SAMPLING ERROR OF THE SUPREMUM OF A LÉVY PROCESS

BY

AO CHEN

DISSERTATION

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Doctoral Committee:

Associate Professor Robert Bauer, Chair
Professor Richard Sowers, Contingent Chair
Professor Renming Song, Co-Director of Research
Assistant Professor Liming Feng, Co-Director of Research

Abstract

This thesis is to study the expected difference of the continuous supremum and discrete maximum of a Lévy process that is often used in finance. We will show that the expected difference is a quantity that highly depends on the variational property of the underlying Lévy process. Two techniques are used with respect to the cases of the complexity of the transition density function of the underlying Lévy process. In particular, we discuss the cases of Merton's jump diffusion, compound Poisson with normal jumps, normal inverse Gaussian process, variance gamma process, Kou's jump diffusion and (symmetric) stable process. A general result on the upper bound estimate for the expected difference is also shown.

To my Father Qifeng Chen,

my Mother Lanying Zhu,

and my Wife Yiyi Huang,

who are always supportive.

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List of Symbols

$\mathbb{R}$	Set of all real numbers
$\mathbb{C}$	Set of all complex numbers
$\mathbb{P}$	Probability measure
Ω	Sample space
$\mathcal{F}$	σ algebra on Ω
X_t or L_t	Lévy process
Π	Lévy measure
B_t	Brownian motion
N_t	Poisson process
Γ	Gamma function
ζ	Riemann-zeta function
$\zeta(z, q)$	Hurwitz-zeta function
B_i	ith Bernoulli number
$B_i(x)$	ith Bernoulli polynomial
$\mathrm{Erf}(x)$	Error function
$\mathrm{Erfc}(x)$	Complementary error function
$\lfloor x \rfloor$	The largest integer that is smaller or equal to x
$K_\alpha(x)$	Modified Bessel function of the second kind of index α
$\mathscr{H}$	Hilbert transform
$\mathscr{F}$	Fourier transform

Chapter 1

Introduction

1.1 Lévy Process and its Supremum

A Lévy process L_t is a stochastic process which admits independent and stationary increments. It has been widely used in numerous areas such as finance, economics, communications, biology, physics, fluid dynamics, quantum dynamics, thermodynamics, etc. For some of these applications, one may refer to the books [7, 16, 60, 61], references therein and numerous papers and journals.

Among subjects related to a Lévy process and its fluctuation theory, the extrema (supremum $\sup_{0\leq s\leq t} L_t$ and infimum $\inf_{0\leq s\leq t} L_t$) of the Lévy process turns out to be one of the central topics and have quite many applications. For instance, in queuing theory, the storage level (or a water level in a dam) can be expressed as the running supremum. An insurance company will be very much interested in the distributional property of the first passage time of its entire capital (reserve) across a certain threshold (usually zero), which is closely related to the infimum of the underlying process. While in finance, the pricing of certain exotic options (e.g. lookback and barrier options) highly depends on the running supremum of the process that drives the dynamics of the asset price.

In the last few decades, many identities involving the running supremum of a Lévy process have been established during the development of the fluctuation theory for Lévy process. The most well known one is the Wiener-Hopf factorization identity. However, the Wiener-Hopf factors are only known explicitly in very few special cases, such as Brownian motion, spectrally negative (positive) Lévy processes. Hence, people have been trying to identify certain classes of Lévy processes for which the Wiener-Hopf factors can be represented in some explicit way. Mordecki [56] and Pistorius [58] studied Lévy processes with positive jumps of phase type. Lewis and Mordecki [52] studied Lévy processes with positive jumps which have a rational Fourier transform. Most recently, Jeannin and Pistorius [39] discussed (generalized) hyper-exponential Lévy processes, and Kuznetsov [46, 47], Kuznetsov, Kyprianou and Pardo [49] dealt with β-family, θ-family and meromorphic Lévy processes. As for the stable processes, Doney [21] and Kuznetsov [48] gave almost complete descriptions for the Wiener-Hopf factors. In addition, Kwasnicki, Malecki and Ryznar [50] developed an estimate of the cumulative distribution functions and some distributional properties of the supremum under certain conditions.

1.2 Random Walk and its Maximum

As early as in 1956, Spitzer [63] studied the maximum of a random walk S_n through an innovative combinatorial argument. The limiting behavior of the S_n as n tends to infinity has been an important topic in the research of random walk theory. A lot of people have made contributions in this area, for instance, Veraverbeke and Teugels [67], Veraverbeke [66], Alam [3], Klass [41], Embrechts and Veraverbeke [24], Grübel [30, 31], Bertoin and Doney [9], Korshunov [43], Blanchet and Glynn [36], Hansen [33], Janssen and VanLeeuwaarden [38], Foss, Konstantopoulos and Zachary[28], etc. A random walk is closely related to a Lévy process, since if we evenly discretize a Lévy process on the time horizon, we will get a random walk. However, due to the fact that we usually study a Lévy process on a finite time interval in practice, the Lévy increment with n and m ($m \neq n$) dividing points would generally have different distributions.

1.3 Connection between Supremum and Maximum

The connection between continuous supremum and discrete maximum has been an interesting subject in both theory and practice. In Monte Carlo simulation of a Lévy process, we usually use a random walk to simulate a Lévy process. Therefore, the supremum of the simulated sample path is nothing but the discrete maximum. So the correction between the maximum and the supremum may be used to achieve the real supremum. One well known result about the simulation error is Asmussen, Glynn and Pitman [5], in which they discussed the discretization error of a one-dimensional reflected Brownian motion. They also derived several important asymptotic results that have been used in many literatures later. For the drifted Brownian motion case, Janssen and Van Leeuwaarden [37] developed a full expansion of the expected difference between continuous supremum and discrete maximum. To the best of our knowledge, no full expansion for the expected difference has been shown for other Lévy process, especially pure jump Lévy processes. Also, Szimayer and Maller [65] collected the first jumps with truncations in space in each interval to approximate a pure jump Lévy process and derived several upper bounds, which could be used to control the difference of continuous and discrete suprema.

In the finance literature, there are quite a few papers on the option pricing under discrete monitoring, especially path-dependent options since the supremum of the underlying asset plays a crucial role on the option price. Broadie, Glasserman and Kou [12] [13] proposed a continuity correction for discretely monitored barrier option and lookback option in the Black-Scholes-Merton model, respectively, both of which are widely used in practice. Borovkov and Novikov [11] developed a new approach to compute the discretely monitored lookback option by integrating the moment generate function with a certain weight, in which they illustrated their approach with the classical Black-Scholes-Merton and variance gamma models. More generally, Petrella and Kou [57] proposed a scheme using Laplace transform that allows to compute the price and hedging parameters of discrete lookback and barrier options at any point in time. Broadie and Yamamoto [14] used a fast Gaussian transform method under the assumption of Black-Scholes-Merton model, while in the general Lévy model, Feng and Linetsky [27] developed a fast computational method based on Hilbert transform. Howison and Steinberg [34] [35] used a perturbation method to get accurate approximations for discrete barrier and Bermudan options. Very recently, Dia and Lamberton [20] generalized Broadie, Glasserman and Kou's result into the jump diffusion model. For a more detailed summary of discrete barrier and lookback options, see Kou [45].

1.4 Setup and Main Results

We consider a general one dimensional Lévy process L_t defined on a finite time horizon $[0, T]$ (without loss of generality, we set $T = 1$ in some cases) and the corresponding supremum $\sup_{0 \le t \le T} L_t$. If we discretize the time horizon into N subintervals of equal length and make the sample path piecewise constant in each subinterval, then it is clear that the discrete maximum converges in mean to the original continuous supremum as N tends to infinity. For different Lévy processes, the convergence rates are different with respect to N. The most well known case is the (drifted) Brownian motion, which admits a convergence rate of leading order $\frac{1}{\sqrt{N}}$, see e.g. Asmussen, Glynn and Pitman [5], Janssen and Van Leeuwaarden [37]. However, if we consider the pure jump cases, the convergence rates might be better than that of the Brownian motion. In this thesis, our main goal is to find the expected difference Δ_N of the continuous supremum and discrete maximum for the Lévy processes widely used in applications, especially in finance. We will develop two techniques depending on the complexity of the transition density function of the underlying Lévy process. Specifically, we will analyze six concrete examples in details: Merton's jump diffusions, compound Poisson processes with normal jumps, NIGs (Normal Inverse Gaussian processes), VGs (Variance Gamma processes), Kou's jump diffusions, (symmetric) α-stable processes with $1 < \alpha < 2$. For Merton's jump diffusions, compound Poisson processes with normal jumps, NIGs and symmetric α-stable processes, we can also derive full asymptotic expansions for the expected difference of the continuous supremum and discrete maximum. For a general Lévy process, we also derive an upper bound for Δ_N, which depends on the Blumenthal-Getoor index. Lastly, we provide a recursive algorithm for computing the discrete maximum using tools in Fourier analysis, and then we show some numerical results.

Chapter 2

Mathematical Background

2.1 Lévy Process and Related Properties

Definition 2.1.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. A cádlág, adapted, one-dimensional real-valued stochastic process L_t with $L_0 = 0$. a.s. is called a Lévy process if the following conditions are satisfied: L_t has

Independent increments: $L_t - L_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t$;

Stationary increments: $L_t - L_s \overset{d}{=} L_{t-s}$ for any $0 \leq s < t$.

One of the most important results in the theory of Lévy process is the celebrated Lévy-Khinchine formula, Bertoin [8]:

Theorem 2.1.2. (Lévy-Khinchine Formula) *For any real-valued Lévy process, we have the following representation:*

$$\mathbb{E}[e^{iuL_t}] = e^{t\psi(u)},$$

where

$$\psi(u) = iub - \frac{1}{2}u^2 c + \int_{\mathbb{R}\backslash\{0\}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|<1\}})\Pi(dx).$$

Here, $b \in \mathbb{R}$ *is called the linear coefficient,* $c \in \mathbb{R}^+$ *is called the Gaussian or diffusion coefficient, the Borel* σ*-finite measure* Π *on* $\mathbb{R}\backslash\{0\}$ *is called the Lévy measure which satisfies*

$$\int_{\mathbb{R}\backslash\{0\}} 1 \wedge x^2 \Pi(dx) < \infty.$$

We call (b, c, Π) *the Lévy triple, which uniquely (in distribution) determines the Lévy process.*

Another important result is the following Lévy-Itô Decomposition, (see, e.g. Bertoin [8] and Kyprianou [51]):

Theorem 2.1.3. (Lévy-Itô Decomposition) *Given any $b \in \mathbb{R}$, $c \in \mathbb{R}^+$, and a Borel σ-finite measure Π on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}\backslash\{0\}} 1 \wedge x^2 \Pi(dx) < \infty$, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which four independent Lévy processes $L^{(1)}$, $L^{(2)}$, $L^{(3)}$, and $L^{(4)}$ exist, i.e.*

$$L_t^{(1)} = bt \text{ with } \psi^{(1)}(u) = iub,$$

$$L_t^{(2)} = \sqrt{c} B_t \text{ with } \psi^{(2)}(u) = -\frac{1}{2}u^2 c,$$

$$L_t^{(3)} = J_t^b \text{ with } \psi^{(3)}(u) = \int_{|x| \geq 1} (e^{iux} - 1)\Pi(dx),$$

$$L_t^{(4)} = J_t^s \text{ with } \psi^{(4)}(u) = \int_{|x| < 1} (e^{iux} - 1 - iux)\Pi(dx).$$

Then, $L_t = L_t^{(1)} + L_t^{(2)} + L_t^{(3)} + L_t^{(4)}$ is a Lévy process with characteristic exponent $\psi(u) = iub - \frac{1}{2}u^2 c + \int_{\mathbb{R}\backslash\{0\}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|<1\}})\Pi(dx)$.

Remark 2.1.4. Actually, $L_t^{(3)}$ is a compound Poisson process and usually called the big jump part, while $L_t^{(4)}$ is a pure jump martingale and usually called the small jump part.

The next few theorems are about the path properties of a Lévy process, see e.g. Kyprianou [51].

Theorem 2.1.5. *Given a Lévy process L_t with Lévy triple (b, c, Π),*

(a) L_t is said to be of finite activity if $\Pi(\mathbb{R}) < \infty$, in which case almost all paths of L have a finite number of jumps in every compact interval.

(b) L_t is said to be of infinite activity if $\Pi(\mathbb{R}) = \infty$, in which case almost all paths of L have a infinite number of jumps in every compact interval.

Theorem 2.1.6. *Let L_t be a Lévy process with Lévy triple (b, c, Π),*

(a) Almost all sample paths of L have finite variation iff $c = 0$ and $\int_{|x| \leq 1} |x|\Pi(dx) < \infty$;

(b) Almost all sample paths of L have infinite variation iff $c \neq 0$ or $\int_{|x| \leq 1} |x|\Pi(dx) = \infty$.

Theorem 2.1.7. *Let L_t be a Lévy process with Lévy triple (b, c, Π),*

(a) $\mathbb{E}|L_t|^p < \infty$ iff $\int_{|x| \geq 1} |x|^p \Pi(dx) < \infty$, in which case we say L_t has finite p-th moment for $p > 0$;

(b) $\mathbb{E}|e^{pL_t}| < \infty$ iff $\int_{|x| \geq 1} e^{px}\Pi(dx) < \infty$, in which case we say L_t has finite p-th exponential moment for $p \in \mathbb{R}$.

For more details about Lévy processes and related properties, one may refer to Bertoin [8], Sato [59], Applebaum [4] and Kyprianou [51].

2.2 Setup

Now let $T \in (0, \infty)$, and we consider a Lévy process L_t defined on the interval $[0, T]$. We partition the interval $[0, T]$ into N equal subintervals and denote the dividing points as t_i's, where $0 \leq i \leq N$. For any given sample path of the Lévy process, the time-discretized version is defined as

$$L_t^D(N) := L_{t_{i-1}}, \text{ for any } t_{i-1} \leq t < t_i, 1 \leq i \leq N,$$

and

$$L_T^D(N) = L_{t_N} := L_T.$$

As mentioned in the introduction, we are concerned with the following continuous supremum and discrete maximum of L_t:

The continuous supremum:

$$\sup_{0 \leq t \leq T} L_t$$

The discrete maximum:

$$\sup_{0 \leq i \leq N} L_{t_i}$$

By our construction, it is clear that the continuous supremum is always no smaller than the discrete maximum. The main goal of the thesis is to investigate the expected difference Δ_N between them, i.e.

$$\Delta_N := \mathbb{E}[\sup_{0 \leq t \leq T} L_t - \max_{0 \leq i \leq N} L_{t_i}]$$

For illustration purpose in some cases and without loss of generality, we set $T = 1$. It is trivial to see that Δ_N tends to zero as N goes to infinity, however, the speed at which it tends to zero varies depending on the underlying Lévy process L_t. In this thesis, we will show that the quantity Δ_N is highly dependent on the variational property of L_t, i.e., Δ_N for those Lévy processes of infinite variation tends to zero slower than that for Lévy processes of finite variation. Before we analyze Δ_N in details, we first review the result for Δ_N in the case of drifted Brownian motion.

A (drifted) Brownian motion is a simple example of Lévy process, which admits continuous sample path almost surely. The following result is due to Janssen and Van Leeuwaarden [37].

Theorem 2.2.1. (Janssen and Leeuwaarden [37]): *Suppose that* $L_t = \mu t + B_t$. *Then for* $\frac{|\mu|}{\sqrt{N}} < 2\sqrt{\pi}$,

$$\begin{aligned}\Delta_N &= \mathbb{E}[\sup_{0\le t\le 1} L_t - \sup_{0\le i\le N} L_{t_i}] \\ &= -\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi N}} - \frac{2g(1)-\mu}{4N} - \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\frac{g^{(2k-1)}(1)}{N^{2k}} \\ &\quad - \frac{1}{\sqrt{2\pi N}} \sum_{r=0}^{\infty} \frac{\zeta(-\frac{1}{2}-r)(-\frac{1}{2})^r}{r!(2r+1)(2r+2)} \Big(\frac{\mu}{\sqrt{N}}\Big)^{2r+2} + O\Big(\frac{1}{N^{2p+2}}\Big),\end{aligned}$$

where ζ *is the Riemann Zeta function,* p *is some positive integer,* B_n *is the* n*th Bernoulli number,* $g(t) = \mu\Phi(\mu\sqrt{t}) + \frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2}\mu^2 t}$ *with* Φ *being the cumulative distribution function of the standard normal random variable,* $g^{(k)}$ *is the* k*th derivative of* g.

First, we note that the result can be easily generalized to the process $L_t = \mu t + \sigma B_t$ defined on $[0,T]$ for $0 < T < \infty$. Second, the leading coefficient is $-\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}} > 0$ (in the case of $L_t = \mu t + \sigma B_t$ defined on $[0,T]$, it is actually $-\frac{\sigma\zeta(\frac{1}{2})\sqrt{T}}{\sqrt{2\pi}}$) since $\zeta(\frac{1}{2}) \approx -1.46035451$. Thus, the positivity of the first term is consistent with the positivity of Δ_N. Moreover, in this case we see that the convergence rate of the discrete maximum to continuous supremum is of order $\frac{1}{\sqrt{N}}$, which is pretty slow due to the fact that Brownian motion is highly irregular and fluctuating.

However, we are more interested in more general Lévy processes, especially those processes with jumps. In chapter 3, we will investigate the following cases: Merton's jump diffusions, compound Poisson processes with normal jumps, Normal Inverse Gaussian processes (NIGs) and Variance Gamma processes (VGs). All of these have explicit transition density functions or transition laws.

Chapter 3

Sampling Error for Lévy Processes with Explicit Transition Density Function/Law

3.1 Merton's Jump Diffusion

Merton's jump diffusion model was first introduced in Merton [55]. It is the sum of a drifted Brownian motion and a compound Poisson process, i.e.,

$$X_t = \mu t + \sigma B_t + \sum_{n=1}^{N_t} Z_n,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, N_t is a Poisson process with intensity $\lambda > 0$, the jump size Z_n's, which can be realized as the arrivals of big news, are i.i.d normal random variables with mean m and variance $s^2 > 0$. Here, B_t, N_t and Z_n are all independent. Since the jump component is a compound Poisson process, Merton's jump diffusion is of finite activity, i.e., there are only finitely many jumps in any compact interval. However, due to the existence of the Brownian motion term, Merton's jump diffusion is of infinite variation.

According to Cont and Tankov [16], for any Borel set A,

$$\mathbb{P}(X_t \in A) = \sum_{k=0}^{\infty} \mathbb{P}(X_t \in A | N_t = k)\mathbb{P}(N_t = k),$$

and $\mathbb{P}(X_t \in \cdot)$ is absolutely continuous with respect to the Lebesgue measure with the density,

$$p(t, x) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \frac{1}{\sqrt{2\pi(\sigma^2 t + ks^2)}} \exp\{-\frac{(x - \mu t - km)^2}{2(\sigma^2 t + ks^2)}\}.$$

Similar to the result on the drifted Brownian motion derived in [37], we are able to establish the full expansion of Δ_N for Merton's jump diffusion as follows:

Theorem 3.1.1. *For Merton's jump diffusion with parameters $\mu, \lambda, \sigma, m, s$ defined on $[0, T]$, with $\frac{(\lambda + \frac{\mu^2}{2\sigma^2})T}{N} < 2\pi$, the expected difference between the continuous supremum and the discrete maximum* *admits the following asymptotic expansion:*

$$\begin{aligned}\Delta_N &= \mathbb{E}[\sup_{0\le t\le T} X_t - \sup_{0\le i\le N} X_{t_i}] \\ &= -\frac{\zeta(\frac{1}{2})\sigma\sqrt{T}}{\sqrt{2\pi}}\frac{1}{\sqrt{N}} + \frac{b_1}{N} + \sum_{r=1}^{\infty}\frac{a_r}{N^{r+\frac{1}{2}}} + \sum_{i=1}^{p}\frac{b_{2i}}{N^{2i}} + O(\frac{1}{N^{2p+2}}),\end{aligned}$$

where

$$\begin{aligned}b_1 &= \frac{\mu T}{4} - \frac{\mu e^{-\lambda T}}{4} Erfc(-\frac{\mu\sqrt{T}}{\sqrt{2}\sigma}) - \frac{\sigma\sqrt{T}}{2\sqrt{2\pi}} e^{-(\lambda+\frac{\mu^2}{2\sigma^2})T} + \frac{\lambda m T}{4} Erfc(-\frac{m}{\sqrt{2}s}) \\ &\quad -\frac{\mu e^{-\lambda T}}{4}\sum_{k=1}^{\infty}\frac{\lambda^k T^{k+1}}{k!} Erfc(-\frac{\mu T + km}{\sqrt{2(\sigma^2 T + ks^2)}}) \\ &\quad -\frac{\lambda m e^{-\lambda T}}{4}\sum_{k=0}^{\infty}\frac{\lambda^k T^{k+1}}{k!} Erfc(-\frac{\mu T + (k+1)m}{\sqrt{2(\sigma^2 T + (k+1)s^2)}}) \\ &\quad +\frac{\lambda s T}{2\sqrt{2\pi}} e^{-\frac{m^2}{2s^2}} - \frac{e^{-\lambda T}}{2\sqrt{2\pi}}\sum_{k=1}^{\infty}\frac{\lambda^k T^k}{k!}\sqrt{\sigma^2 T + ks^2}\, e^{-\frac{(\mu T + km)^2}{2(\sigma^2 T + ks^2)}},\end{aligned}$$

$$a_r = -\frac{\zeta(\frac{1}{2} - r) T^{r+\frac{1}{2}}}{r!\sqrt{2\pi}}\Big[\sigma(-\lambda - \frac{\mu^2}{2\sigma^2})^r + \frac{r\mu}{\sigma}\int_0^{\mu}(-\lambda - \frac{z^2}{2\sigma^2})^{r-1} dz\Big], r \ge 1,$$

$$\begin{aligned}b_{2i} &= -\frac{B_{2i} T^{2i}}{(2i)!}\Big(\frac{\mu}{2}\lambda^{2i-1}(1 - e^{-\lambda T}) + \frac{\mu}{\sqrt{2\pi\sigma^2}}\int_0^{\mu} g^{(2i-1)}(T, z) dz \\ &\quad +\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}(r_k^{(2i-1)}(T) - r_k^{(2i-1)}(0)) + \frac{\sigma}{\sqrt{2\pi}} f_2^{(2i-1)}(T) \\ &\quad +\frac{1}{2}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\big(\mu(q_k^{(2i-1)}(T) - q_k^{(2i-1)}(0)) + mk(p_k^{(2i-1)}(T) - p_k^{(2i-1)}(0))\big)\Big)\end{aligned}$$

and $Erfc(x)$ *is the complementary error function defined as*

$$Erfc(x) = 1 - Erf(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2} dt$$

$\zeta(s)$ *is the Riemann zeta function, in particular* $\zeta(\frac{1}{2}) \approx -1.46035451$; B_i *are ith Bernoulli number, and the functions* $p_k(t), q_k(t), r_k(t), g(t,z)$ *and* $f_2(t)$ *are defined as follows:*

$$p_k(t) = e^{-\lambda t} t^{k-1} Erfc(-\frac{\mu t + km}{\sqrt{2(\sigma^2 t + ks^2)}}), \qquad k \ge 1,$$

$$q_k(t) = tp_k(t), \qquad k \geq 1,$$

$$r_k(t) = e^{-\lambda t} t^{k-1} \sqrt{\frac{\sigma^2 t + ks^2}{2\pi}} \exp\{-\frac{(\mu t + km)^2}{2(\sigma^2 t + ks^2)}\}, \qquad k \geq 1,$$

$$f_2(t) = \frac{1}{\sqrt{t}} e^{-(\lambda + \frac{\mu^2}{2\sigma^2})t},$$

$$g(t, z) = \sqrt{t} e^{-(\lambda + \frac{z^2}{2\sigma^2})t},$$

and $f^{(i)}(t)$ *denotes the ith derivative of* $f(t)$*,* $g^{(i)}(t, z)$ *denotes the ith partial derivative of* $g(t, z)$ *with respect to* t*.*

Before proceeding to the proof of the theorem, we first introduce the three main tools that will be used in the proof , 1) Spitzer's identity, 2) the Euler-Maclaurin summation formula and 3) Bateman's formula for Lerch's transcendent.

Note that equal-time-discretization of a Lévy process produces a random walk and it follows from Spitzer's identity [63] that:

$$\mathbb{E}[\sup_{0 \leq i \leq N} X_{t_i}] = \sum_{n=1}^{N} \frac{1}{n} \mathbb{E}[X^+_{\frac{nT}{N}}].$$

Then by the monotone or dominated convergence theorem, we get that

$$\mathbb{E}[\sup_{0 \leq t \leq T} X_t] = \int_0^T \frac{1}{t} \mathbb{E}[X_t^+] dt.$$

Therefore,

$$\Delta_N = \int_0^T \frac{1}{t} \mathbb{E}[X_t^+] dt - \sum_{n=1}^{N} \frac{1}{n} \mathbb{E}[X^+_{\frac{nT}{N}}],$$

which is in the form of the difference of an integral and a summation. This motivates us to use the second tool, the Euler-Maclaurin summation formula.

The Euler-Maclaurin formula is a generalization of the well known trapezoid summation scheme. More precisely, see (Dahlquist and Björck [18], Chapter 3 Theorem 3.4.10, or De Bruijn [19]):

If $f \in C^{2p+2}([a, b])$, $b - a = Nh$ for some positive integer $N \geq 1$ and $h > 0$, then

$$\int_a^b f(x)dx - h\sum_{n=1}^{N} f(a+nh)$$

$$= \frac{h}{2}(f(a)-f(b)) - \sum_{i=1}^{p}\frac{B_{2i}}{(2i)!}h^{2i}(f^{(2i-1)}(b)-f^{(2i-1)}(a)) - R_{2p+2},$$

where

$$R_{2p+2} = h^{2p+2}\int_a^b (B_{2p+2}-\hat{B}_{2p+2}(\frac{x-a}{h}))\frac{f^{(2p+2)}(x)}{(2p+2)!}dx.$$

Here, $B_i(x)$ is the ith Bernoulli polynomial, i.e., $B_i(x) = \sum_{j=0}^{i}\binom{i}{j}B_j x^{i-j}$, where B_j is the jth Bernoulli number, and $\hat{B}_i(x) = B_i(x-\lfloor x \rfloor)$, $\lfloor x \rfloor$ is the largest integer that is less than or equal to x. Note that $\hat{B}_i(x)$ is the periodically extended version of $B_i(x)$ on $[0,1]$ and is hence absolutely bounded by the absolute value of the corresponding Bernoulli number B_i. In particular, if $f \in C^{2p+2}[a,\infty)$, and $\lim_{b\to\infty} f^{(i)}(b) = 0$ for $0 \le i \le 2p+1$, and $\int_a^\infty |f^{(2p+2)}(x)|dx < \infty$, then the above still holds if we replace $f(b)$ and $f^{(2i-1)}(b)$, $1 \le i \le p$, by zero, and the upper limit b of the integral in R_{2p+2} by ∞.

In addition to the Euler-Maclaurin formula, we also need Bateman's formula (see, for example, Erdelyi, Magnus, Oberhettinger and Tricomi [25]) for Lerch's transcendent, which is defined as the analytic continuation of the following infinite series:

$$\Phi(z,s,v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n,$$

and according to Bateman's formula, we have that

$$\Phi(z,s,v) = \frac{\Gamma(1-s)}{z^v}(-\log z)^{s-1} + z^{-v}\sum_{r=0}^{\infty}\zeta(s-r,v)\frac{(\log z)^r}{r!},$$

which holds for $|\log z| < 2\pi$, $s \ne 1,2,3,...$, $v \ne 0,-1,-2,...$, where Γ, ζ are the Gamma function and the Hurwitz zeta function, respectively.

Proof. By Spitzer's identity [63] and the previous discussion, we have that

$$\Delta_N = \mathbb{E}[\sup_{0\le t\le T} X_t - \sup_{0\le i\le N} X_{t_i}]$$

$$
\begin{aligned}
&= \int_0^T \frac{1}{t}\mathbb{E}[X_t^+]dt - \sum_{n=1}^N \frac{1}{n}\mathbb{E}[X_{\frac{nT}{N}}^+] \\
&= \int_0^T \frac{1}{t}\int_0^\infty xp(t,x)dxdt - \sum_{n=1}^N \frac{1}{n}\int_0^\infty xp(\frac{nT}{N},x)dx,
\end{aligned}
$$

where $X_t^+ = X_t \vee 0 = X_t \mathbb{1}_{\{X_t>0\}}$ and $p(t,x)$ is the transition density function. Then

$$
\begin{aligned}
\int_0^\infty xp(t,x)dx &= \int_0^\infty xe^{-\lambda t}\sum_{k=0}^\infty \frac{(\lambda t)^k}{k!}\frac{\exp\{-\frac{(x-\mu t-km)^2}{2(\sigma^2 t+ks^2)}\}}{\sqrt{2\pi(\sigma^2 t+ks^2)}}dx \\
&= e^{-\lambda t}\sum_{k=0}^\infty \frac{(\lambda t)^k}{k!}\frac{1}{\sqrt{2\pi(\sigma^2 t+ks^2)}}\int_0^\infty x\exp\{-\frac{(x-\mu t-km)^2}{2(\sigma^2 t+ks^2)}\}dx \\
&= e^{-\lambda t}\sum_{k=0}^\infty \frac{(\lambda t)^k}{k!}\Big(\frac{\mu t+km}{2}\mathrm{Erfc}(-\frac{\mu t+km}{\sqrt{2(\sigma^2 t+ks^2)}}) \\
&\quad + \sqrt{\frac{\sigma^2 t+ks^2}{2\pi}}\exp\{-\frac{(\mu t+km)^2}{2(\sigma^2 t+ks^2)}\}\Big),
\end{aligned}
$$

where the second equality follows from the monotone convergence theorem, and the third equality follows from a simple change of variable technique, i.e., $y = \frac{x-\mu t-km}{\sqrt{2(\sigma^2 t+ks^2)}}$. Hence,

$$
\begin{aligned}
&\mathbb{E}[\sup_{0\le t\le T} X_t] \\
=\ & \int_0^T \frac{1}{t}\mathbb{E}[X_t^+]dt \\
=\ & \int_0^T \frac{1}{t}\int_0^\infty xp(t,x)dxdt \\
=\ & \int_0^T \frac{1}{t}e^{-\lambda t}\sum_{k=0}^\infty \frac{(\lambda t)^k}{k!}\Big(\frac{\mu t+km}{2}\mathrm{Erfc}(-\frac{\mu t+km}{\sqrt{2(\sigma^2 t+ks^2)}}) + \sqrt{\frac{\sigma^2 t+ks^2}{2\pi}}\exp\{-\frac{(\mu t+km)^2}{2(\sigma^2 t+ks^2)}\}\Big)dt \\
=\ & \sum_{k=0}^\infty \int_0^T \frac{1}{t}e^{-\lambda t}\frac{(\lambda t)^k}{k!}\Big(\frac{\mu t+km}{2}\mathrm{Erfc}(-\frac{\mu t+km}{\sqrt{2(\sigma^2 t+ks^2)}}) + \sqrt{\frac{\sigma^2 t+ks^2}{2\pi}}\exp\{-\frac{(\mu t+km)^2}{2(\sigma^2 t+ks^2)}\}\Big)dt \\
=\ & \int_0^T e^{-\lambda t}\big[\frac{\mu}{2}\mathrm{Erfc}(-\mu\sqrt{\frac{t}{2\sigma^2}}) + \sqrt{\frac{\sigma^2}{2\pi t}}e^{-\frac{\mu^2 t}{2\sigma^2}}\big]dt + \\
& \sum_{k=1}^\infty \int_0^T \frac{e^{-\lambda t}}{t}\frac{(\lambda t)^k}{k!}\Big(\frac{\mu t+km}{2}\mathrm{Erfc}(-\frac{\mu t+km}{\sqrt{2(\sigma^2 t+ks^2)}}) + \sqrt{\frac{\sigma^2 t+ks^2}{2\pi}}\exp\{-\frac{(\mu t+km)^2}{2(\sigma^2 t+ks^2)}\}\Big)dt \\
=\ & \frac{\mu}{2}\int_0^T e^{-\lambda t}\mathrm{Erfc}(-\mu\sqrt{\frac{t}{2\sigma^2}})dt + \sqrt{\frac{\sigma^2}{2\pi}}\int_0^T \frac{1}{\sqrt{t}}e^{-(\lambda+\frac{\mu^2}{2\sigma^2})t}dt \\
& + \sum_{k=1}^\infty \int_0^T \frac{1}{t}e^{-\lambda t}\frac{(\lambda t)^k}{k!}\Big(\frac{\mu t+km}{2}\mathrm{Erfc}(-\frac{\mu t+km}{\sqrt{2(\sigma^2 t+ks^2)}}) \\
& + \sqrt{\frac{\sigma^2 t+ks^2}{2\pi}}\exp\{-\frac{(\mu t+km)^2}{2(\sigma^2 t+ks^2)}\}\Big)dt \qquad (3.1)
\end{aligned}
$$

where the fourth equality follows from the monotone convergence theorem. The reason that we separate the cases $k=0$ and $k\geq 1$ in (3.1) is that the tools used to deal with these cases will be different due to the smoothness of the integrands at $t=0$. More precisely, as $k=0$, we will use Bateman's formula; as $k\geq 1$, we will use the Euler-Maclaurin formula.

Now we turn to the discrete version of (3.1). It is given by

$$\begin{aligned}
&\frac{\mu}{2}\sum_{n=1}^{N} e^{-\lambda\frac{nT}{N}}\text{Erfc}(-\mu\sqrt{\frac{nT}{2\sigma^2 N}})\frac{T}{N}+\sqrt{\frac{\sigma^2}{2\pi}}\sum_{n=1}^{N}\frac{1}{\sqrt{\frac{nT}{N}}}e^{-(\lambda+\frac{\mu^2}{2\sigma^2})\frac{nT}{N}}\frac{T}{N}\\
&+\sum_{k=1}^{\infty}\sum_{n=1}^{N}\frac{1}{n}e^{-\lambda\frac{nT}{N}}\frac{(\lambda\frac{nT}{N})^k}{k!}\Big(\frac{\mu\frac{nT}{N}+km}{2}\text{Erfc}(-\frac{\mu\frac{nT}{N}+km}{\sqrt{2(\sigma^2\frac{nT}{N}+ks^2)}})\\
&+\sqrt{\frac{\sigma^2\frac{nT}{N}+ks^2}{2\pi}}\exp\{-\frac{(\mu\frac{nT}{N}+km)^2}{2(\sigma^2\frac{nT}{N}+ks^2)}\}\Big).
\end{aligned}$$

Hence, the desired expected difference $\Delta_N=\mathbb{E}[\sup_{0\leq t\leq T}X_t-\sup_{0\leq i\leq N}X_{t_i}]$ can be written as

$$\begin{aligned}
\Delta_N &= \mathbb{E}[\sup_{0\leq t\leq T}X_t-\sup_{0\leq i\leq N}X_{t_i}]\\
&= \frac{\mu}{2}[\int_0^T e^{-\lambda t}\text{Erfc}(-\mu\sqrt{\frac{t}{2\sigma^2}})dt-\sum_{n=1}^{N}e^{-\lambda\frac{nT}{N}}\text{Erfc}(-\mu\sqrt{\frac{nT}{2\sigma^2N}})\frac{T}{N}]\\
&\quad+\sqrt{\frac{\sigma^2}{2\pi}}[\int_0^T\frac{1}{\sqrt{t}}e^{-(\lambda+\frac{\mu^2}{2\sigma^2})t}dt-\sum_{n=1}^{N}\frac{1}{\sqrt{\frac{nT}{N}}}e^{-(\lambda+\frac{\mu^2}{2\sigma^2})\frac{nT}{N}}\frac{T}{N}]\\
&\quad+\sum_{k=1}^{\infty}\Big(\int_0^T\frac{1}{t}e^{-\lambda t}\frac{(\lambda t)^k}{k!}\frac{\mu t+km}{2}\text{Erfc}(-\frac{\mu t+km}{\sqrt{2(\sigma^2t+ks^2)}})dt\\
&\quad-\sum_{n=1}^{N}\frac{1}{n}e^{-\lambda\frac{nT}{N}}\frac{(\lambda\frac{nT}{N})^k}{k!}\frac{\mu\frac{nT}{N}+km}{2}\text{Erfc}(-\frac{\mu\frac{nT}{N}+km}{\sqrt{2(\sigma^2\frac{nT}{N}+ks^2)}})\Big)\\
&\quad+\sum_{k=1}^{\infty}\Big(\int_0^T\frac{1}{t}e^{-\lambda t}\frac{(\lambda t)^k}{k!}\sqrt{\frac{\sigma^2t+ks^2}{2\pi}}\exp\{-\frac{(\mu t+km)^2}{2(\sigma^2t+ks^2)}\}dt\\
&\quad-\sum_{n=1}^{N}\frac{1}{n}e^{-\lambda\frac{nT}{N}}\frac{(\lambda\frac{nT}{N})^k}{k!}\sqrt{\frac{\sigma^2\frac{nT}{N}+ks^2}{2\pi}}\exp\{-\frac{(\mu\frac{nT}{N}+km)^2}{2(\sigma^2\frac{nT}{N}+ks^2)}\}\Big). \qquad (3.2)
\end{aligned}$$

We analyze the above quantity (3.2) term by term. The first difference in (3.2) is given by

$$E_1:=\frac{\mu}{2}[\int_0^T e^{-\lambda t}\text{Erfc}(-\mu\sqrt{\frac{t}{2\sigma^2}})dt-\sum_{n=1}^{N}e^{-\lambda\frac{nT}{N}}\text{Erfc}(-\mu\sqrt{\frac{nT}{2\sigma^2N}})\frac{T}{N}]$$

For the analysis of E_1, we start with the summation term

$$\sum_{n=1}^{N} e^{-\lambda\frac{nT}{N}}\,\mathrm{Erfc}(-\mu\sqrt{\frac{nT}{2\sigma^2 N}})\frac{T}{N}.$$

For simplicity of notation, we set $A := -\sqrt{\frac{nT}{2\sigma^2 N}}$, then

$$\begin{aligned}
\mathrm{Erfc}(-\mu\sqrt{\frac{nT}{2\sigma^2 N}}) &= \int_0^{\mu} \mathrm{Erfc}'(Az)d(Az) + 1 \\
&= -\sqrt{\frac{nT}{2\sigma^2 N}}\int_0^{\mu}(-\frac{2}{\sqrt{\pi}}e^{-\frac{nT}{2\sigma^2 N}z^2})dz + 1 \\
&= \sqrt{\frac{2}{\sigma^2\pi}}\sqrt{\frac{nT}{N}}\int_0^{\mu} e^{-\frac{nT}{2\sigma^2 N}z^2}dz + 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{n=1}^{N} e^{-\lambda\frac{nT}{N}}\,\mathrm{Erfc}(-\mu\sqrt{\frac{nT}{2\sigma^2 N}})\frac{T}{N} \\
=\ &\sum_{n=1}^{N} e^{-\lambda\frac{nT}{N}}\Big(\sqrt{\frac{2}{\sigma^2\pi}}\sqrt{\frac{nT}{N}}\int_0^{\mu} e^{-\frac{nT}{2\sigma^2 N}z^2}dz + 1\Big)\frac{T}{N} \\
=\ &\sqrt{\frac{2}{\sigma^2\pi}}\int_0^{\mu}\sum_{n=1}^{N}\sqrt{\frac{nT}{N}}e^{-(\frac{z^2}{2\sigma^2}+\lambda)\frac{nT}{N}}\frac{T}{N}dz + \sum_{n=1}^{N} e^{-\lambda\frac{nT}{N}}\frac{T}{N}.
\end{aligned}$$

By Fubini's theorem, the corresponding continuous version of the above is given by

$$\sqrt{\frac{2}{\sigma^2\pi}}\int_0^{\mu}\int_0^{T}\sqrt{t}e^{-(\frac{z^2}{2\sigma^2}+\lambda)t}dtdz + \int_0^{T} e^{-\lambda t}dt.$$

Thus,

$$\begin{aligned}
E_1 = \frac{\mu}{2}\Big[\sqrt{\frac{2}{\sigma^2\pi}}\int_0^{\mu}\Big(\int_0^{T}\sqrt{t}e^{-(\frac{z^2}{2\sigma^2}+\lambda)t}dt - \sum_{n=1}^{N}\sqrt{\frac{nT}{N}}e^{-(\frac{z^2}{2\sigma^2}+\lambda)\frac{nT}{N}}\frac{T}{N}\Big)dz \\
+\Big(\int_0^{T} e^{-\lambda t}dt - \sum_{n=1}^{N} e^{-\lambda\frac{nT}{N}}\frac{T}{N}\Big)\Big].
\end{aligned} \tag{3.3}$$

In the terms above, it is easy to see that the second difference above can be analyzed by the Euler-Maclaurin formula, namely, for some positive integer p,

$$\int_0^{T} e^{-\lambda t}dt - \sum_{n=1}^{N} e^{-\lambda\frac{nT}{N}}\frac{T}{N} = \frac{T(1-e^{-\lambda T})}{2N} - \sum_{i=1}^{p}\frac{B_{2i}}{(2i)!}\frac{\lambda^{2i-1}(1-e^{-\lambda T})T^{2i}}{N^{2i}} + O(\frac{1}{N^{2p+2}}).$$

Thus, it suffices to analyze the first term of (3.3), i.e.,

$$\begin{aligned}
&\int_0^T \sqrt{t}e^{-(\frac{z^2}{2\sigma^2}+\lambda)t}dt - \sum_{n=1}^{N}\sqrt{\frac{nT}{N}}e^{-(\frac{z^2}{2\sigma^2}+\lambda)\frac{nT}{N}}\frac{T}{N} \\
= &\Big(\int_0^\infty \sqrt{t}e^{-(\frac{z^2}{2\sigma^2}+\lambda)t}dt - \sum_{n=1}^{\infty}\sqrt{\frac{nT}{N}}e^{-(\frac{z^2}{2\sigma^2}+\lambda)\frac{nT}{N}}\frac{T}{N}\Big) \\
&-\Big(\int_T^\infty \sqrt{t}e^{-(\frac{z^2}{2\sigma^2}+\lambda)t}dt - \sum_{n=N+1}^{\infty}\sqrt{\frac{nT}{N}}e^{-(\frac{z^2}{2\sigma^2}+\lambda)\frac{nT}{N}}\frac{T}{N}\Big). \qquad (3.4)
\end{aligned}$$

The first term of (3.4) involves Lerch's transcendent and Bateman's formula. For simplicity, we define $a(z) := \frac{z^2}{2\sigma^2}+\lambda$, then we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sqrt{\frac{nT}{N}}e^{-(\frac{z^2}{2\sigma^2}+\lambda)\frac{nT}{N}}\frac{T}{N} \\
= &\frac{T}{N}\sum_{n=1}^{\infty}\sqrt{\frac{nT}{N}}e^{-a(z)\frac{nT}{N}} \\
= &\frac{T\sqrt{T}e^{-\frac{a(z)T}{N}}}{N\sqrt{N}}\sum_{n=0}^{\infty}(1+n)^{\frac{1}{2}}(e^{-\frac{a(z)T}{N}})^n \\
= &\frac{T\sqrt{T}e^{-\frac{a(z)T}{N}}}{N\sqrt{N}}\Phi(z=e^{-\frac{a(z)T}{N}}, s=-\frac{1}{2}, v=1) \\
= &\frac{T\sqrt{T}e^{-\frac{a(z)T}{N}}}{N\sqrt{N}}\Big(\frac{\Gamma(\frac{3}{2})}{e^{-\frac{a(z)T}{N}}}(\frac{a(z)T}{N})^{-\frac{3}{2}} + e^{\frac{a(z)T}{N}}\sum_{r=0}^{\infty}\frac{\zeta(-\frac{1}{2}-r)}{r!}(-\frac{a(z)T}{N})^r\Big) \\
= &\Gamma(\frac{3}{2})a(z)^{-\frac{3}{2}} + \frac{T\sqrt{T}}{N\sqrt{N}}\sum_{r=0}^{\infty}\frac{\zeta(-\frac{1}{2}-r)}{r!}(-\frac{a(z)T}{N})^r.
\end{aligned}$$

The fourth equality above follows from Bateman's formula, in which we need $\frac{a(z)T}{N} = \frac{(\frac{z^2}{2\sigma^2}+\lambda)T}{N} < 2\pi$. This is satisfied for sufficiently large N since $z^2 \in (0,\mu^2)$. Clearly,

$$\int_0^\infty \sqrt{t}e^{-(\frac{z^2}{2\sigma^2}+\lambda)t}dt = \Gamma(\frac{3}{2})a(z)^{-\frac{3}{2}}.$$

Thus, the first difference in (3.4) is given by

$$-\frac{T\sqrt{T}}{N\sqrt{N}}\sum_{r=0}^{\infty}\frac{\zeta(-\frac{1}{2}-r)}{r!}(-\frac{a(z)T}{N})^r.$$

With the integration with respect to z as in (3.3), we have the following

$$\sqrt{\frac{2}{\sigma^2\pi}}\int_0^\mu(-\frac{T\sqrt{T}}{N\sqrt{N}}\sum_{r=0}^{\infty}\frac{\zeta(-\frac{1}{2}-r)}{r!}(-\frac{a(z)T}{N})^r)dz$$

$$= -\sqrt{\frac{2}{\sigma^2\pi}}\frac{T\sqrt{T}}{N\sqrt{N}}\sum_{r=0}^{\infty}\frac{\zeta(-\frac{1}{2}-r)}{r!}(-\frac{T}{N})^r\int_0^{\mu}(\frac{z^2}{2\sigma^2}+\lambda)^r dz.$$

The interchange of integral and summation above needs to be justified. More precisely, we need to verify the following

$$\int_0^{\mu}\sum_{r=0}^{\infty}\frac{\zeta(-\frac{1}{2}-r)}{r!}(\frac{-a(z)T}{N})^r dz = \sum_{r=0}^{\infty}\int_0^{\mu}\frac{\zeta(-\frac{1}{2}-r)}{r!}(\frac{-a(z)T}{N})^r dz.$$

First, we recall the following reflection functional equation for the Riemann zeta function, see Abramowitz and Stegun [1], page 807, 23.2.6 or Choudhury [15]:

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\zeta(s)\cos(\frac{1}{2}s\pi), \text{for all } s > 0.$$

Taking $s = r + \frac{3}{2} > 0$, we get

$$\zeta(-\frac{1}{2}-r) = 2(2\pi)^{-r-\frac{3}{2}}\Gamma(r+\frac{3}{2})\zeta(r+\frac{3}{2})\cos(\frac{r+\frac{3}{2}}{2}\pi).$$

By Abramowitz and Stegun [1], page 255, 6.1.12, we know that

$$\Gamma(r+\frac{3}{2}) = \frac{(2r+1)!!}{2^{r+1}}\sqrt{\pi}.$$

Clearly, $\zeta(r+\frac{3}{2}) \leq \zeta(\frac{3}{2}) < \infty$ for all $r \geq 0$. And

$$\frac{(2r+1)!!}{r!2^{r+1}} = (r+1)\frac{(2r+1)!!}{(r+1)!2^{r+1}} = (r+1)\frac{(2r+1)!!}{(2r+2)!!} < r+1.$$

Thus, we obtain

$$|\frac{\zeta(-\frac{1}{2}-r)}{r!}| < 2(2\pi)^{-r-\frac{3}{2}}(r+1)\sqrt{\pi}\zeta(\frac{3}{2})\frac{\sqrt{2}}{2} = (2\pi)^{-r-1}(r+1)\zeta(\frac{3}{2}),$$

and

$$\begin{aligned}\sum_{r=0}^{\infty}|\frac{\zeta(-\frac{1}{2}-r)}{r!}(\frac{-a(z)T}{N})^r| &\leq \sum_{r=0}^{\infty}(2\pi)^{-r-1}(r+1)\zeta(\frac{3}{2})|\frac{a(z)T}{N}|^r \\ &= \sum_{r=0}^{\infty}(2\pi)^{-1}(r+1)\zeta(\frac{3}{2})|\frac{a(z)T}{2\pi N}|^r \\ &< (2\pi)^{-1}\zeta(\frac{3}{2})\sum_{r=0}^{\infty}(r+1)\Big(\frac{(\lambda+\frac{\mu^2}{2\sigma^2})T}{2\pi N}\Big)^r.\end{aligned}$$

Given the condition that $\frac{(\lambda+\frac{\mu^2}{2\sigma^2})T}{2\pi N} < 1$, the above series converges. Now we may apply the Weierstrass M-test to obtain that the series

$$\sum_{r=0}^{\infty} \frac{\zeta(-\frac{1}{2}-r)}{r!}(\frac{-a(z)T}{N})^r$$

converges absolutely and uniformly for all $z^2 \in (0, \mu^2)$, which allows integration with respect to z term by term on the interval $[0, \mu]$, or equivalently the interchange of summation and integration. For $\mu < 0$, the analysis is exactly the same.

Next we return to (3.4) to look at the second difference term. Denote $g(t,z) := \sqrt{t}e^{-(\frac{z^2}{2\sigma^2}+\lambda)t}$. Note that $g(t,z)$, as a function of t, is smooth on $[T,\infty)$, and all its derivatives with respect to t vanish at ∞ and are absolutely integrable on $[T,\infty)$ due to the exponential tail. Therefore, using the Euler-Maclaurin formula, we obtain that

$$\begin{aligned} & \int_T^{\infty} \sqrt{t}e^{-(\frac{z^2}{2\sigma^2}+\lambda)t}dt - \sum_{n=N+1}^{\infty} \sqrt{\frac{nT}{N}}e^{-(\frac{z^2}{2\sigma^2}+\lambda)\frac{nT}{N}}\frac{T}{N} \\ = & \int_T^{\infty} g(t,z)dt - \frac{T}{N}\sum_{n=N+1}^{\infty} g(\frac{nT}{N}, z) \\ = & \frac{T\sqrt{T}e^{-a(z)T}}{2N} + \sum_{i=1}^{p} \frac{B_{2i}}{(2i)!}\frac{g^{(2i-1)}(T,z)T^{2i}}{N^{2i}} - R_{2p+2}(z), \end{aligned}$$

where $g^{(2i-1)}(t,z) := \frac{\partial^{2i-1}}{\partial t}g(t,z)$ and the remainder term is given by

$$R_{2p+2}(z) = \frac{T^{2p+2}}{N^{2p+2}}\int_T^{\infty} \big(B_{2p+2} - \hat{B}_{2p+2}((t-T)\frac{N}{T})\big)\frac{g^{(2p+2)}(t,z)}{(2p+2)!}dt.$$

Note that

$$\Big|\int_0^{\mu} R_{2p+2}dz\Big| \leq \frac{T^{2p+2}}{N^{2p+2}}\int_0^{\mu}\int_T^{\infty} |B_{2p+2} - \hat{B}_{2p+2}((t-T)\frac{N}{T})|\frac{|g^{(2p+2)}(t,z)|}{(2p+2)!}dtdz = O(\frac{1}{N^{2p+2}}),$$

and

$$\int_0^{\mu} e^{-a(z)T}dz = e^{-\lambda T}\int_0^{\mu} e^{-\frac{z^2 T}{2\sigma^2}}dz = \sqrt{\frac{\sigma^2\pi}{2T}}e^{-\lambda T}\mathrm{Erf}(\frac{\mu\sqrt{T}}{\sqrt{2\sigma^2}}).$$

So we obtain that

$$E_1 = \frac{\mu}{2}\Big[\frac{T}{2N}(1-e^{-\lambda}\mathrm{Erfc}(-\frac{\mu\sqrt{T}}{\sqrt{2\sigma^2}})) - \sqrt{\frac{2}{\pi\sigma^2}}\sum_{r=0}^{\infty}\frac{\zeta(-\frac{1}{2}-r)T^{r+\frac{3}{2}}}{r!N^{r+\frac{3}{2}}}\int_0^{\mu}(-\lambda-\frac{z^2}{2\sigma^2})^r dz$$

$$-\sum_{i=1}^{p}\frac{B_{2i}T^{2i}}{(2i)!N^{2i}}\Big(\lambda^{2i-1}(1-e^{-\lambda T}) + \sqrt{\frac{2}{\pi\sigma^2}}\int_0^{\mu} g^{(2i-1)}(T,z)dz\Big)\Big] + O(\frac{1}{N^{2p+2}}).$$

Now we move to the analysis of the second difference in (3.2), called E_2. Define $f_2(t) = \frac{1}{\sqrt{t}}e^{-at}$ and $a = \lambda + \frac{\mu^2}{2\sigma^2}$, then

$$\begin{aligned} E_2 &:= \frac{\sigma}{\sqrt{2\pi}}\Big[\int_0^T f_2(t)dt - \frac{T}{N}\sum_{n=1}^{N} f_2(\frac{nT}{N})\Big] \\ &= \frac{\sigma}{\sqrt{2\pi}}\Big[\Big(\int_0^{\infty} f_2(t)dt - \frac{T}{N}\sum_{n=1}^{\infty} f_2(\frac{nT}{N})\Big) - \Big(\int_T^{\infty} f_2(t)dt - \frac{T}{N}\sum_{n=N+1}^{\infty} f_2(\frac{nT}{N})\Big)\Big]. \end{aligned}$$

Again, we apply Bateman's formula for Lerch's transcendent and use the condition that $\frac{aT}{N} < 2\pi$ to obtain that

$$\int_0^{\infty} f_2(t)dt - \frac{T}{N}\sum_{n=1}^{\infty} f_2(\frac{nT}{N}) = -\frac{\sqrt{T}}{\sqrt{N}}\sum_{r=0}^{\infty}\frac{\zeta(\frac{1}{2}-r)}{r!}(-\frac{aT}{N})^r.$$

For the second component of E_2, we apply the Euler-Maclaurin formula for $f_2(t)$, which is smooth on $[T,\infty)$ with all its derivatives vanishing at ∞ and absolutely integrable on $[T,\infty)$. Thus,

$$\int_T^{\infty} f_2(t)dt - \frac{T}{N}\sum_{n=N+1}^{\infty} f_2(\frac{nT}{N}) = \frac{Tf_2(T)}{2N} + \sum_{i=1}^{p}\frac{B_{2i}T^{2i}}{(2i)!N^{2i}}f_2^{(2i-1)}(T) + O(\frac{1}{N^{2p+2}}).$$

Hence,

$$E_2 = \frac{\sigma}{\sqrt{2\pi}}\Big[-\frac{\zeta(\frac{1}{2})\sqrt{T}}{\sqrt{N}} - \frac{\sqrt{T}e^{-(\lambda+\frac{\mu^2}{2\sigma^2})T}}{2N} - \sum_{r=1}^{\infty}\frac{\zeta(\frac{1}{2}-r)(-\lambda-\frac{\mu^2}{2\sigma^2})^r T^{r+\frac{1}{2}}}{r!N^{r+\frac{1}{2}}}$$

$$-\sum_{i=1}^{p}\frac{B_{2i}T^{2i}}{(2i)!N^{2i}}f_2^{(2i-1)}(T) + O(\frac{1}{N^{2p+2}})\Big].$$

Now we take a look at the third difference in (3.2), i.e.,

$$
\begin{aligned}
E_3 \;:=\; & \sum_{k=1}^{\infty}\Big(\int_0^T \frac{1}{t}e^{-\lambda t}\frac{(\lambda t)^k}{k!}\frac{\mu t+km}{2}\text{Erfc}(-\frac{\mu t+km}{\sqrt{2(\sigma^2 t+ks^2)}})dt \\
& -\sum_{n=1}^{N}\frac{1}{n}e^{-\lambda\frac{nT}{N}}\frac{(\lambda\frac{nT}{N})^k}{k!}\frac{\mu\frac{nT}{N}+km}{2}\text{Erfc}(-\frac{\mu\frac{nT}{N}+km}{\sqrt{2(\sigma^2\frac{nT}{N}+ks^2)}})\Big) \\
=\; & \frac{1}{2}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\Big(\int_0^T[\mu q_k(t)+kmp_k(t)]dt-\frac{T}{N}\sum_{n=1}^{N}[\mu q_k(\frac{nT}{N})+kmp_k(\frac{nT}{N})]\Big) \\
=\; & \frac{\mu}{2}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\Big(\int_0^T q_k(t)dt-\frac{T}{N}\sum_{n=1}^{N}q_k(\frac{nT}{N})\Big)+\frac{m}{2}\sum_{k=1}^{\infty}\frac{k\lambda^k}{k!}\Big(\int_0^T p_k(t)dt-\frac{T}{N}\sum_{n=1}^{N}p_k(\frac{nT}{N})\Big),
\end{aligned}
$$

where we recall the definitions of the functions $p_k(t)$ and $q_k(t)$:

$$
p_k(t)=e^{-\lambda t}t^{k-1}\text{Erfc}(-\frac{\mu t+km}{\sqrt{2(\sigma^2 t+ks^2)}}),\qquad \text{and}\qquad q_k(t)=tp_k(t),\qquad k\geq 1.
$$

Note that any order of the derivatives of $p_k(t)$ and $q_k(t)$ can be computed explicitly. For example,

$$
p_k^{(1)}(t)=\mathbb{1}_{\{k\geq 2\}}(k-1)e^{-\lambda t}t^{k-2}\text{Erfc}(-\frac{\mu t+km}{\sqrt{2(\sigma^2 t+ks^2)}})
$$

$$
-\lambda p_k(t)+e^{-\lambda t}t^{k-1}p(\frac{\mu t+km}{\sqrt{\sigma^2 t+ks^2}})\frac{\mu\sigma^2 t+2\mu ks^2-k\sigma^2 m}{(\sigma^2 t+ks^2)^{\frac{3}{2}}},\qquad k\geq 1,
$$

$$
q_k^{(1)}(t)=p_k(t)+tp_k^{(1)}(t),
$$

$$
q_k^{(2)}(t)=2p_k^{(1)}(t)+tp_k^{(2)}(t),
$$

and so on. Here $p(x)$ is the probability density function of the standard normal distribution. Note that $k\geq 1$, so all the denominators in any order derivatives are nonzero. With the fact that the normal density function $p(x)$ and the complementary error function $\text{Erfc}(x)$ are absolutely bounded, we conclude that for any positive integer i, the i-th order derivatives of $p_k(t)$ and $q_k(t)$ can be bounded by polynomials in k of finite orders on the interval $[0,T]$. Hence, we get that

$$
\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\int_0^T|p_k^{(i)}(t)|dt<\infty,\qquad \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\int_0^T|q_k^{(i)}(t)|dt<\infty,
$$

$$\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}|p_k^{(i)}(T)-p_k^{(i)}(0)|<\infty, \qquad \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}|q_k^{(i)}(T)-q_k^{(i)}(0)|<\infty.$$

Therefore, by the Euler-Maclaurin formula, with $q_k(0)=0$, $k\geq 1$, we obtain that

$$\begin{aligned}
&\frac{\mu}{2}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\Big(\int_0^T q_k(t)dt-\frac{T}{N}\sum_{n=1}^{N}q_k(\frac{nT}{N})\Big)\\
=&\frac{\mu}{2}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\Big(\frac{(q_k(0)-q_k(1))T}{2N}-\sum_{i=1}^{p}\frac{B_{2i}T^{2i}}{(2i)!N^{2i}}\big(q_k^{(2i-1)}(T)-q_k^{(2i-1)}(0)\big)\Big)+O(\frac{1}{N^{2p+2}})\\
=&-\frac{\mu T}{4N}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}q_k(T)-\frac{\mu}{2}\sum_{i=1}^{p}\frac{B_{2i}T^{2i}}{(2i)!N^{2i}}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\big(q_k^{(2i-1)}(T)-q_k^{(2i-1)}(0)\big)+O(\frac{1}{N^{2p+2}}).
\end{aligned}$$

Likewise, with $p_k(0)=0$ for $k\geq 2$, from the Euler-Maclaurin formula again, we know that

$$\begin{aligned}
&\frac{m}{2}\sum_{k=1}^{\infty}\frac{k\lambda^k}{k!}\Big(\int_0^T p_k(t)dt-\frac{T}{N}\sum_{n=1}^{N}p_k(\frac{nT}{N})\Big)\\
=&\frac{mT}{4N}\Big(\lambda p_1(0)-\sum_{k=1}^{\infty}\frac{k\lambda^k}{k!}p_k(T)\Big)-\frac{m}{2}\sum_{i=1}^{p}\frac{B_{2i}T^{2i}}{(2i)!N^{2i}}\sum_{k=1}^{\infty}\frac{k\lambda^k}{k!}\big(p_k^{(2i-1)}(T)-p_k^{(2i-1)}(0)\big)+O(\frac{1}{N^{2p+2}}).
\end{aligned}$$

Note that $q_k(T)=Tp_k(T)$, we get the asymptotic expansion of E_3 by adding the above terms, i.e.,

$$\begin{aligned}
E_3=&\frac{T}{4N}\Big(\lambda m p_1(0)-\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}(\mu T+mk)p_k(T)\Big)\\
&-\frac{1}{2}\sum_{i=1}^{p}\frac{B_{2i}T^{2i}}{(2i)!N^{2i}}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\Big[\mu(q_k^{(2i-1)}(T)-q_k^{(2i-1)}(0))+mk(p_k^{(2i-1)}(T)-p_k^{(2i-1)}(0))\Big]+O(\frac{1}{N^{2p+2}}).
\end{aligned}$$

Lastly, we move to the last difference in (3.2), i.e.,

$$\begin{aligned}
E_4 \quad:=\quad & \sum_{k=1}^{\infty}\Big(\int_0^T\frac{1}{t}e^{-\lambda t}\frac{(\lambda t)^k}{k!}\sqrt{\frac{\sigma^2 t+ks^2}{2\pi}}\exp\{-\frac{(\mu t+km)^2}{2(\sigma^2 t+ks^2)}\}dt\\
&-\sum_{n=1}^{N}\frac{1}{n}e^{-\lambda\frac{nT}{N}}\frac{(\lambda\frac{nT}{N})^k}{k!}\sqrt{\frac{\sigma^2\frac{nT}{N}+ks^2}{2\pi}}\exp\{-\frac{(\mu\frac{nT}{N}+km)^2}{2(\sigma^2\frac{nT}{N}+ks^2)}\}\Big)\\
=\quad & \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\Big(\int_0^T r_k(t)dt-\frac{T}{N}\sum_{n=1}^{N}r_k(\frac{nT}{N})\Big),
\end{aligned}$$

where we recall the definition of the function $r_k(t)$ for $k\geq 1$, i.e.,

$$r_k(t) = e^{-\lambda t} t^{k-1} \sqrt{\frac{\sigma^2 t + ks^2}{2\pi}} \exp\{-\frac{(\mu t + km)^2}{2(\sigma^2 t + ks^2)}\}.$$

The first order derivative of $r_k(t)$ is given by

$$r_k^{(1)}(t) = \mathbb{1}_{\{k\geq 2\}}(k-1)e^{-\lambda t} t^{k-2} \sqrt{\frac{\sigma^2 t + ks^2}{2\pi}} \exp\{-\frac{(\mu t + km)^2}{2(\sigma^2 t + ks^2)}\}$$

$$+\frac{r_k(t)}{\sqrt{2\pi}}\big(-\lambda + \frac{\sigma^2}{2(\sigma^2 t + ks^2)} - \frac{(\mu t + km)(\mu\sigma^2 t + 2\mu ks^2 - km\sigma^2)}{2(\sigma^2 t + ks^2)^2}\big).$$

We can easily compute other higher order derivatives of $r_k(t)$. Again, since $k \geq 1$, then $\sigma^2 t + ks^2 > 0$. In other words, there is no singularities in any order derivatives of $r_k(t)$. Thus, $r_k(t)$ is in fact smooth on $[0, T]$ and for any positive integer i, $r_k^{(i)}(t)$ can be bounded by a polynomial in k of finite orders on $[0, T]$. Hence, we get that

$$\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} |r_k(T)| < \infty, \qquad \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} |r_k^{(i)}(T) - r_k^{(i)}(0)| < \infty.$$

Thus, by the Euler-Maclaurin formula again, with $r_k(0) = 0$ for $k \geq 2$, we get the following asymptotic expansion for E_4:

$$E_4 = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \Big(\frac{(r_k(0) - r_k(T))T}{2N} - \sum_{i=1}^{p} \frac{B_{2i} T^{2i}}{(2i)! N^{2i}} \big(r_k^{(2i-1)}(T) - r_k^{(2i-1)}(0) \big) \Big) + O(\frac{1}{N^{2p+2}})$$

$$= \frac{T}{2N} \big(\lambda r_1(0) - \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} r_k(T) \big) - \sum_{i=1}^{p} \frac{B_{2i} T^{2i}}{(2i)! N^{2i}} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \big(r_k^{(2i-1)}(T) - r_k^{(2i-1)}(0) \big) + O(\frac{1}{N^{2p+2}})$$

$$= \frac{T}{2N} \big(\lambda \sqrt{\frac{s^2}{2\pi}} e^{-\frac{m^2}{2s^2}} - \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} r_k(T) \big) - \sum_{i=1}^{p} \frac{B_{2i} T^{2i}}{(2i)! N^{2i}} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \big(r_k^{(2i-1)}(T) - r_k^{(2i-1)}(0) \big) + O(\frac{1}{N^{2p+2}}).$$

Combining all the E_i's, we obtain the asymptotic expansion of Δ_N for Merton's jump diffusions. □

Remark 3.1.2. (1) The leading term in the asymptotic expansion of Δ_N is given by $-\frac{\sigma\zeta(\frac{1}{2})\sqrt{T}}{\sqrt{2\pi N}}$, which is the same as the leading term of the drifted Brownian motion in Janssen and Van Leeuwaarden [37] as $\sigma = 1$ and $T = 1$. This shows that the leading term in Merton's jump diffusions actually is independent of the jumps, and only depends on the diffusion coefficient σ. One explanation is that as N tends to infinity, the frequent fluctuation pattern of the Brownian motion will be the dominant source for Δ_N, while the jumps within in each small subinterval are very unlikely to occur since compound Poisson is a Lévy process of finite activity, which means there are only finitely many jumps in any compact interval almost surely.

(2) If we simply take the jump intensity $\lambda = 0$ to represent the case of no jumps, through calculations we could find that all the coefficients in the asymptotic expansion are exactly the same as those of Janssen and Van Leeuwaarden [37] when taking $\sigma = 1$ and $T = 1$. In other words, our result is simply a generalization of theirs.

(3) One may think about the limiting case as $\sigma \downarrow 0$, i.e. the case where there is no Brownian motion term but only the compound Poisson part. For that case, the condition stated in Theorem 3.1.1 that $\frac{(\lambda + \frac{\mu^2}{2\sigma^2})T}{N} < 2\pi$ is not satisfied for any N, thus we cannot simply take $\sigma = 0$ in Theorem 3.1.1 and we have to modify the proof somehow, as what will be shown in the next section.

3.2 Compound Poisson Process with Normal Jumps

Let's first recall the process:

$$X_t = \mu t + \sum_{n=1}^{N_t} Z_n,$$

where $\mu \in \mathbb{R}$, N_t is a Poisson process with intensity $\lambda > 0$, the jump size Z_n's are i.i.d normal random variables with mean m and variance $s^2 > 0$. Here, N_t and Z_n are independent. It is clear that the process is of finite activity and of finite variation. For simplicity, we assume that the process is defined on $[0, 1]$.

According to Sato [59] page 175, Remark 27.3: If X_t is a compound Poisson process on $\mathbb{R}^d$ with Lévy measure ν, then the law of X_t is given by

$$P_{X_t} = e^{-t\nu(\mathbb{R}^d)} \sum_{k=0}^{\infty} \frac{t^k \nu^k}{k!},$$

which is not continuous, as $\mathbb{P}(X_t = 0) > 0$. Also, for $t > 0$, $[P_{X_t}]_{\mathbb{R}^d \setminus \{0\}}$ is continuous if and only if ν is continuous. Here, ν^k should be understood as the n-fold convolution of ν and ν^0 is, in particular, understood to be the delta measure δ_0.

Consequently, in our case, the Lévy measure has finite mass on the entire real line (i.e. $\Pi(\mathbb{R}) = \lambda$), and is absolutely continuous with respect to the Lebesgue measure. Also, it is well known that the sum of independent normal random variables is still normally distributed. Thus, taking the drift term into account, we can write up the following,

$$\mathbb{E}[X_t^+] = \int_{\mathbb{R}} x \mathbb{1}_{\{x>0\}} P_{X_t}(dx)$$

$$= e^{-\lambda t} \int_{\mathbb{R}} x \mathbb{1}_{\{x>0\}} \delta_{\mu t}(x) dx + \int_0^\infty x e^{-\lambda t} \sum_{k=1}^\infty \frac{(\lambda t)^k}{k!} \frac{\exp\{-\frac{(x-\mu t-km)^2}{2(\sigma^2 t+ks^2)}\}}{\sqrt{2\pi(\sigma^2 t+ks^2)}} dx$$

$$= e^{-\lambda t} \mu t \mathbb{1}_{\{\mu t>0\}} + \int_0^\infty x e^{-\lambda t} \sum_{k=1}^\infty \frac{(\lambda t)^k}{k!} \frac{\exp\{-\frac{(x-\mu t-km)^2}{2(\sigma^2 t+ks^2)}\}}{\sqrt{2\pi(\sigma^2 t+ks^2)}} dx$$

$$= e^{-\lambda t} \mu t \mathbb{1}_{\{\mu>0\}} + \int_0^\infty x e^{-\lambda t} \sum_{k=1}^\infty \frac{(\lambda t)^k}{k!} \frac{\exp\{-\frac{(x-\mu t-km)^2}{2(\sigma^2 t+ks^2)}\}}{\sqrt{2\pi(\sigma^2 t+ks^2)}} dx.$$

Therefore, using Spitzer's identity, we obtain that

$$\mathbb{E}[\sup_{0\le t\le 1} X_t] = \mu \mathbb{1}_{\{\mu>0\}} \int_0^1 e^{-\lambda t} dt + \int_0^1 \frac{1}{t} \int_0^\infty x e^{-\lambda t} \sum_{k=1}^\infty \frac{(\lambda t)^k}{k!} \frac{\exp\{-\frac{(x-\mu t-km)^2}{2(\sigma^2 t+ks^2)}\}}{\sqrt{2\pi(\sigma^2 t+ks^2)}} dx dt, \tag{3.5}$$

and correspondingly,

$$\mathbb{E}[\sup_{0\le i\le N} X_{\frac{i}{N}}] = \mu \mathbb{1}_{\{\mu>0\}} \sum_{n=1}^N e^{-\lambda \frac{n}{N}} \frac{1}{N} + \sum_{n=1}^N \frac{1}{n} \int_0^\infty x e^{-\lambda \frac{n}{N}} \sum_{k=1}^\infty \frac{(\lambda \frac{n}{N})^k}{k!} \frac{\exp\{-\frac{(x-\mu \frac{n}{N}-km)^2}{2(\sigma^2 \frac{n}{N}+ks^2)}\}}{\sqrt{2\pi(\sigma^2 \frac{n}{N}+ks^2)}} dx.$$

So, we get the Corollary of Theorem 3.1.1:

Corollary 3.2.1. *For compound Poisson processes with normal jumps with jump intensity λ, mean m and variance s^2, the expected difference of continuous supremum and discrete maximum admits the following asymptotic expansion:*

$$\mathbb{E}[\sup_{0\le t\le 1} X_t - \sup_{0\le i\le N} X_{\frac{i}{N}}] = \frac{b_1}{N} + \sum_{i=1}^p \frac{b_{2i}}{N^{2i}} + O(\frac{1}{N^{2p+2}}),$$

where

$$b_1 = \frac{\mu}{2} \mathbb{1}_{\{\mu>0\}} (1-e^{-\lambda}) + \frac{1}{4}\Big(\lambda m p_1(0) - \sum_{k=1}^\infty \frac{\lambda^k}{k!}(\mu+mk) p_k(1)\Big) + \frac{1}{2}\big(\lambda r_1(0) - \sum_{k=1}^\infty \frac{\lambda^k}{k!} r_k(1)\big)$$

$$b_{2i}=-\frac{B_{2i}}{(2i)!}\Big(\mu\mathbb{1}_{\{\mu>0\}}\lambda^{2i-1}(1-e^{-\lambda})+\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}(r_k^{(2i-1)}(1)-r_k^{(2i-1)}(0))$$

$$+\frac{1}{2}\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\big(\mu(q_k^{(2i-1)}(1)-q_k^{(2i-1)}(0))+mk(p_k^{(2i-1)}(1)-p_k^{(2i-1)}(0))\big)\Big),$$

and $p_k(t), q_k(t)$ and $r_k(t)$ are defined the same way as in Theorem 3.1.1, B_n is the nth Bernoulli number.

Therefore, we see that the leading term of the above asymptotic expansion is of order $\frac{1}{N}$. Also, if we consider the very extreme case that the jump intensity $\lambda=0$, then all the coefficients $b_i=0$, which indicates the term $\Delta_N=0$ for any N. This is indeed the case since if $\lambda=\sigma=0$, we have $X_t=\mu t$, a deterministic Lévy process.

Alternatively, to get the above Corollary (3.2.1), we may apply a limit argument ($\sigma\downarrow 0$) in the proof of Theorem 3.1.1. More precisely,

$$\lim_{\sigma\downarrow 0}\mathbb{E}[X_t^+]=\lim_{\sigma\downarrow 0}\int_0^{\infty}xp(t,x)dx.$$

It suffices to check the first two terms in (3.1) as $k=0$, i.e., the terms that do not involve any jumps,

$$\lim_{\sigma\downarrow 0}\int_0^1 e^{-\lambda t}\big[\frac{\mu}{2}\mathrm{Erfc}(-\mu\sqrt{\frac{t}{2\sigma^2}})+\sqrt{\frac{\sigma^2}{2\pi t}}e^{-\frac{\mu^2 t}{2\sigma^2}}\big]dt.$$

$$=\int_0^1 e^{-\lambda t}\big[\frac{\mu}{2}\lim_{\sigma\downarrow 0}\mathrm{Erfc}(-\mu\sqrt{\frac{t}{2\sigma^2}})+\lim_{\sigma\downarrow 0}\sqrt{\frac{\sigma^2}{2\pi t}}e^{-\frac{\mu^2 t}{2\sigma^2}}\big]dt$$

$$=\int_0^1 e^{-\lambda t}\big[\frac{\mu}{2}\lim_{\sigma\downarrow 0}\mathrm{Erfc}(-\mu\sqrt{\frac{t}{2\sigma^2}})\big]dt. \tag{3.6}$$

The first equality above follows from the fact that $\mathrm{Erfc}(x)$ is absolutely bounded and the dominated convergence theorem. Here we have three cases.

Case 1: $\mu=0$, clearly, (3.6) is zero.

Case 2: $\mu<0$, since Erfc($+\infty$)=0, so (3.6) is zero.

Case 3: $\mu>0$, since Erfc($-\infty$)=2, so (3.6) is equal to

$$\mu\int_0^1 e^{-\lambda t}dt.$$

Combine these cases together, we obtain that

$$(3.6) = \mu \mathbb{1}_{\{\mu>0\}} \int_0^1 e^{-\lambda t} dt,$$

which is exactly the first term appearing on the right-hand side of $\mathbb{E}[\sup_{0\le t\le 1} X_t]$ in (3.5).

3.3 Normal Inverse Gaussian (NIG) Process

A Normal Inverse Gaussian process (NIG) is a pure jump Lévy process often used in modeling the log return of asset price, introduced by Barndorff-Nielsen [6]. NIG forms a subclass of the more general class called the hyperbolic Lévy processes. It can be described as a Brownian motion subordinated by an independent inverse Gaussian process (which is called a subordinator). More precisely, the Normal Inverse Gaussian process can be written as

$$X_t = \mu t + B(z_t; \beta, 1),$$

where $B(z_t; \beta, 1)$ is a Brownian motion with drift β and diffusion coefficient 1 subordinated by an independent inverse Gaussian process z_t. Recall that the inverse Gaussian process z_t can be modeled as the first time a standard Brownian motion with linear drift $\gamma > 0$ crosses above level δt ($\delta > 0$), i.e.,

$$z_t = \inf\{s > 0 : B_s + \gamma s > \delta t\}.$$

Denote $\alpha = \sqrt{\beta^2 + \gamma^2}$. The Lévy measure of X_t is

$$\Pi(dx) = \frac{\delta\alpha}{\pi|x|} e^{\beta x} K_1(\alpha|x|) dx,$$

where $K_1(x)$ is the modified Bessel function of the second kind with index 1. The characteristic function of the NIG process is given by

$$\phi_t(\xi) = \exp(i\mu t\xi - \delta t(\sqrt{\alpha^2 - (\beta + i\xi)^2} - \sqrt{\alpha^2 - \beta^2})).$$

The transition density of X_t is given by

$$p_t(x) = \frac{\alpha\delta t}{\pi} \frac{K_1(\alpha\sqrt{\delta^2 t^2 + (x - \mu t)^2})}{\sqrt{\delta^2 t^2 + (x - \mu t)^2}} \exp(\gamma\delta t + \beta(x - \mu t)).$$

By checking the Lévy density, it is easy to show that the NIG process is of infinite activity and infinite variation. For more details about the NIG process, one may refer to Barndorff-Nielsen [6], Cont and Tankov [16], and Feng [26].

Before we move on to the result about the expected difference Δ_N between continuous supremum and discrete maximum , we first need to establish a few lemmas that will be used later.

Lemma 3.3.1. *For integers $k \geq 0$, $a \in \mathbb{R}, T \in \mathbb{R}^+$, $\sum_{n=0}^{N} \frac{T}{N} e^{a\frac{nT}{N}} (\frac{nT}{N})^k$ converges to the integral $\int_0^T e^{at} t^k dt$ as $N \to \infty$. Moreover, we have*

$$\sum_{n=0}^{N} \frac{T}{N} e^{a\frac{nT}{N}} (\frac{nT}{N})^k - \int_0^T e^{at} t^k dt = T^{k+1} \big[\frac{1}{2N} \mathbb{1}_{\{k=0\}} + \frac{1}{2N} e^{aT}$$

$$+ \sum_{m=1}^{p} \frac{B_{2m}}{(2m)!} (g^{(2m-1)}(1) - g^{(2m-1)}(0)) \frac{1}{N^{2m}} \big] + O(\frac{1}{N^{2p+2}}), \tag{3.7}$$

where p is some positive integer, $g(x) = x^k e^{aTx}$, $B_m(x)$ is the Bernoulli polynomial of order m, B_m is the mth Bernoulli number, and $g^{(n)}(x)$ denotes the nth derivative of $g(x)$.

Proof. It is just a simple application of the Euler-Maclaurin formula. □

Lemma 3.3.2. *Let p be some positive integer greater than 1, $T \in (0, \infty)$, then*

$$\sum_{n=1}^{N} \frac{T}{N} \log \frac{nT}{N} - \int_0^T \log t dt = \frac{T \log N}{2N} + \frac{T \log 2\pi}{2N} + \frac{T}{12N^2} + \sum_{q=2}^{p} \frac{T B_{2q}}{2q(2q-1)N^{2q}} + O(\frac{1}{N^{2p+2}}).$$

Proof.

$$\begin{aligned}
& \sum_{n=1}^{N} \frac{T}{N} \log \frac{nT}{N} - \int_0^T \log t dt \\
= \; & T(\sum_{n=1}^{N} \frac{1}{N} \log \frac{n}{N} - \int_0^1 \log t dt) \\
= \; & T\big(\frac{1}{N}(\log(N!) - N \log N) + 1\big) \\
= \; & \frac{T}{N}(\log(N!) - N \log N + N) \\
= \; & \frac{T}{N}(\frac{1}{2} \log(2\pi N) + \frac{1}{12N} + \sum_{q=2}^{p} \frac{B_{2q}}{2q(2q-1)N^{2q-1}} + R_p(N)) \\
= \; & \frac{T \log N}{2N} + \frac{T \log 2\pi}{2N} + \frac{T}{12N^2} + \sum_{q=2}^{p} \frac{T B_{2q}}{2q(2q-1)N^{2q}} + O(\frac{1}{N^{2p+2}}),
\end{aligned}$$

where

$$|R_p(N)| \leq \frac{|B_{2p+2}|}{(2p+1)(2p+2)N^{2p+1}},$$

and the second last equality follows from Abramowitz and Stegen [1] 6.1.40 and 6.1.42, p is some positive integer, B_m denotes the mth Bernoulli number. □

Note that Lemma 3.3.2 actually refines the celebrated Sterling's formula. Before stating the next few lemmas, we first recall a result with regard to the Riemann zeta function, the Hurwitz zeta function and their derivatives.

Lemma 3.3.3. (Elizalde [23]) *Let k, N be positive integers, then the partial derivative of the Hurwitz zeta function (which was called the generalized Riemann zeta function in [23]) with respect to the first argument $\zeta'(z,q) := \frac{\partial}{\partial z}\zeta(z,q)$ admits the following asymptotic expansion:*

$$\zeta'(-k,N) = \frac{N^{k+1}\log N}{k+1} - \frac{N^{k+1}}{(k+1)^2} - \frac{1}{2}N^k \log N + \frac{k}{12}N^{k-1}\log N + \frac{1}{12}N^{k-1} + \sum_{q=1}^{\infty} a_{2q,k}N^{k-(2q+1)},$$

where

$$a_{2q,k} = \begin{cases} \frac{B_{2q+2}}{2q+2}\left[\binom{k}{2q+1}\log N + \sum_{h=0}^{2q}\binom{k}{h}\frac{(-1)^h}{2q-h+1}\right] & \text{if } 2q \leq k-1 \\ \frac{B_{2q+2}}{2q+2}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^h}{2q-h+1} & \text{if } 2q \geq k, \end{cases}$$

and B_{2q+2} is the $(2q+2)$th Bernoulli number.

In the following lemmas, we will deal with the difference between $\sum_{n=1}^{N}\frac{T}{N}(\frac{nT}{N})^k \log\frac{nT}{N}$ and $\int_0^T t^k \log t dt$ for $k \geq 1$. Without loss of generality, we take $T = 1$.

Lemma 3.3.4. *Case (i) $k = 1$:*

$$\sum_{n=1}^{N}\frac{1}{N}\frac{n}{N}\log\frac{n}{N} - \int_0^1 t\log t dt$$

$$= \frac{\log N}{12N^2} + \frac{\log A}{N^2} - \sum_{q=1}^{p}\frac{B_{2q+2}}{2q(2q+1)(2q+2)}\frac{1}{N^{2q+2}} + O(\frac{1}{N^{2p+4}}),$$

where A is the Glaisher-Kinkelin constant and p is some positive integer.

Case (ii) $k = 2$:

$$\sum_{n=1}^{N} \frac{1}{N}(\frac{n}{N})^2 \log \frac{n}{N} - \int_0^1 t^2 \log t dt$$

$$= \frac{1}{12N^2} - \frac{\zeta'(-2)}{N^3} + \sum_{q=1}^{p} \frac{B_{2q+2}}{q(2q-1)(2q+1)(2q+2)} \frac{1}{N^{2q+2}} + O(\frac{1}{N^{2p+4}}).$$

Case (iii) $k = 3$*:*

$$\sum_{n=1}^{N} \frac{1}{N}(\frac{n}{N})^3 \log \frac{n}{N} - \int_0^1 t^3 \log t dt = \frac{1}{12N^2} - \frac{1}{120} \frac{\log N}{N^4} - (\zeta'(-3) + \frac{11}{720}) \frac{1}{N^4}$$
$$- \sum_{q=2}^{p} \frac{3B_{2q+2}}{q(2q-2)(2q-1)(2q+1)(2q+2)} \frac{1}{N^{2q+2}} + O(\frac{1}{N^{2p+4}}).$$

Since the proof of all the cases in Lemma 3.3.4 are pretty much the same, so we only give the proof for the first case in the lemma.

Proof. First we note that the result in case (i) actually gives the Sterling-like representation for the hyperfactorial sequence. It is trivial that $\int_0^1 t \log t dt = -\frac{1}{4}$ through integration by parts. Also,

$$\begin{aligned}
& \sum_{n=1}^{N} \frac{1}{N} \frac{n}{N} \log \frac{n}{N} \\
= & \sum_{n=1}^{N-1} \frac{1}{N} \frac{n}{N} \log \frac{n}{N} \\
= & \frac{1}{N^2} \Big(\sum_{n=1}^{N-1} n \log n - \log N \sum_{n=1}^{N-1} n \Big) \\
= & \frac{1}{N^2} \Big(-\zeta'(-1) + \zeta'(-1, N) - \frac{1}{2} N(N-1) \log N \Big) \\
= & -\frac{1}{N^2}\zeta'(-1) + \frac{1}{2} \log N - \frac{1}{4} - \frac{\log N}{2N} + \frac{\log N}{12N^2} + \frac{1}{12N^2} \\
& - \sum_{q=1}^{p} \frac{B_{2q+2}}{2q(2q+1)(2q+2)} \frac{1}{N^{2q+2}} + R_{p,1}(N) - \frac{1}{2} \log N + \frac{\log N}{2N} \\
= & -\frac{1}{N^2}\zeta'(-1) - \frac{1}{4} + \frac{\log N}{12N^2} + \frac{1}{12N^2} - \sum_{q=1}^{p} \frac{B_{2q+2}}{2q(2q+1)(2q+2)} \frac{1}{N^{2q+2}} + R_{p,1}(N),
\end{aligned}$$

where the third equality follows from Adamchik [2], the fourth equality follows from expression (18) in Elizalde [23], and

$$|R_{p,1}(N)| \leq \frac{B_{2p+4}}{(2p+2)(2p+3)(2p+4)} \frac{1}{N^{2p+4}},$$

and p is some positive integer. It is well known that $\zeta'(-1) = \frac{1}{12} - \log A$, where $A \approx 1.2824271291$ is the Glaisher-Kinkelin constant. Hence,

$$\sum_{n=1}^{N} \frac{1}{N}\frac{n}{N}\log\frac{n}{N} - \int_0^1 t\log t dt = \frac{\log N}{12N^2} + \frac{\log A}{N^2} - \sum_{q=1}^{p} \frac{B_{2q+2}}{2q(2q+1)(2q+2)}\frac{1}{N^{2q+2}} + O(\frac{1}{N^{2p+4}}).$$

□

In general, we have the following lemma about the difference between $\sum_{n=1}^{N}\frac{1}{N}(\frac{n}{N})^k\log\frac{n}{N}$ and $\int_0^1 t^k \log t dt$ for $k \geq 4$.

Lemma 3.3.5. *For $k \geq 4$:*

$$\sum_{n=1}^{N} \frac{1}{N}(\frac{n}{N})^k \log\frac{n}{N} - \int_0^1 t^k \log t dt$$

$$= \frac{1}{12N^2} - \frac{\zeta'(-k)}{N^{k+1}} + \sum_{q=1}^{\lfloor \frac{k}{2} \rfloor} \frac{b_{2q,k}}{N^{2q+2}} + \sum_{q=\lfloor \frac{k}{2} \rfloor+1}^{\infty} \frac{c_{2q,k}}{N^{2q+2}} + \frac{B_{k+1}}{k+1}\frac{\log N}{N^{k+1}},$$

where

$$b_{2q,k} = \frac{B_{2q+2}}{2q+2}\sum_{h=0}^{2q}\binom{k}{h}\frac{(-1)^h}{2q-h+1}, c_{2q,k} = \frac{B_{2q+2}}{2q+2}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^h}{2q-h+1},$$

ζ is the Riemann zeta function, B_m is the mth Bernoulli number, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.

Actually, Lemma 3.3.5 also works for $k = 1, 2, 3$, but the proof will make more sense if $k \geq 4$ and for the illustration purpose, we separately list the cases $k = 1, 2, 3$ in the previous lemma.

Proof. Using integration by parts, we get that $\int_0^1 t^k \log t dt = -\frac{1}{(k+1)^2}$. Next,

$$\sum_{n=1}^{N} \frac{1}{N}(\frac{n}{N})^k \log\frac{n}{N} = \sum_{n=1}^{N-1} \frac{1}{N}(\frac{n}{N})^k \log\frac{n}{N}$$

$$= \frac{1}{N^{k+1}}\Big(\sum_{n=1}^{N-1} n^k \log n - \log N \sum_{n=1}^{N-1} n^k\Big). \tag{3.8}$$

By Adamchik [2], the first term in (3.8) is given by

$$\frac{1}{N^{k+1}}\sum_{n=1}^{N-1} n^k \log n = \frac{1}{N^{k+1}}\Big(-\zeta'(-k) + \zeta'(-k,N)\Big)$$

$$= -\frac{\zeta'(-k)}{N^{k+1}} + \frac{\log N}{k+1} - \frac{1}{(k+1)^2} - \frac{\log N}{2N} + \frac{k}{12}\frac{\log N}{N^2} + \frac{1}{12N^2} + \sum_{q=1}^{\infty} a_{2q,k}\frac{1}{N^{2q+2}},$$

which follows from Lemma 3.3.3. Also, by the power summation formula (Bernoulli's formula or Faulhaber's formula, see e.g. Knuth [42]),

$$\sum_{n=1}^{N-1} n^k = \sum_{n=1}^{N} n^k - N^k = \sum_{l=1}^{k+1} \eta_{k,l} N^l - N^k,$$

where $\eta_{k,l} = \frac{(-1)^{k-l+1} B_{k-l+1} k!}{l!(k-l+1)!}$ and $B_1 = -\frac{1}{2}$. Specifically, $\eta_{k,k+1} = \frac{1}{k+1}$, $\eta_{k,k} = \frac{1}{2}$, $\eta_{k,k-1} = \frac{k}{12}$, and $\eta_{k,k-2r} = 0$ for all integers $1 \le r \le \lfloor \frac{k-1}{2} \rfloor$ since all odd order (starting from 3 and above) Bernoulli numbers are zero. Therefore,

$$\sum_{n=1}^{N-1} n^k = \sum_{q=1}^{\lfloor \frac{k-2}{2} \rfloor} \eta_{k,k-(2q+1)} N^{k-(2q+1)} + \frac{k}{12} N^{k-1} - \frac{1}{2} N^k + \frac{1}{k+1} N^{k+1}.$$

Note that $\lfloor \frac{k-2}{2} \rfloor \ge 1$ implies $k \ge 4$. This is why we need to separate this case ($k \ge 4$) with the previous cases ($k = 1, 2, 3$). Hence, the second term in (3.8) is given by

$$\frac{\log N}{N^{k+1}} \sum_{n=1}^{N-1} n^k = \sum_{q=1}^{\lfloor \frac{k-2}{2} \rfloor} \eta_{k,k-(2q+1)} \frac{\log N}{N^{2q+2}} + \frac{k}{12}\frac{\log N}{N^2} - \frac{\log N}{2N} + \frac{\log N}{k+1}.$$

So we get that

$$\begin{aligned}
&\sum_{n=1}^{N} \frac{1}{N}\Big(\frac{n}{N}\Big)^k \log\frac{n}{N} - \int_0^1 t^k \log t\,dt \\
= \quad & -\frac{\zeta'(-k)}{N^{k+1}} + \frac{\log N}{k+1} - \frac{1}{(k+1)^2} - \frac{\log N}{2N} + \frac{k}{12}\frac{\log N}{N^2} + \frac{1}{12N^2} + \sum_{q=1}^{\infty} a_{2q,k}\frac{1}{N^{2q+2}} \\
& - \sum_{q=1}^{\lfloor \frac{k-2}{2} \rfloor} \eta_{k,k-(2q+1)} \frac{\log N}{N^{2q+2}} - \frac{k}{12}\frac{\log N}{N^2} + \frac{\log N}{2N} - \frac{\log N}{k+1} + \frac{1}{(k+1)^2} \\
= \quad & -\frac{\zeta'(-k)}{N^{k+1}} + \frac{1}{12N^2} + \sum_{q=1}^{\infty} a_{2q,k}\frac{1}{N^{2q+2}} - \sum_{q=1}^{\lfloor \frac{k-2}{2} \rfloor} \eta_{k,k-(2q+1)} \frac{\log N}{N^{2q+2}} \\
= \quad & -\frac{\zeta'(-k)}{N^{k+1}} + \frac{1}{12N^2} + \sum_{q=1}^{\lfloor \frac{k-2}{2} \rfloor} b_{2q,k}\frac{1}{N^{2q+2}} + \sum_{q=\lfloor \frac{k}{2} \rfloor}^{\infty} a_{2q,k}\frac{1}{N^{2q+2}},
\end{aligned} \tag{3.9}$$

where

$$b_{2q,k} = \frac{B_{2q+2}}{2q+2} \sum_{h=0}^{2q} \binom{k}{h} \frac{(-1)^h}{2q-h+1},$$

$$a_{2q,k} = \begin{cases} \frac{B_{2q+2}}{2q+2} \left[\binom{k}{2q+1} \log N + \sum_{h=0}^{2q} \binom{k}{h} \frac{(-1)^h}{2q-h+1} \right] & \text{if } 2q \leq k-1 \\ \frac{B_{2q+2}}{2q+2} \sum_{h=0}^{k} \binom{k}{h} \frac{(-1)^h}{2q-h+1} & \text{if } 2q \geq k. \end{cases}$$

If k is even, then $2q \geq k$ holds for all $q \geq \lfloor \frac{k}{2} \rfloor$, thus we don't get any $\log N$ terms involved in the last summation of (3.9). However, if k is odd, then the leading term in the last summation of (3.9) with $q = \lfloor \frac{k}{2} \rfloor$ implies that $2q = k-1$, which means the $\log N$ term will show up according to the definition of $a_{2q,k}$. Therefore, we conclude that, for even k, i.e., $k = 2m$ where $m \geq 2$,

$$\sum_{n=1}^{N} \frac{1}{N} (\frac{n}{N})^{2m} \log \frac{n}{N} - \int_0^1 t^{2m} \log t dt = \frac{1}{12N^2} - \frac{\zeta'(-2m)}{N^{2m+1}} + \sum_{q=1}^{m-1} b_{2q,2m} \frac{1}{N^{2q+2}} + \sum_{q=m}^{\infty} c_{2q,2m} \frac{1}{N^{2q+2}},$$

where $b_{2q,2m} = \frac{B_{2q+2}}{2q+2} \sum_{h=0}^{2q} \frac{(-1)^h \binom{2m}{h}}{2q-h+1}$, $c_{2q,2m} = \frac{B_{2q+2}}{2q+2} \sum_{h=0}^{2m} \frac{(-1)^h \binom{2m}{h}}{2q-h+1}$; and for odd k, i.e., $k = 2m+1$ where $m \geq 2$,

$$\sum_{n=1}^{N} \frac{1}{N} (\frac{n}{N})^{2m+1} \log \frac{n}{N} - \int_0^1 t^{2m+1} \log t dt$$

$$= \frac{1}{12N^2} - \frac{\zeta'(-2m-1)}{N^{2m+2}} + \sum_{q=1}^{m-1} b_{2q,2m+1} \frac{1}{N^{2q+2}} + \frac{B_{2m+2}}{2m+2} \frac{\log N}{N^{2m+2}}$$

$$+ \frac{B_{2m+2}}{2m+2} \sum_{h=0}^{2m} \binom{2m+1}{h} \frac{(-1)^h}{2m-h+1} \frac{1}{N^{2m+2}} + \sum_{q=m+1}^{\infty} c_{2q,2m+1} \frac{1}{N^{2q+2}}.$$

Collectively, we may write

$$\sum_{n=1}^{N} \frac{1}{N} (\frac{n}{N})^{k} \log \frac{n}{N} - \int_0^1 t^{k} \log t dt = \frac{1}{12N^2} - \frac{\zeta'(-k)}{N^{k+1}} + \sum_{q=1}^{\lfloor \frac{k}{2} \rfloor} \frac{b_{2q,k}}{N^{2q+2}} + \sum_{q=\lfloor \frac{k}{2} \rfloor + 1}^{\infty} \frac{c_{2q,k}}{N^{2q+2}} + \frac{B_{k+1}}{k+1} \frac{\log N}{N^{k+1}},$$

where

$$b_{2q,k} = \frac{B_{2q+2}}{2q+2} \sum_{h=0}^{2q} \binom{k}{h} \frac{(-1)^h}{2q-h+1}$$

$$c_{2q,k} = \frac{B_{2q+2}}{2q+2}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^h}{2q-h+1}.$$

□

Remark 3.3.6. More concisely, we may write the above two lemmas as, for $k \geq 1$,

$$\sum_{n=1}^{N}\frac{1}{N}(\frac{n}{N})^k \log\frac{n}{N} - \int_0^1 t^k \log t dt = \frac{1}{12N^2} - \frac{\zeta'(-k)}{N^{k+1}} + \sum_{q=1}^{\infty}\frac{d_{2q,k}}{N^{2q+2}} + \frac{B_{k+1}}{k+1}\frac{\log N}{N^{k+1}}$$

with

$$d_{2q,k} = \begin{cases} b_{2q,k} = \frac{B_{2q+2}}{2q+2}\sum_{h=0}^{2q}\binom{k}{h}\frac{(-1)^h}{2q-h+1} & \text{if } q \leq \lfloor\frac{k}{2}\rfloor \\ c_{2q,k} = \frac{B_{2q+2}}{2q+2}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^h}{2q-h+1} & \text{if } q \geq \lfloor\frac{k}{2}\rfloor + 1. \end{cases}$$

Remark 3.3.7. Our result in Remark 3.3.6 actually is a very interesting example that is consistent with the result stated in Cruz-Uribe and Neugebauer [17] in the following sense.

Let $f_k(t) = t^k \log t$ defined on $I := [0,1]$ for $k \geq 1$. It is clear that $f_k \in C(I)$, the space of continuous function on $[0,1]$. For all $k \geq 2$, $f_k \in W_2^1(I)$, the Sobolev space that is defined as the space of differentiable functions f such that f' is absolutely continuous and $f'' \in L^2(I)$. Thus, according to Theorem 1.23 in Cruz-Uribe and Neugebauer [17], we exactly get the same leading term $\frac{1}{12N^2}$ as what they got.

The interesting thing is the case when $k = 1$, i.e. $f_1(t) = t\log t$. It is clear that $f_1 \notin W_2^p(I)$ for any $p \geq 1$, but $f_1 \in W_1^1(I)$, which is defined as the Sobolev space of absolutely continuous functions f such that $f' \in L^1(I)$. However, the leading term for the difference between the summation and the integral as $k = 1$ in Remark 3.3.6 is $\frac{\log N}{N^2}$, which contradicts to the Theorem 1.13 in Cruz-Uribe and Neugebauer [17]. This indicates that f_1 lies in the space that is strictly smoother than $W_1^1(I)$.

Fortunately, we find that $f_1 \in W_1^{1,\infty}(I)$, which is the space of absolutely continuous functions f such that f' is in the Lorentz space $L^{1,\infty}(I)$. In terms of smoothness, $W_1^{1,\infty}(I)$ lies between $W_1^1(I)$ and $W_2^1(I)$. For the formal definition of Lorentz space, one may refer to Stein and Weiss [64].

From the previous lemmas, we can derive the following lemma.

Lemma 3.3.8. *For $N \in \mathbb{N}$, $a \in \mathbb{R}, T \in \mathbb{R}^+$,*

$$\sum_{n=1}^{N}\frac{T}{N}e^{a\frac{nT}{N}}\log\frac{nT}{N} \longrightarrow \int_0^T e^{at}\log t dt \quad \text{as} \quad N \to \infty.$$

Moreover, with $\frac{|a|T}{2\pi N} < 1$ satisfied,

$$
\begin{aligned}
&\sum_{n=1}^{N}\frac{T}{N}e^{a\frac{nT}{N}}\log\frac{nT}{N}-\int_0^T e^{at}\log t dt\\
=\;& T\Big(\frac{\log N}{2N}+\frac{\log 2\pi}{2N}+\frac{aT\log N}{12N^2}+(\frac{e^{aT}}{12}+aT\log A-\frac{aT}{12})\frac{1}{N^2}\\
&+\sum_{q=1}^{p}\frac{D(q)}{N^{2q+2}}-\sum_{k=2}^{2p+2}\frac{(aT)^k}{k!}\frac{\zeta'(-k)}{N^{k+1}}+\sum_{m=1}^{p+1}\frac{(aT)^{2m+1}B_{2m+2}}{(2m+2)!}\frac{\log N}{N^{2m+2}}\Big)\\
&+T\log T\Big(\frac{e^{aT}-1}{2N}+\sum_{m=1}^{p+1}\frac{(aT)^{2m-1}(e^{aT}-1)B_{2m}}{(2m)!N^{2m}}\Big)+O(\frac{1}{N^{2p+4}}),
\end{aligned}
$$

where

$$D(q)=\sum_{l=0}^{2q-1}\frac{(aT)^l}{l!}c_{2q,l}+\sum_{l=2q}^{\infty}\frac{(aT)^l}{l!}b_{2q,l}$$

$$b_{2q,l}=\frac{B_{2q+2}}{2q+2}\sum_{h=0}^{2q}\binom{l}{h}\frac{(-1)^h}{2q-h+1}$$

$$c_{2q,l}=\frac{B_{2q+2}}{2q+2}\sum_{h=0}^{l}\binom{l}{h}\frac{(-1)^h}{2q-h+1},$$

and ζ is the Riemann zeta function, B_m is the mth Bernoulli number.

Proof. First, we note that

$$\sum_{n=1}^{N}\frac{T}{N}e^{a\frac{nT}{N}}\log\frac{nT}{N}=T\sum_{n=1}^{N}\frac{1}{N}e^{aT\frac{n}{N}}\log\frac{n}{N}+T\log T\sum_{n=1}^{N}\frac{1}{N}e^{aT\frac{n}{N}},$$

and

$$\int_0^T e^{at}\log t dt=T\int_0^1 e^{aTt}\log t dt+T\log T\int_0^1 e^{aTt}dt.$$

The difference incurred from $T\log T\int_0^1 e^{aTt}dt$ actually is just an application of the Euler-Maclaurin formula. More precisely,

$$T\log T(\sum_{n=1}^{N}\frac{1}{N}e^{aT\frac{n}{N}}-\int_0^1 e^{aTt}dt)=T\log T\Big(\frac{e^{aT}-1}{2N}+\sum_{m=1}^{p+1}\frac{(aT)^{2m-1}(e^{aT}-1)B_{2m}}{(2m)!N^{2m}}+O(\frac{1}{N^{2p+4}})\Big).$$

So we left with the difference from the first terms, namely

$$
\begin{aligned}
& T\sum_{n=1}^{N}\frac{1}{N}e^{aT\frac{n}{N}}\log\frac{n}{N}-T\int_0^1 e^{aTt}\log t dt \\
= \;& T\Big(\sum_{n=1}^{N}\frac{1}{N}\sum_{k=0}^{\infty}\frac{(aT\frac{n}{N})^k}{k!}\log\frac{n}{N}-\int_0^1\sum_{k=0}^{\infty}\frac{(aTt)^k}{k!}\log t dt\Big) \\
= \;& T\sum_{k=0}^{\infty}\frac{(aT)^k}{k!}\Big(\sum_{n=1}^{N}\frac{1}{N}(\frac{n}{N})^k\log\frac{n}{N}-\int_0^1 t^k\log t dt\Big) \\
= \;& T\Big(\frac{\log N}{2N}+\frac{\log 2\pi}{2N}+\frac{1}{12N^2}+\sum_{q=1}^{p}\frac{B_{2q+2}}{(2q+2)(2q+1)N^{2q+2}}+O(\frac{1}{N^{2p+4}})\Big) \\
& +aT^2\Big(\frac{\log N}{12N^2}+\frac{\log A}{N^2}-\sum_{q=1}^{p}\frac{B_{2q+2}}{2q(2q+1)(2q+2)N^{2q+2}}+O(\frac{1}{N^{2p+4}})\Big) \\
& +\frac{T(aT)^2}{2}\Big(\frac{1}{12N^2}-\frac{\zeta'(-2)}{N^3}+\sum_{q=1}^{p}\frac{B_{2q+2}}{q(2q-1)(2q+1)(2q+2)N^{2q+2}}+O(\frac{1}{N^{2p+4}})\Big) \\
& +\frac{T(aT)^3}{3!}\Big(\frac{1}{12N^2}-\frac{1}{120}\frac{\log N}{N^4}-(\zeta'(-3)+\frac{11}{720})\frac{1}{N^4} \\
& -\sum_{q=2}^{p}\frac{3B_{2q+2}}{q(2q-2)(2q-1)(2q+1)(2q+2)}\frac{1}{N^{2q+2}}+O(\frac{1}{N^{2p+4}})\Big) \\
& +T\sum_{k=4}^{\infty}\frac{(aT)^k}{k!}\Big(\frac{1}{12N^2}-\frac{\zeta'(-k)}{N^{k+1}}+\sum_{q=1}^{\lfloor\frac{k}{2}\rfloor}\frac{b_{2q,k}}{N^{2q+2}}+\sum_{q=\lfloor\frac{k}{2}\rfloor+1}^{\infty}\frac{c_{2q,k}}{N^{2q+2}}+\frac{B_{k+1}}{k+1}\frac{\log N}{N^{k+1}}\Big), \quad (3.10)
\end{aligned}
$$

where the second equality follows from the dominated convergence theorem, the third equality follows from Lemma 3.3.2, Lemma 3.3.4, and Lemma 3.3.6, and

$$
b_{2q,k}=\frac{B_{2q+2}}{2q+2}\sum_{h=0}^{2q}\binom{k}{h}\frac{(-1)^h}{2q-h+1}
$$

$$
c_{2q,k}=\frac{B_{2q+2}}{2q+2}\sum_{h=0}^{k}\binom{k}{h}\frac{(-1)^h}{2q-h+1}.
$$

Note that the last infinite summation term in (3.10) could be written as

$$
\begin{aligned}
& \frac{T}{12N^2}\sum_{k=4}^{\infty}\frac{(aT)^k}{k!}-T\sum_{k=4}^{\infty}\frac{(aT)^k}{k!}\frac{\zeta'(-k)}{N^{k+1}}+T\sum_{m=2}^{\infty}\frac{(aT)^{2m+1}}{(2m+1)!}\frac{B_{2m+2}}{2m+2}\frac{\log N}{N^{2m+2}}+\frac{T}{N^4}\sum_{k=4}^{\infty}\frac{(aT)^k}{k!}b_{2,k} \\
& +\frac{T}{N^6}\sum_{k=4}^{\infty}\frac{(aT)^k}{k!}b_{4,k}+\sum_{m=3}^{p}\frac{T}{N^{2m+2}}\Big(\sum_{l=4}^{2m-1}\frac{(aT)^l}{l!}c_{2m,l}+\sum_{l=2m}^{\infty}\frac{(aT)^l}{l!}b_{2m,l}\Big)+O(\frac{1}{N^{2p+4}}). \quad (3.11)
\end{aligned}
$$

Clearly, the first term $\frac{T}{12N^2}\sum_{k=4}^{\infty}\frac{(aT)^k}{k!}$ in (3.11) is finite for any given N. The second term in (3.11) involves derivatives of Riemann zeta function. The finiteness of this term needs to be justified. First, we recall that by the reflection functional equation, see e.g. Choudhury [15],

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\zeta(s)\cos(\frac{1}{2}s\pi),$$

and

$$\frac{\zeta'(1-s)}{\zeta(1-s)} = \log 2\pi + \frac{\pi}{2}\tan(\frac{1}{2}s\pi) - \psi(s) - \frac{\zeta'(s)}{\zeta(s)},$$

where $\psi(1) = -\widetilde{\gamma}$, $\psi(s) = -\widetilde{\gamma} + \sum_{k=1}^{s-1} k^{-1}$, $s \geq 2$, $\widetilde{\gamma} \approx 0.5772156649$ is the Euler-Mascheroni constant. Hence, we get that, for a positive integer n,

$$\zeta'(-2n) = \frac{(-1)^n\zeta(2n+1)(2n)!}{2^{2n+1}\pi^{2n}},$$

and

$$\zeta'(-2n+1) = \zeta(-2n+1)(\log 2\pi - \psi(2n) - \frac{\zeta'(2n)}{\zeta(2n)})$$

$$= 2(2\pi)^{-2n}(2n-1)!(-1)^n(\log 2\pi\zeta(2n) - \psi(2n)\zeta(2n) - \zeta'(2n)),$$

and also

$$\zeta'(s) = -\sum_{k=1}^{\infty}\frac{\log k}{k^s} = -\sum_{k=2}^{\infty}\frac{\log k}{k^s}, \text{ for } Re(s) > 1.$$

Thus,

$$\Big|\sum_{k=4}^{\infty}\frac{(aT)^k}{k!}\frac{\zeta'(-k)}{N^{k+1}}\Big| \leq \sum_{k=4}^{\infty}\frac{(|a|T)^k}{k!}\frac{|\zeta'(-k)|}{N^{k+1}} = \sum_{n=2}^{\infty}\frac{(aT)^{2n}\zeta(2n+1)}{2N(2\pi N)^{2n}}$$

$$+\sum_{n=2}^{\infty}\frac{(|a|T)^{2n+1}}{\pi N(2\pi N)^{2n+1}}\Big|\log 2\pi\zeta(2n+2) - \psi(2n+2)\zeta(2n+2) - \zeta'(2n+2)\Big|.$$

So by the ratio test, the above series converge absolutely if $\frac{|a|T}{2\pi N} < 1$.

In addition, the finiteness of the third term $T\sum_{m=2}^{\infty}\frac{(aT)^{2m+1}}{(2m+1)!}\frac{B_{2m+2}}{2m+2}\frac{\log N}{N^{2m+2}}$ in (3.11) also needs to be established. Due to Abramowitz and Stegun [1] page 805, 23.1.15, for all $m \geq 2$

$$|B_{2m+2}| < \frac{2(2m+2)!}{(2\pi)^{2m+2}} \frac{1}{1-2^{-1-2m}} \leq \frac{2(2m+2)!}{(2\pi)^{2m+2}} \frac{32}{31}.$$

Thus,

$$\begin{aligned}
& \Big| \sum_{m=2}^{\infty} \frac{(aT)^{2m+1}}{(2m+1)!} \frac{B_{2m+2}}{2m+2} \frac{\log N}{N^{2m+2}} \Big| \\
\leq & \sum_{m=2}^{\infty} \frac{(|a|T)^{2m+1}}{(2m+1)!} \frac{|B_{2m+2}|}{2m+2} \frac{\log N}{N^{2m+2}} \\
\leq & \sum_{m=2}^{\infty} \frac{(|a|T)^{2m+1}}{N^{2m+1}} \frac{64}{31(2\pi)^{2m+2}} \frac{\log N}{N}.
\end{aligned}$$

It is clear that the above series converges absolutely if $\frac{|a|T}{2\pi N} < 1$. Also, according to the definition of $b_{2q,k}$, the terms $\sum_{k=4}^{\infty} \frac{(aT)^k}{k!} b_{2,k}$ and $\sum_{k=4}^{\infty} \frac{(aT)^k}{k!} b_{4,k}$ are absolutely convergent.

The last infinite sum term in (3.11) can be bounded by

$$\sum_{l=2m}^{\infty} \frac{(|a|T)^l}{l!} |b_{2m,l}| \leq \frac{|B_{2m+2}|}{2m+2} \sum_{l=2m}^{\infty} \frac{(|a|T)^l}{l!} 2^l \leq \frac{|B_{2m+2}|}{2m+2} e^{\frac{|a|T}{2}},$$

which is also finite. Therefore, (3.11) is a well-defined asymptotic expansion. In conclusion, with $\frac{|a|T}{2\pi N} < 1$,

$$T \sum_{n=1}^{N} \frac{1}{N} e^{aT\frac{n}{N}} \log \frac{n}{N} - T \int_0^1 e^{aTt} \log t dt$$

$$= T\Big(\frac{\log N}{2N} + \frac{\log 2\pi}{2N} + \frac{aT \log N}{12N^2} + (\frac{e^{aT}}{12} + aT \log A - \frac{aT}{12}) \frac{1}{N^2}$$
$$+ \sum_{q=1}^{p} \frac{D(q)}{N^{2q+2}} - \sum_{k=2}^{2p+2} \frac{(aT)^k}{k!} \frac{\zeta'(-k)}{N^{k+1}} + \sum_{m=1}^{p+1} \frac{(aT)^{2m+1} B_{2m+2}}{(2m+2)!} \frac{\log N}{N^{2m+2}} \Big) + O(\frac{1}{N^{2p+4}}),$$

where

$$D(q) = \sum_{l=0}^{2q-1} \frac{(aT)^l}{l!} c_{2q,l} + \sum_{l=2q}^{\infty} \frac{(aT)^l}{l!} b_{2q,l},$$

which completes the proof. □

The next lemma will be focused on the modified Bessel function of the second kind $K_\alpha(x)$, which is an important component in the transition density function of the NIG. We will need the following lemma later.

Lemma 3.3.9. *For any integers $k \geq -1$ and positive real number t, the modified Bessel function of the second kind K_α satisfies*

$$\int_1^\infty K_1(tz)(z^2-1)^{\frac{k}{2}}dz = 2^{\frac{k}{2}}t^{-1-\frac{k}{2}}K_{\frac{k}{2}}(t)\Gamma(1+\frac{k}{2}).$$

Proof. First, we recall Abramowitz and Stegun [1] 9.6.27 and 11.3.27 or 11.3.28, and it is easy to see that the lemma immediately holds for $k = 0$. Also, we get that

$$d(-K_0(tz)) = K_1(tz)d(tz), \qquad d(tzK_1(tz)) = -tzK_0(tz)d(tz).$$

We find the following recurrence relation:

$$S_{\frac{k}{2}}(t) := \int_1^\infty K_1(tz)(z^2-1)^{\frac{k}{2}}dz$$

$$= \frac{1}{t}\int_1^\infty (z^2-1)^{\frac{k}{2}}d(-K_0(tz))$$

$$= \frac{1}{t}\int_1^\infty K_0(tz)\frac{k}{2}(z^2-1)^{\frac{k}{2}-1}2zdz$$

$$= \frac{k}{t^3}\int_1^\infty K_0(tz)tz(z^2-1)^{\frac{k}{2}-1}d(tz)$$

$$= \frac{k}{t^3}\int_1^\infty (z^2-1)^{\frac{k}{2}-1}d(-tzK_1(tz))$$

$$= \frac{k}{t^3}\int_1^\infty tzK_1(tz)(\frac{k}{2}-1)(z^2-1)^{\frac{k}{2}-2}2zdz$$

$$= \frac{k(k-2)}{t^2}\int_1^\infty z^2K_1(tz)(z^2-1)^{\frac{k}{2}-2}dz$$

$$= \frac{k(k-2)}{t^2}\int_1^\infty (z^2-1+1)K_1(tz)(z^2-1)^{\frac{k}{2}-2}dz$$

$$= \frac{k(k-2)}{t^2}\Big(\int_1^\infty K_1(tz)(z^2-1)^{\frac{k}{2}-1}dz + \int_1^\infty K_1(tz)(z^2-1)^{\frac{k}{2}-2}dz\Big)$$

$$= \frac{k(k-2)}{t^2}[S_{\frac{k}{2}-1}(t) + S_{\frac{k}{2}-2}(t)].$$

Through first half of the derivation above (i.e., integration by parts only once, up to the fourth equality), we immediately see that the lemma holds for $k = 2$. Next, we take a look at different cases for k.

Case (i): k is an even integer, i.e., $k = 2m$. So far we have gotten that the lemma holds for $m = 0, 1$. Considering those as induction base, we use the above recurrence relation and apply the classical induction technique to see that

$$\begin{aligned}
S_m(t) &= \frac{4m(m-1)}{t^2}[S_{m-1}(t) + S_{m-2}(t)] \\
&= \frac{4m(m-1)}{t^2}[2^{m-1}t^{-m}K_{m-1}(t)\Gamma(m) + 2^{m-2}t^{1-m}K_{m-2}(t)\Gamma(m-1)] \\
&= \frac{4m(m-1)}{t^2}2^{m-2}t^{1-m}\Gamma(m-1)[\frac{2}{t}(m-1)K_{m-1}(t) + K_{m-2}(t)] \\
&= \frac{4m(m-1)}{t^2}2^{m-2}t^{1-m}\Gamma(m-1)K_m(t) \\
&= 2^m\Gamma(m+1)t^{-1-m}K_m(t),
\end{aligned}$$

where the last equality above follows from Abramowitz and Stegun [1] 9.6.26.

Case (ii): k is an odd integer, i.e., $k = 2m+1$, then the recurrence relation only makes sense for $m \geq 1$, namely,

$$S_{m+\frac{1}{2}}(t) = \frac{(2m+1)(2m-1)}{t^2}[S_{m-\frac{1}{2}}(t) + S_{m-\frac{3}{2}}(t)].$$

In order to do induction, we must show the induction base, i.e., when $m = -1, 0$. Recall that for $m = -1$,

$$S_{-\frac{1}{2}}(t) = \int_1^\infty K_1(tz)(z^2-1)^{-\frac{1}{2}}dz.$$

We claim that:

$$S_{-\frac{1}{2}}(t) = \frac{\pi}{2t}e^{-t}.$$

Proof of the claim: We first show that

$$tS_{-\frac{1}{2}}(t) + (tS_{-\frac{1}{2}}(t))' = 0.$$

Using the dominated convergence theorem for interchange of integration and differentiation, we have

$$\begin{aligned}
S'_{-\frac{1}{2}}(t) &= \int_1^\infty \frac{\partial K_1(tz)}{\partial t}(z^2-1)^{-\frac{1}{2}}dz \\
&= \int_1^\infty (-K_0(tz) - \frac{K_1(tz)}{tz})\frac{z}{\sqrt{z^2-1}}dz.
\end{aligned}$$

Thus,

$$\begin{aligned}
& tS_{-\frac{1}{2}}(t) + (tS_{-\frac{1}{2}}(t))' \\
= \; & tS_{-\frac{1}{2}}(t) + S_{-\frac{1}{2}}(t) + tS'_{-\frac{1}{2}}(t) \\
= \; & tS_{-\frac{1}{2}}(t) + S_{-\frac{1}{2}}(t) - \int_1^\infty tK_0(tz)\frac{z}{\sqrt{z^2-1}}dz - \int_1^\infty \frac{K_1(tz)}{\sqrt{z^2-1}}dz \\
= \; & t(S_{-\frac{1}{2}}(t) - \int_1^\infty K_0(tz)\frac{z}{\sqrt{z^2-1}}dz).
\end{aligned}$$

The above quantity is actually 0 since

$$S_{-\frac{1}{2}}(t) = \int_1^\infty \frac{K_1(tz)}{\sqrt{z^2-1}}dz$$

$$= \int_1^\infty \frac{K_1(tz)}{z}d(\sqrt{z^2-1})$$

$$= -\int_1^\infty \sqrt{z^2-1}\,d(\frac{K_1(tz)}{z})$$

$$= -\int_1^\infty \sqrt{z^2-1}\,d(t\int_1^\infty e^{-tzs}\sqrt{s^2-1}ds)$$

$$= t\int_1^\infty \sqrt{z^2-1}\,ts\int_1^\infty e^{-tzs}\sqrt{s^2-1}dsdz$$

$$= t\int_1^\infty \sqrt{s^2-1}\,ts\int_1^\infty e^{-tsz}\sqrt{z^2-1}dzds$$

$$= t\int_1^\infty \sqrt{s^2-1}K_1(ts)ds \qquad (= tS_{\frac{1}{2}}(t))$$

$$= \int_1^\infty \sqrt{s^2-1}d(-K_0(ts))$$

$$= \int_1^\infty K_0(ts)\frac{s}{\sqrt{s^2-1}}ds,$$

where the fourth, fifth, sixth equalities follow from Abramowitz and Stegun [1] 9.6.23, the dominated convergence theorem and Tonelli's theorem, respectively. Therefore, we can get that

$$tS_{-\frac{1}{2}}(t) + (tS_{-\frac{1}{2}}(t))' = 0 \Rightarrow tS_{-\frac{1}{2}}(t) = ce^{-t}.$$

We show that $c = \frac{\pi}{2}$ using a limit argument as t goes to zero. More precisely, by the dominated convergence theorem

$$\begin{aligned} c &= \lim_{t\to 0} tS_{-\frac{1}{2}}(t) \\ &= \lim_{t\to 0} t\int_1^\infty K_1(tz)(z^2-1)^{-\frac{1}{2}}dz \\ &= \int_1^\infty \lim_{t\to 0} tK_1(tz)(z^2-1)^{-\frac{1}{2}}dz \\ &= \int_1^\infty \frac{1}{z\sqrt{z^2-1}}dz \\ &= \text{arcsec} z\Big|_1^\infty \\ &= \frac{\pi}{2}. \end{aligned}$$

So, we conclude that $S_{-\frac{1}{2}}(t) = \frac{\pi}{2t}e^{-t}$. Moreover, during the derivation, we also get another induction base case, namely, $S_{\frac{1}{2}}(t) = \frac{\pi}{2t^2}e^{-t}$.

According to Abramowitz and Stegun [1] 10.2.16 and 10.2.17,

$$K_{-\frac{1}{2}}(t) = K_{\frac{1}{2}}(t) = \sqrt{\frac{\pi}{2t}}e^{-t}.$$

Therefore, we can rewrite $S_{-\frac{1}{2}}(t)$ and $S_{\frac{1}{2}}(t)$ as follows:

$$S_{-\frac{1}{2}}(t) = \frac{\pi}{2t}e^{-t} = \sqrt{\frac{\pi}{2t}}K_{-\frac{1}{2}}(t) = 2^{-\frac{1}{2}}t^{-\frac{1}{2}}K_{-\frac{1}{2}}(t)\Gamma(\frac{1}{2})$$

$$S_{\frac{1}{2}}(t) = \frac{\pi}{2t^2}e^{-t} = \sqrt{\frac{\pi}{2t^3}}K_{\frac{1}{2}}(t) = 2^{\frac{1}{2}}t^{-\frac{3}{2}}K_{\frac{1}{2}}(t)\Gamma(\frac{3}{2}),$$

which shows that the lemma holds for the two base cases: $m = -1$ and 0, or equivalently, $k = -1$ and 1. Now we use the recurrence relation we just got and apply the induction, i.e.,

$$\begin{aligned}
& S_{m+\frac{1}{2}}(t) \\
= & \frac{(2m+1)(2m-1)}{t^2}[S_{m-\frac{1}{2}}(t) + S_{m-\frac{3}{2}}(t)] \\
= & \frac{(2m+1)(2m-1)}{t^2}[2^{m-\frac{1}{2}}t^{-m-\frac{1}{2}}K_{m-\frac{1}{2}}(t)\Gamma(m+\frac{1}{2}) + 2^{m-\frac{3}{2}}t^{-m+\frac{1}{2}}K_{m-\frac{3}{2}}(t)\Gamma(m-\frac{1}{2})] \\
= & \frac{(2m+1)(2m-1)}{t^2}2^{m-\frac{3}{2}}t^{-m+\frac{1}{2}}\Gamma(m-\frac{1}{2})[2t^{-1}K_{m-\frac{1}{2}}(t)(m-\frac{1}{2}) + K_{m-\frac{3}{2}}] \\
= & \frac{(2m+1)(2m-1)}{t^2}2^{m-\frac{3}{2}}t^{-m+\frac{1}{2}}\Gamma(m-\frac{1}{2})K_{m+\frac{1}{2}}(t) \\
= & 2^{m+\frac{1}{2}}t^{-m-\frac{3}{2}}K_{m+\frac{1}{2}}(t)\Gamma(m+\frac{3}{2}),
\end{aligned}$$

where the second last equality follows from Abramowitz and Stegun [1] 10.2.18. So we complete the proof of the lemma. □

Now we are ready to state our main theorem for the expected difference between continuous suprema and discrete maximum for the NIG. Being the same as Merton's jump diffusion, we still take $T = 1$ and equal-distant partition by N points.

Theorem 3.3.10. *For the Normal Inverse Gaussian (NIG) processes defined on* $[0, 1]$ *with parameters* $\mu, \beta, \delta, \gamma$*, and let* N *be the number of equal-distant dividing points on* $[0, 1]$*, suppose that* $\frac{\alpha\delta}{\pi N} < 1$*,* $\frac{2\gamma\delta}{\pi N} < 1$*,* $\frac{\gamma\delta+|\beta\mu|}{2\pi N} < 1$ *and* $\frac{\alpha\sqrt{\delta^2+\mu^2}}{4\pi N} < 1$*, then the expected difference of the continuous supremum and discrete maximum admits the following asymptotic expansion*

$$\begin{aligned}
\Delta_N &= \mathbb{E}[\sup_{0\le t\le 1} X_t - \sup_{0\le i\le N} X_{t_i}] \\
&= Y_1\frac{\log N}{N} + \frac{Z_1}{N} + Y_2\frac{\log N}{N^2} + \frac{Z_2}{N^2} \\
&\quad + \sum_{q=1}^{p+1}\frac{Z_{2q+1}}{N^{2q+1}} + \sum_{q=1}^{p}\frac{Z_{2q+2}}{N^{2q+2}} + \sum_{q=1}^{p+1}Y_{2q+2}\frac{\log N}{N^{2q+2}} + O(\frac{1}{N^{2p+4}}),
\end{aligned}$$

where all these Y, Z*'s are finite constants independent of* N*, which will be given explicitly during the proof.*

Proof. According to Spitzer's identity:

$$
\begin{aligned}
\Delta_N &= \mathbb{E}[\sup_{0\le t\le 1} X_t - \sup_{0\le i\le N} X_{t_i}] \\
&= \int_0^1 \frac{1}{t}\mathbb{E}[X_t^+]dt - \sum_{n=1}^{N} \frac{1}{n}\mathbb{E}[X_{\frac{n}{N}}^+] \\
&= \int_0^1 \frac{1}{t}\int_0^\infty xp(t,x)dxdt - \sum_{n=1}^{N}\frac{1}{n}\int_0^\infty xp(\frac{n}{N},x)dx.
\end{aligned}
$$

The first term above, the continuous supremum, could be written as $\int_0^1 \frac{1}{t}\int_0^\infty xf(t,x-\mu t)dxdt$, where $f(t,x)$ is the transition density function for the non-drifted process $Y_t = X_t - \mu t = B(z_t;\beta,1)$. So,

$$
\begin{aligned}
& \int_0^1 \frac{1}{t}\int_0^\infty xp(t,x)dxdt \\
=& \int_0^1 \frac{1}{t}\int_0^\infty xf(t,x-\mu t)dxdt \\
=& \int_0^1 \frac{1}{t}\int_{-\mu t}^\infty (y+\mu t)f(t,y)dydt \\
=& \int_0^1 \frac{1}{t}\int_{-\mu t}^\infty yf(t,y)dydt + \mu\int_0^1\int_{-\mu t}^\infty f(t,y)dydt \\
=& \int_0^1 \frac{1}{t}\int_0^\infty yf(t,y)dydt + \int_0^1\frac{1}{t}\int_{-\mu t}^0 yf(t,y)dydt + \mu\int_0^1\int_{-\mu t}^\infty f(t,y)dydt. \qquad (3.12)
\end{aligned}
$$

We see that the first term in (3.12) is actually $\mathbb{E}[\sup_{0\le t\le 1} Y_t]$, then

$$
\begin{aligned}
& \mathbb{E}[\sup_{0\le t\le 1} Y_t] \\
=& \int_0^1 \frac{1}{t}\int_0^\infty yf(t,y)dydt \\
=& \int_0^1 \frac{1}{t}\int_0^\infty y\frac{\alpha\delta t}{\pi}\frac{K_1(\alpha\sqrt{\delta^2t^2+y^2})}{\sqrt{\delta^2t^2+y^2}}\exp(\delta\gamma t+\beta y)dydt \\
=& \frac{\alpha\delta}{\pi}\int_0^1 e^{\delta\gamma t}\int_0^\infty y\frac{K_1(\alpha\sqrt{\delta^2t^2+y^2})}{\sqrt{\delta^2t^2+y^2}}e^{\beta y}dydt \\
=& \frac{\alpha^2\delta}{\pi}\int_0^1 e^{\delta\gamma t}\int_{\alpha\delta t}^\infty \frac{K_1(w)}{w}e^{\frac{\beta}{\alpha}\sqrt{w^2-\alpha^2\delta^2t^2}}\frac{w}{\alpha^2}dwdt \\
=& \frac{\delta}{\pi}\int_0^1 e^{\delta\gamma t}\int_{\alpha\delta t}^\infty K_1(w)e^{\frac{\beta}{\alpha}\sqrt{w^2-\alpha^2\delta^2t^2}}dwdt \\
=& \frac{\alpha\delta^2}{\pi}\int_0^1 te^{\delta\gamma t}\int_1^\infty K_1(\alpha\delta tz)e^{\beta\delta t\sqrt{z^2-1}}dzdt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha\delta^2}{\pi}\int_0^1 te^{\delta\gamma t}\sum_{k=0}^{\infty}\int_1^{\infty} K_1(\alpha\delta tz)\frac{(\beta\delta t\sqrt{z^2-1})^k}{k!}dzdt \\
&= \frac{\alpha\delta^2}{\pi}\int_0^1 te^{\delta\gamma t}\sum_{k=0}^{\infty}\frac{(\beta\delta t)^k}{k!}\int_1^{\infty} K_1(\alpha\delta tz)(z^2-1)^{\frac{k}{2}}dzdt \\
&= \frac{\alpha\delta^2}{\pi}\int_0^1 te^{\delta\gamma t}\sum_{k=0}^{\infty}\frac{(\beta\delta t)^k}{k!}2^{\frac{k}{2}}(\alpha\delta t)^{-1-\frac{k}{2}}K_{\frac{k}{2}}(\alpha\delta t)\Gamma(1+\frac{k}{2})dt \\
&= \frac{\alpha\delta^2}{\pi}\int_0^1 te^{\delta\gamma t}\sum_{m=0}^{\infty}\Big(\frac{(\beta\delta t)^{2m}}{(2m)!}2^m(\alpha\delta t)^{-1-m}K_m(\alpha\delta t)\Gamma(1+m) \\
&\quad +\frac{(\beta\delta t)^{2m+1}}{(2m+1)!}2^{m+\frac{1}{2}}(\alpha\delta t)^{-1-\frac{2m+1}{2}}K_{m+\frac{1}{2}}(\alpha\delta t)\Gamma(1+\frac{2m+1}{2})\Big)dt, \qquad (3.13)
\end{aligned}
$$

where the seventh equality follows from the monotone convergence theorem for $\beta \geq 0$ and the dominated convergence theorem for $\beta < 0$, and the ninth equality follows from Lemma 3.3.9.

According to Abramowitz and Stegun [1] page 375, 9.6.11 and page 444, 10.2.15, the modified Bessel functions of the second kind of integer order and fraction order have different series representations, namely, for any integer n,

$$
K_n(z) = \frac{1}{2}(\frac{1}{2}z)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(-\frac{1}{4}z^2)^k + (-1)^{n+1}\log(\frac{1}{2}z)I_n(z)
$$

$$
+(-1)^n\frac{1}{2}(\frac{1}{2}z)^n\sum_{k=0}^{\infty}(\psi(k+1)+\psi(n+k+1))\frac{(\frac{1}{4}z^2)^k}{k!(n+k)!},
$$

and

$$
\sqrt{\frac{\pi}{2z}}K_{n+\frac{1}{2}}(z) = \frac{\pi}{2z}e^{-z}\sum_{k=0}^{n}H(n+\frac{1}{2},k)(2z)^{-k},
$$

where I_n is the modified Bessel function of the first kind of index n, which can be written as

$$
I_n(z) = (\frac{z}{2})^n\sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{k!\Gamma(n+k+1)}
$$

$$
\psi(1) = -\widetilde{\gamma}, \quad \psi(n) = -\widetilde{\gamma} + \sum_{k=1}^{n-1}k^{-1}, \quad n \geq 2,
$$

where $\widetilde{\gamma}$ is the Euler-Mascheroni constant. The Hankel's symbol $H(n,k)$ is defined as

$$H(n,k) = \frac{\Gamma(\frac{1}{2}+n+k)}{k!\Gamma(\frac{1}{2}+n-k)}.$$

We denote T_1^c the first part in (3.13). By the monotone convergence theorem,

$$\begin{aligned}
T_1^c &= \frac{\alpha\delta^2}{\pi}\int_0^1 te^{\delta\gamma t}\sum_{m=0}^{\infty}\frac{(\beta\delta t)^{2m}}{(2m)!}2^m(\alpha\delta t)^{-1-m}K_m(\alpha\delta t)\Gamma(1+m)dt \\
&= \frac{\alpha\delta^2}{\pi}\sum_{m=0}^{\infty}\frac{m!2^m}{(2m)!}\int_0^1 te^{\gamma\delta t}(\beta\delta t)^{2m}(\alpha\delta t)^{-1-m}K_m(\alpha\delta t)dt \\
&= \frac{\delta}{\pi}\sum_{m=0}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}\int_0^1 e^{\gamma\delta t}(\alpha\delta t)^m K_m(\alpha\delta t)dt \\
&= \frac{\delta}{\pi}\int_0^1 e^{\gamma\delta t}K_0(\alpha\delta t)dt + \frac{\delta}{\pi}\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}\int_0^1 e^{\gamma\delta t}(\alpha\delta t)^m K_m(\alpha\delta t)dt \\
&:= \frac{\delta}{\pi}(T_2^c + T_3^c).
\end{aligned}$$

Using Abramowitz and Stegun [1] page 375, 9.6.12 and 9.6.13, we see that

$$\begin{aligned}
T_2^c &= \int_0^1 e^{\gamma\delta t}K_0(\alpha\delta t)dt \\
&= (\log\frac{2}{\alpha\delta} - \widetilde{\gamma})\int_0^1 e^{\gamma\delta t}dt + \sum_{k=1}^{\infty}(\log\frac{2}{\alpha\delta} + \psi(k+1))\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{(k!)^2}\int_0^1 e^{\gamma\delta t}t^{2k}dt \\
&\quad - \int_0^1 e^{\gamma\delta t}\log t dt - \sum_{k=1}^{\infty}\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{(k!)^2}\int_0^1 e^{\gamma\delta t}t^{2k}\log t dt,
\end{aligned}$$

in which the interchange of integral and summation can be justified by the monotone convergence theorem.

Also, by Abramowitz and Stegun [1] page 375, 9.6.10 and 9.6.11,

$$\begin{aligned}
T_3^c &= \sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}\int_0^1 e^{\gamma\delta t}(\alpha\delta t)^m K_m(\alpha\delta t)dt \\
&= \sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}\Big(2^{m-1}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{k!}(-\frac{1}{4}(\alpha\delta)^2)^k\int_0^1 e^{\gamma\delta t}t^{2k}dt \\
&\quad - \frac{(\alpha\delta)^{2m}}{(-2)^{m+1}}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(m+k+1) - 2\log\frac{\alpha\delta}{2})\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k)!}\int_0^1 e^{\gamma\delta t}t^{2m+2k}dt \\
&\quad + \frac{(-1)^{m+1}(\alpha\delta)^{2m}}{2^m}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k)!}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}\int_0^1 t^{2m+2k+j}\log t dt\Big).
\end{aligned}$$

The corresponding discrete version of the first part in (3.13) is denoted as T_1^d, and

$$\begin{aligned}
T_1^d &= \frac{\alpha\delta^2}{\pi}\sum_{n=1}^{N}\frac{n}{N}e^{\gamma\delta\frac{n}{N}}\sum_{m=0}^{\infty}\frac{(\beta\delta\frac{n}{N})^{2m}}{(2m)!}2^m(\alpha\delta\frac{n}{N})^{-1-m}K_m(\alpha\delta\frac{n}{N})\Gamma(1+m)\frac{1}{N} \\
&= \frac{\alpha\delta^2}{\pi}\sum_{m=0}^{\infty}\frac{m!2^m}{(2m)!}\sum_{n=1}^{N}\frac{n}{N}e^{\gamma\delta\frac{n}{N}}(\beta\delta\frac{n}{N})^{2m}(\alpha\delta\frac{n}{N})^{-1-m}K_m(\alpha\delta\frac{n}{N})\frac{1}{N} \\
&= \frac{\delta}{\pi}\sum_{m=0}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}\sum_{n=1}^{N}e^{\gamma\delta\frac{n}{N}}(\alpha\delta\frac{n}{N})^{m}K_m(\alpha\delta\frac{n}{N})\frac{1}{N} \\
&= \frac{\delta}{\pi}\sum_{n=1}^{N}e^{\gamma\delta\frac{n}{N}}K_0(\alpha\delta\frac{n}{N})\frac{1}{N}+\frac{\delta}{\pi}\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}\sum_{n=1}^{N}e^{\gamma\delta\frac{n}{N}}(\alpha\delta\frac{n}{N})^{m}K_m(\alpha\delta\frac{n}{N})\frac{1}{N} \\
&:= \frac{\delta}{\pi}(T_2^d+T_3^d).
\end{aligned}$$

Similarly, by Abramowitz and Stegun [1] page 375, 9.6.12 and 9.6.13, we see that

$$\begin{aligned}
T_2^d &= \sum_{n=1}^{N}e^{\gamma\delta\frac{n}{N}}K_0(\alpha\delta\frac{n}{N})\frac{1}{N} \\
&= (\log\frac{2}{\alpha\delta}-\widetilde{\gamma})\sum_{n=1}^{N}e^{\gamma\delta\frac{n}{N}}\frac{1}{N} \\
&\quad+\sum_{k=1}^{\infty}(\log\frac{2}{\alpha\delta}+\psi(k+1))\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{(k!)^2}\sum_{n=1}^{N}e^{\gamma\delta\frac{n}{N}}(\frac{n}{N})^{2k}\frac{1}{N}-\sum_{n=1}^{N}e^{\gamma\delta\frac{n}{N}}\log\frac{n}{N}\frac{1}{N} \\
&\quad-\sum_{k=1}^{\infty}\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{(k!)^2}\sum_{n=1}^{N}e^{\gamma\delta\frac{n}{N}}(\frac{n}{N})^{2k}\log\frac{n}{N}\frac{1}{N}.
\end{aligned}$$

A similar calculation shows that

$$\begin{aligned}
T_3^d &= \sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}\Big(2^{m-1}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{k!}(-\frac{1}{4}(\alpha\delta)^2)^k\sum_{n=1}^{N}e^{\gamma\delta\frac{n}{N}}(\frac{n}{N})^{2k}\frac{1}{N} \\
&\quad-\frac{(\alpha\delta)^{2m}}{(-2)^{m+1}}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(m+k+1)-2\log\frac{\alpha\delta}{2})\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k)!}\sum_{n=1}^{N}e^{\gamma\delta\frac{n}{N}}(\frac{n}{N})^{2m+2k}\frac{1}{N} \\
&\quad+\frac{(-1)^{m+1}(\alpha\delta)^{2m}}{2^m}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k)!}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}\sum_{n=1}^{N}(\frac{n}{N})^{2m+2k+j}\log\frac{n}{N}\frac{1}{N}\Big).
\end{aligned}$$

Clearly,

$$T_1^c-T_1^d=\frac{\delta}{\pi}(T_2^c-T_2^d+T_3^c-T_3^d).$$

Combine all the expansion expressions above, we get that

$$
\begin{aligned}
& T_2^c - T_2^d \\
= \quad & (\widetilde{\gamma} - \log\frac{2}{\alpha\delta})\big(\sum_{n=1}^{N} e^{\gamma\delta\frac{n}{N}}\frac{1}{N} - \int_0^1 e^{\gamma\delta t}dt\big) \\
& -\sum_{k=1}^{\infty}(\log\frac{2}{\alpha\delta} + \psi(k+1))\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{(k!)^2}\big(\sum_{n=1}^{N} e^{\gamma\delta\frac{n}{N}}(\frac{n}{N})^{2k}\frac{1}{N} - \int_0^1 e^{\gamma\delta t}t^{2k}dt\big) \\
& +\sum_{n=1}^{N} e^{\gamma\delta\frac{n}{N}}\log\frac{n}{N}\frac{1}{N} - \int_0^1 e^{\gamma\delta t}\log t dt \\
& +\sum_{k=1}^{\infty}\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{(k!)^2}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}\big(\sum_{n=1}^{N}(\frac{n}{N})^{2k+j}\log\frac{n}{N}\frac{1}{N} - \int_0^1 t^{2k+j}\log t dt\big) \\
= \quad & (\widetilde{\gamma} - \log\frac{2}{\alpha\delta})\Big(\frac{e^{\gamma\delta}-1}{2N} + \sum_{m=1}^{p+1}\frac{(\gamma\delta)^{2m-1}(e^{\gamma\delta}-1)B_{2m}}{(2m)!N^{2m}}\Big) \\
& -\sum_{k=1}^{\infty}(\log\frac{2}{\alpha\delta} + \psi(k+1))\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{(k!)^2}\big(\frac{e^{\gamma\delta}}{2N} + \sum_{m=1}^{p+1}\frac{(g_{k,0}^{(2m-1)}(1) - g_{k,0}^{(2m-1)}(0))B_{2m}}{(2m)!N^{2m}}\big) \\
& +\frac{\log N}{2N} + \frac{\log 2\pi}{2N} + \frac{\gamma\delta\log N}{12N^2} + (\frac{e^{\gamma\delta}}{12} + \gamma\delta\log A - \frac{\gamma\delta}{12})\frac{1}{N^2} \\
& +\sum_{q=1}^{p}\frac{D(q)}{N^{2q+2}} - \sum_{k=2}^{2p+2}\frac{(\gamma\delta)^k}{k!}\frac{\zeta'(-k)}{N^{k+1}} + \sum_{m=1}^{p+1}\frac{(\gamma\delta)^{2m+1}B_{2m+2}}{(2m+2)!}\frac{\log N}{N^{2m+2}} \\
& +\sum_{k=1}^{\infty}\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{(k!)^2}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}\Big(\frac{1}{12N^2} - \frac{\zeta'(-(2k+j))}{N^{2k+j+1}} + \sum_{q=1}^{p}\frac{d_{2q,2k+j}}{N^{2q+2}} + \frac{B_{2k+j+1}}{2k+j+1}\frac{\log N}{N^{2k+j+1}}\Big) \\
& +O(\frac{1}{N^{2p+4}}) \\
= \quad & \frac{\log N}{2N} \\
& +\Big[(\widetilde{\gamma} - \log\frac{2}{\alpha\delta})(e^{\gamma\delta}-1) - e^{\gamma\delta}\sum_{k=1}^{\infty}(\log\frac{2}{\alpha\delta} + \psi(k+1))\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{(k!)^2} + \log 2\pi\Big]\frac{1}{2N} \\
& +\frac{\gamma\delta\log N}{12N^2} \\
& +\Big[(\log\frac{2}{\alpha\delta} - \widetilde{\gamma})\frac{\gamma\delta}{12} - \frac{\gamma\delta e^{\gamma\delta}}{12}K_0(\alpha\delta) - \frac{e^{\gamma\delta}}{6}\sum_{k=1}^{\infty}(\log\frac{2}{\alpha\delta} + \psi(k+1))\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(k-1)!} \\
& +\frac{e^{\gamma\delta}}{12}I_0(\alpha\delta) + \gamma\delta(\log A - \frac{1}{12})\Big]\frac{1}{N^2} \\
& -\sum_{q=1}^{p+1}\sum_{k\geq 0, j\geq 0, 2k+j=2q}\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}\frac{(\gamma\delta)^j}{j!}\zeta'(-2q)\frac{1}{N^{2q+1}} + \sum_{q=1}^{p}\frac{C(q)}{N^{2q+2}} \\
& +\sum_{q=1}^{p+1}\sum_{k\geq 0, j\geq 0, 2k+j=2q+1}\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}\frac{(\gamma\delta)^j}{j!}\frac{B_{2q+2}}{2q+2}\frac{\log N}{N^{2q+2}} + O(\frac{1}{N^{2p+4}}),
\end{aligned}
\tag{3.14}
$$

where the second equality follows from Lemma 3.3.2, Lemma 3.3.4 and Lemma 3.3.6, $g_{k,m}(t) =$

$e^{\gamma\delta t}t^{2m+2k}$, I_0 is the modified Bessel function of the first kind with index 0, and

$$\begin{aligned}
C(q) &= \sum_{k=0}^{\infty}(\log\frac{\alpha\delta}{2}-\psi(k+1))\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}(g_{k,0}^{(2q+1)}(1)-g_{k,0}^{(2q+1)}(0))\frac{B_{2q+2}}{(2q+2)!}+D(q)\\
&\quad+\sum_{k=1}^{\infty}\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}d_{2q,2k+j}-\sum_{k,j\geq 0,2k+j=2q+1}\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}\frac{(\gamma\delta)^j}{j!}\zeta'(-(2q+1))\\
&= \sum_{k=0}^{\infty}(\log\frac{\alpha\delta}{2}-\psi(k+1))\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}(g_{k,0}^{(2q+1)}(1)-g_{k,0}^{(2q+1)}(0))\frac{B_{2q+2}}{(2q+2)!}\\
&\quad+\sum_{k=0}^{\infty}\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}d_{2q,2k+j}-\sum_{k,j\geq 0,2k+j=2q+1}\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}\frac{(\gamma\delta)^j}{j!}\zeta'(-(2q+1)),
\end{aligned}$$

where we note that $c_{2q,l}=b_{2q,l}$ if $2q=l$, and $D(q)=\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}d_{2q,j}$.

One subtle issue is that the big-O in (3.14) actually combines a few other big-O's, for instance, the ones incurred from the truncations involving the terms $\zeta'(-2q)$ and B_{2q+2} in the last two lines of (3.14). We need to figure out a condition under which the tails for both terms still converge. Firstly, it suffices to check

$$\sum_{q=p+2}^{\infty}\sum_{k,j\geq 0,2k+j=2q}\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}\frac{(\gamma\delta)^j}{j!}\zeta'(-2q)\frac{1}{N^{2q+1}}<\infty.$$

Note that $\frac{(2k+j)!}{k!k!j!}\leq 3^{2k+j}$ due to the property of multinomial coefficient. Also, we have derived in the proof of Theorem 3.3.8 that

$$\zeta'(-2q)=\frac{(-1)^q\zeta(2q+1)(2q)!}{2^{2q+1}\pi^{2q}}.$$

Thus,

$$\begin{aligned}
&\Big|\sum_{q=p+2}^{\infty}\sum_{k,j\geq 0,2k+j=2q}\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}\frac{(\gamma\delta)^j}{j!}\zeta'(-2q)\frac{1}{N^{2q+1}}\Big|\\
\leq\;& \sum_{q=p+2}^{\infty}\sum_{k,j\geq 0,2k+j=2q}\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}\frac{(\gamma\delta)^j}{j!}|\zeta'(-2q)|\frac{1}{N^{2q+1}}\\
\leq\;& \sum_{q=p+2}^{\infty}\sum_{k,j\geq 0,2k+j=2q}\frac{(\frac{\alpha\delta}{2})^{2k}3^{2k+j}(\gamma\delta)^j}{(2k+j)!}\frac{\zeta(2q+1)(2q)!}{2^{2q+1}\pi^{2q}}\frac{1}{N^{2q+1}}\\
\leq\;& \sum_{q=p+2}^{\infty}\sum_{k,j\geq 0,2k+j=2q}\frac{(\frac{\alpha\delta}{2})^{2k}3^{2k+j}(\gamma\delta)^j}{2^{2k+j+1}\pi^{2k+j}}\frac{\zeta(7)}{N^{2k+j+1}}\\
\leq\;& \frac{\zeta(7)}{2N}\sum_{k=0}^{\infty}(\frac{3\alpha\delta}{4\pi N})^{2k}\sum_{j=0}^{\infty}(\frac{3\gamma\delta}{2\pi N})^j.
\end{aligned}$$

Hence, we need that $0 < \frac{3\alpha\delta}{4\pi N} < 1$ and $0 < \frac{3\gamma\delta}{2\pi N} < 1$. A similar argument works for the other term with B_{2q+2}, which will also be convergent absolutely with these two conditions satisfied. Namely,

$$\Big|\sum_{q=p+2}^{\infty}\sum_{k\geq 0, j\geq 0, 2k+j=2q+1}\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}\frac{(\gamma\delta)^j}{j!}\frac{B_{2q+2}}{2q+2}\frac{\log N}{N^{2q+2}}\Big| < \infty.$$

Also, we note that the coefficient of $\frac{1}{2N}$ in (3.14) can be written in a more concise form, i.e.,

$$\Big[(\widetilde{\gamma}-\log\frac{2}{\alpha\delta})(e^{\gamma\delta}-1) - e^{\gamma\delta}\sum_{k=1}^{\infty}(\log\frac{2}{\alpha\delta}+\psi(k+1))\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{(k!)^2}+\log 2\pi\Big]\frac{1}{2N}$$

$$= \Big(-\widetilde{\gamma}+\log(\frac{4\pi}{\alpha\delta}) - e^{\gamma\delta}K_0(\alpha\delta)\Big)\frac{1}{2N}.$$

Similar to the analysis of $T_2^c - T_2^d$ in above, we have

$$\begin{aligned}
& T_3^c - T_3^d \\
= & \sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}\Big[-2^{m-1}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{k!}(-\frac{1}{4}(\alpha\delta)^2)^k(\frac{e^{\gamma\delta}-\mathbb{1}_{\{k=0\}}}{2N} \\
& +\sum_{i=1}^{p+1}\frac{B_{2i}}{(2i)!}[g_{k,0}^{(2i-1)}(1)-g_{k,0}^{(2i-1)}(0)]\frac{1}{N^{2i}}) \\
& +\frac{(\alpha\delta)^{2m}}{(-2)^{m+1}}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(m+k+1)-2\log\frac{\alpha\delta}{2})\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k)!}\cdot \\
& (\frac{e^{\gamma\delta}}{2N}+\sum_{i=1}^{p+1}\frac{B_{2i}}{(2i)!}[g_{k,m}^{(2i-1)}(1)-g_{k,m}^{(2i-1)}(0)]\frac{1}{N^{2i}}) \\
& +\frac{(-1)^m(\alpha\delta)^{2m}}{2^m}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k)!}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}\Big(\frac{1}{12N^2}-\frac{\zeta'(-(2m+2k+j))}{N^{2m+2k+j+1}} \\
& +\sum_{q=1}^{p}\frac{d_{2q,2m+2k+j}}{N^{2q+2}}+\frac{B_{2m+2k+j+1}}{2m+2k+j+1}\frac{\log N}{N^{2m+2k+j+1}}\Big)\Big]+O(\frac{1}{N^{2p+4}}).
\end{aligned} \tag{3.15}$$

Similarly, we need to find the condition for the term involving the ζ' in (3.15). It suffices to check the following

$$\begin{aligned}
& \Big|\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}\frac{(-1)^m(\alpha\delta)^{2m}}{2^m}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k)!}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}\frac{(2m+2k+j)!}{(2\pi N)^{2m+2k+j}}\Big| \\
\leq & \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{m!(\beta\delta)^{2m}}{(2m)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\frac{(2m+2k+j)!}{(2\pi N)^{2m+2k+j}} \\
\leq & \sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^{1-2m}\sqrt{m}(\beta\delta)^{2m}}{m!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\frac{(2m+2k+j)!}{(2\pi N)^{2m+2k+j}}
\end{aligned}$$

$$\leq 2\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sqrt{m}(\frac{\beta\delta}{2})^{2m}(\frac{\alpha\delta}{2})^{2k}(\gamma\delta)^{j}(\frac{4}{2\pi N})^{2m+2k+j},$$

where the second inequality follows from the fact that $\frac{m!}{(2m)!}\leq\frac{2^{1-2m}\sqrt{m}}{m!}$ and the third inequality is just from the property of multinomial coefficient. Thus, we need that

$$\frac{|\beta|\delta}{\pi N}<1, \frac{\alpha\delta}{\pi N}<1 \text{ and } \frac{2\gamma\delta}{\pi N}<1.$$

Since by the relation $\alpha^2=\beta^2+\gamma^2$, the second inequality actually implies the first one. Also, they immediately imply that $\frac{3\alpha\delta}{4\pi N}<1$ and $\frac{3\gamma\delta}{2\pi N}<1$. So we actually need

$$\frac{\alpha\delta}{\pi N}<1, \frac{2\gamma\delta}{\pi N}<1.$$

The same bounds are required for the convergence of the term involving the Bernoulli numbers $B_{2m+2k+j+1}$. Also, for any given q such that $1\leq q\leq p$, the finiteness of the term $\frac{d_{2q,2m+2k+j}}{N^{2q+2}}$ is also clear since $|d_{2q,2m+2k+j}|\leq\frac{B_{2q+2}}{2q+2}2^{2m+2k+j}$, which yields finite sum under the infinite summation operators with respect to m,k and j.

Note that we still face the same subtle issue with respect to the big-O term in (3.15), which is actually the sum of a few big-O's. We first show that the big-O term in the second component of (3.15) is well-defined with infinite summations. More precisely, the second term in (3.15) is given by

$$\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}\frac{(\beta\delta)^{2m}}{(-2)^{m+1}}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(m+k+1)-2\log\frac{\alpha\delta}{2})\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k)!}(\frac{e^{\gamma\delta}}{2N}$$

$$+\frac{(2(m+k)+\gamma\delta)e^{\gamma\delta}}{12N^2}+\sum_{i=2}^{p+1}\frac{B_{2i}}{(2i)!}[g_{k,m}^{(2i-1)}(1)-g_{k,m}^{(2i-1)}(0)]\frac{1}{N^{2i}}+O(\frac{1}{N^{2p+4}})),$$

where the constant involved in the above big-O is given by

$$\Big|\frac{B_{2p+4}}{(2p+4)!}[g_{k,m}^{(2p+3)}(1)-g_{k,m}^{(2p+3)}(0)]\Big|.$$

Thus, we see that, for any positive but finite p

$$\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}\frac{(\beta\delta)^{2m}}{(-2)^{m+1}}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(m+k+1)-2\log\frac{\alpha\delta}{2})\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k)!}.$$

$$\Big|\frac{B_{2p+4}}{(2p+4)!}[g_{k,m}^{(2p+3)}(1)-g_{k,m}^{(2p+3)}(0)]\Big| = C(p) < \infty.$$

This guarantees that we may take the big-O outside of the summations with respect to k and m. We also have to look at the first component in (3.15). It is given by

$$-\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}2^{m-1}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{k!}(-\frac{(\alpha\delta)^2}{4})^k(\frac{e^{\gamma\delta}}{2N}-\frac{1}{2N}\mathbb{1}_{\{k=0\}}$$

$$+\frac{(2k+\gamma\delta)e^{\gamma\delta}-\mathbb{1}_{\{k=0\}}\gamma\delta}{12N^2}+\sum_{i=2}^{p+1}\frac{B_{2i}}{(2i)!}[g_{k,0}^{(2i-1)}(1)-g_{k,0}^{(2i-1)}(0)]\frac{1}{N^{2i}}+O(\frac{1}{N^{2p+4}})).$$

Since p is a finite integer, we know that

$$\sum_{m=p+3}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}2^{m-1}\sum_{k=p+2}^{m-1}\frac{(m-k-1)!}{k!}(\frac{(\alpha\delta)^2}{4})^k\cdot\Big|\frac{B_{2p+4}}{(2p+4)!}[g_{k,0}^{(2p+3)}(1)-g_{k,0}^{(2p+3)}(0)]\Big|$$

$$\le\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}2^{m-1}(m-1)!\sum_{k=0}^{m-1}\frac{1}{k!}(\frac{1}{4}(\alpha\delta)^2)^k\Big|\frac{B_{2p+4}}{(2p+4)!}[g_{k,0}^{(2p+3)}(1)-g_{k,0}^{(2p+3)}(0)]\Big|$$

$$\le\sum_{m=1}^{\infty}\frac{(2m-2)!!}{(2m-1)!!}(\frac{\beta}{\alpha})^{2m}\sum_{k=0}^{\infty}\frac{1}{k!}(\frac{1}{4}(\alpha\delta)^2)^k\Big|\frac{B_{2p+4}}{(2p+4)!}[g_{k,0}^{(2p+3)}(1)-g_{k,0}^{(2p+3)}(0)]\Big|$$

$$\le C_1(p),$$

since $\alpha=\sqrt{\beta^2+\gamma^2}$. Thus, we can take the big-O outside of the summations with respect to k and m. Meanwhile, by the same reason, we see that all the coefficients of $\frac{1}{N}$, $\frac{1}{N^{2k}}$ with $1\le k\le p+1$ are finite.

Therefore, after simplification, we may conclude that, with $\frac{\alpha\delta}{\pi N}<1$ and $\frac{2\gamma\delta}{\pi N}<1$ satisfied,

$$T_3^c-T_3^d$$

$$=\Big[\sum_{m=1}^{\infty}\frac{(2m-2)!!}{(2m-1)!!}(\frac{\beta}{\alpha})^{2m}-e^{\gamma\delta}\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}(\alpha\delta)^mK_m(\alpha\delta)\Big]\frac{1}{2N}$$

$$+\Big[\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}(-\alpha\delta)^m I_m(\alpha\delta)-\gamma\delta\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}(\alpha\delta)^m K_m(\alpha\delta)$$
$$+\sum_{m=1}^{\infty}\frac{m!(-1)^{m+1}(\beta\delta)^{2m}}{(2m)!}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(m+k+1)-2\log\frac{\alpha\delta}{2})\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k-1)!}$$
$$-\sum_{m=2}^{\infty}\frac{m!2^{2m}}{(2m)!}(\frac{\beta}{\alpha})^{2m}\sum_{k=1}^{m-1}\frac{(m-k-1)!}{(k-1)!}(-\frac{(\alpha\delta)^2}{4})^k+\gamma\delta e^{-\gamma\delta}\sum_{m=1}^{\infty}\frac{(2m-2)!!}{(2m-1)!!}(\frac{\beta}{\alpha})^{2m}\Big]\frac{e^{\gamma\delta}}{12N^2}$$
$$-\sum_{q=1}^{p+1}\sum_{m\geq1,k,j\geq0,2m+2k+j=2q}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\zeta'(-2q)\frac{1}{N^{2q+1}}+\sum_{q=1}^{p}\frac{E(q)}{N^{2q+2}}$$
$$+\sum_{q=1}^{p+1}\sum_{m\geq1,k,j\geq0,2m+2k+j=2q+1}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\frac{B_{2q+2}}{2q+2}\frac{\log N}{N^{2q+2}}$$
$$+O(\frac{1}{N^{2p+4}}), \tag{3.16}$$

where

$$E(q)=$$
$$\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}\Big[-2^{m-1}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{k!}(-\frac{1}{4}(\alpha\delta)^2)^k\frac{B_{2q+2}}{(2q+2)!}[g_{k,0}^{(2q+1)}(1)-g_{k,0}^{(2q+1)}(0)]$$
$$+\frac{(\alpha\delta)^{2m}}{(-2)^{m+1}}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(m+k+1)-2\log\frac{\alpha\delta}{2})\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k)!}\frac{B_{2q+2}}{(2q+2)!}\cdot$$
$$[g_{k,m}^{(2q+1)}(1)-g_{k,m}^{(2q+1)}(0)]+\frac{(-1)^m(\alpha\delta)^{2m}}{2^m}\sum_{k=0}^{\infty}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}d_{2q,2m+2k+j}\Big]$$
$$-\sum_{m\geq1,k,j\geq0,2m+2k+j=2q+1}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\zeta'(-(2q+1)).$$

Combine with $T_2^c-T_2^d$, we obtain that

$$\begin{aligned}
&\frac{\pi}{\delta}(T_1^c-T_1^d)\\
=\;&\frac{\log N}{2N}+\Big(-\widetilde{\gamma}+\log(\frac{4\pi}{\alpha\delta})-e^{\gamma\delta}K_0(\alpha\delta)\Big)\frac{1}{2N}+\frac{\gamma\delta\log N}{12N^2}\\
&+\Big[(\log\frac{2}{\alpha\delta}-\widetilde{\gamma})\frac{\gamma\delta}{12}-\frac{\gamma\delta e^{\gamma\delta}}{12}K_0(\alpha\delta)-\frac{e^{\gamma\delta}}{6}\sum_{k=1}^{\infty}(\log\frac{2}{\alpha\delta}+\psi(k+1))\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(k-1)!}\\
&+\frac{e^{\gamma\delta}}{12}I_0(\alpha\delta)+\gamma\delta(\log A-\frac{1}{12})\Big]\frac{1}{N^2}\\
&-\sum_{q=1}^{p+1}\sum_{k\geq0,j\geq0,2k+j=2q}\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}\frac{(\gamma\delta)^j}{j!}\zeta'(-2q)\frac{1}{N^{2q+1}}+\sum_{q=1}^{p}\frac{C(q)}{N^{2q+2}}\\
&+\sum_{q=1}^{p+1}\sum_{k\geq0,j\geq0,2k+j=2q+1}\frac{(\frac{\alpha\delta}{2})^{2k}}{(k!)^2}\frac{(\gamma\delta)^j}{j!}\frac{B_{2q+2}}{2q+2}\frac{\log N}{N^{2q+2}}
\end{aligned}$$

$$
\begin{aligned}
&+\Big[\sum_{m=1}^{\infty}\frac{(2m-2)!!}{(2m-1)!!}(\frac{\beta}{\alpha})^{2m}-e^{\gamma\delta}\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}(\alpha\delta)^m K_m(\alpha\delta)\Big]\frac{1}{2N}\\
&+\Big[\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}(-\alpha\delta)^m I_m(\alpha\delta)-\gamma\delta\sum_{m=1}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}(\alpha\delta)^m K_m(\alpha\delta)\\
&+\sum_{m=1}^{\infty}\frac{m!(-1)^{m+1}(\beta\delta)^{2m}}{(2m)!}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(m+k+1)-2\log\frac{\alpha\delta}{2})\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k-1)!}\\
&-\sum_{m=2}^{\infty}\frac{m!2^{2m}}{(2m)!}(\frac{\beta}{\alpha})^{2m}\sum_{k=1}^{m-1}\frac{(m-k-1)!}{(k-1)!}(-\frac{(\alpha\delta)^2}{4})^k+\gamma\delta e^{-\gamma\delta}\sum_{m=1}^{\infty}\frac{(2m-2)!!}{(2m-1)!!}(\frac{\beta}{\alpha})^{2m}\Big]\frac{e^{\gamma\delta}}{12N^2}\\
&-\sum_{q=1}^{p+1}\sum_{m\geq 1,k,j\geq 0,2m+2k+j=2q}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\zeta'(-2q)\frac{1}{N^{2q+1}}+\sum_{q=1}^{p}\frac{E(q)}{N^{2q+2}}\\
&+\sum_{q=1}^{p+1}\sum_{m\geq 1,k,j\geq 0,2m+2k+j=2q+1}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\frac{B_{2q+2}}{2q+2}\frac{\log N}{N^{2q+2}}+O(\frac{1}{N^{2p+4}})\\
=\;&\frac{\log N}{2N}+\Big[\log\frac{4\pi}{\alpha\delta}-\widetilde{\gamma}+\sum_{m=1}^{\infty}\frac{(2m-2)!!}{(2m-1)!!}(\frac{\beta}{\alpha})^{2m}-e^{\gamma\delta}\sum_{m=0}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta^2\delta}{\alpha})^m K_m(\alpha\delta)\Big]\frac{1}{2N}\\
&+\frac{\gamma\delta\log N}{12N^2}+\Big[\gamma\delta(\log\frac{2}{\alpha\delta}-\widetilde{\gamma}+12\log A-1+\sum_{m=1}^{\infty}\frac{(2m-2)!!}{(2m-1)!!}(\frac{\beta}{\alpha})^{2m})\\
&+\delta e^{\gamma\delta}\sum_{m=0}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta^2\delta}{\alpha})^m\big(\alpha K_{m-1}(\alpha\delta)-\gamma K_m(\alpha\delta)\big)\Big]\frac{1}{12N^2}\\
&-\sum_{q=1}^{p+1}\sum_{m,k,j\geq 0,2m+2k+j=2q}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\zeta'(-2q)\frac{1}{N^{2q+1}}\\
&+\sum_{q=1}^{p+1}\sum_{m,k,j\geq 0,2m+2k+j=2q+1}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\frac{B_{2q+2}}{2q+2}\frac{\log N}{N^{2q+2}}\\
&+\sum_{q=1}^{p}\frac{C(q)+E(q)}{N^{2q+2}}+O(\frac{1}{N^{2p+4}}).
\end{aligned}
$$

Next we deal with the second half of (3.13), called T_4^c, i.e.,

$$
\begin{aligned}
T_4^c &= \frac{\alpha\delta^2}{\pi}\int_0^1 te^{\gamma\delta t}\sum_{m=0}^{\infty}\frac{(\beta\delta t)^{2m+1}}{(2m+1)!}2^{m+\frac{1}{2}}(\alpha\delta t)^{-1-\frac{2m+1}{2}}K_{m+\frac{1}{2}}(\alpha\delta t)\Gamma(m+\frac{3}{2})dt\\
&= \frac{\delta}{\pi}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{3}{2})2^{m+\frac{1}{2}}}{(2m+1)!}(\frac{\beta}{\alpha})^{2m+1}\int_0^1 e^{\gamma\delta t}(\alpha\delta t)^{m+\frac{1}{2}}K_{m+\frac{1}{2}}(\alpha\delta t)dt\\
&= \frac{\delta}{\pi}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{3}{2})2^{m+\frac{1}{2}}}{(2m+1)!}(\frac{\beta}{\alpha})^{2m+1}\int_0^1\sqrt{\frac{\pi}{2}}e^{(\gamma-\alpha)\delta t}\sum_{k=0}^{m}\frac{H(m+\frac{1}{2},k)}{2^k}(\alpha\delta t)^{m-k}dt\\
&= \frac{\delta}{2}\sum_{m=0}^{\infty}\frac{1}{(2m)!!}(\frac{\beta}{\alpha})^{2m+1}\int_0^1 e^{(\gamma-\alpha)\delta t}\sum_{k=0}^{m}\frac{(m+k)!}{2^k k!(m-k)!}(\alpha\delta t)^{m-k}dt\\
&= \frac{\delta}{2}\sum_{m=0}^{\infty}\frac{1}{(2m)!!}(\frac{\beta}{\alpha})^{2m+1}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!}\int_0^1 e^{(\gamma-\alpha)\delta t}t^{m-k}dt,
\end{aligned}
$$

where the second equality follows from the monotone convergence theorem. The corresponding discrete summation, called T_4^d, is given by

$$= \frac{\delta}{2}\sum_{m=0}^{\infty}\frac{1}{(2m)!!}(\frac{\beta}{\alpha})^{2m+1}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!}\sum_{n=1}^{N}e^{(\gamma-\alpha)\delta\frac{n}{N}}(\frac{n}{N})^{m-k}\frac{1}{N}.$$

Thus, we get that

$$\begin{aligned} T_4^c - T_4^d &= -\frac{\delta}{2}\sum_{m=0}^{\infty}\frac{1}{(2m)!!}(\frac{\beta}{\alpha})^{2m+1}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!}\Big(\frac{e^{(\gamma-\alpha)\delta}}{2N} - \frac{1}{2N}\mathbb{1}_{\{k=m\}} \\ &+\frac{(m-k+(\gamma-\alpha)\delta)e^{(\gamma-\alpha)\delta} - \mathbb{1}_{\{k=m\}}(\gamma-\alpha)\delta}{12N^2} \\ &+\sum_{i=2}^{p+1}\frac{B_{2i}}{(2i)!}[y_{k,m}^{(2i-1)}(1) - y_{k,m}^{(2i-1)}(0)]\frac{1}{N^{2i}}\Big) + O(\frac{1}{N^{2p+4}}), \end{aligned}$$

where $y_{k,m}(t) = e^{(\gamma-\alpha)\delta t}t^{m-k}$. Realize that given any positive integer m, $\frac{(m+k)!}{(2m)!!(2k)!!} \leq 1$ for all $k = 0, 1, 2, .., m$. Therefore, we have achieved the finiteness of the coefficient of $\frac{1}{N}$ as follows,

$$\begin{aligned} &\frac{\delta}{2}\sum_{m=0}^{\infty}\frac{1}{(2m)!!}(\frac{|\beta|}{\alpha})^{2m+1}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!} \\ \leq\ &\frac{\delta}{2}\sum_{m=0}^{\infty}(\frac{|\beta|}{\alpha})^{2m+1}\sum_{k=0}^{m}\frac{(\alpha\delta)^{m-k}}{(m-k)!} \\ \leq\ &\frac{\delta}{2}e^{\alpha\delta}\sum_{m=0}^{\infty}(\frac{|\beta|}{\alpha})^{2m+1} \\ \leq\ &\frac{\delta}{2}e^{\alpha\delta}\frac{\alpha^2}{\alpha^2-\beta^2}\frac{|\beta|}{\alpha} \\ =\ &\frac{\delta\alpha|\beta|}{2\gamma^2}e^{\alpha\delta}. \end{aligned}$$

To establish the finiteness of the coefficient for the term involving $\frac{1}{N^2}$, we need to consider a multiplicative factor $m-k$. However, we still get a finite sum, more precisely,

$$\begin{aligned} &\sum_{m=0}^{\infty}\frac{1}{(2m)!!}(\frac{|\beta|}{\alpha})^{2m+1}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!}(m-k) \\ =\ &\sum_{m=1}^{\infty}\frac{1}{(2m)!!}(\frac{|\beta|}{\alpha})^{2m+1}\sum_{k=0}^{m-1}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k-1)!} \\ \leq\ &\sum_{m=1}^{\infty}(\frac{|\beta|}{\alpha})^{2m+1}\sum_{k=0}^{m-1}\frac{(\alpha\delta)^{m-k}}{(m-k-1)!} \\ \leq\ &\alpha\delta e^{\alpha\delta}\sum_{m=1}^{\infty}(\frac{|\beta|}{\alpha})^{2m+1} < \infty. \end{aligned}$$

The above shows that the coefficients of all $\frac{1}{N^r}$ with $1 \le r \le p < \infty$ in the expansion of $T_4^c - T_4^d$ are finite. Thus, we get that

$$T_4^c - T_4^d = -\frac{\delta}{4N}\sum_{m=0}^{\infty}\frac{1}{(2m)!!}(\frac{\beta}{\alpha})^{2m+1}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!}(e^{(\gamma-\alpha)\delta} - \mathbb{1}_{\{k=m\}})$$

$$-\frac{\delta e^{(\gamma-\alpha)\delta}}{24N^2}\sum_{m=0}^{\infty}\frac{1}{(2m)!!}(\frac{\beta}{\alpha})^{2m+1}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!}\cdot$$

$$\Big(m-k+(\gamma-\alpha)\delta - \mathbb{1}_{\{k=m\}}(\gamma-\alpha)\delta e^{-(\gamma-\alpha)\delta}\Big)$$

$$-\frac{\delta}{2}\sum_{i=2}^{p+1}\sum_{m=0}^{\infty}\frac{(\frac{\beta}{\alpha})^{2m+1}}{(2m)!!}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!}\frac{B_{2i}}{(2i)!}(y_{k,m}^{(2i-1)}(1) - y_{k,m}^{(2i-1)}(0))\frac{1}{N^{2i}} + O(\frac{1}{N^{2p+4}}),$$

where $y_{k,m}(t) = e^{(\gamma-\alpha)\delta t}t^{m-k}$.

In summary, we have the following: With $\frac{\alpha\delta}{\pi N} < 1$ and $\frac{2\gamma\delta}{\pi N} < 1$ satisfied, for some positive integer p,

$$\mathbb{E}[\sup_{0\le t\le 1} Y_t - \sup_{0\le i\le N} Y_{t_i}] = T_1^c - T_1^d + T_4^c - T_4^d$$

$$= G_1\frac{\log N}{N} + \frac{H_1}{N} + G_2\frac{\log N}{N^2} + \frac{H_2}{N^2} + \sum_{q=1}^{p+1}\frac{H_{2q+1}}{N^{2q+1}} + \sum_{q=1}^{p}\frac{H_{2q+2}}{N^{2q+2}} + \sum_{q=1}^{p+1}G_{2q+2}\frac{\log N}{N^{2q+2}} + O(\frac{1}{N^{2p+4}}),$$

where

$$G_1 = \frac{\delta}{2\pi}, G_2 = \frac{\gamma\delta^2}{12\pi}$$

$$H_1 = \frac{\delta}{2\pi}\Big(\log(\frac{4\pi}{\alpha\delta}) - \widetilde{\gamma} + \sum_{m=1}^{\infty}\frac{(2m-2)!!}{(2m-1)!!}(\frac{\beta}{\alpha})^{2m} - e^{\gamma\delta}\sum_{m=0}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta^2\delta}{\alpha})^m K_m(\alpha\delta)\Big)$$

$$-\frac{\delta}{4}\sum_{m=0}^{\infty}\frac{1}{(2m)!!}(\frac{\beta}{\alpha})^{2m+1}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!}(e^{(\gamma-\alpha)\delta} - \mathbb{1}_{\{k=m\}})$$

$$H_2 = \frac{\delta}{12\pi}\Big[\gamma\delta(\log\frac{2}{\alpha\delta} - \widetilde{\gamma} + 12\log A - 1 + \sum_{m=1}^{\infty}\frac{(2m-2)!!}{(2m-1)!!}(\frac{\beta}{\alpha})^{2m})$$

$$+\delta e^{\gamma\delta}\sum_{m=0}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta^2\delta}{\alpha})^m\big(\alpha K_{m-1}(\alpha\delta) - \gamma K_m(\alpha\delta)\big)\Big]$$

$$
-\frac{\delta e^{(\gamma-\alpha)\delta}}{24}\sum_{m=0}^{\infty}\frac{1}{(2m)!!}(\frac{\beta}{\alpha})^{2m+1}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!}(m-k+(\gamma-\alpha)\delta)
$$

$$
+\frac{\delta^2(\gamma-\alpha)}{24}\sum_{m=0}^{\infty}\frac{(2m)!}{((2m)!!)^2}(\frac{\beta}{\alpha})^{2m+1}
$$

$$
\begin{aligned}
H_{2q+1} &= -\frac{\delta}{\pi}\sum_{m,k,j\geq 0,2m+2k+j=2q}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\zeta'(-2q)\\
H_{2q+2} &= \frac{\delta}{\pi}(C(q)+E(q))\\
&\quad -\frac{\delta}{2}\sum_{m=0}^{\infty}\frac{(\frac{\beta}{\alpha})^{2m+1}}{(2m)!!}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!}\frac{B_{2q+2}}{(2q+2)!}(y_{k,m}^{(2q+1)}(1)-y_{k,m}^{(2q+1)}(0)),
\end{aligned}
$$

and

$$
\begin{aligned}
C(q)+E(q) &= \sum_{m=0}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}\Big[-2^{m-1}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{k!}(-\frac{1}{4}(\alpha\delta)^2)^k\frac{B_{2q+2}}{(2q+2)!}\cdot\\
&\quad [g_{k,0}^{(2q+1)}(1)-g_{k,0}^{(2q+1)}(0)]\\
&\quad +\frac{(\alpha\delta)^{2m}}{(-2)^{m+1}}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(m+k+1)-2\log\frac{\alpha\delta}{2})\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k)!}\frac{B_{2q+2}}{(2q+2)!}\cdot\\
&\quad [g_{k,m}^{(2q+1)}(1)-g_{k,m}^{(2q+1)}(0)]\\
&\quad +\frac{(-1)^m(\alpha\delta)^{2m}}{2^m}\sum_{k=0}^{\infty}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}d_{2q,2m+2k+j}\Big]\\
&\quad -\sum_{m,k,j\geq 0,2m+2k+j=2q+1}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\zeta'(-(2q+1))\\
G_{2q+2} &= \sum_{m,k,j\geq 0,2m+2k+j=2q+1}\frac{\delta}{\pi}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\frac{B_{2q+2}}{2q+2}.
\end{aligned}
$$

This concludes the case of non-drifted NIG, i.e., for the case where $\mu = 0$.

Now we go back to see the second term in (3.12). First, note that this term is indeed 0 if $\mu = 0$. By the same calculation as above, we know that, for $\mu > 0$,

$$
\begin{aligned}
&\int_0^1\frac{1}{t}\int_{-\mu t}^0 yf(t,y)dydt\\
&= -\frac{\alpha\delta^2}{\pi}\int_0^1 te^{\gamma\delta t}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}K_1(\alpha\delta tz)e^{-\beta\delta t\sqrt{z^2-1}}dzdt\\
&= -\frac{\alpha\delta^2}{\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\int_0^1 te^{(\gamma\delta-\beta\delta\sqrt{z^2-1})t}K_1(\alpha\delta tz)dtdz
\end{aligned}
$$

$$= \quad -\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\int_0^1 e^{(\gamma\delta-\beta\delta\sqrt{z^2-1})t}\Big(\frac{\delta}{\pi z}+\frac{\alpha^2\delta^3 z}{2\pi}t^2\log t\sum_{k=0}^{\infty}\frac{(\frac{1}{4}(\alpha\delta tz)^2)^k}{k!(k+1)!}$$
$$+\frac{\alpha^2\delta^3 zt^2}{4\pi}\sum_{k=0}^{\infty}(2\log\frac{\alpha\delta z}{2}-\psi(k+1)-\psi(k+2))\frac{(\frac{1}{4}(\alpha\delta tz)^2)^k}{k!(k+1)!}\Big)dtdz,$$

where the second equality follows from Tonelli's theorem, and the third one follows from the series expansion of the modified Bessel function of the second kind with index 1, i.e.,

$$K_1(\alpha\delta tz) \quad = \quad (\alpha\delta tz)^{-1}+\log t(\frac{\alpha\delta tz}{2})\sum_{k=0}^{\infty}\frac{(\frac{1}{4}(\alpha\delta tz)^2)^k}{k!(k+1)!}$$
$$+\frac{\alpha\delta tz}{4}\sum_{k=0}^{\infty}(2\log\frac{\alpha\delta z}{2}-\psi(k+1)-\psi(k+2))\frac{(\frac{1}{4}(\alpha\delta tz)^2)^k}{k!(k+1)!}.$$

The corresponding discretized term is given by

$$= -\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\sum_{n=1}^{N} e^{(\gamma\delta-\beta\delta\sqrt{z^2-1})\frac{n}{N}}\Big(\frac{\delta}{\pi z}+\frac{\alpha^2\delta^3 z}{2\pi}(\frac{n}{N})^2\log\frac{n}{N}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}(\alpha\delta z\frac{n}{N})^2)^k}{k!(k+1)!}$$
$$+\frac{\alpha^2\delta^3 z}{4\pi}(\frac{n}{N})^2\sum_{k=0}^{\infty}(2\log\frac{\alpha\delta z}{2}-\psi(k+1)-\psi(k+2))\frac{(\frac{1}{4}(\alpha\delta z\frac{n}{N})^2)^k}{k!(k+1)!}\Big)\frac{1}{N}dz.$$

For simplicity, we define

$$\rho_{\pm}(z) := \gamma\delta \pm \beta\delta\sqrt{z^2-1}$$

$$\varphi_{\pm}(t) := e^{\rho_{\pm}(z)t},$$

and

$$\phi_{k,\pm}(t) := \varphi_{\pm}(t)t^{2k+2}.$$

Note that $|\rho_{\pm}(z)| \le \gamma\delta + |\beta\mu|$ since $1 \le z \le \sqrt{1+\frac{\mu^2}{\delta^2}}$. By the dominated convergence theorem, the difference of the continuous and discretized terms is given by

$$\int_0^1\frac{1}{t}\int_{-\mu t}^{0} yf(t,y)dydt-\sum_{n=1}^{N}\frac{1}{n}\int_{-\mu\frac{n}{N}}^{0} yf(\frac{n}{N},y)dy\frac{1}{N}$$

$$
\begin{aligned}
&= \int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} \Big[\frac{\delta}{\pi z}\big(\frac{e^{\rho_-(z)}-1}{2N} + \frac{1}{12N^2}[\varphi_-'(1)-\varphi_-'(0)] \\
&\quad + \sum_{i=1}^{p} \frac{B_{2i+2}}{(2i+2)!}[\varphi_-^{(2i+1)}(1)-\varphi_-^{(2i+1)}(0)]\frac{1}{N^{2i+2}}\big) \\
&\quad + \frac{\alpha^2\delta^3 z}{2\pi}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\sum_{j=0}^{\infty}\frac{\rho_-(z)^j}{j!}\Delta_{j,k,N} \\
&\quad + \frac{\alpha^2\delta^3 z}{4\pi}\sum_{k=0}^{\infty}(2\log\frac{\alpha\delta z}{2} - \psi(k+1)-\psi(k+2))\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\big(\frac{e^{\rho_-(z)}}{2N} \\
&\quad + \frac{1}{12N^2}[\phi_{k,-}'(1)-\phi_{k,-}'(0)] + \sum_{i=1}^{p}\frac{B_{2i+2}}{(2i+2)!}[\phi_{k,-}^{(2i+1)}(1)-\phi_{k,-}^{(2i+1)}(0)]\frac{1}{N^{2i+2}}\big)\Big]dz \\
&\quad + O(\frac{1}{N^{2p+4}}) \\
&= \frac{P_1}{N} + \frac{P_2}{N^2} + \sum_{q=1}^{p+1}\frac{P_{2q+1}}{N^{2q+1}} + \sum_{q=1}^{p}\frac{P_{2q+2}}{N^{2q+2}} + \sum_{q=1}^{p+1} Q_{2q+2}\frac{\log N}{N^{2q+2}} + O(\frac{1}{N^{2p+4}}), \qquad (3.17)
\end{aligned}
$$

where

$$
\Delta_{j,k,N} = \frac{1}{12N^2} - \frac{\zeta'(-(2k+j+2))}{N^{2k+j+3}} + \sum_{q=1}^{p}\frac{d_{2q,2k+j+2}}{N^{2q+2}} + \frac{B_{2k+j+3}}{2k+j+3}\frac{\log N}{N^{2k+j+3}},
$$

and similar to above, in order to make sure the term with $\Delta_{j,k,N}$ is well defined, it suffices that $\frac{\gamma\delta+|\beta\mu|}{2\pi N} < 1$ and $\frac{\alpha\sqrt{\delta^2+\mu^2}}{4\pi N} < 1$), and also

$$
\begin{aligned}
P_1 &= \frac{\delta}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} \frac{e^{\gamma\delta-\beta\delta\sqrt{z^2-1}}-1}{z} \\
&\quad + \frac{\alpha^2\delta^2 z}{4}\sum_{k=0}^{\infty}(2\log\frac{\alpha\delta z}{2} - \psi(k+1)-\psi(k+2))\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}e^{\gamma\delta-\beta\delta\sqrt{z^2-1}}dz \\
&= -\frac{\delta}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\frac{1}{z}dz + \frac{\alpha\delta^2}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} e^{\gamma\delta-\beta\delta\sqrt{z^2-1}}K_1(\alpha\delta z)dz \\
&= -\frac{\delta}{4\pi}\log(1+\frac{\mu^2}{\delta^2}) + \frac{\alpha\delta^2}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} e^{\gamma\delta-\beta\delta\sqrt{z^2-1}}K_1(\alpha\delta z)dz \\
P_2 &= \frac{\delta^2}{12\pi}\Big[-\gamma\log\sqrt{1+\frac{\mu^2}{\delta^2}} + \frac{\beta\mu}{\delta} - \beta\,\mathrm{arcsec}\sqrt{1+\frac{\mu^2}{\delta^2}} \\
&\quad + \alpha\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} \rho_-(z)e^{\rho_-(z)}K_1(\alpha\delta z) - \alpha\delta z e^{\rho_-(z)}K_0(\alpha\delta z)dz\Big] \\
P_{2q+1} &= -\frac{\alpha^2\delta^3}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} \sum_{k,j\geq 0, 2k+j=2q-2} z\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{\rho_-(z)^j}{j!}\zeta'(-2q)dz \\
P_{2q+2} &= \int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} \Big[\frac{\delta}{\pi z}\frac{B_{2q+2}}{(2q+2)!}[\varphi_-^{(2q+1)}(1)-\varphi_-^{(2q+1)}(0)] \\
&\quad + \frac{\alpha^2\delta^3 z}{2\pi}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\sum_{j=0}^{\infty}\frac{\rho_-(z)^j}{j!}d_{2q,2k+j+2}
\end{aligned}
$$

$$+\frac{\alpha^2\delta^3 z}{4\pi}\sum_{k=0}^{\infty}(2\log\frac{\alpha\delta z}{2}-\psi(k+1)-\psi(k+2))\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{B_{2q+2}}{(2q+2)!}[\phi_{k,-}^{(2q+1)}(1)-\phi_{k,-}^{(2q+1)}(0)]\Big]$$
$$-\frac{\alpha^2\delta^3 z}{2\pi}\sum_{k,j\geq 0,2k+j=2q-1}\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{\rho_-(z)^j}{j!}\zeta'(-(2q+1))dz,$$

and

$$Q_{2q+2}=\frac{\alpha^2\delta^3}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\sum_{k,j\geq 0,2k+j=2q-1}\frac{z(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{\rho_-(z)^j}{j!}\frac{B_{2q+2}}{2q+2}dz.$$

For $\mu<0$, we have

$$\int_0^1\frac{1}{t}\int_{-\mu t}^0 yf(t,y)dydt=-\frac{\alpha\delta^2}{\pi}\int_0^1 te^{\gamma\delta t}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}K_1(\alpha\delta tz)e^{\beta\delta t\sqrt{z^2-1}}dzdt.$$

So we see that this is actually the same as the case of $\mu>0$ except for the sign of β, which indicates that $\mu<0$ gives a conclusion taking the same form as the case of $\mu>0$. In other words, in (3.17), whenever we encounter a constant or function involving β, we just replace it with $-\beta$, which will give us the conclusion for the case $\mu<0$.

We only left with the last term in (3.12). If $\mu<0$, then it is given by

$$\begin{aligned}
&\mu\int_0^1\int_{-\mu t}^{\infty}f(t,y)dydt\\
=\;&\frac{\mu\alpha\delta}{\pi}\int_0^1 te^{\delta\gamma t}\int_{\sqrt{1+\frac{\mu^2}{\delta^2}}}^{\infty}K_1(\alpha\delta tz)e^{\beta\delta t\sqrt{z^2-1}}\frac{1}{\sqrt{z^2-1}}dzdt\\
=\;&\frac{\mu\alpha\delta}{\pi}\int_0^1 te^{\delta\gamma t}\Big(\int_1^{\infty}K_1(\alpha\delta tz)e^{\beta\delta t\sqrt{z^2-1}}\frac{1}{\sqrt{z^2-1}}dz\\
&-\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}K_1(\alpha\delta tz)e^{\beta\delta t\sqrt{z^2-1}}\frac{1}{\sqrt{z^2-1}}dz\Big)dt.
\end{aligned}\tag{3.18}$$

The first term in the above parenthesis of (3.18) is

$$\begin{aligned}
&\int_1^{\infty}K_1(\alpha\delta tz)e^{\beta\delta t\sqrt{z^2-1}}\frac{1}{\sqrt{z^2-1}}dz\\
=\;&\int_1^{\infty}\sum_{k=0}^{\infty}\frac{(\beta\delta t\sqrt{z^2-1})^k}{k!}\frac{K_1(\alpha\delta tz)}{\sqrt{z^2-1}}dz\\
=\;&\sum_{k=0}^{\infty}\frac{(\beta\delta t)^k}{k!}\int_1^{\infty}K_1(\alpha\delta tz)(z^2-1)^{\frac{k-1}{2}}dz\\
=\;&\sum_{k=0}^{\infty}\frac{(\beta\delta t)^k}{k!}2^{\frac{k-1}{2}}(\alpha\delta t)^{-1-\frac{k-1}{2}}K_{\frac{k-1}{2}}(\alpha\delta t)\Gamma(1+\frac{k-1}{2})
\end{aligned}$$

$$= \quad (2\alpha\delta t)^{-\frac{1}{2}}K_{-\frac{1}{2}}(\alpha\delta t)\sqrt{\pi} + \sum_{k=1}^{\infty}\frac{(\beta\delta t)^k}{k!}2^{\frac{k-1}{2}}(\alpha\delta t)^{-1-\frac{k-1}{2}}K_{\frac{k-1}{2}}(\alpha\delta t)\Gamma(1+\frac{k-1}{2})$$

$$= \quad \frac{\pi}{2\alpha\delta t}e^{-\alpha\delta t} + \beta\delta t\sum_{k=0}^{\infty}\frac{(\beta\delta t)^k}{(k+1)!}2^{\frac{k}{2}}(\alpha\delta t)^{-1-\frac{k}{2}}K_{\frac{k}{2}}(\alpha\delta t)\Gamma(1+\frac{k}{2}),$$

where the second equality follows from the monotone convergence theorem for $\beta \geq 0$ and the dominated convergence theorem for $\beta < 0$, and the third one follows from Lemma 3.3.6. Therefore, by the dominated convergence theorem, the first term in (3.18) can be written as

$$\begin{aligned}
& \frac{\mu}{2}\int_0^1 e^{(\gamma-\alpha)\delta t}dt + \frac{\mu\beta\delta}{\pi}\int_0^1 te^{\gamma\delta t}\sum_{k=0}^{\infty}\frac{(\frac{\beta}{\alpha})^k 2^{\frac{k}{2}}}{(k+1)!}(\alpha\delta t)^{\frac{k}{2}}K_{\frac{k}{2}}(\alpha\delta t)\Gamma(1+\frac{k}{2})dt \\
= \quad & \frac{\mu}{2}\int_0^1 e^{(\gamma-\alpha)\delta t}dt + \frac{\mu\beta\delta}{\pi}\sum_{m=0}^{\infty}\frac{2^m m!}{(2m+1)!}(\frac{\beta}{\alpha})^{2m}\int_0^1 te^{\gamma\delta t}(\alpha\delta t)^m K_m(\alpha\delta t)dt \\
& +\frac{\mu\beta\delta}{\pi}\sum_{m=0}^{\infty}\frac{2^{m+\frac{1}{2}}\Gamma(m+\frac{3}{2})}{(2m+2)!}(\frac{\beta}{\alpha})^{2m+1}\int_0^1 te^{\gamma\delta t}(\alpha\delta t)^{m+\frac{1}{2}}K_{m+\frac{1}{2}}(\alpha\delta t)dt. \qquad (3.19)
\end{aligned}$$

Hence, the corresponding discretized version of (3.19) is given by

$$\begin{aligned}
& \frac{\mu}{2}\sum_{n=1}^{N}e^{(\gamma-\alpha)\delta\frac{n}{N}}\frac{1}{N} + \frac{\mu\beta\delta}{\pi}\sum_{m=0}^{\infty}\frac{2^m m!(\frac{\beta}{\alpha})^{2m}}{(2m+1)!}\sum_{n=1}^{N}\frac{n}{N}e^{\gamma\delta\frac{n}{N}}(\alpha\delta\frac{n}{N})^m K_m(\alpha\delta\frac{n}{N})\frac{1}{N} \\
& +\frac{\mu\beta\delta}{\pi}\sum_{m=0}^{\infty}\frac{2^{m+\frac{1}{2}}\Gamma(m+\frac{3}{2})}{(2m+2)!}(\frac{\beta}{\alpha})^{2m+1}\sum_{n=1}^{N}\frac{n}{N}e^{\gamma\delta\frac{n}{N}}(\alpha\delta\frac{n}{N})^{m+\frac{1}{2}}K_{m+\frac{1}{2}}(\alpha\delta\frac{n}{N})\frac{1}{N} \qquad (3.20)
\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}
& (3.19) - (3.20) \\
= \quad & -\frac{\mu}{2}\Big[\frac{e^{(\gamma-\alpha)\delta}-1}{2N} + \sum_{q=0}^{p}\frac{B_{2q+2}}{(2q+2)!}\frac{((\gamma-\alpha)\delta)^{2q+1}(e^{(\gamma-\alpha)\delta}-1)}{N^{2q+2}}\Big] \\
& -\frac{\mu\beta\delta e^{\gamma\delta}}{2\pi N}\sum_{m=0}^{\infty}\frac{m!2^m}{(2m+1)!}(\frac{\beta}{\alpha})^{2m}(\alpha\delta)^m K_m(\alpha\delta) + \frac{\mu\beta\delta}{12\pi}\frac{\log N}{N^2} \\
& +\frac{\mu\beta\delta}{12\pi N^2}\Big[\sum_{m=0}^{\infty}\frac{m!2^m}{(2m+1)!}(\frac{\beta}{\alpha})^{2m}(\alpha\delta)^m[\alpha\delta e^{\gamma\delta}K_{m-1}(\alpha\delta) - (1+\gamma\delta)e^{\gamma\delta}K_m(\alpha\delta)] \\
& +\log\frac{2}{\alpha\delta} - \widetilde{\gamma} - 12\zeta'(-1) + \sum_{m=1}^{\infty}\frac{(2m-2)!!}{(2m+1)!!}(\frac{\beta}{\alpha})^{2m}\Big] \\
& +\frac{\mu\beta\delta}{\pi}\sum_{q=1}^{p+1}\sum_{m,k,j\geq 0, 2m+2k+j=2q-1}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m+1)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\frac{-\zeta'(-2q)}{N^{2q+1}} \\
& +\frac{\mu\beta\delta}{\pi}\sum_{q=1}^{p}\sum_{m,k,j\geq 0, 2m+2k+j=2q}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m+1)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\frac{B_{2q+2}}{2q+2}\frac{\log N}{N^{2q+2}}
\end{aligned}$$

$$+\frac{\mu\beta\delta}{\pi}\sum_{q=1}^{p}\Big[\sum_{m=0}^{\infty}\frac{2^m m!}{(2m+1)!}(\frac{\beta}{\alpha})^{2m}\Big(-2^{m-1}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{k!}(-\frac{1}{4}(\alpha\delta)^2)^k\cdot$$

$$(a_{k,0}^{(2q+1)}(1)-a_{k,0}^{(2q+1)}(0))\frac{B_{2q+2}}{(2q+2)!}$$

$$+\frac{(\alpha\delta)^{2m}}{(-2)^{m+1}}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(m+k+1)-2\log\frac{\alpha\delta}{2})\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\cdot$$

$$(a_{k,m}^{(2q+1)}(1)-a_{k,m}^{(2q+1)}(0))\frac{B_{2q+2}}{(2q+2)!}$$

$$+\frac{(-1)^m(\alpha\delta)^{2m}}{2^m}\sum_{k=0}^{\infty}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}d_{2q,2m+2k+j+1}\Big)$$

$$-\sum_{m,k,j\geq 0,2m+2k+j=2q}\frac{m!(-1)^m(\beta\delta)^{2m}}{(2m+1)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\zeta'(-(2q+1))\Big]\frac{1}{N^{2q+2}}$$

$$-\frac{\mu\beta\delta e^{(\gamma-\alpha)\delta}}{4N}\sum_{m=0}^{\infty}\frac{(\frac{\beta}{\alpha})^{2m+1}}{(2m+2)!!}\sum_{k=0}^{m}\frac{(m+k)!(\alpha\delta)^{m-k}}{(2k)!!(m-k)!}$$

$$-\frac{\mu\beta\delta}{24N^2}\sum_{m=0}^{\infty}\frac{(\frac{\beta}{\alpha})^{2m+1}}{(2m+2)!!}\sum_{k=0}^{m}\frac{(m+k)!(\alpha\delta)^{m-k}}{(2k)!!(m-k)!}[(m-k+1+(\gamma-\alpha)\delta)e^{(\gamma-\alpha)\delta}]$$

$$+\frac{\mu\beta\delta}{24N^2}\sum_{m=0}^{\infty}\frac{(2m)!}{(2m+2)!!(2m)!!}(\frac{\beta}{\alpha})^{2m+1}$$

$$-\frac{\mu\beta\delta}{2}\sum_{q=1}^{p}\Big[\sum_{m=0}^{\infty}\frac{(\frac{\beta}{\alpha})^{2m+1}}{(2m+2)!!}\sum_{k=0}^{m}\frac{(m+k)!(\alpha\delta)^{m-k}}{(2k)!!(m-k)!}(b_{k,m}^{(2q+1)}(1)-b_{k,m}^{(2q+1)}(0))\Big]\cdot$$

$$\frac{B_{2q+2}}{(2q+2)!}\frac{1}{N^{2q+2}}+O(\frac{1}{N^{2p+4}}),$$

where $a_{k,m}(t)=e^{\gamma\delta t}t^{2m+2k+1}, b_{k,m}(t)=e^{(\gamma-\alpha)\delta t}t^{m-k+1}$.

The second half in (3.18) is given by

$$=-\frac{\mu\alpha\delta}{\pi}\int_0^1 te^{\gamma\delta t}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}K_1(\alpha\delta tz)e^{\beta\delta t\sqrt{z^2-1}}\frac{1}{\sqrt{z^2-1}}dzdt$$

$$=-\frac{\mu\alpha\delta}{\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\int_0^1 tK_1(\alpha\delta tz)e^{(\gamma\delta+\beta\delta\sqrt{z^2-1})t}\frac{1}{\sqrt{z^2-1}}dtdz.$$

Now recall that the second term in (3.12) with $\mu<0$, is given by

$$\int_0^1\frac{1}{t}\int_{-\mu t}^{0}yf(t,y)dydt=-\frac{\alpha\delta^2}{\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\int_0^1 te^{(\gamma\delta+\beta\delta\sqrt{z^2-1})t}K_1(\alpha\delta tz)dtdz.$$

Ignoring the constant multiples, we see that these two terms are very similar, except for that the former integrand contains one more factor $\frac{1}{\sqrt{z^2-1}}$. Thus, all the conclusions for the second term of (3.12) holds here. More precisely, we have

$$-\frac{\mu\alpha\delta}{\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\int_0^1 tK_1(\alpha\delta tz)e^{(\gamma\delta+\beta\delta\sqrt{z^2-1})t}dt\frac{dz}{\sqrt{z^2-1}}$$

$$+\frac{\mu\alpha\delta}{\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\sum_{n=1}^{N}\frac{n}{N}K_1(\alpha\delta\frac{n}{N}z)e^{(\gamma\delta+\beta\delta\sqrt{z^2-1})\frac{n}{N}}\frac{1}{N}\frac{dz}{\sqrt{z^2-1}}$$

$$=\frac{U_1}{N}+\frac{U_2}{N^2}+\sum_{q=1}^{p+1}\frac{U_{2q+1}}{N^{2q+1}}+\sum_{q=1}^{p}\frac{U_{2q+2}}{N^{2q+2}}+\sum_{q=1}^{p+1}V_{2q+2}\frac{\log N}{N^{2q+2}}+O(\frac{1}{N^{2p+4}}), \tag{3.21}$$

where

$$\begin{aligned}
U_1 &= \frac{\mu}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\frac{e^{\gamma\delta+\beta\delta\sqrt{z^2-1}}-1}{z} \\
&\quad+\frac{\alpha^2\delta^2 z}{4}\sum_{k=0}^{\infty}(2\log\frac{\alpha\delta z}{2}-\psi(k+1)-\psi(k+2))\frac{(\frac14(\alpha\delta z)^2)^k}{k!(k+1)!}e^{\gamma\delta+\beta\delta\sqrt{z^2-1}}\frac{dz}{\sqrt{z^2-1}} \\
&= -\frac{\mu}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\frac{1}{z}\frac{dz}{\sqrt{z^2-1}}+\frac{\mu\alpha\delta}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}e^{\gamma\delta+\beta\delta\sqrt{z^2-1}}K_1(\alpha\delta z)\frac{dz}{\sqrt{z^2-1}} \\
&= -\frac{\mu}{2\pi}\mathrm{arcsec}\sqrt{1+\frac{\mu^2}{\delta^2}}+\frac{\mu\alpha\delta}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}e^{\gamma\delta+\beta\delta\sqrt{z^2-1}}K_1(\alpha\delta z)\frac{dz}{\sqrt{z^2-1}} \\
U_2 &= \frac{\mu\delta}{12\pi}\Big[-\gamma\mathrm{arcsec}\sqrt{1+\frac{\mu^2}{\delta^2}}-\beta\log\sqrt{1+\frac{\mu^2}{\delta^2}} \\
&\quad+\alpha\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\rho_+(z)e^{\rho_+(z)}K_1(\alpha\delta z)-\alpha\delta z e^{\rho_+(z)}K_0(\alpha\delta z)\frac{dz}{\sqrt{z^2-1}}\Big] \\
U_{2q+1} &= -\frac{\mu\alpha^2\delta^2}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\sum_{k,j\geq 0,2k+j=2q-2}z\frac{(\frac14(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{\rho_+(z)^j}{j!}\zeta'(-2q)\frac{dz}{\sqrt{z^2-1}} \\
U_{2q+2} &= \int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\Big[\frac{\mu}{\pi z}\frac{B_{2q+2}}{(2q+2)!}[\varphi_+^{(2q+1)}(1)-\varphi_+^{(2q+1)}(0)] \\
&\quad+\frac{\mu\alpha^2\delta^2 z}{2\pi}\sum_{k=0}^{\infty}\frac{(\frac14(\alpha\delta z)^2)^k}{k!(k+1)!}\sum_{j=0}^{\infty}\frac{\rho_+(z)^j}{j!}d_{2q,2k+j+2} \\
&\quad+\frac{\mu\alpha^2\delta^2 z}{4\pi}\sum_{k=0}^{\infty}(2\log\frac{\alpha\delta z}{2}-\psi(k+1)-\psi(k+2))\frac{(\frac14(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{B_{2q+2}}{(2q+2)!}\cdot \\
&\quad[\phi_{k,+}^{(2q+1)}(1)-\phi_{k,+}^{(2q+1)}(0)]\Big] \\
&\quad-\frac{\mu\alpha^2\delta^2 z}{2\pi}\sum_{k,j\geq 0,2k+j=2q-1}\frac{(\frac14(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{\rho_+(z)^j}{j!}\zeta'(-(2q+1))\frac{dz}{\sqrt{z^2-1}},
\end{aligned}$$

and

$$V_{2q+2} = \frac{\mu\alpha^2\delta^2}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\sum_{k,j\geq 0,2k+j=2q-1}\frac{z(\frac14(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{\rho_+(z)^j}{j!}\frac{B_{2q+2}}{2q+2}\frac{dz}{\sqrt{z^2-1}}.$$

All the coefficients in the above are finite since the variable z is in a finite range, and by realizing that $\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} \frac{1}{z\sqrt{z^2-1}}dz = \text{arcsec}\sqrt{1+\frac{\mu^2}{\delta^2}}$. Also, we require that $\frac{\alpha\delta}{\pi N} < 1$, $\frac{2\gamma\delta}{\pi N} < 1$ as above.

Lastly, we have the case $\mu > 0$ for the third term in (3.12).

$$
\begin{aligned}
& \mu \int_0^1 \int_{-\mu t}^{\infty} f(t,y)dydt \\
= \quad & \mu \int_0^1 \int_{-\mu t}^{0} f(t,y)dydt + \mu \int_0^1 \int_0^{\infty} f(t,y)dydt \\
= \quad & \frac{\mu\alpha\delta}{\pi} \int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} \int_0^1 tK_1(\alpha\delta tz)e^{(\gamma\delta-\beta\delta\sqrt{z^2-1})t} \frac{1}{\sqrt{z^2-1}}dtdz \\
& + \frac{\mu\alpha\delta}{\pi} \int_0^1 te^{\delta\gamma t} \int_1^{\infty} K_1(\alpha\delta tz)e^{\beta\delta t\sqrt{z^2-1}} \frac{1}{\sqrt{z^2-1}}dzdt.
\end{aligned}
$$

Now we see that the first term above corresponds to the second term in (3.18) of the case $\mu < 0$ except for the sign of β, while the second term above corresponds to the first term in (3.18), so these two cases are exactly the same. Therefore, we completed all the cases.

In summary, the general NIG process defined on $[0,1]$ with parameters $\alpha, \beta, \gamma, \delta, \mu$ with $\frac{\alpha\delta}{\pi N} < 1$, $\frac{2\gamma\delta}{\pi N} < 1$, $\frac{\gamma\delta+|\beta\mu|}{2\pi N} < 1$ and $\frac{\alpha\sqrt{\delta^2+\mu^2}}{4\pi N} < 1$, then the difference of the continuous supremum and discrete maximum admits the following asymptotic expansion

$$
\begin{aligned}
\Delta_N \quad = \quad & \mathbb{E}[\sup_{0\le t\le 1} X_t - \sup_{0\le i\le N} X_{t_i}] \\
= \quad & Y_1 \frac{\log N}{N} + \frac{Z_1}{N} + Y_2 \frac{\log N}{N^2} + \frac{Z_2}{N^2} \\
& + \sum_{q=1}^{p+1} \frac{Z_{2q+1}}{N^{2q+1}} + \sum_{q=1}^{p} \frac{Z_{2q+2}}{N^{2q+2}} + \sum_{q=1}^{p+1} Y_{2q+2} \frac{\log N}{N^{2q+2}} + O(\frac{1}{N^{2p+4}}),
\end{aligned}
$$

where

$$
Y_1 = \frac{\delta}{2\pi}, \quad Y_2 = \frac{\gamma\delta^2 + \mu\beta\delta}{12\pi}
$$

$$
\begin{aligned}
Z_1 \quad = \quad & \frac{\delta}{2\pi}\Big(\log(\frac{4\pi}{\alpha\delta}) - \widetilde{\gamma} + \frac{|\beta|}{\gamma}\arcsin(\frac{|\beta|}{\alpha}) - e^{\gamma\delta}\sum_{m=0}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta^2\delta}{\alpha})^m K_m(\alpha\delta)\Big) \\
& - \frac{\delta e^{(\gamma-\alpha)\delta}}{4}\sum_{m=0}^{\infty}\frac{(\frac{\beta}{\alpha})^{2m+1}}{(2m)!!}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!} + \frac{\delta\beta}{4\gamma} \\
& - \frac{\delta}{4\pi}\log(1+\frac{\mu^2}{\delta^2}) + \frac{\alpha\delta^2}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} e^{\gamma\delta\mp\beta\delta\sqrt{z^2-1}}K_1(\alpha\delta z)dz
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{\mu}{4}(1-e^{(\gamma-\alpha)\delta}) - \frac{\mu\beta\delta e^{\gamma\delta}}{2\pi}\sum_{m=0}^{\infty}\frac{m!2^m}{(2m+1)!}(\frac{\beta}{\alpha})^{2m}(\alpha\delta)^m K_m(\alpha\delta) \\
& -\frac{\mu\beta\delta e^{(\gamma-\alpha)\delta}}{4}\sum_{m=0}^{\infty}\frac{(\frac{\beta}{\alpha})^{2m+1}}{(2m+2)!!}\sum_{k=0}^{m}\frac{(m+k)!(\alpha\delta)^{m-k}}{(2k)!!(m-k)!} \\
& \pm\frac{\mu}{2\pi}\mathrm{arcsec}\sqrt{1+\frac{\mu^2}{\delta^2}} \mp \frac{\mu\alpha\delta}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} e^{\gamma\delta\mp\beta\delta\sqrt{z^2-1}}K_1(\alpha\delta z)\frac{dz}{\sqrt{z^2-1}} \\
Z_2 \quad = \quad & \frac{\delta}{12\pi}\Big[\gamma\delta(\log\frac{2}{\alpha\delta}-\widetilde{\gamma}+12\log A-1+\frac{|\beta|}{\gamma}\arcsin(\frac{|\beta|}{\alpha})) \\
& +\delta e^{\gamma\delta}\sum_{m=0}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta^2\delta}{\alpha})^m\big(\alpha K_{m-1}(\alpha\delta)-\gamma K_m(\alpha\delta)\big)\Big] \\
& -\frac{\delta e^{(\gamma-\alpha)\delta}}{24}\sum_{m=0}^{\infty}\frac{1}{(2m)!!}(\frac{\beta}{\alpha})^{2m+1}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!}(m-k+(\gamma-\alpha)\delta) \\
& +\frac{\beta\delta^2(\gamma-\alpha)}{24\gamma}+\frac{\delta^2}{12\pi}\Big[-\gamma\log\sqrt{1+\frac{\mu^2}{\delta^2}}\pm\frac{\beta\mu}{\delta}\mp\beta\mathrm{arcsec}\sqrt{1+\frac{\mu^2}{\delta^2}} \\
& +\alpha\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\rho_\mp(z)e^{\rho_\mp(z)}K_1(\alpha\delta z)-\alpha\delta z e^{\rho_\mp(z)}K_0(\alpha\delta z)dz\Big]-\frac{\mu}{24}(e^{(\gamma-\alpha)\delta}-1)(\gamma-\alpha)\delta \\
& +\frac{\mu\beta\delta}{12\pi}\Big[\sum_{m=0}^{\infty}\frac{m!2^m}{(2m+1)!}(\frac{\beta}{\alpha})^{2m}(\alpha\delta)^m[\alpha\delta e^{\gamma\delta}K_{m-1}(\alpha\delta)-(1+\gamma\delta)e^{\gamma\delta}K_m(\alpha\delta)] \\
& +\log\frac{2}{\alpha\delta}-\widetilde{\gamma}+12\log A-\frac{\gamma}{|\beta|}\arcsin(\frac{|\beta|}{\alpha})\Big] \\
& -\frac{\mu\beta\delta}{24}\sum_{m=0}^{\infty}\frac{(\frac{\beta}{\alpha})^{2m+1}}{(2m+2)!!}\sum_{k=0}^{m}\frac{(m+k)!(\alpha\delta)^{m-k}}{(2k)!!(m-k)!}[(m-k+1+(\gamma-\alpha)\delta)e^{(\gamma-\alpha)\delta}] \\
& +\frac{\mu\delta(\alpha-\gamma)}{24}\mp\frac{\mu\delta}{12\pi}\Big[-\gamma\mathrm{arcsec}\sqrt{1+\frac{\mu^2}{\delta^2}}\pm\beta\log\sqrt{1+\frac{\mu^2}{\delta^2}} \\
& +\alpha\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\rho_\mp(z)e^{\rho_\mp(z)}K_1(\alpha\delta z)-\alpha\delta z e^{\rho_\mp(z)}K_0(\alpha\delta z)\frac{dz}{\sqrt{z^2-1}}\Big], \\
Z_{2q+1} \quad = \quad & -\frac{\delta}{\pi}\sum_{m,k,j\geq 0,2m+2k+j=2q}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\zeta'(-2q) \\
& -\frac{\alpha^2\delta^3}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\sum_{k,j\geq 0,2k+j=2q-2} z\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{\rho_\mp(z)^j}{j!}\zeta'(-2q)dz \\
& +\frac{\mu\beta\delta}{\pi}\sum_{m,k,j\geq 0,2m+2k+j=2q-1}\frac{(-1)^m(\beta\delta)^{2m}m!}{(2m+1)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}(-\zeta'(-2q)) \\
& \pm\frac{\mu\alpha^2\delta^2}{2\pi}\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\sum_{k,j\geq 0,2k+j=2q-2} z\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{\rho_\mp(z)^j}{j!}\zeta'(-2q)\frac{dz}{\sqrt{z^2-1}}, \\
Z_{2q+2} \quad = \quad & \frac{\delta}{\pi}(C(q)+E(q)) \\
& -\frac{\delta}{2}\sum_{m=0}^{\infty}\frac{(\frac{\beta}{\alpha})^{2m+1}}{(2m)!!}\sum_{k=0}^{m}\frac{(m+k)!}{(2k)!!}\frac{(\alpha\delta)^{m-k}}{(m-k)!}\frac{B_{2q+2}}{(2q+2)!}(y_{k,m}^{(2q+1)}(1)-y_{k,m}^{(2q+1)}(0))
\end{aligned}
$$

$$+\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\Big[\frac{\delta}{\pi z}\frac{B_{2q+2}}{(2q+2)!}[\varphi_{\mp}^{(2q+1)}(1)-\varphi_{\mp}^{(2q+1)}(0)]+\frac{\alpha^2\delta^3 z}{2\pi}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\sum_{j=0}^{\infty}\frac{\rho_{\mp}(z)^j}{j!}d_{2q,2k+j+2}$$

$$+\frac{\alpha^2\delta^3 z}{4\pi}\sum_{k=0}^{\infty}(2\log\frac{\alpha\delta z}{2}-\psi(k+1)-\psi(k+2))\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\cdot\frac{B_{2q+2}}{(2q+2)!}[\phi_{k,\mp}^{(2q+1)}(1)-\phi_{k,\mp}^{(2q+1)}(0)]\Big]$$

$$-\frac{\alpha^2\delta^3 z}{2\pi}\sum_{k,j\geq 0,2k+j=2q-1}\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{\rho_{\mp}(z)^j}{j!}\zeta'(-(2q+1))dz-\frac{\mu}{2}\frac{B_{2q+2}}{(2q+2)!}((\gamma-\alpha)\delta)^{2q+1}(e^{(\gamma-\alpha)\delta}-1)$$

$$+\frac{\mu\beta\delta}{\pi}\Big[\sum_{m=0}^{\infty}\frac{2^m m!}{(2m+1)!}(\frac{\beta}{\alpha})^{2m}\Big(-2^{m-1}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{k!}(-\frac{1}{4}(\alpha\delta)^2)^k(a_{k,0}^{(2q+1)}(1)-a_{k,0}^{(2q+1)}(0))\frac{B_{2q+2}}{(2q+2)!}$$

$$+\frac{(\alpha\delta)^{2m}}{(-2)^{m+1}}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(m+k+1)-2\log\frac{\alpha\delta}{2})\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}(a_{k,m}^{(2q+1)}(1)-a_{k,m}^{(2q+1)}(0))\frac{B_{2q+2}}{(2q+2)!}$$

$$+\frac{(-1)^m(\alpha\delta)^{2m}}{2^m}\sum_{k=0}^{\infty}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}d_{2q,2m+2k+j+1}\Big)$$

$$-\sum_{m,k,j\geq 0,2m+2k+j=2q}\frac{m!(-1)^m(\beta\delta)^{2m}}{(2m+1)!}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\frac{(\gamma\delta)^j}{j!}\zeta'(-(2q+1))\Big]$$

$$-\frac{\mu\beta\delta}{2}\Big[\sum_{m=0}^{\infty}\frac{(\frac{\beta}{\alpha})^{2m+1}}{(2m+2)!!}\sum_{k=0}^{m}\frac{(m+k)!(\alpha\delta)^{m-k}}{(2k)!!(m-k)!}(b_{k,m}^{(2q+1)}(1)-b_{k,m}^{(2q+1)}(0))\Big]\frac{B_{2q+2}}{(2q+2)!}$$

$$\mp\int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}}\Big[\frac{\mu}{\pi z}\frac{B_{2q+2}}{(2q+2)!}[\varphi_{\mp}^{(2q+1)}(1)-\varphi_{\mp}^{(2q+1)}(0)]+\frac{\mu\alpha^2\delta^2 z}{2\pi}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\sum_{j=0}^{\infty}\frac{\rho_{\mp}(z)^j}{j!}d_{2q,2k+j+2}$$

$$+\frac{\mu\alpha^2\delta^2 z}{4\pi}\sum_{k=0}^{\infty}(2\log\frac{\alpha\delta z}{2}-\psi(k+1)-\psi(k+2))\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{B_{2q+2}}{(2q+2)!}[\phi_{k,\mp}^{(2q+1)}(1)-\phi_{k,\mp}^{(2q+1)}(0)]\Big]$$

$$-\frac{\mu\alpha^2\delta^2 z}{2\pi}\sum_{k,j\geq 0,2k+j=2q-1}\frac{(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!}\frac{\rho_{\mp}(z)^j}{j!}\zeta'(-(2q+1))\frac{dz}{\sqrt{z^2-1}}$$

$$C(q)+E(q)$$

$$=\sum_{m=0}^{\infty}\frac{m!2^m}{(2m)!}(\frac{\beta}{\alpha})^{2m}\Big[-2^{m-1}\sum_{k=0}^{m-1}\frac{(m-k-1)!}{k!}(-\frac{1}{4}(\alpha\delta)^2)^k\frac{B_{2q+2}}{(2q+2)!}[g_{k,0}^{(2q+1)}(1)-g_{k,0}^{(2q+1)}(0)]$$

$$+\frac{(\alpha\delta)^{2m}}{(-2)^{m+1}}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(m+k+1)-2\log\frac{\alpha\delta}{2})\frac{(\frac{1}{4}(\alpha\delta)^2)^k}{k!(m+k)!}\frac{B_{2q+2}}{(2q+2)!}[g_{k,m}^{(2q+1)}(1)-g_{k,m}^{(2q+1)}(0)]$$

$$+\frac{(-1)^m(\alpha\delta)^{2m}}{2^m}\sum_{k=0}^{\infty}\frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!}\sum_{j=0}^{\infty}\frac{(\gamma\delta)^j}{j!}d_{2q,2m+2k+j}\Big]$$

$$-\sum_{m,k,j\geq 0, 2m+2k+j=2q+1} \frac{(-1)^m (\beta\delta)^{2m} m!}{(2m)!} \frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!} \frac{(\gamma\delta)^j}{j!} \zeta'(-(2q+1))$$

$$\begin{aligned}
Y_{2q+2} &= \sum_{m,k,j\geq 0, 2m+2k+j=2q+1} \frac{\delta}{\pi} \frac{(-1)^m (\beta\delta)^{2m} m!}{(2m)!} \frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!} \frac{(\gamma\delta)^j}{j!} \frac{B_{2q+2}}{2q+2} \\
&+ \frac{\alpha^2\delta^3}{2\pi} \int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} \sum_{k,j\geq 0, 2k+j=2q-1} \frac{z(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!} \frac{\rho_{\mp}(z)^j}{j!} \frac{B_{2q+2}}{2q+2} dz \\
&+ \frac{\mu\beta\delta}{\pi} \sum_{m,k,j\geq 0, 2m+2k+j=2q} \frac{(-1)^m (\beta\delta)^{2m} m!}{(2m+1)!} \frac{(\frac{\alpha\delta}{2})^{2k}}{k!(m+k)!} \frac{(\gamma\delta)^j}{j!} \frac{B_{2q+2}}{2q+2} \\
&\mp \frac{\mu\alpha^2\delta^2}{2\pi} \int_1^{\sqrt{1+\frac{\mu^2}{\delta^2}}} \sum_{k,j\geq 0, 2k+j=2q-1} \frac{z(\frac{1}{4}(\alpha\delta z)^2)^k}{k!(k+1)!} \frac{\rho_{\mp}(z)^j}{j!} \frac{B_{2q+2}}{2q+2} \frac{dz}{\sqrt{z^2-1}},
\end{aligned}$$

where A is the Glaisher-Kinkelin constant, $\widetilde{\gamma}$ is the Euler constant, and the upper signs in $\pm$ or $\mp$ correspond to the case $\mu > 0$, while the lower ones correspond to the case $\mu < 0$. Also, we used the following identities to simplify the terms

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{(2m-2)!!}{(2m-1)!!} (\frac{\beta}{\alpha})^{2m} &= \frac{|\beta|}{\gamma} \arcsin(\frac{|\beta|}{\alpha}) \\
\sum_{m=1}^{\infty} \frac{(2m-2)!!}{(2m+1)!!} (\frac{\beta}{\alpha})^{2m} &= 1 - \frac{\gamma}{|\beta|} \arcsin(\frac{|\beta|}{\alpha}) \\
\sum_{m=0}^{\infty} \frac{(2m)!}{((2m)!!)^2} (\frac{\beta}{\alpha})^{2m+1} &= \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} (\frac{\beta}{\alpha})^{2m+1} = \frac{\beta}{\gamma} \\
\sum_{m=0}^{\infty} \frac{(2m)!}{(2m)!!(2m+2)!!} (\frac{\beta}{\alpha})^{2m+1} &= \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m+2)!!} (\frac{\beta}{\alpha})^{2m+1} = \frac{\beta}{\alpha+\gamma},
\end{aligned}$$

in which we should note that $\alpha^2 = \beta^2 + \gamma^2$, and $0!! = 1$, $(-1)!! = 1$, by convention. □

Remark 3.3.11. From the result above, we see that the drift coefficient μ has no effect on the leading term since the order $\frac{\log N}{N}$ is slower than $\frac{1}{N}$. This could also be observed in the result of Janssen and Van Leeuwaarden's, in which the coefficient of $\frac{1}{\sqrt{N}}$ doesn't involve μ, either. Also, if the NIG is defined on $[0, T]$, then the leading term would be $\frac{T\delta \log N}{2\pi N}$.

3.4 Variance Gamma Process

Variance gamma process was first introduced by Madan and Senata [53] in 1990. It is a 3-parameter pure jump Lévy process with the following Lévy density:

$$\Pi(dx) = \mathbb{1}_{\{x<0\}}\frac{C}{|x|}e^{Gx}dx + \mathbb{1}_{\{x>0\}}\frac{C}{x}e^{-Mx}dx,$$

where $C, G, M > 0$. Clearly, the Lévy measure has infinite mass and makes the variance gamma process admit paths of finite variation. Moreover, the variance gamma can be written as the difference of two gamma processes, which may be thought of the overall buy orders and sell orders in terms of logarithmic price scale. For more details about variance gamma, one may refer to Kyprianou [51], Madan, Carr and Chang [54].

Alternatively, with adding another drift term, we may define the variance gamma process as a subordinated Brownian motion. More precisely, the variance gamma process can be written as

$$X_t = \mu t + B(\gamma_t; \theta, s),$$

where $B(\gamma_t; \theta, s)$ is a Brownian motion with drift θ and volatility s subordinated by a gamma process with mean rate 1 and variance rate ν. The correspondence between the triples (C, G, M) and (s, ν, θ) is given by

$$C = \frac{1}{\nu}, G = \sqrt{\frac{\theta^2}{s^4} + \frac{2}{s^2\nu}} + \frac{\theta}{s^2}, M = \sqrt{\frac{\theta^2}{s^4} + \frac{2}{s^2\nu}} - \frac{\theta}{s^2},$$

$$s = \sqrt{\frac{2C}{GM}}, \nu = \frac{1}{C}, \theta = \frac{C}{M} - \frac{C}{G}.$$

According to (6) in Madan, Carr and Chang [54], the transition density function is the following

$$f(t,x) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi u}} \exp\Big(- \frac{(x-\theta u)^2}{2\sigma^2 u}\Big) \frac{u^{\frac{t}{\nu}-1}\exp(-\frac{u}{\nu})}{\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})} du.$$

Just like the NIG case, we first deal with the non-drifted case, i.e. $Y_t = X_t - \mu t$, then the second and third terms, as what was shown in (3.12).

$$\mathbb{E}[\sup_{0\le t\le 1} Y_t] = \int_0^1 \frac{1}{t}\int_0^\infty y f(t,y) dy dt$$

$$= \int_0^1 \frac{1}{t}\int_0^\infty y \int_0^\infty \frac{1}{\sigma\sqrt{2\pi u}} \exp\Big(- \frac{(y-\theta u)^2}{2\sigma^2 u}\Big) \frac{u^{\frac{t}{\nu}-1}\exp(-\frac{u}{\nu})}{\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})} du dy dt$$

$$= \int_0^1 \frac{1}{t} \int_0^\infty \frac{1}{\sigma\sqrt{2\pi u}} \int_0^\infty y \exp\Big(-\frac{(y-\theta u)^2}{2\sigma^2 u}\Big) dy \frac{u^{\frac{t}{\nu}-1}\exp(-\frac{u}{\nu})}{\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})} du dt$$
$$= \int_0^1 \frac{1}{t} \int_0^\infty \frac{1}{\sigma\sqrt{2\pi u}} \Big[\sigma^2 u e^{-\frac{\theta^2 u}{2\sigma^2}} + \theta\sigma u^{\frac{3}{2}}\sqrt{\frac{\pi}{2}}\mathrm{Erfc}(-\frac{\theta}{\sigma}\sqrt{\frac{u}{2}})\Big] \frac{u^{\frac{t}{\nu}-1}\exp(-\frac{u}{\nu})}{\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})} du dt$$
$$= \int_0^\infty \int_0^1 \frac{(\frac{u}{\nu})^{\frac{t}{\nu}}}{\Gamma(\frac{t}{\nu}+1)} dt \frac{1}{\nu\sqrt{2\pi u}} \Big[\sigma e^{-\frac{\theta^2 u}{2\sigma^2}} + \theta u^{\frac{1}{2}}\sqrt{\frac{\pi}{2}}\mathrm{Erfc}(-\frac{\theta}{\sigma}\sqrt{\frac{u}{2}})\Big] e^{-\frac{u}{\nu}} du, \tag{3.22}$$

where both the third and fifth equalities follow from Tonelli's theorem. On the other hand, the discrete version of (3.22), i.e., $\mathbb{E}[\sup_{0\le i\le N} Y_{t_i}]$, is given by

$$= \int_0^\infty \sum_{n=1}^N \frac{(\frac{u}{\nu})^{\frac{n}{N\nu}}}{\Gamma(\frac{n}{N\nu}+1)} \frac{1}{N} \frac{1}{\nu\sqrt{2\pi u}} \Big[\sigma e^{-\frac{\theta^2 u}{2\sigma^2}} + \theta u^{\frac{1}{2}}\sqrt{\frac{\pi}{2}}\mathrm{Erfc}(-\frac{\theta}{\sigma}\sqrt{\frac{u}{2}})\Big] e^{-\frac{u}{\nu}} du. \tag{3.23}$$

Thus,

$$\mathbb{E}[\sup_{0\le t\le 1} Y_t - \sup_{0\le i\le N} Y_{t_i}] = \int_0^\infty \Big[\int_0^1 \frac{(\frac{u}{\nu})^{\frac{t}{\nu}}}{\Gamma(\frac{t}{\nu}+1)} dt - \sum_{n=1}^N \frac{(\frac{u}{\nu})^{\frac{n}{N\nu}}}{\Gamma(\frac{n}{N\nu}+1)} \frac{1}{N}\Big]$$
$$\cdot \frac{1}{\nu\sqrt{2\pi u}} \Big[\sigma e^{-\frac{\theta^2 u}{2\sigma^2}} + \theta u^{\frac{1}{2}}\sqrt{\frac{\pi}{2}}\mathrm{Erfc}(-\frac{\theta}{\sigma}\sqrt{\frac{u}{2}})\Big] e^{-\frac{u}{\nu}} du. \tag{3.24}$$

Using the Euler-Maclaurin formula, we get that

$$\int_0^1 \frac{(\frac{u}{\nu})^{\frac{t}{\nu}}}{\Gamma(\frac{t}{\nu}+1)} dt - \sum_{n=1}^N \frac{(\frac{u}{\nu})^{\frac{n}{N\nu}}}{\Gamma(\frac{n}{N\nu}+1)} \frac{1}{N}$$

$$= \int_0^1 \frac{(\frac{u}{\nu})^{\frac{t}{\nu}}}{\Gamma(\frac{t}{\nu}+1)} dt - \sum_{n=0}^N \frac{(\frac{u}{\nu})^{\frac{n}{N\nu}}}{\Gamma(\frac{n}{N\nu}+1)} \frac{1}{N} + \frac{1}{N}$$

$$= \frac{1}{N} - \Big(\sum_{n=0}^N \frac{(\frac{u}{\nu})^{\frac{n}{N\nu}}}{\Gamma(\frac{n}{N\nu}+1)} \frac{1}{N} - \int_0^1 \frac{(\frac{u}{\nu})^{\frac{t}{\nu}}}{\Gamma(\frac{t}{\nu}+1)} dt\Big)$$

$$= \frac{1}{N} - (\frac{1}{2N} + \frac{1}{2N}\frac{(\frac{u}{\nu})^{\frac{1}{\nu}}}{\Gamma(\frac{1}{\nu}+1)} + \frac{1}{12}(f'(N) - f'(0)) - \int_0^N f''(x)\frac{B_2(x - \lfloor x \rfloor)}{2} dx)$$

$$= \frac{1}{2N} - \frac{1}{2N}\frac{(\frac{u}{\nu})^{\frac{1}{\nu}}}{\Gamma(\frac{1}{\nu}+1)} - \frac{1}{12N^2}(g'(1) - g'(0)) + \int_0^1 g''(x)\frac{B_2(Nx - \lfloor Nx \rfloor)}{2N^2} dx,$$

where $f(x) = \frac{(\frac{u}{\nu})^{\frac{x}{N\nu}}}{\Gamma(\frac{x}{N\nu}+1)}\frac{1}{N} := g(\frac{x}{N})\frac{1}{N}, g(x) = \frac{(\frac{u}{\nu})^{\frac{x}{\nu}}}{\Gamma(\frac{x}{\nu}+1)}$ and $f^{(k)}(x) = g^{(k)}(\frac{x}{N})\frac{1}{N^{k+1}}$.

Note that the function g is infinitely continuously differentiable with respect to x on the interval

$[0,1]$ and $g^{(k)}(1)$, $g^{(k)}(0)$ can be regarded as functions of u. To be more precise, let's take $\nu = 1$, which does not lose any generality of this function g. Then we know that,

$$g(x) = \frac{u^x}{\Gamma(x+1)}, \qquad g'(x) = \frac{u^x \log u \Gamma(x+1) - u^x \Gamma'(x+1)}{\Gamma(x+1)^2}$$

$$g''(x) = \frac{u^x \log^2 u}{\Gamma(x+1)} - \frac{2u^x \log u \Gamma'(x+1)}{\Gamma(x+1)^2} - \frac{u^x \Gamma''(x+1)}{\Gamma(x+1)^2} + \frac{2u^x \Gamma'(x+1)^2}{\Gamma(x+1)^3},$$

where

$$\Gamma'(x+1) = \int_0^\infty z^x \log z e^{-z} dz \text{ and } \Gamma''(x+1) = \int_0^\infty z^x \log^2 z e^{-z} dz.$$

Thus,

$$\Gamma'(x+1)\Big|_{x=0} = \int_0^\infty \log z e^{-z} dz = -\int_{-\infty}^\infty t e^{-e^{-t}} e^{-t} dt = -\gamma,$$

where the last equality follows from the evaluation of the mean of the standard Gumbel distribution. Also,

$$\Gamma'(x+1)\Big|_{x=1} = \int_0^\infty z \log z e^{-z} dz = -\int_0^\infty z \log z d(e^{-z})$$

$$= 1 + \int_0^\infty \log z e^{-z} dz = 1 - \int_{-\infty}^\infty t e^{-e^{-t}} e^{-t} dt = 1 - \gamma.$$

Hence $g'(0) = \log u + \gamma$, $\quad g'(1) = u \log u - u(1-\gamma)$, which means that these two terms integrate to finite values in (3.24). Similarly,

$$|g''(x)| \leq C_1 u^x (\log^2 u + |\log u| + C_2),$$

since all of $\frac{1}{\Gamma(x+1)}$, $\Gamma'(x+1)$, and $\Gamma''(x+1)$ are bounded in absolute value in the interval $x \in (0,1)$. Combining with the fact that $|B_2(Nx - \lfloor Nx \rfloor)| \leq |B_2| = \frac{1}{6}$, if we plug these back into (3.24), we have

$$\mathbb{E}[\sup_{0 \leq t \leq 1} Y_t - \sup_{0 \leq i \leq N} Y_{t_i}] = \frac{C_3}{N} + O(\frac{1}{N^2}),$$

where

$$C_3 = \int_0^\infty (1 - \frac{(\frac{u}{\nu})^{\frac{1}{\nu}}}{\Gamma(\frac{1}{\nu}+1)}) \frac{e^{-\frac{u}{\nu}}}{2\nu\sqrt{2\pi u}} \Big[\sigma e^{-\frac{\theta^2 u}{2\sigma^2}} + \theta u^{\frac{1}{2}} \sqrt{\frac{\pi}{2}} \mathrm{Erfc}(-\frac{\theta}{\sigma}\sqrt{\frac{u}{2}})\Big] du.$$

The above discussion actually finishes the case without drift term μt.

Next, we take a look at the other two terms caused by the drift μ, namely,

$$\mu \int_0^1 \int_{-\mu t}^\infty f(t,y) dy dt + \int_0^1 \frac{1}{t} \int_{-\mu t}^0 y f(t,y) dy dt$$

$$= \mu \int_0^1 \int_{-\mu t}^0 f(t,y) dy dt + \mu \int_0^1 \int_0^\infty f(t,y) dy dt + \int_0^1 \frac{1}{t} \int_{-\mu t}^0 y f(t,y) dy dt. \tag{3.25}$$

We first analyze the middle term in (3.25) whose upper and lower limits do not depend on μt.

$$\begin{aligned}
& \mu \int_0^1 \int_0^\infty f(t,y) dy dt \\
= \;& \mu \int_0^1 \int_0^\infty \frac{1}{\sigma\sqrt{2\pi u}} \sigma \sqrt{\frac{\pi u}{2}} [1 + \mathrm{Erf}(\frac{\theta}{\sigma}\sqrt{\frac{u}{2}})] \frac{u^{\frac{t}{\nu}-1} \exp(-\frac{u}{\nu})}{\nu^{\frac{t}{\nu}} \Gamma(\frac{t}{\nu})} du dt \\
= \;& \frac{\mu}{2} \int_0^1 \int_0^\infty [1 + \mathrm{Erf}(\frac{\theta}{\sigma}\sqrt{\frac{u}{2}})] \frac{u^{\frac{t}{\nu}-1} \exp(-\frac{u}{\nu})}{\nu^{\frac{t}{\nu}} \Gamma(\frac{t}{\nu})} du dt \\
= \;& \frac{\mu}{2} + \frac{\mu}{2} \int_0^1 \int_0^\infty \mathrm{Erf}(\frac{\theta}{\sigma}\sqrt{\frac{u}{2}}) \frac{u^{\frac{t}{\nu}-1} \exp(-\frac{u}{\nu})}{\nu^{\frac{t}{\nu}} \Gamma(\frac{t}{\nu})} du dt \\
= \;& \frac{\mu}{2} + \frac{\mu}{2} \int_0^1 \frac{1}{\nu^{\frac{t}{\nu}} \Gamma(\frac{t}{\nu}+1)} \int_0^\infty \mathrm{Erf}(\frac{\theta}{\sigma}\sqrt{\frac{u}{2}}) \exp(-\frac{u}{\nu}) d(u^{\frac{t}{\nu}}) dt \\
= \;& \frac{\mu}{2} + \frac{\mu}{2} \int_0^1 \frac{1}{\nu^{\frac{t}{\nu}} \Gamma(\frac{t}{\nu}+1)} \Big(\frac{1}{\nu} \int_0^\infty u^{\frac{t}{\nu}} e^{-\frac{u}{\nu}} \mathrm{Erf}(\frac{\theta}{\sigma}\sqrt{\frac{u}{2}}) du - \frac{\theta}{\sigma\sqrt{2\pi}} \int_0^\infty u^{\frac{t}{\nu}} e^{-\frac{u}{\nu}} \frac{e^{-\frac{\theta^2 u}{2\sigma^2}}}{\sqrt{u}} du \Big) dt \\
= \;& \frac{\mu}{2} + \frac{\mu}{2} \int_0^\infty \int_0^1 \frac{(\frac{u}{\nu})^{\frac{t}{\nu}}}{\Gamma(\frac{t}{\nu}+1)} dt \Big(\frac{1}{\nu} \mathrm{Erf}(\frac{\theta}{\sigma}\sqrt{\frac{u}{2}}) - \frac{\theta e^{-\frac{\theta^2 u}{2\sigma^2}}}{\sigma\sqrt{2\pi u}} \Big) e^{-\frac{u}{\nu}} du,
\end{aligned}$$

where the last equality follows from Tonelli's theorem. The discretized version of the above is given by

$$= \frac{\mu}{2} + \frac{\mu}{2} \int_0^\infty \sum_{n=1}^N \frac{(\frac{u}{\nu})^{\frac{n}{N\nu}}}{\Gamma(\frac{n}{N\nu}+1)} \frac{1}{N} \Big(\frac{1}{\nu} \mathrm{Erf}(\frac{\theta}{\sigma}\sqrt{\frac{u}{2}}) - \frac{\theta e^{-\frac{\theta^2 u}{2\sigma^2}}}{\sigma\sqrt{2\pi u}} \Big) e^{-\frac{u}{\nu}} du$$

Hence, the difference of the continuous and discretized versions is

$$\mu \int_0^1 \int_0^\infty f(t,y) dy dt - \mu \sum_{n=1}^N \int_0^\infty f(\frac{n}{N}, y) dy \frac{1}{N}$$

$$= \frac{\mu}{2}\int_0^\infty \Big(\int_0^1 \frac{(\frac{u}{\nu})^{\frac{t}{\nu}}}{\Gamma(\frac{t}{\nu}+1)}dt - \sum_{n=1}^N \frac{(\frac{u}{\nu})^{\frac{n}{N\nu}}}{\Gamma(\frac{n}{N\nu}+1)}\frac{1}{N}\Big)\Big(\frac{1}{\nu}\mathrm{Erf}(\frac{\theta}{\sigma}\sqrt{\frac{u}{2}}) - \frac{\theta e^{-\frac{\theta^2 u}{2\sigma^2}}}{\sigma\sqrt{2\pi u}}\Big)e^{-\frac{u}{\nu}}du. \tag{3.26}$$

By using the same technique in dealing with (3.24), we know that

$$(3.26) = \frac{C_4}{N} + O(\frac{1}{N^2}),$$

where

$$C_4 = \frac{\mu}{4}\int_0^\infty \Big(1 - \frac{(\frac{u}{\nu})^{\frac{1}{\nu}}}{\Gamma(\frac{1}{\nu}+1)}\Big)\Big(\frac{1}{\nu}\mathrm{Erf}(\frac{\theta}{\sigma}\sqrt{\frac{u}{2}}) - \frac{\theta e^{-\frac{\theta^2 u}{2\sigma^2}}}{\sigma\sqrt{2\pi u}}\Big)e^{-\frac{u}{\nu}}du.$$

Now we are left with the other two terms in (3.25). From now on, let's denote

$$g(t) := \int_{-\mu t}^0 f(t,y)dy,$$

and

$$h(t) := \frac{1}{t}\int_{-\mu t}^0 y f(t,y)dy := \frac{a(t)}{t}.$$

Then, the sum of the first and the third terms in (3.25) may be written as

$$\mu\int_0^1 g(t)dt + \int_0^1 h(t)dt. \tag{3.27}$$

Correspondingly, the discrete version of (3.27) is given by

$$\mu\sum_{n=1}^N g(\frac{n}{N})\frac{1}{N} + \sum_{n=1}^N h(\frac{n}{N})\frac{1}{N}. \tag{3.28}$$

Therefore, by applying Euler-Maclaurin formula, we deduce that the difference between (3.27) and (3.28) is

$$\mu\int_0^1 g(t)dt + \int_0^1 h(t)dt - \mu\sum_{n=1}^N g(\frac{n}{N})\frac{1}{N} - \sum_{n=1}^N h(\frac{n}{N})\frac{1}{N}$$

$$= \mu\Big(\int_0^{\frac{1}{N}} g(t)dt - \frac{1}{2N}(g(\frac{1}{N}) + g(1)) - \frac{1}{12N^2}(g'(1) - g'(\frac{1}{N})) + \frac{1}{N^2}\int_{\frac{1}{N}}^1 g''(t)\frac{B_2(Nt - \lfloor Nt\rfloor)}{2}dt\Big)$$

$$+\int_0^{\frac{1}{N}} h(t)dt - \frac{1}{2N}(h(\frac{1}{N}) + h(1)) - \frac{1}{12N^2}(h'(1) - h'(\frac{1}{N})) + \frac{1}{N^2}\int_{\frac{1}{N}}^1 h''(t)\frac{B_2(Nt - \lfloor Nt \rfloor)}{2}dt \tag{3.29}$$

Given the definition of the function $g(t)$, we see that the Leibniz differentiation rule may not be applicable since the function $f(t,y)$ is not continuous on the compact set $[0,1] \times [0,1] \in \mathbb{R}^2$. Therefore, we have to evaluate $g'(t)$ directly.

According to Madan, Carr and Chang [54], given the parameters $\nu, \sigma \in \mathbb{R}^+$ and $\theta \in \mathbb{R}$, the transition density function of variance gamma process could be written as

$$f(t,y) = \frac{2e^{\frac{\theta y}{\sigma^2}}}{\nu^{\frac{t}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{t}{\nu})}\Big(\frac{y^2}{\frac{2\sigma^2}{\nu} + \theta^2}\Big)^{\frac{t}{2\nu} - \frac{1}{4}} K_{\frac{t}{\nu} - \frac{1}{2}}\Big(\frac{1}{\sigma^2}\sqrt{(\frac{2\sigma^2}{\nu} + \theta^2)y^2}\Big),$$

where $K_\alpha(x)$ is the modified Bessel function of the second kind.

For simplicity, let's denote $\lambda := \sqrt{\frac{2\sigma^2}{\nu} + \theta^2} > 0$. Also, since in (3.29), we have the integrals $\int_0^{\frac{1}{N}} g(t)dt$, $\int_0^{\frac{1}{N}} h(t)dt$, thus we may assume that N is large enough such that $0 < t < \frac{1}{N} < \frac{\nu}{2}$, so $\frac{1}{2} - \frac{t}{\nu} > 0$. Then, by Abramowitz and Stegun [1] page 375, 9.6.2 and 9.6.10,

$$\begin{aligned}
f(t,y) &= \frac{2e^{\frac{\theta y}{\sigma^2}}}{\nu^{\frac{t}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{t}{\nu})}\Big(\frac{|y|}{\lambda}\Big)^{\frac{t}{\nu} - \frac{1}{2}}\frac{\pi}{2}\csc((\frac{1}{2} - \frac{t}{\nu})\pi) \cdot \\
&\quad \Big[(\frac{\lambda|y|}{2\sigma^2})^{\frac{t}{\nu} - \frac{1}{2}}\sum_{k=0}^{\infty}\frac{(\frac{\lambda^2 y^2}{4\sigma^4})^k}{k!\Gamma(k + \frac{t}{\nu} + \frac{1}{2})} - (\frac{\lambda|y|}{2\sigma^2})^{\frac{1}{2} - \frac{t}{\nu}}\sum_{k=0}^{\infty}\frac{(\frac{\lambda^2 y^2}{4\sigma^4})^k}{k!\Gamma(k - \frac{t}{\nu} + \frac{3}{2})}\Big] \\
&= \frac{\sqrt{\pi}\sec(\frac{t}{\nu}\pi)}{\sqrt{2}\sigma\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})}(1 + R_y)\Big[(\frac{y^2}{2\sigma^2})^{\frac{t}{\nu} - \frac{1}{2}}\Big(\frac{1}{\Gamma(\frac{t}{\nu} + \frac{1}{2})} + \sum_{k=1}^{\infty}\frac{(\frac{\lambda^2 y^2}{4\sigma^4})^k}{k!\Gamma(k + \frac{t}{\nu} + \frac{1}{2})}\Big) \\
&\quad -(\frac{\lambda^2}{2\sigma^2})^{\frac{1}{2} - \frac{t}{\nu}}\Big(\frac{1}{\Gamma(\frac{3}{2} - \frac{t}{\nu})} + \sum_{k=1}^{\infty}\frac{(\frac{\lambda^2 y^2}{4\sigma^4})^k}{k!\Gamma(k - \frac{t}{\nu} + \frac{3}{2})}\Big)\Big] \\
&= \frac{\sqrt{\pi}\sec(\frac{t}{\nu}\pi)}{\sqrt{2}\sigma\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})}(1 + R_y)\Big[(\frac{y^2}{2\sigma^2})^{\frac{t}{\nu} - \frac{1}{2}}\Big(\frac{1}{\Gamma(\frac{t}{\nu} + \frac{1}{2})} + R^1_{t,y}\Big) - (\frac{\lambda^2}{2\sigma^2})^{\frac{1}{2} - \frac{t}{\nu}}\Big(\frac{1}{\Gamma(\frac{3}{2} - \frac{t}{\nu})} + R^2_{t,y}\Big)\Big].
\end{aligned}$$

Note that the leading term of R_y in the Taylor expansion is linear of y, and since gamma function with positive argument has a minimum attained at a value between 1.46 and 1.47, which implies that $R_y = \frac{\theta}{\sigma^2}y + o(y)$ and $R^1_{t,y} = \frac{\lambda^2 y^2}{4\sigma^4\Gamma(\frac{t}{\nu} + \frac{3}{2})} + o(y^2)$, $R^2_{t,y} = \frac{\lambda^2 y^2}{4\sigma^4\Gamma(\frac{5}{2} - \frac{t}{\nu})} + o(y^2)$. Also, for all $k \geq 0$, the functions $\frac{1}{\Gamma(k + \frac{t}{\nu} + \frac{1}{2})}$ and $\frac{1}{\Gamma(k - \frac{t}{\nu} + \frac{3}{2})}$ have continuous bounded first and second order derivatives for $t \in [0,1]$. Thus, we see that in the above expression, the terms with lowest degrees of y are

$$\frac{\sqrt{\pi}\sec(\frac{t}{\nu}\pi)}{\sqrt{2}\sigma\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})\Gamma(\frac{t}{\nu} + \frac{1}{2})}(\frac{y^2}{2\sigma^2})^{\frac{t}{\nu} - \frac{1}{2}},$$

and

$$-\frac{\sqrt{\pi}\sec(\frac{t}{\nu}\pi)(\frac{\lambda^2}{2\sigma^2})^{\frac{1}{2}-\frac{t}{\nu}}}{\sqrt{2}\sigma\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})\Gamma(\frac{3}{2}-\frac{t}{\nu})}.$$

Let's assume that $\mu > 0$ (the case of $\mu < 0$ would be the same). Besides, the uniform convergence allows us to interchange integration and summation, i.e.,

$$g(t) = \int_{-\mu t}^{0} f(t,y)dy$$

$$= \frac{\sqrt{\pi}\sec(\frac{t}{\nu}\pi)}{(2\sigma^2\nu)^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})\Gamma(\frac{t}{\nu}+\frac{1}{2})}\int_0^{\mu t} y^{\frac{2t}{\nu}-1}dy - \frac{\sqrt{\pi}\sec(\frac{t}{\nu}\pi)(\frac{\lambda^2}{2\sigma^2})^{\frac{1}{2}-\frac{t}{\nu}}}{\sqrt{2}\sigma\nu^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu})\Gamma(\frac{3}{2}-\frac{t}{\nu})}\mu t + R(t)$$

$$= \frac{\sqrt{\pi}\sec(\frac{t}{\nu}\pi)}{2(2\sigma^2\nu/\mu^2)^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu}+1)\Gamma(\frac{t}{\nu}+\frac{1}{2})}t^{\frac{2t}{\nu}} - \frac{\mu\sqrt{\pi}\sec(\frac{t}{\nu}\pi)(\frac{\lambda^2}{2\sigma^2})^{\frac{1}{2}-\frac{t}{\nu}}}{\sqrt{2}\sigma\nu^{\frac{t}{\nu}+1}\Gamma(\frac{t}{\nu}+1)\Gamma(\frac{3}{2}-\frac{t}{\nu})}t^2 + R(t)$$

$$= \frac{\sec(\frac{t}{\nu}\pi)}{2(\sigma^2\nu/2\mu^2)^{\frac{t}{\nu}}\Gamma(\frac{2t}{\nu}+1)}t^{\frac{2t}{\nu}} - \frac{\mu\sqrt{\pi}\sec(\frac{t}{\nu}\pi)(\frac{\lambda^2}{2\sigma^2})^{\frac{1}{2}-\frac{t}{\nu}}}{\sqrt{2}\sigma\nu^{\frac{t}{\nu}+1}\Gamma(\frac{t}{\nu}+1)\Gamma(\frac{3}{2}-\frac{t}{\nu})}t^2 + R(t), \tag{3.30}$$

where the last equality follows from the duplication formula $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$, with $z = \frac{t}{\nu}+\frac{1}{2}$, and $R(t) \in W_2^1([0,1])$, the Sobolev space.

Therefore, at this point can we analyze some terms in (3.29). For example,

$$\begin{aligned}
\mu\int_0^{\frac{1}{N}} g(t)dt &= \mu\int_0^{\frac{1}{N}} \frac{\sec(\frac{t}{\nu}\pi)}{2(\sigma^2\nu/2\mu^2)^{\frac{t}{\nu}}\Gamma(\frac{2t}{\nu}+1)}t^{\frac{2t}{\nu}}dt + O(\frac{1}{N^3}) \\
&= \frac{\mu}{2}\int_0^{\frac{1}{N}} t^{\frac{2t}{\nu}}dt + O(\frac{1}{N^2}) \\
&= \frac{\mu}{2}\int_0^{\frac{1}{N}} (1 + \frac{2t\text{log}t}{\nu} + R_\nu(t))dt + O(\frac{1}{N^2}) \\
&= \frac{\mu}{2}(\frac{1}{N} - \frac{\log N}{\nu N^2}) + O(\frac{1}{N^2}),
\end{aligned}$$

where the second equality follows from Taylor's expansion and the uniform convergence. Also, $|R_\nu(t)| \leq C(\nu)t$. As long as N is large enough, e.g. $\frac{\log N}{N} < \nu$, we can conclude that the above quantity is positive. If $\mu < 0$, we get a similar result. Put them together, we obtain that the first term in (3.29) can be written as

$$\mu\int_0^{\frac{1}{N}} g(t)dt = \frac{|\mu|}{2N} + O(\frac{\log N}{N^2}).$$

Now we take a look at $g(\frac{1}{N})$ and $g(1)$. Clearly, $g(1) = \int_{-\mu}^{0} f(1,y)dy < 1$. Also, from the discussion above, we see that $g(\frac{1}{N}) = \frac{1}{2} + O(\frac{\log N}{N})$. Therefore, with the consideration of the sign of μ, the second term in (3.29) has the following form

$$-\frac{\mu}{2N}(g(\frac{1}{N}) + g(1)) = -\frac{|\mu|}{4N} - \frac{\mu}{2N}\int_{-\mu}^{0} f(1,y)dy + O(\frac{\log N}{N^2}).$$

Next, we deal with the derivatives of g, i.e., $g'(1)$ and $g'(\frac{1}{N})$. From (3.30) and the fact that $(t^{\frac{2t}{\nu}})' = \frac{2}{\nu}t^{\frac{2t}{\nu}}(1+\log t)$ and all the other functions are in C^2, we conclude that the third term of (3.29) can be written as

$$-\frac{\mu}{12N^2}(g'(1) - g'(\frac{1}{N})) = O(\frac{\log N}{N^2}).$$

We will return to the fourth term later. At the time being, we move on to see those terms involving the function h. Again, we first assume that $\mu > 0$.

$$h(t) := \frac{1}{t}\int_{-\mu t}^{0} yf(t,y)dy$$

$$= -\frac{\sqrt{\pi}\sec(\frac{t}{\nu}\pi)\int_0^{\mu t} y^{\frac{2t}{\nu}}dy}{\nu(2\sigma^2\nu)^{\frac{t}{\nu}}\Gamma(\frac{t}{\nu}+1)\Gamma(\frac{t}{\nu}+\frac{1}{2})} + \frac{\sqrt{\pi}\sec(\frac{t}{\nu}\pi)(\frac{\lambda^2}{2\sigma^2})^{\frac{1}{2}-\frac{t}{\nu}}}{\sqrt{2}\sigma\nu^{\frac{t}{\nu}+1}\Gamma(\frac{t}{\nu}+1)\Gamma(\frac{3}{2}-\frac{t}{\nu})}\frac{\mu^2t^2}{2} + \widetilde{R}(t)$$

$$= -\frac{\mu\sqrt{\pi}\sec(\frac{t}{\nu}\pi)(\frac{\mu^2}{2\sigma^2\nu})^{\frac{t}{\nu}}t^{\frac{2t}{\nu}+1}}{(2t+\nu)\Gamma(\frac{t}{\nu}+1)\Gamma(\frac{t}{\nu}+\frac{1}{2})} + \frac{\mu^2\sqrt{\pi}\sec(\frac{t}{\nu}\pi)(\frac{\lambda^2}{2\sigma^2})^{\frac{1}{2}-\frac{t}{\nu}}t^2}{2\sqrt{2}\sigma\nu^{\frac{t}{\nu}+1}\Gamma(\frac{t}{\nu}+1)\Gamma(\frac{3}{2}-\frac{t}{\nu})} + \widetilde{R}(t)$$

$$= -\frac{\mu\sec(\frac{t}{\nu}\pi)(\frac{2\mu^2}{\sigma^2\nu})^{\frac{t}{\nu}}t^{\frac{2t}{\nu}+1}}{(2t+\nu)\Gamma(\frac{2t}{\nu}+1)} + \frac{\mu^2\sqrt{\pi}\sec(\frac{t}{\nu}\pi)(\frac{\lambda^2}{2\sigma^2})^{\frac{1}{2}-\frac{t}{\nu}}t^2}{2\sqrt{2}\sigma\nu^{\frac{t}{\nu}+1}\Gamma(\frac{t}{\nu}+1)\Gamma(\frac{3}{2}-\frac{t}{\nu})} + \widetilde{R}(t), \tag{3.31}$$

where $\widetilde{R}(t) \in W_2^1([0,1])$. Hence, from the above, the leading term of $h(t)$ is given by $-\frac{\mu}{\nu}t$. Combine with the case of $\mu < 0$, we obtain that the fifth term in (3.29) could be written as

$$\int_0^{\frac{1}{N}} h(t)dt = -\frac{|\mu|}{\nu}\int_0^{\frac{1}{N}} tdt + O(\frac{\log N}{N^3}) = -\frac{|\mu|}{2\nu N^2} + O(\frac{\log N}{N^3}).$$

Next, it is trivial that $h(1) = \displaystyle\int_{-\mu}^{0} yf(1,y)dy < \infty$. Also,

$$h(\frac{1}{N}) = N\int_{-\frac{\mu}{N}}^{0} yf(\frac{1}{N},y)dy = -\frac{|\mu|}{\nu N} + O(\frac{\log N}{N^2}).$$

Thus, the sixth term in (3.29) could be written as

$$-\frac{1}{2N}(h(\frac{1}{N})+h(1)) = -\frac{1}{2N}\int_{-\mu}^{0} yf(1,y)dy + \frac{|\mu|}{2\nu N^2} + O(\frac{\log N}{N^3}).$$

From (3.31), it is not hard to see that the seventh term in (3.29)

$$-\frac{1}{12N^2}(h'(1)-h'(\frac{1}{N})) = O(\frac{1}{N^2}).$$

Lastly, we consider the fourth and eighth term in (3.29) together, i.e., define

$$R^{\mu}_{VG} := \frac{1}{N^2}\int_{\frac{1}{N}}^{1}(\mu g''(t)+h''(t))\frac{B_2(Nt-\lfloor Nt\rfloor)}{2}dt.$$

Recall the expression of the transition density function of variance gamma and the series expansion of the modified Bessel function of the second kind $K_{\frac{t}{\nu}-\frac{1}{2}}$, we see that the terms with lowest degree of t involved in g'' and h'' will generate the dominating term in N, i.e., it suffices to check the following,

$$\frac{\mu\sec(\frac{t}{\nu}\pi)}{2(\sigma^2\nu/2\mu^2)^{\frac{t}{\nu}}\Gamma(\frac{2t}{\nu}+1)}t^{\frac{2t}{\nu}} - \frac{\mu\sec(\frac{t}{\nu}\pi)(\frac{2\mu^2}{\sigma^2\nu})^{\frac{t}{\nu}}t^{\frac{2t}{\nu}+1}}{(2t+\nu)\Gamma(\frac{2t}{\nu}+1)}$$

$$= \frac{\mu\sec(\frac{t}{\nu}\pi)(\frac{2\mu^2}{\sigma^2\nu})^{\frac{t}{\nu}}t^{\frac{2t}{\nu}}}{\Gamma(\frac{2t}{\nu}+1)}(\frac{1}{2}-\frac{t}{2t+\nu}) = \frac{\mu\sec(\frac{t}{\nu}\pi)(\frac{2\mu^2}{\sigma^2\nu})^{\frac{t}{\nu}}t^{\frac{2t}{\nu}}}{\Gamma(\frac{2t}{\nu}+1)}\frac{\nu}{2(2t+\nu)}.$$

Recall that $(t^t)' = t^t(\log t+1)$ and $(t^t)'' = t^{t-1}+t^t(1+\log t)^2$. Therefore,

$$|R^{\mu}_{VG}| \le C_5\frac{1}{N^2}\int_{\frac{1}{N}}^{1}\frac{1}{t}dt + O(\frac{1}{N^2}) = O(\frac{\log N}{N^2}).$$

In summary, we reach the conclusion that for the asymmetric case, i.e., $\mu \neq 0$, the leading term in (3.29) can be expressed as the following

$$\frac{|\mu|}{4N} - \frac{\mu}{2N}\int_{-\mu}^{0} f(1,y)dy - \frac{1}{2N}\int_{-\mu}^{0} yf(1,y)dy.$$

In conclusion, for any variance gamma process $X_t = \mu t + B(\gamma_t;\theta,s)$ with transition density function $f(t,y)$, the expected difference of continuous supremum and discrete maximum is given by

$$\mathbb{E}[\sup_{0\le t\le 1} X_t - \sup_{0\le i\le N} X_{t_i}] = \frac{C_6}{N} + \mu O(\frac{\log N}{N^2}) + O(\frac{1}{N^2}),$$

where

$$
\begin{aligned}
C_6 \quad = \quad & \int_0^\infty (1 - \frac{(\frac{u}{\nu})^{\frac{1}{\nu}}}{\Gamma(\frac{1}{\nu}+1)}) \frac{e^{-\frac{u}{\nu}}}{2\nu\sqrt{2\pi u}} \Big[\sigma e^{-\frac{\theta^2 u}{2\sigma^2}} + \theta u^{\frac{1}{2}} \sqrt{\frac{\pi}{2}} \text{Erfc}(-\frac{\theta}{\sigma}\sqrt{\frac{u}{2}})\Big] du \\
& + \frac{\mu}{4} \int_0^\infty \Big(1 - \frac{(\frac{u}{\nu})^{\frac{1}{\nu}}}{\Gamma(\frac{1}{\nu}+1)}\Big) \Big(\frac{1}{\nu} \text{Erf}(\frac{\theta}{\sigma}\sqrt{\frac{u}{2}}) - \frac{\theta e^{-\frac{\theta^2 u}{2\sigma^2}}}{\sigma\sqrt{2\pi u}}\Big) e^{-\frac{u}{\nu}} du \\
& + \frac{|\mu|}{4N} - \frac{\mu}{2N} \int_{-\mu}^0 f(1,y) dy - \frac{1}{2N} \int_{-\mu}^0 y f(1,y) dy.
\end{aligned}
$$

Remark 3.4.1. No matter in which case, symmetric or asymmetric, the leading order of variance gamma is always $\frac{1}{N}$, which is much better than the case when Brownian motion term exists. This conveys the information that the convergence speed is really dependent on the path property. In the next chapter, we will be dealing with the case when the transition density function is not very well expressed.

Chapter 4

Sampling Error for Lévy Processes with Implicit or Complicated Transition Density Function/Law

4.1 Kou's Jump Diffusion

As we have already seen in the above cases, i.e., Merton's jump diffusion, Normal Inverse Gaussian process (NIG) and Variance Gamma process (VG), each of the processes has an explicit expression of its transition density function. However, there exist quite a few processes used in finance that do not admit an explicit expression of transition density function or have very complicated ones, for instance, Kou's jump diffusion.

Kou's jump diffusion model was introduced by Kou in 2002 [44]. It is actually a type of Lévy process, i.e.,

$$X_t = \mu t + \sigma B_t + \sum_{n=1}^{N_t} Z_n,$$

where B_t is a standard Brownian motion, and N_t is a Poisson process with intensity λ, Z_n's are jump sizes that are i.i.d double exponential random variables with probability p of positive jumps with mean positive jump size $\frac{1}{\eta_1}$, and with probability $1-p$ of negative jumps with mean negative jump size $\frac{1}{\eta_2}$. All of the random sources are independent. Note that both η_1 and η_2 are positive, by definition. Thus, we know the Lévy density of Kou's jump diffusion:

$$\Pi(dx) = p\lambda\eta_1 e^{-\eta_1 x}\mathbb{1}_{\{x>0\}}dx + (1-p)\lambda\eta_2 e^{\eta_2 x}\mathbb{1}_{\{x<0\}}dx.$$

The characteristic function of X_t is given by

$$\phi_t(\xi) = \exp\Big[-\frac{1}{2}\sigma^2 t\xi^2 + i\mu t\xi - \lambda t\big(1 - \frac{p\eta_1}{\eta_1 - i\xi} - \frac{(1-p)\eta_2}{\eta_2 + i\xi}\big)\Big].$$

However, the transition density function of Kou's jump diffusion model can not be expressed explicitly. Thus, we may not be able to apply the same technique as what we have achieved for

Merton's jump diffusion, NIG and VG. A new approach with the help of Hilbert transform is needed. Let's first state the result with respect to Kou's jump diffusion:

Theorem 4.1.1. *With Kou's jump diffusion process X_t defined as above on time interval $[0,T]$ with the parameters $\sigma, \mu, p, \lambda, \alpha := \frac{1}{\eta_1}, \beta := \frac{1}{\eta_2}$ and $T > 0$, we have*

$$\mathbb{E}[\sup_{0\le t\le T} X_t - \sup_{0\le i\le N} X_{t_i}] = \frac{F_1}{\sqrt{N}} + \frac{F_2}{N} + \frac{F_3}{N\sqrt{N}} + O(\frac{1}{N^2}),$$

where

$$\begin{aligned}
F_1 &= -\frac{\zeta(\frac{1}{2})\sigma\sqrt{T}}{\sqrt{2\pi}}, \\
F_3 &= \frac{\sigma\zeta(-\frac{1}{2})(\lambda - \frac{\mu^2}{2\sigma^2})T\sqrt{T}}{\sqrt{2\pi}}, \\
F_2 &= -\frac{\sigma^2 T}{2\pi}\int_0^\infty e^{-\frac{1}{2}\sigma^2 T y^2 - \lambda T + DT} cos(\mu T y + ET) dy \\
&\quad -\frac{\mu T}{2\pi}\int_0^\infty \frac{1}{y} e^{-\frac{1}{2}\sigma^2 T y^2 - \lambda T + DT} sin(\mu T y + ET) dy \\
&\quad +\frac{p\alpha\lambda T}{4} - \frac{p\alpha^2\lambda T}{\pi}\int_0^\infty \frac{e^{-\frac{1}{2}\sigma^2 T y^2 - \lambda T + DT} cos(\mu T y + ET)}{(1+\alpha^2 y^2)^2} dy \\
&\quad -\frac{p\alpha\lambda T}{2\pi}\int_0^\infty e^{-\frac{1}{2}\sigma^2 T y^2 - \lambda T + DT} sin(\mu T y + ET)\frac{1}{y}\frac{1-\alpha^2 y^2}{(1+\alpha^2 y^2)^2} dy \\
&\quad +\frac{(1-p)\beta\lambda T}{4} - \frac{(1-p)\beta^2\lambda T}{\pi}\int_0^\infty \frac{e^{-\frac{1}{2}\sigma^2 T y^2 - \lambda T + DT} cos(\mu T y + ET)}{(1+\beta^2 y^2)^2} dy \\
&\quad +\frac{(1-p)\beta\lambda T}{2\pi}\int_0^\infty e^{-\frac{1}{2}\sigma^2 T y^2 - \lambda T + DT} sin(\mu T y + ET)\frac{1}{y}\frac{1-\beta^2 y^2}{(1+\beta^2 y^2)^2} dy,
\end{aligned}$$

and

$$D = \frac{\lambda p}{1+\alpha^2 y^2} + \frac{\lambda(1-p)}{1+\beta^2 y^2},$$

$$E = \frac{\lambda p \alpha y}{1+\alpha^2 y^2} - \frac{\lambda(1-p)\beta y}{1+\beta^2 y^2}.$$

Proof. Recall Spitzer's identity,

$$\mathbb{E}[\sup_{0\le t\le T} X_t - \sup_{0\le i\le N} X_{t_i}] = \int_0^T \frac{1}{t}\mathbb{E}[X_t^+]dt - \sum_{n=1}^N \frac{1}{n}\mathbb{E}[X^+_{\frac{nT}{N}}].$$

Thus, one key issue is to figure out the quantity $\mathbb{E}[X_t^+]$. We know that

$$\begin{aligned}\mathbb{E}[X_t^+] &= \int_{\mathbb{R}} x p_t(x) \mathbb{1}_{\{x>0\}} dx \\ &= (-i)\Big(\mathscr{F}\big(\mathbb{1}_{(0,\infty)} p_t\big)(\xi)\Big)'\Big|_{\xi=0} \\ &= (-i)\Big(\frac{1}{2}\hat{p}_t(\xi) + \frac{i}{2}\mathscr{H}\hat{p}_t(\xi)\Big)'\Big|_{\xi=0} \\ &= -\frac{i}{2}\hat{p}_t(\xi)'\Big|_{\xi=0} + \frac{1}{2}\mathscr{H}\hat{p}_t(\xi)'\Big|_{\xi=0} \\ &= \frac{1}{2}\mathbb{E}[X_t] + \frac{1}{2}\mathscr{H}\hat{p}_t(\xi)'\Big|_{\xi=0},\end{aligned}$$

where the second equality can be justified by the dominated convergence theorem since $\mathbb{E}[|X_t|] < \infty$, and both $\mathscr{F}$ and $\wedge$ denote the Fourier transform, $\mathscr{H}$ denote the Hilbert transform, and p_t is the transition density function. We need to figure out the term $\mathscr{H}\hat{p}_t(\xi)'\Big|_{\xi=0}$. Since the derivative of Hilbert transform is the Hilbert transform of the derivative, thus we obtain that

$$\mathscr{H}\hat{p}_t(\xi)'\Big|_{\xi=0} = \frac{d}{d\xi}\mathscr{H}\hat{p}_t(\xi)\Big|_{\xi=0} = \mathscr{H}\hat{p}_t{}'(\xi)\Big|_{\xi=0} = \mathscr{H}\phi_t'(\xi)\Big|_{\xi=0}$$

$$= \frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int_{\epsilon}^{\infty}\frac{1}{y}\big(\phi_t'(-y) - \phi_t'(y)\big)dy,$$

Therefore,

$$\mathbb{E}[X_t^+] = \frac{1}{2}\mathbb{E}[X_t] + \frac{1}{2\pi}\lim_{\epsilon\downarrow 0}\int_{\epsilon}^{\infty}\frac{1}{y}\big(\phi_t'(-y) - \phi_t'(y)\big)dy,$$

and

$$\mathbb{E}[\sup_{0\le t\le T} X_t] = \int_0^T \frac{1}{2t}\mathbb{E}[X_t]dt + \int_0^T \frac{1}{2\pi t}\lim_{\epsilon\downarrow 0}\int_{\epsilon}^{\infty}\frac{1}{y}\big(\phi_t'(-y) - \phi_t'(y)\big)dydt.$$

From the above discussion, we relate the supremum of the process with its own characteristic function, without using its transition density function, which is exactly the approach we apply.

For simplicity, in Kou's jump diffusion, we denote $\frac{1}{\eta_1} = \alpha$, $\frac{1}{\eta_2} = \beta$. Then we know

$$\phi_t(y) = \exp\Big[-\frac{1}{2}\sigma^2 t y^2 + i\mu t y - \lambda t\big(1 - \frac{p}{1 - i\alpha y} - \frac{1-p}{1 + i\beta y}\big)\Big],$$

and

$$\phi_t'(y) = \phi_t(y)\big(- \sigma^2 ty + i\mu t + \frac{ip\alpha\lambda t}{(1-i\alpha y)^2} - \frac{i(1-p)\beta\lambda t}{(1+i\beta y)^2}\big)$$

$$\phi_t'(-y) = \phi_t(-y)\big(\sigma^2 ty + i\mu t + \frac{ip\alpha\lambda t}{(1+i\alpha y)^2} - \frac{i(1-p)\beta\lambda t}{(1-i\beta y)^2}\big).$$

To simplify our notation, we denote $A \in \mathbb{C}$, $B, C \in \mathbb{R}$, such that

$$A := -\frac{1}{2}\sigma^2 ty^2 + i\mu ty - \lambda t\big(1 - \frac{p}{1-i\alpha y} - \frac{1-p}{1+i\beta y}\big) := B + iC.$$

Thus,

$$B = -\frac{1}{2}\sigma^2 ty^2 - \lambda t + \frac{\lambda pt}{1+\alpha^2 y^2} + \frac{\lambda(1-p)t}{1+\beta^2 y^2} := -\frac{1}{2}\sigma^2 ty^2 - \lambda t + Dt$$

$$C = \mu ty + \frac{\lambda p\alpha ty}{1+\alpha^2 y^2} - \frac{\lambda(1-p)\beta ty}{1+\beta^2 y^2} := \mu ty + Et.$$

Note that since $y > 0$, we have $0 < D < \lambda$ and $|E| < \frac{\lambda}{2}$. Therefore,

$$\phi_t(y) = e^A = e^{B+iC}$$

$$\phi_t'(y) = e^{B+iC}\big(- \sigma^2 ty + i\mu t + \frac{ip\alpha\lambda t}{(1-i\alpha y)^2} - \frac{i(1-p)\beta\lambda t}{(1+i\beta y)^2}\big)$$

$$\phi_t'(-y) = e^{B-iC}\big(\sigma^2 ty + i\mu t + \frac{ip\alpha\lambda t}{(1+i\alpha y)^2} - \frac{i(1-p)\beta\lambda t}{(1-i\beta y)^2}\big).$$

Now we take a look at the terms involved in $\phi_t'(-y) - \phi_t'(y)$, the first of which is

$$\sigma^2 ty(e^{B-iC} + e^{B+iC}) = 2\sigma^2 tye^B \text{cos} C.$$

Hence

$$\int_0^T \frac{1}{2\pi t} \lim_{\epsilon\downarrow 0} \int_\epsilon^\infty \frac{1}{y} 2\sigma^2 tye^B \text{cos} C dydt$$

$$= \frac{\sigma^2}{\pi} \int_0^T \lim_{\epsilon\downarrow 0} \int_\epsilon^\infty e^{-\frac{1}{2}\sigma^2 ty^2 - \lambda t + Dt} \cos(\mu ty + Et) dydt$$

$$
\begin{aligned}
&= \frac{\sigma^2}{\pi}\int_0^T\int_0^\infty e^{-\frac{1}{2}\sigma^2 ty^2-\lambda t+Dt}\cos(\mu ty+Et)dydt \\
&= \frac{\sigma^2}{\pi}\int_0^\infty\int_0^T e^{-\frac{1}{2}\sigma^2 ty^2-\lambda t+Dt}\cos(\mu ty+Et)dtdy \\
&= \frac{\sigma^2}{\pi}\int_0^\infty\int_0^T e^{-\frac{1}{2}\sigma^2 ty^2-\lambda t+Dt}[\cos(\mu ty)+\cos(\mu ty+Et)-\cos(\mu ty)]dtdy \\
&= \frac{\sigma^2}{\pi}\int_0^\infty\int_0^T e^{-\frac{1}{2}\sigma^2 ty^2-\lambda t}(1+e^{Dt}-1)[\cos(\mu ty)-2\sin(\mu ty+\frac{Et}{2})\sin(\frac{Et}{2})]dtdy \\
&= \frac{\sigma^2}{\pi}\int_0^\infty\int_0^T e^{-\lambda t}e^{-\frac{1}{2}\sigma^2 ty^2}\cos(\mu ty)dtdy \\
&\quad +\frac{\sigma^2}{\pi}\int_0^\infty\int_0^T e^{-\lambda t}e^{-\frac{1}{2}\sigma^2 ty^2}(e^{Dt}-1)\cos(\mu ty)dtdy \\
&\quad -\frac{2\sigma^2}{\pi}\int_0^\infty\int_0^T e^{-\lambda t}e^{-\frac{1}{2}\sigma^2 ty^2}e^{Dt}\sin(\mu ty+\frac{Et}{2})\sin(\frac{Et}{2})dtdy, \qquad (4.1)
\end{aligned}
$$

where the second and the third equalities follows from the dominated convergence theorem and Fubini's theorem, respectively. Now we apply Fubini's Theorem to the first double integral in (4.1) to see that

$$
\frac{\sigma^2}{\pi}\int_0^T e^{-\lambda t}\int_0^\infty e^{-\frac{1}{2}\sigma^2 ty^2}\cos(\mu ty)dydt
$$

$$
= \frac{\sigma^2}{\pi}\int_0^T e^{-\lambda t}\sqrt{\frac{\pi}{2}}e^{-\frac{\mu^2 t}{2\sigma^2}}\frac{1}{\sigma\sqrt{t}}dt
$$

$$
= \frac{\sigma}{\sqrt{2\pi}}\int_0^T e^{-\frac{\mu^2 t}{2\sigma^2}-\lambda t}\frac{1}{\sqrt{t}}dt,
$$

where the first equality is obtained through contour integral. According to the analysis of Merton's jump diffusion in chapter 3, we know that the difference incurred from the above term and its corresponding discrete version is exactly the quantity E_2 in section 3.1, i.e.,

$$
\frac{\sigma}{\sqrt{2\pi}}\Big(\int_0^T e^{-\frac{\mu^2 t}{2\sigma^2}-\lambda t}\frac{1}{\sqrt{t}}dt-\sum_{n=1}^N e^{-\frac{\mu^2 nT}{2\sigma^2 N}-\lambda\frac{nT}{N}}\frac{1}{\sqrt{\frac{nT}{N}}}\frac{T}{N}\Big)
$$

$$
= \frac{\sigma}{\sqrt{2\pi}}\Big[-\frac{\zeta(\frac{1}{2})\sqrt{T}}{\sqrt{N}}-\frac{\sqrt{T}e^{-(\lambda+\frac{\mu^2}{2\sigma^2})T}}{2N}-\sum_{r=1}^\infty\frac{\zeta(\frac{1}{2}-r)(-\lambda-\frac{\mu^2}{2\sigma^2})^r T^{r+\frac{1}{2}}}{r!N^{r+\frac{1}{2}}}
$$

$$-\sum_{i=1}^{p}\frac{B_{2i}T^{2i}}{(2i)!N^{2i}}f_2^{(2i-1)}(T)+O(\frac{1}{N^{2p+2}})\Big],$$

where $f_2(t)=e^{-\frac{\mu^2 t}{2\sigma^2}-\lambda t}\frac{1}{\sqrt{t}}$.

To deal with the second term in (4.1), let's define its integrand as

$$g(t):=e^{-\lambda t}e^{-\frac{1}{2}\sigma^2 t y^2}(e^{Dt}-1)\cos(\mu t y),$$

Then it is clear that $g(0)=0$, $g(T)=e^{-\lambda T}e^{-\frac{1}{2}\sigma^2 T y^2}(e^{DT}-1)\cos(\mu T y)$, and

$$\begin{aligned} g'(t) &= (-\lambda-\frac{1}{2}\sigma^2y^2)e^{-\lambda t}e^{-\frac{1}{2}\sigma^2 t y^2}(e^{Dt}-1)\cos(\mu t y) \\ &\quad +De^{-\lambda t}e^{-\frac{1}{2}\sigma^2 t y^2}e^{Dt}\cos(\mu t y)-\mu y e^{-\lambda t}e^{-\frac{1}{2}\sigma^2 t y^2}(e^{Dt}-1)\sin(\mu t y) \\ &:= g_1(t)+g_2(t)-g_3(t). \end{aligned}$$

Recalling the Euler-Maclaurin formula, we obtain that

$$\int_0^\infty\int_0^T g(t)dtdy-\int_0^\infty\sum_{n=1}^{N}g(\frac{nT}{N})\frac{T}{N}dy$$

$$=\int_0^\infty\Big(\int_0^T g(t)dtdy-\sum_{n=1}^{N}g(\frac{nT}{N})\frac{T}{N}\Big)dy$$

$$=\int_0^\infty\Big(-\frac{T(g(0)+g(T))}{2N}+\frac{(g'(0)-g'(T))T^2}{12N^2}+\frac{T^2}{N^2}\int_0^T g''(t)\frac{B_2(\frac{Nt}{T}-\lfloor\frac{Nt}{T}\rfloor)}{2}dt\Big)dy.$$

Now we have

$$\int_0^\infty g(T)dy<\infty,\ \text{and}\ \int_0^\infty g'(T)dy<\infty.$$

Furthermore, since $D=\frac{\lambda p}{1+\alpha^2y^2}+\frac{\lambda(1-p)}{1+\beta^2y^2}$,

$$\int_0^\infty g'(0)dy=\int_0^\infty Ddy=\int_0^\infty\frac{\lambda p}{1+\alpha^2y^2}+\frac{\lambda(1-p)}{1+\beta^2y^2}dy<\infty.$$

We also have to establish the following

$$\int_0^\infty\int_0^T|g''(t)|dtdy<\infty.$$

First, we note that

$$|g''(t)| = |g_1'(t) + g_2'(t) - g_3'(t)| \leq |g_1'(t)| + |g_2'(t)| + |g_3'(t)|,$$

in which

$$\begin{aligned}
|g_1'(t)| &= |(-\lambda - \frac{1}{2}\sigma^2 y^2)^2 e^{-\lambda t} e^{-\frac{1}{2}\sigma^2 t y^2}(e^{Dt} - 1)\cos(\mu t y) \\
&\quad +(-\lambda - \frac{1}{2}\sigma^2 y^2) e^{-\lambda t} e^{-\frac{1}{2}\sigma^2 t y^2} D e^{Dt}\cos(\mu t y) \\
&\quad +(-\lambda - \frac{1}{2}\sigma^2 y^2) e^{-\lambda t} e^{-\frac{1}{2}\sigma^2 t y^2}(e^{Dt} - 1)(-\sin(\mu t y))\mu y| \\
&\leq \Big[(-\lambda - \frac{1}{2}\sigma^2 y^2)^2 e^{-\lambda t} e^{-\frac{1}{2}\sigma^2 t y^2}(e^{Dt} - 1) + (-\lambda - \frac{1}{2}\sigma^2 y^2) e^{-\lambda t} e^{-\frac{1}{2}\sigma^2 t y^2} D e^{Dt}\Big] \\
&\quad +(2\lambda + \sigma^2 y^2) e^{-\lambda t} e^{-\frac{1}{2}\sigma^2 t y^2} D e^{Dt} + (\lambda + \frac{1}{2}\sigma^2 y^2) e^{-\lambda t} e^{-\frac{1}{2}\sigma^2 t y^2}(e^{Dt} - 1)\mu^2 t y^2 \\
&\leq \big((-\lambda - \frac{1}{2}\sigma^2 y^2) e^{-\lambda t} e^{-\frac{1}{2}\sigma^2 t y^2}(e^{Dt} - 1)\big)' + 2\lambda D + D\sigma^2 y^2 e^{-\frac{1}{2}\sigma^2 t y^2} \\
&\quad +\lambda e^{-\lambda t}\frac{1}{2}\sigma^2 t y^2 e^{-\frac{1}{2}\sigma^2 t y^2} e^{\lambda T} D t \frac{2\mu^2}{\sigma^2} + e^{-\lambda t}(\frac{1}{2}\sigma^2 t y^2)^2 e^{-\frac{1}{2}\sigma^2 t y^2} e^{\lambda T} D \frac{2\mu^2}{\sigma^2},
\end{aligned}$$

where the last step above can be justified by the following argument: since $0 < D < \lambda$, so $0 < Dt < \lambda t < \lambda T$. Thus, $e^{Dt} - 1 < e^{\lambda T} Dt$. Note that the functions $f(x) = xe^{-x}$ and $f(x) = x^2 e^{-x}$ are absolutely bounded for all $x \geq 0$. Hence,

$$\int_0^T |g_1'(t)| dt \leq (-\lambda - \frac{1}{2}\sigma^2 y^2) e^{-\lambda t} e^{-\frac{1}{2}\sigma^2 t y^2}(e^{Dt} - 1)\Big|_0^T + 2\lambda D + 2D(1 - e^{-\frac{1}{2}\sigma^2 T y^2}) + MD$$

$$= (-\lambda - \frac{1}{2}\sigma^2 y^2) e^{-\lambda T} e^{-\frac{1}{2}\sigma^2 T y^2}(e^{DT} - 1) + (2\lambda + 2 + M)D,$$

where M is some positive constant. So,

$$\int_0^\infty \int_0^T |g_1'(t)| dt dy \leq \int_0^\infty (-\lambda - \frac{1}{2}\sigma^2 y^2) e^{-\lambda T} e^{-\frac{1}{2}\sigma^2 T y^2}(e^{DT} - 1) + (2\lambda + 2 + M) D dy < \infty.$$

We apply the same technique to the terms $g_2(t)$ and $g_3(t)$ to obtain that

$$\int_0^\infty \int_0^T |g_2'(t)| dt dy < \infty \text{ and } \int_0^\infty \int_0^T |g_3'(t)| dt dy < \infty.$$

Thus,

$$\int_0^\infty \int_0^T |g''(t)| dtdy < \infty,$$

and

$$\frac{\sigma^2}{\pi}\int_0^\infty \int_0^T e^{-\lambda t} e^{-\frac{1}{2}\sigma^2 t y^2} (e^{Dt}-1) \cos(\mu t y) dtdy$$

$$-\frac{\sigma^2}{\pi}\int_0^\infty \sum_{n=1}^{N} e^{-\lambda \frac{nT}{N}} e^{-\frac{1}{2}\sigma^2 \frac{nT}{N} y^2} (e^{D\frac{nT}{N}}-1) \cos(\mu \frac{nT}{N} y) \frac{T}{N} dy$$

$$= -\frac{\sigma^2}{\pi}\frac{T}{2N}\int_0^\infty g(T) dy + O(\frac{1}{N^2}).$$

Now we turn to the third term in (4.1). We apply the same technique as dealing with the second term of (4.1) to obtain the following

$$-\frac{2\sigma^2}{\pi}\int_0^\infty \int_0^T e^{-\lambda t} e^{-\frac{1}{2}\sigma^2 t y^2} e^{Dt} \sin(\mu t y + \frac{Et}{2}) \sin(\frac{Et}{2}) dtdy$$

$$+\frac{2\sigma^2}{\pi}\int_0^\infty \sum_{n=1}^{N} e^{-\lambda \frac{nT}{N}} e^{-\frac{1}{2}\sigma^2 \frac{nT}{N} y^2} e^{D\frac{nT}{N}} \sin(\mu \frac{nT}{N} y + \frac{E\frac{nT}{N}}{2}) \sin(\frac{E\frac{nT}{N}}{2}) \frac{T}{N} dy$$

$$= \frac{\sigma^2 T}{\pi N}\int_0^\infty e^{-\lambda T} e^{-\frac{1}{2}\sigma^2 T y^2} e^{DT} \sin(\mu T y + \frac{ET}{2}) \sin(\frac{ET}{2}) dy + O(\frac{1}{N^2}).$$

Put the above together, we obtain that the difference incurred from the term $2\sigma^2 t y e^B \cos C$ is given by

$$-\frac{\zeta(\frac{1}{2})\sigma\sqrt{T}}{\sqrt{2\pi N}} - \frac{\sigma^2 T}{2\pi N}\int_0^\infty e^{-\frac{1}{2}\sigma^2 T y^2 - \lambda T + DT} \cos(\mu T y + ET) dy + \frac{\sigma\zeta(-\frac{1}{2})(\lambda + \frac{\mu^2}{2\sigma^2})}{\sqrt{2\pi}} \frac{T\sqrt{T}}{N\sqrt{N}} + O(\frac{1}{N^2}).$$

Next, we analyze the second term in $\phi_t'(-y) - \phi_t'(y)$, i.e.

$$i\mu t(e^{B-iC} - e^{B+iC}) = i\mu t(-2ie^B \sin C) = 2\mu t e^B \sin C.$$

Thus,

$$\begin{aligned}
&\int_0^T \frac{1}{2\pi t}\int_0^\infty \frac{1}{y}2\mu t e^B \sin C dydt \\
= \ & \frac{\mu}{\pi}\int_0^T\int_0^\infty \frac{1}{y}e^B\sin C dydt \\
= \ & \frac{\mu}{\pi}\int_0^T\int_0^\infty \frac{1}{y}e^{-\frac{1}{2}\sigma^2ty^2-\lambda t+Dt}\sin(\mu ty+Et)dydt \\
= \ & \frac{\mu}{\pi}\int_0^\infty\int_0^T \frac{1}{y}e^{-\frac{1}{2}\sigma^2ty^2-\lambda t+Dt}\sin(\mu ty+Et)dtdy \\
= \ & \frac{\mu}{\pi}\int_0^\infty\int_0^T \frac{1}{y}e^{-\frac{1}{2}\sigma^2ty^2-\lambda t+Dt}\big(\sin(\mu ty)+\sin(\mu ty+Et)-\sin(\mu ty)\big)dtdy \\
= \ & \frac{\mu}{\pi}\int_0^\infty\int_0^T \frac{1}{y}e^{-\frac{1}{2}\sigma^2ty^2-\lambda t}(1+e^{Dt}-1)\big(\sin(\mu ty)+2\sin(\frac{Et}{2})\cos(\mu ty+\frac{Et}{2})\big)dtdy \\
= \ & \frac{\mu}{\pi}\int_0^\infty\int_0^T \frac{1}{y}e^{-\lambda t}e^{-\frac{1}{2}\sigma^2ty^2}\sin(\mu ty)dtdy \\
& +\frac{\mu}{\pi}\int_0^\infty\int_0^T \frac{1}{y}e^{-\lambda t}e^{-\frac{1}{2}\sigma^2ty^2}(e^{Dt}-1)\sin(\mu ty)dtdy \\
& +\frac{\mu}{\pi}\int_0^\infty\int_0^T \frac{1}{y}e^{-\lambda t-\frac{1}{2}\sigma^2ty^2+Dt}2\sin(\frac{Et}{2})\cos(\mu ty+\frac{Et}{2})dtdy,
\end{aligned} \tag{4.2}$$

where the third equality follows from Fubini's theorem since $|\frac{\sin x}{x}|<1$. We may apply Fubini's theorem to the first double integral in (4.2) as follows

$$\begin{aligned}
&\frac{\mu}{\pi}\int_0^\infty\int_0^T \frac{1}{y}e^{-\lambda t}e^{-\frac{1}{2}\sigma^2ty^2}\sin(\mu ty)dtdy \\
= \ & \frac{\mu}{\pi}\int_0^T e^{-\lambda t}\int_0^\infty \frac{1}{y}e^{-\frac{1}{2}\sigma^2ty^2}\sin(\mu ty)dydt \\
= \ & \frac{\mu}{\pi}\int_0^T e^{-\lambda t}\frac{\pi}{2}\mathrm{Erf}(\frac{\mu}{\sigma}\sqrt{\frac{t}{2}})dt \\
= \ & \frac{\mu}{2}\int_0^T e^{-\lambda t}\mathrm{Erf}(\frac{\mu}{\sigma}\sqrt{\frac{t}{2}})dt \\
= \ & -\frac{\mu}{2}\int_0^T e^{-\lambda t}dt+\frac{\mu}{2}\int_0^T e^{-\lambda t}\mathrm{Erfc}(-\frac{\mu}{\sigma}\sqrt{\frac{t}{2}})dt.
\end{aligned}$$

According to the analysis in Merton's jump diffusions and the term E_1 in section 3.1, we see that the above equals

$$\frac{\mu}{2}\sqrt{\frac{2}{\sigma^2\pi}}\int_0^\mu\int_0^T \sqrt{t}e^{-(\frac{z^2}{2\sigma^2}+\lambda)t}dtdz.$$

Thus, the difference of the above and corresponding discrete version is given by

$$\frac{\mu}{2}\int_0^T e^{-\lambda t}\mathrm{Erf}(\frac{\mu}{\sigma}\sqrt{\frac{t}{2}})dt - \frac{\mu}{2}\sum_{n=1}^{N} e^{-\lambda\frac{nT}{N}}\mathrm{Erf}(\frac{\mu}{\sigma}\sqrt{\frac{nT}{2N}})\frac{T}{N}$$

$$= -\frac{\mu T}{4N}e^{-\lambda T}\mathrm{Erf}(\frac{\mu}{\sigma}\sqrt{\frac{T}{2}}) - \frac{\mu}{2}\Big[\sqrt{\frac{2}{\pi\sigma^2}}\sum_{r=0}^{\infty}\frac{\zeta(-\frac{1}{2}-r)T^{r+\frac{3}{2}}}{r!N^{r+\frac{3}{2}}}\int_0^{\mu}(-\lambda-\frac{z^2}{2\sigma^2})^r dz$$

$$-\sum_{i=1}^{p}\frac{B_{2i}T^{2i}}{(2i)!N^{2i}}\sqrt{\frac{2}{\pi\sigma^2}}\int_0^{\mu} g^{(2i-1)}(T,z)dz\Big] + O(\frac{1}{N^{2p+2}}),$$

where $g(t,z) = \sqrt{t}e^{-(\frac{z^2}{2\sigma^2}+\lambda)t}$.

Now we move to the second and third double integrals of (4.2). Using the same technique as dealing with the second and third terms of (4.1), we actually obtain a similar conclusion. If we combine the above result of the first term of (4.2), we get that the difference of continuous and discrete versions of the second term in $\phi'_{\cdot}(-y) - \phi'_{\cdot}(y)$ is given by

$$\frac{\mu}{\pi}\int_0^T\int_0^{\infty}\frac{1}{y}e^{-\frac{1}{2}\sigma^2 ty^2-\lambda t+Dt}\mathrm{sin}(\mu ty+Et)dydt$$

$$-\frac{\mu}{\pi}\sum_{n=1}^{N}\int_0^{\infty}\frac{1}{y}e^{-\frac{1}{2}\sigma^2\frac{nT}{N}y^2-\lambda\frac{nT}{N}+D\frac{nT}{N}}\mathrm{sin}(\mu\frac{nT}{N}y+E\frac{nT}{N})dy\frac{T}{N}$$

$$= -\frac{\mu T}{2\pi N}\int_0^{\infty}\frac{1}{y}e^{-\frac{1}{2}\sigma^2 Ty^2-\lambda T+DT}\mathrm{sin}(\mu Ty+ET)dy - \frac{\mu^2}{2}\sqrt{\frac{2}{\pi\sigma^2}}\frac{\zeta(-\frac{1}{2})T^{\frac{3}{2}}}{N^{\frac{3}{2}}} + O(\frac{1}{N^2}).$$

The third and fourth terms in $\phi'_t(-y) - \phi'_t(y)$ are essentially the same, therefore it suffices to check the third term only. i.e.,

$$\frac{ip\alpha\lambda t}{(1+i\alpha y)^2}e^{B-iC} - \frac{ip\alpha\lambda t}{(1-i\alpha y)^2}e^{B+iC}$$

$$= 2p\alpha\lambda te^{B}\Big(\frac{2\alpha y\mathrm{cos}C}{(1+\alpha^2y^2)^2} + \frac{1-\alpha^2y^2}{(1+\alpha^2y^2)^2}\mathrm{sin}C\Big). \tag{4.3}$$

The first term of (4.3), with integration with respect to y and t outside, is given by

$$\frac{2p\alpha^2\lambda}{\pi}\int_0^T\int_0^{\infty}\frac{e^B\mathrm{cos}C}{(1+\alpha^2y^2)^2}dydt$$

$$= \frac{2p\alpha^2\lambda}{\pi}\int_0^T\int_0^\infty e^{-\frac{1}{2}\sigma^2 ty^2-\lambda t+Dt}\cos(\mu ty+Et)\frac{1}{(1+\alpha^2y^2)^2}dydt.$$

The same technique as above can be applied here, thus we obtain

$$\frac{2p\alpha^2\lambda}{\pi}\int_0^T\int_0^\infty e^{-\frac{1}{2}\sigma^2 ty^2-\lambda t+Dt}\cos(\mu ty+Et)\frac{1}{(1+\alpha^2y^2)^2}dydt$$

$$-\frac{2p\alpha^2\lambda}{\pi}\sum_{n=1}^N\int_0^\infty e^{-\frac{1}{2}\sigma^2\frac{nT}{N}y^2-\lambda\frac{nT}{N}+D\frac{nT}{N}}\cos(\mu\frac{nT}{N}y+E\frac{nT}{N})\frac{1}{(1+\alpha^2y^2)^2}dy\frac{T}{N}$$

$$= \frac{p\alpha^2\lambda T}{\pi N}\Big(\int_0^\infty\frac{1}{(1+\alpha^2y^2)^2}dy-\int_0^\infty\frac{e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\cos(\mu Ty+ET)}{(1+\alpha^2y^2)^2}dy\Big)+O(\frac{1}{N^2})$$

$$= \frac{p\alpha\lambda T}{4N}-\frac{p\alpha^2\lambda T}{\pi N}\int_0^\infty\frac{e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\cos(\mu Ty+ET)}{(1+\alpha^2y^2)^2}dy+O(\frac{1}{N^2}),$$

where the last equality relies on the identity $\int_0^\infty\frac{1}{(1+\alpha^2y^2)^2}dy=\frac{\pi}{4\alpha}$ by trigonometric substitution. The second term of (4.3), with integration with respect to y and t outside, is given by

$$\frac{p\alpha\lambda}{\pi}\int_0^T\int_0^\infty e^B\sin C\frac{1}{y}\frac{1-\alpha^2y^2}{(1+\alpha^2y^2)^2}dydt$$

$$= \frac{p\alpha\lambda}{\pi}\int_0^T\int_0^\infty e^{-\frac{1}{2}\sigma^2ty^2-\lambda t+Dt}\sin(\mu ty+Et)\frac{1}{y}\frac{1-\alpha^2y^2}{(1+\alpha^2y^2)^2}dydt.$$

Again, we get the following

$$\frac{p\alpha\lambda}{\pi}\int_0^T\int_0^\infty e^{-\frac{1}{2}\sigma^2ty^2-\lambda t+Dt}\sin(\mu ty+Et)\frac{1}{y}\frac{1-\alpha^2y^2}{(1+\alpha^2y^2)^2}dydt$$

$$-\frac{p\alpha\lambda}{\pi}\sum_{n=1}^N\int_0^\infty e^{-\frac{1}{2}\sigma^2\frac{nT}{N}y^2-\lambda\frac{nT}{N}+D\frac{nT}{N}}\sin(\mu\frac{nT}{N}y+E\frac{nT}{N})\frac{1}{y}\frac{1-\alpha^2y^2}{(1+\alpha^2y^2)^2}dy\frac{T}{N}$$

$$= -\frac{p\alpha\lambda T}{2\pi N}\int_0^\infty e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\sin(\mu Ty+ET)\frac{1}{y}\frac{1-\alpha^2y^2}{(1+\alpha^2y^2)^2}dy+O(\frac{1}{N^2}).$$

Hence, the difference of continuous and discrete versions of (4.3), with double integration outside, can be written as

$$\frac{p\alpha\lambda T}{4N} - \frac{p\alpha^2\lambda T}{\pi N}\int_0^\infty \frac{e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\cos(\mu Ty+ET)}{(1+\alpha^2y^2)^2}dy$$

$$-\frac{p\alpha\lambda T}{2\pi N}\int_0^\infty e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\sin(\mu Ty+ET)\frac{1}{y}\frac{1-\alpha^2y^2}{(1+\alpha^2y^2)^2}dy + O(\frac{1}{N^2}).$$

Likewise, the difference from the fourth term in $\phi'_\cdot(-y)-\phi'_\cdot(y)$ is given by

$$\frac{(1-p)\beta\lambda T}{4N} - \frac{(1-p)\beta^2\lambda T}{\pi N}\int_0^\infty \frac{e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\cos(\mu Ty+ET)}{(1+\beta^2y^2)^2}dy$$

$$-\frac{(1-p)(-\beta)\lambda T}{2\pi N}\int_0^\infty e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\sin(\mu Ty+ET)\frac{1}{y}\frac{1-\beta^2y^2}{(1+\beta^2y^2)^2}dy + O(\frac{1}{N^2}).$$

Also, note that for Kou's jump diffusion, $\mathbb{E}[X_t] = \mu t + \lambda t\big(\frac{p}{\eta_1} - \frac{1-p}{\eta_2}\big)$, thus

$$\int_0^T \frac{1}{2t}\mathbb{E}[X_t]dt = \frac{\mu T}{2} + \frac{\lambda T}{2}\Big(\frac{p}{\eta_1} - \frac{1-p}{\eta_2}\Big).$$

Therefore, in summary, the difference of continuous and discrete suprema for Kou's jump diffusion X_t defined on time interval $[0,T]$ admits the following asymptotic expansion

$$\mathbb{E}[\sup_{0\le t\le T} X_t - \sup_{0\le i\le N} X_{t_i}] = \frac{F_1}{\sqrt{N}} + \frac{F_2}{N} + \frac{F_3}{N\sqrt{N}} + O(\frac{1}{N^2}),$$

where

$$\begin{aligned}
F_1 &= -\frac{\zeta(\frac{1}{2})\sigma\sqrt{T}}{\sqrt{2\pi}}\\
F_3 &= \frac{\sigma\zeta(-\frac{1}{2})(\lambda - \frac{\mu^2}{2\sigma^2})T\sqrt{T}}{\sqrt{2\pi}}\\
F_2 &= -\frac{\sigma^2T}{2\pi}\int_0^\infty e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\cos(\mu Ty+ET)dy\\
&\quad -\frac{\mu T}{2\pi}\int_0^\infty \frac{1}{y}e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\sin(\mu Ty+ET)dy\\
&\quad +\frac{p\alpha\lambda T}{4} - \frac{p\alpha^2\lambda T}{\pi}\int_0^\infty \frac{e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\cos(\mu Ty+ET)}{(1+\alpha^2y^2)^2}dy\\
&\quad -\frac{p\alpha\lambda T}{2\pi}\int_0^\infty e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\sin(\mu Ty+ET)\frac{1}{y}\frac{1-\alpha^2y^2}{(1+\alpha^2y^2)^2}dy\\
&\quad +\frac{(1-p)\beta\lambda T}{4} - \frac{(1-p)\beta^2\lambda T}{\pi}\int_0^\infty \frac{e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\cos(\mu Ty+ET)}{(1+\beta^2y^2)^2}dy\\
&\quad +\frac{(1-p)\beta\lambda T}{2\pi}\int_0^\infty e^{-\frac{1}{2}\sigma^2Ty^2-\lambda T+DT}\sin(\mu Ty+ET)\frac{1}{y}\frac{1-\beta^2y^2}{(1+\beta^2y^2)^2}dy,
\end{aligned}$$

and

$$D = \frac{\lambda p}{1+\alpha^2 y^2} + \frac{\lambda(1-p)}{1+\beta^2 y^2}$$

$$E = \frac{\lambda p \alpha y}{1+\alpha^2 y^2} - \frac{\lambda(1-p)\beta y}{1+\beta^2 y^2}$$

□

Remark 4.1.2. From Theorem 4.1.1, we again see that the existence and the coefficient of $\frac{1}{\sqrt{N}}$ entirely depends on the diffusion coefficient σ and length of the time interval T, which is the same as Merton's jump diffusion. Furthermore, just like what was discussed in the remark of Merton's jump diffusion case, if we simply take $\lambda = 0$ to represent the situation of being no jumps, we find that $D = E = 0$ and the coefficients of the terms $\frac{1}{N}$ and $\frac{1}{N\sqrt{N}}$ are consistent with those in Janssen and Van Leeuwaarden [37].

4.2 (Symmetric) Stable Process

Stable process has been used in queuing theory as a limiting process for cumulative inputs under heavy traffic environment, see e.g. Whitt [68], [69], etc. It forms an important sub-family of Lévy process. More precisely, for every $\alpha \in (0, 2]$, a Lévy process with characteristic exponent Ψ is called a stable process with index α if

$$\Psi(k\lambda) = k^\alpha \Psi(\lambda),$$

for every positive k and $\lambda \in \mathbb{R}$, see Bertoin [8]. Also, the scaling property immediately follows, i.e. for every $k > 0$, the rescaled process $(k^{-\frac{1}{\alpha}} X_{kt}, t \geq 0)$ has the same law as X. Due to the Lévy-Khinchine formula, we see that the index α is indeed in the range $(0, 2]$. If $\alpha = 2$, then it is actually a Brownian motion; if $\alpha = 1$, it corresponds to a Cauchy process. Excluding the above special cases, we assume that $\alpha \in (0, 1) \cup (1, 2)$. In such case, the characteristic exponent is given by

$$\Psi(\lambda) = c|\lambda|^\alpha (i\beta \mathrm{sgn}(\lambda) \tan(\frac{\pi\alpha}{2}) - 1),$$

where $\lambda \in (-\infty, \infty)$, $c > 0$, $\beta \in [-1, 1]$. The Lévy measure Π is absolutely continuous with respect to the Lebesgue measure, with density

$$\Pi(dx) = c^+ x^{-\alpha-1}\mathbb{1}_{\{x>0\}}dx + c^-|x|^{-\alpha-1}\mathbb{1}_{\{x<0\}}dx,$$

where c^+ and c^- are two nonnegative reals such that

$$\beta = \frac{c^+ - c^-}{c^+ + c^-}.$$

The process is symmetric when $c^+ = c^-$, or equivalently when $\beta = 0$. In the thesis, we will be focused on the symmetric stable process which admits first moment. Clearly, it has characteristic exponent $\Psi(y) = -c|y|^\alpha$ with $\alpha \in (1, 2]$.

With the same construction as before, we are interested in the quantity $\Delta_N = \mathbb{E}[\sup_{0\le t\le 1} X_t - \max_{0\le n\le N} X_{\frac{n}{N}}]$. Here goes our theorem about the expected difference of the continuous supremum and discrete maximum in the case of symmetric stable process.

Theorem 4.2.1. *For symmetric stable process with index $\alpha \in (1, 2]$ defined on $[0, 1]$, the quantity Δ_N has the asymptotic expansion*

$$\Delta_N = \mathbb{E}[\sup_{0\le t\le 1} X_t - \max_{0\le n\le N} X_{\frac{n}{N}}]$$

$$= -\frac{c^{\frac{1}{\alpha}}\Gamma(1-\frac{1}{\alpha})\zeta(1-\frac{1}{\alpha})}{\pi}\frac{1}{N^{\frac{1}{\alpha}}} - \frac{c^{\frac{1}{\alpha}}\Gamma(1-\frac{1}{\alpha})}{2\pi N} - \frac{c^{\frac{1}{\alpha}}\Gamma(1-\frac{1}{\alpha})}{\pi}\sum_{r=1}^{p}\frac{B_{2r}}{2r}\binom{\frac{1}{\alpha}-1}{2r-1}\frac{1}{N^{2r}} + O(\frac{1}{N^{2p+2}}).$$

where p is some positive integer greater than 1, B_n is the nth Bernoulli number, ζ is the Riemann zeta function.

Proof. Except when X_t is a Brownian motion, a Cauchy process or a stable subordinator with index $\frac{1}{2}$, there does not exist explicit transition density functions. So we have to use the same technique of Hilbert transform as in the previous section to obtain the following

$$\mathbb{E}[\sup_{0\le t\le 1} X_t] = \int_0^1 \frac{1}{2t}\mathbb{E}[X_t]dt + \int_0^1 \frac{1}{2\pi t}\lim_{\epsilon\downarrow 0}\int_\epsilon^\infty \frac{1}{y}\big(\phi_t'(-y) - \phi_t'(y)\big)dydt,$$

where ϕ_t is the characteristic function with

$$\phi_t(y) = \exp(t\Psi(y)) = \exp(-tc|y|^\alpha).$$

So

$$\phi_t'(y)\Big|_{y>0} = e^{-tcy^\alpha}(-tc\alpha y^{\alpha-1})$$

$$\phi_t'(-y)\Big|_{y>0} = e^{-tcy^\alpha}(tc\alpha y^{\alpha-1}).$$

Also, since X_t is symmetric, so $\mathbb{E}[X_t] = 0$ for all $t \geq 0$. Therefore,

$$\begin{aligned}
\mathbb{E}[\sup_{0\leq t\leq 1} X_t] &= \frac{c\alpha}{\pi}\int_0^1 \lim_{\epsilon\downarrow 0}\int_\epsilon^\infty e^{-tcy^\alpha} y^{\alpha-2} dydt \\
&= \frac{c^{\frac{1}{\alpha}}}{\pi}\int_0^1 t^{\frac{1}{\alpha}-1}\lim_{\epsilon\downarrow 0}\int_{tc\epsilon^\alpha}^\infty e^{-z} z^{-\frac{1}{\alpha}} dzdt \\
&= \frac{c^{\frac{1}{\alpha}}}{\pi}\int_0^1 t^{\frac{1}{\alpha}-1}\int_0^\infty e^{-z} z^{-\frac{1}{\alpha}} dzdt \\
&= \frac{c^{\frac{1}{\alpha}}}{\pi}\int_0^1 t^{\frac{1}{\alpha}-1}\Gamma(1-\frac{1}{\alpha})dt \\
&= \frac{c^{\frac{1}{\alpha}}\Gamma(1-\frac{1}{\alpha})}{\pi}\int_0^1 t^{\frac{1}{\alpha}-1}dt \\
&= \frac{c^{\frac{1}{\alpha}}\Gamma(1-\frac{1}{\alpha})\alpha}{\pi},
\end{aligned}$$

where the second equality follows from a change of variable, i.e., $z = tcy^\alpha$, and the third equality follows from the dominated convergence theorem. Note that the above result is consistent with the extreme case that $\alpha = 2$ and $c = \frac{1}{2}$, i.e., the Brownian motion, in which it is trivial to check that $\mathbb{E}[\sup_{0\leq t\leq 1} B_t] = \sqrt{\frac{2}{\pi}}$.

The corresponding expected discrete maximum can be written as

$$\mathbb{E}[\max_{0\leq n\leq N} X_{\frac{n}{N}}]$$

$$= \frac{c^{\frac{1}{\alpha}}\Gamma(1-\frac{1}{\alpha})}{\pi}\sum_{n=1}^N (\frac{n}{N})^{\frac{1}{\alpha}-1}\frac{1}{N}$$

$$= \frac{c^{\frac{1}{\alpha}}\Gamma(1-\frac{1}{\alpha})}{\pi}(\frac{1}{N})^{\frac{1}{\alpha}}\sum_{n=1}^N n^{\frac{1}{\alpha}-1}.$$

According to Kanemitsu and Tsukada [40], page 56, (3.7), we know that

$$\sum_{n=1}^N n^{\frac{1}{\alpha}-1} = \sum_{n=0}^{N-1}(n+1)^{\frac{1}{\alpha}-1}$$

$$= \sum_{r=1}^{l} \frac{(-1)^r}{r} \binom{\frac{1}{\alpha}-1}{r-1} \overline{B}_r(N-1) N^{\frac{1}{\alpha}-r} + O((N-1)^{\frac{1}{\alpha}-1-l}) + \alpha N^{\frac{1}{\alpha}} + \zeta(1-\frac{1}{\alpha}, 1)$$

$$= \sum_{r=1}^{l} \frac{(-1)^r}{r} \binom{\frac{1}{\alpha}-1}{r-1} B_r N^{\frac{1}{\alpha}-r} + O(N^{\frac{1}{\alpha}-1-l}) + \alpha N^{\frac{1}{\alpha}} + \zeta(1-\frac{1}{\alpha}),$$

where $\overline{B}_r(x) = B_r(x - \lfloor x \rfloor)$ and $B_r(x)$ is the rth Bernoulli polynomial, l is some positive integer. Thus,

$$\mathbb{E}[\max_{0 \le n \le N} X_{\frac{n}{N}}] = \frac{c^{\frac{1}{\alpha}} \Gamma(1-\frac{1}{\alpha})}{\pi} \Big[\frac{\zeta(1-\frac{1}{\alpha})}{N^{\frac{1}{\alpha}}} + \sum_{r=1}^{l} \frac{(-1)^r}{r} \binom{\frac{1}{\alpha}-1}{r-1} B_r N^{-r} + O(N^{-1-l}) + \alpha \Big],$$

and

$$\Delta_N = \mathbb{E}[\sup_{0 \le t \le 1} X_t - \max_{0 \le n \le N} X_{\frac{n}{N}}]$$

$$= -\frac{c^{\frac{1}{\alpha}} \Gamma(1-\frac{1}{\alpha}) \zeta(1-\frac{1}{\alpha})}{\pi} \frac{1}{N^{\frac{1}{\alpha}}} + \frac{c^{\frac{1}{\alpha}} \Gamma(1-\frac{1}{\alpha})}{\pi} \sum_{r=1}^{l} \frac{(-1)^{r+1}}{r} \binom{\frac{1}{\alpha}-1}{r-1} B_r \frac{1}{N^r} + O(\frac{1}{N^{l+1}})$$

$$= -\frac{c^{\frac{1}{\alpha}} \Gamma(1-\frac{1}{\alpha}) \zeta(1-\frac{1}{\alpha})}{\pi} \frac{1}{N^{\frac{1}{\alpha}}} - \frac{c^{\frac{1}{\alpha}} \Gamma(1-\frac{1}{\alpha})}{2\pi N} - \frac{c^{\frac{1}{\alpha}} \Gamma(1-\frac{1}{\alpha})}{\pi} \sum_{r=1}^{p} \frac{B_{2r}}{2r} \binom{\frac{1}{\alpha}-1}{2r-1} \frac{1}{N^{2r}} + O(\frac{1}{N^{2p+2}}),$$

where the last equality follows from the fact that all Bernoulli numbers $B_r = 0$ for r odd and greater than or equal to 3, and $B_1 = -\frac{1}{2}$. Also, we take $l = 2p+1$ with $p \ge 1$. □

Remark 4.2.2. Note that if we set $\alpha = 2$ and $c = \frac{1}{2}$, i.e., the special case of standard Brownian motion, and recall the definition of generalized binomial coefficient and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we see that all coefficients are exactly the same as those in Janssen and Van Leeuwaarden [37] when $\mu = 0$. This shows that our result is another generalization of theirs while keeping scaling property.

Chapter 5

Upper Bound for Sampling Error in General

5.1 Construction and Review

All the above cases dealt with concrete examples of commonly-seen Lévy processes in real application, especially in finance. We see that the order of the leading term in the asymptotic expansion entirely depends on the path structure of the underlying Lévy process. More specifically, we might make conjecture that if the Brownian motion term exists in the Itô-Lévy decomposition, then the order of the leading term is $\frac{1}{\sqrt{N}}$. In this section, we will see that this is indeed the case and we will also relate the order of the leading term with the Blumenthal-Getoor index in pure jump Lévy processes.

Again we are given a general real-valued one-dimensional Lévy process L_t defined on the compact interval $[0, T]$with Lévy triple (b, c, Π). For any given sample path, we partition the interval $[0, T]$ equally into N subintervals and define, as above, the time-discrete piecewise constant process

$$L_t^D(N) = L_{t_{i-1}}, \qquad \text{for} \qquad t_{i-1} \leq t < t_i, 1 \leq i \leq N,$$

and for the last point

$$L_T^D(N) := L_T.$$

We define the time step $\Lambda_N := t_i - t_{i-1} = \frac{T}{N}$. If L_t attains its supremum at $t = 0$ or $t = T$, a.s. for example, a scalar multiple of a subordinator, then by our construction, $\sup\limits_{0 \leq t \leq T} L_t - \sup\limits_{0 \leq t \leq T} L_t^D(N) = 0$ a.s. for any N, which immediately implies the expected difference of these two quantities is again zero. From now on, we will exclude this trivial case.

First, let's recall one result derived by Dia and Lamberton [20].

Theorem 5.1.1. *Let L_t be an integrable Lévy process with Lévy triple (b, c, Π), then*

1. *If $c > 0$*

$$\mathbb{E}(\sup_{0\leq t\leq T} L_t - \sup_{0\leq t\leq T} L_t^D(N)) = O(\frac{1}{\sqrt{N}})$$

2. If $c = 0$

$$\mathbb{E}(\sup_{0\leq t\leq T} L_t - \sup_{0\leq t\leq T} L_t^D(N)) = o(\frac{1}{\sqrt{N}})$$

3. If $c = 0$ *and* $\int_{|x|\leq 1} |x|\Pi(dx) < \infty$

$$\mathbb{E}(\sup_{0\leq t\leq T} L_t - \sup_{0\leq t\leq T} L_t^D(N)) = O(\frac{\log N}{N}).$$

In the above theorem, we see that the third case actually is the case of being finite variation.

Actually, for the second case above, the little-o notation is not as informative as the big-O notation's in the case 1 and case 3. Therefore, we are trying to improve their result by coming up with a result with big-O notation for the second case, i.e., the case of being infinite variation but without Brownian motion term.

5.2 Main Result

We consider a general Lévy process which admits the Lévy-Itô decomposition:

$$L_t = \gamma t + \sigma B_t + J_t^s(N) + J_t^b(N),$$

where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, $J_t^s(N)$ denotes the small jump martingale and $J_t^b(N)$ is the big jump part, i.e. the compound Poisson process. For each positive integer N, we use $\theta_N < 1$ to denote the truncation threshold to distinguish small jumps with large jumps.

Note that in the usual setting where the truncation threshold is set to be 1, the characteristic exponent is given by

$$\Psi(u) = iub - \frac{1}{2}u^2 c + \int_{\mathbb{R}\setminus\{0\}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|<1\}})\Pi(dx).$$

Now if we reset the truncation threshold to be θ_N, we should modify the above characteristic exponent to the following

$$\Psi(u) = iu\big(b - \int_{\mathbb{R}\setminus\{0\}} x\mathbb{1}_{\{\theta_N\leq|x|<1\}}\Pi(dx)\big) - \frac{1}{2}u^2 c$$

$$+\int_{0<|x|<\theta_N}(e^{iux}-1-iux)\Pi(dx)+\int_{|x|\geq\theta_N}(e^{iux}-1)\Pi(dx).$$

Therefore, each term above corresponds to one term in the Lévy-Itô decomposition of L_t. More precisely, we have

$$\gamma = b - \int_{\mathbb{R}\setminus\{0\}} x\mathbb{1}_{\{\theta_N\leq|x|<1\}}\Pi(dx)$$

$$\sigma = c,$$

and $J_t^s(N)$ and $J_t^b(N)$ have characteristic exponents $\Psi_3(u) := \int_{0<|x|<\theta_N}(e^{iux}-1-iux)\Pi(dx)$ and $\Psi_4(u) := \int_{|x|\geq\theta_N}(e^{iux}-1)\Pi(dx)$, respectively.

Due to the equivalence between σ and c, we may replace the Lévy triple (b,c,Π) of L_t by (b,σ,Π), and the drift term associated with the truncation threshold θ_N is given by $\gamma = b - \int_{\mathbb{R}\setminus\{0\}} x\mathbb{1}_{\{\theta_N\leq|x|<1\}}\Pi(dx)$.

Before stating our main theorem, we first need to recall the Blumenthal-Getoor index β, which is defined as follows, see [10],

$$\beta = \inf\{\alpha > 0 : \int_{|x|<1}|x|^\alpha\Pi(dx) < \infty\}.$$

It is clear that for any Lévy process, this index is always in $[0,2)$.

Now we are ready to state our main result of the expected difference between continuous supremum and discrete maximum when the underlying Lévy process is a general one. Similarly, we also have three cases.

Theorem 5.2.1. *Consider a one dimensional real-valued Lévy process L_t with Lévy triple (b,σ,Π) defined on $[0,T]$ where $T<\infty$, and assume that the first moment of L_t is finite, i.e.*

$$\int_{|x|\geq 1}|x|\Pi(dx) < \infty,$$

then the expected difference of continuous supremum and discrete maximum takes the following asymptotic behavior:

Case (i) $\sigma > 0$,

$$\mathbb{E}(\sup_{0\le t\le T} L_t - \sup_{0\le t\le T} L_t^D(N)) = O(\frac{1}{\sqrt{N}})$$

Case (ii) L_t *is of finite variation,*

$$\mathbb{E}(\sup_{0\le t\le T} L_t - \sup_{0\le t\le T} L_t^D(N)) = O(\frac{\log N}{N})$$

Case (iii) $\sigma = 0$ *and* L_t *is of infinite variation,*

$$\mathbb{E}(\sup_{0\le t\le T} L_t - \sup_{0\le t\le T} L_t^D(N)) = O(\frac{1}{N^r}),$$

where $\frac{1}{2} < r = \frac{1}{\beta} - \epsilon < \frac{1}{\beta} \le 1$ *for any small* $\epsilon > 0$.

Note that we may achieve $r = \frac{1}{\beta}$ (i.e., $\epsilon = 0$) in some cases, which will be shown later. The proof of the theorem will need the following lemma, which gives the upper bound estimate of the expected supremum of L_t.

Lemma 5.2.2. *Let* L_t *be a one-dimensional real-valued Lévy process with Lévy triple* (b, c, Π) *defined on* $[0, T]$ *where* $T < \infty$. *For any* $0 < t < T$, *we have the following three cases:*

Case 1: if $c > 0$, *then*

$$\mathbb{E}[\sup_{0\le s\le t} L_s] \le C_1\sqrt{t}$$

Case 2: if L_t *is of finite variation, then*

$$\mathbb{E}[\sup_{0\le s\le t} L_s] \le C_2 t$$

Case 3: if $c = 0$ *and* L_t *is of infinite variation, then*

$$\mathbb{E}[\sup_{0\le s\le t} L_s] \le C_3 t^r,$$

where $\frac{1}{2} < r = \frac{1}{\beta} - \epsilon < \frac{1}{\beta} \le 1$ *for any fixed small* $\epsilon > 0$ *and* β *is the Blumenthal-Getoor index defined as above.*

Proof. For any $0 < t < T$, we first define the truncation threshold θ_N used in the Lévy-Itô decomposition above as

$$\theta_N = (\frac{t}{T})^k.$$

It's clear that $0 < \theta_N < 1$. Now

$$\mathbb{E}[\sup_{0\le s\le t} L_s] = \mathbb{E}[\sup_{0\le s\le t}(\gamma s + \sigma B_s + J_s^s(N) + J_s^b(N))]$$

$$\le \mathbb{E}\Big(\sup_{0\le s\le t}\gamma s + \sup_{0\le s\le t}\sigma B_s + \sup_{0\le s\le t} J_s^s(N) + \sup_{0\le s\le t} J_s^b(N)\Big)$$

$$\le |\gamma| t + \sigma\mathbb{E}[\sup_{0\le s\le t} B_s] + \mathbb{E}[\sup_{0\le s\le t} J_s^s(N)] + \mathbb{E}[\sup_{0\le s\le t} J_s^b(N)], \tag{5.1}$$

where $\gamma = b - \int_{\mathbb{R}\setminus\{0\}} x\mathbb{1}_{\{\theta_N\le|x|<1\}}\Pi(dx)$, $\theta_N = (\frac{t}{T})^k < 1$, and k is to be determined. We need to analyze (5.1) term by term. Firstly,

$$|\gamma| t \le \big(|b| + \int_{\mathbb{R}\setminus\{0\}} |x|\mathbb{1}_{\{\theta_N\le|x|<1\}}\Pi(dx)\big)t = \big(|b| + \int_{\theta_N\le|x|<1} |x|\Pi(dx)\big)t.$$

Secondly, we know from Shreve[62], page 297, Corollary 7.2.2, that, $\forall x \ge 0$

$$\mathbb{P}(\sup_{0\le s\le t} B_s \le x) = \mathbb{P}(\max_{0\le s\le t} B_s \le x) = 2\Phi(\frac{x}{\sqrt{t}}) - 1,$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}du$. Thus, the density is $f_{max_{0\le s\le t}B_s}(x) = \frac{2}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$ for $x \ge 0$, and equal to zero for negative x. Hence, if we integrate, we get the second term of (5.1)

$$\sigma\mathbb{E}(\sup_{0\le s\le t} B_s) = \sigma\int_0^\infty x f_{max_{0\le s\le t}B_s}(x)dx = \sigma\sqrt{\frac{2t}{\pi}} = \sigma\sqrt{\frac{2}{\pi}}\sqrt{t}.$$

Thirdly, by Doob-Dubins-Schwarz's Martingale Inequality [22],

$$\mathbb{E}(\sup_{0\le s\le t} J_s^s(N)) \le \|J_t^s(N)\|_2 = (\mathbb{E}[J_t^s(N)^2])^{\frac{1}{2}} = \sqrt{t\int_{|x|\le\theta_N} x^2\Pi(dx)}.$$

Recall the definition of the Blumenthal-Getoor index:

$$\beta = \inf\left\{\alpha > 0 : \int_{|x|<1} |x|^\alpha\Pi(dx) < \infty\right\}.$$

So for any given α such that $2 > \alpha > \beta$,

$$
\begin{aligned}
\sqrt{t\int_{|x|\le\theta_N} x^2\Pi(dx)} &= t^{\frac{1}{2}}\Big(\int_{|x|\le\theta_N}|x|^{\alpha}|x|^{2-\alpha}\Pi(dx)\Big)^{\frac{1}{2}}\\
&\le t^{\frac{1}{2}}\Big(\theta_N^{2-\alpha}\int_{|x|\le\theta_N}|x|^{\alpha}\Pi(dx)\Big)^{\frac{1}{2}}\\
&= t^{\frac{1}{2}}\theta_N^{1-\frac{\alpha}{2}}\Big(\int_{|x|\le\theta_N}|x|^{\alpha}\Pi(dx)\Big)^{\frac{1}{2}}\\
&= \sqrt{T}(\frac{t}{T})^{\frac{1}{2}}(\frac{t}{T})^{(1-\frac{\alpha}{2})k}\Big(\int_{|x|\le(\frac{t}{T})^k}|x|^{\alpha}\Pi(dx)\Big)^{\frac{1}{2}}\\
&\le \sqrt{T}(\frac{t}{T})^{\frac{1}{2}}(\frac{t}{T})^{(1-\frac{\alpha}{2})k}\Big(\int_{|x|<1}|x|^{\alpha}\Pi(dx)\Big)^{\frac{1}{2}}\\
&\le \sqrt{T}C_{\alpha}(\frac{t}{T})^{\frac{1}{2}+(1-\frac{\alpha}{2})k},
\end{aligned}
$$

where $C_\alpha := \Big(\int_{|x|<1}|x|^{\alpha}\Pi(dx)\Big)^{\frac{1}{2}}$. Lastly, we deal with the large jump part in (5.1), i.e., the compound Poisson part.

$$
\begin{aligned}
\mathbb{E}[\sup_{0\le s\le t} J_s^b(N)] &\le \mathbb{E}\big[\sum_{i=1}^{N_t} J^i(N)\mathbb{1}_{J^i(N)>0}\big]\\
&= \mathbb{E}(N_t)\mathbb{E}(J\mathbb{1}_{J>0})\\
&= t\overline{\Pi}(\theta_N)\int_{x>\theta_N} xF(dx)\\
&= t\int_{x>\theta_N} x\Pi(dx)\\
&= t\int_{x>(\frac{t}{T})^k} x\Pi(dx)\\
&= t\int_{(\frac{t}{T})^k<x<1} x\Pi(dx) + t\int_{x\ge1} x\Pi(dx),
\end{aligned}
$$

where the first equality above follows from the fact that all the jumps have the same law, and $\overline{\Pi}(\theta_N) := \int_{|x|>\theta_N}\Pi(dx)$, and recall that $F(dx) = \frac{\Pi(dx)}{\overline{\Pi}(\theta_N)}$. Now, if we combine all the four terms together, we obtain the following

$$
\begin{aligned}
\mathbb{E}[\sup_{0\le s\le t} L_s] \le\ & \big(|b| + \int_{(\frac{t}{T})^k\le|x|<1}|x|\Pi(dx)\big)t + \sigma\sqrt{\frac{2}{\pi}}\sqrt{t} + \sqrt{T}C_{\alpha}(\frac{t}{T})^{\frac{1}{2}+(1-\frac{\alpha}{2})k}\\
&+t\int_{(\frac{t}{T})^k<x<1} x\Pi(dx) + t\int_{x\ge1} x\Pi(dx). \qquad (5.2)
\end{aligned}
$$

First, we have to say that if we stick to using the classical truncation threshold 1 under any circumstances, we may simply get the result of proposition 3.4 of Dia and Lamberton [20] with a little improvement. i.e.,

If L_t is an integrable Lévy process with Lévy triple (b, σ, Π), then

$$\mathbb{E}[\sup_{0\le s\le t} L_s] \le (b^+ + \int_{x>1} x\Pi(dx))t + \Big(\sigma\sqrt{\frac{2}{\pi}} + \sqrt{\int_{|x|\le 1} x^2\Pi(dx)}\Big)\sqrt{t}.$$

The reason that we are approaching the problem using floating truncation threshold is to find a better bound for the third case, which is a pure jump Lévy process with infinite variation.

From now on, we assume L_t is a pure jump Lévy process with infinite variation, in which we know that the Brownian motion term vanishes and the Blumenthal-Getoor index $\beta \ge 1$, thus for $2 > \alpha > \beta \ge 1$ and (5.2), we can derive that

$$\begin{aligned}
&\mathbb{E}[\sup_{0\le s\le t} L_s] \\
\le\ & \big(T|b| + T\int_{x\ge 1} x\Pi(dx)\big)\frac{t}{T} + \sqrt{T}C_\alpha(\frac{t}{T})^{\frac{1}{2}+(1-\frac{\alpha}{2})k} + 2T\frac{t}{T}\int_{(\frac{t}{T})^k\le|x|<1}|x|\Pi(dx) \\
\le\ & \big(T|b| + T\int_{x\ge 1} x\Pi(dx)\big)\frac{t}{T} + \sqrt{T}C_\alpha(\frac{t}{T})^{\frac{1}{2}+(1-\frac{\alpha}{2})k} + 2T(\frac{t}{T})^{1+k(1-\alpha)}\int_{(\frac{t}{T})^k\le|x|<1}|x|^\alpha\Pi(dx) \\
\le\ & \big(T|b| + T\int_{x\ge 1} x\Pi(dx)\big)\frac{t}{T} + \sqrt{T}C_\alpha(\frac{t}{T})^{\frac{1}{2}+(1-\frac{\alpha}{2})k} + 2C_\alpha^2T(\frac{t}{T})^{1+k(1-\alpha)} \\
\le\ & \big(T|b| + T\int_{x\ge 1} x\Pi(dx)\big)\frac{t}{T} + D_\alpha(\frac{t}{T})^{\frac{1}{2}+(1-\frac{\alpha}{2})k} + D_\alpha(\frac{t}{T})^{1+k(1-\alpha)} \\
\le\ & \big(T|b| + T\int_{x\ge 1} x\Pi(dx)\big)\frac{t}{T} + 2D_\alpha(\frac{t}{T})^{\eta(k)},
\end{aligned}$$

where

$$D_\alpha := \max(\sqrt{T}C_\alpha, 2C_\alpha^2T)$$

$$\eta(k) := \min(\frac{1}{2} + (1-\frac{\alpha}{2})k, 1 + k(1-\alpha)).$$

In order to achieve an optimal (or say least) upper bound, we would like to maximize $\eta(k)$ over all k. Note that as functions of k, $\frac{1}{2} + (1-\frac{\alpha}{2})k$ and $1 + k(1-\alpha)$ are both linear and the maximum is achieved as they intersect. In other words, we'd equate them to solve for k as follows

$$\frac{1}{2} + (1-\frac{\alpha}{2})k = 1 + k(1-\alpha) \Longrightarrow k = \frac{1}{\alpha},$$

and

$$\max_k \eta(k) = \frac{1}{2} + (1 - \frac{\alpha}{2})\frac{1}{\alpha} = 1 + \frac{1}{\alpha}(1-\alpha) = \frac{1}{\alpha} \in (\frac{1}{2}, 1).$$

So, we complete the proof of the case 3. □

Remark 5.2.3. Remark: During the proof of the third case, we used quite a few inequalities, which could be optimized to equality if we know more details about the Lévy measure. As we will see later, in certain more specific examples, we actually would be able to make the $\epsilon = 0$ in case 3.

Proof. (of Theorem 5.2.1) Now we begin to show the proof of Theorem 5.2.1, which is based on the proof of [20] with some improvement for the case of pure jump Lévy process of infinite variation. Hence, we only show this particular case. In the following, we fix a small positive number ϵ, with $\frac{1}{2} < \frac{1}{\beta} - \epsilon < \frac{1}{\beta} \leq 1$.

Denote $\Lambda(N) := \frac{T}{N}$, by Spitzer's identity,

$$\mathbb{E}[\sup_{0\leq t\leq T} L_t - \sup_{0\leq t\leq T} L_t^D(N)]$$

$$= \int_0^{\Lambda(N)} \Big(\frac{\mathbb{E}[L_s^+]}{s} - \frac{\mathbb{E}[L_{\Lambda(N)}^+]}{\Lambda(N)}\Big)ds + \sum_{j=2}^{N}\int_{(j-1)\Lambda(N)}^{j\Lambda(N)} \Big(\frac{\mathbb{E}[L_s^+]}{s} - \frac{\mathbb{E}[L_{j\Lambda(N)}^+]}{j\Lambda(N)}\Big)ds. \tag{5.3}$$

The first term in (5.3) only concerns with the behavior of the Lévy process in time interval $[0, \Lambda(N)]$. Actually, it is always positive and equals

$$\mathbb{E}[\sup_{0\leq t\leq \Lambda(N)} L_t] - \mathbb{E}[L_{\Lambda(N)}^+] \leq \mathbb{E}[\sup_{0\leq t\leq \Lambda(N)} L_t]$$

$$\leq C_\epsilon(\Lambda(N))^{\frac{1}{\beta}-\epsilon} = \frac{D_\epsilon}{N^{\frac{1}{\beta}-\epsilon}}.$$

According to [20], the second term of (5.3) could be computed as

$$\sum_{j=2}^{N}\int_{(j-1)\Lambda(N)}^{j\Lambda(N)} \Big(\frac{\mathbb{E}[L_s^+]}{s} - \frac{\mathbb{E}[L_{j\Lambda(N)}^+]}{j\Lambda(N)}\Big)ds$$

$$\leq \sum_{j=2}^{N} \frac{\mathbb{E}[L_{j\Lambda(N)}^+]}{j(j-1)} + \Lambda(N)\mathbb{E}[L_1]\sum_{j=2}^{N}\frac{1}{j-1}$$

$$
\begin{aligned}
&\leq \sum_{j=2}^{N} \frac{\mathbb{E}[L^+_{j\Lambda(N)}]}{j(j-1)} + T\mathbb{E}[L_1]\frac{\log N}{N} \\
&\leq \sum_{j=2}^{N} \frac{\mathbb{E}[\sup_{0\leq t\leq j\Lambda(N)} L_t]}{j(j-1)} + T\mathbb{E}[L_1]\frac{\log N}{N} \\
&\leq \sum_{j=2}^{N} \frac{C_\epsilon (j\Lambda(N))^{\frac{1}{\beta}-\epsilon}}{j(j-1)} + T\mathbb{E}[L_1]\frac{\log N}{N} \\
&= D_\epsilon \frac{1}{N^{\frac{1}{\beta}-\epsilon}} \sum_{j=2}^{N} \frac{1}{j^{1+\epsilon-\frac{1}{\beta}}(j-1)} + T\mathbb{E}[L_1]\frac{\log N}{N}.
\end{aligned}
$$

Since $1+\epsilon-\frac{1}{\beta} > 0$, so $\sum_{j=2}^{N} \frac{1}{j^{1+\epsilon-\frac{1}{\beta}}(j-1)} < \infty$. Moreover, since $\frac{1}{2} < \frac{1}{\beta} - \epsilon < 1$, it is clear that $\frac{\log N}{N} < \frac{1}{N^{\frac{1}{\beta}-\epsilon}}$ for sufficiently large N. Therefore, we reach the conclusion that the second term of (5.3) is

$$
\sum_{j=2}^{N} \int_{(j-1)\Lambda(N)}^{j\Lambda(N)} \Big(\frac{\mathbb{E}[L^+_s]}{s} - \frac{\mathbb{E}[L^+_{j\Lambda(N)}]}{j\Lambda(N)}\Big) ds = O(\frac{1}{N^{\frac{1}{\beta}-\epsilon}}).
$$

Combined with the first term of (5.3), we complete the proof of Theorem 5.2.1. □

Remark 5.2.4. (i) Our result regarding the pure jump Lévy process of infinite variation is more informative and illustrative compared to Dia and Lamberton's result in the corresponding case, since it clearly distinguishes the Lévy processes that are "close" (in terms of Blumenthal-Getoor index) to finite variation Lévy process with those that fluctuate a lot and "close" to Brownian motion.

(ii) We see that in any case, the existence of the Brownian motion will cause the slowest rate of convergence, which is $\frac{1}{\sqrt{N}}$.

(iii) As for the leading term of convergence, in some cases where we already have seen, i.e., Merton's or Kou's jump diffusions (diffusion coefficient $\sigma \neq 0$), we see that we actually are able to achieve the leading convergence rates, which are $\frac{1}{\sqrt{N}}$. However, in some other cases, for instance, Variance Gamma and NIG, which is of case two and three in Theorem 5.2.1 respectively, the leading orders are in fact better than the upper bound estimations.

5.3 Case Study

Now we turn to some concrete examples to show how the above results apply. In all these examples, we again assume, without loss of generality, that the equally spaced partition of the interval $[0,1]$ and $\Lambda(N) = \frac{1}{N}$. However, θ_N will vary case by case.

5.3.1 Jump diffusions

A general jump diffusion process is actually the sum of a compound Poisson process and a Brownian motion with drift, where the two processes are independent. i.e.

$$L_t = \gamma t + \sigma B_t + \sum_{i=1}^{N_t} \xi_i,$$

where N_t is a Poisson process and $\{\xi_i\}_{i\geq 1}$ are i.i.d. random variables denoting the jumping sizes, which are independent of N_t and B_t. In order for the process to be integrable, we need to assume that

$$\mathbb{E}[\xi_i] < \infty,$$

so that

$$\mathbb{E}[L_1] = \gamma + \lambda \mathbb{E}[\xi_i] < \infty.$$

Since the Brownian motion term exists, we simply obtain that

$$\Delta_N = \mathbb{E}(\sup_{0\leq t\leq 1} L_t - \sup_{0\leq t\leq 1} L_t^D(N)) \leq \frac{C}{\sqrt{N}}.$$

Note that our results about Merton's and Kou's jump diffusions are consistent with the above conclusion and the leading rate is achieved exactly.

5.3.2 Variance Gamma Process

We already have seen the definition of variance gamma process. Through checking the Lévy density, it is clear that variance gamma process is of finite moments, finite variation, and infinite activity. Note that VG's Blumenthal-Getoor index $\beta = 0$. Thus, by our theorem, we have

$$\mathbb{E}(\sup_{0\leq t\leq T} L_t - \sup_{0\leq t\leq T} L_t^D(N)) \leq C\frac{\log N}{N}.$$

However, through previous calculation in chapter 3, we know that the actual order of the leading term is $\frac{1}{N}$, which is slightly better than $\frac{\log N}{N}$.

5.3.3 CGMY Process

CGMY process is one of the most widely used pure jump Lévy models in financial world. CGMY process is a generalization of variance gamma process introduced by Carr, Geman, Madan, and Yor [32]. The Lévy measure of CGMY process is given by

$$\Pi(dx) = \mathbb{1}_{\{x<0\}} \frac{C}{|x|^{1+Y}} e^{Gx} dx + \mathbb{1}_{\{x>0\}} \frac{C}{x^{1+Y}} e^{-Mx} dx,$$

where $C, G, M > 0$ and $Y \in (-\infty, 2)$.

If $Y = 0$, we simply get the variance gamma process. It can be checked that the process has finite activity iff $Y \in (-\infty, 0)$ and the paths have finite variation iff $Y \in (-\infty, 1)$. Y cannot be larger than or equal to 2, otherwise the natural condition of Lévy measure $\int_{\mathbb{R}\setminus\{0\}} 1 \wedge x^2 \Pi(dx) < \infty$ will not be satisfied. Also, the process has finite first moment for all applicable Y due to the exponential tail in the Lévy density. We denote $\theta_N = \frac{1}{N^k}$ for some k to be determined.

Case (a): $Y < 0$.

The condition $Y < 0$ implies both finite activity and finite variation. The Blumenthal index is $\beta = 0$. It's essentially a compound poisson case, thus, according to [20] Theorem 3.5

$$\Delta_N = \mathbb{E}(\sup_{0\le t\le 1} L_t - \sup_{0\le t\le 1} L_t^D(N)) = O(\frac{1}{N}).$$

Case (b): $Y \in (0, 1)$. We exclude the case of $Y = 0$ since it was discussed as variance gamma process as above.

In this case, the CGMY process is of finite variation and infinite activity. Suggested by Geman [29], pure jump Lévy processes with finite variation but infinite activity are good candidate of modeling the dynamics of log return of stock prices. Therefore, the case when $Y \in [0, 1)$ is of particular interest. Note that the Blumenthal-Getoor index is $\beta = Y$. Then theorem 3.12 of [20] indicates that

$$\Delta_N = \mathbb{E}(\sup_{0\le t\le 1} L_t - \sup_{0\le t\le 1} L_t^D(N)) = O(\frac{1}{N}).$$

Case (c): $Y = 1$, which means that the Blumenthal-Getoor index $\beta = 1$. Then, using the technique in the proof of Theorem 5.2.1, we have

$$\mathbb{E}[\sup_{0\le s\le t} L_s] \le t \int_{\theta_N<|x|<1} |x|\Pi(dx) + \sqrt{t \int_{|x|\le\theta_N} x^2\Pi(dx)} + t \int_{x>\theta_N} x\Pi(dx)$$

$$
\begin{aligned}
&= t\Big(\int_{|x|>\theta_N} |x|\Pi(dx) - \int_{|x|\geq 1} |x|\Pi(dx)\Big) \\
&\quad + \sqrt{tC}\Big(\int_0^{\theta_N} (e^{-Mx} + e^{-Gx})dx\Big)^{\frac{1}{2}} + tC\Big[\int_{\theta_N}^{\infty} x^{-1}e^{-Mx}dx\Big] \\
&= tC\Big(\int_{\theta_N}^{\infty} x^{-1}e^{-Mx}dx + \int_{\theta_N}^{\infty} x^{-1}e^{-Gx}dx\Big) - t\int_{|x|\geq 1} |x|\Pi(dx) \\
&\quad + \sqrt{tC}\Big(\int_0^{\theta_N} (e^{-Mx} + e^{-Gx})dx\Big)^{\frac{1}{2}} + tC\Big[\int_{\theta_N}^{\infty} x^{-1}e^{-Mx}dx\Big] \\
&\leq 2tC\Big[\int_{\theta_N}^{\infty} x^{-1}e^{-Mx}dx\Big] + tC\Big[\int_{\theta_N}^{\infty} x^{-1}e^{-Gx}dx\Big] - t\int_{|x|\geq 1} |x|\Pi(dx) + \sqrt{tC}\Big(\int_0^{\theta_N} 2dx\Big)^{\frac{1}{2}} \\
&= 2tC\Big[-\gamma - \log(M\theta_N) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(M\theta_N)^m}{mm!}\Big] \\
&\quad + tC\Big[-\gamma - \log(G\theta_N) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(G\theta_N)^m}{mm!}\Big] + \sqrt{2tC\theta_N} - t\int_{|x|\geq 1} |x|\Pi(dx) \\
&= 2tC\Big[-\gamma - \log M - \log t + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(Mt)^m}{mm!}\Big] \\
&\quad + tC\Big[-\gamma - \log G - \log t + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(Gt)^m}{mm!}\Big] + t\sqrt{2C} - t\int_{|x|\geq 1} |x|\Pi(dx) \\
&\leq D_1(-t\log t)\mathbb{1}_{\{0<t<\frac{1}{e}\}} + D_2 t\mathbb{1}_{\{\frac{1}{e}\leq t<1\}},
\end{aligned}
$$

where we used the fact that for $t \in (0,1)$, $-t\log t \leq t \iff t \in [\frac{1}{e}, 1)$. Therefore, for any integer $N \geq 3$, we know in particular that

$$
\mathbb{E}[\sup_{0\leq t\leq \frac{1}{N}} L_t] \leq D\frac{\log N}{N},
$$

where D is independent of N. So we combine them together to get the following

$$
\begin{aligned}
&\mathbb{E}(\sup_{0\leq t\leq 1} L_t - \sup_{0\leq t\leq 1} L_t^D(N)) \\
&\leq \mathbb{E}[\sup_{0\leq t\leq \frac{1}{N}} L_t] + \sum_{j=2}^{N} \frac{\mathbb{E}[L^+_{\frac{j}{N}}]}{j(j-1)} + \mathbb{E}[L_1]\frac{\log N}{N} \\
&\leq (D + \mathbb{E}[L_1])\frac{\log N}{N} + \sum_{j=2}^{\lfloor \frac{N}{e} \rfloor} \frac{\mathbb{E}[L^+_{\frac{j}{N}}]}{j(j-1)} + \sum_{j=\lfloor \frac{N}{e} \rfloor+1}^{N-1} \frac{\mathbb{E}[L^+_{\frac{j}{N}}]}{j(j-1)} + \frac{\mathbb{E}[L^+_1]}{N(N-1)} \\
&\leq (D + \mathbb{E}[L_1])\frac{\log N}{N} + D_1\sum_{j=2}^{\lfloor \frac{N}{e} \rfloor} \frac{\frac{j}{N}\log\frac{N}{j}}{j(j-1)} + D_2\sum_{j=\lfloor \frac{N}{e} \rfloor+1}^{N-1} \frac{\frac{j}{N}}{j(j-1)} + \frac{\mathbb{E}[L^+_1]}{N(N-1)} \\
&\leq (D + \mathbb{E}[L_1])\frac{\log N}{N} + D_1\frac{\log N}{N}\sum_{j=2}^{\lfloor \frac{N}{e} \rfloor} \frac{1}{j-1} + \frac{D_2}{N}\sum_{j=\lfloor \frac{N}{e} \rfloor+1}^{N-1} \frac{1}{j-1} + \frac{\mathbb{E}[L^+_1]}{N(N-1)}
\end{aligned}
$$

$$
\begin{aligned}
&\le (D+\mathbb{E}[L_1])\frac{\log N}{N} + D_1\frac{\log N}{N}\sum_{j=2}^{N}\frac{1}{j-1} + \frac{D_2}{N}\sum_{j=2}^{N-1}\frac{1}{j-1} + \frac{\mathbb{E}[L_1^+]}{N(N-1)} \\
&= O(\frac{(\log N)^2}{N}).
\end{aligned}
$$

Note that this result is actually consistent with our Theorem 5.2.1 since $Y = 1$ implies that the process is of infinite variation and $\frac{(\log N)^2}{N} = O(\frac{1}{N^{1-\epsilon}})$ for any $\epsilon > 0$.

Case (d): $Y \in (1, 2)$, which also implies the Blumenthal index $\beta = Y$. According to (5.2), we have for $0 < t < 1$,

$$
\begin{aligned}
&\mathbb{E}[\sup_{0\le s\le t} L_s] \\
\le\ & t\int_{\theta_N<|x|<1}|x|\Pi(dx) + \sqrt{t\int_{|x|\le\theta_N}x^2\Pi(dx)} + t\int_{x>\theta_N}x\Pi(dx) \\
=\ & t\Big(\int_{|x|>\theta_N}|x|\Pi(dx) - \int_{|x|\ge 1}|x|\Pi(dx)\Big) + \sqrt{tC}\Big(\int_0^{\theta_N}x^{1-Y}(e^{-Mx}+e^{-Gx})dx\Big)^{\frac{1}{2}} \\
& +tC\Big[\int_{\theta_N}^{\infty}x^{-Y}e^{-Mx}dx\Big] \\
\le\ & tC\Big(\int_{\theta_N}^{\infty}x^{-Y}(e^{-Mx}+e^{-Gx})dx - \int_{|x|\ge 1}|x|\Pi(dx)\Big) + \sqrt{tC}\Big(\int_0^{\theta_N}2x^{1-Y}dx\Big)^{\frac{1}{2}} \\
& +tC\int_{\theta_N}^{\infty}x^{-Y}e^{-Mx}dx \\
\le\ & \frac{2tC}{1-Y}\int_{\theta_N}^{\infty}e^{-Mx}d(x^{1-Y}) + \frac{tC}{1-Y}\int_{\theta_N}^{\infty}e^{-Gx}d(x^{1-Y}) - tC\int_{|x|\ge 1}|x|\Pi(dx) + \sqrt{\frac{2tC}{2-Y}}\theta_N^{1-\frac{Y}{2}} \\
=\ & \frac{2tC}{Y-1}\Big[\theta_N^{1-Y}e^{-M\theta_N} - M\int_{\theta_N}^{\infty}x^{1-Y}e^{-Mx}dx\Big] + \frac{tC}{Y-1}\Big[\theta_N^{1-Y}e^{-G\theta_N} - G\int_{\theta_N}^{\infty}x^{1-Y}e^{-Gx}dx\Big] \\
& -tC\int_{|x|\ge 1}|x|\Pi(dx) + \sqrt{\frac{2tC}{2-Y}}\theta_N^{1-\frac{Y}{2}} \\
\le\ & \frac{3C}{Y-1}t^{1+k(1-Y)} + \sqrt{\frac{2C}{2-Y}}t^{\frac{1}{2}+k(1-\frac{Y}{2})} - tC_k,
\end{aligned}
$$

where the last step follows from the fact that $\int_{\theta_N}^{\infty}x^{1-Y}e^{-Mx}dx < \int_0^{\infty}x^{1-Y}e^{-Mx}dx = M^{Y-2}\Gamma(2-Y)$ since $1 < Y < 2$. Also, recall that $\theta_N = t^k$ for some $k > 0$. C_k depends on k but finite. According to the same technique in the proof of the theorem, we get the optimal rate of convergence $t^{\frac{1}{Y}}$ when we set $k = \frac{1}{Y}$. So with defining $C_1 = \max(\frac{3C}{Y-1}, \sqrt{\frac{2C}{2-Y}})$, the above is

$$
\le C_1 t^{\frac{1}{Y}}.
$$

In particular, if we set $t = \frac{1}{N}$, then

$$\mathbb{E}[\sup_{0\le t\le \frac{1}{N}}] \le C_1 \frac{1}{N^{\frac{1}{Y}}}.$$

Again, we combine the terms together to obtain the following

$$\begin{aligned}
& \mathbb{E}(\sup_{0\le t\le 1} L_t - \sup_{0\le t\le 1} L_t^D(N)) \\
\le\ & \mathbb{E}[\sup_{0\le t\le \frac{1}{N}} L_t] + \sum_{j=2}^{N} \frac{\mathbb{E}[L^+_{\frac{j}{N}}]}{j(j-1)} + \mathbb{E}[L_1]\frac{\log N}{N} \\
\le\ & C_1 \frac{1}{N^{\frac{1}{Y}}} + C_1 \sum_{j=2}^{N-1} \frac{(\frac{j}{N})^{\frac{1}{Y}}}{j(j-1)} + \frac{\mathbb{E}[L_1^+]}{N(N-1)} + \mathbb{E}[L_1]\frac{\log N}{N} \\
=\ & \frac{C_1}{N^{\frac{1}{Y}}} + \frac{C_1}{N^{\frac{1}{Y}}} \sum_{j=2}^{N-1} \frac{1}{j^{1-\frac{1}{Y}}(j-1)} + \frac{\mathbb{E}[L_1^+]}{N(N-1)} + \mathbb{E}[L_1]\frac{\log N}{N} \\
=\ & O(\frac{1}{N^{\frac{1}{Y}}}).
\end{aligned}$$

Through this case, we see that we can actually achieve that $\epsilon = 0$ when we have more detailed information with regard to the Lévy measure. We can imagine that if the Blumenthal-Getoor index Y is close to 2, then the behavior of the convergence should somewhat close to the Brownian motion, which is of order $O(\frac{1}{\sqrt{N}})$, while if the Blumenthal-Getoor index is close to 1, then the convergence should be $O(\frac{1}{N^{1-\epsilon}})$ and clearly $\frac{\log N}{N} = O(\frac{1}{N^{1-\epsilon}})$, which again indicates that the smaller the index, the faster the convergence.

5.3.4 Normal Inverse Gaussian (NIG) Process

We already introduced NIG in chapter 3. We recall that the Lévy measure of NIG process is given by

$$\Pi(dx) = \mathbb{1}_{\{x<0\}} e^{\beta x} \frac{\delta\alpha}{\pi|x|} K_1(-\alpha x)dx + \mathbb{1}_{\{x>0\}} e^{\beta x} \frac{\delta\alpha}{\pi x} K_1(\alpha x)dx,$$

where $0 \le |\beta| < \alpha$, $\delta > 0$, K_1 is the modified Bessel function of second kind of index 1. Through calculation, it is not hard to see that NIG has finite first moment, but is of infinite activity and infinite variation. Also, the Lévy measure tells us that the Blumenthal-Getoor index of NIG is exactly 1. From Abramowitz and Stegun [1], page 375, 9.6.10 and 9.6.11, we know the series expansion of $K_1(\alpha x)$ as follows:

$$K_1(\alpha x) = (\alpha x)^{-1} + \log\frac{\alpha x}{2}\frac{\alpha x}{2}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}(\alpha x)^2)^k}{k!(k+1)!} - \frac{\alpha x}{4}\sum_{k=0}^{\infty}(\psi(k+1)+\psi(k+2))\frac{(\frac{1}{4}(\alpha x)^2)^k}{k!(k+1)!}.$$

Then, from (5.2), we obtain that for any $0 < t < 1$,

$$\mathbb{E}[\sup_{0\le s\le t} L_s] \le t\int_{\theta_N<|x|<1}|x|\Pi(dx) + \sqrt{t\int_{|x|\le\theta_N}x^2\Pi(dx)} + t\int_{x>\theta_N}x\Pi(dx)$$

$$\le 2t\int_{\theta_N<|x|<1}|x|\Pi(dx) + \sqrt{t\int_{|x|\le\theta_N}x^2\Pi(dx)} + t\int_{x>1}x\Pi(dx). \tag{5.4}$$

The second term in (5.4) could be written as

$$\Big(\frac{t\delta}{\pi}\int_0^{\theta_N}(e^{\beta x}+e^{-\beta x})\alpha x K_1(\alpha x)dx\Big)^{\frac{1}{2}} \le \Big(\frac{t\delta}{\pi}\int_0^{\theta_N}D_1 dx\Big)^{\frac{1}{2}} = D_2 t,$$

where D_1, D_2 are constants depending on α and β only. In the above, we let $\theta_N = t < 1$.

The third term in (5.4) is less than or equal to $D_3 t$, since $\int_{x>1}x\Pi(dx) < \infty$. We only left with the first term in (5.4).

$$2t\int_{\theta_N<|x|<1}|x|\Pi(dx) = \frac{2t\delta}{\pi}\int_{\theta_N}^1(e^{\beta x}+e^{-\beta x})\alpha K_1(\alpha x)dx$$

$$\le \frac{2t\delta}{\pi}\int_{\theta_N}^1\frac{1}{x}dx + D_4 t$$

$$= -\frac{2\delta}{\pi}t\log t + D_4 t.$$

Then, we follow the proof of CGMY for $Y = 1$ in exactly the same way to obtain that

$$\Delta_N = \mathbb{E}(\sup_{0\le t\le 1}L_t - \sup_{0\le t\le 1}L_t^D(N)) = O(\frac{(\log N)^2}{N}).$$

Remark 5.3.1. (1) In the NIG, even if we add the drift term μt, the conclusion stays the same since the drift will incur a difference of order $\frac{1}{N}$, which is smaller than $\frac{(\log N)^2}{N}$.

(2) Note that $\frac{(\log N)^2}{N} = O(\frac{1}{N^{1-\epsilon}})$, so our result above is consistent with, but slightly better than what is stated in our Theorem 5.2.1 since we have more detailed information for the Lévy measure. Furthermore, Theorem 3.3.10 gives the exact rate, i.e., the leading term is $\frac{\log N}{N}$, this is the best

result since we don't have any inequalities in the proof of Theorem 3.3.10, but all equalities. Also, the consistency again follows since $\frac{\log N}{N} < \frac{(\log N)^2}{N} < \frac{1}{N^{1-\epsilon}}$ for large N.

(3) The equivalence of the results for NIG and CGMY (as $Y = 1$) can be justified through checking the asymptotic behavior of $K_1(x)$ for small x and comparing the Lévy measures of the two. Correspondingly, both Blumenthal-Getoor indices are 1 and both processes are of infinite activity and of infinite variation.

Chapter 6

Numerical Result

6.1 A Recursive Algorithm for Computation of Discrete Maximum

Given $\mathbb{E}(\sup_{0\le t\le T} L_t - \sup_{0\le t\le T} L_t^D(N))$, if we could figure out an algorithm to calculate the discrete maximum, we would have a good estimate on the continuous supremum.

With the same process construction as the previous sections, we are given a Lévy process L_t defined on a finite time interval $[0,1]$, and we divide the unit interval equally into N subintervals, each of which has length denoted by Λ_N. Let $M_j = \max(L_0, L_{\Lambda_N}, L_{2\Lambda_N} ... L_{j\Lambda_N})$. For simplicity, we also use $L_j := L_{j\Lambda_N}$. Clearly, $M_0 = L_0 = 0$ a.s., and M_N is the overall maximum under N-discretization. We are interested in $\mathbb{E}[M_N]$.

Inspired by Feng and Linetsky [27], we show the following derivation of the desired algorithm. For any j, realize that the random variable $M_j - L_j$ has a distribution of mixed type, i.e. a continuous density $f_j(x)$ defined for all $x > 0$ and a point mass at 0, called c_j. For technical convenience, we define $f_j(x) = 0$ for all $x \le 0$. These two quantities are related as follows:

$$c_j = 1 - \int_{\mathbb{R}} f_j(x)dx.$$

Initially, $f_0(x) = 0$ for all x, and $c_0 = 1$. Note that, for $1 \le j \le N$,

$$M_{j-1} - L_j = M_{j-1} - L_{j-1} - (L_j - L_{j-1}) = M_{j-1} - L_{j-1} + (-\Delta L_j).$$

Since $M_{j-1} - L_{j-1}$ has a continuous density $f_{j-1}(x)$ and a point mass c_{j-1}, then the continuous density function of $M_{j-1} - L_j$, called $g_j(x)$, is given by

$$g_j(x) = c_{j-1}p(-x) + \int_{\mathbb{R}} f_{j-1}(y)p(y-x)dy,$$

where $p(x)$ is the transition density function of $\Delta L_j = L_{\Lambda_N}$, which is independent of j by stationary increment property of Lévy process. Therefore, the continuous density function $f_j(x)$ is given by

$$f_j(x) = \mathbb{1}_{(0,\infty)}(x) g_j(x),$$

and the corresponding point mass $c_j = 1 - \int_{\mathbb{R}} f_j(x)dx$.

To facilitate the calculation, we need the help of Fourier analysis. We use conventional notation $\hat{f}$ to denote the Fourier transform of a L^1 function f. From above, we see that

$$\hat{g}_j(\xi) = (c_{j-1} + \hat{f}_{j-1}(\xi))\phi(-\xi),$$

where $\phi(\xi)$ is the Fourier transform of $p(x)$. Thus, due to Feng and Linetsky [27], we obtain

$$\hat{f}_j(\xi) = \mathscr{F}(\mathbb{1}_{(0,\infty)} g_j)(\xi) = \frac{1}{2}\hat{g}_j(\xi) + \frac{i}{2}(\mathscr{H}\hat{g}_j)(\xi),$$

and

$$c_j = 1 - \hat{f}_j(0),$$

where $\mathscr{H}$ denotes the Hilbert transform. To evaluate $\mathbb{E}[M_j - L_j]$, we also need $\hat{f}'_j$. Thus, as we differentiate the above expression, we obtain the following

$$\hat{g}'_j(\xi) = (c_{j-1} + \hat{f}_{j-1}(\xi))\phi'(-\xi)(-1) + \hat{f}'_{j-1}(\xi)\phi(-\xi),$$

and note that the derivative of a Hilbert transform of a function is the Hilbert transform of the derivative of the function, thus

$$\hat{f}'_j(\xi) = \frac{1}{2}\hat{g}'_j(\xi) + \frac{i}{2}(\mathscr{H}(\hat{g}'_j))(\xi).$$

Initially, we have

$$\hat{f}_0(\xi) = 0, \hat{f}'_0(\xi) = 0, c_0 = 1.$$

Hence, we come up with a recursive algorithm for computing the discrete maximum.

Theorem 6.1.1. *Denote $\phi(\xi)$ the characteristic function of $X_{\frac{1}{N}}$. With initial values as follows*

$$f_0(x) = 0, \hat{f}_0(\xi) = 0, \hat{f}_0'(\xi) = 0, c_0 = 1.$$

Recursively, we define and calculate

$$\begin{aligned}\hat{g}_j(\xi) &= (c_{j-1} + \hat{f}_{j-1}(\xi))\phi(-\xi)\\ \hat{g}_j'(\xi) &= (c_{j-1} + \hat{f}_{j-1}(\xi))\phi'(-\xi)(-1) + \hat{f}_{j-1}'(\xi)\phi(-\xi),\end{aligned}$$

and

$$\begin{aligned}\hat{f}_j(\xi) &= \frac{1}{2}\hat{g}_j(\xi) + \frac{i}{2}(\mathscr{H}\hat{g}_j)(\xi)\\ \hat{f}_j'(\xi) &= \frac{1}{2}\hat{g}_j'(\xi) + \frac{i}{2}(\mathscr{H}(\hat{g}_j'))(\xi).\end{aligned}$$

Then

$$-i\hat{f}_N'(0) = \mathbb{E}[M_N - L_N],$$

and the expected discrete maximum of X_t on $[0,1]$ with N-equal partition is given by

$$\mathbb{E}[M_N] = \mathbb{E}[L_N] - i\hat{f}_N'(0) = \mathbb{E}[L_1] - i\hat{f}_N'(0).$$

In particular, if the Lévy process is symmetric, then we have

$$\mathbb{E}[M_N] = -i\hat{f}_N'(0).$$

The scheme of computation of the discrete Hilbert transform can be found in Feng and Linetsky [27].

6.2 Case study

6.2.1 Merton's Jump Diffusion

Let's pick $\mu = 2, \sigma = 1, \lambda = 1, m = 1, s = 1, T = 1$. Also, we know from Feng and Linetsky [27], the discrete Hilbert transform approximation and fast Hilbert transform requires a truncation value M, which is set to be large enough for each N to achieve the 9th decimal precision, respectively.

It is trivial to check that the condition $\frac{\lambda+\frac{\mu^2}{2\sigma^2}}{N} < 2\pi$ is satisfied for all $N \geq 1$. Then, use the above recursive algorithm, we obtain the following table

N	Maximum
1	3.007719914
2	3.028566332
4	3.061174902
8	3.097834414
16	3.132295934
32	3.161519807
64	3.184802944
128	3.202637760
256	3.215954626
512	3.225730617
1024	3.232825295

Table 6.1: Discrete maximum for Merton's jump diffusion

According to Theorem 3.1.1, we see that the coefficients of $\frac{1}{\sqrt{N}}$ and $\frac{1}{N}$ should be constants independent of N, which indicates that we may apply Richardson's extrapolation to get a much better estimate for the continuous supremum. i.e., the twice extrapolation generates the following table of maximum values:

N	Maximum	1st Extrapolation	2nd Extrapolation
1	3.007719914	3.078894038	3.200903832
2	3.028566332	3.139898955	3.232777657
4	3.061174902	3.186338306	3.244648498
8	3.097834414	3.215493402	3.248651553
16	3.132295934	3.232072478	3.249954341
32	3.161519807	3.241013409	3.250376220
64	3.184802944	3.245694815	3.250513954
128	3.202637760	3.248104385	3.250559510
256	3.215954626	3.249331947	3.250574779
512	3.225730617	3.249953363	
1024	3.232825295		

Table 6.2: Richardson's extrapolation results for Merton's jump diffusion

Note that the theoretical value (up to tenth decimal precision) is actually 3.2505826874. In order to see more clearly the approximations, we give the table of errors as follows,

N	Maximum	1st Extrapolation	2nd Extrapolation
1	0.242862765	0.171688641	0.049678807
2	0.222016346	0.110683724	0.017805022
4	0.189407776	0.064244373	0.005934180
8	0.152748264	0.035089276	0.001931125
16	0.118286744	0.018510201	0.000628338
32	0.089062871	0.009569269	0.000206458
64	0.065779734	0.004887864	0.000068724
128	0.047944918	0.002478294	0.000023169
256	0.034628052	0.001250731	0.000007900
512	0.024852061	0.000629316	
1024	0.017757387		

Table 6.3: Errors of Richardson's extrapolation for Merton's jump diffusion

Meanwhile, we have the table of ratios of errors:

Ratios of	Maximum	1st Extrapolation	2nd Extrapolation
E1/E2	1.093895871	1.551164298	2.790157043
E2/E4	1.172160673	1.722854766	3.000418155
E4/E8	1.239999533	1.830883372	3.072913010
E8/E16	1.291338812	1.895672374	3.073387622
E16/E32	1.328126329	1.934337964	3.043413923
E32/E64	1.353956081	1.957761060	3.004161641
E64/E128	1.371985532	1.972269609	2.966234280
E128/E256	1.384568726	1.981475806	2.932846824
E256/E512	1.393367409	1.987447059	
E512/E1024	1.399533976		

Table 6.4: Error ratios of Richardson's extrapolation for Merton's jump diffusion

From Table 6.4, we see that the extrapolation method works very well and the first column of error ratio gets close to $\sqrt{2}$, the second getting closer to 2 and the third getting closer to $2\sqrt{2}$, which ensures the asymptotic results derived from Theorem 3.1.1. Also we note that the 1024-discrete maximum is even farther away from the continuous supremum, compared to most results of second extrapolation. This indicates that the extrapolation is indeed a very powerful tool to obtain a good approximation.

6.2.2 Normal Inverse Gaussian (NIG) Process

Take $T = 1, \alpha = 15, \beta = -5, \mu = 1, \delta = 0.5$, (then $\gamma = \sqrt{200}$). The expected continuous supremum is calculated through a double integral as 0.842496621. Note that all $N \geq 5$ satisfy the conditions of the theorem. Now we see the following table if we apply Theorem 6.1.1.

N	Maximum	Error
5	0.827311105	0.015185516
10	0.830937837	0.011558784
20	0.834427144	0.008069477
40	0.837217781	0.005278840
80	0.839205389	0.003291232
160	0.840516571	0.001980050

Table 6.5: Discrete maximum for NIG

From Theorem 3.3.10, the extrapolation method may not be applicable here. So we make corrections to the discrete maximum by adding the first few error terms, since we know all coefficients explicitly. In Table 6.5, we see that the pattern of the error decreasing is not very clear. Now by apply Theorem 3.3.10, i.e., make a $\frac{\log N}{N}$ correction with a coefficient $\frac{\delta}{2\pi} = \frac{1}{4\pi}$, we may achieve the following table

N	Maximum	Up to $\frac{\log N}{N}$ Correction	Error	Error Ratio
5	0.827311105	0.852926105	-0.010429484	1.5418252
10	0.830937837	0.849261227	-0.006764606	1.7569661
20	0.834427144	0.846346784	-0.003850163	1.8690547
40	0.837217781	0.844556573	-0.002059952	1.9294263
80	0.839205389	0.843564271	-0.001067650	1.9621086
160	0.840516571	0.843040755	-0.000544134	

Table 6.6: Error analysis of corrected discrete maximum for NIG up to $\frac{\log N}{N}$ term

After $\frac{\log N}{N}$ correction, the errors tend to a pattern that is more evident, i.e., if we double the number of discrete points, the error is cut in half, which ensures that the leading error term after removing the $\frac{\log N}{N}$ term is $\frac{1}{N}$. Furthermore, we may apply more corrections, i.e., $\frac{1}{N}$ and $\frac{\log N}{N^2}$ corrections with coefficients $Z_1 \approx -0.088966931$ and $Y_2 \approx 0.027468389$, respectively. We get that

N	Maximum	Up to $\frac{1}{N}$ Correction	Error
5	0.827311105	0.835132719	0.007363902
10	0.830937837	0.840364534	0.002132087
20	0.834427144	0.841898437	0.000598184
40	0.837217781	0.842332400	0.000164221
80	0.839205388	0.842452185	0.000044436
160	0.840516571	0.842484712	0.000011909

Table 6.7: Error analysis of corrected discrete maximum for NIG up to $\frac{1}{N}$ term

N	Maximum	Up to $\frac{\log N}{N^2}$ Correction	Error	Error Ratio
5	0.827311105	0.836901065	0.005595556	3.731355236
10	0.830937837	0.840997017	0.001499604	3.821000637
20	0.834427144	0.842104157	0.000392464	3.889975702
40	0.837217781	0.842395730	0.000100891	3.936633500
80	0.839205388	0.842470992	0.000025629	3.965235567
160	0.840516571	0.842490158	0.000006463	

Table 6.8: Error analysis of corrected discrete maximum for NIG up to $\frac{\log N}{N^2}$ term

Now it is much more clear that after we make corrections of the first three terms, the error is reduced to one quarter of the previous one as N gets doubled. This is a clear sign of order $\frac{1}{N^2}$, which verifies our Theorem 3.3.10.

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