let x'_1 and x'_2 be neighbors of y_2 on C, so $x'_1y_2, x'_2y_2 \notin F$. Let $P_1 = C - x'_1y_2$ and $P_2 = C - x'_2y_2$. Let $Q_1 = \langle P_1(x'_1, y_2), x_1, y_1 \rangle$ and $Q_2 = \langle P_2(x'_2, y_2), x_1, y_1 \rangle$. Now $|E_{\mathsf{odd}}(Q_1) \cap F| + |E_{\mathsf{odd}}(Q_2) \cap F| = |F| = t_1 + 2t_2$, so Lemma 2.4.1 applies to Q_1 or Q_2. □

Theorem 2.6.3. Let G be an n-vertex balanced X, Y-bigraph, and let F be a linear forest in G. If $\sigma(G) \geq n/2 + \tau(F)$, then G has a spanning cycle through F.

Proof. Lemma 2.6.2 and Lemma 2.2.3. □

Theorem 2.6.4. Let G be an n-vertex balanced X, Y-bigraph, and let F be a linear forest in G. If $\delta(G) \geq n/4 + \tau(F)/2$, then G has a spanning cycle through F. Moreover the inequality is sharp.

Proof. Sufficiency is trivial by Theorem 2.6.3.

The previous sharpness constructions also remain valid. The graphs in Figure 2.1(a), Figure 2.1(b), Figure 2.2 and Figure 2.3 all satisfy $\delta(G) = n/4 + \tau(F)/2 - 1/2$ and none of them has a spanning cycle through its specified edges. □

CHAPTER 3

THE DOMINATION GAME

The domination number is a well-studied graph parameter; books have been devoted to this concept and its variations [16, 15]. There is more than one interesting way to design a two-player game involving the notion of domination number of graphs.

In [2], N. Alon, J. Balogh, B. Bollobás, and T. Szabó introduced the following game. Two players, *Dominator* and *Avoider*, take turns to orient the edges of an undirected graph G, till all the edges are oriented. A dominating set S in the resulting graph is a subset of vertices such that any vertex v not in S has some neighbor u in S with edge uv oriented towards v. Dominator's goal is to orient in such a way that the resulting graph has a small dominating set, and Avoider's goal is to make such a dominating set as large as possible. The size of the smallest dominating set in the resulting graph when both players play optimally is defined as the *game domination number* of G.

In this chapter we consider the variation proposed by Brešar et al. [4]. For a graph G and a subset A of vertices in G, an instance $\mathcal{DS}(G, A)$ of the *domination game* is a two-player game played by two players named *Dominator* (abbreviated $\mathcal{D}$) and *Staller* (abbreviated $\mathcal{S}$). At the beginning of the game all the vertices in A are marked *dominated*. At any moment of the game, a vertex is a *valid* move if its closed neighborhood is not completely dominated. Playing a vertex will make all vertices in its closed neighborhood dominated. Dominator is the first player and Staller is the second player (as the notation $\mathcal{DS}$ suggests). Players alternate playing a valid move till all vertices are dominated. Dominator's goal is to finish the game as soon as possible, and Staller's goal is to prolong it as much as possible. We will use $\gamma_g(G, A)$ to indicate the total number of moves in $\mathcal{DS}(G, A)$ when both players play optimally. The *game domination number* of a graph G is $\gamma_g(G, \emptyset)$, abbreviated as $\gamma_g(G)$. The game $\mathcal{SD}(G, A)$ is completely similar to $\mathcal{DS}(G, A)$ with the only difference that Staller starts the game. The number of moves in $\mathcal{SD}(G, A)$ under optimal play by both players is denoted by $\gamma'_g(G, A)$. The *Staller-start game domination number* of a graph G is $\gamma'_g(G, \emptyset)$, abbreviated as $\gamma'_g(G)$. We also use these abbreviations: $\mathcal{DS}(G) = \mathcal{DS}(G, \emptyset)$ and $\mathcal{SD}(G) = \mathcal{SD}(G, \emptyset)$. Brešar et al. [4] explicitly defined just the games $\mathcal{DS}(G)$ and $\mathcal{SD}(G)$, but they used the more general $\mathcal{DS}(G, A)$ and $\mathcal{SD}(G, A)$ implicitly in their proofs.

3.1 Some Properties of Game Domination Number

The following two inequalities appear in [4]:

$$\gamma(G) \le \gamma_g(G) \le 2\gamma(G) - 1 \tag{3.1}$$

$$\gamma_g(G) - 1 \le \gamma'_g(G) \le \gamma_g(G) + 2 \tag{3.2}$$

The lower bound in (3.1) is immediate from the definition, and the upper bound holds since Dominator can insist on playing vertices in some fixed dominating set of G.

Brešar et al. [4] prove the lower bound in (3.2) by building a Dominator strategy for $\mathcal{DS}(G)$ using an optimal Dominator strategy for $\mathcal{SD}(G)$ in which essentially the moves of Dominator in the former game are copied from the later. We mention the following simpler copycat strategy by Bill Kinnersley: Dominator imagines a virtual instance of $\mathcal{SD}(G)$ played by some optimal Dominator strategy. The real game is an instance of $\mathcal{DS}(G)$. Dominator plays an arbitrary vertex in $\mathcal{DS}(G)$ to start. After each play by Dominator in $\mathcal{DS}(G)$, Staller makes a move in $\mathcal{DS}(G)$. When Staller makes a move in $\mathcal{DS}(G)$, Dominator imagines Staller making the same move also in the virtual game $\mathcal{SD}(G)$, which is possible since the set of dominated vertices of G in the virtual game will always be a subset of the set of dominated vertices in the real game. The optimal Dominator in $\mathcal{SD}(G)$ then responds in the virtual game. When the optimal Dominator makes a move in the virtual game, Dominator tries to copy that move to the real game $\mathcal{DS}(G)$. However that move might not be a valid move in the real game; in that case, Dominator just makes an arbitrary move if the real game is not already finished. It is easy to observe that after every copied move, the set of dominated vertices in the real game $\mathcal{DS}(G)$ is a superset of dominated vertices in the virtual game $\mathcal{SD}(G)$. The number of moves in the real game is always at most one more than the virtual game. Since the virtual game $\mathcal{SD}(G)$ is played by an optimal Dominator strategy, its length is at most $\gamma'_g(G)$. Hence the real game $\mathcal{DS}(G)$ takes no more than $\gamma'_g(G) + 1$ moves, so $\gamma_g(G) \le \gamma'_g(G) + 1$. The symmetric argument implies $\gamma'_g(G) \le \gamma_g(G) + 1$, which improves the upper bound in (3.2). Therefore

$$|\gamma_g(G) - \gamma'_g(G)| \le 1. \tag{3.3}$$

Considering (3.1), a natural question that one might ask is whether $\gamma_g(G)$ can assume every possible value from $\gamma(G)$ up to $2\gamma(G) - 1$ among graphs G with a fixed value of $\gamma(G)$. In [4] an existential argument proves that for any k and r with $0 \le r \le k - 1$, there is a connected graph G with $\gamma(G) = k$ and $\gamma_g(G) = 2k - 1$. Analogously, one can prove that $\gamma(G) \leq \gamma'_g(G) \leq 2\gamma(G)$ and that for any k and r with $0 \leq r \leq k$ there is a connected graph with $\gamma(G) = k$ and $\gamma'_g(G) = k + r$. The same question could be posed restricted to a certain class of graphs, for example one might ask how much of the rang $[\gamma(G), 2\gamma(G)-1]$ is realizable by the graphs with large minimum degree or large connectivity. The constructions in [4] have lots of pendant vertices so they cannot be used for answering the previous questions. We present constructions with large minimum degree and large domination number.

In light of (3.3), a natural goal is to find all values k and $i \in \{0, 1, -1\}$ for which there is a graph G such that $\gamma_g(G) = k$ and $\gamma'_g(G) = k + i$. Brešar et al. [4] defined such a pair $(k, k + i)$ to be *realizable*. They showed that the pairs (k, k), $(k, k + 1)$ and $(2k + 1, 2k)$ are realizable for $k \geq 1$. However, only for (k, k) are their general constructions connected. We present connected graphs that realize pairs $(k, k + 1)$, $(2k + 1, 2k)$ and $(2k + 2, 2k + 1)$ for $k \geq 1$. It is easy to see that $(2, 1)$ is not realizable. We conclude that all the pairs (k, k') with $k, k' \geq 1$ and $|k - k'| \leq 1$ are realizable by connected graphs except for the pair $(2, 1)$.

3.2 Graphs with Large Game Domination Number and Large Minimum Degree

Let D_k be a graph whose vertex set is the $k \times (k + 1)$ matrix $\{(a, b)\colon 1 \leq a \leq k, 0 \leq b \leq k\}$ and whose edge set is $\{(a, b)(a', b)\colon 1 \leq a, a' \leq k, 0 \leq b \leq k, a \neq a'\} \bigcup \{(a, 0)(a, b)\colon 1 \leq a, b \leq k\}$. Figure 3.1 shows a drawing of this graph for $k = 4$. The vertices enclosed in each ellipse form a clique and the higher ellipse collects vertices in column 0.

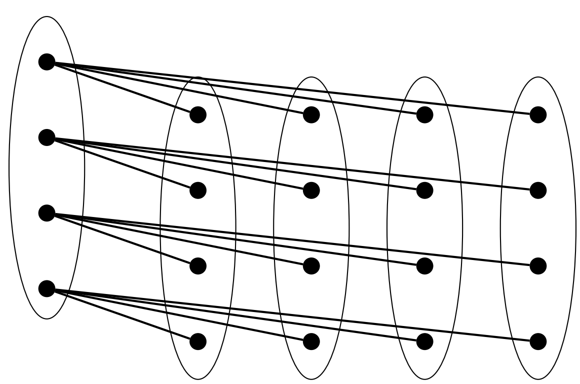


Figure 3.1: The graph D_4.

The k vertices in column 0 form a dominating set for D_k. Any set of size at most $k - 1$ does not intersect a column j among the columns $1, \ldots, k$, and does not contain a vertex $(i, 0)$

among the k vertices in column 0, so it will not dominate vertex (i, j). Hence $\gamma(D_k) = k$ and $\gamma_g(D_k) \leq 2k - 1$. The k vertices in column 0 form a cutset of D_k. Let S be a cutset in D_k. For each $(i, 0) \notin S$, at least one neighbor of $(i, 0)$ from columns $1,\ldots,k$ has to be in S, since otherwise all vertices are reachable from $(i, 0)$. Thus $|S| \geq k$. Hence the connectivity of D_k is k.

Now we present a Staller strategy to prove that $2k - 1 \leq \gamma_g(D_k)$: If Dominator does not play in column 0, then Staller plays some vertex in column 0, and if Dominator plays a vertex $(i, 0)$, then Staller finds a column j with none of its vertices played and plays (i, j).

Consider an instance of $\mathcal{DS}(D_k)$ where Staller plays with the above strategy. Let d_i be the ith vertex that Dominator plays in that game, and let s_i be the ith vertex that Staller plays. With the above strategy, each pair of moves (d_i, s_i) adds 1 to the number of played vertices in column $k + 1$, and it decreases by at most 1 the number of columns containing no played vertices. Therefore, right before Dominator plays d_i, we have $k - (i - 1)$ unplayed vertices in column $k+1$ and at least $k-(i-1)$ columns with no played vertices. Thus Staller can prolong the game till he plays s_{k-1}. At this point still there is one unplayed vertex in column $k + 1$ and also there is at least one column having no played vertices, so the game is not finished. Thus Staller has made the game to continue for $2(k - 1) + 1 = 2k - 1$ steps.

Let n be the number of vertices in D_k, that is $n = k(k + 1)$. Hence D_k satisfies $\gamma_g(D_k) = 2\gamma(D_k)-1$, and both its domination number and minimum degree are as large as $\sqrt{n}+o(\sqrt{n})$.

3.3 Game Domination Number for Paths and Cycles

Paths and cycles demonstrate realizability for some pairs $(k, k + i)$. We first prove a lemma relating game domination numbers of graphs and induced subgraphs.

Lemma 3.3.1. Let G be a fixed induced subgraph of a graph H. If for every vertex $u \in V(H) - V(G)$ there is some vertex $v \in V(G)$ such that $N_H(u) \cap V(G) \subseteq N_G(v)$, then

$$\gamma_g(G) \leq \gamma_g(H) \tag{3.4}$$

$$\gamma'_g(G) \leq \gamma'_g(H) \tag{3.5}$$

Proof. To prove (3.4), we build a strategy for the Staller $\mathcal{S}_H$ in $\mathcal{DS}(H)$ from an optimal strategy for the Staller $\mathcal{S}_G$ in $\mathcal{DS}(G)$. Player $\mathcal{S}_H$ considers an imaginary instance of the game $\mathcal{DS}(G)$ played using an optimal Staller strategy on an auxiliary copy G' of G. Whenever $\mathcal{D}_H$

plays a vertex u in $V(G)$, imagined player $\mathcal{D}_{G'}$ plays the copy of that vertex in G' if possible; otherwise $\mathcal{D}_{G'}$ makes some arbitrary move. Whenever $\mathcal{D}_H$ plays a vertex u in $V(H) - V(G)$, player $\mathcal{D}_{G'}$ finds a vertex $v \in V(G)$ such that $N_H(u) \cap V(G) \subseteq N_G(v)$ and plays the copy of it in G' if possible, otherwise $\mathcal{D}_{G'}$ makes an arbitrary move. It is easy to see that the set of dominated vertices in G' is always a superset of the set of dominated vertices in G. Therefore $\mathcal{DS}(G)$ finishes no later than $\mathcal{DS}(H)$, hence $\gamma_g(H) \leq \gamma_g(G)$. The proof for (3.5) is similar. □

For a predicate $\mathcal{P}$, the notation $[\mathcal{P}]$ equals 1 if $\mathcal{P}$ is true; otherwise $[\mathcal{P}] = 0$.

Lemma 3.3.2. For $n = 4q + r$ with $0 \leq r \leq 3$, we have

$$\gamma_g(P_n) = \lceil n/2 \rceil - [r = 3] \tag{3.6}$$
$$\gamma'_g(P_n) = \lceil n/2 \rceil \tag{3.7}$$
$$\gamma_g(C_n) = \lceil n/2 \rceil - [r = 3], \text{ when } n \geq 3 \tag{3.8}$$
$$\gamma'_g(C_n) = \lceil n/2 \rceil - [r \neq 0], \text{ when } n \geq 3 \tag{3.9}$$

Proof. These relations are easy to see for $n \leq 3$, so we may assume $n \geq 4$. Let G be the graph under study, so $G \in \{P_n, C_n\}$. If $G = P_n$, then let a and b be its endpoints; otherwise, let ab be an arbitrary edge of the cycle G. At any point during the game (before the end), let H be the subgraph of G induced by its dominated vertices. Let t be the number of components of H that do not contain endpoints of G (when G is a cycle, this includes all components of H; except that at the end we set $t = 0$), and let t_0 be the number of components of G that include endpoints of G (when G is a cycle, always $t_0 = 0$).

We first verify that in each equation the right side is an upper bound for the value. To do so, we build an explicit Dominator strategy. The following simple Dominator strategy works in all these games: Dominator plays to extend one component of H as much as possible. If H has no components, then Dominator plays a neighbor of a.

At any moment in the game, let n' indicate the number of vertices of the graph that have been dominated and m indicate the number of moves played so far. We are going to track a "potential" value p as Dominator and Staller play. At any moment, let $p = n' - 2t - t_0 - 2m$. The potential value p is designed so that Staller cannot reduce it by more than 1 on any move. If Staller creates a new component, then the number of new vertices minus the contribution from that component is at least 1. A reduction in the number of components or conversion of a component to one that contains an endpoint increases p. Incorporating the change in $2m$ now yields $\Delta p \geq -1$ for any Staller's move. Also, any move by Dominator that does not start or finish the game extends a component (possibly combining components) and achieves $\Delta p \geq 1$ (it does not increase t or t_0, and it either increases n' by 3 or decreases t).

Let $k = \lfloor M/2 \rfloor$ and $i = M - 2k$, where M is the total number of moves when the game is finished. Now we consider each game separately.

The game for (3.6). Here the first move of Dominator is the first move of the game and increases t_0; for this move, $\Delta p = 0$. If Dominator plays the last move of the game, then that move does not increase t_0, so for that move $\Delta p \geq -1$. Combining Dominator and Staller moves in rounds, we have $p \geq -1 - i$ at the end of the game (when $i = 1$, Dominator finishes the game). Since at the end of the game $t = 0$, $t_0 = 1$, $n' = n$, and $m = 2k + i$, we have $n \geq 4k + i$. Hence $k \leq \lfloor n/4 \rfloor$. Also, if $k = \lfloor n/4 \rfloor$ and $r = 0$, then $i = 0$. Therefore, $\gamma_g(G) \leq m = 2k + i \leq 2\lfloor n/4 \rfloor + [r \neq 0] \leq \lceil n/2 \rceil - [r = 3]$. This proves the upper bound for (3.6).

The game for (3.8). For the first move of the game played by Dominator we have $\Delta p = -1$. For the other moves of Dominator we have $\Delta p \geq 1$. For Staller's moves $\Delta p \geq -1$. For the move that finishes the game we always have $\Delta p \geq 1$. Thus at the end of the game we have $p \geq -i$. Since at that point $t = 0$, $n' = n$ and $m = 2k + i$, we have $n \geq 4k + i$, the same inequality as in the previous case. Therefore $\gamma_g(G) \leq m \leq \lceil n/2 \rceil - [r = 3]$, which proves the upper bound for (3.8).

The game for (3.7). Similar to the previous cases, for a move of Dominator that does not finish the game we have $\Delta p \geq 1$. For a move of Dominator that finishes the game we have $\Delta p \geq -1$. For Staller's moves we have $\Delta p \geq -1$. Immediately after the first $2k$ moves we have $p \geq -2 + 2i$, that is $n' \geq 4k - 1 + 2i$. Hence $m = 2k + i \leq (n' + 1)/2$ and $\gamma'_g(G) \leq m \leq \lceil n/2 \rceil$.

The game for (3.9). Here Dominator's move net effect is $\Delta p \geq 1$, and Staller's move net effect is $\Delta p \geq -1$. Therefore immediately after the first $2k$ moves we have $p \geq 0$. Since at that moment $p = n' - 2i - 4k$, we have $n' \geq 4k + 2i$. This implies $k \leq q$. If $i = 1$, then $n > n'$. Hence if $k = q$ and $i = 1$, then $r = 3$. This proves the upper bound for (3.9).

Lower bound arguments. To show the lower bounds we present a strategy for Staller: If Staller starts the game, then his first move is on a. For his other moves, Staller plays to extend a component of H by one more vertex.

Staller's strategy guarantees that on every round, except for the first round of the games started by Staller, no more than four new vertices are marked dominated. Therefore in the games for (3.6) and (3.8), since we start by $n = 4q + r$ undominated vertices, at least $2q + [r \neq 0]$ moves are needed. In the game for (3.9) the first round may dominate at most six new vertices, hence $2q + [r \neq 3]$ moves are necessary to dominated all vertices.

For the game in (3.7) a slightly different Staller strategy works: If Staller starts the game and $r \in \{0,3\}$, then his first move is on a. If Staller starts the game and $r \in \{1,2\}$, then his first move is on a vertex with distance 2 from a. If Staller starts the game, it never plays to dominate a, except if a is the only undominated vertex left. For the other moves, Staller plays to extend a component of H by one more vertex.

Let the total length of the game in (3.7) be $2k+i$ moves (where $i \in \{0,1\}$). If $r \in \{0,3\}$, then the first k rounds will dominate at most $4k+1$ vertices. If $i=1$, then Staller finishes the game. Hence at the end of the game at most $4k+1+i$ vertices are dominated. We have $4k+1+i \geq n = 4q+r$, hence $k \geq q+i$. This proves the lower bound for (3.7) when $r \in \{0,3\}$. Now we may assume $r \in \{1,2\}$. If Staller dominates a, then Staller finishes the game and $i=1$. The first $2k$ moves of the game dominate at most $4k+2$ vertices, hence $k \geq q$ and the game length is at least $2q+1$. So we may assume Dominator dominates a. If Dominator does so in his first move, then the first round of the game dominates four new vertices. Otherwise the round in which Dominator dominates a, marks at most two new vertices dominated. Hence in both cases the first $2k$ moves dominate at most $4k$ vertices. Therefore $k \geq q$ and if $k=q$, then $i=1$. Hence the game length is again at least $2q+1$. □

Lemma 3.3.3. For $n = 4q+r$ with $0 \leq r \leq 3$, we have

$$\gamma_g(P_n + P_1) = 2q + 1 + [r \neq 0] \tag{3.10}$$

$$\gamma'_g(P_n + P_1) = 1 + \lceil n/2 \rceil \tag{3.11}$$

Proof. The equalities are easy to observe for $n < 4$, so we may assume $n \geq 4$. Let t, t_0, n', m and p have the same meanings as in the proof of Lemma 3.3.2, and let the length of the game be $2k+i$ for $i \in \{0,1\}$.

First we show the upper bound for (3.10) by presenting Dominator's strategy in $\mathcal{DS}(P_n + P_1)$. If $r \in \{0,1,2\}$, then Dominator starts by playing the only vertex of P_1, reducing the game to $\mathcal{SD}(P_n)$. Therefore Dominator will be able to finish the game in $1 + \gamma'_g(P_n) = 1 + \lceil n/2 \rceil = 2q + 1 + [r \neq 0]$ steps. So we may assume $r = 3$. In that case Dominator plays by his optimal strategy on P_n as in the proof of Lemma 3.3.2 and avoids playing on the vertex of P_1, except at the end of the game. Therefore if the last move of the game happens to be in P_1, then the game finishes in $1 + \gamma_g(P_n) = 2q+2$ steps, as desired. So we may assume that the move in P_1 happens before the end of the game, hence it should be Staller's move. Again for the first Dominator's move $\Delta p = 0$, and if Dominator finishes the game, then for that move $\Delta p \geq -1$. For every other Dominator's move $\Delta p \geq 1$ and for all Staller moves $\Delta p \geq -1$. Since the game finishes in exactly $2k+i$ ($i \in \{0,1\}$) steps, at the end of the game we have $p \geq -2 - i$. At the end of the game $n' = n + 1$, $t_0 = 1$, $t = 0$ and $m = 2k + i$, hence $n \geq 4k + i - 1$. Since $n = 4q + 3$, we have $k \leq q + 1$ and if $k = q + 1$, then $i = 0$. Hence $\gamma_g(P_n + P_1) = m = 2k + i \leq 2q + 2$ as desired.

To establish the lower bound for (3.10) we present a strategy for Staller. If the first move of Dominator is in P_1, then the game reduces to $\mathcal{SD}(P_n)$, hence Staller is able to finish in $1 + \gamma_g'(P_n) = 1 + \lceil n/2 \rceil = 2q + 1 + [r \neq 0]$ moves. So we may assume Dominator starts by playing in P_n. In that case Staller always plays in P_n to dominate only one more vertex. If P_n is completely dominated, then Staller will play in P_1. Now if the last move of the game is in P_1, then we know that the game in P_n is already finished and Staller has played optimally in it, hence the total number of moves is at least $\gamma_g(P_n) + 1 = 2q + [r \neq 0] + 1$. So we may assume that the move in P_1 is not the first or the last move the game. Hence it is a Dominator's move. Every round of the game dominates at most four new vertices of P_n, except for the round when Dominator plays in P_1 which dominates only one vertex of P_n. The k rounds of the game dominate at most $4k - 2$ vertices. Hence at the end of the game we have at most $4k - 2 + 3i$ vertices dominated. Since the graph has $n + 1$ vertices, we have $4k - 2 + 3i \geq n + 1 = 4q + r + 1$. Hence $k \geq q$ and if $k = q$ then $(r, i) = (0, 1)$. So $\gamma_g'(P_n + P_1) \geq 2k + i \geq 2q + 1 + [r \neq 0]$.

Now we build Dominator's strategy for (3.11). Again we look at the potential $p = n' - 2t - t_0 - 2m$. For every move of Dominator that does not finish the game we have $\Delta p \geq 1$. For any other move in the game we have $\Delta p \geq -1$. At the end of the game we have $p \geq -2 + i$, that is $(n+1) - 2 \times 0 - 1 - 2(2k+i) \geq -2 + i$. Hence $m = 2k + i \leq (n+i)/2 + 1 \leq \lceil n/2 \rceil + 1$ as desired.

Finally we present Staller's strategy for (3.11). Staller starts by playing an endpoint of P_n. In his other turns, Staller plays in P_n to dominate only one more vertex. If P_n is completely dominated, then Staller plays in P_1. If the last move of the game is in P_1, then the game length is at least $1 + \gamma_g'(P_n) = 1 + \lceil n/2 \rceil$ as desired. So we may assume that P_1 is played before the last move, hence it should be played by Dominator's move. The first round of the game dominates at most five new vertices, and the round when P_1 is played dominates at most two new vertices. Every other round dominates at most four new vertices. Hence at the end of the game we have at most $4k - 1 + i$ dominated vertices (when $i = 1$ Staller finishes the game so that last move will dominate only one new vertex). We have $4k - 1 + i \geq n + 1 = 4q + r + 1$. Hence $k \geq q + 1$, and if $(k, r) = (q + 1, 3)$, then $i = 1$. Therefore $\gamma_g'(P_n + P_1) \geq 2k + i \geq 2q + 2 + [r = 3] \geq 1 + \lceil n/2 \rceil$ as desired.

□

An (h, s)-*broom*, $B(h, s)$, is a graph built out of a path P_h together with s additional vertices all adjacent to the same endpoint of that path.

Lemma 3.3.4. For $h \geq 3$ and $s \geq 2$, let $h+1 = 4q + r$ where $0 \leq r \leq 3$. We have

$$\gamma_g(B(h,s)) = \gamma_g(P_{h+1}) = 2q + [r \neq 0], \tag{3.12}$$

$$\gamma'_g(B(h,s)) = 2q + 1 + [r = 1 \text{ and } s \geq 3, \text{ or } r \in \{2,3\}]. \tag{3.13}$$

Proof. Let u be the vertex of maximum degree in $B(h,s)$, P be the path of order h ending at u, and $v_1, \ldots, v_s$ be the vertices attached to u. Let Q be the path $P + uv_1$. By Lemma 3.3.1 we have that $\gamma_g(P_{h+1}) \leq \gamma_g(B(h,s))$. The proof of Lemma 3.3.2 presents an optimal Dominator strategy for the game $\mathcal{DS}(Q)$ with the first move on u. If Dominator plays with that strategy in $\mathcal{DS}(B(h,s))$, then his first move dominates all $v_1, \ldots, v_s$ and the rest of the states of the game is exactly as in $\mathcal{DS}(Q)$. Hence Dominator can finish the game in at most $\gamma_g(Q) = \gamma_g(P_{h+1})$ moves. This finishes the proof of (3.12).

Now we build a strategy for Dominator in the game of (3.13). If $(r,s) \neq (1,2)$, then Dominator ignores the first move of Staller and uses his strategy in the previous part to finish the game in $2q + [r \neq 0]$ steps. Hence together with the first move of Staller we have $\gamma'_g(B(h,s)) \leq 2q + 1 + [r \neq 0]$ in this case. So we may assume $(r,s) = (1,2)$. In this case if the first move of Staller is not one of v_1 or v_2, then again Dominator can start by playing on u and continue by his optimal strategy in $\mathcal{SD}(Q)$. After Dominator's move the game on the broom and $\mathcal{SD}(Q)$ will be the same, hence $\gamma'_g(B(h,s)) \leq \gamma'_g(Q) = \gamma'_g(P_{h+1}) = 2q+1$. So we may assume that the first move of Staller is on v_1 or v_2, by symmetry we may assume it is on v_1. Dominator will play on the vertex of P with distance 2 from u. Note that after this move the state of the game is the same as the state of the game in $\mathcal{DS}(P_{4q-1} + P_1)$ after Dominator of this game makes his first optimal move. Hence Dominator is able to finish the game on the broom in at most $1 + \gamma_g(P_{4q-1} + P_1) = 1 + 2q$ steps as desired.

Finally we present Staller's strategy for (3.13). Let the length of the game be $2k + i$ for $i \in \{0,1\}$. First we consider $s \geq 3$. In this case Staller makes his first move on v_1 and on his subsequent turns it plays to dominate only one new vertex. The move on vertex u will dominate at most s vertices for Dominator. All the other moves for Dominator dominate at most four new vertices. Hence at most $(s + 3(k-1)) + (2 + (k-1) + i) = 4k - 2 + s + i$ vertices are dominated at the end of the game. Since all vertices are dominated at the end, we have $4k - 2 + s + i \geq n = 4q + r + s - 1$. So $k \geq q$, and if $k = q$, then $i = 1$ and $r = 0$. Hence $\gamma'_g(B(h,s)) \geq 2k + i \geq 2q + 1 + [r \neq 0]$ as needed.

If $s = 2$, then Staller starts by playing on v_1. In his subsequent moves, Staller keeps playing a vertex of P to dominate only one new vertex. If P is completely dominated, then Staller plays on v_2. A move of Dominator that dominates v_2 will dominate at most two new vertices. Other moves of Dominator dominate at most three vertices. Hence if Dominator dominates v_1, then at the end of the game at most $(2+3(k-1))+(2+(k-1)+i) = 4k+i$ vertices are dominated. We have $4k+i \geq 4q+r+1$. Hence $k \geq q$, and if $k = q$ then $i = 1$ or $r = 0$. Therefore in this case $\gamma_g'(B(h,s)) \geq 2k+i \geq 2q+1+[r \neq 0]$ as needed. If Staller dominates v_1, then it will dominate it by his last move, hence $i = 1$. At the end of the game at most $(3k)+(2+(k-1)+1) = 4k+2$ vertices are dominated. We have $4k+2 \geq 4q+r+1$. Hence again $\gamma_g'(B(h,s)) \geq 2k+1 \geq 2q+1$ as needed. □

Corollary 3.3.5. For $k \geq 1$, the pairs $(k, k+1)$ and $(2k+1, 2k)$ are realizable by connected graphs.

Proof. By Lemma 3.3.2 we have $\gamma_g(C_{4k+1}) = 2k+1$ and $\gamma_g'(C_{4k+1}) = 2k$, hence $(2k+1, 2k)$ is realizable by a cycle of order $4k+1$.

By Lemma 3.3.2 we have $\gamma_g(C_{4q+3}) = 2q+1$ and $\gamma_g'(C_{4q+3}) = 2q+2$ for $q \geq 0$. Therefore if k is odd, say $k = 2q+1$, then the pair $(k, k+1)$ is realized by a cycle of order $2k+1$. By Lemma 3.3.4 we have $\gamma_g(B(4q-1,2)) = 2q$ and $\gamma_g'(B(4q-1,2)) = 2q+1$ for $q \geq 1$. Therefore if k is even, say $k = 2q$, then the pair $(k, k+1)$ is realized by $B(2k-1, 2)$. □

3.4 The Difficult Case $(2k, 2k-1)$

We introduce a sequence of graphs, where L_k will satisfy $\gamma_g(L_k) = 2k+4$ and $\gamma_g'(L_k) = 2k+3$.

Definition 3.4.1. L_k is the graph composed of blocks $Q_0, \ldots, Q_k$ with $Q_0 \cong P_2 \Box P_4$, and $Q_i \cong P_2 \Box P_3$ for $i \geq 1$. For $0 < i < k$, the block Q_i shares exactly two of its degree-two vertices that are in distance two from each other with other blocks, one with the block Q_{i-1} and the other with Q_{i+1} (see Figure 3.2).

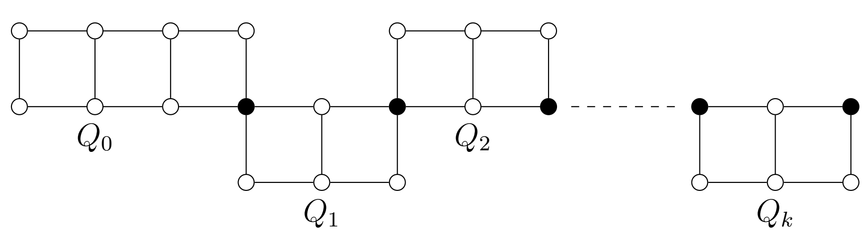


Figure 3.2: The graph L_k

First we prove that $\gamma_g(L_k) \geq 2k + 4$. We present an explicit strategy for the Staller. By *the root block*, we mean Q_0, the block of L_k isomorphic to $P_2 \Box P_4$. At any moment during the course of the game, each of the blocks of L_k isomorphic to $P_2 \Box P_3$ is configured as one of the eleven types below, where a filled dot is a dominated vertex, a hollow dot is an undominated vertex, a diamond is a "don't care" vertex (dominated or undominated), and vertices belonging to neighboring blocks are indicated by a short horizontal line (the block to the far right misses one neighbor). Vertices a and c of each of the blocks Q_i, where $i > 0$, are called *corners*; in the figure for L_k above, corners are shown by bold dots. The long horizontal path of length $2k + 3$ that passes through all corners of L_k will be refered to as the *spine*.

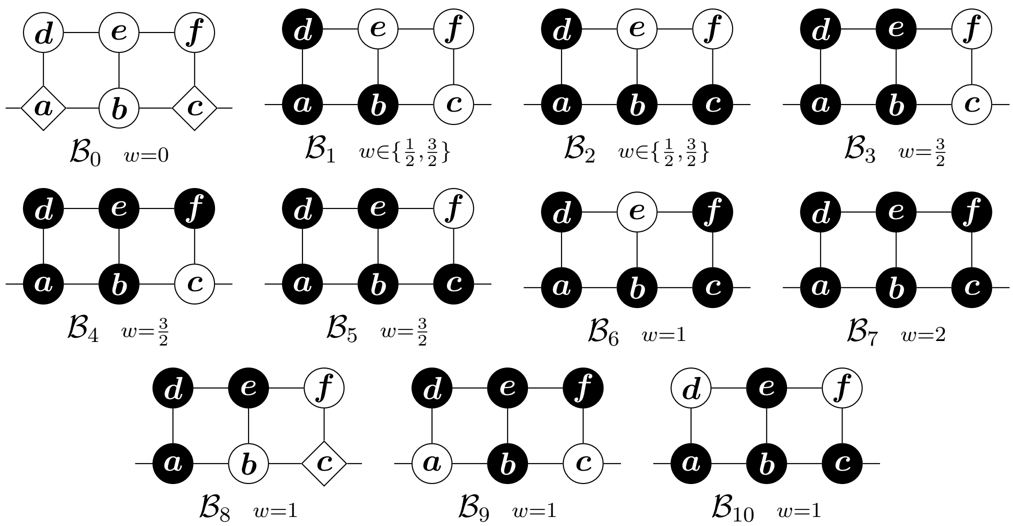


Figure 3.3: Configurations that may occur during the game.

To any state of the game, we assign weight W that is the sum of the current weights assigned to the blocks and the corners. After each move, the weights of a few blocks or corners may change; the others remain unchanged. Weights for the blocks isomorphic to $P_2 \Box P_3$ are indicated in Figure 3.3. Each corner u (except the one to the far right) is part of two 4-cycles. Corner u gets weight $-\frac{1}{2}$ if both of the 4-cycles containing u are completely dominated except for one neighbor of u (see $\mathcal{C}_2$ and $\mathcal{C}_3$ in Figure 3.4). Corner u gets weight $\frac{1}{2}$ if one of the 4-cycles containing u is completely dominated and its two neighbors in the other 4-cycle are not dominated (see $\mathcal{C}_0$ in Figure 3.4), or if one of the 4-cycles containing u and its neighbor located on the spine in the other cycle are dominated and the remaining two vertices of the other 4-cycle are not dominated (see $\mathcal{C}_1$ in Figure 3.4). The special corner to the far right has weight $\frac{1}{2}$ when the only 4-cycle that contains it is completely dominated. All corners in other configurations have weight 0. Note that if a corner has been played, then its weight thereafter is always 0.

At any moment of the game, we use $\mathcal{B}_i$ to indicate the set of blocks in G that have type i at that time. Also $\mathcal{B}_i(x)$ indicates the subset of $\mathcal{B}_i$ whose members have weight x (this is relevant only for $\mathcal{B}_1$ and $\mathcal{B}_2$). The notation $\mathcal{C}_i$ is defined similarly.

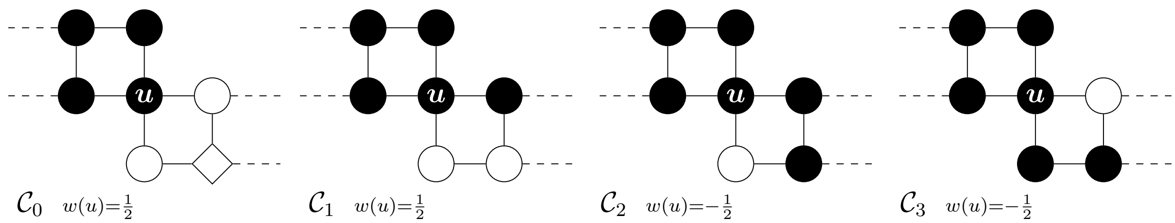


Figure 3.4: Corner u in the configurations with nonzero weight.

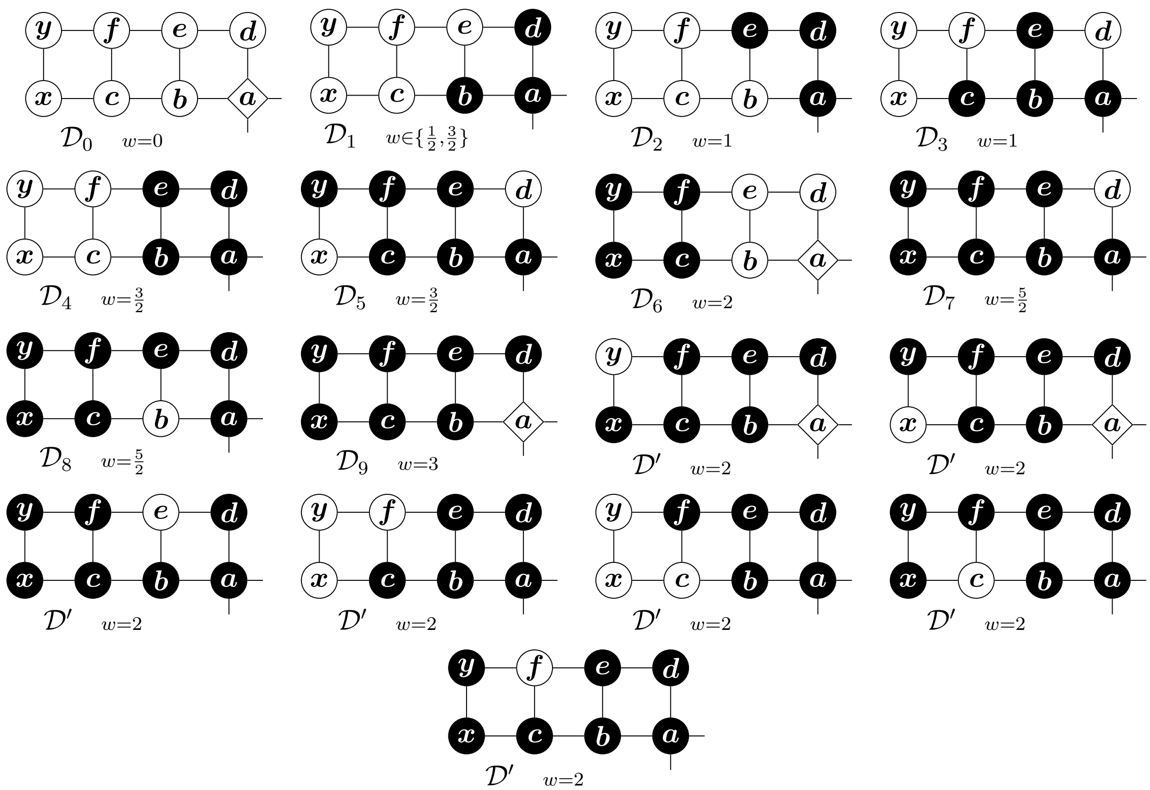


Figure 3.5: Root block configurations.

We also assign a weight to the root block Q_0 in a similar way. We say that Q_0 is in state $\mathcal{D}'$ if $Q_0 - \{a\}$ contains at least one undominated vertex, all its undominated vertices can be dominated by one move played inside Q_0 and every move that dominates some new vertex in $Q_0 - \{a\}$ is in $Q_0 - \{a\}$. When Q_0 is in $\mathcal{D}'$ it has weight 2; a few instances of $\mathcal{D}'$ are shown in Figure 3.5. We also assign a weight to Q_0 when it is in one of the configurations $\mathcal{D}_0, \ldots, \mathcal{D}_9$, shown with their weights in Figure 3.5. Notations $\mathcal{D}_i$ and $\mathcal{D}_i(x)$ are defined similarly to $\mathcal{B}_i$ and $\mathcal{B}_i(x)$.

We now specify how to choose the weights for the members of $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{D}_1$ (these blocks may have weight $\frac{1}{2}$ or $\frac{3}{2}$). For $B \in \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{D}_1$, corner a of B must have been played (since d is dominated and e is not). If immediately before a being played, a as a corner had weight $\frac{1}{2}$, then upon playing a block B gets weight $\frac{3}{2}$; otherwise, a did not have weight $\frac{1}{2}$, and B gets weight $\frac{1}{2}$ when a is played. The analogous conditions specify the weight of the root block upon playing a to put it into $\mathcal{D}_1$. During subsequent plays, we may change the weight of a block in $\mathcal{B}_1$ or $\mathcal{D}_1$ from $\frac{1}{2}$ to $\frac{3}{2}$ (moving that block from $\mathcal{B}_1(\frac{1}{2})$ to $\mathcal{B}_1(\frac{3}{2})$ or from $\mathcal{D}_1(\frac{1}{2})$ to $\mathcal{D}_1(\frac{3}{2})$). Formally we define *readjustment* as the operation of changing the weight of a single block in $\mathcal{B}_1(\frac{1}{2}) \cup \mathcal{D}_1(\frac{1}{2})$ from $\frac{1}{2}$ to $\frac{3}{2}$.

During the game, Staller's moves preserve or establish the following invariants:

I0 For $Q_i \in \mathcal{B}_{10}$, either Q_i is adjacent to two members of $\mathcal{B}_5$, or $i = 1$ and $Q_0 \in \mathcal{D}_2$.

I1 $\mathcal{B}_8 = \mathcal{B}_9 = \emptyset$.

I2 $|\mathcal{D}_1(\frac{1}{2})| + |\mathcal{B}_1(\frac{1}{2})| + \epsilon \leq |\mathcal{B}_3|$, where ϵ is a flag set by Staller that can be 0 or 1 (initially it is 0).

I3 $m \geq W$, where m is the number of moves played so far in the game and W is the sum of the weights of all blocks and corners.

For a block B of type i, we use the language "vertex v of B" to refer to the particular vertex of B that coincides with vertex named $v \in \{a, b, \ldots, f\}$ in the figure of $\mathcal{B}_i$. Also, $(Q_h, v) \to \mathcal{B}_j$ for $Q_h \in \mathcal{B}_i$ means "play on vertex v of block Q_h of type i, converting Q_h to type j". We use a similar notation for playing inside the root block.

Note that Dominator can put a block B into $\mathcal{B}_8$, $\mathcal{B}_9$ or $\mathcal{B}_{10}$ only by playing on vertex d, e or b of B, respectively. Since each such vertex lies in only one block, Dominator can only create one such block on a turn. Therefore **I1** is preserved by lines 3a, 3b and 3c of Staller's strategy. **I0** is preserved by lines 2c, 2g, 3e, 3d, and 3f.

We present a prioritized list of rules that Staller applies to respond to Dominator's play. In many cases, the situation that Staller is responding to arises only when Dominator has just played a particular vertex that is not named in the rule; the argument that the rule works will discuss and make use of that. In addition, the argument that application of a particular rule preserves or restores the desired invariants may use the hypothesis that none of the earlier rules apply. We present the list first as a summary of the analysis that follows. The reader may in fact prefer to read the list along with the subsequent analysis.

Given that Staller can preserve the invariants, the total weight at the end of the game is $2k + 3 + \frac{1}{2}$ (3 for the root block, 2 for other blocks, and $\frac{1}{2}$ for the rightmost corner), and then **I3** implies that at least the desired number $(2k + 4)$ of moves have been made.

Strategy for Staller

1. Staller plays according to the first instruction that can be implemented and ignores the rest.
2. If Dominator's last play was in Q_0, then

   (a) $(Q_1, d) \to \mathcal{B}_3$ if Dominator's play moved Q_0 into $\mathcal{D}_1(\frac{1}{2})$ and Q_1 into $\mathcal{B}_1(\frac{1}{2})$.

   (b) $(Q_h, d) \to \mathcal{B}_3$ for some $Q_h \in \mathcal{B}_1(\frac{1}{2})$ if Dominator's play removed Q_1 from $\mathcal{B}_3$.

   (c) $(Q_1, b) \to \mathcal{B}_{10}$ if $Q_1 \in \mathcal{B}_0$, $\mathcal{B}_1(\frac{1}{2}) = \emptyset$ and Dominator's play put Q_0 in $\mathcal{D}_2$.

   (d) $(Q_h, d) \to \mathcal{B}_3$ for some $Q_h \in \mathcal{B}_1(\frac{1}{2})$ and set $\epsilon = 1$, if $Q_1 \in \mathcal{B}_0$, $\mathcal{B}_1(\frac{1}{2}) \neq \emptyset$ and Dominator's play put Q_0 in $\mathcal{D}_2$.

   (e) $(Q_0, a) \to \mathcal{D}_4$ if $Q_1 \notin \mathcal{B}_0$ and Dominator's play put Q_0 in $\mathcal{D}_2$.

   (f) If Dominator's play was the first move in Q_0, then play $f(u)$, where u $(\neq d)$ is the vertex of Q_0 played by Dominator (see Figure 3.6). This puts Q_0 in $\mathcal{D}'$, $\mathcal{D}_4$, $\mathcal{D}_5$ or $\mathcal{D}_6$.

   (g) If $Q_1 \in \mathcal{B}_{10}$ and Dominator's play removed Q_0 from $\mathcal{D}_2$ but not Q_1 from $\mathcal{B}_{10}$, then play vertex a of Q_0 to put Q_1 in $\mathcal{B}_5$ and Q_0 in $\mathcal{D}'$.

   (h) $(Q_h, e) \to \mathcal{B}_4$ for some $Q_h \in \mathcal{B}_3$ and set $\epsilon = 0$, if $Q_1 \in \mathcal{B}_0$ and Dominator moves Q_0 from $\mathcal{D}_2$ to $\mathcal{D}'$ by playing on vertex c.

   (i) If Dominator's play removed Q_0 from $\mathcal{D}_2$, then play a non-cut-vertex in Q_0 to put it into $\mathcal{D}'$ and if not possible, $\mathcal{D}_9$.

   (j) $(Q_h, e) \to \mathcal{B}_4$ for some $Q_h \in \mathcal{B}_3$ if Dominator's play moved Q_0 from $\mathcal{D}_1(\frac{1}{2})$ to $\mathcal{D}'$.

   (k) If Dominator's play removed Q_0 from $\mathcal{D}_1$, then play a non-cut-vertex in Q_0 to put it into $\mathcal{D}'$.

   (l) If Dominator's play removed Q_0 from $\mathcal{D}_3$, then play a non-cut-vertex in Q_0 to put it into $\mathcal{D}'$ or $\mathcal{D}_9$ (putting Q_0 in $\mathcal{D}_9$ happens only if Dominator plays y; Staller plays $(Q_0, d) \to \mathcal{D}_9$).

   (m) If Dominator's play removed Q_0 from $\mathcal{D}_4$, then play a non-cut-vertex in Q_0 to put it into $\mathcal{D}_9$.

(n) $(Q_0, e) \to \mathcal{D}_9$ if Dominator's play changed Q_0 from $\mathcal{D}_6$ by a play other than vertex a. Readjust if **I2** become violated.

(o) If Dominator played in Q_0, with $Q_0 \in \mathcal{D}'$ before and after, then play to put Q_0 in $\mathcal{D}_9$.

3. If Dominator's last play was not in Q_0 or no rule above applies, then

(a) $(Q_h, e) \to \mathcal{B}_4$ for some $Q_h \in \mathcal{B}_8$ if vertex c of Q_h is undominated. Readjust if **I2** become violated.

(b) $(Q_h, b) \to \mathcal{B}_5$ for some $Q_h \in \mathcal{B}_8$ if vertex c of Q_h is dominated. Readjust if **I2** become violated.

(c) $(Q_h, d) \to \mathcal{B}_4$ for some $Q_h \in \mathcal{B}_9$. Readjust if **I2** become violated.

(d) $(Q_h, d) \to \mathcal{B}_5$ if Dominator's play put one block Q_h in $\mathcal{B}_{10}$ and removed at most one block Q from $\mathcal{B}_3$; where vertex a of Q is the common vertex of Q_h and Q (if such a block Q exist). Readjust if **I2** become violated.

(e) $(Q_h, d) \to \mathcal{B}_3$ for some $Q_h \in \mathcal{B}_1(\frac{1}{2})$ if Dominator's play put one block in $\mathcal{B}_{10}$ and removed two blocks from $\mathcal{B}_3$; or $(Q_0, d) \to \mathcal{D}_4$ if $\mathcal{B}_1(\frac{1}{2}) = \emptyset$ and **I2** is violated.

(f) $(Q_h, d) \to \mathcal{B}_5$ if block Q_h was in $\mathcal{B}_{10}$ and was adjacent to two blocks in $\mathcal{B}_5$ before Dominator's play, and Dominator's play didn't change the type of Q_h but it changed the type of a neighboring block.

(g) $(Q_h, e) \to \mathcal{B}_4$ for some $Q_h \in \mathcal{B}_3$ if Dominator's play moved a block from $B_1(\frac{1}{2})$ to $\mathcal{B}_7$ and didn't remove any block from $\mathcal{B}_3$.

(h) $(Q_h, d) \to \mathcal{B}_3 \cup \mathcal{D}_4$ for some $Q_h \in \mathcal{B}_1(\frac{1}{2}) \cup \mathcal{D}_1(\frac{1}{2})$. (if Dominator's last move violated **I2**, then here is the last chance to restore it)

(i) $(Q_h, d) \to \mathcal{B}_3$ for some $Q_h \in \mathcal{B}_1$.

(j) $(Q_h, d) \to \mathcal{B}_5$ for some $Q_h \in \mathcal{B}_2$.

(k) $(Q_h, e) \to \mathcal{B}_4$ for some $Q_h \in \mathcal{B}_3$.

(l) $(Q_h, f) \to \mathcal{B}_7$ for some $Q_h \in \mathcal{B}_4 \cup \mathcal{B}_5 \cup \mathcal{B}_6$.

(m) $(Q_h, a) \to \mathcal{B}_1(\frac{3}{2})$ for some $Q_h \in \mathcal{B}_0$ with vertex a of Q_h having weight $\frac{1}{2}$.

(n) $(Q_1, d) \to \mathcal{B}_5$ (using this requires $Q_1 \in \mathcal{B}_9$).

(o) $(Q_0, a) \to \mathcal{D}_1(\frac{3}{2})$ if $Q_0 \in \mathcal{D}_0$.

(p) $(Q_0, d) \to \mathcal{D}_4$ if $Q_0 \in \mathcal{D}_1$.

(q) $(Q_0, a) \to \mathcal{D}_4$ if $Q_0 \in \mathcal{D}_2$.

(r) $(Q_0, f) \to \mathcal{D}_5$ if $Q_0 \in \mathcal{D}_3$.

(s) $(Q_0, e) \to \mathcal{D}_9$ if $Q_0 \in \mathcal{D}_6$.

(t) $(Q_0, u) \to \mathcal{D}_9$ for some non-corner vertex u, if $Q_0 \in \mathcal{D}' \cup \mathcal{D}_7 \cup \mathcal{D}_8$.

(u) $(Q_0, c) \to \mathcal{D}'$ if $Q_0 \in \mathcal{D}_4$.

(v) $(Q_0, x) \to \mathcal{D}_7$ if $Q_0 \in \mathcal{D}_5$.

u	y	x	e	c	f	b	a
$f(u)$	x	y	c	e	e	f	d

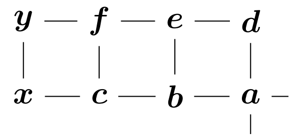


Figure 3.6: The root block and function f.

Lemma 3.4.2. Staller strategy preserves **I0**, **I1**, **I2** and **I3**.

Proof. We have already observed (before the formal statement of the Strategy) that Staller's moves preserves **I0** and **I1**. Observe that Staller never puts a block in $\mathcal{B}_1(\frac{1}{2}) \cup \mathcal{D}_1(\frac{1}{2})$. Therefore if **I2** is true before Staller's move, then **I2** could be violated only if Staller sets ϵ to 1 or if Staller's move reduces $|\mathcal{B}_3|$. Staller sets ϵ to 1 only in line 2d, but that line also increase $|\mathcal{B}_3|$, therefore **I2** is preserved. No action of Staller can reduce $|\mathcal{B}_3|$ by more than 1, and reducing it by 1 only happens in line 2c or one of the lines containing a readjustment. In line 2c, reducing $|\mathcal{B}_3|$ can happen only if $Q_2 \in \mathcal{B}_3$ before Staller's move with the common vertex of Q_1 and Q_2 undominated. Since $\mathcal{B}_1(\frac{1}{2}) = \emptyset$, $Q_0 \in \mathcal{D}_2$, and $\epsilon = 0$, **I2** will remain valid. Lines containing a readjustment also remove a block from $\mathcal{B}_1(\frac{1}{2}) \cup \mathcal{D}_1(\frac{1}{2})$ to restore **I2**, if necessary.

Now consider what happens when Dominator's move invalidates **I2**. Dominator's move may have one of these effects:

- *Increasing* $|\mathcal{B}_1(\frac{1}{2}) \cup \mathcal{D}_1(\frac{1}{2})|$ *by 2.* Dominator's move does not change $\mathcal{B}_3$. Staller replies by line 2a or 3h, decreasing $|\mathcal{B}_1(\frac{1}{2}) \cup \mathcal{D}_1(\frac{1}{2})|$ and increasing $|\mathcal{B}_3|$.

- *Increasing* $|\mathcal{B}_1(\frac{1}{2}) \cup \mathcal{D}_1(\frac{1}{2})|$ *by 1 and decreasing* $|\mathcal{B}_3|$ *by 1.* Staller replies by line 2b or 3h, decreasing $|\mathcal{B}_1(\frac{1}{2}) \cup \mathcal{D}_1(\frac{1}{2})|$ and increasing $|\mathcal{B}_3|$.

- *Increasing* $|\mathcal{B}_1(\frac{1}{2}) \cup \mathcal{D}_1(\frac{1}{2})|$ *by 1 and keeping* $|\mathcal{B}_3|$ *unchanged.* Staller replies by line 2f or 3h, decreasing $|\mathcal{B}_1(\frac{1}{2}) \cup \mathcal{D}_1(\frac{1}{2})|$.

- *Not increasing $|\mathcal{B}_1(\frac{1}{2}) \cup \mathcal{D}_1(\frac{1}{2})|$, but decreasing $|\mathcal{B}_3|$.* Either Staller decreases $|\mathcal{B}_1(\frac{1}{2}) \cup \mathcal{D}_1(\frac{1}{2})|$ in one of the lines 2b, 3e or 3h, or Staller restores **I2** via readjustment in one of the lines 3a, 3b, 3c, or 3d.

So, in all cases **I2** is satisfied by Staller's reply.

To check **I3** we need some observations.

O1. Consider a block B and a corner v of B. The weight of B increases when v becomes dominated by one (or more) moves outside B only if previously $B \in \mathcal{B}_4 \cup \mathcal{B}_9$. Since v is previously undominated in these cases, its weight as a corner is zero. By considering Figure 3.4, dominating v from outside in these cases cannot increase the weight of v. Therefore, if v is the only newly dominated vertex of B during some moves played outside of B, for the net effect of those moves we have $\Delta w(v) + \Delta w(B) \leq \frac{1}{2}$.

O2. When corner v of a block is played, the weight of the block increases by at most $\frac{1}{2}$ unless the block previously was in $\mathcal{D}_3$ or was in $\mathcal{B}_0 \cup \mathcal{D}_0$ with v having weight $\frac{1}{2}$ before the play.

O3. When $\mathcal{B}_8 = \emptyset$, a move played on a corner will not increase the weight of other corners.

For a block B with corner vertex a, we use B^a to indicate its neighboring block that shares a with B. We allow B^a to be a "Null" block, so we do not need to add proof exceptions for the terminal blocks. All vertices of a "Null" block that do not exist are considered as "Null Vertices". A Null block and its non-existent corner always have weight 0.

Let $\alpha = m - W$. We will verify that for the most of Dominator and Staller's moves, $\Delta\alpha \geq 0$. Sometimes for a particular Dominator's move we cannot guarantee that $\Delta\alpha \geq 0$, in those cases we will verify that the game continues with a Staller's reply and for that pair of moves $\Delta\alpha \geq 0$ is maintained. Here we study the quantity $\Delta\alpha$ under several possible moves played by the players.

When Staller plays corner a of Q_0 by line 2g. Considering this move together with its previous Dominator's move we have $\Delta m = 2$, $\Delta w(Q_1) = \frac{1}{2}$, $\Delta w(a) = 0$ and $\Delta w(Q_0) = 1$. Therefore $\Delta\alpha \geq 0$. (**)

When a player plays a corner v, excepting the case discussed above. Let B_1 and B_2 be the blocks containing v and let u_1 and u_2 be the other corners of B_1 and B_2, respectively (We allow B_2 and u_2 to be "Null").

If $B_1 \in \mathcal{D}_3$, then this should be Dominator's move, since Staller's move inside Q_0 when $Q_0 \in \mathcal{D}_3$ only happens in lines 2f and 3r and none of them prescribe a move on a corner.

Observe that $B_2 \in \mathcal{B}_5$. Such a Dominator's move makes Staller to reply by line 2l. For the net effect of these two moves $\Delta m = 2$, $\Delta w(B_1) = 1$, $\Delta w(B_2) = \frac{1}{2}$ and $\Delta w(v) = 0$. Therefore $\Delta\alpha \geq 0$ and we may assume $B_1, B_2 \notin \mathcal{D}_3$.

If $\mathcal{B}_8 \neq \emptyset$, then this is Staller's move (by **I1**). As soon as Dominator puts a block in $\mathcal{B}_8$, Staller puts it out by line 3a or 3b. None of these two lines prescribes a corner move, therefore we may assume $\mathcal{B}_8 = \emptyset$.

Now having these assumptions, for a move on corner v we have $\Delta m = 1$, $\Delta w(u_1), \Delta w(u_2) \leq 0$ (by O3), $\Delta w(B_1) + \Delta w(B_2) \leq 1$ (by O2), and $\Delta w(v) \leq \frac{1}{2}$. We observe that $\Delta w(v) = \frac{1}{2}$ happens only when corner v changes its weight from $-\frac{1}{2}$ to 0, hence only when $\Delta w(B_1) = 0$ or $\Delta w(B_2) = 0$, so $\Delta w(B_1) + \Delta w(B_2) \leq \frac{1}{2}$ (by O2). Therefore $\Delta\alpha \geq 0$.

When a player plays a non-corner vertex v. We consider a block B in each of the possible configurations and we compute $\Delta\alpha$ resulting from playing v (sometimes we consider the play on v with the subsequent move together). Implied by **I1**, when Dominator starts to play, every block is in $(\bigcup_{i=0}^{7} \mathcal{B}_i) \cup \mathcal{B}_{10} \cup (\bigcup_{i=0}^{9} \mathcal{D}_i) \cup \mathcal{D}'$. Also note that Staller never puts a block in $\mathcal{B}_8$, $\mathcal{B}_9$, $\mathcal{B}_1(\frac{1}{2})$, $\mathcal{D}_2$, $\mathcal{D}_1(\frac{1}{2})$ or $\mathcal{D}_3$.

When $B \in \mathcal{B}_0$ and $v = b$. By O1 we can observe that $\Delta w(B^a) + \Delta w(a) \leq 0$ and $\Delta w(B^c) + \Delta w(c) \leq 0$. $\Delta w(B) = 1$ and $\Delta m = 1$. Hence $\Delta\alpha \geq 0$.

When $B \in \mathcal{B}_0$ and $v \in \{d, f\}$. By symmetry we may assume $v = d$. We have $\Delta m = 1$, $\Delta w(B^a) + \Delta w(a) \leq 0$ (by O1), $\Delta w(B^c) = \Delta w(c) = 0$ and $\Delta w(B) = 1$. Hence $\Delta\alpha \geq 0$.

When $B \in \mathcal{B}_0$ and $v = e$. We have $\Delta m = 1$, $\Delta w(a), \Delta w(c) \leq 0$, $\Delta w(B^a) = \Delta w(B^c) = 0$ and $\Delta w(B) = 1$. Hence $\Delta\alpha \geq 0$.

When $B \in \mathcal{B}_1 \cup \mathcal{B}_2$ and $v = b$. Hence a is already played. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(a) = 0$, $\Delta w(B^c) + \Delta w(c) \leq 0$ (by O1) and $\Delta w(B) \leq 1$. Hence $\Delta\alpha \geq 0$.

When $B \in \mathcal{B}_1$ and $v \in \{d, e\}$. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(a) = \Delta w(B^c) = \Delta w(c) = 0$ and $\Delta w(B) \leq 1$. Hence $\Delta\alpha \geq 0$.

When $B \in \mathcal{B}_1(\frac{3}{2}) \cup \mathcal{B}_3$ and $v = f$. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(a) = 0$, $\Delta w(B^c) + \Delta w(c) \leq \frac{1}{2}$ (by O1) and $\Delta w(B) = \frac{1}{2}$. Hence $\Delta\alpha \geq 0$.

When $B \in \mathcal{B}_1(\frac{1}{2})$, $B^c \in \mathcal{B}_3$ and $v = f$. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(a) = 0$, $\Delta w(B^c) = 0$, $\Delta w(c) = -\frac{1}{2}$ and $\Delta w(B) = \frac{3}{2}$. Hence $\Delta\alpha \geq 0$.

When $B \in \mathcal{B}_1(\frac{1}{2})$, $B^c \notin \mathcal{B}_3$ and $v = f$. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(a) = 0$, $\Delta w(B^c) + \Delta w(c) \leq \frac{1}{2}$ (by O1) and $\Delta w(B) = \frac{3}{2}$. Hence $\Delta\alpha \geq -1$.

This is a Dominator's move. **I2** implies that after this move $\mathcal{B}_3 \neq \emptyset$, so the game will not finish. We want to consider the effect of Staller's reply here. Considering **I1**, Staller will reply by line 3g. In the subsequent item marked with $(*)$ we will see that for such a move $\Delta\alpha \geq 1$. Therefore these two moves together will not decrease the value of α.

When $B \in \mathcal{B}_2$ and $v = d$. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(a) = \Delta w(B^c) = 0$, $\Delta w(c) \leq 0$, and $\Delta w(B) \leq 1$. Hence $\Delta\alpha \geq 0$.

When $B \in \mathcal{B}_2$ and $v \in \{e, f\}$. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(a) = \Delta w(B^c) = 0$, $\Delta w(c) \leq -\frac{1}{2}$, and $\Delta w(B) \leq \frac{3}{2}$. Hence $\Delta\alpha \geq 0$.

When $B \in \mathcal{B}_3$ and $v = b$. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(a) = 0$, $\Delta w(B^c) + \Delta w(c) \leq 0$ (by O1) and $\Delta w(B) = 0$. Hence $\Delta\alpha \geq 1$.

When $B \in \mathcal{B}_3$ and $v = e$. We have $\Delta m = 1$ and $\Delta w(B^a) = \Delta w(a) = \Delta w(B^c) = \Delta w(c) = \Delta w(B) = 0$. Hence $\Delta\alpha \geq 1$. $(*)$

When $B \in \mathcal{B}_4$ and $v \in \{b, f\}$. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(a) = 0$, $\Delta w(B^c) + \Delta w(c) \leq \frac{1}{2}$ (by O1) and $\Delta w(B) = \frac{1}{2}$. Hence $\Delta\alpha \geq 0$.

When $B \in \mathcal{B}_5$ and $v \in \{e, f\}$. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(a) = \Delta w(B^c) = 0$, $\Delta w(c) \leq \frac{1}{2}$ and $\Delta w(B) = \frac{1}{2}$. Hence $\Delta\alpha \geq 0$.

When $B \in \mathcal{B}_6$ and $v \in \{b, d, e, f\}$. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(a) = \Delta w(B^c) = \Delta w(c) = 0$ and $\Delta w(B) = 1$. Hence $\Delta\alpha \geq 0$.

When $B \in \mathcal{B}_8$. By **I1** this is a Staller's move. If Staller replies by line 3a, then we have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(B^c) = \Delta w(c) = 0$, $\Delta w(a) \leq \frac{1}{2}$ and $\Delta w(B) = \frac{1}{2}$, hence $\Delta\alpha \geq 0$. If Staller replies by line 3b, then we have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(B^c) = 0$, $\Delta w(c) \leq 0$, $\Delta w(a) \leq \frac{1}{2}$ and $\Delta w(B) = \frac{1}{2}$, hence $\Delta\alpha \geq 0$. Note that for both of these possible Staller replies a readjustment may become necessary. That only happens if $B^a \in \mathcal{B}_5$, and it implies $\Delta w(a) \leq -\frac{1}{2}$, hence in both cases $\Delta\alpha \geq 1$. Now if a readjustment happens, W decreases by 1 and we would have $\Delta\alpha \geq 0$ after readjustment.

When $B \in \mathcal{B}_9$. By **I1** this is a Staller's move (line 3c). We have $\Delta m = 1$, $\Delta w(B^a) + \Delta w(a) \leq \frac{1}{2}$ (by O1), $\Delta w(B^c) = \Delta w(c) = 0$ and $\Delta w(B) = \frac{1}{2}$. Hence $\Delta\alpha \geq 0$. If $B^a \in \mathcal{B}_3$, then $\Delta w(B^a) = 0$ and $\Delta w(a) = -\frac{1}{2}$, hence $\Delta\alpha \geq 1$ that pays for the possibly required readjustment.

When $B \in \mathcal{B}_{10}$ and $v \in \{d, f\}$. By symmetry we may assume $v = d$. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(B^c) = \Delta w(c) = 0$, $\Delta w(a) \leq \frac{1}{2}$ and $\Delta w(B) = \frac{1}{2}$. Again in case of $B^a \in \mathcal{B}_5$, we have $\Delta w(a) \leq -\frac{1}{2}$ hence $\Delta\alpha \geq 1$. Therefore we would have enough difference to pay for a possible readjustment (line 3d).

When $B \in \mathcal{B}_{10}$ and $v = e$. Staller strategy never prescribes this action, so this is a Dominator's move. **I0** implies two possibilities, either $B^a, B^c \in \mathcal{B}_5$, or $B = Q_1$ and $Q_0 \in \mathcal{D}_2$. In both cases $\Delta w(a) + \Delta w(c) \leq 0$. We have $\Delta m = 1$, $\Delta w(B^a) = \Delta w(B^c) = 0$ and $\Delta w(B) = 1$. Hence $\Delta\alpha \geq 0$.

Now we consider the cases where $B = Q_0$ and a non-corner vertex v inside it is played.

When $Q_0 \in \mathcal{D}_0$ and $v \in \{b, d\}$. We have $\Delta m = 1$, $\Delta w(Q_1) + \Delta w(a) \leq 0$ and $\Delta w(Q_0) = 1$. Hence $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_0$ and $v \in \{c, e, f, x, y\}$. This is a Dominator's move. Staller replies by line 2f, playing $f(v)$ to put Q_0 in $\mathcal{D}'$ or $\mathcal{D}_6$. For both moves together we have $\Delta m = 2$, $\Delta w(Q_1) = 0$, $\Delta w(a) \leq 0$ and $\Delta w(Q_0) = 2$. Hence $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_1$ and $v = d$. Hence a is played. We have $\Delta m = 1$, $\Delta w(Q_1) = \Delta w(a) = 0$, $\Delta w(Q_0) \leq 1$. Hence $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_1(\frac{3}{2})$ and $v \in \{b, c, e, f, x, y\}$. Hence a is played. Q_0 moves to $\mathcal{D}'$. We have $\Delta m = 1$, $\Delta w(Q_1) = \Delta w(a) = 0$, $\Delta w(Q_0) = \frac{1}{2}$. Hence $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_1(\frac{1}{2})$ and $v \in \{b, c, e, f, x, y\}$. Hence a is played. This is a Dominator's move that puts Q_0 in $\mathcal{D}'$ and Staller replies by line 2j. **I2** guarantees that $\mathcal{B}_3$ is not empty and so the game continues. For Dominator's move we have $\Delta m = 1$, $\Delta w(Q_1) = \Delta w(a) = 0$ and $\Delta w(Q_0) \leq \frac{3}{2}$, so $\Delta\alpha \geq -\frac{1}{2}$. For Staller's reply we have $\Delta\alpha \geq 1$ by $(*)$. Hence for both moves together $\Delta\alpha \geq 0$.

In the following notice that Staller never plays on a non-corner vertex of Q_0 when $Q_0 \in \mathcal{D}_2$, therefore the play is a Dominator's move. Also note that if in Dominator's turn $Q_0 \in \mathcal{D}_2$ then $Q_1 \in \mathcal{B}_0 \cup \mathcal{B}_5 \cup \mathcal{B}_{10}$.

When $Q_0 \in \mathcal{D}_2$, $Q_1 \in \mathcal{B}_5$ and $v \in \{b, c, e, f, x, y\}$. Staller replies by line 2i. We have $\Delta m = 2$, $\Delta w(Q_1) = 0$, $\Delta w(a) = -\frac{1}{2}$ and $\Delta w(Q_0) \leq 2$. Hence $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_2$, $Q_1 \in \mathcal{B}_0$ and $v \in \{b, e, f, x, y\}$. Staller replies by line 2i moving Q_0 to $\mathcal{D}'$ (and not $\mathcal{D}_9$). We have $\Delta m = 2$, $\Delta w(Q_1) = 0$, $\Delta w(a) \leq \frac{1}{2}$ and $\Delta w(Q_0) \leq 1$. Hence $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_2$, $Q_1 \in \mathcal{B}_0$ and $v = c$. Staller replies by line 2h. Note that ϵ is set to 1 by line 2d and hasn't changed after that, therefore $\mathcal{B}_3 \neq \emptyset$ by **I2**. For the Dominator's move we have $\Delta m = 1$, $\Delta w(a) \leq \frac{1}{2}$, $\Delta w(Q_1) = 0$ and $\Delta w(Q_0) = 1$, so $\Delta\alpha \geq -\frac{1}{2}$. For Staller's reply $\Delta\alpha \geq 1$ by $(*)$. Hence for both moves together $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_2$, $Q_1 \in \mathcal{B}_{10}$ and $v \in \{b, e, c, f, x, y\}$. Staller replies by line 2g. This is discussed before in $(**)$.

When $Q_0 \in \mathcal{D}_3$, $v \in \{c, d, e, f, x, y\}$. This is Dominator's move since Staller never plays on a non-corner vertex in the root block when $Q_0 \in \mathcal{D}_3$. Note that always $Q_1 \in \mathcal{B}_5$ when Dominator observes $Q_0 \in \mathcal{D}_3$. Staller replies by line 2l. We have $\Delta m = 2$, $\Delta w(Q_1) = 0$, $\Delta w(a) = -\frac{1}{2}$ and $\Delta w(Q_0) \leq 2$. Hence for both moves together $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_4$, $v \in \{b, c, e, f, x, y\}$. This puts Q_0 in $\mathcal{D}'$. We have $\Delta m = 1$, $\Delta w(Q_1) = \Delta w(a) = 0$ and $\Delta w(Q_0) = \frac{1}{2}$. Hence $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_5$, $v \in \{d, e\}$. This puts Q_0 in $\mathcal{D}'$. We have $\Delta m = 1$, $\Delta w(Q_1) = 0$, $\Delta w(a) \leq \frac{1}{2}$ and $\Delta w(Q_0) = \frac{1}{2}$. Hence $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_5$, $v \in \{c, x, y\}$. This puts Q_0 in $\mathcal{D}_7$. We have $\Delta m = 1$, $\Delta w(Q_1) = \Delta w(a) = 0$ and $\Delta w(Q_0) = 1$. Hence $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_6$, $v \in \{b, c, e, f, d\}$. If $v = e$, then this move puts Q_0 in $\mathcal{D}_9$ and we have $\Delta m = 1$, $\Delta w(Q_1) = 0$, $\Delta w(a) \leq 0$ and $\Delta w(Q_0) = 1$, so $\Delta\alpha \geq 0$. Note that since the configuration of Q_1 does not change, readjustment will not become necessary.

If $v \neq e$, then this is Dominator's move since Staller deals with $Q_0 \in \mathcal{D}_6$ by playing e in line 3s. Since Q_0 is not completely dominated yet, Staller will reply to this move by line 2n. For the net effect of these two moves we have $\Delta m = 2$, $\Delta w(Q_1) + \Delta w(a) \leq \frac{1}{2}$ (by O1), $\Delta w(Q_0) = 1$. Hence $\Delta\alpha \geq 0$. A readjustment in line 2n would become necessary if $Q_1 \in \mathcal{B}_3$. In that case $\Delta w(Q_1) = 0$ and $\Delta w(a) = -\frac{1}{2}$, hence $\Delta\alpha \geq 1$. Therefore after readjustment we would have $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_7$ and $v \in \{d, e\}$, or $Q_0 \in \mathcal{D}_8$ and $v \in \{b, c, e\}$. We have $\Delta m = 1$, $\Delta w(Q_1) = 0$, $\Delta w(a) \leq \frac{1}{2}$ and $\Delta w(Q_0) = \frac{1}{2}$. Hence $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}_9$ and $v \in \{b, d\}$. We have $\Delta m = 1$, $\Delta w(Q_1) + \Delta w(a) \leq \frac{1}{2}$ (by O1) and $\Delta w(Q_0) = 0$. Hence $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}'$ and playing v puts Q_0 in $\mathcal{D}'$. By definition of D' this move will not change the configuration of Q_1. We have $\Delta m = 1$, $\Delta w(Q_1) = 0$, $\Delta w(a) \leq \frac{1}{2}$ and $\Delta w(Q_0) = 0$. Hence $\Delta\alpha \geq 0$.

When $Q_0 \in \mathcal{D}'$ and playing v puts Q_0 in $\mathcal{D}_9$. If a is already played then $\Delta w(a) = 0$. If not, then by the definition of $\mathcal{D}'$ since b is dominated, e should be dominated too, so $\Delta w(a) = 0$. Definition of $\mathcal{D}'$ also implies $\Delta w(Q_1) = 0$. We have $\Delta m = 1$ and $\Delta w(Q_0) = 1$, hence $\Delta\alpha \geq 0$.

Therefore we have proved that for a move played in the game either $\Delta\alpha \geq 0$ or it is Dominator's move necessarily followed by a reply from Staller with a net effect of $\Delta\alpha \geq 0$. Therefore since at the beginning of the game $\alpha = m - W = 0 - 0 = 0$, at the end of the game $\alpha = m - W \geq 0$. At the end of the game the root block is in $\mathcal{D}_9$ and all the other blocks are in $\mathcal{B}_7$ and the only corner with nonzero weight is the far right corner with weight $\frac{1}{2}$; so $W = 2k + 3 + \frac{1}{2}$. Since m is integer and $m \geq W$, we have $m \geq 2k + 4$. □

Lemma 3.4.3. Let L_k be the graph in Figure 3.2, we have $\gamma'_g(L_k) \geq 2k + 3$.

Proof. The proof is as before with a change in the first move. At the beginning of the game Staller plays the far right corner and puts Q_k in $\mathcal{B}_1$ or $\mathcal{D}_1$. However the analysis assigns weight $\frac{3}{2}$ (instead of $\frac{1}{2}$) to Q_k so $Q_k \in \mathcal{B}_1(\frac{3}{2})$ or $Q_k \in \mathcal{D}_1(\frac{3}{2})$. For this first move we have $\Delta\alpha = -\frac{1}{2}$. Note that invariants **I0**, **I1** and **I2** are satisfied. From now on Staller plays according to the strategy in the previous lemma. The same analysis as before implies that the rest of the moves together do not decrease α, hence at the end of the game we have $\alpha \geq -\frac{1}{2}$. Also at the end of the game $W = 2k + 3 + \frac{1}{2}$, therefore $m \geq 2k + 3$. □

Lemma 3.4.4. Let L_k be the graph in Figure 3.2, we have $\gamma'_g(L_k) \leq 2k + 3$.

Proof. We devise an explicit Dominator strategy.

Strategy for Dominator

1. If Staller's play was the first move inside Q_0, then play another vertex of Q_0 to leave at most one vertex of it undominated.

2. $(Q_h, u) \rightarrow \mathcal{B}_7$ for some $Q_h \notin \mathcal{B}_7$ and a vertex u in Q_h.

3. $(Q_h, e) \rightarrow \mathcal{B}_4$ for some Q_h.

4. Play a vertex of Q_0 to completely dominate it.

5. $(Q_0, e) \to \mathcal{D}'$.

Line 1 would be executed at most once and if it gets executed then line 5 would never be executed. Now consider a round of the game (pair of Staller's move and its subsequent Dominator's reply), it falls into one of these types:

1. Staller's play was in Q_0 and dominated some undominated vertex of Q_0. Dominator's reply could be one of these three possibilities:

   (a) If Staller's move was the first move ever in Q_0, then Dominator replies by line 1.

   (b) If in previous rounds Staller has played the first move in Q_0 (and so Dominator has replied by line 1), then Staller's current move finishes Q_0 and Dominator replies by line 2 or line 3, increasing $|\mathcal{B}_7|$ or $|\mathcal{B}_4|$.

   (c) If in previous rounds Dominator has played the first move in Q_0 (by line 5) we have $Q_0 \in \mathcal{D}'$. So the current round will finish Q_0.

2. Otherwise Staller's play was in Q_i for some $i > 0$ and dominated some undominated vertex of Q_i. If $Q_i \notin \mathcal{B}_7$ after Staller's move, then Q_i was not in $\mathcal{B}_4$ before that move and also Dominator replies by line 2 and puts Q_i in $\mathcal{B}_7$. Therefore in this case always $|\mathcal{B}_7|$ and $|\mathcal{B}_4 \cup \mathcal{B}_7|$ increases.

   If $Q_i \in \mathcal{B}_7$ after Staller's move but $Q_i \notin \mathcal{B}_4$ beforehand, then this round increases both $|\mathcal{B}_7|$ and $|\mathcal{B}_4 \cup \mathcal{B}_7|$.

   If $Q_i \in \mathcal{B}_7$ after Staller's play and $Q_i \in \mathcal{B}_4$ beforehand, then $|\mathcal{B}_7|$ increases and Dominator replies by lines 2, 3, 4 or 5. We have these cases:

   (a) If the reply is by line 2, then $|\mathcal{B}_7|$ increases again. So $|\mathcal{B}_7|$ increases at least by 2 during this round.

   (b) If the reply is by line 3, then $|\mathcal{B}_4|$ increases. So both $|\mathcal{B}_7|$ and $|\mathcal{B}_4 \cup \mathcal{B}_7|$ increase in this round.

   (c) If the reply is by line 4, then this round finishes the game.

   (d) If the reply is by line 5, then Dominator moves in Q_0 first.

Note that types 1a and 1b cannot coexist with 1c, and we have at most one round of each of these types, and that all of them are not empty. Now we consider two cases.

When Dominator moves in Q_0 first. Therefore we have a round of type 1c and all the other rounds are type 2. Each round of type 2 increase $|\mathcal{B}_7|$, therefore we have at most k rounds of that type. Hence the entire game has no more than $k + 1$ rounds.

When Staller moves in Q_0 first. Therefore we have a round of type 1a and no round of type 1c and 2d. If we have no round of type 1b or we have a round of type 2a, then the increase in $|\mathcal{B}_7|$ implies that the game has no more than $k+1$ rounds. Hence we may assume we have one round of type 1b and no round of type 2a. If we have a round of type 2c, then it is the last round of the game and no round is of type 1b. Since all rounds, except the one of type 1a, increase $|\mathcal{B}_7|$, the entire game has no more than $k+1$ rounds. So we may assume that no round is of type 2c too. Now the remaining possible types, except type 1a, increase $|\mathcal{B}_4 \cup \mathcal{B}_7|$, therefore we conclude again that the game has no more than $k+1$ rounds.

Only the last move in the game could belong to no round. Since the game has at most $k+1$ rounds, it has no more than $2k+3$ moves. □

Claim 3.4.5. Let L_k be the graph in Figure 3.2, we have $\gamma_g(L_k) = 2k+4$ and $\gamma_g'(L_k) = 2k+3$.

Proof. Lemma 3.4.3 and 3.4.4 imply that $\gamma_g'(L_k) = 2k+3$. Since $\gamma_g(L_k) \leq \gamma_g'(L_k)+1 = 2k+4$, lemma 3.4.2 implies that $\gamma_g(L_k) = 2k+4$. □

CHAPTER 4

THE GAME OF REVOLUTIONARIES AND SPIES

The revolutionaries and spies game has been attributed to Józef Beck (see [17]). It enjoys the same movement rules as those in the famous cops and robber[1] pursuit game [3, 10]. Formally, $\mathcal{RS}(G, m, r, s)$ is an instance of the *game of revolutionaries and spies* played on a graph G with integer parameters m, r and s as follows. The game is played by two players $\mathcal{R}$ and $\mathcal{S}$. Player $\mathcal{R}$ controls r pieces called *revolutionaries* and player $\mathcal{S}$ controls s pieces called *spies*. Player $\mathcal{R}$ starts the game (as the name $\mathcal{RS}$ suggests) by placing his pieces on vertices of G. Then player $\mathcal{S}$ places his pieces. At each subsequent round, player $\mathcal{R}$ can move each of his revolutionaries from their current vertex to a neighboring vertex. Then player $\mathcal{S}$ has the same option moving his pieces. If after $\mathcal{S}$ finishes his move there is a vertex with at least m revolutionaries and no spy, then $\mathcal{R}$ wins. If $\mathcal{S}$ can prevent such a situation indefinitely, then $\mathcal{S}$ wins.

The primary question of interest is: given an instance of this game, who wins? Note that if the graph has at most s vertices, then spies can always win by occupying all vertices. Hence we may always assume that the graph has at least $s+1$ vertices. At the start of the game revolutionaries can form $\lfloor r/m \rfloor$ distinct meetings (if $\lfloor r/m \rfloor \le |V(G)|$); hence if $s < \lfloor r/m \rfloor$, then $\mathcal{R}$ wins right away. On the other hand, if $r - m + 1 \le s \le r$, then $\mathcal{S}$ matches his spies to revolutionaries and instructs every spy to follow the revolutionary it is matched to. The remaining $m-1$ revolutionaries cannot form an unguarded meeting. Hence if $s \ge r - m + 1$, then spies will prevent a meeting forever by this strategy.

Howard et al. [17] proved that on an acyclic graph with at least $s+1$ vertices, $\mathcal{S}$ wins if $s \ge \lceil r/m \rceil$ and sometimes when $s = \lfloor r/m \rfloor$ (including on all trees). They also considered the game played on a graph whose vertices are arranged in a d-dimensional grid. Let $\sigma(G, m, r)$ be the minimum number of spies that are needed to beat r revolutionaries on a graph G. Let G_d be the graph with vertex set $\mathbb{Z}^d$ and edge set $\{(a_1, \ldots, a_d)(b_1, \ldots, b_d)\colon |a_i - b_i| \le 1 \text{ for all } i\}$. Let $\sigma(G_d, m, r)$ be the minimum number of spies needed to defeat r revolutionaries on G_d with meeting size m. They proved that $\liminf_{r\to\infty} \sigma(G_d, 2, r)/r \ge 3/4$.

[1] also known as prey and predator game

4.1 Basic Properties of Revolutionaries-Spies Game

As mentioned before, $\mathcal{S}$ wins $\mathcal{RS}(G, m, r, s)$ only if $s \geq \lfloor r/m \rfloor$. If for a graph G this necessary condition is also sufficient, then G is *spy-good*.

Lemma 4.1.1. Any graph G with a dominating vertex is spy-good.

Proof. We present a strategy for $\mathcal{S}$ to win $\mathcal{RS}(G, m, r, s)$ when $s \geq \lfloor r/m \rfloor$. Let v be a dominating vertex in G. Player $\mathcal{S}$ always keeps a single spy on every meeting at a vertex in $V(G) - \{v\}$ and keeps the rest of his spies on v. Call such a position *stable*. Since $s \geq \lfloor r/m \rfloor$, player $\mathcal{S}$ can start with a stable position. If $\mathcal{S}$ can reach a stable position at the end of each round, then the revolutionaries never win.

Suppose $\mathcal{S}$ has achieved a stable position at the end of some round. We will show that he can reach a stable position at the end of the next round. A meeting is *normal* if it is located on a vertex other than v. Assume there are k normal meetings at the end of round t, name these meetings $\mu_1, \ldots, \mu_k$. Call exactly m revolutionaries from each of these meetings as *bound*, the other revolutionaries are *free*. There is one bound spy at each meeting μ_i for $1 \leq i \leq k$ and the rest of the spies are on v. Name these $s-k$ spies $\sigma_1, \ldots, \sigma_{s-k}$. Now at the beginning of round $t+1$ revolutionaries move; let the number of normal meetings be k' after this move. Name these meetings $\nu_1, \ldots, \nu_{k'}$. Let $X = \{\mu_1, \ldots, \mu_k\}$, $X' = \{\sigma_1, \ldots, \sigma_{s-k}\}$ and $Y = \{\nu_1, \ldots, \nu_{k'}\}$. Let H be a bipartite graph with partite sets $X \cup X'$ and Y. We make all X' adjacent to all Y in H reflecting the fact that every spy in X' can move to every meeting in Y. We make ν_i and μ_j adjacent if and only if at least one bound revolutionary of μ_j is inside the meeting ν_i. If H admits a matching that covers Y, then in response to revolutionaries moves in round $t+1$ corresponding spies can travel along the edges of the matching to cover meetings in Y; the rest of the spies will move to v.

We can see by Hall's Theorem that H actually contains such a matching. Let A be a subset of Y. Since X' is adjacent to all Y, we have $|N_H(A)| = |X'| + |N_H(A) \cap X|$. We have $r - km$ free revolutionaries and the number of revolutionaries in the meeting of A is at least $m|A|$. Therefore at least $m|A| - (r - km)$ bound revolutionaries are in the meetings of A. Those revolutionaries have traveled from a meeting in X, and since each meeting in X contains exactly m of them, we have $|N_H(A) \cap X| \geq \lceil (m|A| - (r - km))/m \rceil = |A| + k - \lfloor r/m \rfloor$. Since $s \geq \lfloor r/m \rfloor$, we have

$$|N_H(A)| = |X'| + |N_H(A) \cap X| = (s - k) + (|A| + k - \lfloor r/m \rfloor) \geq |A|.$$

Hence H satisfies Hall's criterion and contains a matching that saturates Y. □

Corollary 4.1.2. Player $\mathcal{S}$ wins $\mathcal{RS}(G, m, r, s)$ if $s \geq \gamma(G)\lfloor r/m \rfloor$.

Proof. Let D be a minimum dominating set of G. For every vertex $v \in D$, let $G_v = G[N_G[v]]$. Player $\mathcal{S}$ will assign $\lfloor r/m \rfloor$ spies to each G_v. Spies assigned to each G_v will play according to some optimal strategy for $\mathcal{RS}(G_v, m, r, \lfloor r/m \rfloor)$; whenever a revolutionary leaves subgraph G_v spies will imagine that the revolutionary has moved to v and whenever that revolutionary comes back to some vertex u of G_v, spies will imagine that it is moved from v to u. Therefore spies assigned to each G_v can ensure that there is no meeting inside G_v. Since $\cup_v G_v = G$, spies will prevent a meeting in entire G. □

Corollary 4.1.3. For $0 \leq i \leq \binom{n}{2}$ and any meeting size m, there exists a spy-good n-vertex graph G_i with i edges.

Proof. For $i < n-1$, let G_i be the disjoint union of a star with i edges and $n-i-1$ singleton vertices. Player $\mathcal{S}$ has to put a spy on a singleton vertex if and only if m revolutionaries choose to stay on that vertex in the first round (and hence that spy and revolutionaries will not move forever). Therefore if $\mathcal{R}$ puts r' revolutionaries on the star, then $\mathcal{S}$ has at least $\lfloor r'/m \rfloor$ spies left to use on the star after he covers all the meetings formed on the singleton vertices. Hence by Lemma 4.1.1 spies can prevent a meeting on the star forever.

For $i \geq n-1$, let G_i be a connected graph with a dominating vertex and i edges. Lemma 4.1.1 implies that G_i is spy-good. □

4.2 The Game on a Large Bipartite Graph

An *m-large complete bipartite graph* is a complete bipartite graph with at least m vertices in each of its partite sets. In this section we consider the game played on an $r+s$-large complete bipartite graph. The point is that each part is large enough such that each revolutionary and spy can be placed on distinct vertices in each of the partite sets, if necessary. We will refer to the partite sets of such a graph as part 1 and part 2. We consider the game for $m = 2$ and $m = 3$ separately.

Lemma 4.2.1. Let G be an $r+s$-large complete bipartite graph. Player $\mathcal{R}$ wins $\mathcal{RS}(G, 2, r, s)$ if $s < \lfloor 7r/2 \rfloor/5 - 3/5$.

Proof. We present a strategy for Player $\mathcal{R}$ to win. Player $\mathcal{R}$ puts all his r revolutionaries on distinct vertices in part 1. In response, player $\mathcal{S}$ has to put at least $\lfloor r/2 \rfloor$ spies in part 1, otherwise revolutionaries in part 1 could swarm towards part 2 and generate as much meetings as possible on spy-free vertices and there will not be enough spies in part 1 to follow them, therefore $\mathcal{R}$ will win easily in that case. In the next round $\mathcal{R}$ moves $\lfloor r/2 \rfloor$ revolutionaries from part 1 to part 2 to leave at least $\lfloor r/2 \rfloor$ spies in part 1 *lonely* (a spy is lonely if it is on a vertex with no revolutionaries). In response $\mathcal{S}$ moves his spies leaving s_i spies in part i (for $i \in \{1, 2\}$). Note that at least $\lfloor r/2 \rfloor - s_2$ spies are still lonely in part 1. Now it is again $\mathcal{R}$'s turn. If he makes his pieces swarm part 1, then he can generate at least $\lfloor \frac{r - s_1 + (\lfloor r/2 \rfloor - s_2)}{2} \rfloor$ meetings in that part; note that lonely spies in part 1 could not endanger any meeting. Therefore $\mathcal{R}$ wins unless there is enough spies in part 2 to cover these possible meetings, so we may assume $s_2 \geq \lfloor \frac{r - s_1 + (\lfloor r/2 \rfloor - s_2)}{2} \rfloor$. Similarly $\mathcal{R}$ wins by swarming his pieces to part 2 unless $s_1 \geq \lfloor \frac{r - s_2 - 1}{2} \rfloor$, so we may assume that is the case. Now adding the first inequality to twice the second inequality we have $5s = 5(s_1 + s_2) \geq \lfloor 7r/2 \rfloor - 3$. Hence $s \geq \lfloor 7r/2 \rfloor / 5 - 3/5$, which contradicts the assumption of this lemma. Therefore revolutionaries can always win in at most three rounds. □

Lemma 4.2.2. Let G be an $r+s$-large complete bipartite graph. Player $\mathcal{R}$ wins $\mathcal{RS}(G, m, r, s)$ if $s \leq \lfloor \lfloor r/2 \rfloor / \lceil m/3 \rceil \rfloor - 2$.

Proof. Player $\mathcal{R}$ places at least $\lfloor r/2 \rfloor$ revolutionaries on each part. He groups those $\lfloor r/2 \rfloor$ revolutionaries into colonies of size $\lceil m/3 \rceil$ and places these colonies on distinct vertices. Then $\mathcal{S}$ places s_j spies on part j for $j \in \{1, 2\}$. Note that he can make $\lfloor \lfloor r/2 \rfloor / \lceil m/3 \rceil \rfloor$ colonies in each part; let $c = \lfloor \lfloor r/2 \rfloor / \lceil m/3 \rceil \rfloor$. By symmetry we may assume $s_1 \leq s_2$. Therefore at least $c - s_2$ colonies in part 2 remain unguarded. At the next round $\mathcal{R}$ moves all his colonies in part 1 to form as much as meetings as possible in part 2. We consider two cases.

When $\lfloor c/2 \rfloor < c - s_2$ (that is $s_2 < \lceil c/2 \rceil$). Here every two colonies in part 1 could be merge with one of the unguarded colonies of part 2 to form a meeting of size $3\lceil m/3 \rceil \geq m$. Therefore $\mathcal{R}$ makes at least $\lfloor c/2 \rfloor$ colonies. Now $\mathcal{R}$ wins unless $s_1 \geq \lfloor c/2 \rfloor$, thus we may assume so. Since $\lfloor c/2 \rfloor \leq s_1 \leq s_2 < \lceil c/2 \rceil$, we have $s_1 = s_2 = \lfloor c/2 \rfloor$ which contradicts $s_1 + s_2 = s \leq c - 2$.

When $\lfloor c/2 \rfloor \geq c - s_2$ (that is $s_2 \geq \lceil c/2 \rceil$). Here $\mathcal{R}$ can make a meeting at all of those $c - s_2$ unguarded colonies in part 2, and he will still have $\lfloor c/2 \rfloor - (c - s_2)$ extra colonies from part 1 that he will use for making new meetings at some empty vertices of part 2. Using those extra colonies, he can form at least $\lfloor 2\lceil m/3 \rceil (s_2 - \lceil c/2 \rceil)/m \rfloor$ meetings. So $\mathcal{R}$ can make at least $c - s_2 + \lfloor 2\lceil m/3 \rceil (s_2 - \lceil c/2 \rceil)/m \rfloor$ meetings in part 2. Player $\mathcal{R}$ wins unless s_1 is at least that quantity, that is

$$\begin{aligned} s_1 = s - s_2 &\geq c - s_2 + \lfloor 2\lceil m/3 \rceil (s_2 - \lceil c/2 \rceil)/m \rfloor \geq c - s_2 + \lfloor (2/3)(s_2 - \lceil c/2 \rceil) \rfloor \\ &\geq c - s_2 + (2/3)(s_2 - \lceil c/2 \rceil) - (2/3) \geq c - s_2/3 - (2/3)\lceil c/2 \rceil - (2/3). \end{aligned}$$

Hence $s-(2/3)s_2 \geq c-(2/3)\lceil c/2\rceil-(2/3)$. Since $s_2 \geq s/2$, we have $s \geq (3/2)c-\lceil c/2\rceil-1 = c+(c/2-\lceil c/2\rceil)-1 \geq c-3/2$. This contradicts $s \leq c-2$. □

In the following lemmas we repeatedly use these notations: Any statement that includes index j is considered true for $j \in \{1,2\}$. The number of revolutionaries and spies in part j at the end of round t is denoted by r_j and s_j, respectively. At the end of round t, the number of revolutionaries in part j that are located on a vertex guarded with a spy is denoted by c_j. Notations r'_j, s'_j and c'_j denote the similar quantities at the end of round $t+1$. A spy on a vertex in part j is *new* if in the previous round that spy was not in part j; a spy is *old* if it is not new. A meeting located on a vertex in part j is *new* if in the previous round there was no meeting on that vertex; a meeting is *old* if it is not new. Revolutionaries *swarm* a part j in a round, if they all move towards part j in that round.

Lemma 4.2.3. Let G be an $r+s$-large complete bipartite graph. For $r \geq 11$, player $\mathcal{S}$ wins $\mathcal{RS}(G,2,r,s)$ if $s \geq \lceil 7r/10\rceil$.

Proof. Let $e=\lceil r/5\rceil$ if r is even, otherwise $e=\lceil (r+2)/5\rceil$. We present a winning strategy for player $\mathcal{S}$ using $\lfloor r/2\rfloor+e$ spies. Since $\lceil 7r/10\rceil \geq \lfloor r/2\rfloor+e$, we may assume $s=\lfloor r/2\rfloor+e$.

Note that if revolutionaries swarm part $3-j$, then they can generate at most $\min\{r_j, \lfloor\frac{r-c_{3-j}}{2}\rfloor\}$ uncovered meetings. The spy strategy will ensure that always $s_j \geq \min\{r_j, \lfloor\frac{r-c_{3-j}}{2}\rfloor\}$, hence all the new meetings formed in part $3-j$ at round $t+1$ could be potentially covered by the spies in part j at the end of that round.

At each round, in response to moves of player $\mathcal{R}$, player $\mathcal{S}$ moves x_j spies from part j to part $3-j$. More specifically, he first picks x_j spies from part j so that he uncovers as few revolutionaries as possible. Afterwards he places those x_j spies in part $3-j$ such that together with spies currently in that part, they cover as much revolutionaries as possible. To finish describing the spy strategy we should show how to compute x_js.

Player $\mathcal{S}$ considers three main cases in order for computing x_js.

Case 1: If $r'_i \leq e$ for some i, then $x_{3-i}=e$ and $x_i=s_i$.

Case 2: If $s_i \geq \min\{r'_{3-i}, 2s-r\}$, then $x_i=\min\{r'_{3-i}, 2s-r\}$ and $x_{3-i}=s_{3-i}$.

Case 3: Otherwise $x_i=s_i$ and $x_{3-i}=s_{3-i}$.

We claim that if $\mathcal{S}$ plays with this strategy, then at the end of each round these invariants are satisfied:

(A) $s_i \geq e$ for $i \in \{1,2\}$.

(B) $s_i \geq \min\{r_i, \lfloor\frac{r-c_{3-i}}{2}\rfloor\}$ for $i \in \{1,2\}$.

(C) All meetings are covered.

Note that invariant (C) implies that spies will prevent an uncovered meeting forever, hence we just need to prove that those invariants are maintained. Trivially player $\mathcal{S}$ is able to satisfy the invariants at the round 0. Now assuming that the invariants are satisfied at round t, we will show that they continue to hold at round $t+1$.

Invariant (A) is preserved. If $\mathcal{S}$ plays by Case 1, then $s'_i = e$ and $s'_{3-i} = \lfloor r/2 \rfloor$, hence (A) is true. If $\mathcal{S}$ plays by Case 2 or Case 3 then $r'_i, r'_{3-i} > e$. If $\mathcal{S}$ plays by Case 2, then $s'_{3-i} = \min\{r'_{3-i}, 2s-r\} \geq r'_{3-i} > e$ and $s'_i = s - s'_{3-i} = s - \min\{r'_{3-i}, 2s-r\} \geq s-(2s-r) = r - s = \lceil r/2 \rceil - e \geq e$ (the last inequality is true for $r \geq 11$). Finally if $\mathcal{S}$ plays by Case 3, then we have $s'_j = s_{3-j} \geq e$.

Invariant (B) is preserved. Again if $\mathcal{S}$ plays by Case 1, then $s'_i = e \geq r'_i \geq \min\{r'_i, \lfloor\frac{r-c'_{3-i}}{2}\rfloor\}$ and $s'_{3-i} = \lfloor r/2 \rfloor \geq \lfloor\frac{r-c'_{3-i}}{2}\rfloor \geq \min\{r'_i, \lfloor\frac{r-c'_{3-i}}{2}\rfloor\}$. If $\mathcal{S}$ plays by Case 2 or Case 3 then $r'_i, r'_{3-i} > e$. Assume $\mathcal{S}$ plays by Case 2, we first show that (B) is true for part $3-i$. We have $s'_{3-i} = \min\{r'_{3-i}, 2s-r\}$ so if $s'_{3-i} = r'_{3-i}$, then (B) is trivially true for part $3-i$, hence we may assume $s'_{3-i} = 2s-r$. Since $\mathcal{S}$ has moved x_{3-i} spies to part i and $x_{3-i} = s_{3-i} \geq e$ and since $r'_i > e$, we know that $c'_i \geq e$. Therefore $\min\{r'_{3-i}, \lfloor\frac{r-c'_i}{2}\rfloor\} \leq \lfloor\frac{r-c'_i}{2}\rfloor \leq \lfloor\frac{r-e}{2}\rfloor \leq 2s-r = s'_{3-i}$.

Now we show that (B) is true for part i. Since player $\mathcal{S}$ has moved x_i spies to part $3-i$ and $x_i \leq r'_{3-i}$, we have $c'_{3-i} \geq x_i = s'_{3-i}$. Since $s'_{3-i} \leq 2s-r$, we have $r \leq 2s - s'_{3-i}$, hence $\min\{r'_i, \lfloor\frac{r-c'_{3-i}}{2}\rfloor\} \leq \lfloor\frac{r-c'_{3-i}}{2}\rfloor \leq \lfloor\frac{2s-s'_{3-i}-s'_{3-i}}{2}\rfloor = s - s'_{3-i} = s_i$.

Finally if $\mathcal{S}$ plays by Case 3, then $s'_j \leq r'_j$ and $s'_j \leq 2s-r$. Since all spies move in this case and $s'_j \leq r'_j$, we have $c'_j \geq s'_j$. Moreover $r \leq 2s - s'_j$. Therefore similar to the previous case we have $\min\{r'_j, \lfloor\frac{r-c'_{3-j}}{2}\rfloor\} \leq \lfloor\frac{r-c'_{3-j}}{2}\rfloor \leq \lfloor\frac{2s-s'_{3-j}-s'_{3-j}}{2}\rfloor = s - s'_{3-j} = s_j$.

Invariant (C) is preserved. We have $r_j \geq c_j$. Hence by invariant (B) we have $s_j \geq \lfloor\frac{r-c_{3-j}}{2}\rfloor = \lfloor\frac{r_j+(r_{3-j}-c_{3-j})}{2}\rfloor \geq \lfloor\frac{r_j}{2}\rfloor$. Therefore at the end of each round the number of spies in each part is no fewer than the number of meetings in that part. We observe that at each round player $\mathcal{S}$ sends all the spies of one part to the other part. Thus if all the spies that $\mathcal{S}$ leaves on part j are sent from part $3-j$, all the meetings in part j will be covered since those spies will choose their locations so as to maximize their coverage. If all the s'_j spies that end up in part j are not sent from part $3-j$, then all the spies which were in part $3-j$ are sent to part j. By invariant (B) all the spies which were in part $3-j$ suffice to cover all the meetings on the uncovered vertices in part j. Hence if there is an uncovered meeting in part j after spies move, it is located on some previously covered vertex u in that part. This implies that a spy has left vertex u and since spies are chosen to leave so as to uncover as little revolutionaries as possible, it implies that all the old spies that are not chosen to leave part j are covering a meeting. Since a meeting on vertex u is uncovered, the spies migrated to part j have not chosen to cover u which means that all of them are covering other meetings in part j. Hence all the spies in part j are covering meetings and a meeting on u is uncovered. This is a contradiction since we know that the number of spies in each part is no fewer than the number of meetings. □

We say that player $\mathcal{S}$ uses a *greedy migration strategy* if his strategy is the following. At round $t+1$ after $\mathcal{R}$ moves revolutionaries, $\mathcal{S}$ computes his desired number of spies to place on part j, s'_j. By symmetry we may assume $s'_1 \leq s_2$. He removes s'_1 spies from part 2 such that the total number of revolutionaries that are uncovered is as few as possible. Afterwards he moves all spies from part 1 to part 2 and place them such that together with the spies that are already there they cover as much revolutionaries as possible. Finally he places s'_1 revolutionaries on part 1 so as to maximize coverage.

Note that if $\mathcal{S}$ uses a greedy migration strategy, then at the end of each round either all spies on part j are new, or all spies that were on part $3-j$ in the previous round have migrated to part j.

Lemma 4.2.4. Let G be an $r+s$-large complete bipartite graph. If $\mathcal{S}$ has a greedy migration strategy in $\mathcal{RS}(G, m, r, s)$ to prevent revolutionaries win by swarming a part, then $\mathcal{S}$ in fact wins the game using that strategy.

Proof. Since the r_j revolutionaries on part j can swarm part $3-j$ to form at least $\lfloor r_j/m \rfloor$ new meetings, the greedy migration strategy always satisfies $s_j \geq \lfloor r_j/m \rfloor$, which implies that if every spy on part j covers a meeting then all meetings in part j are covered. Hence if all the spies on part j are new, then since the greedy migration places them so as maximize coverage, all meetings in part j will be covered. So we may assume that all spies on part j are not new. Hence all the spies on part $3-j$ in the previous round, have migrated to part j. The number of those spies is at least the number of new meetings in part j, since otherwise revolutionaries can win by swarming part j. Hence if old meetings in part j continue to be covered by old spies, then all meetings in part j will be covered. So we may assume that one of the old spies leaves an old meeting in part j. Since the greedy migration picks leaving spies so as to minimize the number of uncovered revolutionaries, all remaining old spies on part j are covering meetings. All the new spies are placed on part j so as to maximize coverage. So if there is an uncovered meeting in part j at the end, then every spy on part j is covering a meeting. But since always $s_j \geq \lfloor r_j/m \rfloor$, this is impossible. □

Lemma 4.2.5. Let G be an $r+s$-large complete bipartite graph. Player $\mathcal{S}$ wins $\mathcal{RS}(G, 3, r, s)$ if $s \geq \lfloor r/2 \rfloor$.

Proof. We present a greedy migration strategy for spies that prevents revolutionaries win through swarming, which by Lemma 4.2.4 is also a winning strategy. Let u_j be the size of a largest colony of revolutionaries in part j that is not covered by a spy at the end of round t, and let u'_j be the similar quantity for round $t+1$. Note that as far as the game is not finished we have $u_j, u'_j \leq 2$. Let $e = \lfloor r/6 \rfloor$ and $d = s - \lfloor (r-e)/3 \rfloor$. Player $\mathcal{S}$ computes values s'_j, the number of spies that should be placed on part j at round $t+1$. Then he uses greedy migration to actually place s'_j spies on part j.

Case 1: If $r'_i \leq e$ for some $i \in \{1,2\}$, then $s'_i = e$ and $s'_{3-i} = s - s'_i$.

Case 2: If $e < r'_i \leq d$ for some $i \in \{1,2\}$, then $s'_i = r'_i$ and $s'_{3-i} = s - s'_i$.

Case 3: If $d < r'_i \leq 2d$ for some $i \in \{1,2\}$, then $s'_i = d$ and $s'_{3-i} = s - s'_i$.

Case 4: If $2d < r'_i \leq \lfloor r/2 \rfloor$ for some $i \in \{1,2\}$, then $s'_i = \lfloor r'_i/2 \rfloor$ and $s'_{3-i} = s - s'_i$.

Let $f(j) = \min\{\lfloor \frac{r-c_{3-j}}{3} \rfloor, \lfloor \frac{r_j}{3-u_{3-j}} \rfloor\}$. We prove the following two invariants. Since the number of new meetings in part j is at most $f(j)$, invariant (B) implies that revolutionaries cannot win by swarming part j.

(A) $s_i \geq e$ for $i \in \{1,2\}$.

(B) $s_i \geq f(i)$ for $i \in \{1,2\}$.

Trivially $\mathcal{S}$ is able to satisfy the invariants at round 0. Now assuming that they are satisfied at round t, we prove that they will be preserved at round $t+1$. Let $f'(j) = \min\{\lfloor \frac{r-c'_{3-j}}{3} \rfloor, \lfloor \frac{r'_j}{3-u'_{3-j}} \rfloor\}$. Note that the strategy moves at least e spies from each part at each round. Since in Cases 2, 3 and 4 each part contains at least e revolutionaries, we have $c'_j \geq e$ in those cases. Also note that by the strategy $s'_j \geq \lfloor r'_j/3 \rfloor$. Hence if $u'_{3-j} = 0$, then $s'_j \geq \lfloor r'_j/3 \rfloor = \lfloor r'_j/(3-u'_{3-j}) \rfloor \geq f'(j)$, that is invariant (B) holds in that case. So we may assume $u'_j \in \{1,2\}$. Finally since the strategy moves at least e spies to part j and places them so as to maximize the coverage, and since there is a colony of uncovered revolutionaries in part j with size u'_j at the end of round $t+1$, all of those e spies are covering colonies of size at least u'_j, hence $c'_j \geq u'_j e$.

Invariant (A) is preserved. Since the four cases in the strategy put at least e and at most $\lceil \lfloor r/2 \rfloor /2 \rceil$ spies on side i, this invariant holds.

Invariant (B) is preserved. If $\mathcal{S}$ plays by Case 1, then $s'_i = e \geq r'_i \geq f'(i)$ and $s'_{3-i} = s - e \geq \lfloor r/3 \rfloor \geq f'(3-i)$.

If $\mathcal{S}$ plays by Case 2, then $s'_i = r'_i \geq f'(i)$. Moreover $c'_i \geq e$ and $s'_{3-i} = s - r'_i \geq s - d = \lfloor (r-e)/3 \rfloor \geq \lfloor (r-c'_i)/3 \rfloor \geq f'(3-i)$.

If $\mathcal{S}$ plays by Case 3, then again $c'_i \geq e$ and $s'_{3-i} = s - d = \lfloor (r-e)/3 \rfloor \geq \lfloor (r-c'_i)/3 \rfloor \geq f'(3-i)$.

If $\mathcal{S}$ plays by Case 3 or Case 4, then $s'_i \geq \lfloor r'_i/2 \rfloor$. If $u'_{3-i} = 1$, then $s'_i \geq \lfloor r'_i/2 \rfloor = \lfloor r'_i/(3-u'_{3-i}) \rfloor \geq f'(i)$. If $u'_{3-i} = 2$, then $c'_{3-i} \geq 2e$. Hence $s'_i \geq \lfloor (r-2e)/3 \rfloor \geq \lfloor (r-c'_{3-i})/3 \rfloor \geq f'(i)$.

If $\mathcal{S}$ plays by Case 4, then $s'_{3-i} = s - \lfloor r'_i/2 \rfloor$. If $u'_i = 1$, then $s'_{3-i} = s - \lfloor r'_i/2 \rfloor \geq \lfloor r'_{3-i}/2 \rfloor = \lfloor r'_{3-i}/(3-u'_i) \rfloor \geq f'(3-i)$. If $u'_i = 2$, then $c'_i \geq 2e$. Hence $s'_{3-i} = s - \lfloor r'_i/2 \rfloor \geq \lfloor r/2 \rfloor - \lfloor \lfloor r/2 \rfloor /2 \rfloor = \lceil \lfloor r/2 \rfloor /2 \rceil \geq \lfloor (r-2e)/3 \rfloor \geq \lfloor (r-c'_i)/3 \rfloor \geq f'(3-i)$.

□

Lemma 4.2.6. Let G be an $r{+}s$-large complete bipartite graph. Player $\mathcal{S}$ wins $\mathcal{RS}(G, m, r, s)$ with $\sqrt{3}/(\sqrt{3}-1) \leq r/m$ if

$$s \geq (1 + \frac{1}{\sqrt{3}})\frac{r}{m} + 1.$$

Proof. Again we present a greedy migration strategy for $\mathcal{S}$ that prevents revolutionaries win through swarming a part. After revolutionaries move in round $t+1$, player $\mathcal{S}$ computes four numbers x, α, u_1 and u_2 (not necessarily integers) such that

$$x \leq \lfloor r/m \rfloor, \quad x + r/m + 1 \leq s, \text{ and} \tag{4.1}$$

$$\alpha = x + r/m - \frac{r - u_1 x}{m} = x + r/m - \frac{r'_2}{m - u_1} = \frac{r'_1}{m - u_2} = \frac{r - u_2 x}{m}. \tag{4.2}$$

We will show that he is always able to find such numbers. Now numbers s'_j are computed according to the following cases.

Case 1: If $\alpha \leq x$, then $s'_1 = \lceil x \rceil$ and $s'_2 = s - s'_1$.

Case 2: If $\alpha > \lfloor r/m \rfloor$, then $s'_1 = \lfloor r/m \rfloor$ and $s'_2 = s - s'_1$.

Case 3: If $x < \alpha \leq \lfloor r/m \rfloor$, then $s'_1 = \lceil \alpha \rceil$ and $s'_2 = s - s'_1$.

Note that this strategy guarantees $s'_j \geq x$, hence the greedy migration places at least $\lceil x \rceil$ new spies in each part at each round. Now we show that revolutionaries cannot win by swarming a part at the end of round $t+1$. If at the end of round $t+1$, the size of a largest uncovered colony of revolutionaries on part j is at most u_j, then a swarm towards part j in the next round will generate at most $r'_{3-j}/(m - u_j)$ new meetings. On the other hand if the size of a largest uncovered colony on part j is more than u_j, since part j includes at least

x new spies who have not selected to cover that colony (and so by the greedy choice, each of them has covered at least than u_j revolutionaries), then a swarm towards part j forms at most $(r - u_j x)/m$ new meetings. Hence swarming part j in round $t + 2$ is ineffective if $s'_{3-j} \geq \max\{r'_{3-j}/(m - u_j), (r - u_j x)/m\}$. This condition is equivalent to $s'_1 \geq \alpha$ for $j = 2$ and equivalent to $s'_2 \geq x + r/m - \alpha$ for $j = 1$. Since $s - 1 \geq x + r/m$, it is enough to prove $s'_2 \geq s - 1 - \alpha$ to show that swarming part 1 is ineffective.

If $\mathcal{S}$ plays by Case 1, then $s'_1 = \lceil x \rceil$ and $s'_2 > r/m$. Here $s'_1 \geq x \geq \alpha$, hence swarming part 2 is ineffective. Swarming part 1 is also ineffective since the total number of meetings is always at most $\lfloor r/m \rfloor$.

If $\mathcal{S}$ plays by Case 2, then $s'_1 = \lfloor r/m \rfloor < \alpha$ and $s'_2 = s - s'_1$. Swarming part 2 is again ineffective since the total number of meetings is always at most $\lfloor r/m \rfloor$. We have $s'_2 = s - s'_1 > s - \alpha \geq s - 1 - \alpha$, hence swarming part 2 is ineffective too.

Finally if $\mathcal{S}$ plays by Case 3, then $s'_1 = \lceil \alpha \rceil$ and $s'_2 = s - s'_1$. Thus $s'_1 \geq \alpha$ and swarming part 2 is ineffective. We have $s'_2 = s - s'_1 = s - \lceil \alpha \rceil \geq s - \alpha - 1$, so swarming part 1 is also ineffective.

It remains to show that such numbers exist. Solving (4.2) one obtains

$$
\begin{aligned}
x &= \frac{\sqrt{9r^2 + 12r'_1 r - 12{r'_1}^2}}{6m}, \\
u_1 &= \frac{r + mx - \sqrt{r^2 + 2rxm + x^2m^2 - 4xr'_1 m}}{2x}, \text{ and} \\
u_2 &= \frac{r + mx - \sqrt{r^2 - 2rxm + x^2m^2 + 4xr'_1 m}}{2x}.
\end{aligned}
$$

We have $x \leq r/(\sqrt{3}m)$. Hence for $\sqrt{3}/(\sqrt{3}-1) \leq r/m$, the inequalities in (4.1) are true. □

4.3 The Game with the Initial Positioning Given

In this section we consider a version of the game of revolutionaries and spies in which the initial positioning of the pieces on the game board (i.e. the graph) is given. We show that the problem of deciding who wins the new game is NP-Hard.

We reduce from 3-SAT. Let ϕ be an instance of 3-SAT with k clauses $c_1, \ldots, c_k$ over p binary variables $x_1, \ldots, x_p$. Construct graph G_ϕ on the vertex set $\{x_1, y_1, \neg x_1, \ldots, x_k, y_k, \neg x_k\} \cup \{c_1, \ldots, c_p\} \cup \{u, u_1, u_2, v, v_1, v_2, w\}$. For $i \in \{1, 2\}$, make v_i adjacent to u_i and $c_1, \ldots, c_p$ (see Figure 4.1). For $i \in \{1, \ldots, p\}$, make c_i adjacent to the three literals composing it. For $i \in \{1, \ldots, k\}$, make y_i adjacent to both x_i and $\neg x_i$. Make w adjacent to all vertices other than itself and u. Finally make u and v adjacent. The description of G_ϕ is complete. At the start of the game we put a single spy on each vertex y_i for $i \in \{1, \ldots, k\}$. Additionally we place $2k + p + 7$ spies on vertex u.

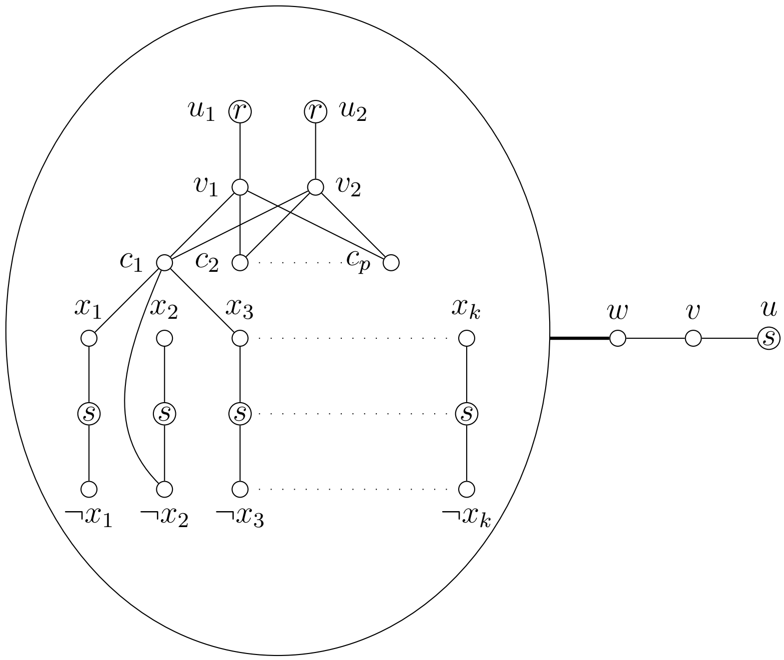


Figure 4.1: $c_1 = x_1 \vee \neg x_2 \vee x_3$.

We consider the game played on G_ϕ with the specified initial settings and meeting size 2 (i.e. $m = 2$). The total number of spies is equal to the number of vertices. The spies on u can move to w and from w to all the other vertices in three rounds, hence if revolutionaries win the game, then they will win it in at most two rounds. There is no way that the two revolutionaries, which are located on vertices u_1 and u_2, can create a meeting in one round. They can create a meeting on one of the vertices c_1,...,c_p at the end of round 2. Spies can successfully cover such a meeting if and only if they can move such that at the end of round 1 they cover at least one literal of each vertex c_i. Such a coverage is equivalent to a truth assignment that satisfies ϕ. Hence spies win the game if and only if ϕ is satisfiable.

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